Compartmental models in epidemiology

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Purpose

- explain simple (indeed very simple) mathematical models in epidemiology
- probably too simplistic for detailed analysis of COVID-19 but should still be useful to learn the basics of epidemiology and get some insight into what is happening now
- ref: "Compartmental models in epidemiology" in wikipedia and references therein (and some others)

Contents

- introduction of some models
  - SIR model
  - SEIR model
  - SEIRS model
- implications for COVID-19

SIR model (w/o vital dynamics)

Susceptible $\xrightarrow{\beta SI/N}$ **Infectious** $\xrightarrow{\gamma I}$ **Removed**

(incl. recovered and dead)

$$\frac{dS}{dt} = -\frac{\beta SI}{N}$$

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$$N = S + I + R = \text{const}\,.$$

β $[T^{-1}]$: time scale of contacts/infections

γ $[T^{-1}]$: time scale of recovery

$\boxed{R_0 = \frac{\beta}{\gamma}}$ controls the dynamics

R_0: basic reproduction number

Reproduction numbers

- basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$

contact/infection time scale
→ depends on environmental conditions
(culture, lockdown, social distancing, …)

recovery time scale
→ (mainly) biological constant

※ however, it is often defined as a value in absence of any deliberate intervention in disease transmission

- effective reproduction number

$$R_e = \frac{\beta}{\gamma}\frac{S}{N}$$

change with time, $R_e \approx R_0$ when $S \approx N$

- **here I mostly use R_0 and regards it as a variable** (unlike the convention in this field)

R_0 of well-known diseases

Values of R_0 of well-known infectious diseases[1]

	Disease	Transmission	R_0
はしか	Measles	Aerosol	12–18[2]
水痘	Chickenpox (varicella)	Aerosol	10–12[3]
おたふく風邪	Mumps	Respiratory droplets	10–12[4]
ポリオ	Polio	Fecal–oral route	5–7[citation needed]
風疹	Rubella	Respiratory droplets	5–7[citation needed]
百日咳	Pertussis	Respiratory droplets	5.5[5]
	COVID-19	Respiratory droplets	3.8–8.9[6]
天然痘	Smallpox	Respiratory droplets	3.5–6[7]
	HIV/AIDS	Body fluids	2–5[citation needed]
	SARS	Respiratory droplets	3.1–4.2[8]
	Common cold	Respiratory droplets	2–3[9]
ジフテリア	Diphtheria	Saliva	1.7–4.3[10]
	Influenza (1918 pandemic strain)	Respiratory droplets	1.4–2.8[11]
	Ebola (2014 Ebola outbreak)	Body fluids	1.5–1.9[12]
	Influenza (2009 pandemic strain)	Respiratory droplets	1.4–1.6[13]
	Influenza (seasonal strains)	Respiratory droplets	0.9–2.1[14]
	MERS	Respiratory droplets	0.3–0.8[15]

(from wikipedia)

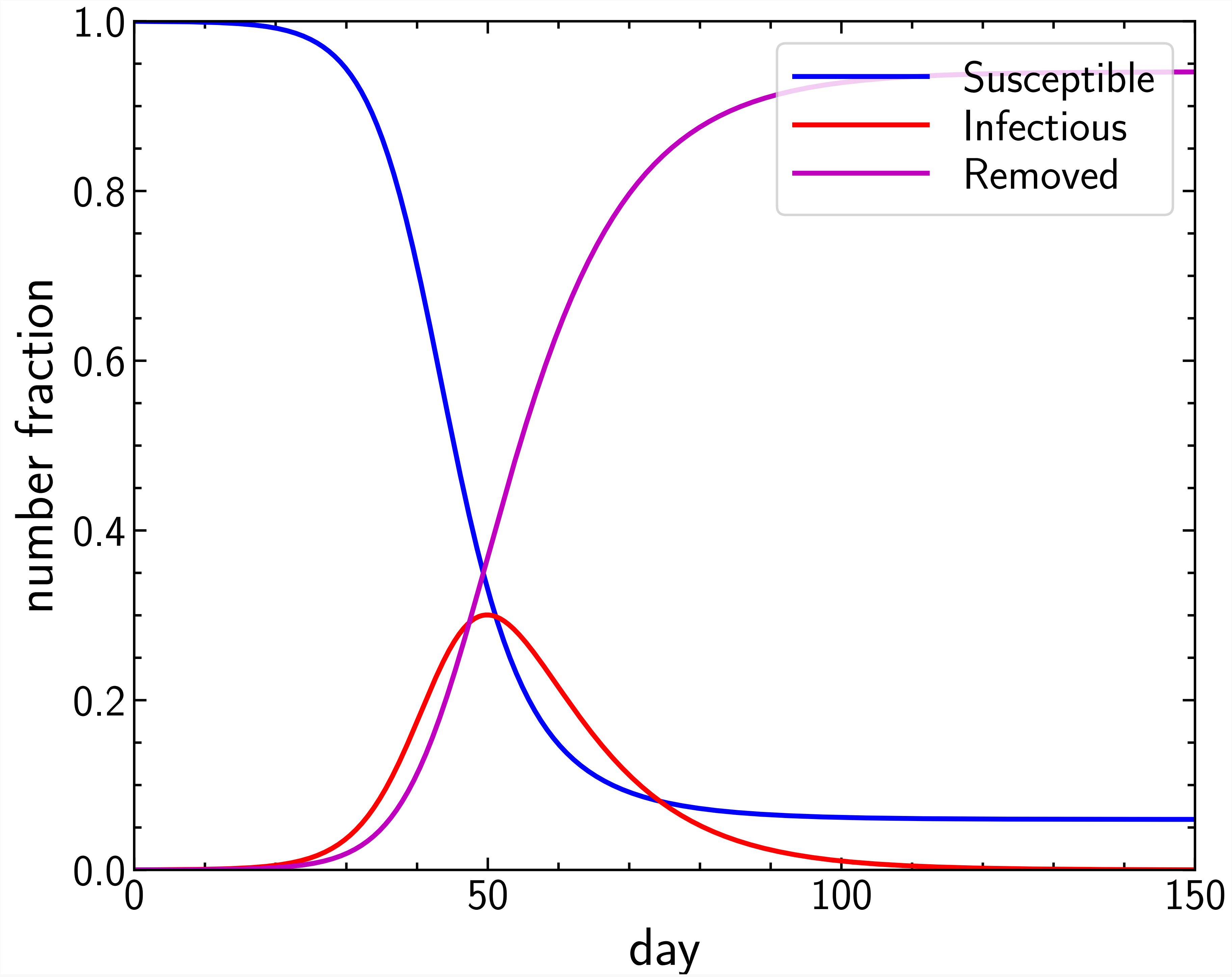
SIR model: example

β=0.3day^{-1}

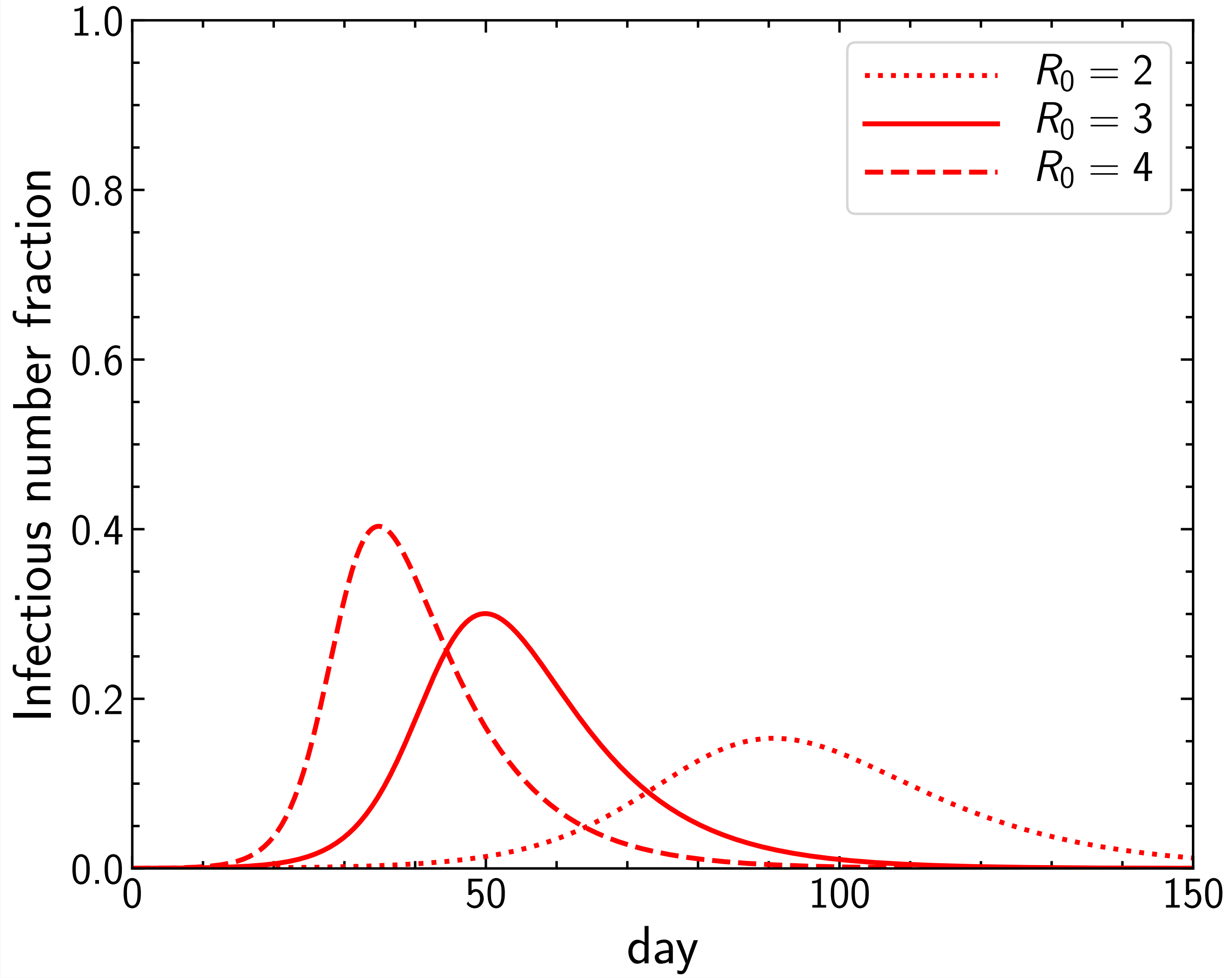
γ=0.1day^{-1}

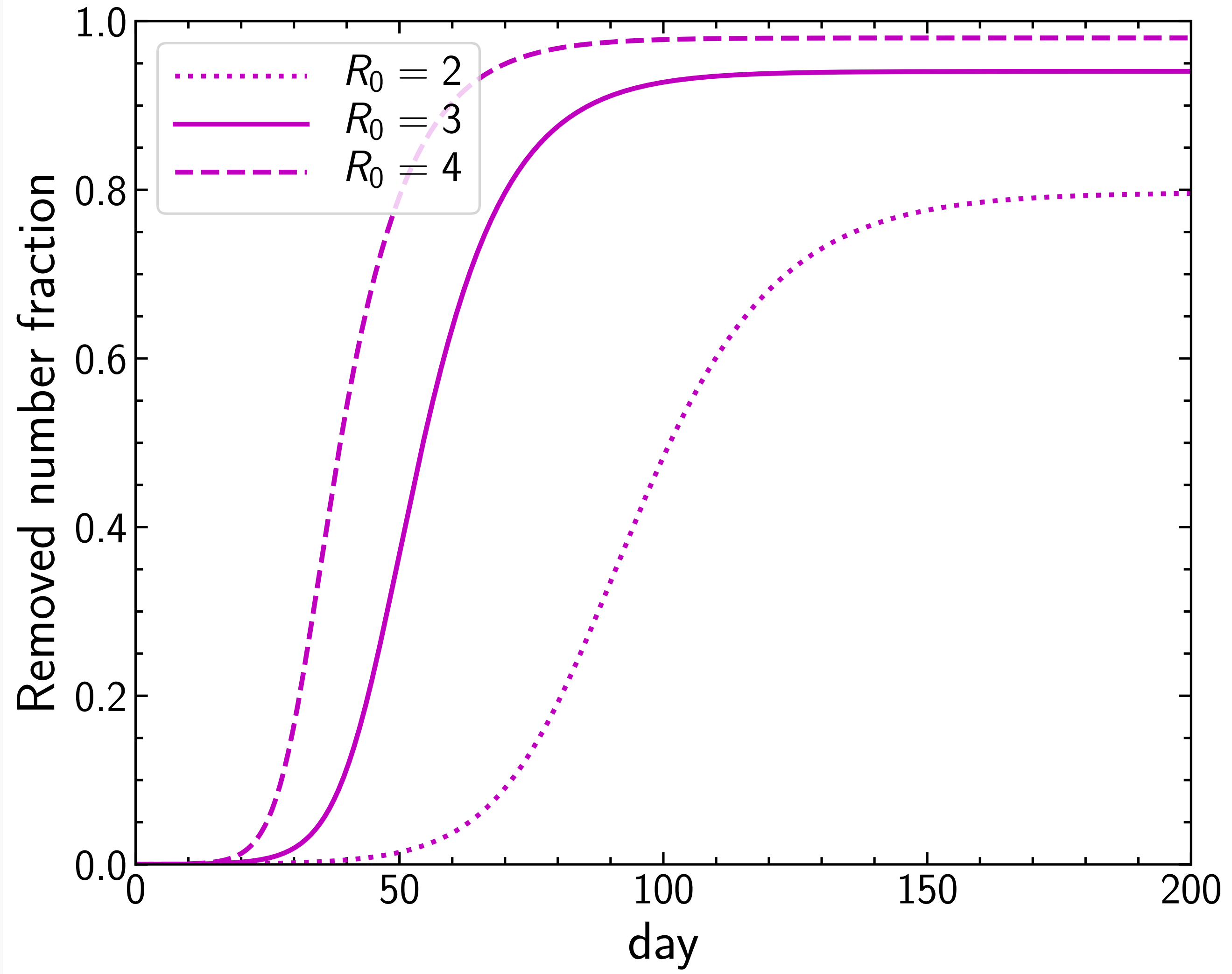
R_0=3

>90% people experiences infections in the end



Effect of different R_0 (β)





smaller R_0 → lower peak at later time ("flatten the curve")

converged values of fraction of Removed persons depend on R_0

Limiting behaviors

- at early stage when S≈N

$$\frac{dI}{dt} \sim \gamma(R_0 - 1)I \quad \Rightarrow \quad I \propto e^{\gamma(R_0-1)t}$$

slope depends on R_0

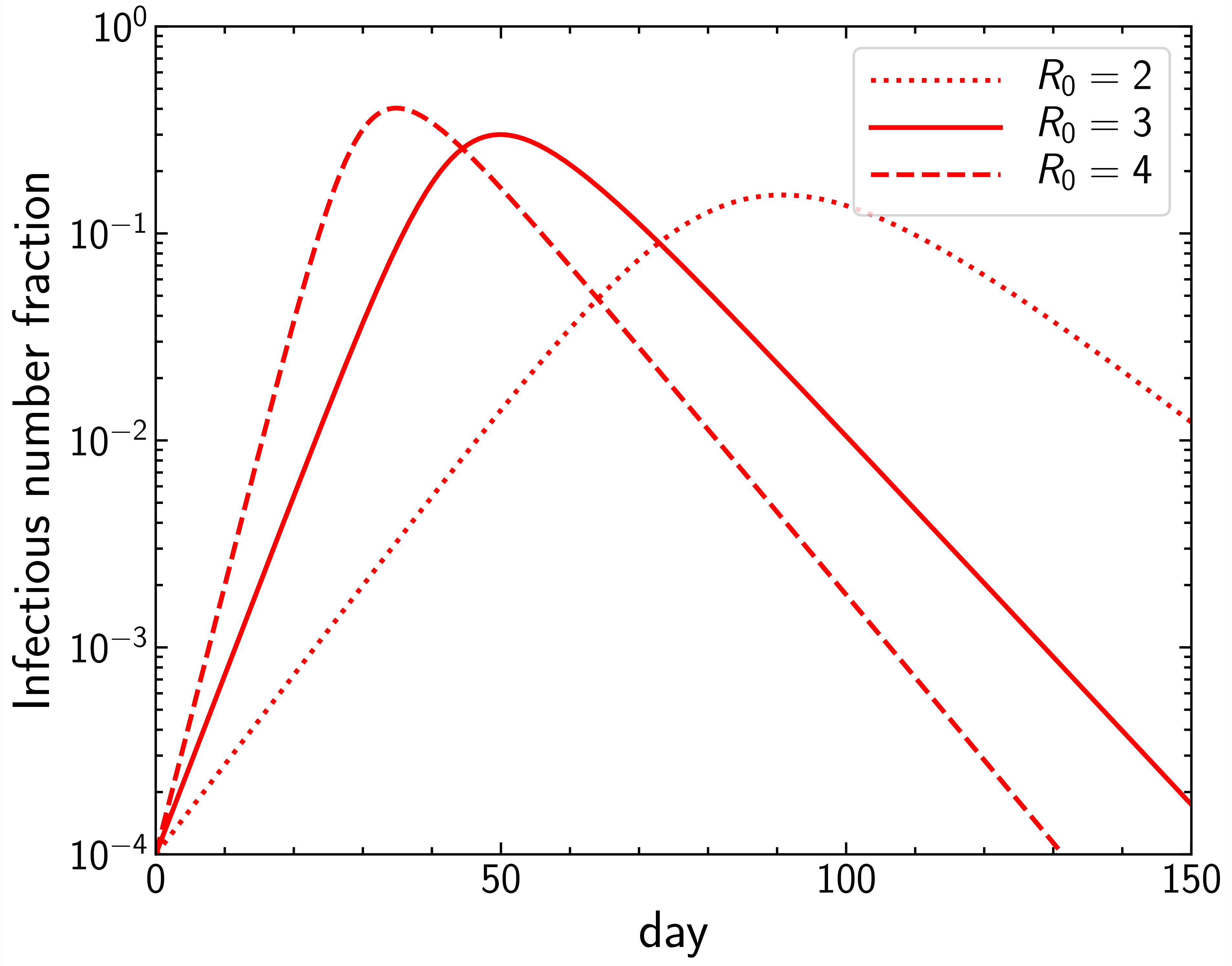
- at t→∞
  - from the original differential equations

$$\frac{dS}{S} = -R_0\frac{dR}{N} \quad \Rightarrow \quad \frac{S(\infty)}{S(0)} = e^{-R_0\left\{R(\infty) - R(0)\right\}/N}$$

$$\xrightarrow[S(\infty)+R(\infty)=N]{R(0)=0} \quad \frac{S(\infty)}{S(0)} = e^{-R_0\left\{1 - S(\infty)/N\right\}}$$

convergence value can be derived from this equation

Exponential growth



R_0 may be inferred from slope

Importance of R_0 (or R_e)

at early stage $I \propto e^{\gamma(R_0-1)t}$

or at any time $\dfrac{dI}{dt} = \gamma(R_\mathrm{e} - 1)I$

R_0 or R_e >1 → exponential grow
R_0 or R_e <1 → exponential decay

SEIR model (w/o vital dynamics)

Susceptible $\xrightarrow{\beta SI/N}$ **Exposed** $\xrightarrow{\alpha E}$ **Infectious** $\xrightarrow{\gamma I}$ **Removed** (incl. recovered and dead)

$$\frac{dS}{dt} = -\frac{\beta SI}{N}$$

$$\frac{dE}{dt} = \frac{\beta SI}{N} - \alpha E$$

$$\frac{dI}{dt} = \alpha E - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$$N = S + E + I + R = \text{const}\,.$$

β $[T^{-1}]$: time scale of contacts/infections

α $[T^{-1}]$: time scale of incubation

γ $[T^{-1}]$: time scale of recovery

$$R_0 = \frac{\beta}{\gamma}$$ (will be discussed later)

SEIR model: example

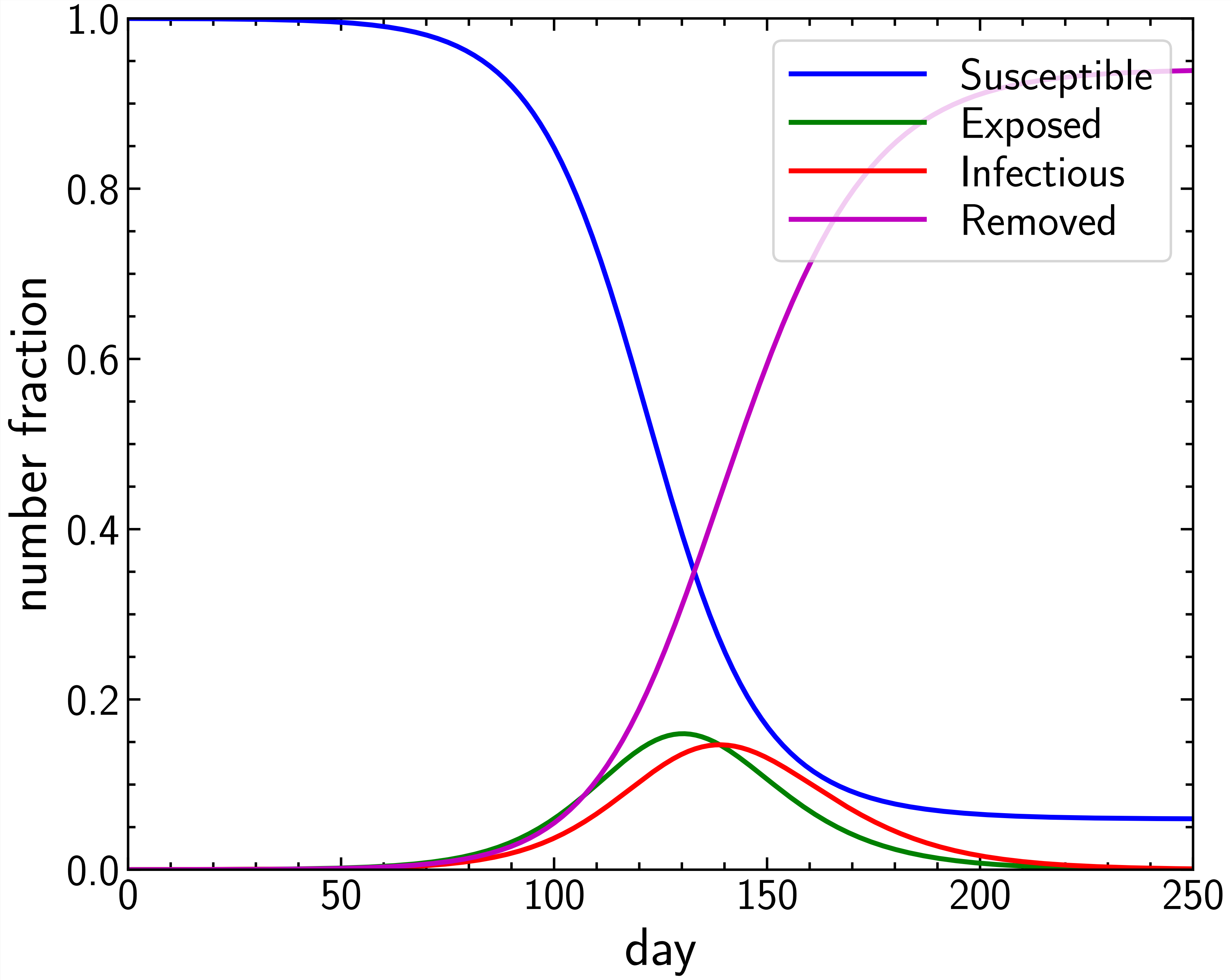
$\beta = 0.3\,\mathrm{day}^{-1}$

$\alpha = 0.1\,\mathrm{day}^{-1}$

$\gamma = 0.1\,\mathrm{day}^{-1}$

$R_0 = 3$

>90% people experiences infections in the end



Limiting behavior at t→∞

$$\frac{dS}{dt} = -\frac{\beta SI}{N}$$

$$\frac{dR}{dt} = \gamma I$$

$$\frac{dS}{S} = -R_0 \frac{dR}{N} \quad \Rightarrow \quad \frac{S(\infty)}{S(0)} = e^{-R_0\{1 - S(\infty)/N\}}$$

same as in SIR model

(determined only by R_0)

Limiting behavior at early stage (1/2)

- at early stage when S≈N

$$\frac{d}{dt}\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\gamma \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix} = \gamma\begin{pmatrix} -\alpha' & R_0 \\ \alpha' & -1 \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix}$$

$$\alpha' = \frac{\alpha}{\gamma} \quad R_0 = \frac{\beta}{\gamma}$$

  - characteristic equation

$$\begin{vmatrix} -\alpha' - \lambda & R_0 \\ \alpha' & -1 - \lambda \end{vmatrix} = 0$$

$$\lambda_\pm = \frac{-(1+\alpha') \pm \sqrt{(1+\alpha')^2 + 4\alpha'(R_0 - 1)}}{2}$$

Limiting behavior at early stage (2/2)

– "growing" mode

$$\lambda_+ = \frac{-(1+\alpha') + \sqrt{(1+\alpha')^2 + 4\alpha'(R_0 - 1)}}{2} \quad \Rightarrow \quad I \propto e^{\gamma\lambda_+ t}$$

$\lambda_+ > 0$ if $R_0 > 1$, $\lambda_+ < 0$ if $R_0 < 1$

→ R_0 can be regarded as basic reproduction number

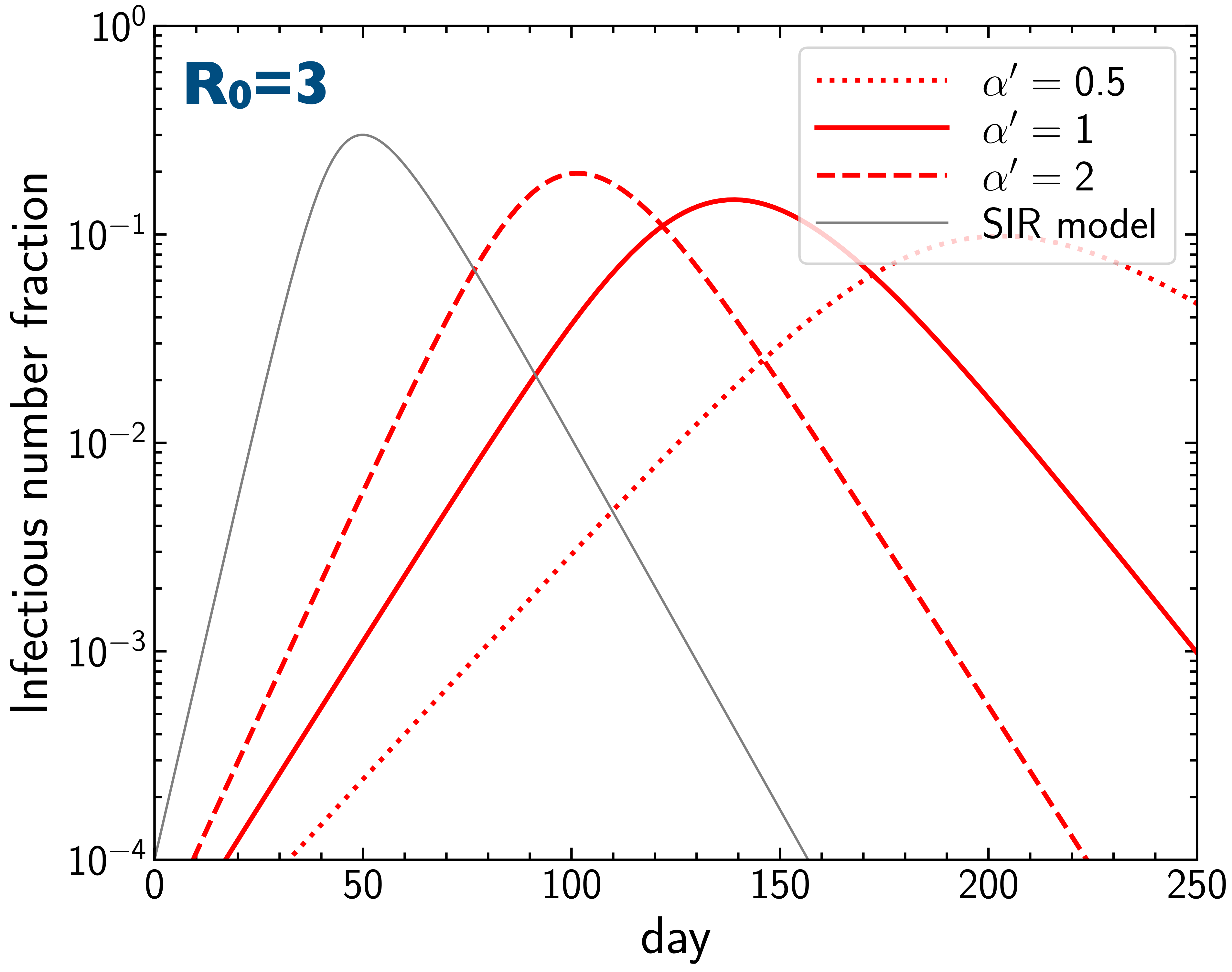
$\lambda_+ \sim R_0 - 1$ at $\alpha' \gg 1$

→ $\boxed{I \propto e^{\gamma(R_0-1)}t}$ (same as SIR)

$\lambda_+ \sim \alpha'(R_0 - 1)$ at $\alpha' \ll 1$

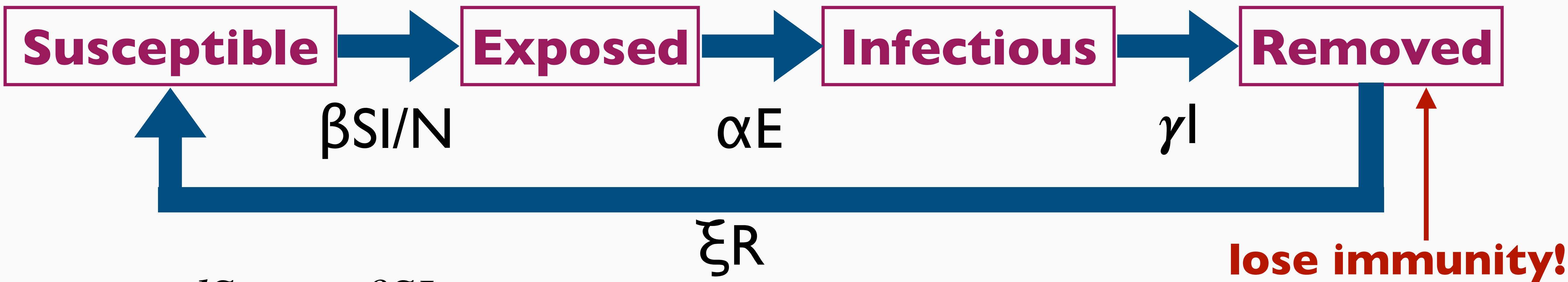
→ $\boxed{I \propto e^{\alpha(R_0-1)t}}$ (slower than SIR)

Exponential growth



slope gets shallower for smaller α'

SEIRS model (w/o vital dynamics)



$$\frac{dS}{dt} = -\frac{\beta SI}{N} + \xi R$$

$$\frac{dE}{dt} = \frac{\beta SI}{N} - \alpha E$$

$$\frac{dI}{dt} = \alpha E - \gamma I$$

$$\frac{dR}{dt} = \gamma I - \xi R$$

$$N = S + E + I + R = \text{const}.$$

β [T^{-1}] : time scale of contacts/infections

α [T^{-1}] : time scale of incubation

γ [T^{-1}] : time scale of recovery

ξ [T^{-1}] : time scale of losing immunity

$$R_0 = \frac{\beta}{\gamma}$$

SEIRS model: example

$\beta = 0.3\,\text{day}^{-1}$

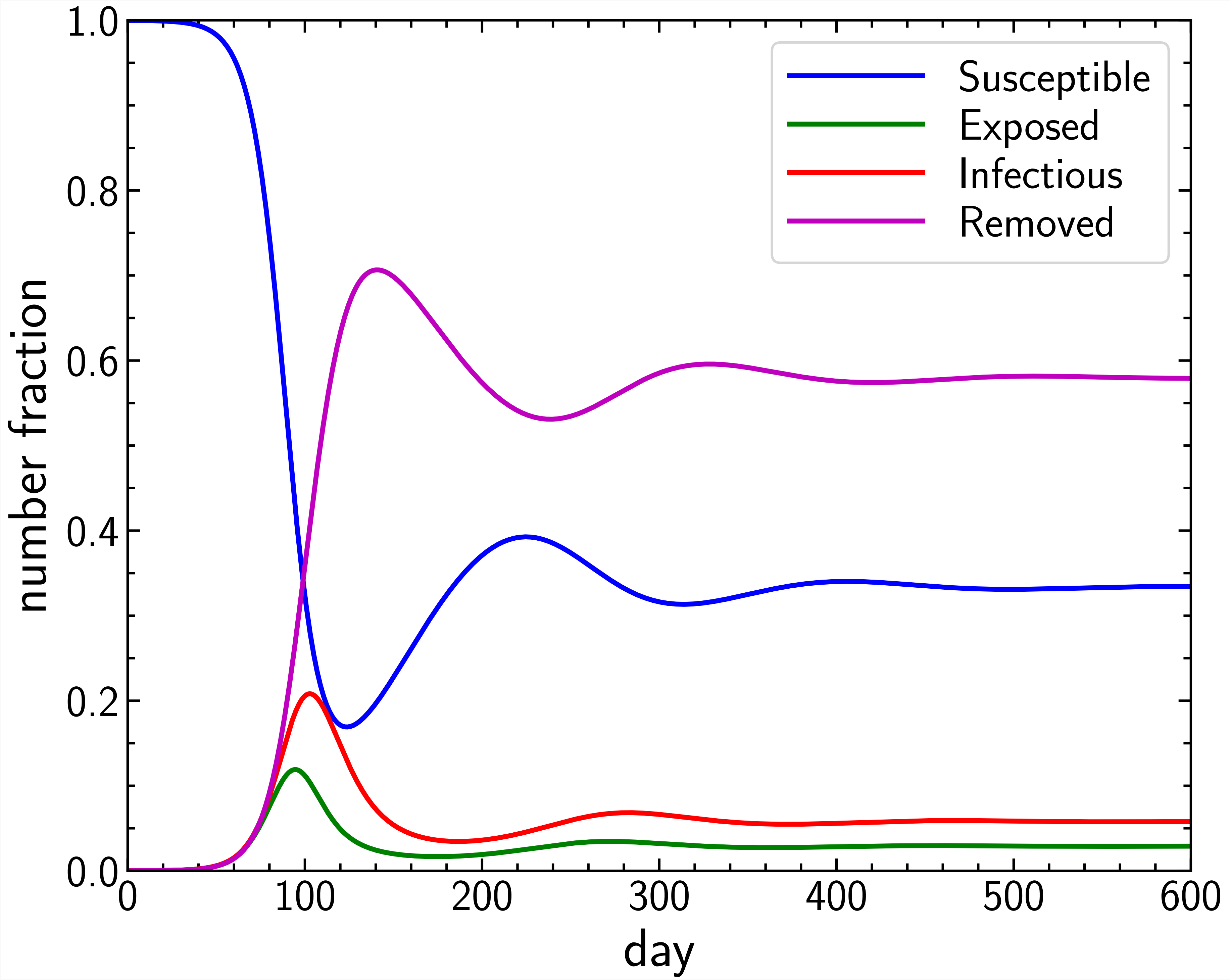
$\alpha = 0.2\,\text{day}^{-1}$

$\gamma = 0.1\,\text{day}^{-1}$

$\xi = 0.01\,\text{day}^{-1}$

$R_0 = 3$

all converge to non-zero values (**endemic**)



Limiting behavior at early stage

- at early stage when S≈N

$$\frac{d}{dt}\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\gamma \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix} = \gamma\begin{pmatrix} -\alpha' & R_0 \\ \alpha' & -1 \end{pmatrix}\begin{pmatrix} E \\ I \end{pmatrix}$$

same as in SEIR model

Limiting behavior at t→∞

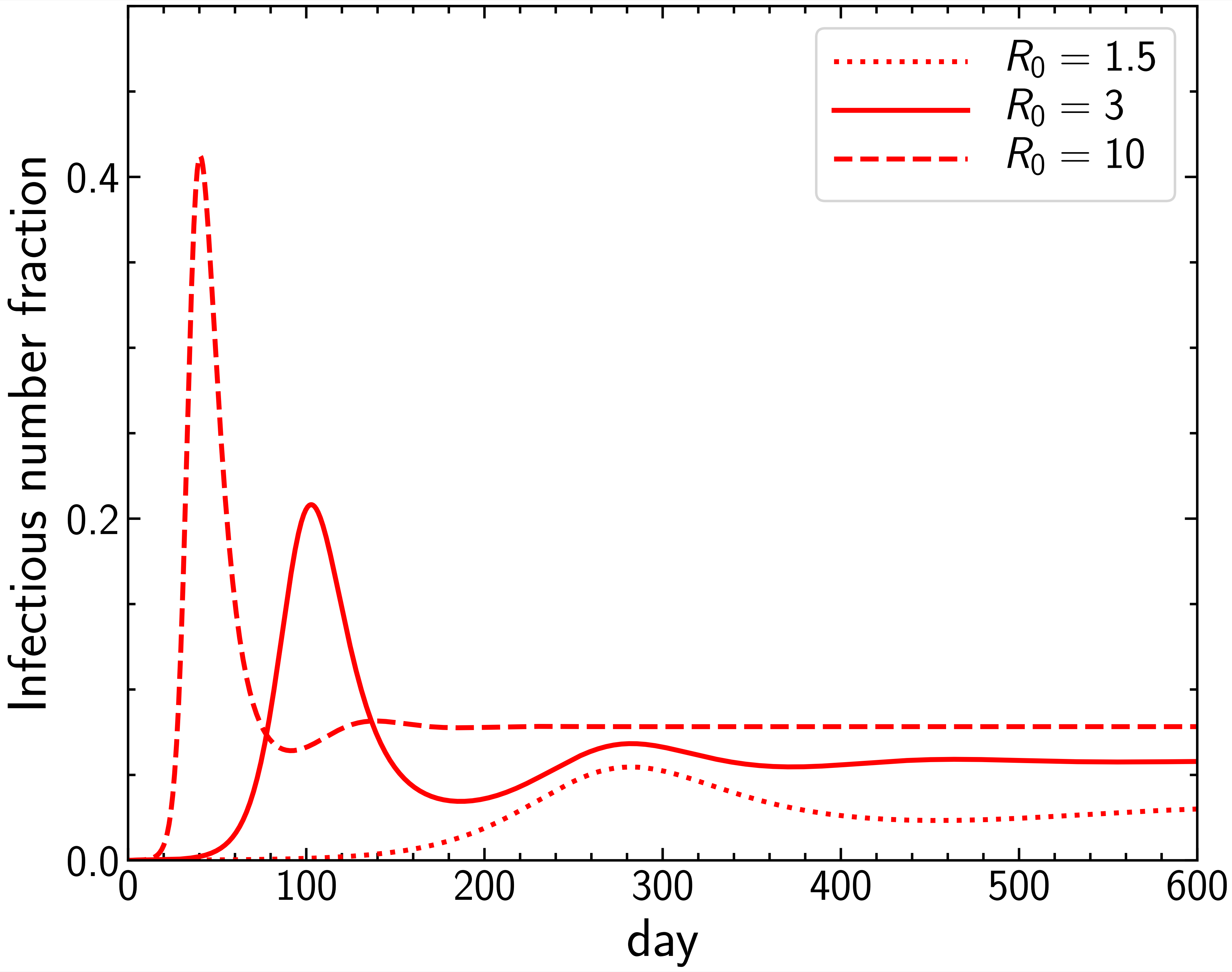
- at equilibrium

$$\frac{dS}{dt} = \frac{dE}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$$

$$\left[\begin{array}{l} \dfrac{S(\infty)}{N} = \dfrac{1}{R_0} \\ \dfrac{E(\infty)}{N} = \dfrac{1}{\alpha}\left(\dfrac{1 - 1/R_0}{1/\alpha + 1/\gamma + 1/\xi}\right) \\ \dfrac{I(\infty)}{N} = \dfrac{1}{\gamma}\left(\dfrac{1 - 1/R_0}{1/\alpha + 1/\gamma + 1/\xi}\right) \\ \dfrac{R(\infty)}{N} = \dfrac{1}{\xi}\left(\dfrac{1 - 1/R_0}{1/\alpha + 1/\gamma + 1/\xi}\right) \end{array}\right.$$

Infectious proportional to $1-1/R_0$

Some examples

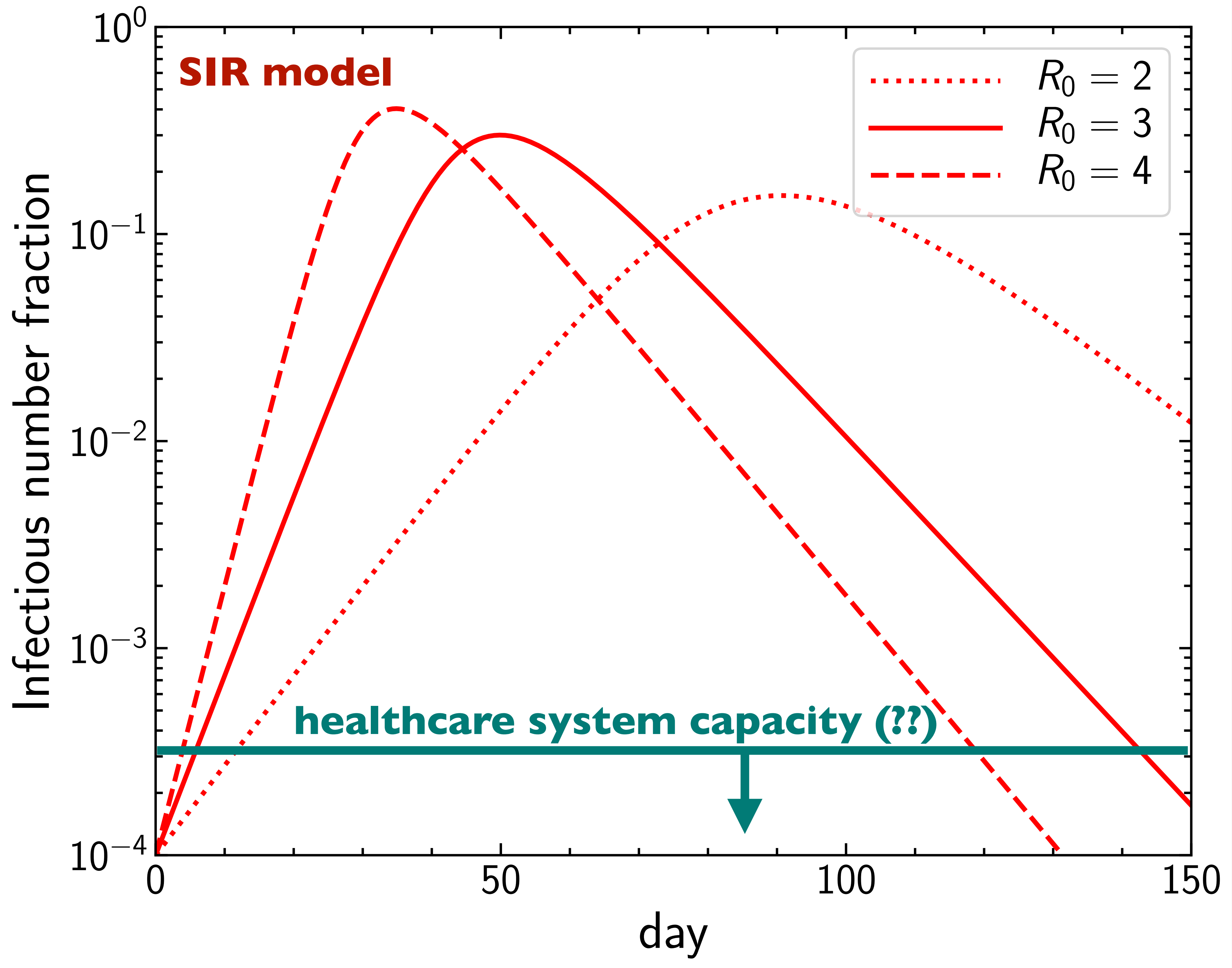


Infectious at $t \to \infty$ proportional to $1 - 1/R_0$

Introduction to models: summary

- these models are very simple yet capture the essence of epidemiology
- the models can be extended in several ways, e.g., adding more compartments (isolated, vaccination, deceased, …), considering birth and death, introducing stochasticity in model parameters, etc.

Implications for COVID-19



# of infected person need to be below capacity

disaster if $R_0 > 1$

reducing R_0 to $R_0 < 1$ is needed

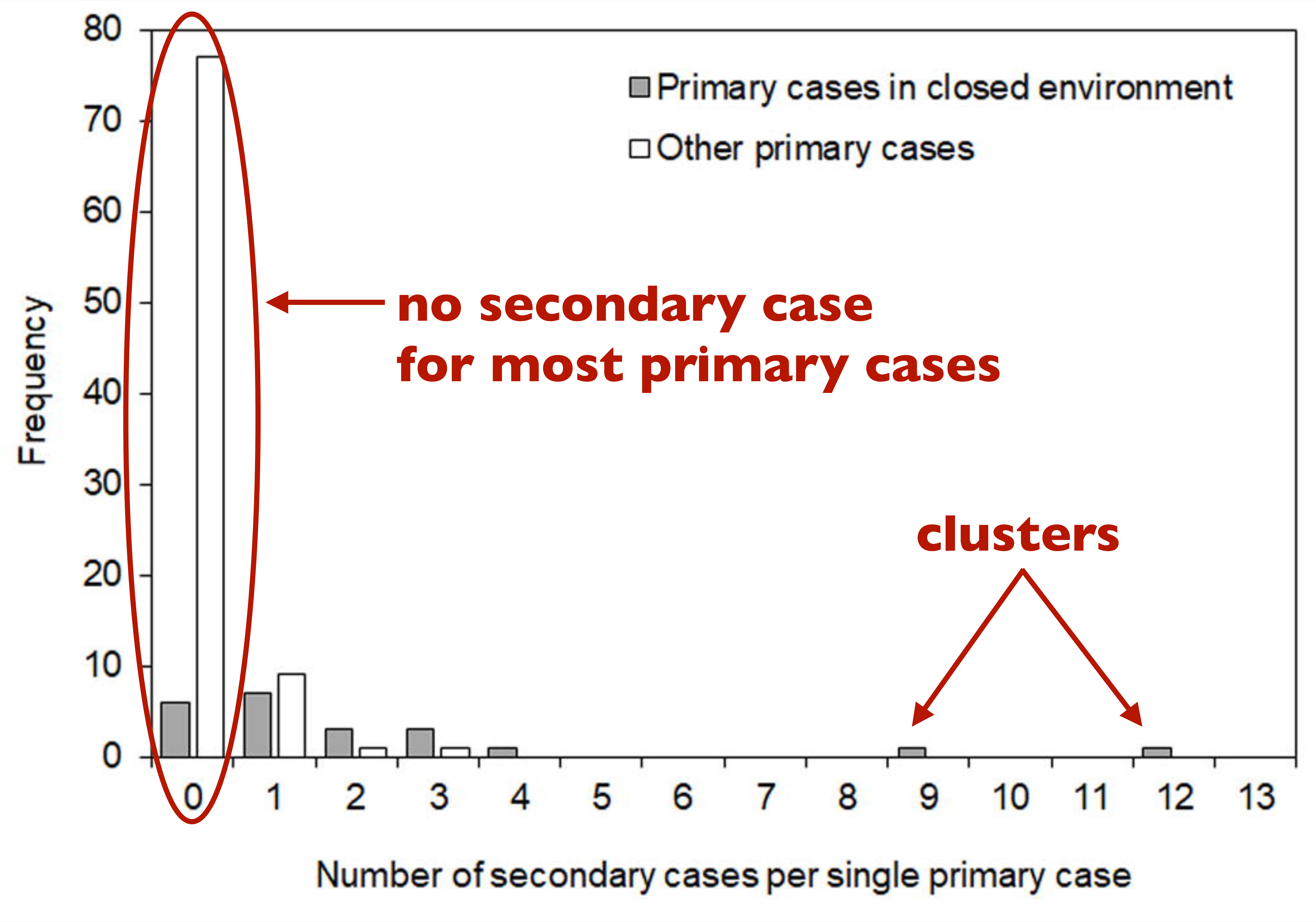
How to reduce R_0

- it’s a highly complicated problem
  - vaccine
  - isolation of infected persons
  - social distancing
  - wearing mask
  - lockdown/“lockdown”
  - …

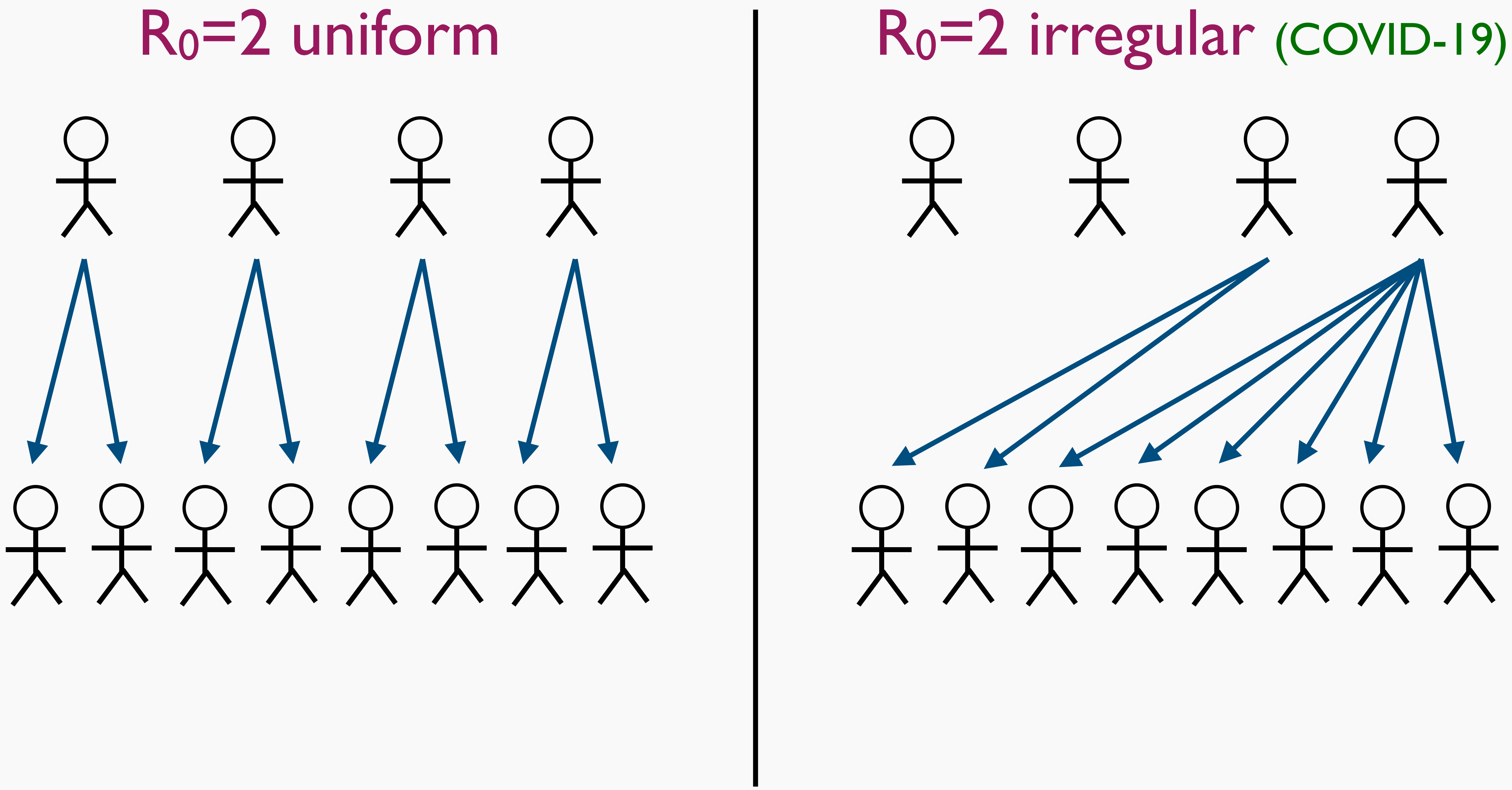
Nishiura+2020

Cluster strategy

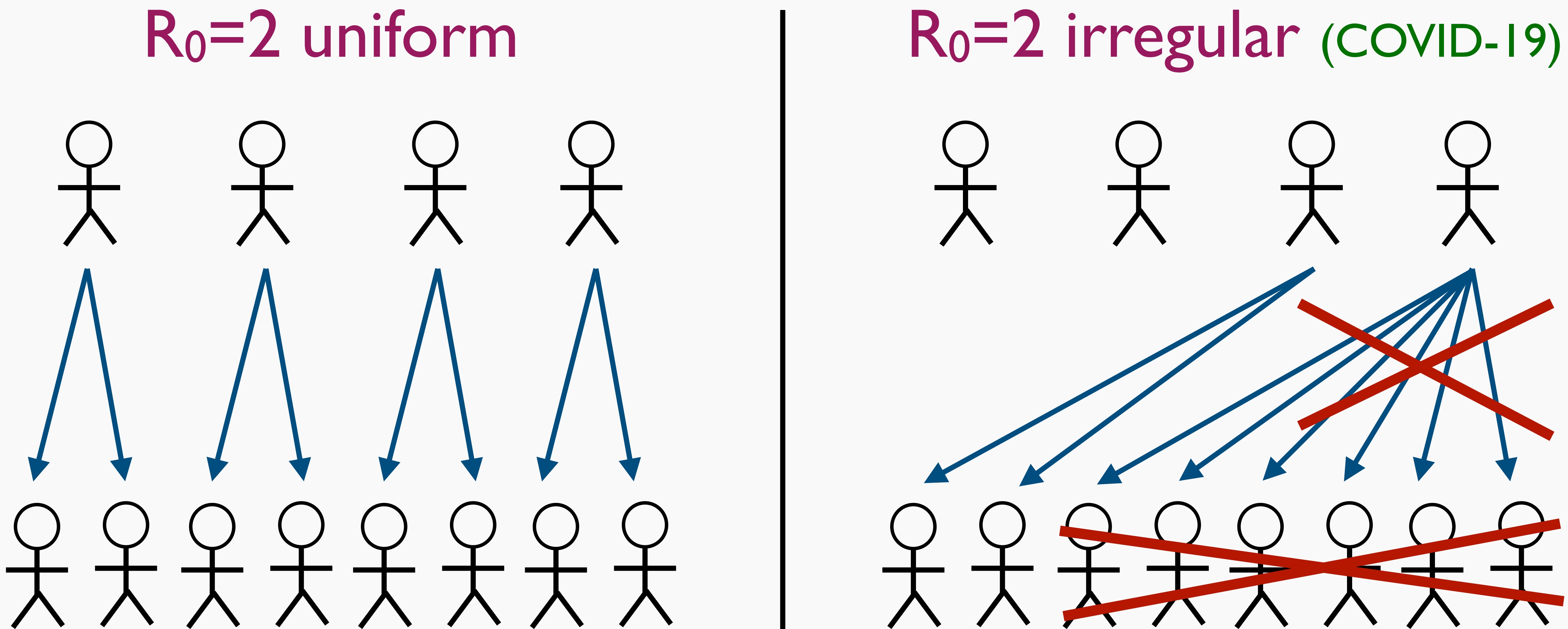
- analysis of 110 cases in Japan as of Feb. 28



Removing clusters



Removing clusters

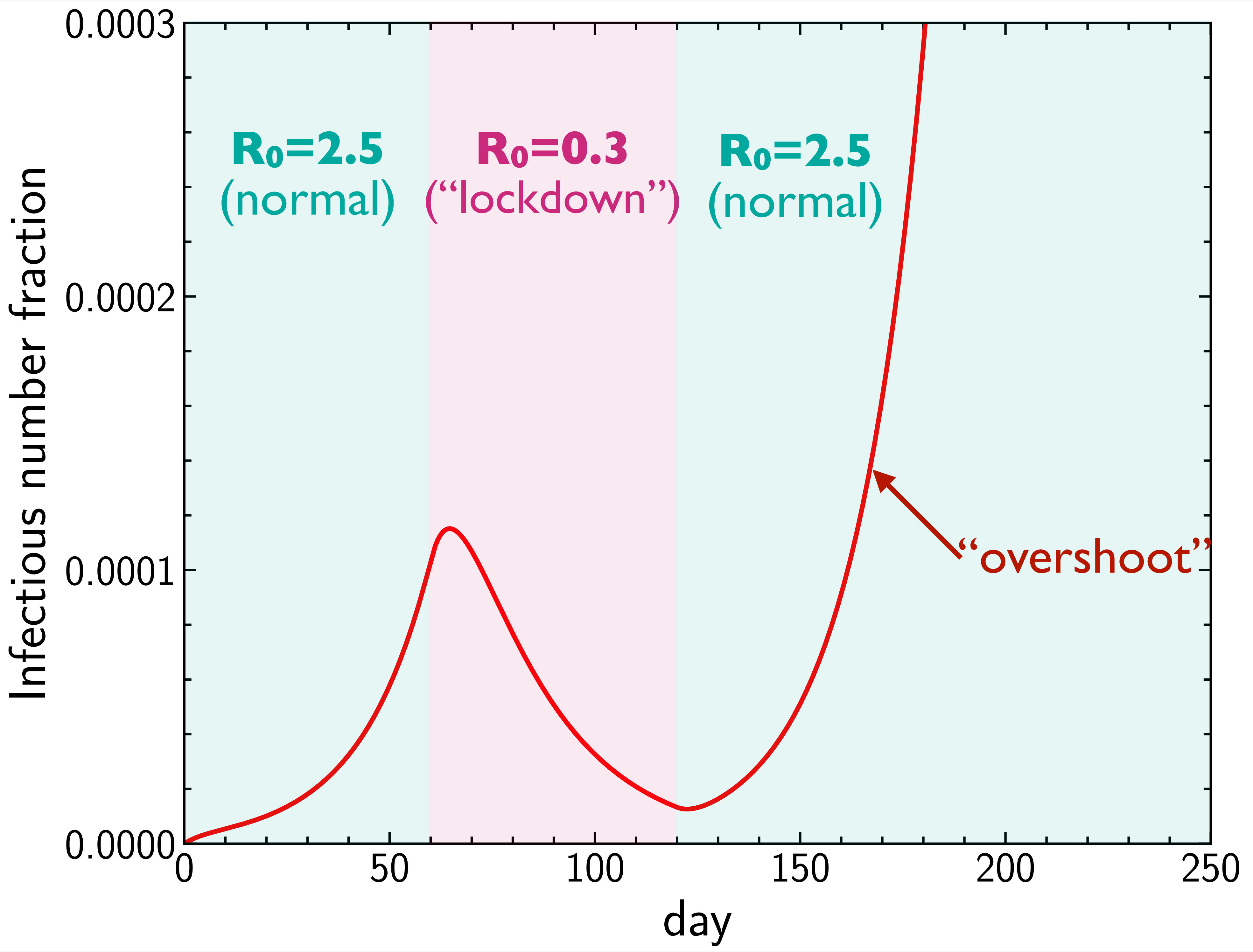


- "avoid three Cs" campaign
- find/isolate clusters

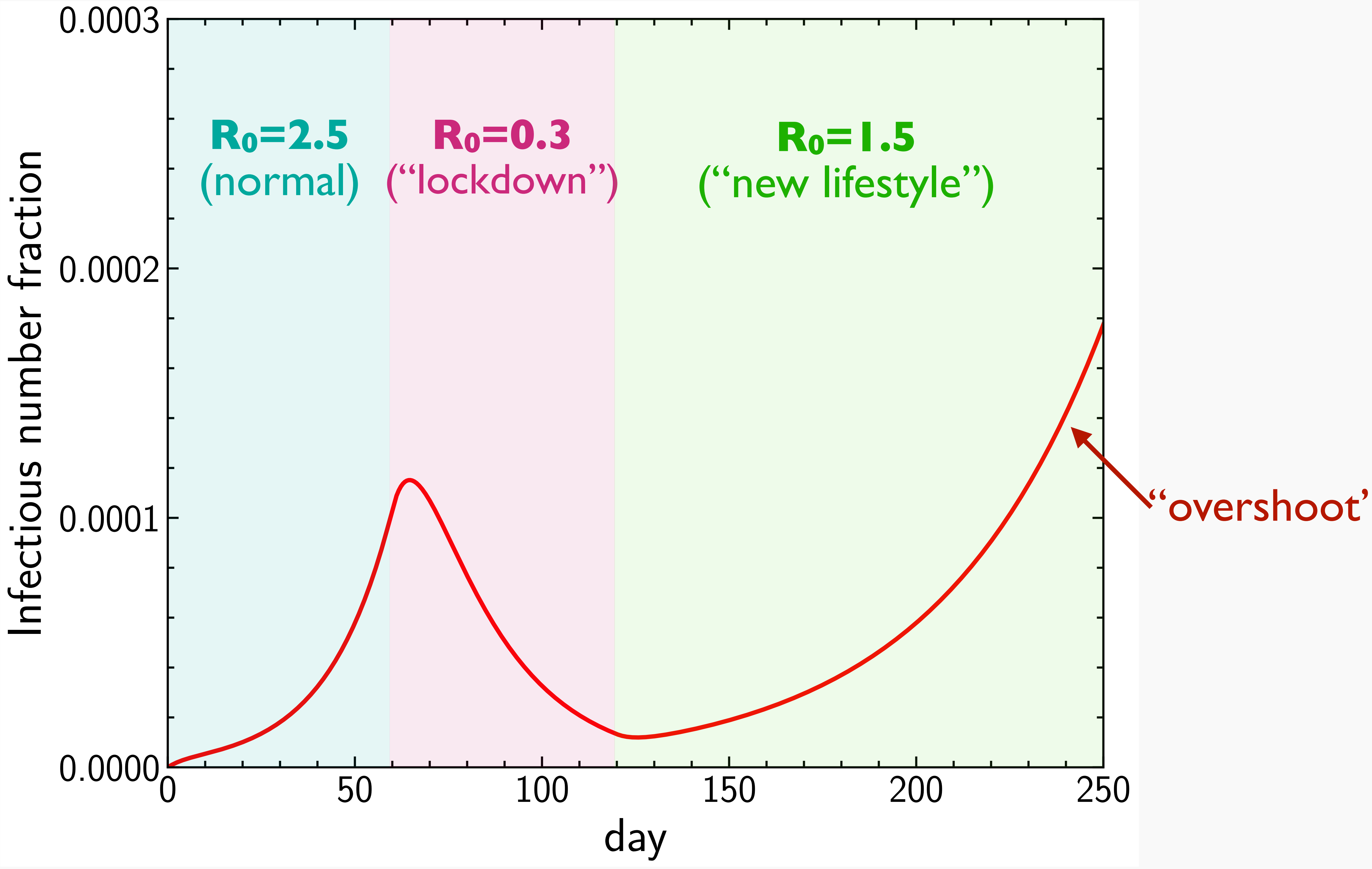
significantly reduce R_0

to some extent worked (?)

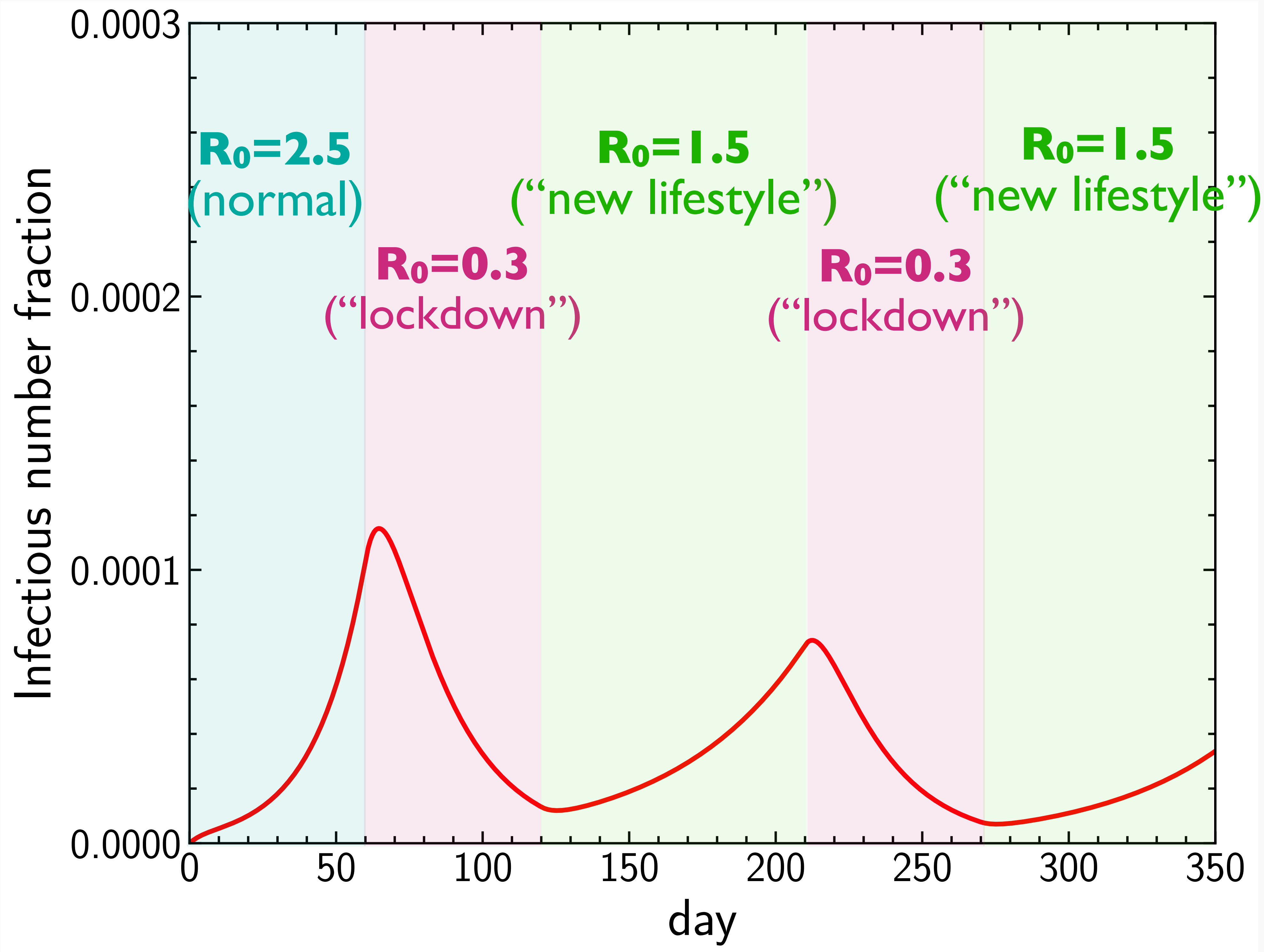
Reducing R_0: SEIR simulations (1/3)



Reducing R_0: SEIR simulations (2/3)



Reducing R_0: SEIR simulations (3/3)



plausible scenario?

Summary

- compartmental models in epidemiology discussed in this talk are simple and easy to understand, yet they are useful to get qualitative understanding of epidemic
- for quantitative analysis of COVID-19 pandemic more sophisticated modes are (should be) used