The Strum - Llouville Problem

the ordinary differential equation $X'' + \lambda X = 0$. $0 < x < L$; $\lambda > 0$ with boundary conditions $X(0) = X(L) = 0$ or $X'(0) = X'(L) = 0$. This is an example of "a boundary value problem". Unlike initial-value problems, the present boundary-value problem has an infinite solution.

e.g. $X(0) = 0 = X(L)$, then we have $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ with $\lambda_n = \frac{n^2\pi^2}{L^2}$; $n = 1, 2, \ldots$

the λ_n's are called "the eigenvalues" and the X_n's are the corresponding eigenfunctions of this Strum - Llouville boundary-value problem.

Eigenvalues and Eigenfunctions

The second order differential equation we solve,

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0 \qquad a \leq x \leq b$$

with boundary conditions

$$\alpha y(a) + \beta y'(a) = 0 \quad \text{and} \quad \gamma y(b) + \delta y'(b) = 0$$

This system is called "a regular Strum-Llouville problem. If $p(x)$ and $r(x)$ vanishes at one of the end points of the interval $[a, b]$ or when the interval is of infinite length, the problem becomes "a singular Strum - Llouville problem".

Theorem: For a regular Strum - Llouville problem with $p(x) > 0$, all of the eigenvalues are real if $p(x)$, $q(x)$ and $r(x)$ are real functions and the eigenfunctions are differentiable and continuous.

If there is only one independent eigenfunction for each eigenvalue, that eigenvalue is "simple". When more than one eigenfunction belongs to a single eigenvalue the problem is degenerate".

Theorem! The regular Strum - Llouville problem has infinitely many real and simple eigenvalues λ_n, $n = 1, 2, 3\ldots$, which can be arranged in a monotonically increasing sequence $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Every eigenfunction $y_n(x)$ associated with the corresponding λ_n has exactly n zeros in the interval (a, b). For each eigenvalue there exists only one eigenfunction (up to a multiplicative constant).

Example: $y'' + \lambda y = 0$; $y'(0) = 0 = y(L)$

$$y(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$y'(0) = c_2\sqrt{\lambda} = 0 \rightarrow c_2 = 0, \lambda \neq 0$$

Eigenvalues ↓   Eigenfunctions. ↑

$$y(L) = c_1\cos\underbrace{\sqrt{\lambda}L}_{0} = 0 \quad \sqrt{\lambda}L = \left(\frac{2n-1}{2}\right)\pi \rightarrow \boxed{\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}} \quad \boxed{y_n(x) = \cos\left(\frac{(2n-1)\pi}{2L}x\right)}$$

Example: $y'' + \lambda y = 0$; $y(0) + y'(0) = 0$ and $y(\pi) + y'(\pi) = 0$

① $\lambda = -m^2 < 0$

$y'' - m^2 y = 0 \rightarrow y(x) = c_1 e^{mx} + c_2 e^{-mx}$, $y'(x) = c_1 m e^{mx} - c_2 m e^{-mx}$

$y(0) + y'(0) = c_1 + c_2 + c_1 m - c_2 m = 0 \rightarrow c_1(1+m) = c_2(m-1)$

$y(\pi) + y'(\pi) = c_1 e^{m\pi} + c_2 e^{-m\pi} + c_1 m e^{m\pi} - c_2 m e^{-m\pi} = 0$

$= c_1 e^{m\pi}(1+m) + c_2 e^{-m\pi}(1-m) = 0$

$= c_1 e^{m\pi}(1+m) - e^{-m\pi}(\underbrace{c_2(m-1)}_{c_1(1+m)}) = 0$

$= c_1(1+m)\left[e^{m\pi} - e^{-m\pi}\right] = 0$

$m = -1$

$c_1(1+m) = c_2(1-m) \rightarrow c_1 + c_1 m + c_2 m - c_2 = 0$

$m(c_1 + c_2) = c_2 - c_1 \rightarrow m = \frac{c_2 - c_1}{c_1 + c_2} \rightarrow c_2 - \cancel{c_1} = -\cancel{c_1} - c_2$

$\boxed{m = -1}$ $\boxed{c_2 = 0}$

$m = -1$ & $c_2 = 0. \rightarrow \boxed{\lambda_0 = 1 \text{ and } y_0(x) = e^{-x}}$

② $\lambda = 0$

$y'' = 0 \rightarrow y(x) = B + Ax \quad y'(x) = A$

$y(0) + y'(0) = B + A = 0 \rightarrow B = -A$

$y(\pi) + y'(\pi) = B + A\pi + A = 0 \rightarrow A\pi + A - A = 0 \rightarrow A\pi = 0 \rightarrow \boxed{A = 0, B = 0}$

No solution!

③ $\lambda = m^2 > 0$

$y'' + m^2 y = 0 \rightarrow y(x) = c_1 \cos(mx) + c_2 \sin(mx)$

$y'(x) = -c_1 m \sin(mx) + c_2 m \cos(mx)$

$$\left.\begin{array}{l} y(0) + y'(0) = c_1 + c_2 m = 0 \\ y(\pi) + y'(\pi) = c_1 \cos(m\pi) + c_2 m \cos(m\pi) = 0 \end{array}\right\} c_1 = -c_2 m$$

$(c_1 + c_2 m)\underbrace{\cos(m\pi)}_{0} = 0 \qquad y(x) = -c_2 m \cos(mx) + c_2 \sin(mx)$

$m\pi = n\pi \rightarrow m = n \qquad y_n(x) = \sin(nx) - n\cos(nx)$

$\boxed{\lambda_n = n^2,\ y_n(x) = \sin(nx) - n\cos(nx)}$

Example: $y^{(iv)} + \lambda y = 0 \quad y(0) = y''(0) = 0, \; y(L) = y''(L) = 0$ ③ 01/09/2023

$\lambda = -m^4 < 0$

$\mu^4 - m^4 = 0 \rightarrow (\mu^2 - m^2)(\mu^2 + m^2) = 0 = (\mu - m)(\mu + m)(\mu - im)(\mu + im)$

$y(x) = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos(mx) + c_4 \sin(mx)$

$y'(x) = c_1 m e^{mx} - c_2 m e^{-mx} - c_3 m \sin(mx) + c_4 m \cos(mx)$

$y''(x) = c_1 m^2 e^{mx} + c_2 m^2 e^{-mx} - c_3 m^2 \cos(mx) - c_4 m^2 \sin(mx)$

$y(0) + y''(0) = c_1 + c_2 + c_3 + c_1 m^2 + c_2 m^2 - c_3 m^2 = 0$

$= c_1(1+m^2) + c_2(1+m^2) + c_3(1-m^2) = 0$

$= (c_1 + c_2)(1+m^2) + c_3(1-m^2) = 0$

$y(L) + y''(L) = c_1 e^{mL} + c_2 e^{-mL} + c_3 \cos(mL) + c_4 \sin(mL)$

$+ c_1 m^2 e^{mL} + c_2 m^2 e^{-mL} - c_3 m^2 \cos(mL) - c_4 m^2 \sin(mL) = 0$

$= c_1 e^{mL}(1+m^2) + c_2 e^{-mL}(1+m^2) + c_3 \cos(mL)(1-m^2) + c_4 \sin(mL)(1-m^2) = 0$

$= (1+m^2)(c_1 e^{mL} + c_2 e^{-mL}) + (1-m^2)(c_3 \cos(mL) + c_4 \sin(mL)) = 0$

$(c_1 = c_2 = c_3 = 0)$

$= (1+m^2)(0) + (1-m^2)(c_4 \sin(mL)) = 0$

$= (1-m^2) c_4 \sin(\underbrace{mL}_{n\pi}) = 0 \qquad \boxed{m = \frac{n\pi}{L}}$

$$\boxed{\lambda_n = -m^4 = -\frac{n^4 \pi^4}{L^4} \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right)}$$

Example: $y'' + \lambda y = 0$; $y(0) = 0$, $y(\pi) + y'(\pi) = 0$

1. $\lambda = -m^2 < 0$

$y'' - m^2 y = 0 \rightarrow y(x) = c_1 e^{mx} + c_2 e^{-mx}$ $\quad y'(x) = c_1 m e^{mx} - c_2 m e^{-mx}$

$y(0) = c_1 + c_2 = 0 \rightarrow c_1 = -c_2$

$y(\pi) + y'(\pi) = c_1 e^{m\pi} + c_2 e^{-m\pi} + c_1 m e^{m\pi} - c_2 m e^{-m\pi} = 0$

$= c_1 e^{m\pi}(1+m) + c_2 e^{-m\pi}(1-m) = 0$

$= c_1 \left[e^{m\pi}(1+m) - e^{-m\pi}(1-m) \right] = 0 \rightarrow c_1 \left[m(\underbrace{e^{m\pi} + e^{-m\pi}}_{\neq 0}) + (\underbrace{e^{m\pi} - e^{-m\pi}}_{\neq 0}) \right] = 0$

$\boxed{c_1 = 0 \rightarrow c_2 = 0}$ No solution!

2. $\lambda = m = 0$

$y'' = 0 \rightarrow y(x) = B + Ax \rightarrow y(0) = B = 0.$

$y'(x) = A$

$y(\pi) + y'(\pi) = \underbrace{B}_{0} + A\pi + A = 0 \rightarrow A\underbrace{(\pi+1)}_{\neq 0} = 0 \rightarrow \boxed{A = 0,\ B = 0}$

No solution!

3. $\lambda = m^2 > 0$

$y'' + m^2 y = 0 \rightarrow y(x) = c_1 \cos(mx) + c_2 \sin(mx),\ y'(x) = -c_1 m \sin(mx) + c_2 m \cos(mx)$

$y(0) = c_1 = 0$

$y(\pi) + y'(\pi) = c_2 \sin(m\pi) + c_2 m \cos(m\pi) = 0$

$= c_2 [\underbrace{\sin(m\pi) + m\cos(m\pi)}_{0}] = 0$

$\rightarrow m = -\tan(m\pi)$ then $\boxed{k_n = -\tan(k_n \pi) \text{ and } \lambda_n = k_n^2}$

Eigenvalues

$\boxed{y_n(x) = \sin(k_n x)}$ $\leftarrow$ Eigenfunctions.

Example: $\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{\lambda}{x}y=0 \quad y(1)=y(e)=0$ ⑤ 01/09/2023

$$=x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{\lambda}{x}y=0$$

$\delta=\ln(x)$ transformation

$$\begin{bmatrix} x & & \delta \\ 1 & \to & 0 \\ e & \to & 1 \end{bmatrix}$$

$$\frac{dy}{dx}=\frac{dy}{d\delta}\frac{d\delta}{dx}=\frac{1}{x}\frac{dy}{d\delta}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\frac{dy}{d\delta}+\frac{1}{x}\frac{d^2y}{d\delta^2}\frac{d\delta}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\frac{dy}{d\delta}+\frac{1}{x^2}\frac{d^2y}{d\delta^2}\to\frac{d^2y}{dx^2}=\frac{1}{x^2}\frac{d^2y}{d\delta^2}-\frac{1}{x^2}\frac{dy}{d\delta} \text{ and } \frac{dy}{dx}=\frac{1}{x}\frac{dy}{d\delta}$$

$$x\left(\frac{1}{x^2}\right)\left(\frac{d^2y}{d\delta^2}-\frac{dy}{d\delta}\right)+\frac{1}{x}\frac{dy}{d\delta}+\frac{\lambda}{x}y=0$$

$$\frac{1}{x}\frac{d^2y}{d\delta^2}-\cancel{\frac{1}{x}\frac{dy}{d\delta}}+\cancel{\frac{1}{x}\frac{dy}{d\delta}}+\frac{\lambda}{x}y=0\to\frac{1}{x}\Big[\underbrace{\frac{d^2y}{d\delta^2}+\lambda y}_{0}\Big]=0$$

we get, $\frac{d^2y}{d\delta^2}+\lambda y=0$ with $y(0)=0$, $y(1)=0$.

$y''+\lambda y=0 \quad \lambda=-m^2<0$

$y(\delta)=c_1\cos(\delta m)+c_2\sin(\delta m)\to y(0)=c_1=0$ ✓

$y(1)=c_2\sin(\underbrace{m}_{n\pi})=0\to m=n\pi$

$$\boxed{\begin{aligned}\lambda_n&=n^2\pi^2\\ y_n(\delta)&=\sin(n\pi\delta)\end{aligned}}$$

Now turning back to "x" variable by back-subs.

$$\boxed{\lambda_n=n^2\pi^2 \text{ and } y_n(x)=\sin(n\pi\ln(x))}$$

Orthogonality of Eigenfunctions

Theorem: let the functions $p(x)$, $q(x)$ and $r(x)$ of the regular Sturm–Liouville problem be real and continuous on the interval $[a,b]$. If $y_n(x)$ and $y_m(x)$ are continuously differentiable eigenfunctions corresponding to the distinct eigenvalues λ_n and λ_m, respectively, then $y_n(x)$ and $y_m(x)$ satisfies the orthogonality condition:

$$\int_a^b r(x)\,y_n(x)\,y_m(x)\,dx = 0 \quad \text{with respect to the weight function } r(x).$$

Example: (1) $y'' + \lambda y = 0$ $\quad y'(0) = 0 = y'(L)$ Verify orthogonality of $y_0(x) = 1$ and $y_n(x) = \cos(n\pi x/L)$

$r(x) = 1$

$$\int_0^L 1.1.\cos\left(\frac{n\pi x}{L}\right)dx = \left\{\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right\}_0^L = \frac{L}{n\pi}\left(\sin(n\pi) - 0\right) = 0.$$

(2) $y'' + \lambda y = 0$, $y(0) = 0 = y(L)$ and $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

$r(x) = 1$ $\quad y_1(x) = \sin\left(\frac{\pi x}{L}\right)$ $\quad y_2(x) = \sin\left(\frac{2\pi x}{L}\right)$ $\quad \left[\begin{matrix} n = 1, 2 \\ n \neq m,\ y_n(x)\ \&\ y_m(x)\end{matrix}\right]$

$$\int_0^L \sin\left(\frac{\pi x}{L}\right)\cdot\sin\left(\frac{2\pi x}{L}\right)dx = \int_0^L 2\sin^2\left(\frac{\pi x}{L}\right)\cdot\cos\left(\frac{\pi x}{L}\right)dx$$

$$= \int 2u^2\cdot\left(\frac{L}{\pi}\right)du = \frac{2}{3}\frac{L}{\pi}\sin^3\left(\frac{\pi x}{L}\right)\Big|_0^L = \frac{2}{3}\frac{L}{\pi}\left(\sin^3(\pi) - 0\right) = 0$$

Expansion in Series of Eigenfunction

$f(x)$ is defined on $a \le x \le b$. $y_n(x)$'s are the eigenfunctions given by a regular "Sturm–Liouville problem" the function $f(x)$ can be represented by

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n y_n(x)} \quad (+)$$

By using orthogonality of $y_n(x)$'s, we may write

$$\int_a^b r(x) f(x) y_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b r(x) y_n(x) y_m(x)\,dx$$

if $m \neq n$, $y_n(x) y_m(x) = 0$ then we get

$$\int_a^b r(x) f(x) y_m(x)\,dx = c_m \int_a^b r(x) y_m(x) y_m(x)\,dx$$

$$\rightarrow \boxed{c_n = \frac{\int_a^b r(x) f(x) y_n(x)\,dx}{\int_a^b r(x) y_n^2(x)\,dx}} \quad (*)$$

Those series is named as "Generalized Fourier series" of the function $f(x)$ with respect to the eigenfunction $y_n(x)$. The coefficients c_n are called "the Fourier coefficients".

Example: $f(x) = x$ on $0 < x < \pi$ and the regular Sturm–Liouville problem is $y'' + \lambda y = 0$ and $y(0) = 0 = y(\pi)$.

The eigenfunctions of the system is $y_n(x) = \sin(nx)$ $n = 1, 2, 3, \ldots$

$r(x) = 1$. Then we may compute c_n's by using

$$c_n = \frac{\int_0^\pi x \sin(nx)\,dx}{\int_0^\pi \underbrace{\sin^2(nx)}_{\frac{1-\cos(2nx)}{2}}\,dx} = \frac{\left\{-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right\}_0^\pi}{\left\{\frac{x}{2} - \frac{1}{4n}\sin(2nx)\right\}_0^\pi}$$

$$\begin{array}{lll} +x & & \sin(nx) \\ -1 & & -\frac{1}{n}\cos(nx) \\ +0 & & -\frac{1}{n^2}\sin(nx) \end{array}$$

$$c_n = \frac{-\frac{\pi}{n}(-1)^n}{\frac{\pi}{2}} = -\frac{2}{n}(-1)^n \quad \text{then}$$

the generalized Fourier series of $f(x)$; $\boxed{f(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)}$

Example $y''+\lambda y=0 \quad y(0)=y'(L)=0$ and $y_n(x)=\sin\left[\frac{(2n-1)\pi x}{2L}\right]$ ⑧ 01/09/2023

Find the eigenfunction expansion of $f(x)=x$.

$$c_n=\frac{\int_0^L x\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx}{\int_0^L \sin^2\left[\frac{(2n-1)\pi x}{2L}\right]dx}$$

$$=\frac{\left\{\frac{-2Lx}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2L}\right]+\frac{4L^2}{(2n-1)^2\pi^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\right\}_0^L}{\left\{\frac{x}{2}-\frac{L}{2(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{L}\right]\right\}_0^L}$$

$$=\frac{4L^2}{(2n-1)^2\pi^2}(-1)^n\Big/\frac{L}{2}=\frac{8L}{(2n-1)^2\pi^2}(-1)^n$$

$$\boxed{c_n=\frac{8L}{(2n-1)^2\pi^2}(-1)^n}$$

then $f(x)=\sum_{n=1}^{\infty}\left[\frac{8L(-1)^n}{(2n-1)^2\pi^2}\right]\sin\left[\frac{(2n-1)\pi x}{2L}\right]=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]$

$+x \qquad \sin\left[\frac{(2n-1)\pi x}{2L}\right]$

$-1 \qquad -\frac{2L}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2L}\right]$

$+0 \qquad -\frac{4L^2}{(2n-1)^2\pi^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]$

$$\frac{1}{2}-\frac{1}{2}\cos\left[\frac{(2n-1)\pi x}{L}\right]$$

$$\frac{x}{2}-\frac{L}{2(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{L}\right]$$