The Laplace Transform

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A function $f(t)$ s.t. $f(t)=0;\ t<0$. Then The Laplace Integral

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$ defines the Laplace Transform of $f(t)$

Not all function has the Laplace Transform. A function should satisfy followings;

→ $f(t)=0;\ t<0$

→ $f(t)$ is continuous or piece-wise continuous in every interval

→ $t^n|f(t)|<\infty$ as $t\to 0$ for some number n, where $n<1$.

→ $e^{-s_0t}|f(t)|<\infty$ as $t\to\infty$ for some number s_0

Example! Find the Laplace transform of 1, e^{at}, $\sin(at)$, $\cos(at)$, and t^n.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \left\{-\frac{1}{s}e^{-st}\right\}_0^\infty = 0-\left(-\frac{1}{s}\right) = \frac{1}{s}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left\{\frac{-1}{s-a}e^{-(s-a)t}\right\}_\infty^\infty = \frac{1}{s-a}\ ;\ s>a.$$

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty \sin(at)e^{-st}dt = \frac{-e^{st}}{s^2+a^2}\left[s\sin(at)+a\cos(at)\right]_0^\infty = \frac{a}{s^2+a^2}\ ;\ s>0$$

$$\mathcal{L}\{\cos(at)\} = \int_0^\infty \cos(at)e^{-st}dt = \frac{e^{-st}}{s^2+a^2}\left[-s\cos(at)+a\sin(at)\right]_0^\infty = \frac{s}{s^2+a^2}\ ;\ s>0$$

and

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = n!\,e^{-st}\sum_{m=0}^{A}\frac{t^{n-m}}{(n-m)!\,s^{m+1}}\Bigg|_0^\infty = \frac{n!}{s^{n+1}}\ ;\ s>0.$$

① Linearity Property: $\mathcal{L}\{c_1f(t)+c_2g(t)\} = c_1\mathcal{L}(f(t))+c_2\mathcal{L}(g(t))$

② $\mathcal{L}(f^{(n)}(t)) = s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

③ $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = F(s)/s$

Example! $f(t) = 2\sin t-\cos 2t+\cos(3)-t$. Find $F(s)$.

$$F(s) = \mathcal{L}(f(t)) = 2\frac{1}{s^2+1}-\frac{s}{s^2+4}+\cos(3)\cdot\frac{1}{s}-\frac{1}{s^2}$$

$$\left[F(s) = \frac{2}{s^2+1}-\frac{s}{s^2+4}+\frac{\cos(3)}{s}-\frac{1}{s^2}\right]$$

Example

1) $\mathcal{L}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

2) $\mathcal{L}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$

3) $\mathcal{L}\left\{\frac{2}{s^2+1} - \frac{15}{s^3} + \frac{2}{s+1} - \frac{6s}{s^2+4}\right\} = 2\sin t - \frac{15}{2}t^2 + 2e^{-t} - 6\cos(2t)$

The Heaviside Step and Dirac Delta Functions

Heaviside Step Function $H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$

where $a > 0$ $\mathcal{L}\{H(t-a)\} = \int_a^\infty e^{-st}\,dt = \frac{e^{-as}}{s}$, $s > 0$

Dirac delta Function (Impulse function) $\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$; $\int_0^\infty \delta(t-a)\,dt = 1$. given $a \geq 0$

A popular way of visualizing the delta function

$$\delta(t-a) = \lim_{\varepsilon \to 0} \begin{cases} 1/\varepsilon, & 0 < |t-a| < \varepsilon/2 \\ 0, & |t-a| > \varepsilon/2 \end{cases} \quad \text{where } \varepsilon > 0, \text{ and } a > 0.$$

The laplace Transform of Dirac Delta Function

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty \delta(t-a) e^{-st}\,dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{a-\varepsilon/2}^{a+\varepsilon/2} e^{-st}\,dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon s}\left(e^{-as+\varepsilon s/2} - e^{-as-\varepsilon s/2}\right)$$

$$= \lim_{\varepsilon \to 0} \frac{e^{-as}}{\varepsilon s}\left(1 + \frac{\varepsilon s}{2} + \frac{\varepsilon^2 s^2}{8} + \cdots - 1 + \frac{\varepsilon s}{2} - \frac{\varepsilon^2 s^2}{8} + \cdots\right)$$

$$= e^{-as}$$

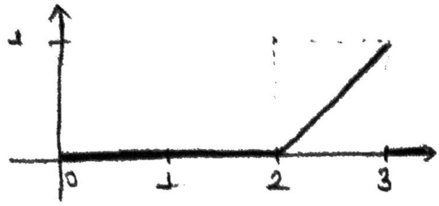
Special case: if $a = 0$, $\mathcal{L}(\delta(t)) = 1$

Integration: $\int_0^t \delta(\tau - a)\,d\tau = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

"Integration & Differentiation" $\int_0^t \delta(\tau-a)\,d\tau = H(t-a) \iff \frac{d}{dt}[H(t-a)] = \delta(t-a)$

Example

$$f(t) = \begin{cases} 0, & 0 \le t \le 2 \\ t-2, & 2 \le t < 3 \\ 0, & 3 < t \end{cases}$$



$f(t) = (t-2)H(t-2) - (t-2)H(t-3) \rightarrow$ Heaviside step Function.

Example Write following DEs by using Heaviside step Function,

① $y'' + 3y' + 2y = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t \end{cases} \rightarrow y'' + 3y' + 2y = H(t-1)$

② $y'' + 4y' + 4y = \begin{cases} 0, & 0 < t < 2 \\ t, & 2 < t \end{cases} \rightarrow y'' + 4y' + 4y = t\,H(t-1)$

③ $y'' - 3y' + 2y = \begin{cases} 0, & 0 < t < 2 \\ e^{t}, & t > 2 \end{cases} \rightarrow y'' - 3y' + 2y = e^{-t}H(t-2)$

④ $y'' + y = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & \pi \le t \end{cases} \rightarrow y'' + y = \sin(t)H(t) - \sin t\,H(t-\pi)$

Some Useful Theorems

First Shifting Theorem

Consider the transform of the function $e^{-at}f(t)$, where a is any real number. Then, by definition,

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\,dt = \int_0^\infty e^{-(s+a)t}f(t)\,dt \quad \text{or}$$

$$\boxed{\mathcal{L}[e^{-at}f(t)] = F(s+a)} \quad (*)$$

That is, if $F(s)$ is the transform of $f(t)$, and a is a constant, then $F(s+a)$ is the transform of $e^{-at}f(t)$.

e.g. $\mathcal{L}[\sin(bt)] = \dfrac{b}{s^2+b^2}$ then

$$\mathcal{L}[e^{-at}\sin(bt)] = \frac{b}{(s+a)^2+b^2} \quad \text{by } (*)$$

Example: Find the inverse of the Laplace transform

$$F(s) = \frac{s+2}{s^2+6s+1}$$

$$\frac{s+2}{(s+3)^2-8} = \frac{s+3}{(s+3)^2-8} - \frac{1}{(s+3)^2-8}$$

$$= e^{-3t}\cosh(2\sqrt{2}t) - \frac{e^{-3t}}{2\sqrt{2}}\sinh(2\sqrt{2}t)$$

Second Shifting Theorem

If $F(s)$ is the transform of $f(t)$, then $e^{-bs}F(s)$ is the transform of $f(t-b)H(t-b)$ where b is real and positive. To show this, consider the Laplace transform of $f(t-b)H(t-b)$. Then, from the definition,

$$\mathcal{L}[f(t-b)H(t-b)] = \int_0^\infty f(t-b)\,H(t-b)\,e^{-st}\,dt = \int_b^\infty f(t-b)e^{-st}\,dt$$

$$= \int_0^\infty e^{-bs}e^{-sx}f(x)\,dx = e^{-bs}\int_0^\infty e^{-sx}f(x)\,dx \quad \text{then}$$

$$\mathcal{L}[f(t-b)H(t-b)] = e^{-bs}F(s)$$

Example:

1) $\mathcal{L}^{-1}\left(\frac{1-e^{-s}}{s}\right) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}}{s}\right] = H(t) - H(t-1)$

2) $\mathcal{L}\left[(t^2-1)H(t-1)\right] = \mathcal{L}\left[\left((t-1+1)^2-1\right)H(t-1)\right]$

$$= \mathcal{L}\left[\left((t-1)^2+2(t-1)\right)H(t-1)\right]$$

$$= \mathcal{L}\left[(t-1)^2H(t-1)\right] + 2\,\mathcal{L}\left[(t-1)H(t-1)\right]$$

$$= \frac{2e^{-s}}{s^3} + \frac{2.e^{-s}}{s^2}$$

Example: $y'' + 3y' + 2y = tH(t) - tH(t-1)$ $\quad y(0)=0 \quad y'(0)=1$

$= tH(t) - (t-1)H(t-1) - H(t-1)$

$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0)$, $\mathcal{L}[y'] = sY(s) - y(0)$, $\mathcal{L}[y] = Y(s)$

$$s^2Y(s) - 1 + 3sY(s) + 2Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$Y(s)(s^2+3s+2) = 1 + \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \rightarrow \boxed{Y(s) = \frac{1}{(s+1)(s+2)} + \frac{1}{s^2(s+1)(s+2)} - \frac{e^{-s}}{s^2(s+1)(s+2)} - \frac{e^{-s}}{s(s+1)(s+2)}}$$

Laplace transform of $t^n f(t)$

by using this, we get $\boxed{F^{(n)}(s) = (-1)^n \mathcal{L}[t^n f(t)]}$

Laplace transform of $f(t)/t$

$$\int_s^\infty F(z)\,dz = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

Example:

① $\mathcal{L}[t\sin(at)] = -\frac{d}{ds}\{\mathcal{L}[\sin(at)]\} = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$

$$= \frac{2as}{(s^2+a^2)^2}$$

② $\mathcal{L}\left[\frac{1-\cos(at)}{t}\right] = \int_s^\infty \mathcal{L}[1-\cos(at)]\,dz = \int_s^\infty \left[\frac{1}{z} - \frac{z}{z^2+a^2}\right] dz$

$$= \ln(z) - \frac{1}{2}\ln(z^2+a^2)\Big|_s^\infty = \ln\left|\frac{z}{\sqrt{z^2+a^2}}\right|_s^\infty$$

$$= -\ln\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

Initial-value Theorem

Let $f(t)$ and $f'(t)$ possess Laplace transform.

$$\int_0^\infty f'(t)e^{-st}\,dt = sF(s) - f(0)$$

let $s \to \infty$, $sF(s) - f(0) = 0$ then $\lim_{s\to\infty} sF(s) = f(0)$. this is the initial-value theorem.

e.g. $f(t) = e^{3t}$ $\quad \mathcal{L}[f(t)] = \frac{1}{s-3}$

$f(0) = 1.$ $\quad \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s}{s-3} = 1.$

Final-value Theorem

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$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ holds.

e.g. $f(t) = t \quad F(s) = 1/s^2$

$\lim_{t\to\infty} f(t) = \infty$ and $\lim_{s\to 0} s.F(s) = \infty$

Example: Laplace transform of $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2)$

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t + \delta(t - \pi/2)\}$$

$$s^2 Y(s) + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} + e^{-s\pi/2}$$

$$Y(s) = \frac{s}{(s^2+2s+2)(s^2+1)} + \frac{1}{s^2+2s+2} e^{-s\pi/2}$$

① $\mathcal{L}\{e^{-t}\sin 2t\} = F(s) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$

② $\mathcal{L}\{t^2 H(t-1)\} = \mathcal{L}\{[(t-1)^2 + 2(t-1) + 1] H(t-1)\}$

$$= F(s) = \frac{2}{s^3} e^{-s} + \frac{2}{s^2} e^{-s} + \frac{e^{-s}}{s}$$

③ $\mathcal{L}\{te^t + e^t \sin 3t + e^{2t}\cos 5t\}$

$$F(s) = -\frac{d}{ds}\left[\frac{1}{s-1}\right] + \frac{3}{(s-1)^2+9} + \frac{s-2}{(s-2)^2+25}$$

$$= \frac{1}{(s-1)^2} + \frac{3}{s^2-2s+10} + \frac{s-2}{s^2-4s+29}$$

④ $\mathcal{L}[te^{-3t}\sin(2t)] = \frac{-d}{ds}\left[\frac{2}{(s+3)^2+4}\right] = \frac{4(s+3)}{(s^2+6s+13)^2}$

⑤ $f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \pi \le t \end{cases}$

$$f(t) = \sin t\, H(t) - \sin t\, H(t-\pi)$$
$$= \sin t\, H(t) - \sin(\pi - t)\, H(t-\pi)$$
$$= \sin t\, H(t) + \sin(t-\pi)\, H(t-\pi)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-\pi s}$$

Example: ① $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{(s+2)^2+1}\right\}$

$$= f(t) = e^{-2(t-4)}\sin(t-4)H(t-4)$$

② $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\} = e^{-t}\cos t - e^{-t}\sin t$

Example:

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4-t, & 2 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$$

$$f(t) = t + (4-2t)H(t-2) + (t-4)H(t-4)$$

$$f(t) = t - 2(t-2)H(t-2) + (t-4)H(t-4)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-2s} + \frac{1}{s^2}e^{-4s}$$

Example:

① $y'' + 3y' + 2y = H(t-1)$: $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{H(t-1)\}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{e^{-s}}{s} \rightarrow \boxed{Y(s) = \frac{e^{-s}}{s(s+1)(s+2)}}$$

② $y'' + 4y' + 4y = tH(t-2)$: $y(0) = 0$, $y'(0) = 2$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{(t-2)H(t-2) + 2H(t-2)\}$$

$$s^2Y(s) - 2 + 4sY(s) + 4Y(s) = \frac{e^{2t}}{s^2} + \frac{2e^{2t}}{s}$$

$$Y(s) = \frac{2}{(s+2)^2} + e^{-2t}\frac{1}{s^2(s+2)^2} + 2e^{-2t}\frac{1}{s(s+2)^2}$$

Example: $f(t) = te^{-t}$ $F(s) = \frac{1}{(s+1)^2}$

IVT

$$\lim_{s \to \infty} sF(s) = f(0)$$

$$\lim_{t \to \infty} te^{-t} = 0$$

$$\lim_{s \to \infty} \frac{s}{(s+1)^2} = 0 = f(0) \checkmark$$

Example: $F(s) = \frac{1}{s-1}$

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$$

$\lim_{t\to\infty} e^t = \infty$ but $\lim_{s\to 0} \frac{s}{s-1} = 0$ NO! FVT does not apply!

Example: $F(s) = \frac{2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$

$$2 = A(s^2+3s+2) + B(s^2+2s) + C(s^2+s)$$

$A+B+C=0$ $\quad 3A+2B+C=0$ $\quad A=1$

$$\left.\begin{aligned} B+C &= -1 \\ 2B+C &= -3 \end{aligned}\right\} \quad B=-2 \quad C=1$$

$$\mathcal{L}^{-1}\{F(s)\} = 1 - 2e^{-t} + e^{-2t} = f(t)$$

$\lim_{t\to\infty} f(t) = 1 = \lim_{s\to 0} s.F(s)$ Yes. FVT applies!

Convolution

the mathematical operation of the convolution product is defined by

$$\boxed{f(t)*g(t) = \int_0^t f(t-x)g(x)dx = \int_0^t f(x)g(t-x)dx}$$

Example:

① $\cos t * \sin t = \int_0^t \cos(t-x)\sin x\,dx = \int_0^t \cos x \sin(t-x)dx$

$$= \frac{1}{2}\int_0^t [\sin t + \sin(t-2x)]dx$$

$$= \frac{1}{2} t \sin(t)$$

② $t^2 * \sin t = \int_0^t (t-x)^2 \sin x\,dx$

$$= t^2 + 2\cos t - 2$$

Solution of Linear Differential Equations with Constant Coeff

Example:

(1) $y'' + 2y' = 8t \; ; \; y(0) = 0 ; \; y'(0) = 0$

$\mathcal{L}\{y'' + 2y'\} = \mathcal{L}\{8t\}$

$(s^2 + 2s)\,Y(s) = 8\frac{1}{s^2} \rightarrow Y(s) = \frac{8}{s^2(s^2+2s)}$

$\frac{8}{s^3(s+2)} = \frac{4}{s^3} - \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+2} = G(s)$

$\mathcal{L}^{-1}\{G(s)\} = 2t^2 - 2t + 1 - e^{-2t}$

(2) $y'' - 4y' + 3y = e^{2t} \; : \; y(0) = 0 ; \; y'(0) = 1$

$s^2Y(s) - 1 - 4sY(s) + 3Y(s) = \frac{1}{s-2}$

$Y(s) = \frac{1}{(s-1)(s-3)} + \frac{1}{(s-1)(s-2)(s-3)}$

$= \frac{1}{(s-1)(s-3)}\left[1 + \frac{1}{s-2}\right]$

$= \frac{1}{(s-1)(s-3)} \; \frac{s-1}{s-2} = \frac{1}{(s-2)(s-3)} = \frac{1}{s-3} - \frac{1}{s-2} = G(s)$

$$\boxed{\mathcal{L}^{-1}\{G(s)\} = e^{3t} - e^{-2t}}$$