Heat Equation PDE, Numerical Solutions with Finite Difference Method

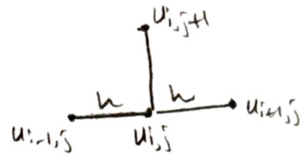
Parabolic Equation $\underbrace{\frac{\partial u}{\partial t}}_{\text{Forward Difference}} = \alpha \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{Center Difference}}$ 1-D Heat Conduction Equation 1

09/08/2023

1. Explicit Method

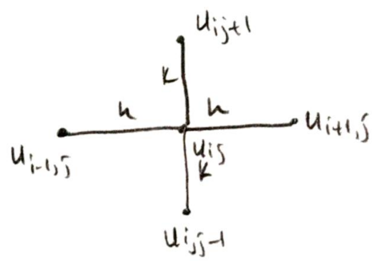
$$\frac{u_{i,j+1} - u_{i,j}}{k} = \alpha\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) \qquad \gamma = \frac{k\alpha}{h^2}$$

we get, Bender-Schmidt Formula



$$\boxed{u_{i,j+1} = \gamma u_{i-1,j} + (1-2\gamma)u_{ij} + \gamma u_{i+1,j}}$$

2. Implicit Method



$$\frac{u_{ij} - u_{i,j-1}}{k} = \alpha\left(\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}\right)$$

$$\gamma = \frac{\alpha k}{h^2}$$

$$\boxed{u_{ij} = -\gamma u_{i-1,j+1} + (1+2\gamma)u_{i,j+1} - \gamma u_{i+1,j+1}}$$

Example (1D-Heat Conduction) → Explicit Method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\alpha = 1.0$$

$$\gamma = \frac{\alpha . k}{h^2} = \frac{(1.0)(0.02)}{(0.2)^2} = 0.5$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x) \quad , 0 \leq x \leq 1$$

Using Bender-Schmidt Formula by taking $h = 0.2$ and $k = 0.02$.
Find all values of u from $t=0$ to $t=0.06$

$$u_{i,j+1} = \gamma u_{i-1,j} + (1-2\gamma) u_{i,j} + \gamma u_{i+1,j} \rightarrow \gamma = 1/2 \Rightarrow u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$$

$u(0,t) = 0$ $\quad$ $u(1,t) = 0$

t \ x	0	0.2	0.4	0.6	0.8	1
0.06	(0)	(0.3115)	(0.503)	(0.503)	(0.3115)	(0)
0.04	(0)	(0.3845)	(0.623)	(0.623)	(0.3845)	(0)
0.02	(0)	(0.4755) (*)	(0.769)	(0.769)	(0.4755)	(0)
$t=0$, $u(x,0)$	(0)	(0.587)	(0.951)	(0.951)	(0.587)	(0)

(*)

$$u(0.2, 0.02) = \frac{1}{2}[u(0,0.02) + u(0.4,0)]$$

$$= \frac{1}{2}(0 + 0.951)$$

$$= 0.4755$$

Example (1D Heat Equation) Implicit Method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \alpha = 1.0$$

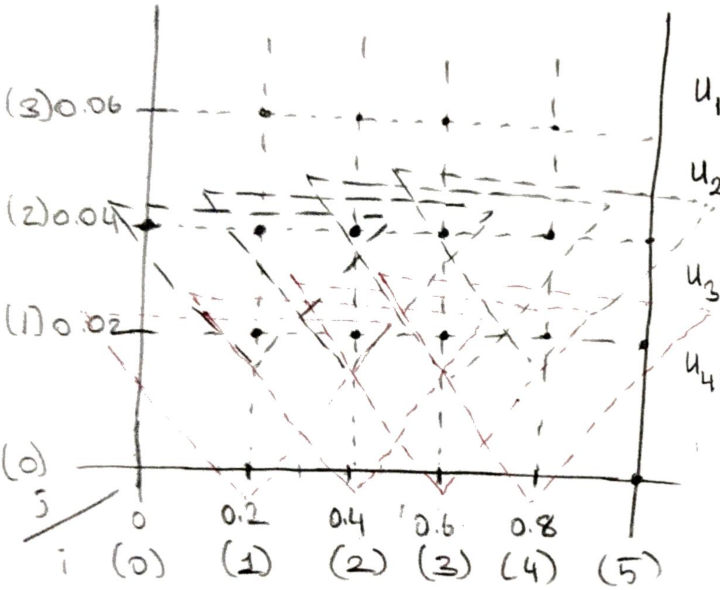
$$\gamma = \frac{\alpha k}{h^2} = \frac{(1.0)(0.02)}{(0.2)^2} = 0.5$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1$$

$h = 0.2$, $k = 0.02$ and find all u values from $t = 0.0$ to $t = 0.06$

Formula $\boxed{u_{ij} = -\frac{1}{2}u_{i-1,j+1} + 2u_{i,j+1} - \frac{1}{2}u_{i+1,j+1}}$



$$u_{11} = -\frac{1}{2}u_{02} + 2u_{12} - \frac{1}{2}u_{22}$$

$$u_{21} = -\frac{1}{2}u_{12} + 2u_{22} - \frac{1}{2}u_{32}$$

$$u_{31} = -\frac{1}{2}u_{22} + 2u_{32} - \frac{1}{2}u_{42}$$

$$u_{41} = -\frac{1}{2}u_{32} + 2u_{42} - \frac{1}{2}u_{52}$$

$$u_{10} = -\frac{1}{2}u_{01} + 2u_{11} - \frac{1}{2}u_{21}$$

$$u_{20} = -\frac{1}{2}u_{11} + 2u_{21} - \frac{1}{2}u_{31}$$

$$u_{30} = -\frac{1}{2}u_{21} + 2u_{31} - \frac{1}{2}u_{41}$$

$$u_{40} = -\frac{1}{2}u_{31} + 2u_{41} - \frac{1}{2}u_{51}$$

$$\begin{bmatrix} 2 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{bmatrix} = \begin{bmatrix} u_{10} + \frac{1}{2}u_{01} \\ u_{20} \\ u_{30} \\ u_{40} + \frac{1}{2}u_{51} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -1/2 & 0 & 0 \\ -1/2 & 2 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1/2 \\ 0 & 0 & -1/2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{bmatrix}}_{u_s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} u_{10} \\ u_{20} \\ u_{30} \\ u_{40} \end{bmatrix}}_{u_i} + \frac{1}{2} \underbrace{\begin{bmatrix} u_{01} \\ 0 \\ 0 \\ u_{51} \end{bmatrix}}_{u_b}$$

$$u_s = A^{-1} u_i + \frac{1}{2} A^{-1} u_b = A^{-1}\left(u_i + \frac{1}{2} u_b\right)$$

$$\begin{bmatrix} 2 & -1/2 & 0 & 0 \\ -1/2 & 2 & -1/2 & 0 \\ 0 & -1/2 & 2 & -1/2 \\ 0 & 0 & -1/2 & 2 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} u_{02} \\ 0 \\ 0 \\ u_{52} \end{bmatrix}$$