Finite Element Method for Ordinary Differential Equations

① 01/09/2023

"G. Evans, Numerical Methods for PDEs"

Introduction

Finite Element Methods depend on a much more enlarged backgrounds of mathematics than the finite difference method.

To take a practical approach, consider the typical problem of finding the approximate solution of

$$(*) \quad \frac{d^2u}{dx^2} + u = -x \;;\; u(0) = 0,\ u(1) = 0.$$

An alternative for the finite difference method is to seek an approximate solution of the form

$$\boxed{u_N = \sum_{i=1}^{N} a_i \beta_i(x) = a_1\beta_1(x) + a_2\beta_2(x) + \cdots + a_N\beta_N(x)}$$

where the functions $\beta_i(x)$ are called "the basis functions". The objective is to choose the coefficients to minimize the resulting error in the computed solution.

If u_N is an approximated solution to the above example, then a measure of the error E; $E = \frac{d^2u_N}{dx^2} + u_N + x$

It is important to ensure continuity conditions across element boundaries. To determine the coefficients, there are methods which are called "the collocation method", "the least squares method", "the weighted residual method" and "the Galerkin Method".

1. The Collocation Method

In this method, the values of $a_1, a_2, \ldots, a_N$ are chosen so that $E=0$ at a set of given points.

Consider the problem $(*)$ with a solution of the form $u_N = x(1-x)(a_1 + a_2x + \cdots + a_Nx^{N-1})$ which satisfies the boundary conditions for any choice of a_i's.

The basis functions are $\beta_1 = x(1-x)$, $\beta_2 = x^2(1-x)$, $\beta_3 = x^3(1-x)$

Restricting up to the two basis functions, then $u_2 = a_1\beta_1 + a_2\beta_2$ or

$$u_2 = a_1[x(1-x)] + a_2[x^2(1-x)] = a_1(x-x^2) + a_2(x^2-x^3)$$

The Error $E = \frac{d^2u_2}{dx^2} + u_2 + x = 0$

$$= a_1(-2) + a_2(2-6x) + a_1(x-x^2) + a_2(x^2-x^3) + x = 0$$

$$= a_1(-2+x-x^2) + a_2(2-6x+x^2-x^3) + x = 0$$

Set $E=0$ for $x = 1/4$ and $x = 1/2$ to determine a_1 and a_2.

$$\Rightarrow a_1\left(-2+\frac{1}{4}-\frac{1}{16}\right) + a_2\left(2-6\cdot\frac{1}{4}+\frac{1}{16}-\frac{1}{64}\right) = -\frac{1}{4}$$

$$\Rightarrow a_1\left(-2+\frac{1}{2}-\frac{1}{4}\right) + a_2\left(2-6\frac{1}{2}+\frac{1}{4}-\frac{1}{8}\right) = -\frac{1}{2}$$

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$$\left.\begin{aligned} -\frac{29}{32}a_1 + \frac{35}{64}a_2 &= -\frac{1}{4} \\ -\frac{7}{4}a_1 - \frac{9}{8}a_2 &= -\frac{1}{2} \end{aligned}\right\} \quad \begin{bmatrix} 29/32 & -35/64 \\ 7/4 & 7/8 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}$$

then $a_1 = \frac{1}{3}$ $a_2 = -\frac{2}{21}$ then $u_2 = \frac{1}{3}x(1-x) - \frac{2}{21}x^2(1-x)$

$$u_2 = \frac{x(1-x)[7-2x]}{21} \Rightarrow \boxed{\frac{x(1-x)(7-2x)}{21} = u_2}$$

__Example:__ the IVP : $u'' + u = x$; $u(0) = 0$, $u'(0) = 2$.

Find the solution of the IVP of the form $u_N = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$u_N(0) = a_0 = 0$ $u_N'(0) = a_1 = 2$ then $\boxed{u_3 = 2x + a_2 x^2 + a_3 x^3}$

$$E = \frac{d^2}{dx^2}u_3 + u_3 = x \to 2a_2 + 6a_3 x + 2x + a_2 x^2 + a_3 x^3 = x$$
$$(2 + x^2)a_2 + (6x + x^3)a_3 = -x$$

Use $x = 1/2$ and $x = 1$ as collocation points, then

$$\left(2 + \frac{1}{4}\right)a_2 + \left(6\frac{1}{2} + \frac{1}{8}\right)a_3 = -\frac{1}{2} \to \frac{9}{4}a_2 + \frac{25}{8}a_3 = -\frac{1}{2}$$

$$(2+1)a_2 + (6.1 + 1)a_3 = -1 \to 3a_2 + 7a_3 = -1$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9/4 & 25/8 \\ 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} -1/2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/17 \\ -2/17 \end{bmatrix}$$

then $\boxed{u_3 = 2x - \frac{1}{17}x^2 - \frac{2}{17}x^3}$ as a solution.

the analytical solution of IVP is $u(x) = \sin(x) + x$ then for $x = 1/2$

$u(1/2) = 0.9794$ and $u_3(1/2) = 0.9705$.

Example: the BVP: $u'' + 2u + x^2 = 0$; $u(0) = 0$ and $u(1) = 0$ ③ 02/09/2023

(1) $x = 1/3$ and $x = 2/3$ as two collocation point.

$$u_N = x(1-x)(a_0 + a_1 x + \cdots + a_N x^{N-1})$$

$\beta_1 = x(1-x)$ $\beta_2 = x^2(1-x)$, $\beta_3 = x^3(1-x)$

$$u_2 = a_1 x(1-x) + a_2 x^2(1-x) = a_1(x - x^2) + a_2(x^2 - x^3)$$

$$u_2'' + 2u + x^2 = a_1(-2) + a_2(2 - 6x) + 2a_1(x - x^2) + 2a_2(x^2 - x^3) + x^2$$

$$= a_1(2x - 2x^2 - 2) + a_2(2 - 6x + 2x^2 - 2x^3) + x^2$$

$$x = \frac{1}{3} \rightarrow a_1\left(\frac{2}{3} - \frac{2}{9} - \frac{2}{1}\right) + a_2\left(2 - 2 + \frac{2}{9} - \frac{2}{27}\right) + \frac{1}{9} = 0$$

$$-\frac{14}{9}a_1 + \frac{4}{27}a_2 = -\frac{1}{9}$$

$$x = \frac{2}{3} \rightarrow a_1\left(\frac{4}{3} - \frac{8}{9} - \frac{2}{1}\right) + a_2\left(\frac{2}{1} - \frac{4}{1} + \frac{8}{9} - \frac{16}{27}\right) + \frac{4}{9}$$

$$-\frac{14}{9}a_1 - \frac{46}{27}a_2 = -\frac{4}{9}$$

$$\begin{bmatrix} -14/9 & 4/27 \\ -14/9 & -46/27 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ -4/9 \end{bmatrix}$$

$$a_1 = 0.08857 \quad a_2 = 0.18$$

$$u_2(x) = 0.08857[x - x^2] + 0.18[x^2 - x^3]$$

$u_2(0.5) = 0.0446425 \rightarrow$ Approximate value of $u(0.5)$

$u(x) = c_1 \sin(\sqrt{2}x) + c_2 \cos(\sqrt{2}x) - \frac{x^2}{2} + \frac{1}{2} \rightarrow$ Exact solution

$$u(0) = c_2 + \frac{1}{2} = 0 \rightarrow \boxed{c_2 = -\frac{1}{2}}$$

$$u(1) = c_2 \sin(\sqrt{2}) - \frac{1}{2}\cos(\sqrt{2}) = 0$$

$$\boxed{c_2 = \frac{1}{2}\cot(\sqrt{2})}$$

$$u(x) = \frac{1}{2}\cot(\sqrt{2})\sin(\sqrt{2}x) - \frac{1}{2}\cos(\sqrt{2}x) - \frac{x^2}{2} + \frac{1}{2}$$

$$u(0.5) = \frac{1}{2}\underbrace{\cot(\sqrt{2})}_{0.157875}\underbrace{\sin(\sqrt{2}/2)}_{0.6496} - \frac{1}{2}\underbrace{\cos(\sqrt{2}/2)}_{0.7603} - \frac{1}{8} + \frac{1}{2}$$

$$= 0.0461278.$$

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Exact value of $u(0.5)$

Now $u_3(x) = a_1 x(1-x) + a_2 x^2(1-x) + a_3 x^3(1-x)$ because we have three collocation points $x = 1/4$, $x = 1/2$, and $x = 3/4$ ④ 04/09/2023

$$u_3(x) = a_1(x-x^2) + a_2(x^2-x^3) + a_3(x^3-x^4)$$

$$u_3'' + 2u_3 + x^2 = [-2a_1 + a_2(2-6x) + a_3(6x-12x^2)]$$
$$+ 2a_1(x-x^2) + 2a_2(x^2-x^3) + 2a_3(x^3-x^4) + x^2$$
$$= a_1[-2+2x-2x^2] + a_2[2-6x+2x^2-2x^3] + a_3[6x-12x^2+2x^3-2x^4] + x^2$$

$x = 1/4$, $-1.625a_1 + 0.59375a_2 + 0.7734375\, a_3 = -\frac{1}{16}$

$x = 1/2$, $-1.5a_1 - 0.75a_2 + 0.125 = -\frac{1}{4}$

$-1.625a_1 - 2.21875a_2 - 2.0390625a_3 = -\frac{9}{16}$

$a_1 = 0.10947$
$a_2 = 0.12354$
$a_3 = 0.054237$

$$u_3(x) = 0.10942x(1-x) + 0.12354x^2(1-x) + 0.054237x^3(1-x)$$

$u_3(0.5) = 0.046187312 \rightarrow$ Approximate value of $u(0.5)$

2. The Least Squares Method

An approximate solution of the form $u_N = \sum_{i=1}^{N} a_i \phi_i$ is found by choosing the parameters so that $F = \int_a^b E^2 dx$ is minimized where $[a,b]$ is the range of interest.

the ODE: $u'' + u = -x$; $u(0) = 0$, $u(1) = 0$

$u_2 = a_1(x - x^2) + a_2(x^2 - x^3)$ then $E = u_2'' + u_2 + x = a_1[-2 + x - x^2] + a_2[2 - 6x + x^2 - x^3]$

$$F = \int_0^1 [a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)]^2 dx$$

$$\frac{\partial F}{\partial a_1} = \int_0^1 2[a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)](-2 + x - x^2)\, dx = 0$$

$$\frac{\partial F}{\partial a_2} = \int_0^1 2[a_1(-2 + x - x^2) + a_2(2 - 6x + x^2 - x^3)](2 - 6x + x^2 - x^3)\, dx = 0$$

Expanding and evaluating the above integrals gives

$$\left.\begin{aligned} 202a_1 + 101a_2 &= 55 \\ 707a_1 + 1572a_2 &= 399 \end{aligned}\right\} \text{ with solution } \begin{aligned} a_1 &= 0.1875 \\ a_2 &= 0.1695 \end{aligned}$$

then we get $0.1875x(1-x) + 0.1695x^2(1-x) \Rightarrow u_2(x) = x(1-x)[0.1875 + 0.1695x]$

the exact value $u(0.5) = 0.070$ and $u_2(0.5) = 0.068$ as an approximate value.

Example: the IVP: $u'' + u = x$ with $u(0) = 0$ and $u'(0) = 2$. Using the method of least squares to find an approximate solution in the form of $u_N = a_0 + a_1 x + a_2 x^2 + \dots$

u_N should satisfy $u(0) = 0$ and $u'(0) = 2$.

$u_N(0) = a_0 = 0$, $u_N'(0) = a_1 = 2$ then $u_N(x) = 2x + a_2 x + a_3 x^2 + \dots$

$u_3(x) = 2x + a_2 x^2 + a_3 x^3$, $E = (2a_2 + 6a_3 x) + (2x + a_2 x^2 + a_3 x^3) - x$

$E = (2 + x^2)a_2 + (6x + x^3)a_3 + x$ then $F = \int_0^1 [(2 + x^2)a_2 + (6x + x^3)a_3 + x]^2 dx$

$$\frac{\partial F}{\partial a_2} = \int_0^1 2[(2 + x^2)a_2 + (6x + x^3)a_3 + x](2 + x^2)dx = 0$$

$$= \int_0^1 2[(4 + 4x^2 + x^4)a_2 + (x^5 + 8x^3 + 12x)a_3 + 2x + x^3]\, dx = 0$$

$$4x + \frac{4}{3}x^3 + \frac{x^5}{5}\Big|_0^1 a_2 + \frac{x^6}{6} + 2x^4 + 6x^2\Big|_0^1 a_3 = -x^2 - \frac{x^4}{4}\Big|_0^1$$

$$= \boxed{\frac{83}{15}a_2 + \frac{49}{6}a_2 = -\frac{5}{4}}$$

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$$\frac{\partial F}{\partial a_3} = \int_0^1 2\left[(2+x^2)a_2 + (6x+x^3)a_3 + x\right](6x+x^3)\,dx$$

$$= \int_0^1 2\left[(x^5+8x^3+12x)a_2 + (36x^2+12x^4+x^6)a_2 + 6x^2+x^4\right]dx$$

$$= \frac{x^6}{6} + 2x^4 + 6x^2\Big|_0^1 a_2 + 12x^3 + \frac{12}{5}x^5 + \frac{x^7}{7}\Big|_0^1 a_2 = -2x^3 - \frac{1}{5}x^5\Big|_0^1$$

$$\boxed{\frac{49}{6}a_2 + \frac{509}{35}a_3 = -\frac{11}{5}}$$

the matrix form

$$\begin{bmatrix} 83/15 & 49/6 \\ 49/6 & 509/35 \end{bmatrix}\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -5/4 \\ -11/5 \end{bmatrix} \qquad \begin{bmatrix} a_2 = -0.0153821 \\ a_3 = -0.1426390 \end{bmatrix}$$

then $u_3(x) = 2x - 0.01538x^2 - 0.1426x^3$. The analytical solution of the IVP is $u(x) = \sin(x) + x$, then "$u(0.5) = 0.9794$" and "$u_3(0.5) = 0.97833$" from the method of least squares. "$u_3(0.5) = 0.9705$" from the collocation method.

3. The Galerkin Method

The method in this chapter is "a weighted residual method." The weighted error is defined by "$\int_0^1 E v\,dx$" where $E = u_N'' + u_N + x$ for "the BVP" (*) $u'' + u = -x;\ u(0)=0;\ u(1)=1$

The aim is to equate $\int_0^1 E v\,dx = 0$ where v is called the weight function or the test function. In the Galerkin Method, the test functions are taken to be identical to the basis functions.

The Galerkin Approximation is found by calculating a_i's such that

$$\boxed{\int_0^1 E \beta_i\,dx = 0\,,\quad i = 1,2,3,\dots,N}$$

Each β_i's should satisfy the boundary conditions and β_i's are forced to be perpendicular to E.

$u_N = a_1 x(1-x) + a_2 x^2(1-x)$ with $\beta_1 = x(1-x)$ and $\beta_2 = x^2(1-x)$.

$$E = a_1(-2) + a_2(2-6x) + a_1(-x^2+x) + a_2(x^2-x^3) + x$$

$$\boxed{E = a_1(-2+x-x^2) + a_2(2-6x+x^2-x^3) + x}\ \text{for the BVP (*)}$$

$$\int_0^1 E\beta_1\,dx = \int_0^1 \left[a_1(-2+x-x^2) + a_2(2-6x+x^2-x^3) + x\right](x-x^2)\,dx$$

$$= \int_0^1 \left[a_1(x^2-x+2)(x^2-x) + a_2(x^3-x^2+6x-2)(x^2-x) - x^3 + x^2\right]dx$$

$$= \int_0^1 a_1(x^4-2x^3+3x^2-2x) + a_2(x^5-2x^4+7x^3-8x^2+2x)\,dx = -\int_0^1 (x^3-x^2)\,dx$$

$$\left.\frac{x^5}{5} - \frac{1}{2}x^4 + x^3 - x^2\right|_0^1 a_1 + \left.\frac{1}{6}x^6 - \frac{2}{5}x^5 + \frac{7}{4}x^4 - \frac{8}{3}x^3 + x^2\right|_0^1 a_2 = +\left(\left.\frac{x^4}{4} - \frac{x^3}{3}\right|_0^1\right)$$

$$\boxed{\frac{3}{10}a_1 + \frac{3}{20}a_2 = \frac{1}{12}}\ \text{(a)}$$

$$\int_0^1 E\beta_2\,dx = \int_0^1 \left[a_1(-2+x-x^2) + a_2(2-6x+x^2-x^3) + x\right](x^2-x^3)\,dx$$

$$= \int_0^1 \left[a_1(x^5-2x^4+3x^3-2x^2) + a_2(x^6-2x^5+7x^4-8x^3+2x^2)\right]dx = \int_0^1 (x^4-x^3)\,dx$$

$$\left.\frac{1}{6}x^6 - \frac{2}{5}x^5 + \frac{3}{4}x^4 - \frac{2}{3}x^3\right|_0^1 a_1 + \left.\frac{1}{7}x^7 - \frac{2}{6}x^6 + \frac{7}{5}x^5 - \frac{8}{4}x^4 + \frac{2}{3}x^3\right|_0^1 a_2 = \left.\frac{x^5}{5} - \frac{x^4}{4}\right|_0^1$$

$$\boxed{\frac{3}{20}a_1 + \frac{13}{105}a_2 = \frac{1}{20}}\ \text{(b)}$$

By solving (a) & (b) together, we get $a_1 = \frac{71}{369}$ and $a_2 = \frac{7}{41}$

then $\boxed{u_3(x) = x(1-x)\left[\frac{71}{369} + \frac{7}{41}x\right]}$ $\quad u_3(0.5) = 0.0689$ and $u_{exact}(0.5) = 0.0697$

The Galerkin Method for a Poisson's Equation

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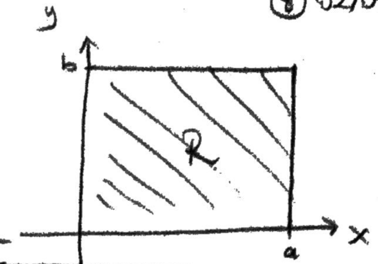

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y),\quad (x,y)\in \underbrace{(0,a)\times(0,b)}_{R}$$

$f(x,y)=c$ and with boundary conditions $u=0$

$u_n(x,y)=\alpha\beta(x,y)$ as a general form. This u_n must satisfy all boundary conditions. Choose $\boxed{u_1(x,y)=\alpha x(x-a)y(y-b)}=\alpha(x^2-ax)(y^2-by)$

The error E is defined by $E=\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}-c$; $\boxed{\beta_1(x,y)=x(x-a)y(y-b)}$

$$E=\alpha(2)(y^2-by)+\alpha(x^2-ax)(2)-c$$

$\boxed{E=2\alpha[x^2+y^2-ax-by]-c}$ then over the region R, we get the integral

$$I=\int_0^b\int_0^a\{2\alpha[x^2+y^2-ax-by]-c\}\,x(x-a)y(y-b)\,dx\,dy=0=\int_R E\beta\,dx\,dy$$

which yields $\alpha=-\frac{5c}{2(a^2+b^2)}$

The resulting approximate solution is $\boxed{u_1=\frac{-5c}{2(a^2+b^2)}x(x-a)y(y-b)}$

Example the BVP $-u''+u-x=0$ with $u(0)=0=u(1)$. ⑨ 02/09/2023

Use the Galerkin Method by defining $u_3=\sum_{i=1}^{3} a_i\phi_i(x)$ and $\phi_i=\sin(i\pi x)$ to find the approximate constants a_i.

$$\boxed{u_3(x)=a_1\sin(\pi x)+a_2\sin(2\pi x)+a_3\sin(3\pi x)}$$

$$\begin{bmatrix}\phi_1(x)=\sin(\pi x)\\ \phi_2(x)=\sin(2\pi x)\\ \phi_3(x)=\sin(3\pi x)\end{bmatrix}$$

$$E=-(-a_1\pi^2\sin(\pi x)-a_2 4\pi^2\sin(2\pi x)-a_3 9\pi^2\sin(3\pi x))+a_1\sin(\pi x)+a_2\sin(2\pi x)+a_3\sin(3\pi x)-x$$

$$E=(\pi^2+1)\sin(\pi x)a_1+(4\pi^2+1)\sin(2\pi x)a_2+(9\pi^2+1)\sin(3\pi x)a_3-x$$

$$\int_0^1 E\phi_1(x)dx=\int_0^1\left[(\pi^2+1)\sin(\pi x)a_1+(4\pi^2+1)\sin(2\pi x)a_2+(9\pi^2+1)\sin(3\pi x)a_3-x\right]\sin(\pi x)\,dx=0$$

$$\left[m\neq n,\ \int\sin(mx)\sin(nx)dx=0\right]$$

$$\Rightarrow\int_0^1(\pi^2+1)\sin^2(\pi x)a_1\,dx=\int_0^1 x\sin(\pi x)dx$$

$+x$ $\quad\sin(\pi x)$

-1 $\quad-\frac{1}{\pi}\cos(\pi x)$

$+0$ $\quad-\frac{1}{\pi^2}\sin(\pi x)$

$$a_1(\pi^2+1)\int_0^1\sin^2(\pi x)dx=\int_0^1 x\sin(\pi x)dx$$

$1-2\sin^2(\pi x)=\cos(2\pi x)$

$\frac{1-\cos(2\pi x)}{2}=\sin^2(\pi x)$

$$a_1(\pi^2+1)\left\{\frac{x}{2}-\frac{1}{4\pi}\sin(2\pi x)\right\}_0^1=\left\{-\frac{x}{\pi}\cos(\pi x)+\frac{1}{\pi^2}\sin(\pi x)\right\}_0^1$$

$$a_1(\pi^2+1)\left(\frac{1}{2}\right)=\left\{-\frac{1}{\pi}(-1)\right\}\rightarrow\boxed{a_1=\frac{2}{\pi(\pi^2+1)}}\rightarrow\boxed{a_1=0.058568807}$$

$$\int_0^1 E\phi_2(x)dx=\int_0^1\left[(\pi^2+1)\sin(\pi x)a_1+(4\pi^2+1)\sin(2\pi x)a_2+(9\pi^2+1)\sin(3\pi x)a_3-x\right]\sin(2\pi x)\,dx=0$$

$+x$ $\quad\sin(2\pi x)$

-1 $\quad-\frac{1}{2\pi}\cos(2\pi x)$

$+0$ $\quad-\frac{1}{4\pi^2}\sin(2\pi x)$

$$\Rightarrow a_2(4\pi^2+1)\int_0^1\sin^2(2\pi x)dx=\int_0^1 x\sin(2\pi x)dx$$

$$a_2(4\pi^2+1)\left\{\frac{x}{2}-\frac{1}{8\pi}\sin(4\pi x)\right\}_0^1=\left\{\frac{-x}{2\pi}\cos(2\pi x)+\frac{1}{4\pi^2}\sin(2\pi x)\right\}_0^1$$

$$a_2(4\pi^2+1)\left(\frac{1}{2}\right)=\left\{-\frac{1}{2\pi}\right\}\rightarrow\boxed{a_2=-\frac{1}{\pi(4\pi^2+1)}}\quad\boxed{a_2=-0.0078637}$$

$+x$ $\quad\sin(3\pi x)$

-1 $\quad-\frac{1}{3\pi}\cos(3\pi x)$

$+0$ $\quad-\frac{1}{9\pi^2}\sin(3\pi x)$

$$\int_0^1 E\phi_3(x)dx=\int_0^1\left[(\pi^2+1)\sin(\pi x)a_1+(4\pi^2+1)\sin(2\pi x)a_2+(9\pi^2+1)\sin(3\pi x)a_3-x\right]\sin(3\pi x)\,dx=0$$

$\frac{1-\cos(6\pi x)}{2}=\sin^2(3\pi x)$

$$\Rightarrow a_3(9\pi^2+1)\int_0^1\sin^2(3\pi x)dx=\int_0^1 x\sin(3\pi x)dx$$

$$a_3(9\pi^2+1)\left\{\frac{x}{2}-\frac{1}{12\pi}\sin(6\pi x)\right\}_0^1=\left\{-\frac{x}{3\pi}\cos(3\pi x)+\frac{1}{9\pi^2}\sin(3\pi x)\right\}_0^1$$

$$a_3(9\pi^2+1)\frac{1}{2}=\frac{1}{3\pi}\rightarrow\boxed{a_3=\frac{2}{3\pi(9\pi^2+1)}}\quad\boxed{a_3=0.0023624}$$

$$\text{then}\left[u_3(x)=\frac{2}{\pi(\pi^2+1)}\sin(\pi x)-\frac{1}{\pi^2(4\pi^2+1)}\sin(2\pi x)+\frac{2}{3\pi(9\pi^2+1)}\sin(3\pi x)\right]$$

4. Symmetric Variational Formulation

the test function $\boxed{v = \sum_{i=1}^{N} \gamma_i \beta_i}$ → "the test function" where v is now a linear combination of the basis functions and β_i are chosen to be linearly independent and satisfy the boundary conditions.

The Galerkin statement,

$\int_a^b E v\,dx = 0$, $\forall v \in \text{span}\{\beta_1, \beta_2, \dots, \beta_N\}$ and since the β_i are independent then this actually gives N conditions as each coefficient of β must be zero. These conditions are used to solve for the N unknowns a_i in the solution

$$\boxed{u_N = \sum_{i=1}^{N} \alpha_i \beta_i} \leftarrow \text{"the trial function"}$$

Consider the BVP: $u'' + u + x = 0$, $u(0) = 0 = u(1)$

Choosing β_i's to satisfy $\beta_i(0) = 0$ and $\beta_i(1) = 0$. The objective is to find values of a_i for which

$$\int_0^1 E v\,dx = \int_0^1 (u_N'' + u_N + x) v\,dx = 0; \quad \forall v \in \text{span}\{\beta_1, \beta_2, \dots, \beta_N\}$$

where $u_N = \sum_{i=1}^{N} \alpha_i \beta_i(x)$ is known as the variational formulation of the equation.

$$\int_0^1 u_N'' v\,dx = [u_N' v]_0^1 - \int_0^1 u_N' v'\,dx = -\int_0^1 u_N' v'\,dx, \quad \forall v \in \text{span}\{\beta_1, \beta_2, \dots, \beta_N\}$$

with boundary conditions $v(0) = v(1) = 0$.

Hence,

$$\int_0^1 (u_N'' + u_N + x) v\,dx = \int_0^1 (-u_N' v' + u_N v + x v)\,dx \quad \forall v \in \text{span}\{\beta_1, \beta_2, \dots, \beta_N\} \text{ and}$$

$u'' + u + x = 0$, $u(0) = 0 = u(1)$ replaced by $\int_0^1 (-u_N' v' + u_N v + x v)\,dx$ $\forall v \in \text{span}\{\beta_1, \beta_2, \dots, \beta_N\}$

This is the symmetric variational formulation.

the following four points are noted;

1) The order of the derivative in the variational formulation is less than in the original problem

2) The derivation takes places restriction on v' and so we are not allowed to place any additional restrictions on v'. For example we cannot require $v'(0)$ or $v'(1)$ to have a specific value.

3) If the boundary conditions are of the form $u = u_0$ at $x = 0$ and $u = u_1$ at $x = 1$ the test function v still satisfies the homogeneous boundary conditions. ($v = 0$ at $x = 0, 1$) These are called "essential boundary conditions".

4) If the boundary conditions involve derivatives of u (called natural boundary conditions) then we do not specify the appropriate v, that is, if $u_0'(0)$ is involved then $v(0)$ is not specified

the variational formulation $\int_0^1 (-u_N'v'+u_Nv+x\cdot 1)dx \overset{(*)}{=} 0$ with $u_N(0)=0=u_N(1)$ and $v(0)=0=v(1)$. Let $v=\sum_{i=1}^N \gamma_i\beta_i$, $u_N=\sum_{j=1}^N a_j\beta_j$ and $\beta_i(0)=0=\beta_i(1)$

(*) yields $\left[\sum_i \gamma_i\left(\sum_j\left\{\int_0^1(-\beta_i'\beta_j'+\beta_i\beta_j)dx\right\}a_j+\int_0^1 x\beta_i dx\right)=0\right]$ $\forall i=1,2,\dots,N$

Define $k_{ij}=\int_0^1(-\beta_i'\beta_j'+\beta_i\beta_j)dx$, $F_i=-\int_0^1 x\beta_i dx$ then

$$\sum_{i=1}^N \gamma_i\left(\sum_{j=1}^N k_{ij}a_j-F_i\right)=0$$ γ_i's are arbitrary then to find a_i's

we use $\left[\sum_{j=1}^N k_{ij}a_j=F_i \text{ for } i=1,2,\dots,N\right]$ or "$Ka=F$"

where
$$K=\begin{pmatrix}k_{11} & k_{12} & \dots & k_{1N}\\ k_{21} & k_{22} & & k_{2N}\\ \vdots & & & \\ k_{N1} & k_{N2} & & k_{NN}\end{pmatrix}\quad a=\begin{pmatrix}a_1\\a_2\\ \vdots\\ a_N\end{pmatrix}\text{ and } F=\begin{pmatrix}F_1\\F_2\\ \vdots\\ F_N\end{pmatrix}$$

K is called "the stiffness matrix" and F is called "the load vector."

Example: $\frac{d^2u}{dx^2}+u=x^2;\ 0<x<1$

$u(0)=0,\ u(1)=0.$

E is defined by $E=u_N''+u_N-x^2$, for any sufficiently smooth function v,

$$\int_0^1 Ev\,dx=\int_0^1(u_N''+u_N-x^2)v\,dx=u_N'v\Big|_0^1-\int_0^1 u_N'v'dx+\int_0^1 u_Nv\,dx-\int_0^1 x^2v\,dx$$
$$=\int_0^1(-u_N'v'+u_Nv)dx+u_N'v\Big|_0^1-\int_0^1 x^2v\,dx$$

The boundary conditions require $\boxed{v(0)=v(1)=0}$ and allow $\boxed{u_N(0)=u_N(1)=0}$

then $\int_0^1 Ev\,dx=\int_0^1(-u_N'v'+u_Nv)dx-\int_0^1 vx^2dx=0$ (■)

Now set $u_N=\sum_{j=1}^N a_j\beta_j(x)$ where $\beta_j(0)=\beta_j(1)=0$ for which $\boxed{\beta_j(x)=x^j(1-x)}$

Take $v=\sum_{i=1}^N\gamma_i\beta_i(x)$ then substitute into (■) to give

$$\int_0^1\left\{-\left(\sum_j a_j\beta_j'(x)\right)\left(\sum\gamma_i\beta_i'\right)+\left(\sum_j a_j\beta_j\right)\left(\sum_j\gamma_i\beta_i\right)\right\}dx-\int_0^1 x^2\left(\sum_i\gamma_i\beta_i(x)\right)dx=0$$

which reduces to $\sum_i\gamma_i\left(\sum_j k_{ij}a_j-F_i\right)=0$ where

$k_{ij}=\int_0^1(-\beta_i'\beta_j'+\beta_i\beta_j)dx$ and $F_i=\int_0^1 x^2\beta_i dx$. then briefly we may rewrite all as

$$\boxed{\sum_j k_{ij}a_j=F_i}$$

$$K_{ij} = \int_0^1 -(ix^{i-1}-(i+1)x^i)(jx^{j-1}-(j+1)x^j) + (x^i - x^{i+1})(x^j - x^{j+1})\,dx$$

$$= -\frac{2ij}{(i+j)((i+j)^2-1)} + \frac{2}{(i+j+1)(i+j+2)(i+j+3)} \quad \text{and}$$

$$F_i = \int_0^1 x^2 x^i (1-x)\,dx = \frac{1}{(i+3)(i+4)}$$

Take $N=1$, $K_{11} = -\frac{2}{6} + \frac{2}{(3)(4)(5)} = -\frac{1}{3} + \frac{1}{30} = -\frac{9}{30} = -\frac{3}{10}$

$F_1 = \frac{1}{(4)(5)} = \frac{1}{20}$ $\quad K_{11}\cdot a_1 = F_1 \rightarrow \boxed{a_1 = \frac{1/20}{-3/10} = -\frac{1}{6}}$

$N=1 \rightarrow u_1 = \alpha_1 x(1-x)$ then $\boxed{u_1 = -\frac{1}{6}x(1-x)}$

Take $N=2$, $\boxed{u_2 = \alpha_1\beta_1 + \alpha_2\beta_2 = \alpha_1 x(1-x) + \alpha_2 x^2(1-x)}$

then $K_{11} = -\frac{3}{10}$ $\quad K_{12} = -\frac{3}{20} = K_{21}$ $\quad K_{22} = \frac{-13}{105}$ and $F_1 = \frac{1}{20}$, $F_2 = \frac{1}{30}$

$$\begin{bmatrix} -3/10 & -3/20 \\ -3/10 & -13/105 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1/20 \\ 1/30 \end{bmatrix} \rightarrow \alpha_1 = -0.0813 \text{ and } \alpha_2 = -0.1707$$

then $\boxed{u_2(x) = -0.0813x(1-x) - 0.1707x^2(1-x)}$

the same method yields for $N=3$

$$\boxed{u_3(x) = -0.0952x(1-x) - 0.1005x^2(1-x) - 0.0702x^3(1-x)}$$

and the exact solution of the ODE: $\boxed{u(x) = \frac{\sin x + 2\sin(1-x)}{\sin(1)} + x^2 - 2}$

For comparison of the approximate values and the actual value,

	u_1	u_2	u_3	u_{exact}
0.5	0.4167	0.4167	0.4076	0.4076

Now consider $u(0)=0$, and $u'(1)=1$ as BCs of the BVP. ⑬ 03/09/2023

So, this gives $u(0)=0 \to v(0)=0$

$$\int_0^1 Ev\,dx = \int_0^1 (-u_N'v' + u_N v)\,dx + \underbrace{u_N'(1)}_{1} v(1) - \underbrace{u_N'(0)v(0)}_{0} - \int_0^1 x^2 v\,dx \quad (*)$$

$$= \int_0^1 (-u_N'v' + u_N v)\,dx + v(1) - \int_0^1 x^2 v\,dx$$

In this case, the β_i should be selected to satisfy only the essential boundary condition that is $\beta_i(0)=0$. The function $\beta_i = x^i$ satisfies this condition.

Let $v = \sum_{i=1}^{N} \gamma_i \beta_i(x)$ and $u_N = \sum_{j=1}^{N} a_j \beta_j(x)$ putting into $(*)$

then we get, $\sum_i \gamma_i \left(\sum_j K_{ij} a_j - F_i \right) = 0 \to \sum_j K_{ij} a_j = F_i \quad i=1,2,\ldots$ where

$\boxed{K_{ij} = \int_0^1 (-\beta_i' \beta_j' + \beta_i \beta_j)\,dx}$ and $\boxed{F_i = \int_0^1 x^2 \beta_i\,dx - 1}$

we choose $\beta_i = x^i$ then $K_{ij} = \int_0^1 \left[-i x^{i-1} j x^{j-1} + x^i x^j \right] dx$

$$= \int_0^1 \left[-ij\, x^{i+j-2} + x^i x^j \right] dx = \frac{-ij}{i+j-1} + \frac{1}{i+j+1}$$

$\boxed{K_{ij} = \frac{-ij}{i+j-1} + \frac{1}{i+j+1}}$ and for F_i's ; $F_i = \int_0^1 x^2 x^i\,dx - 1 = \frac{1}{i+3} - 1$

$\boxed{F_i = \frac{1}{i+3} - 1}$

$N=1$, we have $\beta_1(x) = x$, $\alpha_1 = ?$ $\boxed{K_{11}\alpha_1 = F_1}$ and $\boxed{u_1(x) = \alpha_1 \beta_1(x)}$

$F_1 = \frac{1}{4} - 1 = -\frac{3}{4}$, $K_{11} = \frac{-1}{1} + \frac{1}{3} = -\frac{2}{3}$ then $\alpha_1 = \frac{3}{4}\,\frac{3}{2} = \frac{9}{8}$

then $\boxed{u_1(x) = \frac{9}{8}x}$

$N=2$, we have $\beta_1(x) = x$, $\beta_2(x) = x^2$ and $\boxed{u_2(x) = \alpha_1 x + \alpha_2 x^2}$

$F_1 = -\frac{3}{4}$, $F_2 = -\frac{4}{5}$, $K_{11} = -\frac{2}{3}$, $K_{12} = -\frac{3}{4}$, $K_{21} = -\frac{3}{4}$, $K_{22} = -\frac{17}{15}$

$$\begin{bmatrix} -2/3 & -3/4 \\ -3/4 & -17/15 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -4/5 \end{bmatrix} \to \begin{matrix} \alpha_1 = 1.295 \\ \alpha_2 = -0.1511 \end{matrix}$$ then $\boxed{u_2(x) = 1.295x - 0.1511x^2}$

$N=3$, we get $\boxed{u_3(x) = 1.283x - 0.1142x^2 - 0.02442x^3}$

and for comparison, $u_{exact}(x) = \frac{2\cos(1-x) - \sin x}{\cos(1)} + x^2 - 2$