1. $P(s) = \frac{N(s)}{D(s)}$ $\quad N(s) = \frac{1}{s+1}$ $\quad D(s) = \frac{(s-1)(s-2)}{(s+1)(s+2)}$

$$P(s) = \frac{(s+2)}{(s-1)(s-2)}$$

(a) $G^*G = \frac{1}{1-s^2} + 1 = \frac{2-s^2}{1-s^2}$ $\quad G(s) = \frac{\sqrt{2}+s}{1+s}$ $\quad G(-s) = \frac{\sqrt{2}-s}{1-s}$

(b) $N_n = \left(\frac{1}{\sqrt{2}+s}\right)$ $\quad D_n = \frac{(s-1)(s-2)}{(\sqrt{2}+s)(s+2)}$: normalized coprime factorization

Bezout equation: $Y_n = \frac{1 - X_n N_n}{D_n}$ $\quad \begin{cases} X_n(1) = 1/N_n(1) = (1+\sqrt{2}) \\ X_n(2) = 1/N_n(2) = (2+\sqrt{2}) \end{cases}$

$X_n(s) = \frac{as+b}{(s+1)}$ $\quad \left.\begin{array}{l} a+b = 2(1+\sqrt{2}) \\ 2a+b = 3(2+\sqrt{2}) \end{array}\right\} \Rightarrow \begin{array}{l} a = (4+\sqrt{2}) \\ b = (-2+\sqrt{2}) \end{array}$

$$Y_n(s) = \frac{1 - \frac{(as+b)}{(s+1)} \cdot \left(\frac{1}{\sqrt{2}+s}\right)}{\frac{(s-1)(s-2)}{(\sqrt{2}+s)(s+2)}} = \frac{\left((s+1)(s+\sqrt{2}) - (as+b)\right)(s+2)}{(s+1)(s-2)(s-1)} = \frac{(s+2)}{(s+1)}\left(\cancel{\frac{s^2+(1+\sqrt{2})s+\sqrt{2}-as-b}{(s-2)(s-1)}}^{\,1}\right)$$

Summary: $X_n(s) = \left(\frac{as+b}{s+1}\right)$ $\quad a = 4+\sqrt{2}$ $\quad b = -2+\sqrt{2}$; $Y_n(s) = \left(\frac{s+2}{s+1}\right)$

$$R = N_n^* Y_n - D_n^* X_n = R_s + R_u \longrightarrow R_u(s) \cong \frac{-30.142\,(s-0.64)}{(s-2)(s-\sqrt{2})}$$

Nehari Problem Solution: $Q_{opt} \cong \frac{-0.52\,(s+\sqrt{2})(s+2)}{(s+1)(s+0.48)} = Q_c(s)$

$\gamma_0 = 4.8941$ $\quad b_{max} = 1/\sqrt{1+\gamma_0^2} = 0.2002$

$$C_{opt}(s) \cong \frac{4.8941\,(s-0.27)}{(s+2)}$$

Verify that there is no unstable pole-zero cancellation in $P(s)C_{opt}(s)$

$S_{opt}(s) = \frac{(s-2)(s-1)}{(s+0.48)(s+\sqrt{2})}$ $\quad$ feedback system is stable

(2)

a different $X_n(s) = \left(\frac{cs+d}{s+2}\right)$

$X_n(1) = 1+\sqrt{2} \Leftrightarrow c+d = 3(1+\sqrt{2})$

$X_n(2) = 2+\sqrt{2} \Leftrightarrow 2c+d = 4(2+\sqrt{2})$

$c = 5+\sqrt{2} \qquad d = -2+2\sqrt{2}$

$Y_n(s) = 1$ (check this via Matlab)

$R = N_n^* Y_n - D_n^* X_n = R_s + R_u$ with $R_u = \frac{-30.142(s-0.64)}{(s-2)(s-\sqrt{2})}$

$R_u(s)$ is same as before so γ_0 and b_{max} does not change

Optional computations:

$$Q_c(s) = \frac{-1.5201\,(s+\sqrt{2})}{(s+0.48)}$$

$$C_{opt}(s) = \frac{4.8941\,(s-0.27)}{(s+2)}$$ $\rightarrow$ same as before

```
%plant set up
%

Nn=tf(1,[1,sqrt(2)]);
Nnstar=tf(-1,[1,-sqrt(2)]);
Dn=tf([1,-3,2],[1,2+sqrt(2),2*sqrt(2)]);
Dnstar=tf([1,3,2],[1,-(2+sqrt(2)),2*sqrt(2)]);

Pnn=Nn/Dn;
Pn=minreal(Pnn);
zpk(Pn)


ans =

     (s+2)
  -----------
  (s-2) (s-1)

Continuous-time zero/pole/gain model.
```

solution of Bezout equation for Xn, Yn

```
a=4+sqrt(2);
b=-2+sqrt(2);
Xn=tf([a,b],[1,1]); %first order with pole at -1
Xn=zpk(Xn)
Ynn=(1-Nn*Xn)/Dn;
Yn=minreal(Ynn);
Yn=zpk(Yn)
Bezout=Nn*Xn+Dn*Yn; %check Nn*Xn+Dn*Yn=1
BB=minreal(Bezout);
BezoutEqnRHS=zpk(BB)


Xn =

  5.4142 (s-0.1082)
  -----------------
        (s+1)

Continuous-time zero/pole/gain model.


Yn =

  (s+2)
  -----
  (s+1)

Continuous-time zero/pole/gain model.
```

```
BezoutEqnRHS =

  (s+1)^2 (s+1.414)^2
  -------------------
  (s+1.414)^2 (s+1)^2

Continuous-time zero/pole/gain model.
```

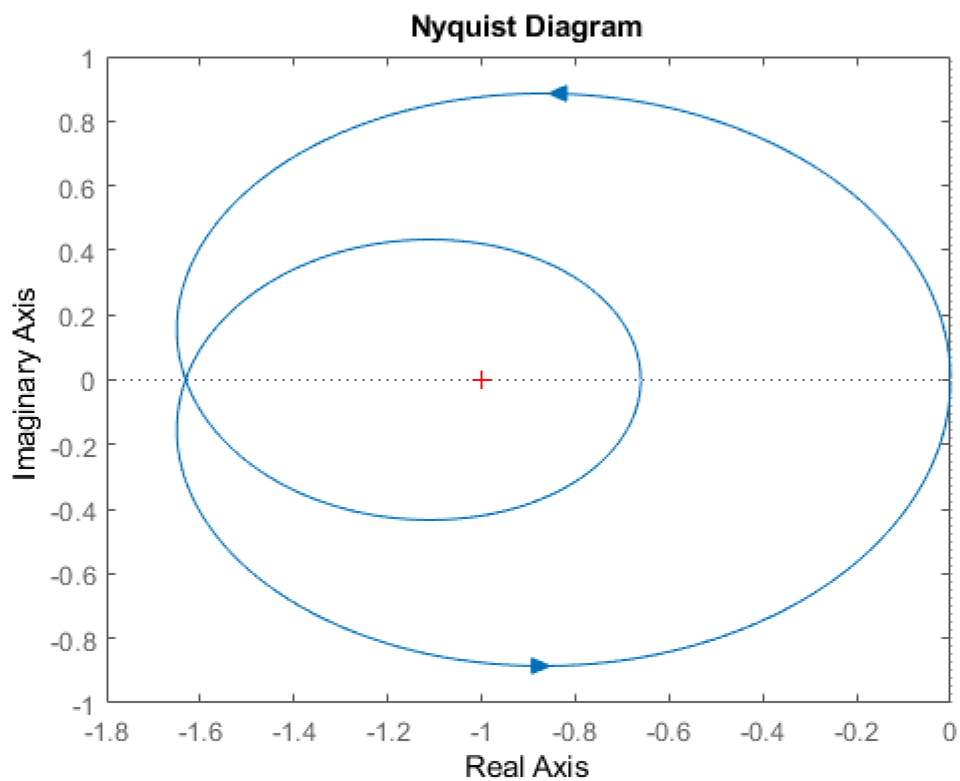
Nehari problem set-up and solution

```
Rr=Nnstar*Yn-Dnstar*Xn;
R=minreal(Rr)
[Rs,Ru]=stabsep(R);
%unstable part of R is Rum
Rum=minreal(Ru);
Rum=zpk(Rum)
[Au,Bu,Cu,Du]=ssdata(Rum);

Wc=lyap(-Au,Bu*Bu');
Wo=lyap(-Au',Cu'*Cu);

gopt=sqrt(max(eig(Wc*Wo)))
[V,D]=eig(Wc*Wo) % check that the largest eig value is the 2nd one
x=V(:,2)
y=(1/gopt)*Wo*x;
QFn=ss(Au,x,Cu,0);
QFd=ss(-Au',y,Bu',0);
FF=gopt*(QFn/QFd);
F=minreal(FF);
Q=R-F;
Qopt=zpk(minreal(Q));
Qc=Qopt
Copt=(Xn+Dn*Qc)/(Yn-Nn*Qc);
Copt=zpk(minreal(Copt))

om=logspace(-3,3,500);
figure(1)
nyquist(Copt*Pn,om) % verify theat the feedback system is stable
allmargin(Copt*Pn) %verify that the feedback system is stable
Sopt=1/(1+Copt*Pn);
Sopt=zpk(minreal(Sopt))
```

```
R =

  -5.4142 (s-0.3377) (s+1.414) (s+2)
  ----------------------------------
        (s-2) (s-1.414) (s+1)

Continuous-time zero/pole/gain model.


Rum =
```

```
  -30.142 (s-0.6408)
  ------------------
   (s-1.414) (s-2)

Continuous-time zero/pole/gain model.


gopt =

    4.8941


V =

   -0.6854    0.1195
    0.7282   -0.9928


D =

    2.1900         0
         0   23.9521


x =

    0.1195
   -0.9928

2 states removed.
4 states removed.

Qc =

  -0.52012 (s+2) (s+1.414)
  ------------------------
      (s+1) (s+0.4799)

Continuous-time zero/pole/gain model.


Copt =

  4.8941 (s-0.27)
  ---------------
       (s+2)

Continuous-time zero/pole/gain model.


ans =

  struct with fields:
```

```
 GainMargin: [1.5136 0.6130]
GMFrequency: [0 1.0904]
PhaseMargin: [-23.1338 55.8336]
PMFrequency: [0.3460 4.3387]
DelayMargin: [16.9946 0.2246]
DMFrequency: [0.3460 4.3387]
     Stable: 1


Sopt =

      (s-2) (s-1)
  --------------------
  (s+0.4799) (s+1.414)

Continuous-time zero/pole/gain model.
```



a different choice for Xn

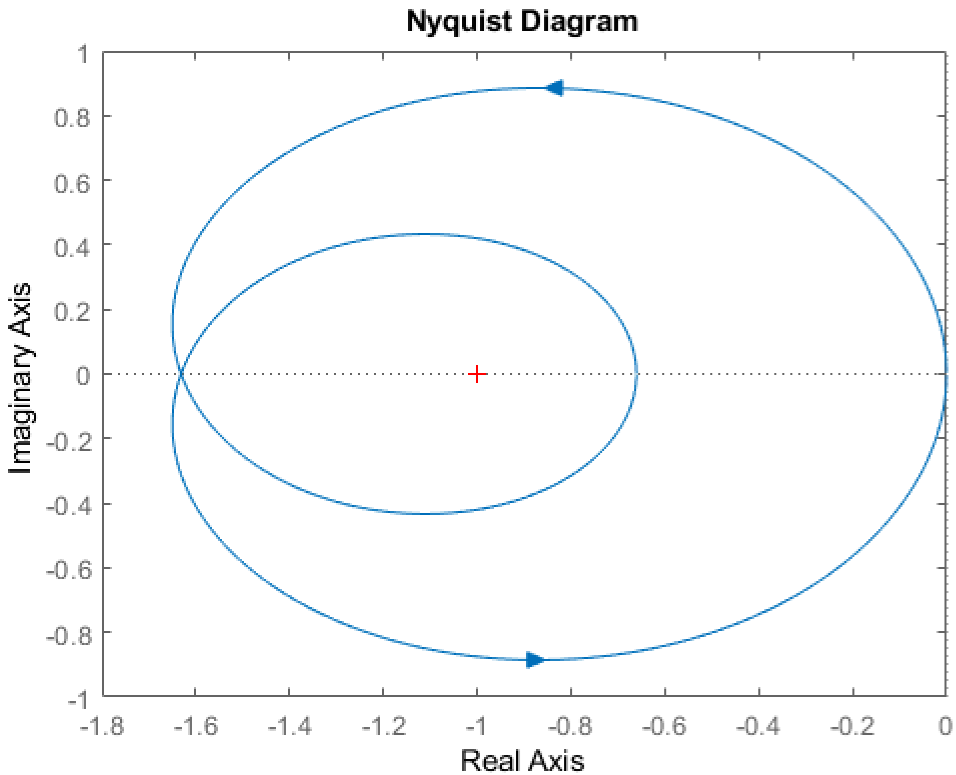
```
aa=5+sqrt(2);
bb=-2+2*sqrt(2);
Xn=tf([aa,bb],[1,2]); %put a pole at -2
Xn=zpk(Xn)
Ynn=(1-Nn*Xn)/Dn;
Yn=minreal(Ynn);
Yn=zpk(Yn)
```

```
Bezout=Nn*Xn+Dn*Yn; %check Nn*Xn+Dn*Yn=1
BB=minreal(Bezout);
BezoutEqnRHS=zpk(BB)

Rr=Nnstar*Yn-Dnstar*Xn;
R=minreal(Rr)
[Rs,Ru]=stabsep(R);
%unstable part of R is Rum
Rum=minreal(Ru);
Rum=zpk(Rum)
[Au,Bu,Cu,Du]=ssdata(Rum);

Wc=lyap(-Au,Bu*Bu');
Wo=lyap(-Au',Cu'*Cu);

gopt=sqrt(max(eig(Wc*Wo)))
[V,D]=eig(Wc*Wo) % check that the largest eig value is the 2nd one
x=V(:,2)
y=(1/gopt)*Wo*x;
QFn=ss(Au,x,Cu,0);
QFd=ss(-Au',y,Bu',0);
FF=gopt*(QFn/QFd);
F=minreal(FF);
Q=R-F;
Qopt=zpk(minreal(Q))
Qc=Qopt
Copt=(Xn+Dn*Qc)/(Yn-Nn*Qc);
Copt=zpk(minreal(Copt))

om=logspace(-3,3,500);
figure(2)
nyquist(Copt*Pn,om) % verify theat the feedback system is stable
allmargin(Copt*Pn) %verify that the feedback system is stable
Sopt=1/(1+Copt*Pn);
Sopt=zpk(minreal(Sopt))
```

```
Xn =

  6.4142 (s+0.1292)
  -----------------
        (s+2)

Continuous-time zero/pole/gain model.


Yn =

  1

Static gain.


BezoutEqnRHS =
```

```
  (s+1.414)^2 (s+2)^2
  -------------------
  (s+1.414)^2 (s+2)^2

Continuous-time zero/pole/gain model.


R =

  -6.4142 (s-0.1292) (s+1.414)
  ----------------------------
        (s-2) (s-1.414)

Continuous-time zero/pole/gain model.


Rum =

  -30.142 (s-0.6408)
  ------------------
   (s-1.414) (s-2)

Continuous-time zero/pole/gain model.


gopt =

    4.8941


V =

   -0.6854    0.1195
    0.7282   -0.9928


D =

    2.1900         0
         0   23.9521


x =

    0.1195
   -0.9928

2 states removed.
4 states removed.

Qopt =

  -1.5201 (s+1.414)
```

```
  -----------------
     (s+0.4799)

Continuous-time zero/pole/gain model.


Qc =

  -1.5201 (s+1.414)
  -----------------
     (s+0.4799)

Continuous-time zero/pole/gain model.


Copt =

  4.8941 (s+2)^2 (s-0.27)
  -----------------------
          (s+2)^3

Continuous-time zero/pole/gain model.


ans =

  struct with fields:

     GainMargin: [1.5136 0.6130]
    GMFrequency: [0 1.0904]
    PhaseMargin: [-23.1338 55.8336]
    PMFrequency: [0.3460 4.3387]
    DelayMargin: [16.9946 0.2246]
    DMFrequency: [0.3460 4.3387]
         Stable: 1


Sopt =

      (s+2)^2 (s-2) (s-1)
  ----------------------------
  (s+0.4799) (s+1.414) (s+2)^2

Continuous-time zero/pole/gain model.
```



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