The Laplace Equation

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The partial differential equation that describes the steady state for two dimensional heat conduction is Laplace Equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ is the general form of Laplace's Equation. It is also classified as "elliptic equation". This equation governs physical process where "equilibrium" has been reached.

Unlike the heat and wave equations, there are no initial conditions and the boundary conditions completely specify the solution.

In polar coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

In spherical coordinates, where $x = r\cos(\varphi)\sin\theta$, $y = r\sin(\varphi)\sin\theta$, $z = r\cos\theta$ and $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial u}{\partial \theta}\right] = 0$$

Separation of Variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad 0 < x < L; \quad 0 < y < z_0$$

$$u(x, z_0) = g z_0 + g c x$$

$$u_x(0, y) = u_x(L, y) = 0$$

$$u_y(x, 0) = 0$$

$$u(x,y) = X(x)Y(y) \rightarrow X''Y + Y''X = 0 \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = -m^2$$

① $X'' - m^2 X = 0 \rightarrow X(x) = A\cosh(mx) + B\sinh(mx)$

② $Y'' + m^2 Y = 0 \rightarrow Y(y) = C\cos(my) + D\sin(my)$

$$u_x(0,y) = u_x(L,y) = 0 \Rightarrow X'(0) = 0 = X'(L)$$

$$u_y(x,0) = X(x)Y'(0) = 0 \Rightarrow Y'(0) = 0$$

$$X'(0) = Bm = 0 \rightarrow B = 0. \quad X'(L) = -mA\sinh(mL) = 0, \quad m = \frac{n\pi}{L}$$

$$X_n(x) = \cosh\left(\frac{n\pi}{L}x\right), \quad Y_n(x) = C_n\cos\left(\frac{n\pi}{L}y\right) + D_n\sin\left(\frac{n\pi}{L}y\right)$$

$$Y_n'(x) = -C_n\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi}{L}y\right) + D_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi}{L}y\right); \quad Y_n'(0) = 0 \Rightarrow D_n = 0 \ \forall n$$

When separation constant (μ) is equal to zero, the problem has trivial soln.
Then the general solution becomes

$$u(x,y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)$$

$$u(x,z_0) = gz_0 + gcx = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}z_0\right)$$

$$A_0 = \frac{2}{L}\int_0^L (gz_0 + gcx)\,dx = 2gz_0 + gcL$$

$$\cosh\left(\frac{n\pi z_0}{L}\right) A_n = \frac{2}{L}\int_0^L (gz_0 + gcx)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = -\frac{2gcL\left[1-(-1)^n\right]}{n^2\pi^2\cosh(n\pi z_0/L)}$$

then the final solution

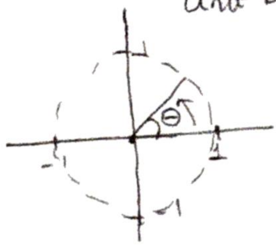
$$u(x,y) = gz_0 + \frac{gcL}{2} - \frac{4gcL}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos\left[(2m-1)\pi x/L\right]\cosh\left[(2m-1)\pi y/L\right]}{(2m-1)^2\cosh\left[(2m-1)\pi z_0/L\right]}$$

Poisson's Integral Formula 24/08/2023

Laplace's Equation within a unit disc.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad 0 \le r < 1,\ 0 \le \theta \le 2\pi$$

Unit Disc.



$$u(1,\theta) = f(\theta)$$

Separation of variables $u(r,\theta) = R(r)\,\phi(\theta)$

$$\frac{r^2 R'' + rR'}{R} = -\frac{\phi''(\theta)}{\phi(\theta)} = k^2$$

① $\phi''(\theta) + k^2\phi(\theta) = 0 \rightarrow \phi(\theta) = A\cos(k\theta) + B\sin(k\theta)$

② $r^2R'' + rR' - k^2R = 0 \rightarrow r^a(a^2 - a + a - k^2) = 0$

$R(r) = r^a$ $\qquad r^a(a^2 - k^2) = 0 \rightarrow a = \pm k$

then $R(r) = Cr^k + Dr^{-k}$ $D = 0$ because the solution should be bounded.

then the general solution $u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)]r^n$

$$u(1,\theta) = f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta,\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\left\{\frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos[n(\theta-\varphi)]\right\}d\theta$$

$|r| < 1$,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi$$

Poisson' Equation on a Rectangle

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \quad ; \quad 0<x<a, \ 0<y<b$$

$$u(0,y) = u(a,y) = 0$$

$$u(x,0) = u(x,b) = 0$$

By applying $u(x,y) = X(x)Y(y)$ we get as a solution

$$u(x,y) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_{nm}}{n^2\pi^2/a^2 + m^2\pi^2/b^2} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

where $a_{nm} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$.