**EE444/544 Spring 2020 Homework #4**

**Due: April 19. You must work alone, no collaboration is permitted.**
**Matlab is allowed; submit your code with the solution.**

**Problem 1**. Consider the standard feedback system with the nominal plant

$$P(s) = \frac{N(s)}{D(s)}\ , \quad \text{whare} \quad N(s) = \frac{1}{s+1}\ , \quad D(s) = \frac{(s-1)\ (s-2)}{(s+1)\ (s+2)}.$$

$P = \frac{N}{D} = \frac{s+2}{(s-1)(s-2)}$

(a) Perform the following spectral factorization (find $G$ satisfying):

$$G(-s)G(s) = N(-s)N(s) + D(-s)D(s)\ , \quad G, G^{-1} \in \mathcal{H}_\infty.$$

(b) Now define $N_n = NG^{-1} \in \mathcal{H}_\infty$ and $D_n = DG^{-1} \in \mathcal{H}_\infty$. Note that $P = N_n/D_n$ is another coprime representation of the plant; this special representation is called the normalized coprime factorization. Find $X_n, Y_n \in \mathcal{H}_\infty$ such that the Bezout equation is satisfied, i.e. $N_n X_n + D_n Y_n = 1$.

$N_n^* N_n + D_n^* D_n = 1$

**Problem 2**. Consider the following uncertain plant

$$P_\Delta = \frac{N_n + \Delta_N}{D_n + \Delta_D}\ , \qquad \|[\Delta_D \quad \Delta_N]\|_\infty < b\ .$$

Now, from the lecture on April 7, we deduce that the feedback system with this plant and controller $C = (X_n + D_n Q_c)/(Y_n - N_n Q_c)$ is stable if

$$\left\| \begin{bmatrix} Y_n - N_n Q_c \\ X_n + D_n Q_c \end{bmatrix} \right\|_\infty \leq 1/b\ . \qquad (*)$$

In this setting, maximum allowable $b$, denoted by $b_{\max}$, is $b_{\max} = 1/\gamma_{\text{opt}}$, where

$$\gamma_{\text{opt}} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \begin{bmatrix} Y_n - N_n Q_c \\ X_n + D_n Q_c \end{bmatrix} \right\|_\infty .$$

Although this problem looks like a two-block $\mathcal{H}_\infty$ problem (a special case of the mixed sensitivity minimization), it can be reduced to a one-block problem as follows. Define

$$L = \begin{bmatrix} N_n^* & -D_n^* \\ D_n & N_n \end{bmatrix}\ ,$$

and verify that $L^* L = L L^* = I$, i.e. $L$ is unitary. Therefore,

$$\gamma_{\text{opt}} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| L \left( \begin{bmatrix} Y_n \\ X_n \end{bmatrix} - \begin{bmatrix} N_n \\ -D_n \end{bmatrix} Q_c \right) \right\|_\infty = \left\| \begin{bmatrix} N_n^* Y_n - D_n^* X_n - Q_c \\ 1 \end{bmatrix} \right\|_\infty . \qquad (**)$$

Constant

Thus $\gamma_{\text{opt}} = \sqrt{1 + \gamma_o^2}$ where

$$\boxed{\gamma_o = \inf_{Q_c \in \mathcal{H}_\infty} \|R - Q_c\|_\infty \quad \text{where} \quad R = N_n^* Y_n - D_n^* X_n.}$$

One block problem

(a) Using the Nehari approach compute $b_{\max}$.

$R = R_s + \underline{\underline{R_u}}$

$R_u(s) = C(sI-A)^{-1}B$

(b) Again, using the Nehari method compute the optimal $Q_{c,\text{opt}}$ leading to maximal allowable uncertainty level $b_{\max}$; and find the corresponding optimal controller.

(c) Go back to part (b) of Problem 1, and choose a different set of $X_n, Y_n$ satisfying the Bezout equation. Now with the newly defined $R = N_n^* Y_n - D_n^* X_n$, verify that $\gamma_o$ computed in part (a) does not change.

$R = R_s + \underline{\underline{R_u}}$