Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

The Fourier series converges to the function "in the mean" over the entire interval $(-L, L)$. This convergence generally improves as more sines and cosines (harmonics) are included.

$-L \leq x \leq L$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi x}{L}\right)dx$$

Dirichlet's Theorem: If for the interval $[-L, L]$, the function $f(t)$ (1) is single-valued, (2) is bounded, (3) has at most a finite number of maxima and minima and (4) has only a finite number of discontinuities (piecewise-continuous) and if (5) $f(t+2L) = f(t)$ for values of t outside of $[-L, L]$ then Fourier series of $f(t)$ converges to $f(t)$ as $N \to \infty$ at values of t for which $f(t)$ is continuous and to $\frac{1}{2}[f(t^-) + f(t^+)]$ at points of discontinuity.

Example)

$$f(t) = \begin{cases} 0, & -\pi < t \leq 0 \\ t, & 0 \leq t \leq \pi \end{cases}$$

Find Fourier series of $f(t)$

$L = \pi$, $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{0}^{\pi} t\,dt = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2} \to \boxed{a_0 = \pi/2}$

$$a_n = \frac{1}{\pi}\int_0^{\pi} t\cos\left(\frac{n\pi t}{\pi}\right)dt = \frac{1}{\pi}\int_0^{\pi} t\cos(nt)\,dt$$

$$\begin{array}{ll} +t & \cos(nt) \\ -1 & \frac{1}{n}\sin(nt) \\ +0 & -\frac{1}{n^2}\cos(nt) \end{array}$$

$$= \frac{1}{\pi}\left\{\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right\}_0^{\pi} = \frac{1}{\pi}\left\{\frac{\cos(n\pi) - 1}{n^2}\right\}$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \to \boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$$

$$\begin{array}{ll} +t & \sin(nt) \\ -1 & -\frac{1}{n}\cos(nt) \\ +0 & -\frac{1}{n^2}\sin(nt) \end{array}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\sin\left(\frac{n\pi t}{\pi}\right)dt = \frac{1}{\pi}\int_0^{\pi} t\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left\{-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right\}_0^{\pi} = \frac{1}{\pi}\left\{-\frac{\pi}{n}\cos(n\pi)\right\}$$

$$= \frac{(-1)^{n+1}}{n} \to \boxed{b_n = \frac{(-1)^{n+1}}{n}}$$

$$\boxed{f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n - 1}{n^2\pi}\right)\cos(nt) + \left(\frac{(-1)^{n+1}}{n}\right)\sin(nt)\right]}$$

Example: $f(t) = |t|$, $-\pi \le t \le \pi$ Find the Fourier series of $f(t)$. ② 31/08/2023

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -t\,dt + \int_{0}^{\pi} t\,dt\right\}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{t^2}{2}\right\}_{-\pi}^{0} + \left\{\frac{t^2}{2}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \left(-\frac{\pi^2}{2}\right) + \frac{\pi^2}{2} - 0\right]$$

$$= \pi \longrightarrow \boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(nt)\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -t\cos(nt)\,dt + \int_{0}^{\pi} t\cos(nt)\,dt\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right\}_{-\pi}^{0} + \frac{1}{\pi}\left\{\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right\}_{0}^{\pi}$$

$$= -\frac{1}{\pi}\left\{\frac{1}{n^2} - \frac{1}{n^2}(-1)^n\right\} + \frac{1}{\pi}\left\{\frac{1}{n^2}(-1) - \frac{1}{n^2}\right\}$$

$$= \frac{2}{n^2\pi}\left((-1)^n - 1\right) \longrightarrow \boxed{a_n = \frac{2\left[(-1)^n - 1\right]}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\underbrace{|t|\sin(nt)}_{\text{odd}}}^{g(t)}\,dt = 0$$

$$g(-t) = |-t|\sin(n(-t)) = \underbrace{-|t|\sin(nt)}_{-g(t)} = -g(t)$$

g is an odd function.

$$|t| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n - 1\right)}{n^2\pi}\cos(nt) = \boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)t}{(2m-1)^2} = |t|}$$

for $t \in [-\pi, \pi]$

③ 31/08/2023

$$f(t) = \begin{cases} 0, & -a < t < 0 \\ 2t; & 0 < t < a \end{cases}$$ Find the Fourier series of $f(t)$.

$$a_0 = \frac{1}{a}\int_{-a}^{a} f(t)\,dt = \frac{1}{a}\int_0^a 2t\,dt = \frac{1}{a}t^2\Big|_0^a = \frac{a^2}{a} = a \rightarrow a_0 = a$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(t)\cos\left(\frac{n\pi t}{a}\right)dt = \frac{1}{a}\int_0^a 2t\cos\left(\frac{n\pi t}{a}\right)dt$$

$$\begin{array}{ll} +2t & \cos\left(\frac{n\pi t}{a}\right) \\ -2 & \frac{a}{n\pi}\sin\left(\frac{n\pi t}{a}\right) \\ +0 & -\frac{a^2}{n^2\pi^2}\cos\left(\frac{n\pi t}{a}\right) \end{array}$$

$$= \frac{1}{a}\left\{2t\frac{a}{n\pi}\sin\left(\frac{n\pi t}{a}\right) + \frac{2a^2}{n^2\pi^2}\cos\left(\frac{n\pi t}{a}\right)\right\}_0^a$$

$$= \frac{1}{a}\left(\frac{2a^2}{n^2\pi^2}(-1)^n - \frac{2a^2}{n^2\pi^2}\right) = \frac{2a}{n^2\pi^2}\left[(-1)^n - 1\right] \rightarrow a_n = \frac{2a}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(t)\sin\left(\frac{n\pi t}{a}\right)dt = \frac{1}{a}\int_0^a 2t\sin\left(\frac{n\pi t}{a}\right)dt$$

$$\begin{array}{ll} +2t & \sin\left(\frac{n\pi t}{a}\right) \\ -2 & -\frac{a}{n\pi}\cos\left(\frac{n\pi t}{a}\right) \\ +0 & -\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi t}{a}\right) \end{array}$$

$$= \frac{1}{a}\left\{-\frac{2ta}{n\pi}\cos\left(\frac{n\pi t}{a}\right) + \frac{2a^2}{n^2\pi^2}\sin\left(\frac{n\pi t}{a}\right)\right\}_0^a$$

$$= \frac{1}{a}\left\{-\frac{2a^2}{n\pi}(-1)^n\right\} = \frac{2a}{n\pi}(-1)^{n+1} \Rightarrow b_n = -\frac{2a}{n\pi}(-1)^n$$

$$f(t) = \frac{a}{2} + \sum_{n=1}^{\infty}\frac{2a}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi t}{a}\right) - \sum_{n=1}^{\infty}\frac{2a}{n\pi}(-1)^n\sin\left(\frac{n\pi t}{a}\right)$$

$$\boxed{f(t) = \frac{a}{2} - \frac{4a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left[\frac{(2m-1)\pi t}{a}\right] - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi t}{a}\right)}$$

Properties of Fourier Series

1) Differentiation of Fourier Series

A function $f(t)$ whose derivative $f'(t)$ is continuous except for a finite number of discontinuities and $f(\tau) = f(\tau + 2L)$ then

$$f'(t) = \sum_{n=1}^{\infty} \frac{n\pi}{L}\left[b_n \cos\left(\frac{n\pi}{L}t\right) - a_n \sin\left(\frac{n\pi}{L}t\right)\right]$$

2) Integration of Fourier Series

$$\int_0^t f(\tau)\,d\tau = \frac{a_0 t}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin(n\pi t/L) - b_n \cos(n\pi t/L)}{n\pi/L}$$

where $A_n = \frac{-b_n}{n\pi L}$ and $B_n = \frac{a_n}{n\pi/L}$

Example! for $-\pi < t < \pi$

$$f(t) = t = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nt)$$

$$\left.\frac{\tau^2}{2}\right|_0^t = \left. 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\tau)\right|_0^t$$

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt) - 2\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}$$

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt) - 2\left(-\frac{\pi^2}{12}\right)$$

$$\boxed{t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nt)}$$

3) Parseval's Equality

$$\frac{1}{L}\int_{\tau}^{\tau+2L} f^2(t)\,dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$f(t)$ is a function whose period is $2L$. Parseval's Equality sums squares of Fourier coefficients.

e.g. $\frac{1}{\pi}\int_{-\pi}^{\pi} t^4\,dt = \frac{4\pi^4}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \rightarrow \frac{1}{\pi}\frac{2\pi^5}{5} - \frac{4\pi^4}{18} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}$

$$\left[\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4}\right]$$

Half-Range Expansion

A Fourier series representation for a function $f(x)$ that applies over the interval $(0,L)$ rather than $(-L,L)$. We know that if $f(x)$ is an even function then $b_n=0$ for all n. Similarly, $f(x)$ is an odd function, then $a_0, a_n = 0$ for all n.

If we extend $f(x)$ as an even function, we will get a half range cosine series; if we extend $f(x)$ as an odd function, we obtain a half-range sine series.

For any $f(x)$ we can construct either a Fourier sine or cosine series over the interval $(-L,L)$. Both of these series will give the correct answer over the interval of $(-L,L)$.

Example: $f(x)=1, \quad 0<x<\pi$

$$\tilde{f}(x) = \begin{cases} -1, & -\pi<x<0 \\ 1, & 0<x<\pi \end{cases} \quad \text{Odd extension of } f(x).$$

$\tilde{f}(x+2\pi) = \tilde{f}(x)$ and $\tilde{f}(x)$ is odd then $a_n = a_0 = 0$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1.\sin(nx)dx = -\frac{2}{n\pi}\left[(-1)^n - 1\right]$$

then $$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n}\sin(nx) = 4\sum_{m=1}^{\infty}\frac{\sin\left[(2m-1)x\right]}{2m-1}.$$

$\tilde{f}(x) = 1, \quad -\pi<x<\pi$ Even extension of $f(x)$ and $\tilde{f}(x+2\pi)=\tilde{f}(x)$

then $b_n=0$, for all n.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} 1\,dx = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos(nx)dx = 0$$

then the half-range cosine expansion equals the single term

$$f(x) = 1, \quad 0<x<\pi.$$