The Fourier Transform

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A periodic function $f(t)$ of period $2T$ that satisfies the so-called Dirichlet's conditions. If the integral $\int_a^b |f(t)|dt$ exists, then the complex Fourier series

$$(*)\ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T}$$

$$c_n = \frac{1}{2T}\int_{-T}^{T} f(t)\, e^{-in\pi t/T}$$

$$\boxed{f(t) = \frac{1}{2T}\sum_{n=-\infty}^{\infty} e^{in\pi t/T}\int_{-T}^{T} f(x) e^{-in\pi x/T} dx}$$

A jump discontinuity at t $(*)$ equals to $\frac{1}{2}[f(t^+)+f(t^-)]$ where $f(t^+) = \lim_{x\to t^+} f(t)$ and $f(t^-) = \lim_{x\to t^-} f(t)$.

Introduce $w_n = n\pi/T$ and $\Delta w = w_{n+1} - w_n = \pi/T$

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(w_n) e^{iw_n t}\Delta w_n \quad \text{where} \quad F(w_n) = \int_{-T}^{T} f(x) e^{-iw_n x} dx$$

As $T \to \infty$, $\Delta w_n \to dw$ then

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w) e^{iwt} dw} \rightarrow \text{the Inverse Fourier Transform of } F(w)$$

$$\boxed{F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt} \rightarrow \text{the Fourier Transform of } f(t).$$

<u>Example</u>: $f(t) = \begin{cases} 1, & |t|<a \\ 0, & |t|>a \end{cases}$

the FT: $F(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt = \int_{-a}^{a} 1 e^{-iwt} dt = -\frac{1}{iw}\left\{e^{-iwt}\right\}_{-a}^{a} = -\frac{1}{iw}\left(e^{-iwa} - e^{iwa}\right)$

$$F(w) = \frac{e^{iwa} - e^{-iwa}}{iw} = \frac{2\sin(wa)}{w} = 2\sin c(wa) \rightarrow \boxed{F(w) = 2\sin c(wa)}$$

the IFT: $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(wa)}{w} e^{iwt} dw = \begin{cases} 1, & |t|<a \\ 0, & |t|>a \end{cases}$

$$f(a) = \frac{1}{2}[f(a^+) + f(a^-)] = \frac{1}{2} \quad \text{and} \quad f(-a) = \frac{1}{2}[f(-a^+) + f(-a^-)] = \frac{1}{2}$$

<u>Dirac Delta Function</u>

$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt} dw$, $\int_{-\infty}^{\infty} \delta(t) e^{-iwt} dt = 1$ and $\int_{-\infty}^{\infty} \delta(t) dt = 1$. Note: the delta function is an even function.

the shifting property $\Rightarrow \int_a^b f(t)\delta(t - t_0)\, dt = f(t_0)$

<u>the delta function may be:</u>

① $\delta(t) = \lim_{\varepsilon\to 0}\begin{cases} 1/\varepsilon, & |t|<\varepsilon/2 \\ 0, & |t|>\varepsilon/2 \end{cases}$ or $\delta(t) = \lim_{\varepsilon\to 0}\begin{cases} \frac{1}{\varepsilon}\left(1 - \frac{1}{|\varepsilon|}\right), & |t|<\varepsilon \\ 0, & |t|>\varepsilon \end{cases}$ and

the Gaussian function: $\delta(t) = \lim_{\varepsilon\to 0}\frac{\exp(-\pi t^2/\varepsilon)}{\sqrt{\varepsilon}}$

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Multiple Fourier Transform

the double FT of $f(x,y)$ → $\boxed{F(\xi,\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(\xi x+\eta y)}\,dx\,dy}$

the double reversing FT of $f(x,y)$ → $\boxed{f(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)\, e^{i(\xi x+\eta y)}\,d\xi\,d\eta}$

Properties of Fourier Transform

1) $\mathcal{F}[c_1 f(x)+c_2 g(x)] = c_1\mathcal{F}[f(x)]+c_2\mathcal{F}[g(x)]$ "Linearity"

2) $\mathcal{F}[f(x-\tau)] = e^{-i\omega\tau}\mathcal{F}[f(x)]$ "Time shifting"

3) $\mathcal{F}[f(kx)] = F(\omega/k)/|k|$; $k\in\mathbb{R}$ "Scaling Factor"

4) $\mathcal{F}[f^{(n)}(x)] = (i\omega)^n F(\omega)$

5) $\int_{-\infty}^{\infty}|f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega$ "Parseval's Equality"

6) $\sum_{k=-\infty}^{\infty} f(ak) = \frac{1}{a}\sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{a}\right)$ "Poisson's Summation Formula"