The Wave Equation

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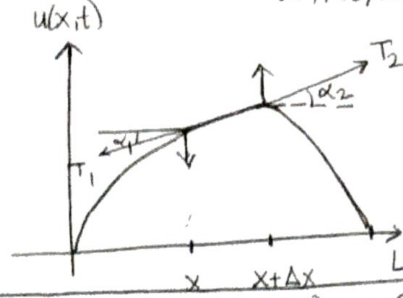
The one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$



The Vibrating String

Length: L, density: ρ

Equating x-direction forces to zero by writing $\boxed{-T(x)\cos(\alpha_1) + T(x+\Delta x)\cos(\alpha_2) = 0}$

$$T(x)\cos(\alpha_1) = T(x+\Delta x)\cos(\alpha_2) = T \text{ (constant)}$$

Equating y-direction forces to acceleration. (Newton's Second Law)
Assuming only external force is gravity.

$$-T(x)\sin(\alpha_1) + T(x+\Delta x)\sin(\alpha_2) - mg = m\frac{\partial^2 u}{\partial t^2} \quad (*)$$

We can write $T(x) = \frac{T}{\cos\alpha_1}$ and $T(x+\Delta x) = \frac{T}{\cos\alpha_2}$ then $(*)$ becomes

$$-T\tan(\alpha_1) + T\tan(\alpha_2) - \rho g\Delta x = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

$$\tan(\alpha_1) = \frac{\partial u(x,t)}{\partial x}, \quad \tan(\alpha_2) = \frac{\partial u(x+\Delta x,t)}{\partial x}$$

$$T\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] = \rho\Delta x\left(\frac{\partial^2 u}{\partial t^2} + g\right)$$

$$\frac{T}{\Delta x}\left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] = \rho\left(\frac{\partial^2 u}{\partial t^2} + g\right) \quad (**)$$ becomes as $\Delta x \to 0$

$$T\frac{\partial^2 u(x,t)}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} + \rho g \rightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\rho g}{T}$$

then $c^2 = T/\rho$ we get $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{g}{c^2}$

By neglecting last term, we get $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$ which is called one-dimensional wave equation.

Cauchy problem: Finding solutions that satisfy the initial conditions (initial data) is called the "Cauchy Problem". For the Wave Equation, we are required to specify two conditions because the equation has two time derivatives.

Separation of Variables 24/08/2023

Assuming $u(x,t) = X(x)T(t)$ then solve the following problem.

$$u_{tt} = c^2 u_{xx}; \quad 0 < x < L, \ t > 0$$

$$u(x,0) = f(x); \quad 0 < x < L$$

$$u_t(x,0) = g(x); \quad 0 < x < L$$

$$u(0,t) = u(L,t) = 0; \quad t > 0$$

$$XT'' = c^2 X'' T \rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda = -\mu^2$$

$u(0,t) = X(0)T(t) = 0$

$u(L,t) = X(L)T(t) = 0$

$X(0) = X(L) = 0.$

$\lambda = \mu^2 > 0$

For X

$$X'' + \mu^2 X = 0 \rightarrow X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$$

$$X(0) = c_1 = 0 \checkmark \quad X(L) = c_2 \sin(\underbrace{\mu L}_{n\pi}) = 0 \Rightarrow \mu = \frac{n\pi}{L}$$

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \quad \lambda = \frac{n^2\pi^2}{L^2}$$

For t $\quad T'' + c^2\mu^2 T = 0 \rightarrow T'' + \frac{c^2 n^2 \pi^2}{L^2} T = 0$

$$T_n(t) = A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)$$

$$u_n(x,t) = X_n T_n = \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$ and the derivative is

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[-A_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi c}{L}t\right) + B_n\left(\frac{n\pi c}{L}\right)\cos\left(\frac{n\pi c}{L}t\right)\right]$$

Now apply the initial conditions on $u(x,t)$ and $u_t(x,t)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

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$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

$$B_n \left(\frac{n\pi c}{\cancel{L}}\right) = \frac{2}{\cancel{L}} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$

As a numerical example

$$f(x) = \begin{cases} 0, & 0 < x \leq L/4 \\ 4h\left(\frac{x}{L} - \frac{1}{4}\right), & L/4 \leq x \leq L/2 \\ 4h\left(\frac{3}{4} - \frac{x}{L}\right), & L/2 \leq x \leq 3L/4 \\ 0, & 3L/4 \leq x \leq L \end{cases} \quad \text{and} \quad g(x) = 0$$

Compute A_n and B_n. $B_n = 0$ $\forall n$ because $g(x) = 0$. We can only find $A_n \neq 0$.

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\left(\int_{L/4}^{L/2} 4h\left(\frac{x}{L} - \frac{1}{4}\right) \sin\left(\frac{n\pi x}{L}\right) dx + \int_{L/2}^{3L/4} 4h\left(\frac{3}{4} - \frac{x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \right)$$

doing some calculations, we get $A_n = \frac{32h}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{8}\right)$

Then the solution,

$$\boxed{u(x,t) = \frac{32h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{n\pi}{8}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right)}$$

D'Alembert's Formula

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The Homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad , -\infty < x < \infty, t > 0$$

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad -\infty < x < \infty$$

Introducing new variables $\xi = x + ct$ and $\eta = x - ct$ to transfrom $u(x,t)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial}{\partial \eta} \underbrace{\frac{\partial \eta}{\partial x}}_{1} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial^2}{\partial \xi \partial \eta} \underbrace{\frac{\partial \eta}{\partial x}}_{1} + \frac{\partial^2}{\partial \eta^2} \underbrace{\frac{\partial \eta}{\partial x}}_{1} + \frac{\partial^2}{\partial \eta \partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{1} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial t^2} = c\left(\frac{\partial^2}{\partial \xi^2} \frac{\partial \xi}{\partial t} + \frac{\partial^2}{\partial \xi \partial \eta} \frac{\partial \eta}{\partial t}\right) - c\left(\frac{\partial^2}{\partial \eta^2} \frac{\partial \eta}{\partial t} + \frac{\partial^2}{\partial \eta \partial \xi} \cdot \frac{\partial \xi}{\partial t}\right)$$

$$= c\left(\frac{\partial^2}{\partial \xi^2} c - \frac{\partial^2}{\partial \xi \partial \eta} \cdot c\right) - c\left(\frac{\partial^2}{\partial \eta^2} (-c) + \frac{\partial^2}{\partial \eta \partial \xi} \cdot c\right)$$

$$= c^2 \frac{\partial^2}{\partial \xi^2} - 2c^2 \frac{\partial^2}{\partial \xi \partial \eta} + c^2 \frac{\partial^2}{\partial \eta^2} = c^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)$$

Putting $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$ into the equation

$$c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) = c^2\left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)$$

$$\rightarrow \quad \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad \text{then} \quad u(\xi, \eta) = F(\xi) + G(\eta)$$

$$\xi \rightarrow x + ct, \quad \eta \rightarrow x - ct$$

$$u(x,t) = F(x + ct) + G(x - ct)$$

Now, we should impose the initial values.

$$u(x,t) = F(x+ct) + G(x-ct) \qquad u_t(x,t) = cF'(x+ct) - cG'(x-ct)$$

$$u(x,0) = F(x) + G(x) = f(x) \qquad u_t(x,0) = cF'(x) - cG'(x) = g(x)$$

$$\begin{array}{rl} & cF'(x) + cG'(x) = cf'(x) \\ \mp & cF'(x) - cG'(x) = g(x) \\ \hline \end{array}$$

$$\begin{cases} 2cF'(x) = cf'(x) + g(x) \rightarrow F'(x) = \frac{1}{2}f'(x) + \frac{1}{2c}g(x) \\ \boxed{F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\tau)\,d\tau + \frac{c}{2}} \end{cases}$$

$$\begin{cases} 2cG'(x) = cf'(x) - g(x) \rightarrow G'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x) \\ \boxed{G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\tau)\,d\tau - \frac{c}{2}} \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau \rightarrow \text{D'Alembert's Formula.}$$

As a numerical example

$$u(x,0) = f(x) = \frac{1}{x^2+1} \quad \text{and} \quad u_t(x,0) = g(x) = e^x$$

$$u(x,t) = \frac{1}{2}\left[\frac{1}{(x+ct)^2+1} + \frac{1}{(x-ct)^2+1}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} e^{\tau}\,d\tau$$

$$u(x,t) = \frac{1}{2}\left[\frac{1}{(x+ct)^2+1} + \frac{1}{(x-ct)^2+1}\right] + \frac{1}{2c}\left[e^{x+ct} - e^{x-ct}\right]$$