The Heat Equation

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(1)

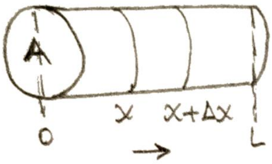
$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (*)$$ is the linear parabolic differential equation

The general form of the linear PDE is given by

$$a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial^2 u}{\partial x \partial t} + c(x,t)\frac{\partial^2 u}{\partial t^2} = f\left(x,t,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right)$$

$b^2 - 4ac = (0)^2 - 4(a^2)(0) = 0$. Therefore, $(*)$ is a parabolic equation.

Derivation of Heat Equation



Uniform cross-section A

Length L

Constant density ρ

$u(x,t)$: Temperature

the specific heat of the rod c

the thermal conductivity of rod K

the amount of heat $\Rightarrow$ $Q = \int_x^{x+\Delta x} c\rho A u(s,t)\,ds$

the gradient of the temperature $\Rightarrow$ $-KA\frac{\partial u(x,t)}{\partial x} \rightarrow \underbrace{+KA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]}_{\text{Heat flow through the cross section}}$

$$\frac{\partial Q}{\partial t} = \int_x^{x+\Delta x} c\rho A \frac{\partial u(s,t)}{\partial t}\,ds = KA\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]$$

$$\left(\int_x^{x+\Delta x} \frac{\partial u(s,t)}{\partial t}\,ds = \frac{\partial u(\xi,t)}{\partial t}\Delta x \quad \text{where} \quad x < \xi < x+\Delta x\right)$$

$$\frac{\partial u(\xi,t)}{\partial t} = \frac{KA}{c\rho A}\underbrace{\left[\frac{\partial u(x+\Delta t,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]\cdot\frac{1}{\Delta x}}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{K}{c\rho}\frac{\partial u^2(x,t)}{\partial x^2} \rightarrow \boxed{\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}} \quad \text{where } a^2 = \frac{K}{c\rho}$$

a^2 is the diffusivity within the solid (constant)

Nonhomogeneous version of the heat equation may be derived by adding the term $\int_x^{x+\Delta x} f(s,t)\,ds$ to the time derivative part of the equation. Then, the equation below is reached.

$$\frac{\partial u(x,t)}{\partial t} - a^2\frac{\partial^2 u(x,t)}{\partial x^2} = F(x,t)$$

where the source of density $F(x,t) = \frac{f(x,t)}{c\rho}$

Initial and Boundary Conditions

A Boundary Value Problem with Dirichlet Boundary Conditions

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad ; 0 < x < L, \ t > 0$$

$$u(x,0) = f(x) \ ; \ 0 < x < L$$

$$u(0,t) = u(L,t) = 0; \ t > 0$$

The boundary conditions may be defined by $u_x(0,t) = u_x(L,t) = 0$. This time, the problem becomes "Neumann type boundary conditions' BVP."

The last type of the definition of the boundary conditions,

$$\frac{\partial u(0,t)}{\partial x} - h u(0,t) = \text{constant} \qquad h > 0$$

$$\frac{\partial u(L,t)}{\partial x} + h u(L,t) = \text{another constant}$$

are called Robin problem. (Linear combination of Dirichlet and Neuman Conditions)

Separation of Variables

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It is a ability to express the solution $u(x,t) = X(x)T(t)$.

Example (Homogeneous Heat Equation)

$$u_t = a^2 u_{xx} \; ; \; 0 < x < L, \; t > 0$$

$$u(x,0) = f(x) \; ; \; 0 < x < L$$

$$u(0,t) = u(L,t) = 0 \; ; \; t > 0$$

Boundary conditions

$$u(0,t) = X(0)T(t) = 0$$
$$u(L,t) = X(L)T(t) = 0$$
$$\Rightarrow X(0) = X(L) = 0.$$

$$u(x,t) = X(x)T(t) \rightarrow u_t = XT' \text{ and } u_{xx} = X''T$$

$$u_t = a^2 u_{xx} \rightarrow XT' = a^2 X''T \rightarrow \frac{T'}{a^2T} = \frac{X''}{X} = -\lambda$$

We should check (3) possible values of λ

1) $\lambda = 0$

$$\frac{T'}{a^2T} = \frac{X''}{X} = 0 \longrightarrow X(x) = mx + n$$
$$X(0) = n$$
$$X(L) = Lm = 0$$ m or L is equal to zero

This gives a trivial solution.

2) $\lambda < 0$, $\lambda = -\mu^2$

$$\frac{T'}{a^2T} = \frac{X''}{X} = -(-\mu^2) = \mu^2 \rightarrow X'' - \mu^2 X = 0 \rightarrow X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$X(0) = c_1 + c_2 = 0 \rightarrow c_1 = -c_2$$

$$X(L) = c_1 e^{\mu L} + c_2 e^{-\mu L} = c_1 \underbrace{(e^{\mu L} - e^{-\mu L})}_{\neq 0} = 0 \rightarrow c_1 = 0, c_2 = 0.$$

This also gives a trivial solution.

3) $\lambda > 0$, $\lambda = \mu^2$

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\mu^2 \rightarrow X'' + \mu^2 X = 0 \rightarrow X(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x)$$

$$X(0) = c_2 \underbrace{\cos(\mu 0)}_{1} = 0 \rightarrow c_2 = 0$$

$$X(L) = c_1 \underbrace{\sin(\mu L)}_{\pi n} = 0 \quad \mu L = n\pi \quad n = 1, 2, 3, \ldots$$

$$\mu = \frac{n\pi}{L}$$

We find $\mu = \frac{n\pi}{L}$ and $\lambda = \mu^2 = \frac{n^2\pi^2}{L^2}$ as "eigenvalues" of the problem (3)

We can also write $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$. Now, we are ready to find "time"

$$\frac{T'}{a^2 T} = -\mu^2 \rightarrow \frac{T'}{T} = -\mu^2 a^2 \rightarrow T(t) = e^{-\mu^2 a^2 t} \rightarrow T_n(t) = e^{-a^2\left(\frac{n^2\pi^2}{L^2}\right)t}$$

We defined a solution $u_n(x,t) = T_n(t)\, u_n(x) = e^{-a^2\left(\frac{n^2\pi^2}{L^2}\right)t} \cdot \sin\left(\frac{n\pi}{L}x\right)$ which is also called "eigenfunctions" of the problem.

The linear sum of the eigenfunctions of the problem gives the solution.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-a^2\left(\frac{n^2\pi^2}{L^2}\right)t}.$$

By imposing the initial condition, we may find B_n values.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

By using a Fourier half-range sine series for $f(x)$ on $(0,L)$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

As a numerical example, taking $L=\pi$, $u(x,0) = x(\pi - x)$

then we may calculate B_n's $B_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx$

$$B_n = \frac{2}{n}\left\{(x^2-\pi x)\frac{1}{n}\cos(nx) + (\pi - 2x)\frac{1}{n^2}\sin(nx) - \frac{2}{n^3}\cos(nx)\right\}\Bigg|_0^{\pi}$$

$+\ (\pi x - x^2)$	$\sin(nx)$
$-\ (\pi - 2x)$	$-\frac{1}{n}\cos(nx)$
$+\ (-2)$	$-\frac{1}{n^2}\sin(nx)$
$-\ 0$	$+\frac{1}{n^3}\cos(nx)$

$$= \frac{2}{\pi}\left\{-\frac{2}{n^3}\cos(n\pi) + \frac{2}{n^3}\right\} = B_n$$

$$B_n = \frac{4}{n^3\pi}\left\{1 - \cos(n\pi)\right\} = \frac{4}{n^3\pi}\left\{1 - (-1)^n\right\}$$

then $u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4}{n^3\pi}\left\{1-(-1)^n\right\}\right]\sin(nx)\, e^{-a^2 n^2 t}$ by putting $2n-1 \rightarrow n$

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{8}{(2n-1)^3\pi}\right]\sin\left((2n-1)x\right) e^{-a^2(2n-1)^2 t}$$

Example (Insulated by one side)

$$u_t = a^2 u_{xx}\,;\ 0<x<L,\ t>0$$

$$u(x,0) = x\,;\ 0<x<L$$

$$u_x(0,t) = u(L,t) = 0,\ t>0$$

The condition $u_x(0,t)=0$ express mathematically the constraint that no heat flows through the left boundary.

$$u(x,t) = X(x)T(t) \rightarrow X'(0) = 0 \text{ and } X(L) = 0.$$

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\lambda \rightarrow X''+\mu^2X=0 \rightarrow X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$$

$$X'(x) = -c_1\mu\sin(\mu x) + c_2\mu\cos(\mu x)$$

$$X'(0) = c_2\mu = 0 \rightarrow c_2 = 0$$

$$X(L) = c_1\cos(\underbrace{\mu L}_{\frac{(2n-1)}{2}\pi}) = 0 \rightarrow \mu = \frac{(2n-1)}{2}\cdot\frac{\pi}{L} \rightarrow \lambda = \mu^2 = \frac{(2n-1)^2}{4L^2}\pi^2$$

$$X_n(x) = \cos\left(\frac{(2n-1)}{2L}\pi x\right)$$

$$\frac{T'}{aT} = -\lambda = -\frac{(2n-1)^2}{4L^2}\pi \rightarrow T_n(t) = e^{-a\frac{(2n-1)^2}{4L^2}\pi^2 t} \text{ then } u_n(x,t) = \cos\left(\frac{(2n-1)}{2L}\pi x\right)e^{-\frac{a(2n-1)^2\pi^2 t}{4L^2}}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n\cos\left(\frac{2n-1}{2L}\pi x\right)e^{-\frac{a(2n-1)^2\pi^2}{4L^2}t}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n\cos\left(\frac{2n-1}{2L}\pi x\right) = x.$$

$$B_n = \frac{\int_0^L x\cos\left(\frac{2n-1}{2L}\pi x\right)dx}{\int_0^L \cos^2\left(\frac{2n-1}{2L}\pi x\right)dx} = \frac{-8L}{(2n-1)^2\pi^2} - \frac{4L(-1)^n}{(2n-1)\pi}$$

The final solution: $u(x,t) = -\frac{4L}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{(2n-1)^2\pi} + \frac{(-1)^n}{2n-1}\right]\cos\left[\frac{(2n-1)\pi x}{2L}\right]e^{-\frac{(2n-1)^2\pi^2a^2t}{4L^2}}$

Example1 $u_t = a^2 u_{xx}$; $0 < x < \pi$, $t > 0$

$u(0,t) = u(\pi,t) = 0$, $t > 0$

$u(x,0) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.$

$u(x,t) = X(x)T(t)$ then $X(0) = X(\pi) = 0$

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\lambda = -\mu^2$$

$X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$

$X(0) = c_1 = 0$, $X(\pi) = c_2\sin(\mu\pi) = 0$ $\quad \mu\pi = n\pi \rightarrow \mu = n,\ \lambda = n^2$

$$\left.\begin{array}{l} X_n(x) = \sin(nx) \\ \frac{T'}{a^2T} = -n^2 \rightarrow \quad T_n(t) = e^{-a^2n^2t} \end{array}\right\} \quad u_n(x,t) = e^{-a^2n^2t}\sin(nx)$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-a^2n^2t}\sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n\sin(nx) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$B_1 = 3/4 \quad B_3 = -1/4$

then $\boxed{u(x,t) = \frac{3}{4}\sin(x)\,e^{-a^2t} - \frac{1}{4}\sin(3x)\,e^{-9a^2t}}$

The Fourier Transformation Method for The Heat Equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \; t > 0$$

$$u(x,0) = f(x)$$

The solution is given by $u(x,t) = \left(\frac{1}{4a^2\pi t}\right)^{1/2} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(\xi - x)^2}{4a^2 t}}\, d\xi$

is the final form for the temperature distribution.

Example 1. $u(x,0) = \begin{cases} T_0, & t > 0 \\ -T_0, & t < 0 \end{cases}$

$$\tau = \frac{\xi - x}{2a\sqrt{t}}$$

$$u(x,t) = \frac{-T_0}{\sqrt{4a^2\pi t}} \int_{-\infty}^{0} e^{-\frac{(\xi-x)^2}{4a^2 t}}\, d\xi + \frac{T_0}{\sqrt{4a^2\pi t}} \int_{0}^{\infty} e^{-\frac{(\xi-x)^2}{4a^2 t}}\, d\xi$$

$$= \frac{T_0}{\sqrt{\pi}} \left[\int_{-x/2a\sqrt{t}}^{\infty} e^{-\tau^2}\, d\tau - \int_{x/2a\sqrt{t}}^{\infty} e^{-\tau^2}\, d\tau \right]$$

$$= \frac{T_0}{\sqrt{\pi}} \left[\int_{-x/2a\sqrt{t}}^{+x/2a\sqrt{t}} e^{-\tau^2}\, d\tau \right] = \frac{2T_0}{\sqrt{\pi}} \int_{0}^{x/2a\sqrt{t}} e^{-\tau^2}\, d\tau$$

$$= T_0 \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right)$$

Example 2

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad ; -\infty < x < \infty, t > 0$$

$$u(x,0) = \begin{cases} 1, & |x| < b \\ 0, & |x| > b \end{cases}$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

$$u(x,t) = \frac{1}{\sqrt{4a^2\pi t}} \int_{-b}^{b} e^{-\frac{(\xi - x)^2}{4a^2 t}} d\xi$$

$$\frac{\xi - x}{2a\sqrt{t}} = \tau$$

$$\frac{d\xi}{2a\sqrt{t}} = d\tau$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{-b-x}{2a\sqrt{t}}}^{0} e^{-\tau^2} dz + \frac{1}{\sqrt{\pi}} \int_0^{\frac{b-x}{2a\sqrt{t}}} e^{-z^2} d\tau$$

$$= \frac{1}{2} \Bigg[\underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\frac{b+x}{2a\sqrt{t}}} e^{-z^2} dz}_{} + \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\frac{b-x}{2a\sqrt{t}}} e^{-z^2} dz}_{} \Bigg]$$

$$\frac{1}{2} \operatorname{erf}\left(\frac{b+x}{2a\sqrt{t}}\right) + \frac{1}{2} \operatorname{erf}\left(\frac{b-x}{2a\sqrt{t}}\right)$$

Example 3

Solve $\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) \quad 0 \le r < \infty,\ t > 0$

1) Assume $v(r,t) = r\,u(r,t)$

$$\frac{\partial v}{\partial t} = r\frac{\partial u}{\partial t}, \quad \frac{\partial v}{\partial r} = u(r,t) + r\frac{\partial u}{\partial r}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2} = 2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}$$

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial v}{\partial t}, \quad \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2}$$

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial v}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = a^2\frac{1}{r}\frac{\partial^2 v}{\partial r^2}$$

$$\rightarrow \frac{1}{r}\frac{\partial v}{\partial t} = a^2\frac{1}{r}\frac{\partial^2 v}{\partial r^2} \Rightarrow \boxed{\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial r^2} \quad \text{with } v(r,0) = r u_0(r)}$$