EEE 444 Robust Feedback Theory HW3 Report
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Problem 1)

a)

We are given an unstable plant

$$P(s) = \frac{4(s-2)}{(s^2-2s+2)}\,.$$

Our aim is to find a characterization of the set of all controller stabilizing the feedback system (C,P). First, we write plant as ratio of two coprime functions such that

$$P(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial of the plant transfer function. $N(s)$ and $D(s)$ both belong to H_∞ and they do not have common zeros in right half plane (RHP) of complex plane including $+\infty$. We know that RHP poles of $P(s)$ must be zeros of $D(s)$. Also, we have to divide $D(s)$ by a stable second order polynomial. Therefore, we get

$$D(s) = \frac{(s^2-2s+2)}{(s+a)(s+b)} \qquad a, b > 0$$

Choosing $a = 1$ and $b = 2$, $D(s)$ becomes

$$D(s) = \frac{(s^2-2s+2)}{(s+1)(s+2)}$$

Also, we have

$$N(s) = D(s)P(s) = \frac{4(s-2)}{(s+1)(s+2)}$$

As we now have $N(s)$ and $D(s)$, we can find $X(s)$ and $Y(s)$ belonging both to H_∞ satisfying Bézout equation

$$N(s)X(s) + D(s)Y(s) = 1$$

Therefore, we can express $Y(s)$ as

$$Y(s) = \frac{1 - N(s)X(s)}{D(s)}$$

For $Y(s)$ to be stable, we must have zeros of $D(s)$ appear also as the zeros of $Y(s)$ so that pole-zero cancellation occurs. $D(s)$ have two zeros, which are $(1+j)$ and $(1-j)$.

We obtain two equations, two interpolation conditions, for $X(s)$:

$$X(1+j) = 1/N(1+j)$$

$$X(1-j) = 1/N(1-j)$$

Now, we need to find a stable transfer function $X(s)$ so that $X(s) \in H_\infty$ and satisfies these two interplolation conditions. As a general case, we can write $X(s)$ as

$$X(s) = \frac{x_1 s + x_2}{s + r_0} \quad r_0 > 0$$

As r_0 is an arbitrary value, let $r_0 = 1$. Then the general form of the $X(s)$ becomes

$$X(s) = \frac{x_1 s + x_2}{s + 1}$$

Solving these two interpolation conditions to find two unknowns x_1 and x_2, we obtain

$$x_1 = -2.5 \quad \& \quad x_1 = 3.75$$

$$X(s) = \frac{-2.5s + 3.75}{s + 1}$$

Once we found $X(s)$, we can compute $Y(s)$ by using the equation of $Y(s)$ written above and doing pole-zero cancellations, we obtain

$$Y(s) = \frac{s + 16}{s + 1}$$

Hence $X(s),\ Y(s),\ N(s)$ and $D(s)$ are computed, we can write a characterization $C(s)$ as

$$C(s) = \frac{X(s) + D(s)Q_c(s)}{Y(s) - N(s)Q_c(s)} \quad Q_c(\mathrm{s}) \in H_\infty$$

b)

In this part, we will find controller $C(s)$ stabilizing (C, P) and satisfying the following steady state performance conditions:

- Steady state error for a unit step reference input is zero
- Steady state error for a sinusoidal input of the form given below is zero
$$r(t) = \sin(3t), \qquad t \geq 0$$

As we have characterization of the contoller $C(s)$ is given above in part a, we need to find the $Q_c(s)$ to construct the controller $C(s)$ stabilizing (C, P). The general form of the $Q_c(s)$ can be written as

$$Q_c(s) = \frac{q_2 s^2 + q_1 s + q_0}{(s+3)^2}$$

To find three unknowns q_0, q_1 and q_2, we use the conditions for steady state error, given above.

In other words, we need to have

- ess (Steady state error) for $R(s) = \frac{1}{s}$ is zero => controller has a pole at $s = 0$
- ess (Steady state error) for $r(t) = \sin(3t)$ is zero => controller has poles at $s = \pm j3$

Using these conditions, we get three equations

$$Y(0) - N(0)Q(0) = 0$$
$$Y(3j) - N(3j)Q(3j) = 0$$
$$Y(-3j) - N(-3j)Q(-3j) = 0$$

Solving these three equations, we find unkowns q_0, q_1 and q_2 and obtain $Q_c(s)$.

$$Q_c(s) = \frac{-10.81s^2 + 13.38s - 36}{s^2 + 6s + 9} \qquad Q_c(\mathrm{s}) \in H_\infty$$

Finally, our controller $C(s)$ can be formed as

$$C(s) = \frac{-13.31s^{10} - 127.6s^9 - 539.4s^8 - 1554s^7 - 3353s^6 - 3673s^5 + 1637s^4 + 8409s^3 + 7453s^2 + 2048s - 81}{s^{10} + 78.23\,s^9 + 796.5\,s^8 + 1.434e04\,s^6 + 38480\,s^5 + 70320\,s^4 + 77520\,s^3 + 45540s^2 + 10890\,s}$$

We can test our controller using SimuLink and see that performance conditions are satisfied.

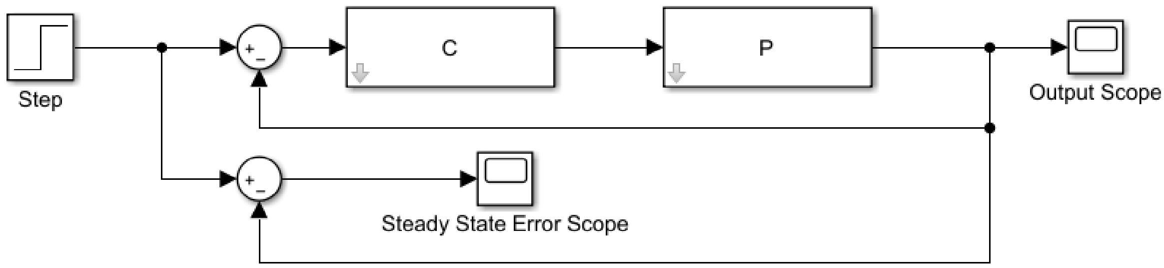

Figure 1: Closed Loop Feedback System with Unit Step Input

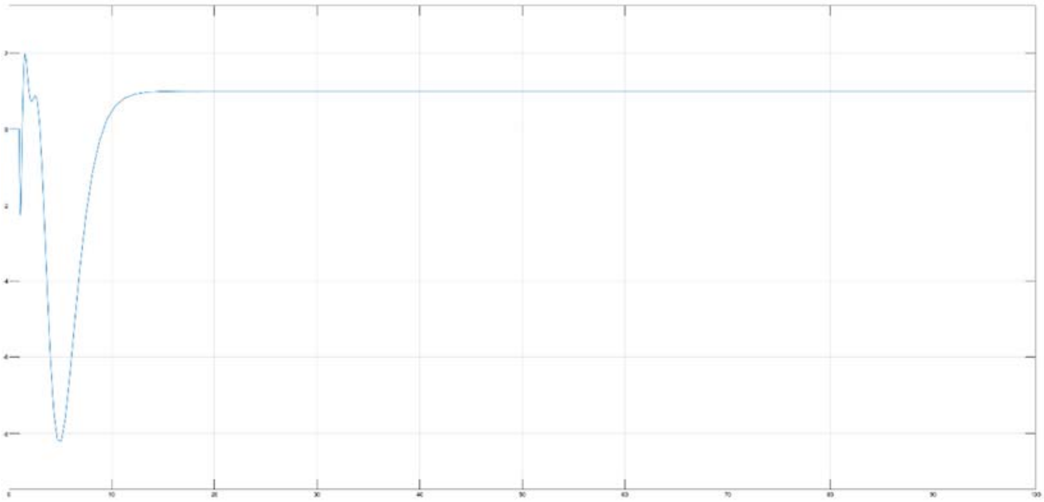

Figure 2: Output of the System with Unit Step Input

As it can be seen, the output of the system becomes 1, as the input is unit step, in steady state.

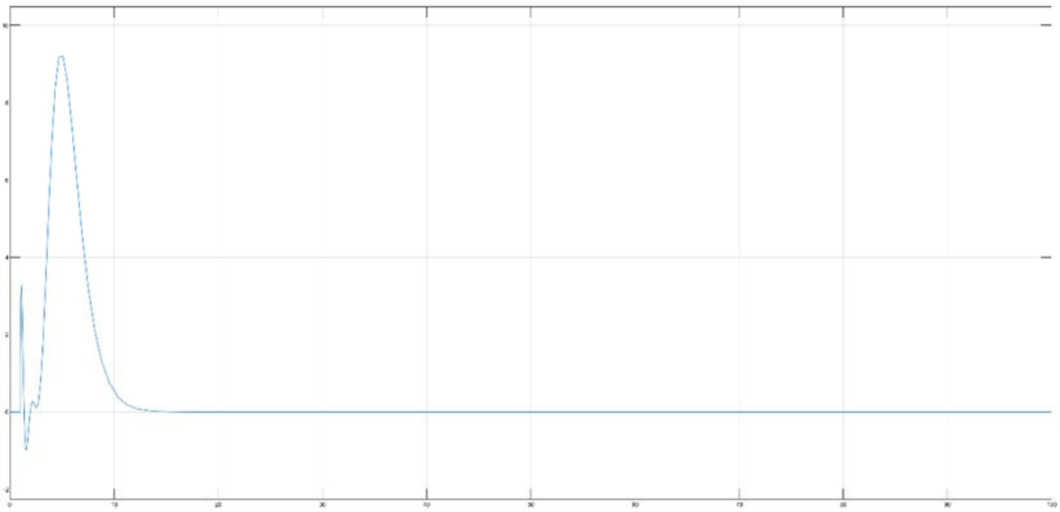

Figure 3: Error of the System with Unit Step Input

Error becomes 0 in the steady state.

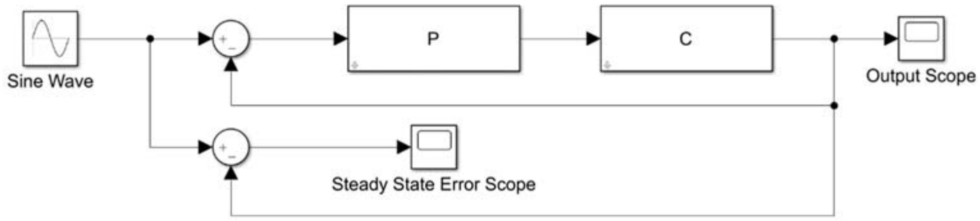

Figure 4: Closed Loop Feedback System with Sinusoidal Input

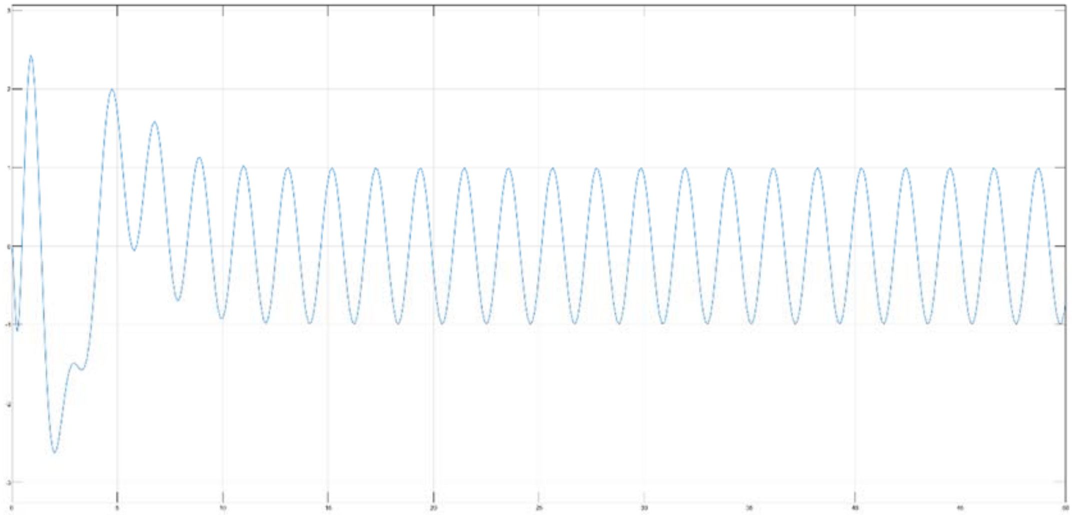

Figure 5: Output of the System with Sinusoidal Input

As it can be seen, the output of the system becomes sin(3t), as the input is sinusoidal sin(3t), in steady state.

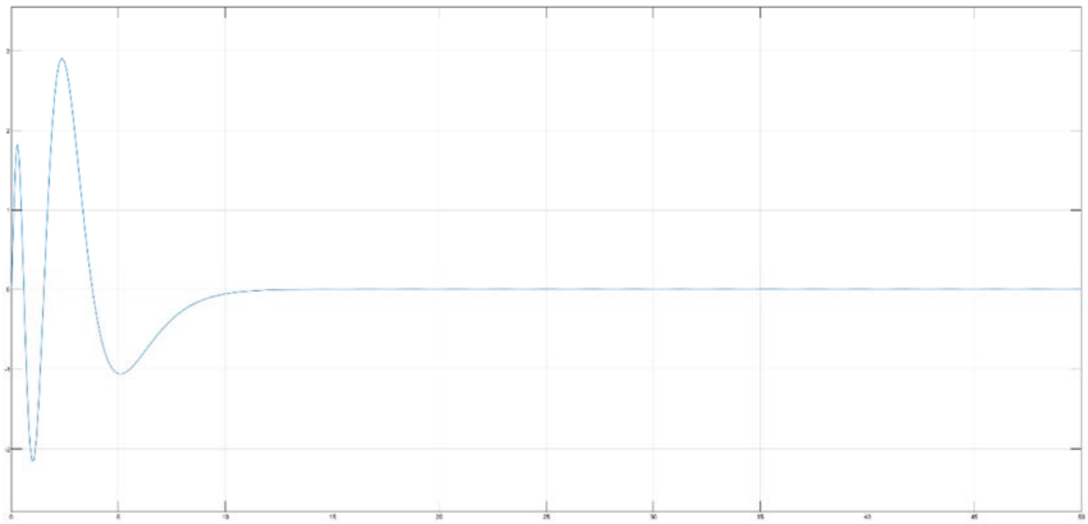

Figure 6: Error of the System with Sinusoidal Input

Error becomes 0 in the steady state.

Problem 2)

For the nominal plant fiven in Problem 1, we have the set of uncertain plants:

$$\mathrm{P} = \{P_\Delta = P(1+\Delta_\mathrm{m}) : P_\Delta\ has\ 2\ poles\ in\ C_+, |\Delta_\mathrm{m}(j\omega)| < |\mathrm{W_m}(j\omega)|, \forall\omega$$

where

$$W_m(s) = \delta(s+1)$$

a) Our aim is to find the largest $\delta > 0$ for which there exists a controller $C(s)$ stabilizing (C, P_Δ) for all $P_\Delta \in \mathrm{P}$ and determine the corresponding optimal controller $C_{opt}(s)$.

We want a robustly stable system. Therefore,

$$\left|\left|W_m T\right|\right|_\infty \leq 1$$

where we can write T as

$$T = N(s)(X(s) + D(s)Q_c(s))$$

Therefore, our condition for robust stability becomes

$$\left|\left|\delta(s+1)N(s)(X(s) + D(s)Q_c(s))\right|\right|_\infty \leq 1$$

As δ is a constant, we can take it out and obtain

$$\left|\left|(s+1)N(s)(X(s) + D(s)Q_c(s))\right|\right|_\infty \leq \frac{1}{\delta}$$

We now that

$$\gamma_{opt} = \left|\left|(s+1)N(s)(X(s) + D(s)Q_c(s))\right|\right|_\infty$$

Therefore, we can see that maximum of δ is equal to

$$\delta_{max} = \frac{1}{\gamma_{opt}}$$

$N(s)$ can be factorized as inner and outer terms

$$N(s) = N_{out}(s)N_{in}(s)$$

thus

$$N_{out}(s) = \frac{4(s+2)}{(s^2+3s+2)}$$

so, the equation becomes

$$\gamma_{opt} = \left|\left|(s+1)N_{outer}(s)X(s) + (s+1)D(s)N_{outer}(s)Q_c(s))\right|\right|_\infty$$

We can convert this equation to the form

$$\gamma_{opt} = \left\|W(s) - M(s)Q(s)\right\|_{\infty}$$

Thus, we get the equalities

$$W(s) = (s+1)N_{out}(s)X(s)$$

$$M(s) = \frac{(s^2 - 2s + 2)}{(s^2 + 2s + 2)}$$

Also, we can define $F(s)$ as

$$F(s) = W(s) - M(s)Q(s)$$

With this form, we can use Neanlinna-Pick Interpolation Theorem, as $M(s)$ has more than one zero.

The zeros of $M(s)$ are $(1+j)$ and $(1-j)$. These two zeros construct $\boldsymbol{\alpha}$ vector i.e

$$\boldsymbol{\alpha} = [\alpha_1 \; \alpha_2] = [(1+j) \; (1-j)]$$

For the β values, we simply plug these zeros to $W(s)$ so that

$$\boldsymbol{\beta} = [\beta_1 \; \beta_2] = [W(1+j) \; W(1-j)]$$

By MATLAB, we compute optimal $F(s)$ and γ as

$$F_{opt}(s) = -\frac{12.071(s - 1.414)}{(s + 1.414)}$$

$$\gamma_{opt} = 12.0711$$

Then, maximum of δ is

$$\delta_{max} = \frac{1}{\gamma_{opt}} = 0.0828$$

As we have found $F_{opt}(s)$ and γ_{opt}, we can use them to form $C_{opt}(s)$. We first find $Q_{opt}(s)$

$$Q_{opt}(s) = \frac{W(s) - F_{opt}(s)}{M(s)}$$

Then we find $\mathrm{Q_{copt}(s)}$ using equation

$$Q_{copt}(s) = \frac{-Q_{opt}M(s)}{(s+1)N_{out}(s)D(s)}$$

Finally, we can construct $C_{opt}(s)$ using $Q_{copt}(s)$

$$C_{opt}(s) = \frac{X(s) + D(s)Q_{copt}(s)}{Y(s) - N(s)Q_{copt}(s)}$$

where

$$Q_{opt}(\mathrm{s}) \in H_\infty \qquad Q_{copt}(\mathrm{s}) \in H_\infty$$

Our optimal controller, constructed by MATLAB is

$$C_{opt}(s) = -\frac{1.9822(s - 1.593)(s + 2.007)}{(s + 13.34)(s + 2.007)}$$

b) Now, using the largest δ computed above, we pick an arbitrary element $P_\Delta \neq P$ in the set P and prove that $\left(C_{opt}(s), P_\Delta(s)\right)$ is stable. I chose $0.9\delta_{max}$ as the δ and the corresponding Nyquist plot of the open-loop system is:

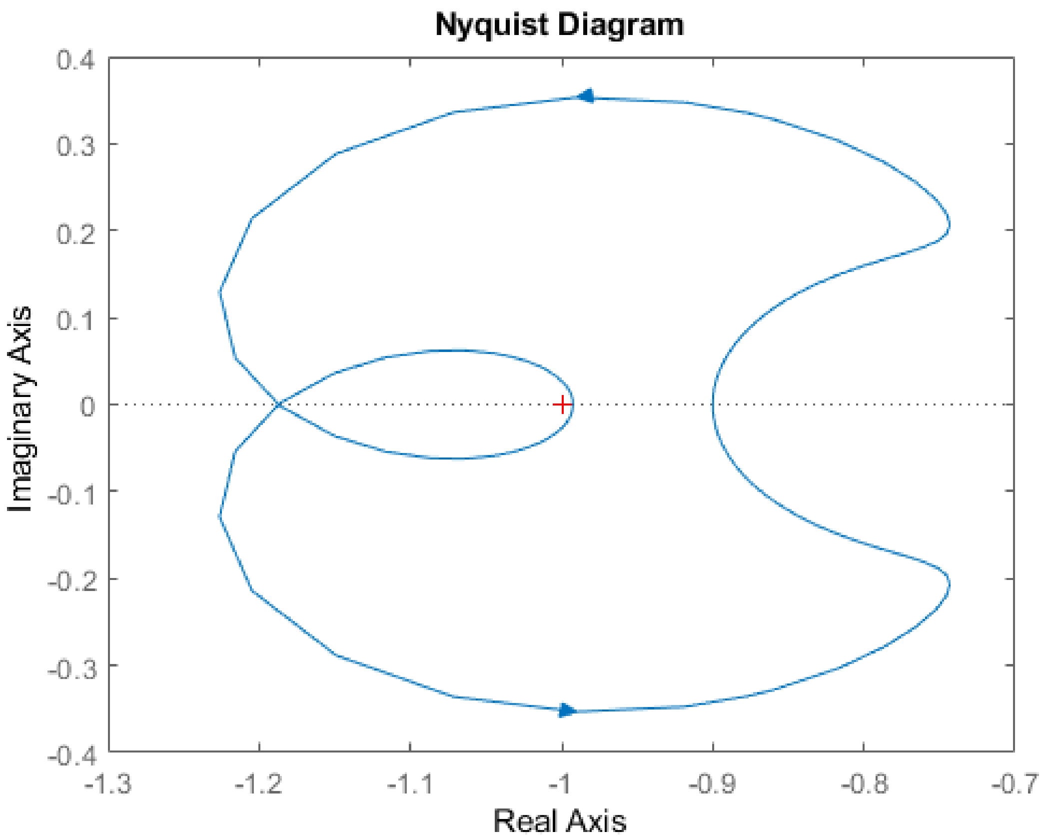

Figure 7: Nyquist Plot of the System

Looking at the margins of the system, MATLAB gives following results:

1x1 struct with 7 fields

Field	Value
GainMargin	[1.0077,0.8421,1.1111]
GMFrequency	[0,0.8633,Inf]
PhaseMargin	[-1.5237,20.5343]
PMFrequency	[0.1437,2.4376]
DelayMargin	[43.5542,0.1470]
DMFrequency	[0.1437,2.4376]
Stable	1

Table 1: Margins

It can be seen from both the Nyquist plot and margins that the system $(C_{opt},\ P_\Delta)$ is stable.

The poles of the closed loop system are calculated by MATLAB:

$$p_1 = -1 \;\; p_2 = -0.0436 \;\; p_3 = -64.8264$$

All of the poles are on the left half side of complex plane. Thus, the system $(C_{opt},\ P_\Delta)$ is stable.

APPENDIX

```
clear; clc;

%Problem 1 part a
P = tf([4,-8],[1,-2,2]);
X = tf([-2.5,3.75],[1,1]);
N = tf([4,-8],[1,3,2]);
D = tf([1,-2,2],[1,3,2]);
Y = minreal((1-N*X)/D);

%Problem 1 part b
Qc = tf([-10.8077,13.3846,-36],[1,6,9]);
C = (X+D*Qc)/(Y-N*Qc);

%Problem 2 part a
Wm = tf([1 1],1);
M = tf([1 -2 2],[1 2 2]);
N_out = 4*tf([1,2],([1,3,2]));
W = Wm*N_out*X;
W = minreal(W);
b = [evalfr(W,1+i) evalfr(W,1-i)];
a = [1+i 1-i];
[gopt,Fopt] = NevPickNew(a,b);
Qopt = minreal((-Fopt+W)/M);
deltamax = 1/gopt;
Qcopt = minreal(Qopt/(minreal(Wm*N_out*D/M)));
Copt = minreal((X+D*Qcopt)/(Y-N*Qcopt));

%Problem 2 part b
delta = 0.5*deltamax;
P_unc = P + P*Wm*delta;
tf_openloop = minreal(Copt*P_unc);
nyquist(tf_openloop);
margins = allmargin(tf_openloop);
```