Complex Variables Ogulcan. O. 27/08/2023

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the complex variable $z = x + iy$

the Real part $\mathbb{R}(z) = x$; the complex part $\mathrm{Im}(z) = y$

the conjugate $\bar{z}$ or z^* ; $\bar{z} = x - iy$

Addition $a + ib + c + id = (a+c) + i(b+d)$

Subtraction $(a+ib) - (c+id) = (a-c) + i(b-d)$

Multiplication $(a+ib)(c+id) = ac - bd + i(ad+bc)$

Division $\frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

The absolute value or modulus for $a+ib$; $|a+ib| = \sqrt{a^2+b^2}$

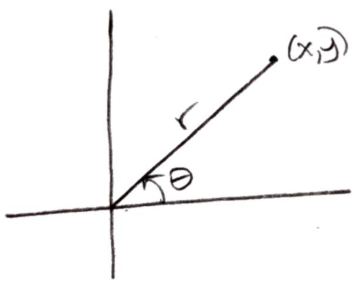
Properties: 1) $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$

2) $|z_1/z_2| = |z_1|/|z_2|$ given $|z_2| \neq 0$

3) $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$

4) $|z_1| - |z_2| \leq |z_1 + z_2|$

The Complex Plane and Representation



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \end{aligned}\right\} \quad z = x + iy = r[\cos\theta + i\sin\theta]$$

Euler's Formula

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(\theta i)^k}{k!} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin\theta}$$

then we get, $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

we know $z = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$$z^n = r^n e^{i\theta n} = r^n[\cos(n\theta) + i\sin(n\theta)]$$

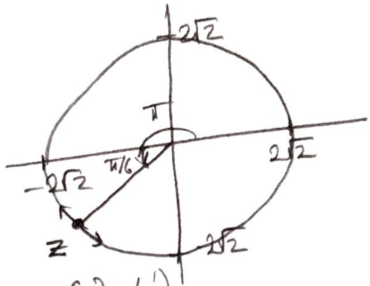
This $\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$ is De Moivre's Theorem

Example: $z = -\sqrt{6} - i\sqrt{2}$

$r = \sqrt{6+2} = 2\sqrt{2}$ $\quad \theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$ or $7\pi/6$ (3^{rd} quadrant)

$-\sqrt{6} - i\sqrt{2} = e^{7\pi i/6}(2\sqrt{2})$ Not unique!

$z = 2\sqrt{2}\, e^{\left(\frac{7\pi}{6} + 2\pi n\right)i} = -\sqrt{6} - i\sqrt{2}$



Example: $\dfrac{5+5i}{3-4i} + \dfrac{20}{3+4i} = \dfrac{(5+5i)(3+4i) + 20(3-4i)}{25}$

$3+4i \quad 3-4i$

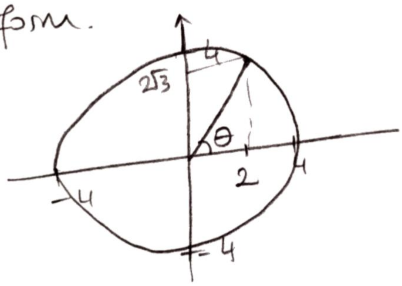
$= \dfrac{-5+35i+60-80i}{25} = \dfrac{55-45i}{25} = \dfrac{11-9i}{5}$ ☒

Example: Represent $z = 2+2\sqrt{3}i$ as polar form.

$r = |z| = \sqrt{2^2 + (2\sqrt{3})^2} = 4$

$\tan^{-1}(\sqrt{3}) = \pi/6$

$z = 4e^{i(\pi/6 + 2\pi n)} = 2 + 2\sqrt{3}i$



Example: Find cosine and sine formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)} = e^{i\alpha}\cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta - \sin\alpha\sin\beta$

$= \underbrace{\cos\alpha\cos\beta - \sin\alpha\sin\beta}_{\cos(\alpha+\beta)} + i\underbrace{(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}_{\sin(\alpha+\beta)}$

then $\boxed{\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}}$

<u>Finding Roots</u> 27/08/2023

(3)

$w^n = z$ where $z = re^{i\varphi}$ and $w = Re^{i\phi}$

$w^n = R^n e^{in\phi} = re^{i\varphi} = z \rightarrow R^n = r,\ n\phi = \varphi + 2\pi k \quad k = 0, \pm 1, \pm 2 \ldots$

$R = r^{1/n}$ and $\phi = \frac{\varphi}{n} + \frac{2\pi k}{n} \quad k = 0, \pm 1, \pm 2, \ldots$

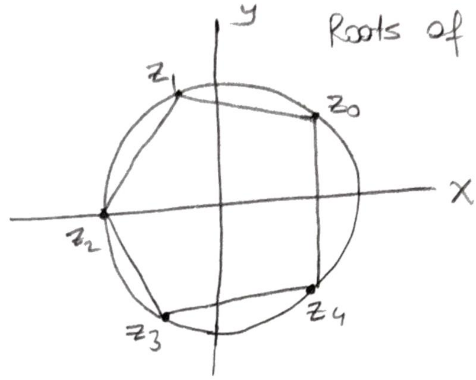
<u>Example</u>

$z^5 = -32 = 32 e^{i\pi} = 2^5 e^{\pi i} = R^5 e^{i5\phi}$

$n = 5 \quad \phi = \frac{\pi}{5} + \frac{2\pi k}{5}$

$z_k = 2e^{i\left(\frac{\pi}{5} + \frac{2\pi}{5}k\right)}; \ k = 0, 1, 2, 3, 4.$

Roots of $z^5 = -32$



<u>Example</u>

Solve all roots of $z^4 + 6iz^2 + 16 = 0$

$z^4 + 6iz^2 + 16 = (z^2 - 2i)(z^2 + 8i) = 0$

1) $z^2 = 2i \Rightarrow r^2 e^{i2\phi} = 2\left(i\sin\left(\frac{\pi}{2}\right)\right) = 2\left(e^{i\frac{\pi}{2}}\right)$

$r^2 = 2,\ r = \sqrt{2} \qquad 2\phi = \frac{\pi}{2} + 2\pi k$

$\phi = \frac{\pi}{4} + \pi k \quad k = 0, 1.$

$z_1 = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$

$z_2 = \sqrt{2}e^{i\frac{5\pi}{4}} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = -1 - i$

$\boxed{z_{1,2} = \pm(1+i)}$ ✓

2) $z^2 = -8i \Rightarrow z^2 = 8\left(i\sin\left(\frac{3\pi}{2}\right)\right) = 8e^{i\frac{3\pi}{2}} = r^2 e^{i2\phi}$

$r^2 = 8 \rightarrow r = 2\sqrt{2} \quad \phi = \frac{3\pi}{4} + \pi k;\ k = 0, 1$

$z_3 = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2(-1+i) = -2(1-i)$

$z_4 = 2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 2(1-i) = 2(1-i)$

$\boxed{z_{3,4} = \pm 2(1-i)}$

$\{\pm(1+i), \pm 2(1-i)\}$

$$z = x+iy \xrightarrow{f} w = u+iv$$

Example: $w = e^{-z^2} = e^{-(x+iy)^2} = e^{-x^2+y^2-2ixy} = e^{y^2-x^2} \cdot e^{-2ixy}$

$$= e^{y^2-x^2}\left[\cos(-2xy) + i\sin(2xy)\right]$$

$$= e^{y^2-x^2}\cos(2xy) - ie^{y^2-x^2}\sin(2xy) = u(x,y) + iv(x,y)$$

then $u(x,y) = e^{y^2-x^2}\cos(2xy)$; $v(x,y) = -e^{y^2-x^2}\sin(2xy)$

Example: $w = \sqrt{z}$, $z = re^{i\phi}$

$$w = (re^{i\phi})^{1/2} = r^{1/2}e^{i\phi/2} = \sqrt{r}\,(\cos(\phi/2) + i\sin(\phi/2))$$

$$u(x,y) = \sqrt{r}\cos(\phi/2) \quad v(x,y) = \sqrt{r}\sin(\phi/2)$$

$r = \sqrt{x^2+y^2}$ and $\phi = \tan^{-1}(y/x)$

Derivative

$$\frac{dw}{dz} = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0}\frac{f(z+\Delta z) - f(z)}{\Delta x}$$

A function of a complex variable that has a derivative at every point within a region of the complex plane is said to be "analytic" (or regular or holomorphic) over that region.

Rules (from Ordinary Calculus)

→ $\frac{d}{dz}\left[cf(z)\right] = cf'(z)$ $c \in \mathbb{R}$

→ $\frac{d}{dz}\left[f(z) \pm g(z)\right] = f'(z) \pm g'(z)$

→ $\frac{d}{dz}\left[f(z)g(z)\right] = f'(z)g(z) + f(z)g'(z)$

→ $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$

→ $\frac{d}{dz}\left[f(g(z))\right] = f'(g(z))g'(z)$. "Chain Rule"

L'Hospital Rule as an example

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1} \underset{LH}{=} \lim_{z\to i}\frac{10z^9}{6z^5} = \frac{5}{3}$$

Cauchy - Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are necessary but sufficient to ensure that a function is differentiable

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{OR}$$

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Example: $\frac{d}{dz}[\sin(z)] = \cos(z)$

$$\frac{d}{dz}[\sin(z)] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos x\cosh y - i\sin x\sinh y = \cos(x+iy) = \cos(z).$$

$$\sin(x+iy) = \sin x.\cos(iy) + \cos x.\sin(iy) = \sin x\cosh y + i\cos x\sinh y$$

$$\cos(iy) = \frac{1}{2}\left[e^{i(iy)} + e^{-i(-iy)}\right] = \frac{1}{2}\left[e^{y} + e^{-y}\right] = \cosh(y)$$

$$\sin(iy) = \frac{1}{2i}\left[e^{i(iy)} - e^{-i(-iy)}\right] = -\frac{1}{2i}\left[e^{y} - e^{-y}\right] = i\sinh(y)$$

$$u(x,y) = \sin x\cosh y \quad v(x,y) = \cos x\sinh y.$$

$$\frac{\partial u}{\partial x} = \cos x\cosh y \;;\; \frac{\partial v}{\partial x} = -\sin x\sinh y$$

$$\frac{\partial u}{\partial y} = +\sin x\sinh y \;;\; \frac{\partial v}{\partial y} = \cos x\cosh y.$$

* by cross differentiating the Cauchy - Riemann equations

$$\left.\begin{array}{l} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\partial y} = -\frac{\partial^2 u}{\partial y^2} \rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; (*) \\ \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x\partial y} = -\frac{\partial^2 v}{\partial y^2} \rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \; (*) \end{array}\right\} \text{Laplace's Equation}$$

Any function that has continuous partial derivatives of second order and satisfies Laplace's Equation is called "harmonic function". Both $u(x,y)$ and $v(x,y)$ satisfy Laplace's Equation if $f(z) = u + iv$ is analytic $u(x,y)$ and $v(x,y)$ are called "conjugate harmonic functions".

Line Integrals

The result of a line integral is a complex number or expression. Generally, integration in the complex plane is an intermediate process with physically realizable quantity occurring only after we take its real or imaginary part.

$$\int_C f(z)\,dz = \int_C [u(x,y)+iv(x,y)][dx+i\,dy] \quad \text{where} \begin{Bmatrix} f(z)=u(x,y)+iv(x,y) \\ dz = dx + i\,dy \end{Bmatrix}$$

$$= \int_C u(x,y)dx - v(x,y)dy + i\int_C v(x,y)dx + u(x,y)dy$$

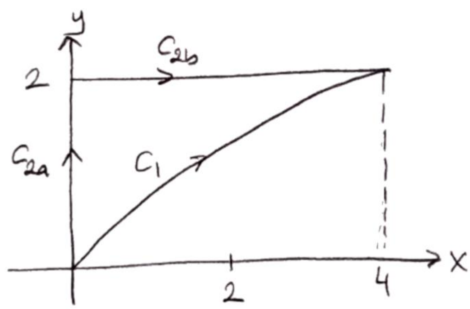
Properties

1) $\displaystyle\int_C f(z)dz = -\int_{C'} f(z)dz$ where C' is the contour in the opposite direction of C

2) $\displaystyle\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$

Example

$\displaystyle\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$



Parametric equation: $z = t^2 + it$

On C_1

$$\int_0^2 \overline{(t^2+it)}\, d(t^2+it) = \int_0^2 (t^2-it)(2t+i)\,dt$$

$$= \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}$$

On C_2

$$\int_{C_2} \bar{z}dz = \int_{C_{2a}} \bar{z}dz + \int_{C_{2b}} \bar{z}dz$$

On C_{2a},

$$\int_{C_{2a}} \bar{z}\,dz = \int_0^2 (x-iy)(dx+i\,dy) = \int_0^2 y\,dy = 2.$$
$(dx=0)$

On C_{2b},

$$\int_{C_{2b}} \bar{z}\,dz = \int_0^4 (x-iy)(dx+i\,dy) = \int_0^4 (x-i2)\,dx$$
$(dy=0)$ $\qquad y=2$

$$= \int_0^4 x\,dx - i\int_0^4 2\,dx = 8-8i$$

$$\int_{C_2} \bar{z}dz = 2+8-8i = 10-8i$$

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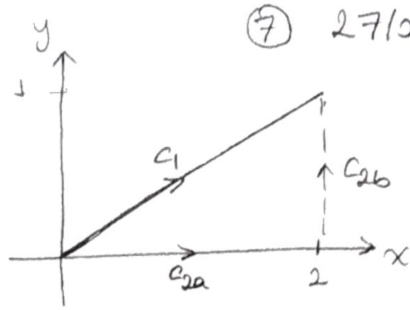
Example:

$$\int_C z^2 dz \qquad z = 2t + it$$



On C_1

$$\int_0^1 (2t+it)^2 (2dt + idt)$$

$$\int_0^1 (4t^2 + 4it^2 - t^2)(2dt + idt) = \int_0^1 (3t^2 + 4it^2)(2dt + idt)$$

$$= \int_0^1 (3t^2 + 4it^2)(2+i)dt = \int_0^1 (6t^2 + 3it^2 + 8it^2 - 4t^2)\, dt$$

$$= \int_0^1 (11it^2 + 2t^2) dt = \frac{2}{3} + \frac{11i}{3}.$$

On C_2

$$\int_{C_2} z^2 dz = \int_{C_{2a}} z^2 dz + \int_{C_{2b}} z^2 dz$$

C_{2a} //

$$\int_0^2 (x+iy)^2 (dx + idy) = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$$

$dy = 0$

C_{2b}

$$\int_0^1 (x+iy)^2 (dx + idy) = \int_0^1 (2+iy)^2 dy = \int_0^1 (4 + 4iy - y^2)\, dy$$

$dx = 0$
$x = 2$

$$= i\left\{ 4y + 2iy^2 - \frac{y^3}{3} \right\}_0^1 = \left(4 + 2i - \frac{1}{3}\right) i = 4i - \frac{1}{3}i - 2$$

$$= \frac{11}{3} i - 2.$$

$$\int_{C_2} z^2 dz = -2 + \frac{11}{3}i + \frac{8}{3} = \frac{2}{3} + \frac{11}{3} i$$

The Cauchy – Goursat Theorem

Let $f(z)$ be analytic in a domain D and let C be a simple Jordan Curve* inside D so that $f(z)$ is analytic on and inside C. Then $\oint_C f(z)dz = 0$.

The principle of deformation of contours:

The value of a line integral of an analytic function around any simple closed contour remains unchanged if we deform the contour in such a manner that we do not pass over a nonanalytic point.

Finally, suppose that we have a function $f(z)$ such that $f(z)$ is an analytic in some domain. Furthermore, let us introduce the analytic function $F(z)$ such that $f(z)=F'(z)$. We would like to evaluate $\int_a^b f(z)dz = F(b)-F(a)$.

Examples:

1) $$\int_0^{\pi i} z\sin(z^2)dz = -\frac{1}{2}\cos(z^2)\Big|_0^{\pi i} = \frac{1}{2}\left[\cos(0)-\cos(-\pi^2)\right]$$
$$= \frac{1}{2}\left[1-\cos(\pi^2)\right]$$

2) $$\int_{1-\pi i}^{2+3\pi i} e^{-2z}dz = -\frac{1}{2}\left\{e^{-2z}\right\}_{1-\pi i}^{2+3\pi i} = \frac{1}{2}\left\{e^{2\pi i-2} - e^{-6\pi i-4}\right\}$$
$$= \frac{1}{2}\left[e^{-2}(\underbrace{\cos(2\pi)}_{1} + \underbrace{i\sin(2\pi)}_{0}) - e^{-4}(\underbrace{\cos(-6\pi)}_{1} + \underbrace{i\sin(-6\pi)}_{0})\right]$$
$$= \frac{1}{2}\left(e^{-2}-e^{-4}\right)$$

3) $$\int_0^{\pi}\sin^2(z)dz = \int_0^{\pi}\frac{1-\cos 2z}{2}dz = \frac{z}{2}-\frac{1}{4}\sin 2z\Big|_0^{\pi} = \frac{\pi}{2} \quad \square$$

Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz \quad (*)$$

By taking n derivatives of (*), we can extend Cauchy's Integral Formula to

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz. \quad (**)$$

For $n = 1, 2, 3, \ldots$ the following formula may be written from (**) as

$$\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Example

Evaluate $\oint \frac{f(z)\,dz}{(z-1)(z-2)}$ where C is the circle $|z| = 5$

$$\oint_C f(z)\left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz = \underbrace{\oint_C f(z)\frac{1}{z-2}\,dz}_{2\pi i} - \underbrace{\oint_C f(z)\frac{1}{z-1}\,dz}_{-2\pi i} \quad \text{from } (**)$$

$$= 4\pi i$$

Example:

1) Evaluate $I = \oint_{|z|=2} \frac{e^z}{(z-1)^2(z-3)}\,dz = \oint \frac{(e^z/z-3)}{(z-1)^2}\,dz$

$n = 1$

$$= \frac{2\pi i}{1!}\left(\frac{e^1(-3)}{(-2)^2}\right) = -\frac{3\pi i e}{2}$$

$f(z) = \frac{e^z}{z-3} \quad f^{(1)}(z) = \frac{e^z(z-4)}{(z-3)^2}$

2) Evaluate $I = \oint_{|z|=1} \frac{\sin^6(z)}{(z-\pi/6)}\,dz = \frac{2\pi i}{0!} f^{(0)}(\pi/6) = 2\pi i \left(\frac{1}{2}\right)^6 = \frac{\pi i}{32}$

3) Evaluate $I = \oint_{|z|=1} \frac{1}{z(z^2+4)}\,dz = \frac{2\pi i}{0!} f^{(0)}(0) = \frac{2\pi i}{0!}\,\frac{1}{4} = \frac{\pi i}{2}$

4) Evaluate $I = \oint_{|z|=5} \frac{e^{z^2}}{z^3}\,dz = \frac{2\pi i}{2} f^{(2)}(0) = \frac{2\pi i}{2}.2 = 2\pi i.$

$f(z) = e^{z^2} \quad f'(z) = e^{z^2}(2z) \quad f''(z) = e^{z^2}(4z^2) + e^{z^2}(2)$

$f''(0) = 2$

Taylor and Laurent Expansion and Singularities

We can expand any analytic function into a Taylor Series. The radius of convergence of this series is equal to the distance between z_0 and the nearest nonanalytic point of $f(z)$.

Taylor Expansion of $f(z)$ is given by

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \dots + \frac{(z-z_n)^n}{n!} f^{(n)}(z_0) + \dots$$ at the point $z = z_0$

e.g $f(z) = \sin z$ at $z_0 = 0$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Radius of convergence is $|z-0| < \infty \rightarrow -\infty < z < \infty$. Because $f(z) = \sin(z)$ is an entire function for all z.

Example: $f(z) = \frac{1}{1-z}$ at $z_0 = 0$ RofC: $|z-0| < 1$ ↳ first nonanalytic point.

$$f(z) = \frac{1}{1-z} = f(0) + \frac{f'(0)}{1!}(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \dots$$

$$= 1 + z + z^2 + \dots$$

Laurent Expansion of $f(z)$ is given by

$$f(z) = \frac{a_1}{z-z_0} + \frac{a_2}{(z-z_0)^2} + \dots + \frac{a_n}{(z-z_0)^n} + \dots + b_0 + b_1(z-z_0) + \dots + b_n(z-z_0)^n + \dots$$

1. If $f(z)$ is analytic at z_0, then all $a_1 = a_2 = \dots = a_n = \dots = 0$. and the Laurent expansion reduces to a "Taylor expansion".
2. If z_0 is a singularity of $f(z)$, then the Laurent expansion will include both positive and negative powers.
3. The coefficient of the $(z-z_0)^{-1}$, a_1, is "the residue".
4. No straightforward method for obtaining a Laurent Series.

Isolated Singularities

1. Essential Singularities:

$$f(z) = \cos\left(\frac{1}{z}\right) = 1 - \underbrace{\frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \cdots}_{\text{An infinite number of inverse powers of } z - z_0.} \quad \text{for } 0 < |z| < \infty$$

The residue $(a_1) = 0$

2. Removable Singularities

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

The residue $(a_1) = 0$.

3. Pole of order n:

Consider $f(z) = \dfrac{1}{(z-1)^3(z+1)}$ two singularities at $z = -1$ and $z = 1$

Consider only $z = 1$,

$$= \frac{1}{(z-1)^3}\,\frac{1}{2+(z-1)} = \frac{1}{2}\,\frac{1}{(z-1)^3}\cdot\frac{1}{1+(z-1)/2}$$

$$= \frac{1}{2}\,\frac{1}{(z-1)^3}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \frac{(z-1)^3}{8} + \cdots\right]$$

$$= \frac{1}{2}\,\frac{1}{(z-1)^3} - \frac{1}{4}\,\frac{1}{(z-1)^2} + \frac{1}{8(z-1)} - \frac{1}{16} + \cdots$$

for $0 < |z-1| < 2$. The residue is $1/8$ (the coefficient of $(z-1)^{-1}$)

The largest inverse (negative) power is "three".

Ex: $f(z) = \dfrac{1}{z(z-2)} = \dfrac{1}{z-2}\cdot\dfrac{1}{2+(z-2)} = \dfrac{1}{2}\cdot\dfrac{1}{z-2}\cdot\dfrac{1}{1+(z-2)/2}$

$$= \frac{1}{2}\cdot\frac{1}{z-2}\left[1 - \frac{(z-2)}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \cdots\right]$$

The residue $= 1/2$

$$= \frac{1}{2}\left[\frac{1}{z-2} - \frac{1}{2} + \frac{(z-2)}{4} - \frac{(z-2)^2}{8} + \cdots\right]$$

Ex: $f(z) = z^{10}e^{-1/z} = z^{10}\displaystyle\sum_{k=0}^{\infty}\frac{(-1/z)^k}{k!}$

The residue $= -\dfrac{1}{11!}$

$$= z^{10}\left(1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \cdots + \frac{1}{10!\,z^{10}} - \frac{1}{11!\,z^{11}} + \cdots\right)$$

$$= z^{10} - z^9 + \frac{z^8}{2} - \frac{z^7}{6} + \cdots + \frac{1}{10!} - \frac{1}{11!\,z} + \cdots)$$

Theory of Residues

Cauchy's residue theorem: If $f(z)$ is analytic inside and on a closed contour C except at points $z_1, z_2, \ldots, z_n$ where $f(z)$ has singularities, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}[f(z); z_j]$$

where $\operatorname{Res}[f(z), z_j]$ denotes the residue of the jth isolated singularity of $f(z)$ located at $z = z_j$

the residue of a pole of order n by

$$\operatorname{Res}[f(z); z_j] = \frac{1}{(n-1)!} \lim_{z \to z_j} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_j)^n f(z) \right]$$

Example 1

Evaluate $\oint \frac{e^{iz}}{z^2+a^2}\,dz$

$$= 2\pi i \left[\operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2} ; ai \right) + \operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2} ; -ai \right) \right]$$

$$\operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2} ; ai \right) = \lim_{z \to ai} (z-ai) \frac{e^{iz}}{(z-ai)(z+ai)} = \frac{e^{-a}}{2ia}$$

$$\operatorname{Res}\left(\frac{e^{iz}}{z^2+a^2} ; -ai \right) = \lim_{z \to ai} (z+ai) \frac{e^{iz}}{(z+ai)(z-ai)} = -\frac{e^{+a}}{2ai}$$

$$= 2\pi i \left[\frac{e^{-a}}{2ia} - \frac{e^{-a}}{2ai} \right] = -\frac{2\pi}{a} \sinh(a)$$

Evaluate $\oint_{|z|=1} \frac{z+1}{z^3(z-2)}\,dz$

$$= 2\pi i \left[\operatorname{Res}\left(\frac{z+1}{z^3(z-2)} ; 0 \right) + \operatorname{Res}\left(\frac{z+1}{z^3(z-2)} ; 2 \right) \right]$$

$$\operatorname{Res}\left(\frac{z+1}{z^3(z-2)} ; 0 \right) = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[z^3 \frac{(z+1)}{z^3(z-2)} \right] = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left[\frac{z+1}{z-2} \right] = -\frac{3}{4}$$

$$\frac{d}{dz} \left[\frac{z+1}{z-2} \right] = \frac{-3}{(z-2)^2}, \quad \frac{d^2}{dz^2} \left[\frac{z+1}{z-2} \right] = \frac{6}{(z-2)^3}$$

$$\operatorname{Res}\left(\frac{z+1}{z^3(z-2)} ; 2 \right) = \lim_{z \to 2} \left[\frac{z+1}{z^3(z-2)} (z-2) \right] = \frac{3}{8}$$

$$= 2\pi i \left[-\frac{3}{4} + \frac{3}{8} \right] = -2\pi i \cdot \frac{3}{8} = -\frac{3\pi i}{4} \quad \square$$