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Homework 1

CS 545
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Problem 1

(a) We can write vectors a and b as,

$$a = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_\mathrm{n} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_\mathrm{m} \end{pmatrix} \tag{1}$$

Then, the left side of the equation becomes,

$$vec(a \cdot b^\mathrm{T}) = vec(\begin{bmatrix} a_1b_1 & a_1b_2 & \cdot & \cdot & \cdot & a_1b_\mathrm{m} \\ a_2b_1 & & & & & \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ a_\mathrm{n}b_1 & & & & & \end{bmatrix}) = \begin{pmatrix} a_1b_1 \\ . \\ . \\ . \\ a_\mathrm{n}b_1 \\ a_2b_1 \\ . \\ . \\ . \\ a_\mathrm{n}b_\mathrm{m} \end{pmatrix} \tag{2}$$

Similarly, the right side of the equation becomes,

$$b \otimes a = \begin{pmatrix} ab_1 \\ ab_2 \\ . \\ . \\ . \\ ab_\mathrm{m} \end{pmatrix} = \begin{pmatrix} a_1b_1 \\ . \\ . \\ . \\ a_\mathrm{n}b_1 \\ a_2b_1 \\ . \\ . \\ . \\ a_\mathrm{n}b_\mathrm{m} \end{pmatrix} \tag{3}$$

Investigating (2) and (3) concludes that,

$$vec(a \cdot b^\mathrm{T}) = b \otimes a \tag{4}$$

(b) The matrices A and B can be written as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{m1} & \cdot & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ b_{m1} & \cdot & \cdot & \cdot & \cdot & b_{mn} \end{bmatrix} \tag{5}$$

Then, the left side of the equation can be written as,

$$vec(A)^{\mathrm{T}} vec(B) = sum(\begin{pmatrix} a_{11} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \\ a_{12} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{pmatrix} \odot \begin{pmatrix} b_{11} \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} \\ b_{12} \\ \cdot \\ \cdot \\ \cdot \\ b_{m2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{mn} \end{pmatrix}) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} b_{ij} \tag{6}$$

Similarly, the right side of the equation can be written as,

$$tr(A^{\mathrm{T}} B) = tr(\begin{bmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{1n} & \cdot & \cdot & \cdot & \cdot & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ b_{m1} & \cdot & \cdot & \cdot & \cdot & b_{mn} \end{bmatrix}) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} b_{ij} \tag{7}$$

Investigating (6) and (7), reveals that,

$$vec(A)^{\mathrm{T}} vec(B) = tr(A^{\mathrm{T}} B) \tag{8}$$

(c) We need to prove symmetry first, so taking the transpose

$$X^{\mathrm{T}}X = XX^{\mathrm{T}} = (X^{\mathrm{T}}X)^{\mathrm{T}} \tag{9}$$

Since the transpose of the multiplied quantity is equal to itself we can say that $\mathrm{X}^{\mathrm{T}}\mathrm{X}$ is symmetric. Similarly, we know that the initial matrix is real and the resulting matrix is,

$$(X^{\mathrm{T}}X)_{\mathrm{ij}} = {x_{\mathrm{ij}}}^2 \tag{10}$$

Since, the resulting matrix is real and symmetric and that makes our eigenvalues positive as well. Hence, we can write,

$$(X^{\mathrm{T}}X)a = \lambda a \tag{11}$$

Multiplying with a^{T},

$$a^{\mathrm{T}}(X^{\mathrm{T}}X)a = a^{\mathrm{T}}\lambda a \tag{12}$$

Since we have a positive λ, then we infer from equation (12) that $\mathrm{X}^{\mathrm{T}}\mathrm{X}$ is positive semi-definite.

Problem 2

We are asked to calculate P(C | M), then

$$P(C|M) = \frac{P(M|C)P(C)}{P(M)} \tag{13}$$

We already know that P(M|C)= 0.9 and P(C)=0.007. So we need to calculate P(M) with total probability rule.

$$P(M) = P(M|C)P(C) + P(M|C^{'})P(C^{'}) \tag{14}$$

Where P(M | $\mathrm{C}^{'}$) and P($\mathrm{C}^{'}$) represents when the patient is not cancer. Then,

$$P(M) = P(M|C)P(C) + P(M|C^{'})P(C^{'}) = 0.9 \cdot 0.007 + 0.08 \cdot 0.993 = 0.08574 \tag{15}$$

Then,

$$P(C|M) = \frac{P(M|C)P(C)}{P(M)} = \frac{0.9 \cdot 0.007}{0.08574} = 0.0735 = 7.35\% \tag{16}$$

Problem 3

1. (a) We need to multiply our matrix with the following vector (M with a size of Kx1) to take the mean for each pixel;

$$M = \frac{1}{K}\begin{pmatrix}1\\1\\.\\.\\.\\1\end{pmatrix} \tag{17}$$

Then, the mean image (MI) becomes (also converting with vec-transpose);

$$MI = vec(I\frac{1}{K}\begin{pmatrix}1\\1\\.\\.\\.\\1\end{pmatrix})^{(\mathrm{M})} \tag{18}$$

(b) First, we need vectors that will extract the top and bottom average of the image from M $\times$ N vector. For that purpose, we construct the following matrix:

$$S = \begin{pmatrix}1 & 0\\1 & 0\\. & .\\0 & 1\\0 & 1\\. & .\\1 & 0\\1 & 0\\. & .\\0 & 1\\0 & 1\end{pmatrix} \tag{19}$$

Here, the first column extracts the upper half of the image since it is always 1 between the rows (i-1)$\times$M+1 and i$\times$M/2 where i is changing from 1 to N. Similarly, the second column extracts the lower half of the image since it is always 1 between the rows i$\times$M/2+1 and i$\times$M where i is changing from 1 to N. Then, we calculate the average top and bottom half image (Z with a size of K$\times$ 2);

$$Z = I^{\mathrm{T}} \frac{2}{M \times N} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ . & . \\ 0 & 1 \\ 0 & 1 \\ . & . \\ 1 & 0 \\ 1 & 0 \\ . & . \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{20}$$

We divide by M×N/2 to find the mean. Then, we calculate the mean values of the K images.

$$\mu = Z^{\mathrm{T}} \frac{1}{K} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \tag{21}$$

We can now calculate the covariance matrix easily as,

$$\sigma = Z^{\mathrm{T}} Z \frac{1}{K} - \mu \mu^{\mathrm{T}} \tag{22}$$

2. (a) We first convert our tensor (CI) into vector by,

$$CIV = vec(CI) \tag{23}$$

Then, we reformulate our vector as ((M×N)×(K×3)),

$$CIN = vec(CIV)^{(3\mathrm{K})} \tag{24}$$

We can now calculate the mean as (M_3 is 3Kx1),

$$M_3 = \frac{1}{3K} \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \tag{25}$$

$$\text{All channel mean} = CINM_3 \tag{26}$$

We then reshape our image as,

$$\text{All channel mean}_{\text{M}\times\text{N}} = vec(CINM_3)^{(\text{M})} \tag{27}$$

(b) For the red channel, we can use the same vectors as in the part (a) with a slight modification. Since, first channel will be red, we just need a vector that looks like,

$$M_{\text{red}} = \frac{1}{K}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \tag{28}$$

Then,

$$\text{Red channel mean} = CINM_{\text{red}} \tag{29}$$

We then reshape our image as,

$$\text{Red channel mean}_{\text{M}\times\text{N}} = vec(CINM_{\text{red}})^{(\text{M})} \tag{30}$$

Problem 4

To calculate the windowed DFT, we first create a matrix with a size of (N/32-1)×64 by n where n is the size of the input vector. This matrix contains (N/32-1) 64x64 DFT matrices that are shifted from each other by 64 in the vertical direction (row) and 32 in the horizontal direction (column). To implement the Hann function, we also scale the rows of the DFT matrix by the Hann function which can be written as:

$$DFT(i,:)_{\text{scaled}} = DFT(i,:) \odot Hann \tag{31}$$

This element-wise scaling helps us to implement the Hann function. We then plot our A matrix in Fig. 1.

After multiplying the vector with matrix A, we can convert the resulting vector into a matrix by vec-transpose function,

$$S_{\text{mat}} = vec(S_{\text{vec}})^{(\text{Number of Windows})} \tag{32}$$

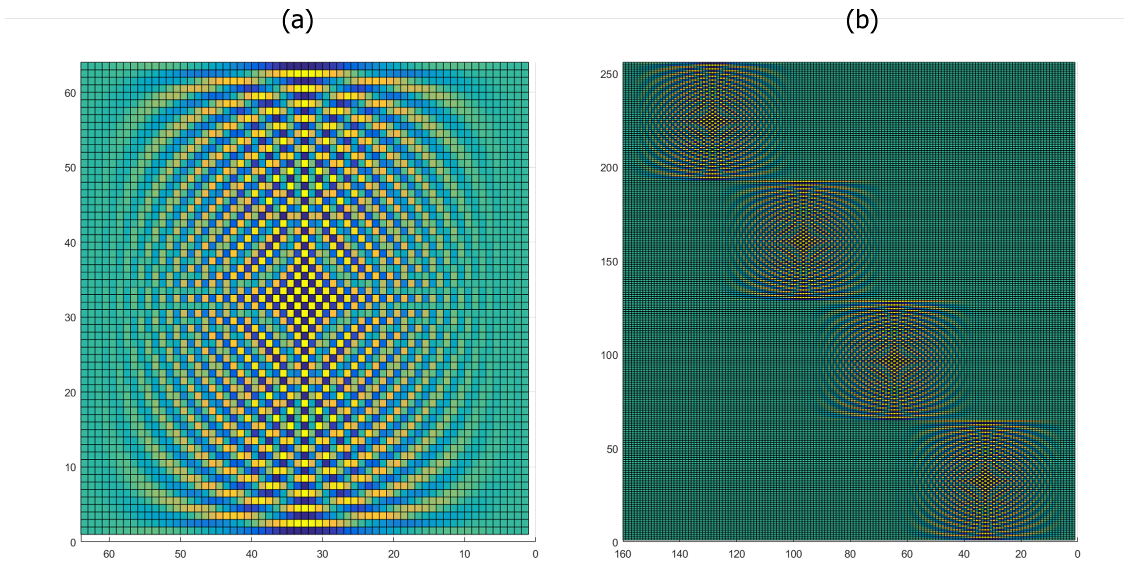

Figure 1: (a) Single unit cell of the A matrix (real part) (b) A matrix showing the placement of the individual scaled DFT matrices

The resulting matrix has columns for different windows , and then we can calculate the spectrogram by multiplying the each element of the matrix by its conjugate and then converting to log scale,

$$S_{\text{spec}} = 10log_{10}(S_{\text{mat}} \odot {S_{\text{mat}}}^{*}) \tag{33}$$

Then, I tried to test my code with two examples and they are given in Fig. 2 and Fig. 3. They are compared with the Matlab's built in spectrogram function. We see that our matrix produces very close spectrograms (actually the same) with the built-in functions.

We also plot the sound signal below,

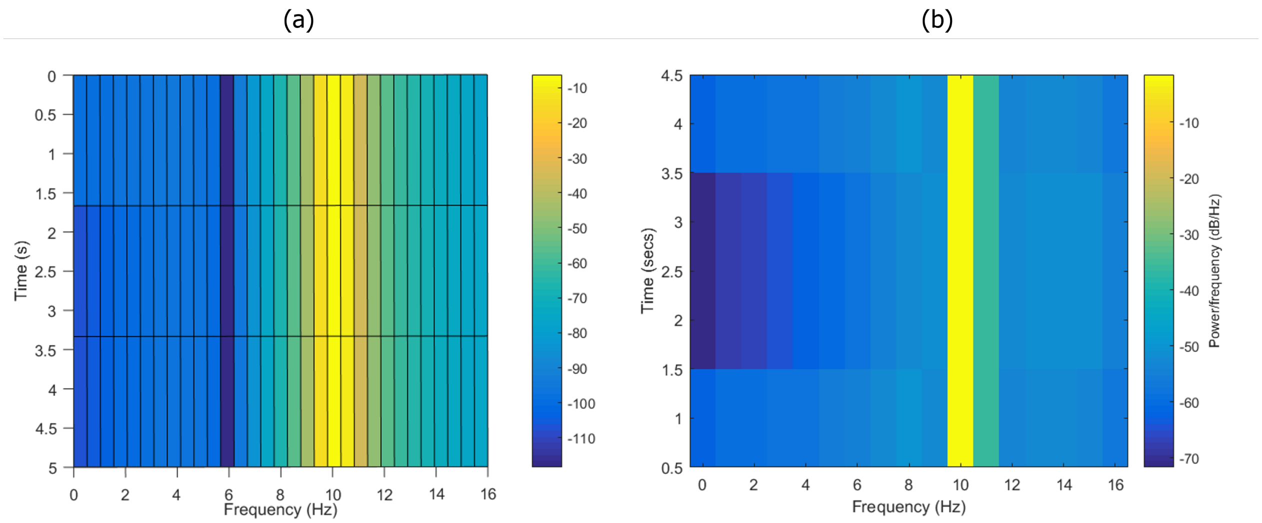

Figure 2: (a) Spectrogram of a sine wave with 10 Hz calculated with our matrix (b) Matlab spectrogram

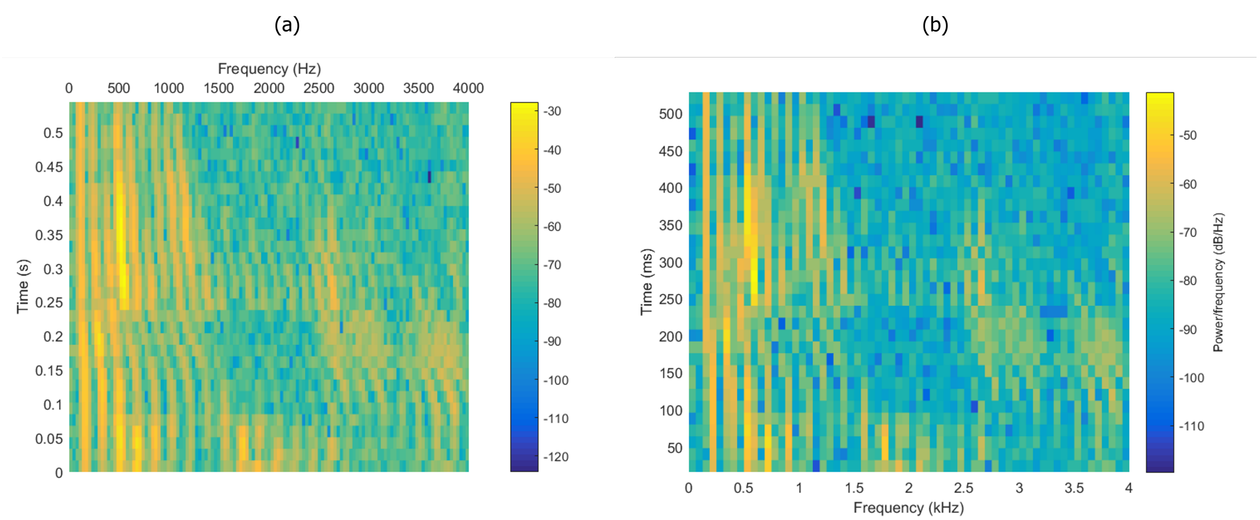

Figure 3: (a) Spectrogram of a "Hello" speech calculated with our matrix (b) Matlab spectrogram

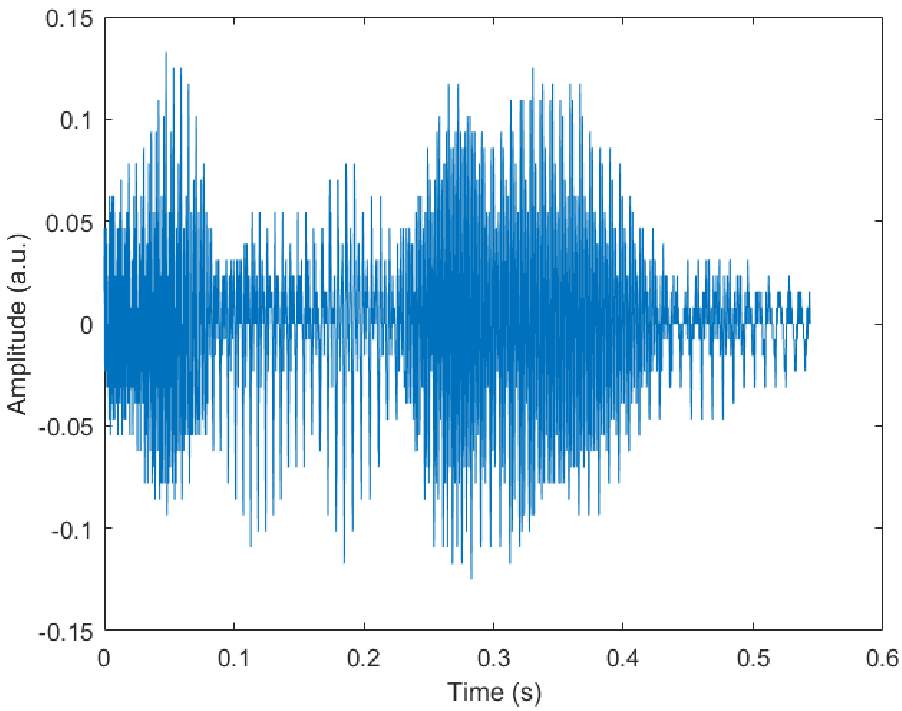

Figure 4: (a) Input sound