EEE 444 Robust Feedback Theory HW4 Report
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Problem 1)

a) We are given a nominal plant

$$P(s) = \frac{N(s)}{D(s)} \text{ where } N(s) = \frac{1}{s+1}, D(s) = \frac{(s-1)(s-2)}{(s+1)(s+2)}$$

We perform following spectral factorization and find staisfying G:

$$G(-s)G(s) = N(-s)N(s) + D(-s)D(s), \quad G, G^{-1} \in H_\infty$$

Calculating right hand side of this factorization equation, we get

$$G(-s)G(s) = \frac{s^2-2}{s^2-1}$$

We know that $G \in H_\infty$, therefore after factorization of $G(-s)G(s) = \frac{s^2-2}{s^2-1}$, we expect to see that left hand side poles and zeros belong to $G(s)$. Therefore, we obtain

$$G(s) = \frac{s+\sqrt{2}}{s+1}$$

b) Defining $N_n(s) = NG^{-1} \in H_\infty$ and $D_n(s) = DG^{-1} \in H_\infty$, which is normalized coprime factorization, we will find corresponding $X_n(s) \in H_\infty$ and $Y_n(s) \in H_\infty$ so that the Bezout equation is satisfied i.e. $N_n(s)X_n(s) + D_n(s)Y_n(s) = 1$.

$$N_n(s) = \frac{1}{s+\sqrt{2}}, D_n(s) = \frac{s^2-3s+2}{s^2+3.414s+2.828}$$

Therefore, we can express $Y_n(s)$ as

$$Y_n(s) = \frac{1 - N_n(s)X_n(s)}{D_n(s)}$$

For $Y_n(s)$ to be stable, we must have zeros of $D_n(s)$ appear also as the zeros of $Y_n(s)$ so that pole-zero cancellation occurs. $D_n(s)$ have two zeros, which are 1 and 2.

We obtain two equations, two interpolation conditions, for $X(s)$:

$$X_n(1) = 1/N_n(1)$$

$$X_n(2) = 1/N_n(2)$$

Now, we need to find a stable transfer function $X_N(s)$ so that $X_n(s) \in H_\infty$ and satisfies these two interplolation conditions. As a general case, we can write $X_n(s)$ as:

$$X_n(s) = \frac{x_1 s + x_2}{s + r_0} \in H_\infty \quad r_0 > 0$$

As r_0 is an arbitrary value, let $r_0 = 1$. Then the general form of the $X_n(s)$ becomes

$$X_n(s) = \frac{x_1 s + x_2}{s + 1} \in H_\infty$$

Solving these two interpolation conditions to find two unknowns x_1 and x_2, we obtain

$$x_1 = 5.414 \quad \& \quad x_2 = -0.5858$$

$$X_n(s) = \frac{5.414s - 0.5858}{s + 1} \in H_\infty$$

Once we found $X_n(s)$, we can compute $Y_n(s)$ by using the equation of $Y_n(s)$ written above and doing pole-zero cancellations, we obtain

$$Y_n(s) = \frac{s + 2}{s + 1} \in H_\infty$$

Problem 2)

a) In this problem, we aim to compute b_{max} using Nehari approach. Using the one-block problem which is converted from the original two-block problem:

$$\gamma_o = \ inf\left\|R - Q_C\right\|_\infty \text{ where } R = N_n^* Y_n - D_n^* X_n$$

Calculating R, we get

$$R = \frac{-5.414\, s^5 - \ 14.41\, s^4 + \ 5.48\, s^3 + \ 32.48s^2 + \ 10.68s \ - \ 7.313}{s^5 \ - \ 2.828s^4 \ - \ 1.001s^3 \ + \ 6.484s^2 \ - \ 0.3422s \ - \ 3.999}$$

We can factorize R as the stable and antistable part, namely

$$R = \ R_s + R_u$$

where $R_S \in H_\infty$ and $R_u(-s) \in H_\infty$, $\mathrm{R_u}$ is an antistable transfer function.

Using MATLAB, we get

$$R_s = \frac{-5.414s - 5}{s + 1}$$

$$R_u = \frac{-30.14s^2 + \ 61.93s \ - \ 27.31}{s^3 - \ 4.828s^2 \ + \ 7.655s \ - \ 3.999}$$

We can express R_u in state-space representation such that

$$R_u = C(sI - A)^{-1}B$$

$$x' = Ax + Bu$$
$$y = CX$$

assuming that (A, B) is controllable and (C, A) is observable.

Using MATLAB, we compute A, B and C using $R_u(s)$. Using Nehari approach, as a first step, we solve following Lyapunov equations to get W_C and W_o:

$$AW_C + W_c A^T = BB^T$$
$$A^T W_o + W_o A = C^T C$$

After finding W_C and W_o, we can find γ_{opt} , γ_o and b_{max}:

$$\gamma_{opt} = \sqrt{\lambda_{max}(W_C W_o)} = 4.8940$$

$$b_{max} = \frac{1}{\gamma_{opt}} = 0.2043$$

$$\gamma_0 = 4.7908$$

b) λ_{max} is the corresponding vector. Using equations:

$$\sigma_{max}^2 = \gamma_{opt}^2$$

$$\sigma_{max}^2 x_{max} = W_C W_o x_{max}$$

$$R = \frac{W}{M} \text{ and } F_{opt} = W - MQ_{c,opt}$$

we can express $Q_{c,opt}$ as:

$$Q_{c,opt} = R - \frac{\gamma_{opt} C(sI - A)^{-1} x_{max}}{B^T (sI - A)^{-1} y_{max}} \in H_\infty$$

where

$$y_{max} = \gamma_{opt}^{-1} W_o x_{max}$$

We compute $Q_{c,opt}$ as:

$$Q_{c,opt}(s) = \frac{-0.51997\ (s + 1.414)(s + 2)}{(s + 1)\ (s + 0.4798)} \in H_\infty$$

We have characterization of the contoller such as:

$$C_{opt}(s) = \frac{X_n + D_n Q_{c,opt}}{Y_n - N_n Q_{c,opt}}$$

Then our controller turns out to be:

$$C_{opt}(s) = \frac{4.8942\ (s - 0.27)(s + 2)}{(s + 2)\ (s + 2)}$$

To see that $Q_{c,opt}(s)$ is stable, we can also look at its poles which are -1.00 and -0.4798, they are on the left hand side of the complex plane.

c) Choosing different X_n and Y_n:

As a general case, we can write $X_n(s)$ as:

$$X_n(s) = \frac{x_1 s + x_2}{s + r_0} \;\in H_\infty \quad r_0 > 0$$

As r_0 is an arbitrary value, this time, we let $r_0 = 2$. Then the general form of the $X_n(s)$ becomes

$$X_n(s) = \frac{x_1 s + x_2}{s + 2} \;\in H_\infty$$

Again using two interpolation conditions:

$$X_n(1) = 1/N_n(1)$$

$$X_n(2) = 1/N_n(2)$$

Solving these two interpolation conditions to find two unknowns x_1 and x_2, we obtain

$$x_1 = 6.414 \quad \& \quad x_2 = 0.8283$$

$$X_n(s) = \frac{6.414\,s \; + \; 0.8283}{s + 2} \in H_\infty$$

Once we found $X_n(s)$, we can compute $Y_n(s)$ by using the equation of $Y_n(s)$ and doing pole-zero cancellations, we obtain

$$Y_n(s) = 1 \;\in H_\infty$$

Solving the rest of the problem using the new X_n and Y_n, we find

$$\gamma_0 = 4.7908$$

Meaning that the γ_0 is found same as the one in the original values of X_n and Y_n. In addition, we also get

$$R_u = \frac{-30.14s^2 + \; 61.93s \; - \; 27.31}{s^3 - \; 4.828s^2 \; + \; 7.655s \; - \; 3.999}$$

We can see that antistable part R_u is again same as the one in the original values of X_n and Y_n. As a conclusion, we can state that γ_0 and R_u does not depend on X_n and Y_n.

APPENDIX

```
clear all;
clc;

N = tf(1,[1 1]);
D = tf([1 -3 2],[1 3 2]);
P = minreal(N/D);

Nc = tf(1,[-1 1]);
Dc = tf([1 3 2],[1 -3 2]);

rhs = N*tf(1,[-1 1])+D*tf([1 3 2],[1 -3 2]);

G = tf([1 sqrt(2)],[1 1]);

Nn = minreal(N/G);
Dn = minreal(D/G);

% X(s) = (x1s+x2)/(s+1)

x1 = 1/evalfr(Nn,1);
x2 = 1/evalfr(Nn,2);

Xn = tf([5.414 -0.5858],[1 1]); % we find Xn(s)

% Yn = minreal((1 - Xn*Nn)/Dn); % we find Yn(s)
Yn = tf([1 2],[1 1]);

% Problem 2

Nnc = tf(1,[-1 1.414]);
Dnc = tf([1 3 2],[1 -3.414 2.828]);

R = Nnc*Yn-Dnc*Xn;
[Rs,Ru] = stabsep(minreal(R));

[A,B,C,~] = ssdata(tf(Ru));

Wc = lyap(A,-B*B');
Wo = lyap(A',-C'*C);

gopt = sqrt(max(eig(Wc*Wo)));
bmax = 1/gopt;
[V,D] = eig(Wc*Wo);
xmax = V(:,1);
ymax = (1/gopt)*Wo*xmax;
bmax = 1/gopt;

Qopt = R - gopt*(ss(A,xmax,C,0)/ss(-A',ymax,B',0));
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```
Qopt = zpk(minreal(Qopt));

Qcopt = Qopt;
Copt = (Xn + Dn*Qcopt)/(Yn - Nn*Qcopt);
Copt = minreal(Copt);

[Qcnum, Qcden] = tfdata(Qcopt, 'v');
isstable(Qcopt)


%% different Xn and Yn

clear all;
clc;

N = tf(1,[1 1]);
D = tf([1 -3 2],[1 3 2]);
P = minreal(N/D);

Nc = tf(1,[-1 1]);
Dc = tf([1 3 2],[1 -3 2]);

G = tf([1 sqrt(2)],[1 1]);

Nn = minreal(N/G);
Dn = minreal(D/G);

% X(s) = (x1s+x2)/(s+2)

x1 = 1/evalfr(Nn,1);
x2 = 1/evalfr(Nn,2);

Xn = tf([6.4143 0.8283],[1 2]); % we find Xn(s)

% Yn = minreal((1 - Xn*Nn)/Dn); % we find Yn(s)
Yn = tf([1],[1]); % we find Yn(s)

Nnc = tf(1,[-1 sqrt(2)])
Dnc = tf([1 3 2],[1 -(2 + sqrt(2)) sqrt(8)]);

Nnc = tf(1,[-1 sqrt(2)])
Dnc = tf([1 3 2],[1 -(2 + sqrt(2)) sqrt(8)]);

Nnc = tf(1,[-1 1.414]);
Dnc = tf([1 3 2],[1 -3.414 2.828]);

R = Nnc*Yn-Dnc*Xn;
[Rs,Ru] = stabsep(minreal(R));
```

```
[A,B,C,~] = ssdata(tf(Ru));

Wc = lyap(A,-B*B');
Wo = lyap(A',-C'*C);

gopt = sqrt(max(eig(Wc*Wo)));
bmax = 1/gopt;
[V,D] = eig(Wc*Wo);
xmax = V(:,1);
ymax = (1/gopt)*Wo*xmax;
bmax = 1/gopt;

Qopt = R - gopt*(ss(A,xmax,C,0)/ss(-A',ymax,B',0));

Qopt = zpk(minreal(Qopt));

Qcopt = Qopt;
Copt = (Xn + Dn*Qcopt)/(Yn - Nn*Qcopt);
Copt = minreal(Copt);

[Qcnum, Qcden] = tfdata(Qcopt, 'v');
isstable(Qcopt)
```