Mathematical Physics

Hard problem (HP)

① Insert ϵ $HP(\epsilon)$

② Assume answer : $ANS(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$

③ Add up series.

Reducing HP to relatively easy problem!

Example: $x^5+x=1$ "HP" find real root? 0.755.

① $x^5+\epsilon x=1$ "strong coupling expansion"

$\epsilon=0$, the problem is "unperturbed" problem

$x^5=1 \rightarrow x=1$

② $ANS(\epsilon) = 1+a\epsilon+b\epsilon^2+c\epsilon^3+\dots$

$(1+s)^5 = 1+5s+10s^2+\dots$; $s=a\epsilon+b\epsilon^2+c\epsilon^3+\dots$

$(1+s)^5 = 1+5(a\epsilon+b\epsilon^2+c\epsilon^3+\dots)+10(a^2\epsilon^2+2ab\epsilon^3+\dots)$

$(x^5+\epsilon x=1)$ $= \underbrace{1+5a\epsilon+\epsilon^2(5b+10a^2)+\epsilon^3(5c+20ab)}_{x^5}+\dots+\underbrace{\epsilon+\epsilon^2 a+\epsilon^3 b+\dots}_{\epsilon x}=1$

ϵ^0: 1

ϵ^1. $5a+1=0$ $\quad a=-1/5$

ϵ^2. $5b+10a^2+a=0$ $\quad b=-1/25$ $\quad c=-\frac{1}{125}$

$ANS(\epsilon) = 1-\frac{1}{5}\epsilon-\frac{1}{25}\epsilon^2-\frac{1}{125}\epsilon^3+\dots$ ← Radius of convergence: 1.64...

③ $ANS(\epsilon=1) = 1-0.2-0.04-0.008=0.752$.

Example! ① $x^3+\epsilon x=1$ $(\epsilon=1)$ real roots?

② $ANS(\epsilon) = \sum_{n=0}^{\infty} a_n\epsilon^n = 1+a\epsilon+b\epsilon^2+c\epsilon^3+\dots$ ($\epsilon=0$)

$x=1+s=1+a\epsilon+b\epsilon^2+c\epsilon^3+\dots$

$x^3=(1+s)^3=1+3s+3s^2+s^3 = 1+(3a\epsilon+3b\epsilon^2+3c\epsilon^3+\dots)+3(a^2\epsilon^2+2ab\epsilon^3+\dots)$

$\epsilon x = \epsilon+a\epsilon^2+b\epsilon^3+c\epsilon^4+\dots$

$x^3+\epsilon x = 1+(1+3a)\epsilon+(3b+3a^2+a)\epsilon^2+(3c+2ab+b)\epsilon^3+\dots=1$

ϵ^0. $1=1$.

ϵ^1 $1+3a=0 \rightarrow a=-\frac{1}{3}$

ϵ^2. $3b+3a^2+a=0 \rightarrow 3b+3.\frac{1}{9}-\frac{1}{3}=0 \rightarrow b=0$

ϵ^3: $3c+2ab+b=0 \rightarrow c=0$

$\Big\}$ $ANS(\epsilon) = 1-\frac{1}{3}\epsilon$

③ $ANS(\epsilon=1) = 1-\frac{1}{3} = \frac{2}{3} = 0.666\dots$

Example: ① $\epsilon x^5 + x = 1$

$\epsilon = 0 \rightarrow x = 1$

② $x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \dots$

$$\epsilon\left(1 + 5a\epsilon + \epsilon^2(5b + 10a^2) + \dots\right) + 1 + a\epsilon + b\epsilon^2 + c\epsilon^3 + \dots = 1$$

$\epsilon^0: 1 = 1$

$\epsilon^1: 1 + a = 0 \rightarrow a = -1$

$\epsilon^2: 5a + b = 0 \rightarrow b = +5$

$\epsilon^3: 5b + 10a^2 + c = 0 \rightarrow c = -35$

$x(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + \dots$

$x(\epsilon = 1) = -25 \rightarrow$ "Diverges!"

Radius of convergence 0.08192

Note: $x^5 + \epsilon x = 1$ does not make any problem even if $\epsilon = 0$, it has five roots but $\epsilon x^5 + x = 1$ creates problem if $\epsilon = 0$ it has only one root!

Asymptotics "$\sim$" $\rightarrow$ "is asymptotic to"

$f(x) \sim g(x)$ as $x \rightarrow x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$

complicated function, simple

$\rightarrow$ Nothing asymptotic to "0"

i.e. $\sin x \sim x$ as $x \rightarrow 0$

$e^x \sim x$ as $x \rightarrow 0$

"$<<$" $\rightarrow$ "is neglig compared with" $\boxed{x << 1 \text{ as } x \rightarrow 0}$

$f(x) << g(x)$ as $x \rightarrow x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

General statement: $f(x) + g(x) = h(x)$ if $g(x) << f(x)$ as $x \rightarrow x_0$ then $f(x) \sim h(x)$

$\epsilon x^5 + x = 1$ "method of dominant balance"

$(\epsilon \rightarrow 0)$

① $\epsilon x^5 \sim -x$ as $\epsilon \rightarrow 0$

$\epsilon x^4 \sim -1$ $(\epsilon \rightarrow 0)$

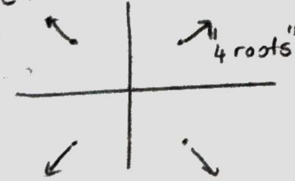
$x^4 \sim -\frac{1}{\epsilon}$ $(\epsilon \rightarrow 0)$

$x \sim \left(-\frac{1}{\epsilon}\right)^{1/4}$ $(\epsilon \rightarrow 0)$

$x \sim \frac{\omega}{\epsilon^{1/4}}$ $\boxed{\omega^4 = -1}$ as $\epsilon \rightarrow 0$

$1 << \frac{\omega}{\epsilon^{1/4}}$

$\epsilon \rightarrow 0$



OK ✓

② $\epsilon x^5 \sim 1$ as $\epsilon \rightarrow 0$

$x^5 \sim \frac{1}{\epsilon}$ as $\epsilon \rightarrow 0$

$x^5 \sim \frac{1}{\epsilon^{1/5}}$ as $\epsilon \rightarrow 0$

NO ✗

Not valid!

③ $x \sim 1$ as $\epsilon \rightarrow 0$

OK ✓

Now, BCs: $a_n(0)=0$; $a_n'(0)=0$ $n>0$ then taking integral to $a_n''(x) = -Q(x)\,a_{n-1}(x)$ gives the form $\boxed{a_n(x) = -\int_0^x dT \int_0^t ds\, Q(s)\, a_{n-1}(s)}$ is the general solution.

then the last term $a_n(x) = (-1)^n \underbrace{\int\int Q \int\int Q \int\int Q \int\int Q(\alpha+\beta x)}_{n \text{ times}}$ then ($\int_a^b f(x)dx \leq |b-a|M$, $f(x) \leq M$)

$|a_n(x)| \leq M^n n \underbrace{\int\int\int \dots}_{2n \text{ Integrals}} \rightarrow \boxed{|a_n(x)| \leq \frac{M^n n\, x^{2n}}{(2n)!}} \rightarrow$ Radius of convergence is infinity for ϵ.

This gives a solution for a finite domain but this does not work for an infinite domain

Eigenvalue Problem

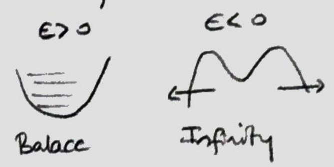
$$\left(-\frac{d^2}{dx^2} + V(x)\right)\Psi(x) = E\,\Psi(x)$$

Harmonic oscillator $V(x) = x^2/4$; $E_n = n + \frac{1}{2}$

Hard problem $\rightarrow$ $\left(-\frac{d^2}{dx^2} + \underbrace{\frac{x^2}{4} + \frac{x^4}{4}}_{V(x)}\right)\Psi(x) = E\,\Psi(x)$ $\boxed{F(x) = -\frac{dV}{dx} = -\left(\frac{x}{2} + x^3\right)}$

Inserting ϵ, $\left(-\frac{d^2}{dx^2} + \frac{x^2}{4} + \epsilon\frac{x^4}{4}\right)\psi(x) = E(\epsilon)\Psi(x) \rightarrow$ Perturbation, "$\epsilon=0$; Unperturbed problem"

Assume $E(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$ and $\Psi(x) = \sum_{n=0}^{\infty} \Psi_n(x)\epsilon^n$



Ground state: the lowest eigenvalue

$a_0 = \frac{1}{2}$ $\Psi_0(x) = e^{-x^2/4}$

$\epsilon = 0$; Singularity!

$E_{\text{ground state}} = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 - \frac{333}{16}\epsilon^3 + \dots$ (diverges)

$$\boxed{(\text{as } n\to\infty) \quad a_n \sim (-1)^{n+1}\frac{\sqrt{6}}{\pi^{3/2}}\, 3^n\, \Gamma\left(n+\tfrac{1}{2}\right)}$$

roughly $E(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n n!\, 3^n (-1)^n$

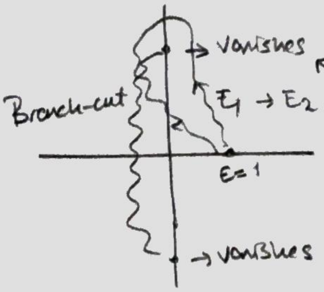
$\downarrow$ Radius of convergence is "0"

Hamiltonian $H = \underbrace{\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}}_{\begin{pmatrix}1\\0\end{pmatrix}_a \ \begin{pmatrix}0\\1\end{pmatrix}_b} + \underset{\downarrow \text{ Perturbation}}{\epsilon}\begin{pmatrix} 0 & c \\ c & 0\end{pmatrix} \rightarrow H = \begin{pmatrix} a & \epsilon c \\ \epsilon c & b\end{pmatrix}$

$$\det\begin{pmatrix} a-E & \epsilon c \\ \epsilon c & b-E\end{pmatrix} = E^2 - (a+b)E + ab - \epsilon^2 c^2$$

"Radius of convergence"

$$E = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab + 4\epsilon^2c^2}}{2} = \frac{a+b \pm \sqrt{(a-b)^2 + 4\epsilon^2 c^2}}{2}$$



$\epsilon = \pm\frac{(a-b)}{2c}i$

(singular points)

Quantum mechanics does not have a quantized energy level.

Smoothly from an energy level to another continuously!

Analytic continuation!

$y'' + Q(x) y = 0$ ← Schrödinger Equation

③ 07/10/2023

lecture 2

$y'' + a(x) y' + b(x) y = 0 \quad (*)$ $\quad y = u(x) z(x)$ subs. into $(*)$

$y' = u'z + uz'$ and $y'' = u''z + 2u'z' + uz''$

$u''z + \underline{2u'z'} + uz'' + a(x)u'z + \underline{a(x)uz'} + b(x)uz = 0$

$2u' + a(x) u = 0 \rightarrow \frac{u'}{u} = -\frac{a(x)}{2} \rightarrow \boxed{u(x) = e^{-\int^x \frac{a(s)}{2} ds}}$ then reduces $(*)$ to Schrödinger Equation.

$\frac{d}{dx} = D$

$(D^2 + a(x) D + b(x)) y(x) = 0 = (D + A(x)) \underbrace{(D + B(x)) y(x)}_{w(x)} = \boxed{w' + A(x) w = 0}$

$w \checkmark \rightarrow \boxed{y' + B(x) y = w(x)}$

$(D^2 + (A+B) D + B' + AB) y(x) = (D^2 + a(x) D + b(x)) y(x).$

$A + B = a(x) \rightarrow A = a(x) - B$

$B' + AB = b(x) \rightarrow aB - B^2 + B' = b \rightarrow B' = B^2 - aB + b$ "Riccati Equation"

Riccati Equation: $y' = \alpha y^2 + \beta y + \gamma \quad y = Q \frac{w'}{w}$

$$Q' \frac{w'}{w} + Q \frac{w''}{w} - Q \frac{(w')^2}{w^2} = \alpha Q^2 \frac{(w')^2}{w^2} + \beta Q \frac{w'}{w} + \gamma$$

$$-Q = \alpha Q^2 \rightarrow \boxed{Q = -\frac{1}{\alpha}} \rightarrow \left[Q(x) = -\frac{1}{\alpha(x)} \right]$$

$$\boxed{Q' w' + Q w'' = \beta Q w' + \gamma w}$$ → Linearized!

$\rightarrow B' = B^2 - aB + b$

$1 \rightarrow Q = -\frac{1}{1} \Rightarrow \boxed{Q = -1}$ then $B = -\frac{w'}{w}$

$$-\frac{w''}{w} + \cancel{\frac{(w')^2}{w^2}} = \cancel{\frac{(w')^2}{w^2}} + a \frac{w'}{w} + b \rightarrow \boxed{w'' + a w' + b w = 0}$$

Hard problem! $y'' + Q(x) y = 0 \quad y(0) = \alpha, \ y'(0) = \beta$

→ Inserting ϵ, $y'' + \epsilon Q(x) y = 0$

Unperturbed problem $(\epsilon = 0)$: $y_0'' = 0 \rightarrow y_0(x) = \alpha + \beta x$

Assume $y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$ where $a_0(x) = \alpha + \beta x$ then inserting into $y'' + \epsilon Q(x) y = 0$

$$\sum_{n=0}^{\infty} a_n''(x) \epsilon^n + \sum_{n=0}^{\infty} Q(x) a_n(x) \epsilon^{n+1} = 0 \quad (n \rightarrow n-1) \rightarrow \sum_{n=0}^{\infty} a_n''(x) \epsilon^n + \sum_{n=1}^{\infty} Q(x) a_{n-1}(x) \epsilon^n = 0$$

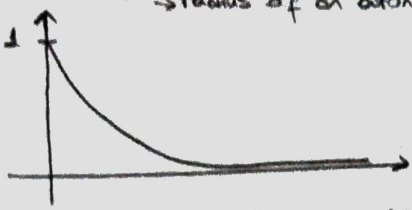
ϵ^0: $a_0''(x) = 0 \rightarrow a_0(x) = \alpha + \beta x$

$\epsilon^n, n > 0$: $a_n''(x) = -Q(x) a_{n-1}(x)$ → Initial conditions: $a_n(0) = 0 \quad a_n'(0) = 0, n > 0$ because $y(x) = \alpha + \beta x + a_1(x)\epsilon + a_2(x)\epsilon^2 + \dots$ already satisfies BCs!

Thomas Fermi Equation "Charge Distribution"

$$y'' = \frac{y^{3/2}}{\sqrt{x}} \quad y(0)=1,\ y(\infty)=0 \quad \text{"BVP"}$$

$\sqrt{x}$ → radius of an atom



Unperturbed problem

$$y_0'' + y_0 = 0 : y(0) = 1,\ y(\infty) = 0$$

$$y_0(x) = e^{-x}$$

ϵ : the measure of non-linearity! ϵ increases nonlinearity increases!

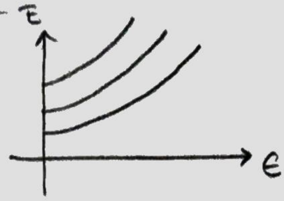
then $y'' = y\left(\frac{y}{x}\right)^{\epsilon}$ $\quad y = \sum_{n=0}^{\infty} y_n(x)\,\epsilon^n$ where $y_0(x) = e^{-x}$

KdV Equation (Nonlinear wave equation)

$$u_t + u u_x + u_{xxx} = 0 \rightarrow u_t + u^{\epsilon} u_x + u_{xxx} = 0\ ;\ \epsilon = 0,\ \text{linear!}$$

Harmonic Oscillator

$$H = p^2 + x^2 (ix)^{\epsilon}$$



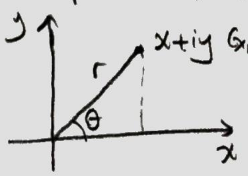
Eigenvalue Problem

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \epsilon \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} = \begin{pmatrix} a & \epsilon c \\ \epsilon c & b \end{pmatrix}$$

$$E = \frac{a + b + \sqrt{(a-b)^2 + 4\epsilon^2 c^2}}{2}$$

$$\boxed{x \rightarrow z = x + iy = re^{i\theta}} \ (?)$$ Is it definable?

$$f(z) = z,\ z^2,\ \frac{z+1}{z^2+1},\ \ldots,\ \sqrt{z}$$

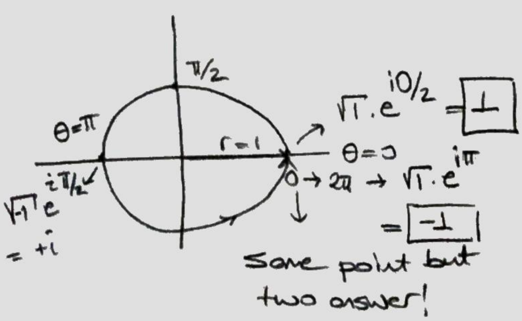


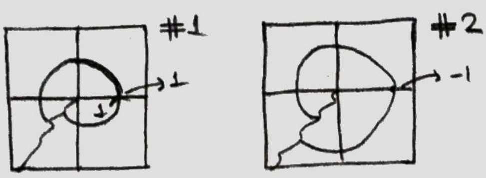
$$z_1 = r_1 e^{i\theta_1},\ z_2 = r_2 e^{i\theta_2}$$

$$\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}}$$

$$\sqrt{z} \rightarrow \sqrt{r}\, e^{i\theta/2} \Rightarrow \left(\sqrt{r}\, e^{i\theta/2}\right)^2 = re^{i\theta}$$

defining theta θ is important!





Riemann Surface!

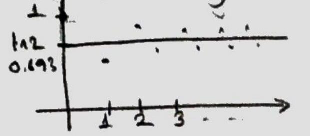
Series Sum

$$S = \sum_{n=0}^{\infty} a_n \qquad S_N = a_0 + a_1 + \cdots + a_N$$

$$\lim_{N\to\infty} S_N = S. \ \text{(Convergent)}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = \ln 2.$$

Oscillating Series.



Shanks (Model)

$$S_{N-1} = L + AB^{N-1}$$

$$S_N = L + AB^N$$

$$S_{N+1} = L + AB^{N+1}$$

$$\frac{S_N - L}{S_{N-1} - L} = B = \frac{S_{N+1} - L}{S_N - L}$$

$$S_N^2 - 2S_N L + \cancel{L^2} = S_{N+1} S_{N-1} - L(S_{N+1} + S_{N-1}) + \cancel{L^2}$$

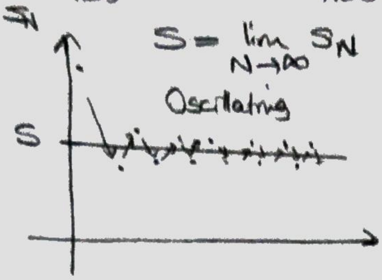
$$\boxed{L = \frac{S_N^2 - S_{N+1} S_{N-1}}{2S_N - S_{N+1} - S_{N-1}}}$$

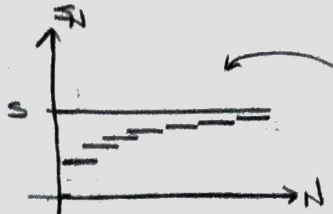
Shanks Transformation

⑥ 08/10/2023
Lecture 4

$$S = \sum_{n=0}^{\infty} a_n \qquad S_N = \sum_{n=0}^{N} a_n$$

$$S = \lim_{N\to\infty} S_N$$





Example!

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$S - S_N = \sum_{N+1}^{\infty} \frac{1}{n^2} \sim \int_{N+1}^{\infty} \frac{dn}{n^2} \sim \frac{1}{N}$$
(as $N\to\infty$)

Richardson Extrapolation $\quad S_N \to S$

① $S_N = S + \frac{a}{N} + \frac{b}{N^2} + \frac{c}{N^3} + \cdots$

$$\left.\begin{array}{ll} S_N \sim S + \frac{a}{N} & \to \; N S_N \sim NS + a \\ S_{N+1} \sim S + \frac{a}{N+1} & \quad (N+1)S_{N+1} \sim (N+1)S + a \end{array}\right\} (N+1)S_{N+1} - NS_N = S$$

OR $(N+1)S_{N+1} - NS_N \equiv R_1 \sim S$ (as $N\to\infty$)

② $S_N \sim S + \frac{a}{N} + \frac{b}{N^2} \to N^2 S_N \sim N^2 S + aN + b \qquad \times 1$

$S_{N+1} \sim S + \frac{a}{N+1} + \frac{b}{(N+1)^2} \to (N+1)^2 S_{N+1} \sim (N+1)^2 S + a(N+1) + b \quad \times(-2)$

$S_{N+2} \sim S + \frac{a}{N+2} + \frac{b}{(N+2)^2} \to (N+2)^2 S_{N+2} \sim (N+2)^2 S + a(N+2) + b \quad \times 1$

$$\left.\begin{array}{l} \dfrac{(N+2)^2 S_{N+2} - 2(N+1)^2 S_{N+1} + N^2 S_N}{2} \equiv R_2 \sim S \text{ (as } N\to\infty) \\[2ex] \dfrac{(N+3)^3 S_{N+3} - 3(N+2)^3 S_{N+2} + 3(N+1)^3 S_{N+1} - N^3 S_N}{6} \equiv R_3 \end{array}\right\} \text{"Binomial coefficients with changing signs"}$$

Summing Divergent Series

A $\quad (a+b)+c = a+(b+c)$ } only valid finitely!
C $\quad a+b = b+c$
D $\quad c(a+b) = ac+bc.$ } Valid also for infinitely

Euler Summation

$\sum_{n=0}^{\infty} a_n$ not conv. $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n \quad |x|<1$

Define $E \equiv \lim_{x\to 1} f(x)$

Consider $1-1+1-1+\cdots$ OR $f(x) = 1 - x + x^2 - x^3 \cdots = \frac{1}{1+x} \quad |x|<1$

$$\lim_{x\to1} f(x) = \frac{1}{1+1} = \frac{1}{2} = 0.5 \Rightarrow \boxed{E = \frac{1}{2}}$$

Borel Summation

$$\sum_{n=0}^{\infty} a_n \qquad \underline{\text{note:}} \int_0^\infty e^{-t} t^n \, dt = n! \rightarrow \frac{\int_0^\infty e^{-t} t^n \, dt}{n!} = 1$$

$$\downarrow \sum_{n=0}^{\infty} a_n \cdot \frac{\int_0^\infty e^{-t} t^n \, dt}{n!} \rightarrow B = \int_0^\infty e^{-t} \left(\sum_{n=0}^{\infty} \frac{t^n a_n}{n!} \right) dt$$

$$B(1-1+1-1+\dots) = \int_0^\infty dt \, e^{-t} \left(\sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \right) = \int_0^\infty e^{-2t} dt = \left\{ -\frac{1}{2} e^{-2t} \right\}_0^\infty = \frac{1}{2}$$

Generic Summation $\mathcal{S}$

$\#1.$ $\mathcal{S}(a_0+a_1+a_2+\dots) = S = a_0 + \mathcal{S}(a_1+a_2+a_3+\dots)$

$\#2.$ $\mathcal{S}(\sum(\alpha a_n + \beta b_n)) = \alpha \mathcal{S}(\sum a_n) + \beta \mathcal{S}(\sum b_n)$ "Linearity"

Ex.1) $\mathcal{S}(1-1+1-1+\dots) = s$

$$s = \mathcal{S}(1-1+1-1+\dots) \overset{\#1}{=} 1 + \mathcal{S}(-1+1-1+\dots) \overset{\#2}{=} 1 - \mathcal{S}(1-1+1-1+\dots) = 1-s$$

$$s = 1-s \rightarrow s = \frac{1}{2}$$

Ex.2) $1+0-1+1+0-1+1+0-1+\dots$

Euler Summation

$$f(x) = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \dots$$

$$= \underbrace{(1+x^3+x^6+x^9+\dots)}_{\frac{1}{1-x^3}} - \underbrace{(x^2+x^5+x^8+\dots)}_{\frac{x^2}{1-x^3}} = \frac{1-x^2}{1-x^3}$$

$$\lim_{x \to 1} \frac{1-x^2}{1-x^3} = \frac{2}{3}.$$

Generic Sum

$s = \mathcal{S}(1+0-1+1+)$

$s = 1 + \mathcal{S}(0-1+1+\dots)$ $\#1$

$s = 1 + \mathcal{S}(-1+1+\dots)$ $\#1$

$$3s = 2 + \mathcal{S}(\underbrace{0+0+\dots+0+\dots}_{0})$$

$$3s = 2 \longrightarrow s = \frac{2}{3}.$$

⑧ 08/10/2023
Lecture 5

1. $S = \mathcal{S}(1+2+4+8+\dots)$

$= 1 + \mathcal{S}(2+4+8+\dots)$

$= 1 + 2\mathcal{S}(1+2+3+\dots)$

$S = 1 + 2S \longrightarrow \boxed{S = -1}$

In complex plane, $\boxed{\cancel{z_1 < z_2}}$ not possible!

Order relation does not work in $\mathbb{C}$.

2. $S = \mathcal{S}(1+1+\dots)$

$S = 1 + \mathcal{S}(1+1+1+\dots)$

$S = 1 + S \rightarrow S \triangleq \infty$

3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots + \dots = \zeta(1)$ "Zeta"

$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad Re(z) > 1$

$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

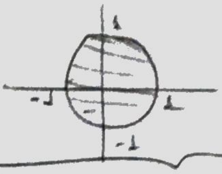
$\zeta(0) = -1/2$

Zeta Summation

<u>Functions</u>

$f(x) = \frac{1}{1-x} \quad x \neq 1$

$f(x) = \sum_{n=0}^{\infty} x^n \quad |x| < 1$



Different Representation

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \; ; \; \Gamma(n) = (n-1)!$

$\boxed{Re\, x > 0}$

Real part of x should be non-negative

Different Reps.

$\Gamma(x) = \int_1^\infty e^{-t} t^{x-1} dt + \int_0^1 e^{-t} t^{x-1} dt = \int_1^\infty e^{-t} t^{x-1} dt + \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\right) t^{x-1} dt$

$= \int_1^\infty e^{-t} t^{x-1} dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 t^{n+x-1} dt = \int_0^1 e^{-t} t^{x-1} dt + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+x)}$

Valid for $x < 0$ but $x \neq -1, -2,$

<u>H.P.</u> <u>Ex:</u> $H = p^2 + \frac{x^2}{4} + \frac{x^4}{4}$

$H\Psi = E\Psi \rightarrow \left(-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{x^4}{4}\right)\Psi(x) = E\Psi(x) \; ; \; \Psi(x) \rightarrow 0$ as $x \rightarrow \pm\infty$

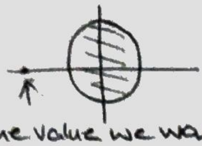
$HP \rightarrow EP_0 + EP_1 + EP_2 + \dots$

$\underbrace{E(\epsilon) \sim \sum a_n \epsilon^n}$ $\quad a_n \sim C\, 3^n\, n!\, (-1)^{n+1}$ as $n \rightarrow \infty$

Not equal but asymptotic! $\quad \mathcal{S}(\epsilon) \rightarrow$ Ans i.e. $E(\epsilon)$

Continued Functions

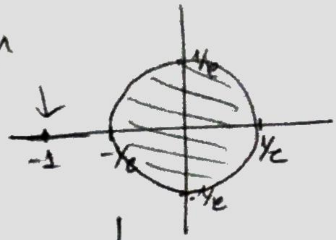
$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad |x| < R$$



$$= b_0 e^{b_1 x e^{b_2 x e^{\cdot^{\cdot^{\cdot}}}}}$$

$a_0 \to b_0$

$$\frac{(n+1)^{n-1}}{n!} \sim \frac{n^n}{n^n \cdot e^{-n}} = e^n$$

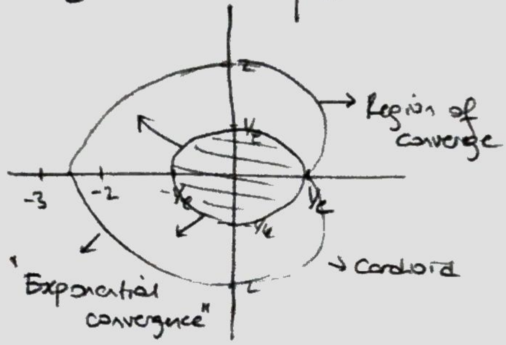


Ex: $e^{xe^{xe^{x\cdots}}} = \sum_{n=0}^{\infty} \frac{(n+1)^{(n-1)}}{n!} x^n$ converges if $|x| < \frac{1}{e}$

$b_n = 1$

$\hookrightarrow e^x, \; e^{xe^x}, \; e^{xe^{xe^x}}, \; \ldots, \{A_n\} \to L$

$A_1, A_2, A_3, \ldots$



Outside of the cordioid, the sequence does not converge but converge to multi limit, oscillates, or infinite limit

Continued Exponentials

$a_0 e^{a_1 z e^{a_2 z \cdots}}$; $\sum_{n=0}^{\infty} c_n z^n$

$c_0 = a_0$

$c_1 = a_1 a_0$

$c_2 = a_0 a_1 a_2 + \frac{1}{2} a_0 a_1^2$

$c_3 = a_0 a_1 a_2 a_3 + \frac{1}{2} a_0 a_1 a_2^2 + a_0 a_1^2 a_2 + \frac{1}{6} a_0 a_1^3$

seq of approximants: $a_0, \; a_0 e^{a_1 z}, \; a_0 e^{a_1 z e^{a_2 x}}$

Does this sequence converge? where it is? Even Taylor-like series does not converge, this series converge! } Series like this converges much more rapid than Taylor-like series.

Example:

$$e^{ze^{ze^z}} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n \quad \text{Radius of converge} = \frac{1}{e} \to |x| < \frac{1}{e}$$

Continued Fraction

$$\sum a_n x^n = \cfrac{b_0}{1 - \cfrac{b_1 x}{1 - \cfrac{b_2 x}{1 - b_3 \cdots}}} \to b_0, \; \frac{b_0}{1 - b_1 x}, \; \cfrac{b_0}{1 - \cfrac{b_1 x}{1 - b_2}}, \ldots \to L \text{ (Limit)}$$

Transformation

\# 1: Given $\{a_n\}$, imagine that $a_n = \int_{-L}^{L} w(x) x^{2n} dx$ (moment) $\quad L, w(x)$ unknown!

\# 2: If you have $\{b_n\}$, $P_0(x) = 1$, $P_1(x) = x$; $P_{n+1}(x) \equiv x P_n(x) - b_n P_{n-1}(x)$

$P_2(x) = x^2 - b_1 \qquad P_4(x) = x^4 - (b_1 + b_2 + b_3) x^2 + b_2 b_1$ "Monic"

$P_3(x) = x^3 - b_1 x - b_2 x$

\# 3: $\int_{-L}^{L} w(x) P_m(x) P_n(x) dx = 0$; $m \neq n$ "Orthogonal"

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Lecture 5

$P_0 \perp P_2$: $a_1 - b_1 = 0 \rightarrow \boxed{a_1 = b_1}$

$P_1 \perp P_3$: $a_2 - (b_1+b_2)\underset{b_1}{a_1} = 0 \rightarrow \boxed{a_2 = (b_1+b_2)\, b_1}$

$P_4 \perp P_0$: $a_2 - (b_1+b_2-b_3)\, b_1 + b_1 b_3 = 0 \rightarrow \boxed{a_2 = (b_1+b_2)\, b_1}$

$P_4 \perp P_2$: $a_3 - (b_1+b_2+b_3)a_2 + b_3 b_1^2 = 0$

$$\boxed{a_3 = b_1(b_1+b_2)^2 + b_1 b_2 b_3}$$

$a_1 = 1 \rightarrow b_1 = 1$
$a_2 = 5 \rightarrow b_2 = 4$
$a_3 = 61 \rightarrow b_3 = 9$
$a_4 = 1385 \leftarrow b_4 = 16$

* if $a_n \sim n! \rightarrow b_n \sim n$
  if $a_n \sim (2n)! \rightarrow b_n \sim n^2$
  as $n \to \infty$

$\Big\}$ $E_n\,(1, 5, 61, 1385, \ldots) \sim (2n)!\, C^n$ as $n \to \infty$, $b_n \sim n^2$ as $n \to \infty$

Pade Sequence

$$\sum a_n x^n \rightarrow b_0,\ \frac{b_0}{1-b_1 x},\ \frac{b_0}{1-\frac{b_1 x}{1-b_2 x}}, \ldots$$

$$\frac{P_0}{P_0}, \frac{P_0}{P_1}, \frac{P_1}{P_1}, \frac{P_1}{P_2}, \frac{P_2}{P_2}, \ldots \Big\}\ \text{Pade Sequence}$$

$$\underbrace{\sum_{n=0}^{p+q} a_n x^n}_{p+q+1} = \frac{S_p \leftarrow p+1}{S_q \leftarrow q} \equiv P^p_q \ \text{(Pade)}$$

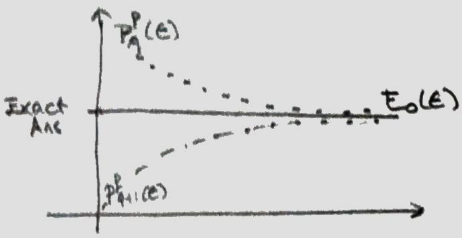
	p_1	p_2	p_3
q_1	0		
q_2		0	0
q_3			0

Typically, this "Pade series" converges.

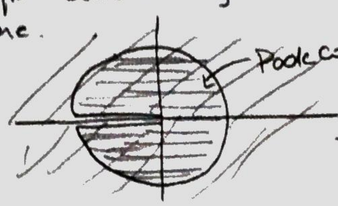
Example 1

$H = p^2 + \frac{x^2}{4} + \epsilon \frac{x^4}{4}$ $\quad E_0(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 + \ldots$ $\quad a_n \sim n!\, 3^n$ as $n \to \infty$

"Stieltjes"



Taylor Series converges in a circle. Pade converges a cut plane.



Pade has poles and zeros to compensate poles of a function i.e. $\Gamma(x)$ poles at negative integer.

Stieltjes Functions

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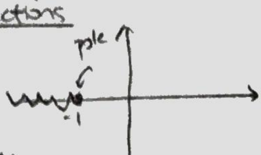
Lecture 7

Poles are "mocking up" the cut!

$\frac{\ln(1+x)}{x}$
$\downarrow$
$P^N_N(x), P^N_{N+1}(x)$



$$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}\left(1 + \frac{1}{12x} + \frac{1}{244x^2} - \right) \text{ as } x \to \infty$$

Stirling

Asymptotic Series

$$\left(\underbrace{f(x)}_{\text{a complicated function}} \sim \underbrace{g(x)}_{\text{a simple function}} \text{ as } x \to x_0 \text{ then } \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1 \right)$$

$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$ as an example for Asymptotic Series

means that

$$\left[\begin{array}{l} f(x) - \sum_{n=0}^{N} a_n x^n \sim a_{N+1} x^{N+1} \text{ as } x \to 0 \text{ for all } N \\ \lim_{x \to 0} \dfrac{f(x) - \sum_{n=0}^{N} a_n x^N}{x^{N+1}} = a_{N+1} \text{ for all } N. \quad a_n\text{'s are unique!} \end{array} \right.$$

Is this $\to \sum_{n=0}^{\infty} (-1)^n n! x^n$ an asymptotic series? Asymptotic is a relative property

Is $\sum a_n$ is convergent series? Yes or No $\leftarrow$ Meaningful question.

Example: $x^3 y'' = y$ Local analysis near $x = 0$.

$a(x)y'' + b(x)y' + c(x)y = 0 \to y'' + B(x)y' + C(x)y = 0$
dividing by $a(x)$

1) $A(x)$ and $B(x)$ are analytic near $x = x_0$.

Fuchs: All solutions have a Taylor series about $x = x_0$
(Theorem)
$$y(x) = \sum_{n=0}^{\infty} (x - x_0)^n a_n$$
then we say x_0 is a "Regular point".

2) Frobeius: (not both A and B are analytic) but
(Theorem)
$(x-x_0)A(x)$ and $(x-x_0)^2 B(x)$ are analytic then x_0 is an "Regular Singular" point.

One soln in the form of $y_1(x) = \sum_{n=0}^{\infty} (x - x_0)^{n+\alpha} a_n$

Other soln is in the form $y_2(x) = \log(x)\, y_1(x)$

3) $x = x_0$ is an Irregular singular point.

$y'' = \dfrac{y}{x^3} \qquad A(x) = 0 \quad B(x) = \dfrac{1}{x^3}$ but try Frobenius Series.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} \quad (a_0 \neq 0)$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}$$

$x^3 y'' = y$

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha+1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$$\sum_{n=1}^{\infty} a_{n-1} (n+\alpha-1)(n+\alpha-2) x^{n+\alpha} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$n=0 \rightarrow a_n x^{n+\alpha} = 0 \rightarrow a_n = 0$ contradiction!

① $y' = y/2$ $\quad y(x) = c e^{x/2}$

② $y' = y/2x$ $\quad y(x) = c\sqrt{x}$, $\alpha = 1/2$

③ $y' = \frac{y}{2x^2}$ $\quad y = c e^{-\frac{1}{2x}}$

<u>Green</u>

Try $y(x) = e^{f(x)}$ $\qquad f(x)$ is singular at $x = x_0$.

$y(x) = e^{ax^b}$; $b < 0$

$y' = f'(x) e^{f(x)}$

$y'' = f''(x) e^{f(x)} + (f'(x))^2 e^{f(x)} = e^{f(x)} [f''(x) + (f'(x))^2]$

$x^3 e^{f(x)} [f''(x) + (f'(x))^2] = e^{f(x)}$

$x^3 [f''(x) + \underbrace{(f'(x))^2}_{T}] = 1 \rightarrow x^3 \underbrace{[T' + T^2]}_{\text{Riccati Equation}} = 1$

$$\left[\begin{array}{ll} S(x) = ax^b & \\ S'(x) = abx^{b-1} & (S')^2 = a^2 b^2 x^{2b-2} \\ S''(x) = ab(b-1)x^{b-2} & \text{if } b<0 \quad S'' << (S')^2 \text{ as } x \to 0 \end{array}\right] (*)$$

then by $(*)$ $\quad x^3 (f'(x))^2 \sim 1$ as $x \to 0$

$(f'(x))^2 \sim \frac{1}{x^3}$ as $x \to 0$

$f'(x) \sim \pm \frac{1}{x^{3/2}}$ as $x \to 0$

$f(x) \sim \mp \frac{2}{x^{1/2}}$ as $x \to 0$

→ If $f(x) \sim g(x)$ as $x \to x_0$. Is $e^{f(x)} \sim e^{g(x)}$ as $x \to x_0$?

Only true if $f(x) - g(x) < 1$ as $x \to x_0$. then $y(x) \sim e^{\frac{2}{\sqrt{x}} + \text{something}}$ may write but we do not say it is asymptotic expansion.

$$y(x) = \begin{cases} e^{2/\sqrt{x}} & \text{as } x \to \infty \\ e^{-2/\sqrt{x}} & \text{as } x \to \infty \end{cases}$$

We will look, $x^3 y'' = y$

$$y(x) \sim e^{\pm 2/\sqrt{x}} x^{3/4} \underbrace{\sum_{n=0}^{\infty} a_n x^{n/2}}_{\text{"Divergent Series"}} \text{ as } x \to x_0.$$

$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$.

means:

For all N $\left(f(x) - \sum_{n=0}^{N} a_n x^n\right) \sim a_{N+1} x^{N+1}$ as $x \to 0$

Ex: $x^3 y'' = y$; $y(x) \sim ?$ as $x \to 0$

$y = e^{S(x)} \leftarrow$ Green's Technique

$S'' \ll (S')^2$ as $x \to 0$ S is something like ax^b $(b<0)$

$x^3((S')^2 + S'') = 1 \longrightarrow x^3 (S')^2 \sim 1$ as $x \to 0$ $(*)$

$\boxed{S(x) \sim \mp \frac{2}{\sqrt{x}}}$ from $(*)$

$S(x) \sim \mp \frac{2}{\sqrt{x}} + C(x)$ $C(x) \ll \frac{1}{\sqrt{x}}$ as $x \to 0$ (1)

$C'(x) \ll \frac{1}{x^{3/2}}$ as $x \to 0$ (2)

$C''(x) \ll \frac{1}{x^{5/2}}$ as $x \to 0$ (3)

$\to x^3((S')^2 + S'') = 1$

$S = \mp \frac{2}{\sqrt{x}} + C(x)$

$S' = \pm \frac{1}{x^{3/2}} + C'(x)$

$S'' = \mp \frac{3}{2x^{5/2}} + C''(x)$

$x^3\left[\cancel{\frac{1}{x^3}} + (C')^2 \pm 2\frac{1}{x^{3/2}} C'(x) \mp \frac{3}{2}\frac{1}{x^{5/2}} + C''(x)\right] = \cancel{1}$

$\underbrace{\overset{(2)}{\cancel{(C'(x))^2}} \pm \frac{2}{x^{3/2}} C'(x)}_{(2)} \underbrace{\mp \frac{3}{2x^{5/2}} + \overset{(3)}{\cancel{C''(x)}}}_{(3)} = 0.$

$\cancel{\pm} \frac{2}{x^{3/2}} C'(x) \sim \cancel{\pm} \frac{3}{2x^{5/2}}$ as $x \to 0$

Now, we may write

$\boxed{S(x) \sim \mp \frac{2}{\sqrt{x}} + \frac{3}{4}\ln(x) \text{ as } x \to 0}$

$C'(x) \sim \left(\frac{3}{4}\right)\frac{1}{x}$ as $x \to 0$

$C(x) \sim \frac{3}{4}\ln(x)$ as $x \to 0$

$S = \mp \frac{2}{\sqrt{x}} + \frac{3}{4}\ln(x) + D(x)$; $D(x) = d + \underbrace{e\sqrt{x} + fx + gx^{3/2} + \dots}$

$y(x) \sim K_{\pm} \underbrace{e^{\mp \frac{2}{\sqrt{x}}}}_{\text{Irregular Singular Points}} \underbrace{x^{3/4}}_{\text{Regular Singular Points}} \underbrace{\sum_{n=0}^{\infty} a_n (\sqrt{x})^n}_{\text{Regular Points.}}$ as $x \to 0$

$y'' + V(x)y = Ey \rightarrow y'' = \underbrace{(V(x) - E)}_{Q} y \rightarrow y'' = Q(x)y$

$Q(x) = x^2 - E$

Let $y = e^{S(x)}$

$(S')^2 + S'' \overset{(*)}{=} Q(x) \rightarrow (S')^2 \sim Q(x) \rightarrow S(x) \sim \pm \int_0^x \sqrt{Q(t)}\,dt \quad x \to a$

(1) $C \ll \int^x \sqrt{Q(t)}\,dt$

(2) $C' \ll \sqrt{Q(x)}$ as $x \to a$

(3) $C' \ll \frac{Q'(x)}{\sqrt{Q(x)}}$

for $S = \pm \int^x \sqrt{Q(t)}\,dt + C(x)$

$S' = \pm \sqrt{Q(x)} + C'(x)$

$S'' = \pm \frac{Q'(x)}{2\sqrt{Q(x)}} + C''(x)$

Plugin $(*)$

$\left(\pm\sqrt{Q(x)} + C'(x)\right)^2 + \left(\pm \frac{Q'(x)}{2\sqrt{Q(x)}} + C''(x)\right) = Q(x)$

$\cancel{Q(x)} + \underbrace{(C'(x))^2 \pm 2C'(x)\sqrt{Q(x)}}_{\pm 2C'(x)\sqrt{Q(x)} \text{ by } (2)} \underbrace{\pm \frac{Q'(x)}{2\sqrt{Q(x)}} + C''(x)}_{\pm \frac{Q'(x)}{2\sqrt{Q(x)}} \text{ by } (3)} = \cancel{Q(x)}$ ①

then we may write,

$2C'(x)\sqrt{Q(x)} \sim -\frac{Q'(x)}{2\sqrt{Q(x)}} \rightarrow C'(x) \sim -\frac{1}{4}\frac{Q'(x)}{Q(x)} \quad$ as $x \to a$

$C(x) \sim -\frac{1}{4}\ln(Q(x))$ as $x \to a$

then

$S = \pm \int^x \sqrt{Q(t)}\,dt - \frac{1}{4}\ln Q(x)\ ;\ y = e^{S(x)}$

then $\boxed{y \sim K_\pm \frac{e^{\int^x \sqrt{Q(t)}\,dt}}{[Q(x)]^{1/4}} \quad \text{as } x \to a}$ → WKB Approximation

Ex: $y \sim e^{\frac{2}{\sqrt{x}}} x^{3/4} \underbrace{\sum a_n x^{n/2}}_{w(x)}$ as $x \to 0$ $(*)$

D.E $x^3 y'' = y$ by putting $(*)$ then $a_n \approx n!$

$x^3 y'' = y$ y behave near $x=0$? (Local Analysis)

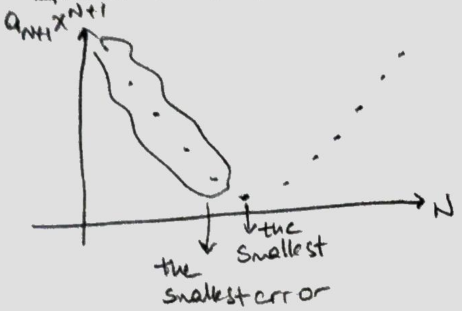
$y(x) \sim e^{\frac{2}{\sqrt{x}}} x^{3/4} \sum_{n=0}^{\infty} a_n x^{n/2}$ $a_n \approx n!$ as $x \to 0$

$a_1 = 1,\ a_2 = \frac{3}{16}, \ldots$

$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0$

for all N $f(x) - \sum_{n=0}^{N} a_n x^n \sim \underbrace{a_{N+1} x^{N+1}}_{\text{"measure of the error"}}$ as $x \to 0$

Optimal Asymptotic Approximation



$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$ How does $f(x)$ behave as $x \to \infty$?

Find a differential equation which $f(x)$ is a solution of it.

$f'(x) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{(n!)^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!\, n!} \to x f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!\, n!}$;

$(xf'(x))' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!\,(n-1)!} \to \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \boxed{(x f'(x))' = f(x).}$

$f' + x f'' = f$ we get as a differential equation.

$x = \infty$ is an irregular singular point $\left(x = \frac{1}{t}\right)$

$f(x) = e^{S(x)}$

$x(S'' + (S')^2) + S' = 1$

$x(S')^2 + S' \sim 1$ as $x \to \infty$ $S'' \ll (S')^2$

Dominant balance $x(S')^2 \sim 1 \to (S')^2 \sim \frac{1}{x} \to S' \sim \pm \frac{1}{\sqrt{x}} \to S(x) \sim \pm 2\sqrt{x}$ as $x \to \infty$

(1) $S' \sim 1 \to S(x) \sim x$ as $x \to \infty$

(2) $x(S')^2 \sim -S' \to S' \sim \frac{-1}{x} \to S(x) \sim -\ln(x)$ as $x \to \infty$

We do not write $f(x) \sim e^{\pm 2\sqrt{x}}$ as $x \to \infty$

Let $S = 2\sqrt{x} + C(x)$ with (1) $C \ll \sqrt{x}$ as $x \to \infty$

(2) $C' \ll \frac{1}{\sqrt{x}}$ as $x \to \infty$

(3) $C'' \ll \frac{1}{x^{3/2}}$ as $x \to \infty$

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Lecture 9

$$\left.\begin{aligned} S'(x) &= \frac{1}{\sqrt{x}} + c'(x) \\ S''(x) &= \frac{-1}{2x^{3/2}} + c''(x) \end{aligned}\right\} \quad xf'' + f' = f \Rightarrow x\left(S'' + (S')^2\right) + S' = 1$$

$$x\left(\frac{-1}{2x^{3/2}} + c''(x) + \frac{1}{x} + (c'(x))^2 + \frac{2c'(x)}{\sqrt{x}}\right) + \frac{1}{\sqrt{x}} + c'(x) = 1$$

$$-\frac{1}{2x^{3/2}} + \underbrace{\cancel{c''(x)}}_{\text{by }(3)} + \underbrace{\cancel{(c'(x))^2}}_{\text{by }(2)} + \frac{2c'(x)}{\sqrt{x}} + \frac{1}{x^{3/2}} + \cancel{\frac{c'(x)}{x}}$$

$$\frac{2c'(x)}{\sqrt{x}} \sim -\frac{1}{2x^{3/2}} \quad \text{as } x \to \infty$$

$$c'(x) \sim -\frac{1}{4x} \quad \text{as } x \to \infty$$

$$c(x) \sim -\frac{1}{4}\ln x \quad \text{as } x \to \infty$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$f(0) = 1$$

but $x \to \infty$ we do not use this information to determine "C" in (*)

$$f(x) = e^{S(x)} \qquad S(x) \sim 2\sqrt{x} - \frac{1}{4}\ln x \cdots \quad \text{as } x \to \infty$$

$$(*) \quad f(x) \sim C e^{2\sqrt{x} - \frac{1}{4}\ln x \cdots} \quad \text{as } x \to \infty$$

Airy Equation

$$y'' = xy \qquad \left.\begin{matrix} Ai(x) \\ Bi(x) \end{matrix}\right\} \text{ Linearly independent solution. as } x \to \infty$$

$$Ai(x) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \quad \text{as } x \to \infty$$

$$y'' = Q(x)y \quad \text{as } x \to \infty,\ Q(x) \to \infty$$

$$y(x) \sim K_{\pm} Q^{-1/4} e^{\pm\int\sqrt{Q}}$$

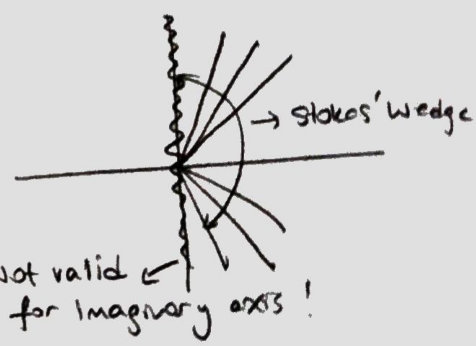
$$Bi(x) \sim \frac{1}{\sqrt{\pi}} \frac{e^{+\frac{2}{3}x^{3/2}}}{x^{1/4}} \quad \text{as } x \to \infty$$

Solution of Airy! $\quad y(x) \sim \dfrac{K_{\pm} e^{\pm\frac{2}{3}x^{3/2}}}{x^{1/4}}$

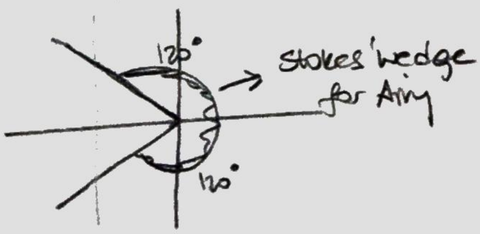
$$\sinh x \sim \frac{1}{2}e^{x} \quad \text{as } |x| \to \infty : -\frac{\pi}{2} < \arg x < \frac{\pi}{2}$$

$$\sinh x \sim -\frac{1}{2}e^{-x} \quad \text{as } |x| \to \infty ; \frac{\pi}{2} < \arg x < \frac{3\pi}{2}$$

$$\sinh x = \frac{1}{2}(e^{x} + e^{-x})$$



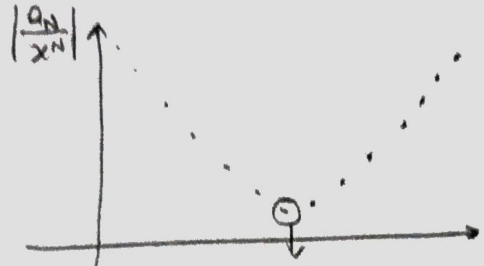
For Airy's Equation



Bessel Function

$$I_n(x) \sim e^x \frac{1}{\sqrt{x}} \times \sum_{n=0}^{\infty} \frac{a_n}{x^n} \quad \text{as } n \to \infty$$

Optimal Asymptotic Approximation



Optimal

Example $\quad y'' = xy + 1 \quad (*)$ How does $y(x)$ behave as $x \to \infty$? $\boxed{K x^{-1/4} e^{\frac{2}{3}x^{3/2}}}$

$$y'' + a(x)y' + b(x)y = c(x)$$

$$y_{gen}(x) = \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{Homogeneous}} + y_{part}(x)$$

$$\left.\begin{aligned} Ai(x) &\sim C e^{-\frac{2}{3}x^{3/2}} x^{-1/4} \\ Bi(x) &\sim D e^{+\frac{2}{3}x^{3/2}} x^{-1/4} \end{aligned}\right\} \text{Solution of homogeneous part of } (*)$$

Method of Dominant Balance

① (✗)

$y'' \sim 1$ as $x \to \infty$

$y' \sim x$ as $x \to \infty$

$y \sim \frac{x^2}{2}$

② ✓

$xy \sim -1$ as $x \to \infty$

$y \sim -\frac{1}{x}$ as $x \to \infty$

$y = -\frac{1}{x} + C(x) \qquad C \ll \frac{1}{x}$ as $x \to \infty$

$$\left.\begin{aligned} y'' &= -\frac{2}{x^3} + C''(x) \\ y' &= \frac{1}{x^2} + C'(x) \end{aligned}\right\} \text{put into equation } (*)$$

$$-\frac{2}{x^3} + \cancel{C''(x)} = \cancel{-1} + xC(x) \cancel{+1}$$

$$C(x) \sim -\frac{2}{x^4} \quad \text{as } x \to \infty$$

$$y_{part} \sim -\frac{1}{x} - \frac{2}{x^4} + \frac{9}{x^7} + \cdots$$

(Ai(x) and $y_{part}(x)$ are "subdominant" compared to Bi(x). Then Bi(x) determines the behavior of DE as $x \to \infty$)

$$y + \frac{1}{x} \sim -\frac{2}{x^4} \quad \text{as } x \to \infty$$

Subdominance

$y'' = xy$

$y(x) = \underbrace{C\,Ai(x)}_{\text{Subdom.}} + \underbrace{D\,Bi(x)}_{\text{Dominant}}, \quad x \to \infty$

$y(+\infty) = 0$ implies $D = 0$. ← "Poincaré^N Asymptotic"

Hyper-asymptotics → asymptotics beyond all errors!

Subdominant solution is to take in Physics!

$y'' = Q(x)y \Rightarrow y \sim C_\pm Q^{-1/4} e^{\pm\int^x \sqrt{Q(s)}\,ds}$

This subdominance is required to go "zero" at infinity (Not blown up!)

Rigorous Theory (Very limited)

Stieltjes functions

$f(x) = \int_0^\infty \frac{\rho(t)}{1+xt}\,dt \qquad \rho(t) \geq 0$ is a weight function for $0 < t < \infty$

n^{th} moment of ρ exists $a_n = \int_0^\infty \rho(t)t^n\,dt$.

Stieltjes series

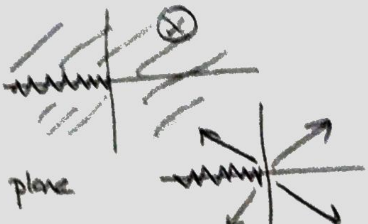
$\int_0^\infty \frac{\rho(t)}{1+xt}\,dt = \int_0^\infty dt\,\rho(t)\sum_{n=0}^\infty (-1)^n x^n t^n \to \sum_{n=0}^\infty (-1)^n a_n x^n \to$ divergent

$f(x) \sim \sum_{n=0}^\infty (-1)^n a_n x^n$ as $x \to 0$

Ex1 $\rho(t) = e^{-t} \quad a_n = n! \quad f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \to \sum_{n=0}^\infty (-1)^n n!\, x^n$

Ex2 $\rho(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t \geq 1 \end{cases} \quad a_n = \frac{1}{n+1} \to$ Series $\sum_{n=0}^\infty \frac{x^n(-1)^n}{n+1}$; $f(x) = \frac{\log(1+x)}{x}$

A Stieltjes function has 4 properties



1) $f(x)$ is analytic in cut x plane

2) $f(x) \to 0$ as $|x| \to \infty$ in cut x plane

3) $f(x) \sim \sum_{n=0}^\infty a_n x^n (-1)^n$ as $|x| \to 0$ in cut plane

4) $-f(x)$ is "Herglotz"

$\mathrm{Im}\,g(x)$ same sign as $\mathrm{Im}\,x$ → HERGLOTZ

Cauchy Theorem (complex)

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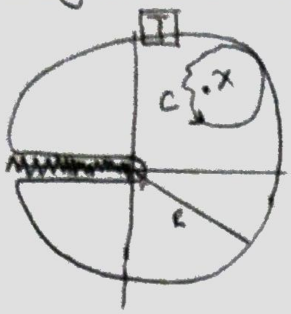
Lecture 11



$R \to \infty;\ r \to 0$

$$\boxed{f(x) = \frac{1}{2\pi i} \oint_C \frac{dt\, f(t)}{t - x}}$$

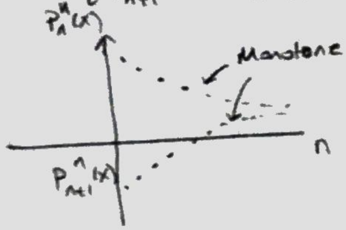
(In real numbers, this is not possible!

$f(x)$? cannot be determined by $f(a)$ and $f(b)$.)

Stieltjes showed

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \xrightarrow{\text{Padé}} \begin{cases} P_n^n(x) \longrightarrow \text{converges in cut plane as } n \to \infty \\ P_{n+1}^n(x) \longrightarrow \text{converges in cut plane as } n \to \infty \end{cases}$$

Also for real x,



$P_{n+1}^n(x) < P_M^M(x)$ for all M, n.

#5 If "the moment problem" has a unique solution then $L_1 = L_2 = f(x)$ in complex plane Padé converges to $f(x)$

$$a_n = \int_0^\infty \rho(t)\, t^n\, dt \quad \text{find } \rho(t) \text{ s.t. } \rho(t) \geq 0$$

given $n = 0, 1, 2, \ldots, \infty$

Ex: $a_n = (8n+7)!$ $\quad n = 0, 1, 2, \ldots$ find $\rho(t)$

$$\rho(t) = \underbrace{\frac{e^{-t^{1/8}}}{8} + a\, e^{-t^{1/4}} \sin(t^{1/4})}_{\geq 0 \text{ if } a \in (-0.14761,\, 1.21584)}; \quad -0.14761 < a < 1.21584$$

Carleman Condition

If a_n grows no faster than $(2n)!$ then solution is unique! to the moment problem

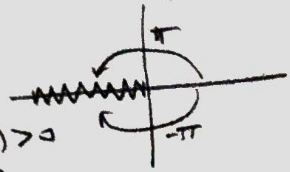
Stieltjes Functions $f(x)$

$$f(x) = \int_0^\infty \frac{dt\, \rho(x)}{1+xt} \; ; \; \rho(x) \geq 0 \text{ for all } t \geq 0 \text{ and } a_n = \int_0^\infty dt\, \rho(t) t^n \text{ exists}$$

1) $f(x)$ is analytic in cut-x plane ("analytic" means $f'(x)$ exists as a complex derivative)

2) $f(x) \to 0$ as $|x| \to \infty$

3) $f(x) \sim \sum_{n=0}^\infty (-1)^n a_n x^n$ as $x \to 0$ $\quad -\pi < \arg x < \pi$



4) $-f(x)$ is Herglotz. In upper half plane, $\operatorname{Im} x > 0$, $\operatorname{Im}(-f(x)) > 0$
In lower half plane, $\operatorname{Im} x < 0$, $\operatorname{Im}(-f(x)) < 0$

$$\Rightarrow ① \; f'(x) = -\int_0^\infty \frac{dt\, \rho(t)\, t}{(1+xt)^2}$$

$$\Rightarrow ④ \; x = a + ib \quad \text{then} \quad f(x) = \int_0^\infty \frac{dt\, \rho(x)}{1+at+ibt} = \int_0^\infty \frac{dt\, \rho(t)(1+at-ibt)}{(1+at)^2 + b^2t^2}$$

$$f(x) = \int_0^\infty \frac{dt\, \rho(t)(1+at-ibt)}{(1+at)^2+b^2t} \to \operatorname{Im}(-f(x)) = b > 0 \text{ if } \operatorname{Im} x > 0.$$

Herglotz is powerful!

→ "Entire" means analytic function all finite complex x.

ex. polynomials, e^x, $\sin x$, - - , $Ai(x)$, ..

If $f(x)$ is entire and Herglotz then

Entire → $f(x) = \sum_{n=0}^\infty a_n x^n$ (all x)

$$x = re^{i\theta} \quad \text{then} \quad f(x) = \sum_{n=0}^\infty a_n r^n e^{in\theta}$$

$$\operatorname{Im} f(x) = \sum_{n=0}^\infty a_n r^n \sin(n\theta)$$

Herglotz: Sign of $\sum_{n=0}^\infty a_n r^n \sin(n\theta)$ same as sign of $\sin\theta$

$n \sin(\theta) \geq \sin(n\theta)$ for $0 \leq \theta < \pi$

$n\sin\theta \pm \sin(n\theta)$ same sign as $\sin\theta$

$$\int_0^\pi d\theta \sin(n\theta)\sin(m\theta) = \begin{cases} 0, & n \neq m \\ \pi/2, & n = m \end{cases}$$

$\frac{\pi}{2} a_1 r \pm \frac{\pi}{2} a_n r^n \qquad n\pi/2 \; (+)$

↑ $a_n = 0$ for all $n > 1$ (!!)

$f(x) = a + bx$ is only function both "entire" and "Herglotz".

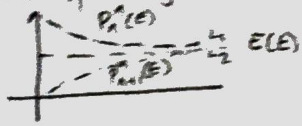
$$\left(\int \sum_{n=0}^\infty a_n r^n \sin(n\theta)(n\sin\theta \pm \sin(n\theta)), \int \sin\theta (n \sin\theta \pm \sin(n\theta)) \right)$$

$$\left(-\frac{d^2}{dx^2} + V(x) + \underset{\epsilon}{\uparrow} W(x)\right)\Psi(x) = E(\epsilon)\Psi(x)$$

$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \text{ as } \epsilon \to 0 \rightarrow \text{"Divergent series"}$$

$E(\epsilon)$ has properties of 1,2,3 and 4 $\Rightarrow E(\epsilon)$ is a function of Stieltjes

$\Rightarrow$ Padé's converges : $\sum_{n=0}^{\infty} a_n\epsilon^n \rightarrow \begin{cases} P^n_n(\epsilon) \to L_1 \downarrow \\ P^n_{n+1}(\epsilon) \to L_2 \uparrow \end{cases}$



then $\boxed{P^n_{n+1}(\epsilon) < E(\epsilon) < P^n_n(\epsilon)}$

$\Rightarrow$ If $\{a_n\}$ grows no faster than $(2n)!$ then $L_1 = L_2$

If $V(x) = x^2$, $W(x) = x^4$ then $\boxed{a_n \sim n!\, c^n}$

$$E(\epsilon) \sim \sum a_n\epsilon^n = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 + \dots \quad \text{Not a Stieltjes Series (Alternating)}$$

$$\frac{E(\epsilon) - 1/2}{\epsilon} \sim \frac{3}{4} - \frac{21}{8}\epsilon + \frac{333}{16}\epsilon^2 - \dots \quad \text{Stieltjes Series}$$

$$E(\epsilon) = \frac{1}{2\pi i}\oint_C \frac{dt}{t-\epsilon} E(t) = \frac{1}{2\pi i}\oint \frac{dt}{t-\epsilon}\, 2i\,\mathrm{Im}(E) =$$

$$\frac{1}{t-\epsilon} = \frac{1}{t}\,\frac{1}{1-\epsilon/t} = \boxed{\frac{1}{t}\sum_{n=0}^{\infty}\frac{\epsilon^n}{t^n}}$$

$$\boxed{a_n = \int_{-\infty}^{0} \frac{dt}{t^{n+1}}\,\frac{\mathrm{Im}(E)}{\pi}}$$

$\frac{x^2}{4} + \epsilon\frac{x^4}{4}$ want a_n $\sum a_n\epsilon^n$ $\left\{ a_n \sim (-1)^{n+1}\,\Gamma\left(n+\frac{1}{2}\right)\frac{3^n\sqrt{6}}{\pi^{3/2}} \text{ as } n\to\infty \right\}$

$$a_n = \frac{1}{\pi}\int_{-\infty}^{0} dt\, \frac{\mathrm{Im}(E)}{t^{n+1}}$$

WKB Theory

$$-y'' + V(x)y = Ey \rightarrow y'' = -(V(x) - E)y \rightarrow \epsilon^2 y'' = Q(x)y$$

$$y(x) = e^{S(x)} \quad S(x) \sim \frac{1}{\epsilon}S_0 + S_1 + \epsilon S_2 + \dots \sim \frac{1}{\epsilon}\sum S_n\epsilon^n$$

$$y(x) \sim e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n\epsilon^n} \quad \epsilon S_2 \ll 1 \text{ as } \epsilon\to 0 \qquad \epsilon^2 S_3 \ll \epsilon S_2 \text{ as } \epsilon \to 0$$

$$\epsilon^2 y'' = Qy$$

$$\epsilon^2\Bigg[\underbrace{\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n''}_{S''} + \underbrace{\frac{1}{\epsilon^2}\left(\sum_{n=0}^{\infty}\epsilon^n S_n'\right)^2}_{(S')^2}\Bigg]\cancel{e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty}S_n\epsilon^n}} = Q(x)\,\cancel{e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty}S_n\epsilon^n}}$$

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Lecture 12

$$\epsilon^2\left[\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n'' + \frac{1}{\epsilon^2}\left(\sum_{n=0}^{\infty}\epsilon^n S_n'\right)^2\right] = Q(x)$$

$$\epsilon\sum_{n=0}^{\infty}\epsilon^n S_n'' + \left(\sum_{n=0}^{\infty}\epsilon^n S_n'\right)^2 = Q(x)$$

$$\boxed{\begin{aligned}&\underbrace{\epsilon\sum_{n=0}^{\infty}\epsilon^n S_n''}\ + \sum_{n=0}^{\infty}\epsilon^n\sum_{j=0}^{n} S_j' S_{n-j}' = Q\\ &\sum_{n=1}^{\infty}\epsilon^n S_{n-1}'' + \sum_{n=0}^{\infty}\epsilon^n\sum_{j=0}^{n} S_j' S_{n-j}' = Q\end{aligned}} \rightarrow \text{WKB Theory.}$$

$\epsilon^0:\ (S_0')^2 = Q \rightarrow S_0 = \pm\int^x\sqrt{Q(s)}\,ds$

$\epsilon^1:\ S_0'' + 2S_0'S_1' = 0 \rightarrow$

$\epsilon^2:\ S_1'' + 2S_0'S_2' + (S_1')^2 = 0 \rightarrow$

$\vdots$

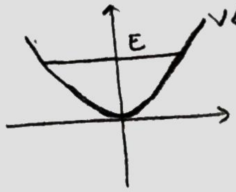
$\epsilon^n:\ S_{n-1}'' + \sum_{j=0}^{n} S_j' S_{n-j}' = 0$

$$\boxed{y(x) \sim e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n\epsilon^n} \quad \text{as } \epsilon\to 0}$$

Lecture 13

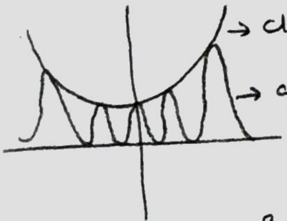
$-y'' + V(x)y = Ey \leftrightarrow y'' = \underbrace{Q(x)}_{V(x)-E}y$ "Time Independent Schrödinger Equation"

$V(x) = x^2$ "Harmonic Oscillator"



Probability

$\underbrace{y^2dx}$



$\epsilon^2 y'' = Q(x)y$

$y = e^{\frac{1}{\epsilon}\sum_{n=0}^{\infty} S_n\epsilon^n}$

$S_0 = \pm\int^x\sqrt{Q(s)}\,ds$

$S_1 = -\frac{1}{4}\ln(Q)$

$S_2 = \pm\int^x \frac{1}{8\sqrt{Q}}\left(\frac{Q''}{Q} - \frac{5}{4}\frac{(Q')^2}{Q}\right)dt$

$\epsilon^2 y'' = xy \leftrightarrow Q(x) = x$

$S_2 = \pm\int^x \frac{dt}{8\sqrt{t}}\left(\frac{-5}{4t^2}\right) = \pm\frac{5}{32}\,\frac{2}{3}\,t^{-3/2}\Big|_0^x = \frac{-5}{48}x^{-3/2}$

$$y = \underbrace{\underbrace{\underbrace{e^{\pm\frac{1}{\epsilon}S_0 + S_1}}_{\text{Geometrical optics}}}_{\text{physical optics}}}_{\text{All have been done!}} \cdot \underbrace{e^{\epsilon S_2}}_{1+\epsilon S_2 \ll 1}$$

↗ Expected to small!

$x = 10$

$\frac{5}{48(10)^{3/2}} \cong \frac{1}{300}$ ✓

Accurate!

Strum-Liouville Problem "Regular" $Q(x) > 0$

$$y'' + Q(x)Ey = 0 \quad y(0) = 0 \,;\, y(\pi) = 0$$

$E : \{E_n\}, y_n(x)$

$y_m(x)$

$$y_n'' + Q(x)E_n y_n(x) y(x) = 0$$

$$\int_0^\pi y_n'' y_m + Q(x) E_n y_n(x) y_m(x)\, dx = -\int_0^\pi y_n'(x) y_m'(x)\,dx + \int_0^\pi E_n(x) y_n(x) y_m(x) Q(x)\,dx$$

$$= \underbrace{\int_0^\pi y_m'' y_n\,dx}_{} + \int_0^\pi E_n y_n(x) y_m(x) Q(x)\,dx$$

$$-\int_0^\pi Q E_m y_m y_n\,dx + \int_0^\pi E_n y_n y_m Q(x)\,dx = \int_0^\pi \underbrace{Q(x) y_n(x) y_m(x)}_{0} \underbrace{[E_m - E_n]}_{\neq 0 \to E_m \neq E_n} = 0$$

Orthogonality with respect to $Q(x)$

$E_n \gg 1 \to E = \frac{1}{\epsilon^2}$

Strum L. problem $\nearrow$ $y'' + Q(x)\underbrace{\left(\frac{1}{\epsilon^2}\right)}_{\to E} y = 0$

$$\epsilon^2 y'' + Q(x) y = 0 \qquad \text{WKB}$$

$$\boxed{y(x) \sim A \sin\left(\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt\right) \cdot Q^{-1/4}(x) + B \sin\left(\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt\right) Q^{-1/4}(x)} \quad \text{as } \epsilon \to 0$$

$$y(0) = 0 \Rightarrow B = 0$$
$$y(\pi) = 0 \Rightarrow y(\pi) - \frac{A^{\neq 0}}{(Q(0))^{1/4}} \sin\left(\underbrace{\frac{1}{\epsilon}\int_0^\pi \sqrt{Q(t)}\,dt}_{n\pi}\right) = 0$$

makes this equal to zero

$\frac{1}{\epsilon}\int_0^\pi \sqrt{Q(t)}\,dt = n\pi \quad n = 1, 2, 3, \ldots$ then

$$\boxed{E_n = \frac{n^2\pi^2}{\left(\int_0^\pi \sqrt{Q(t)}\,dt\right)^2} \quad \text{and} \quad y(x) \sim A \sin\left(\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt\right)(Q(x))^{-1/4} \; ; n = 1, 2, 3 \ldots \text{ as } n \to \infty}$$

$$A = ? \to \int_0^\pi Q(x) y_n^2(x)\,dx = 1$$

$$A^2 \int_0^\pi Q(x) \frac{1}{\sqrt{Q(x)}} \sin^2\left(\underbrace{\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt}_{z}\right) dx = 1. \qquad z = \frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt$$

$x = 0,\, z = 0$
$x = \pi,\, z = \frac{1}{\epsilon}\int_0^\pi \sqrt{Q(t)}\,dt = n\pi$

$dz = \frac{1}{\epsilon}\sqrt{Q(t)}\,dx$ then.

$$A^2 \epsilon \int_0^{n\pi} \sin^2(z) = 1 \to \epsilon A^2 \left\{\frac{1}{2} n\pi\right\} = 1 \to \boxed{A^2 = \frac{2}{\epsilon n \pi}}$$

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Lecture 13

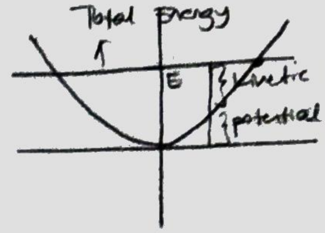
$\epsilon^2 y'' = Q(x) y \; ; \; Q(x) = V(x) - E \; ; \; y(-\infty) = 0 \, , \; y(+\infty) = 0$

$$y \sim A \frac{e^{\frac{1}{\epsilon}\int^x \sqrt{Q(t)}\,dt}}{(Q(x))^{1/4}} + B \frac{e^{-\frac{1}{\epsilon}\int^x \sqrt{Q(t)}\,dt}}{(Q(x))^{1/4}} \quad \text{as } \epsilon \to 0$$

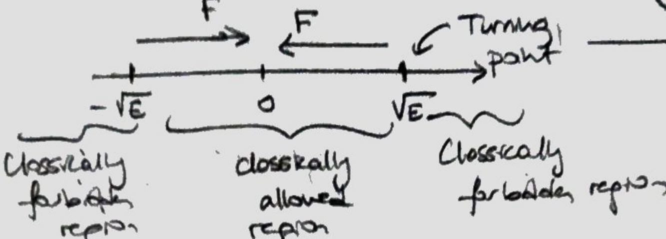
no good when $Q(x) = 0$

<u>Harmonic Oscillator</u>

$H = \underset{\text{speed}}{p^2} + \underset{\text{distance}}{x^2} = E$ (Energy) $\qquad F = -V' = -2x$



when $p = 0, \; x = \pm\sqrt{E}$



$x > E$

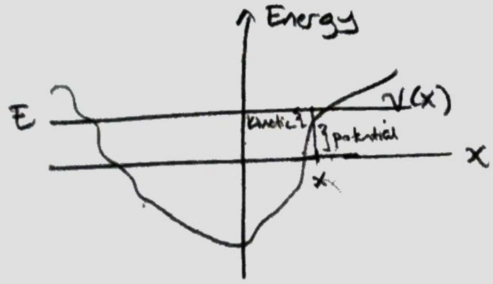
$x^2 > E^2$ then $p^2 < 0$

Lecture 14

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\Psi(x) = E\,\Psi(x)$$

$\Psi(x) \to 0$ as $x \to \infty$

$\Psi(x) \to 0$ as $x \to -\infty$



Let $V(x) - E = Q(x)$

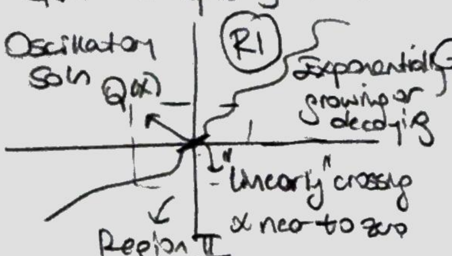
$\epsilon^2 \Psi''(x) = Q(x)\Psi(x) \quad \epsilon \ll 1$

WKB solution fail to give solution when $V(x) = E$ because $Q(x) = 0$ and WKB solution blows up.

<u>One Turning Problem</u>

$\epsilon^2 y'' = Q(x) y$

$Q(x) = 0$ for just "one" value of x, which is $x = 0$. $Q(0) = 0$.



$Q(x) \sim ax$ as $x \to 0$ and $a > 0$

$$Q(x) = Q(\alpha) + V'(\alpha)(x - \alpha) + \frac{V''(\alpha)}{2!}(x-\alpha)^2 + \cdots$$

$x > 0$

$$y(x) \sim \frac{C_{\pm} e^{\pm\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt}}{Q^{1/4}(x)} \quad (\epsilon \to 0) \quad \text{R1}$$

<u>Region I</u> we want to $y(+\infty) = 0$ then $\left[\, y(x) \sim \dfrac{C e^{-\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt}}{Q^{1/4}(x)} \right]$

Region II (x near 0) $x << 1$

$$\epsilon^2 y'' = (ax + \underbrace{bx^2 + cx^3 + \cdots}_{\text{as } 0})y \rightarrow \epsilon^2 y'' = axy$$

Let $x = \gamma t$ $\qquad \frac{a\gamma^3}{\epsilon^2} = 1 \qquad \gamma = \frac{\epsilon^{2/3}}{a^{1/3}}$

$$\frac{\epsilon^2}{\gamma^2}\frac{d^2}{dt^2}y = a\gamma t y \longrightarrow \frac{d^2y}{dt^2} = ty \quad \text{Airy Equation}$$

decaying required

$$y(t) = D_1 Ai(t) + \cancel{D_2 Bi(t)} \longrightarrow \boxed{y(t) = D\,Ai(t)} \rightarrow y(x) \sim D\,Ai\left(x\frac{a^{1/3}}{\epsilon^{2/3}}\right)$$

$$y_I(x) = C[Q(x)]^{-1/4}\exp\left[-\frac{1}{\epsilon}\int_0^x \sqrt{Q(t)}\,dt\right] \quad x>0,\ x \gg \epsilon^{2/3},\ \epsilon \to 0^+$$

$$y_{II}(x) = 2\sqrt{\pi}\,(a\epsilon)^{-1/6} C\,Ai\left(\epsilon^{-2/3}a^{1/3}x\right),\ |x| << 1,\ \epsilon \to 0^+$$

$$y_{III}(x) = 2C[-Q(x)]^{-1/4}\sin\left[\frac{1}{\epsilon}\int_x^0 \sqrt{-Q(t)}\,dt + \frac{\pi}{4}\right],\ x<0,\ (-x) \gg \epsilon^{2/3},\ \epsilon \to 0^+$$