Vector Calculus

$a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

The dot product. $a \cdot b = |a||b|\cos(\theta) = a_1b_1 + a_2b_2 + a_3b_3 \qquad 0 \le \theta \le \pi$

The cross product. $a \times b = |a||b|\sin(\theta)\, n$ $\quad n$ is a unit vector perpendicular to the plane of a and b.

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$a \cdot b = 0 \rightarrow$ a and b are perpendicular to each other.

$a \times b = 0 \rightarrow$ a and b are parallel to each other.

A vector function: $V = u(x,y,z)\hat{i} + v(x,y,z)\hat{j} + w(x,y,z)\hat{k}$

the vector differential operator, "del" or "nabla",

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

the multivariable differentiable scalar function $F(x,y,z)$, gradient of F is given by

$$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

Example: $f(x,y,z) = x^2z^2\sin(4y)$

$$\nabla f = \frac{\partial}{\partial x}(x^2z^2\sin(4y))\hat{i} + \frac{\partial}{\partial y}(x^2z^2\sin(4y))\hat{j} + \frac{\partial}{\partial z}(x^2z^2\sin(4y))\hat{k}$$

$$\nabla f = 2xz^2\sin(4y)\hat{i} + 4x^2z^2\cos(4y)\hat{j} + 2x^2z\sin(4y))\hat{k}$$

Example: (the Unit Normal) $f(x,y,z) = x^2 + y^2 + z^2 = 1$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \qquad n = \frac{\nabla f}{\|\nabla f\|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i} + y\hat{j} + z\hat{k}$$

Example: The solution of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ yields the streamlines.

$F = \sin(z)\hat{j} + e^y\hat{k} \qquad \dfrac{dx}{0} = \dfrac{dy}{\sin(z)} = \dfrac{dz}{e^y}$ gives the point $(2,0,0)$

$dx = 0 \rightarrow x = c_1 \qquad \boxed{x = 2}$

$e^y dy = \sin(z)dz \qquad y=0,\ z=0$

$e^y = -\cos(z) + c_2 \rightarrow 1 = -1 + c_2 \rightarrow c_2 = 2$

$\boxed{e^y = 2 - \cos(z)}$

Example: Find ∇f $f(x,y,z)$ is a "scalar" function.

$$f(x,y,z) = \ln(x^2+y^2+z^2)$$

$$\nabla f = \frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{2z}{x^2+y^2+z^2}\hat{j}$$

Example! Find the unit normal of $f(x,y,z)$

$$z = x^2+y^2 \rightarrow f(x,y,z) = x^2+y^2-z.$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$n = \frac{\nabla f}{\|\nabla f\|} = \frac{(2x\hat{i}+2y\hat{j}-\hat{k})}{\sqrt{4x^2+4y^2+1}} = \frac{2x}{\sqrt{1+4z}}\hat{i} + \frac{2y}{\sqrt{1+4z}}\hat{j} - \frac{1}{\sqrt{1+4z}}\hat{k}$$

Example: Find the streamlines of $F(x,y,z)$ given the point $(2,0,4)$

$$F(x,y,z) = (1/x)\hat{i} + e^{y}\hat{j} - k$$

$$\frac{dx}{1/x} = \frac{dy}{e^y} = \frac{dz}{-1} \rightarrow \frac{dz}{-1} = \frac{dx}{1/x} \rightarrow \frac{dz}{dx} = -x \quad z = -\frac{1}{2}x^2 + c_2$$

$$\underbrace{4 = -2 + c_2 \rightarrow c_2 = 6}$$

$$\left.\begin{aligned} x\,dx &= e^{-y}dy \\ -e^{-y} &= \frac{x^2}{2} + c_1 \end{aligned}\right\} \quad \left.\begin{aligned} -1 &= 2 + c_1 \\ c_1 &= -3 \end{aligned}\right\} \quad \frac{x^2}{2} + e^{-y} = 3 \quad \boxed{y = -\ln\left(3 - \frac{x^2}{2}\right)} \qquad \boxed{z = 6 - \frac{1}{2}x^2}$$

Divergence

A vector field v defined in some region of three-dimensional space.

$v(r) = u(x,y,z)\hat{i} + v(x,y,z)\hat{j} + w(x,y,z)\hat{k}$ then

$$\operatorname{div}(v) = \nabla \cdot v = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(u\hat{i} + v\hat{j} + w\hat{k}\right) = u_x + v_y + w_z.$$

Example: Compute the divergence of F, $F = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$

$$\operatorname{div} F = 2xz + (-6y^2z^2) + xy^2 = 2xz - 6y^2z^2 + xy^2$$

Curl

$$\operatorname{curl}(v) = \nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = (w_y - v_z)\hat{i} + (u_z - w_x)\hat{j} + (v_x - u_y)\hat{k}$$

Example Show that $\nabla . \nabla \times F$ where $F = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y)\hat{i} - (-3xz^2)\hat{j} + (-4xyz\hat{k}$$

$$= (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

$$\underbrace{\nabla \times F}_{G} \rightarrow \nabla G = (4xy) + (0) - 4xy = 0 \rightarrow \nabla . \nabla \times F = 0.$$

Example: Compute $\nabla . F$, $\nabla \times F$, $\nabla . (\nabla \times F)$, $\nabla(\nabla . F)$

1) $F = x^2z\hat{i} + yz^2\hat{j} + xy^2\hat{k}$

$$\nabla F = 2xz + z^2 + 0 = \boxed{2xz + z^2 = \nabla F}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & yz^2 & xy^2 \end{vmatrix} = (2xy - 2yz)\hat{i} - (y^2 - x^2)\hat{j} + (0 - 0)\hat{k}$$

$$\boxed{\nabla \times F = (2xy - 2yz)\hat{i} + (x^2 - y^2)\hat{j}}$$

$$\nabla . (\nabla \times F) = \frac{\partial}{\partial x}(2xy - 2yz) + \frac{\partial}{\partial y}(x^2 - y^2) = 2y - 2y = \boxed{0 = \nabla(\nabla \times F)}$$

$$\boxed{\nabla(\nabla F) = 2z\hat{i} + (2x + 2z)\hat{k}} = \frac{\partial}{\partial x}(2xz + z^2)\hat{i} + \frac{\partial}{\partial y}(2xz + z^2)\hat{j} + \frac{\partial}{\partial y}(2xz + z^2)\hat{k}$$

2) $F = xe^{-y}\hat{i} + yz^2\hat{j} + 3e^{-z}\hat{k}$

$$\nabla . F = \frac{\partial}{\partial x}(xe^{-y}) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(3e^{-z}) = \boxed{e^{-y} + z^2 - 3e^{-z} = \nabla F}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^{-y} & yz^2 & 3e^{-z} \end{vmatrix} = (0 - 2yz)\hat{i} - (0 - 0)\hat{j} + (0 - (-xe^{-y}))\hat{k}$$

$$= \boxed{-2yz\hat{i} + xe^{-y}\hat{k} = \nabla \times F}$$

$$\nabla . (\nabla \times F) = \frac{\partial}{\partial x}(-2yz) + \frac{\partial}{\partial z}(xe^{-y}) = \boxed{0 = \nabla(\nabla \times F)}$$

$$\boxed{\nabla . (\nabla F) = -e^{-y}\hat{j} + (2z + 3e^{-z})\hat{k}} = \frac{\partial}{\partial y}(-e^{-y})\hat{j} + \frac{\partial}{\partial z}(z^2 - 3e^{-z})\hat{k}$$

Line Integrals

$$\int_C \vec{F}\,dr = \int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

If the starting and ending points are the same so that the contour is closed, then this closed contour integral will be denoted by $\oint_C$

Example: $F = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ Evaluate $\int_C \vec{F}.dr$ along $x=t, y=t^2$ and $z=t^3$ from $(0,0,0)$ to $(1,1,1)$.

$$\int_C F dr = \int_0^1 (3t^2+6t^2)(dt) - 14.t^2t^3 d(t^2) + 20.(t)(t^6)d(t^3)$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt = 3t^2 - 4t^7 + 6t^{10}\Big|_0^1 = 5.$$

Another path $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

$dx = dt \qquad dy = dt \qquad dz = dt$

$$\int_C F.dr = \int_0^1 3x^2 dx + \int_0^1 ((3.1+6y)(0) - 14y.0) + \int_0^1 20z^2 dz$$

$$= 1 + \frac{20}{3} = \frac{23}{3}.$$

Another path $x=y=z=t$; $0 \le t \le 1$

$$\int_C F.dr = \int_0^1 (3t^2+6t)dt - 14t^2 dt + 20t^3 dt = \int_0^1 (20t^3 - 11t^2 + 6t)dt$$

$$= \frac{13}{3}.$$

In each case, we obtained different results, Because the field F is not conservative. In conservative fields, the results are path independent.

Example: $F = y\sin(\pi z)\hat{i} + x^2e^y\hat{j} + 3xz\hat{k}$, the curve $x=t, y=t^2$ and $z=t^3$ from $(0,0,0)$ to $(1,1,1)$.

$$\int_C \vec{F}.dr = \int_0^1 t^2\sin(\pi t^3)dt + t^2e^{t^2}2t.dt + 3t.t^3.3t^2dt = \int_0^1 (9t^6 + 2t^3e^{t^2} + t^2\sin(\pi t^3))dt$$

$$= \frac{16}{9} + \frac{2}{3\pi}.$$

The Potential Function

the curl operation applied to a gradient produces the zero vector $\nabla \times \nabla \varphi = 0$
Consequently, If we have a vector field F such that $\nabla \times F = 0$ everywhere then that vector field is called "a conservative" field and we can compute a potential φ such that $F = \nabla \varphi$

Example: $F = ye^{xy}\cos(z)\hat{i} + xe^{xy}\cos(z)\hat{j} - e^{xy}\sin(z)\hat{k}$

$\nabla \times F = 0$?

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy}\cos(z) & xe^{xy}\cos(z) & -e^{xy}\sin(z) \end{vmatrix}$$

$$= \hat{i}\left(-e^{xy}x\sin(z) - (-xe^{xy}\sin(z)\right) - \hat{j}\left(-ye^{xy}\sin z - (-ye^{xy}\sin z)\right)$$
$$+ \hat{k}\left(\cos(z)(e^{xy} + xye^{xy}) - \cos z(e^{xy} + xye^{xy})\right)$$

$$= 0.$$

Then F is a conservative field.

$$\nabla\varphi = F \rightarrow \left.\begin{aligned} \varphi_x &= ye^{xy}\cos(z) \\ \varphi_y &= xe^{xy}\cos(z) \\ \varphi_z &= -e^{xy}\sin(z) \end{aligned}\right\} \quad \begin{aligned} &\varphi(x,y,z) = e^{xy}\cos(z) + \phi(y,z) \\ &\varphi_y = x e^{xy}\cos(z) + \underbrace{\frac{\partial \phi(y,z)}{\partial y}}_{0} = xe^{xy}\cos(z) \end{aligned}$$

$$\varphi_z = -e^{xy}\sin(z) + \underbrace{h'(z)}_{0} = -e^{xy}\sin(z) \rightarrow \varphi(x,y,z) = e^{xy}\cos(z) + \underset{\text{Constant.}}{\underset{\downarrow}{C}}$$

For Line Integrals,

$$\int_C F \cdot dr = \int_C \varphi_x dx + \varphi_y dy + \varphi_z dz = \int_C d\varphi = \varphi(B) - \varphi(A) \quad \text{(Path Independence)}$$

if F is a conservative field

e.g. $\displaystyle\int_C F\,dr = \left[e^{xy}\cos(z) + C\right]_{(0,0,0)}^{(-1,2,\pi)} = -1 - e^{-2}$ ☒

from $(0,0,0)$ to $(-1,2,\pi)$.

Example! $F = e^{2z}\hat{\imath} + 3y^2\hat{\jmath} + 2xe^{2z}\hat{k}$

$$\nabla F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{2z} & 3y^2 & 2xe^{2z} \end{vmatrix} = (0-0)\hat{\imath} - (2e^{2z} - 2e^{2z})\hat{\jmath} + (0-0)\hat{k} = 0$$

F is a conservative field. then $\varphi_x = e^{2z}$, $\varphi_y = 3y^2$ and $\varphi_z = 2xe^{2z}$

$$\varphi(x,y,z) = xe^{2z} + \phi(y,z) \rightarrow \varphi_y = \frac{\partial}{\partial y}\phi(y,z) = 3y^2 \rightarrow \phi(y,z) = y^3 + h(z)$$

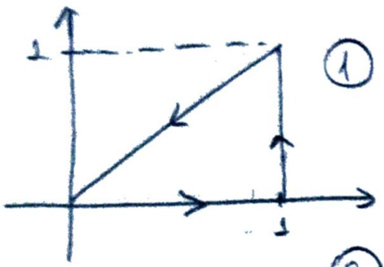
$$\varphi(x,y,z) = xe^{2z} + y^3 + h(z) \rightarrow \varphi_z = 2xe^{2z} + h'(z) = 2xe^{2z} \Rightarrow h'(z) = 0, \quad h(z) = C$$

then $\boxed{\varphi(x,y,z) = xe^{2z} + y^3 + C}$

Green's Lemma

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA \quad \text{or} \quad \oint_C F dr = \iint_R \nabla\times F\, k\, dA$$

Example! $F = -y^2\hat{\imath} + x^2\hat{\jmath}$ and $x=1$, $y=0$ and $y=x$ (Bounded region)

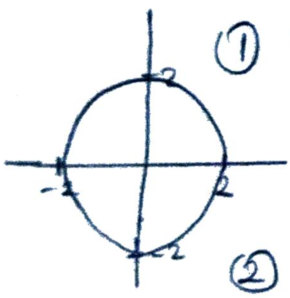


① $$\oint_C F dr = \int_0^1 0\,dx + \int_0^1 1.dy + \int_1^0 -x^2dx + x^2dx = 1$$

② $$\int_0^1\int_0^x (2x+2y)dydx = \int_0^1 3x^2dx = x^3\Big|_0^1 = 1$$

$$\left\{2xy + y^2\right\}_0^x = 3x^2$$

Example $F = -y^2\hat{\imath} + x^2\hat{\jmath}$ and $x^2+y^2=4$ $\quad x = 2\cos t, y = 2\sin t$



① $$\int_C F.dr = \int_0^{2\pi} 4\sin^2 t(-2\sin t)dt + 4\cos^2 t.2\cos t\,dt$$

$$= \int_0^{2\pi} 8(\cos^2 t - \sin^2 t)dt = \int_0^{2\pi} 8.\cos 2t\,dt = 4\sin 2t\Big|_0^{2\pi} = 0$$

② $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x+2y)dydx = \int_{-2}^{2}\left\{2xy + y^2\right\}_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 2x(2\sqrt{4-x^2})dx = 0$$