Homework #3 Problem 2 solution

(a) Robust Stability condition: $\|W_m T\|_\infty \le 1$ $\quad W_m = \delta(s+1)$

$$T = \frac{PC}{1+PC} = \frac{(N/D)\left(\frac{X+DQ_c}{Y-NQ_c}\right)}{1+\left(\frac{N}{D}\right)\left(\frac{X+DQ_c}{Y-NQ_c}\right)} = N(X+DQ_c)\ ; \qquad \|W_m N(X+DQ_c)\|_\infty \le 1$$

largest allowable δ is : $\delta_{max} = 1/\gamma_{opt}$ where

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| (s+1)\, N(s)\left(X(s) + D(s)Q_c(s)\right)\right\|_\infty$$

$$N(s) = \frac{4(s-2)}{(s^2+2s+2)} \qquad X(s) = \left(\frac{-2s+3}{s+1}\right) \qquad D(s) = \left(\frac{s^2-2s+2}{s^2+2s+2}\right)$$

$$N(s) = \underbrace{\frac{(s-2)}{(s+2)}}_{N_i} \cdot \underbrace{\left(\frac{4(s+2)}{s^2+2s+2}\right)}_{N_o} \quad : \text{inner-outer factorization}$$

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \Bigg\| \underbrace{\frac{4(s+2)(s+1)}{(s^2+2s+2)} \cdot \left(\frac{-2s+3}{s+1}\right)}_{W(s)} - \underbrace{\left(\frac{s^2-2s+2}{s^2+2s+2}\right)}_{M(s)} \underbrace{\overbrace{\frac{4(s+2)(s+1)}{(s^2+2s+2)}}^{-T_{20}(s)} Q_c}_{Q(s)} \Bigg\|_\infty$$

Nevanlinna-Pick problem: find $F \in \mathcal{H}_\infty$ with minimal $\|F\|_\infty$ such that $F(\alpha_i) = \beta_i$ $\quad i = 1, 2$ $\quad \alpha_i = 1 \pm j$ $\quad \beta_i = W(1 \pm j)$

$$\left.\begin{matrix} \alpha_1 = 1+j & \alpha_2 = 1-j \\ \beta_1 = 5j & \beta_2 = -5j \end{matrix}\right\} \Rightarrow \gamma_{opt} = 12.071 \qquad F_{opt} = \frac{-12.071\,(s-1.414)}{(s+1.414)}$$

$$Q_{opt} = \frac{W - F_{opt}}{M} \cong \frac{4.0711\,(s-0.02469)}{s+1.414}\ ; \quad Q_c = T_{20}^{-1} Q_{opt} \qquad C_{opt} = \frac{X+DQ_c}{Y-NQ_c}$$

The result is $\displaystyle C_{opt} = -3.0178\left(\frac{s-1.414}{s+18.49}\right)$

$$T_{opt} = \frac{PC_{opt}}{1+PC_{opt}} \quad : \quad \text{poles are } \{-1, -2, -1.414\}$$

$\Rightarrow$ feedback system is stable
(also note: there is no unstable pole-zero cancellation in PC_{opt})

Note: the feedback system is 3rd order, controller is first order
plant is second order

(b) Now with C_{opt} as found above check robust stability

$\| W_m T_{opt} \|_\infty \leq 1$ for $\delta \leq \delta_{max} = 1/12.071$ equivalently

we must have $\| (s+1) T_{opt}(s) \|_\infty \leq 12.071$

Obtain a plot of $|(j\omega+1) T_{opt}(j\omega)|$ and observe that it is constant $= 12.071$

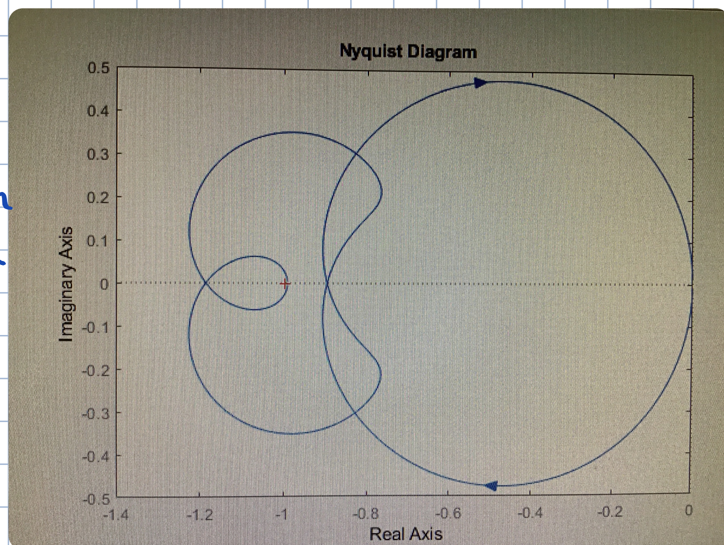
$\delta_{max} = 1/12.071 = 0.0828$ Let's take $\delta = 0.075$ and define

$$P_\Delta(s) = P(s)\left(1 + \underbrace{\delta\left(\frac{s+1}{0.01s+1}\right)}_{\Delta_m(s)}\right) = \frac{34\,(s+12.65)(s-2)}{(s+100)(s^2-2s+2)}$$ $|\Delta_m(j\omega)| < |W_m(j\omega)| \; \forall \omega$

P_Δ and P have 2 poles in $\mathbb{C}_+$, so $P_\Delta \in \mathcal{P}$ and $P_\Delta \neq P$

$S_\Delta = \dfrac{1}{1 + P_\Delta C_{opt}}$ $\Rightarrow$ closed loop poles are $\{-0.041, -1, -6.42 \pm j\,24.73\}$

$\Rightarrow$ feedback system is stable



The Nyquist graph is as shown $\rightarrow$
it encircles $\{-1\}$ twice in ccw direction
verifying stability of the feedback system
"allmargin" command gives stability margins
but the results must be interpreted
together with the Nyquist graph

GM $= 1.0073$, PM $= 1.487°$, DM $= 0.145$ sec

$VM \cong 1 - \dfrac{1}{1.0073} = \dfrac{0.0073}{1.0073}$ achieved at $\omega \cong 0$.

Obviously the uncertainty magnitude is taken very large pushing the limits of robust stability. Check that when $\delta \to 0$ margins increase: GM $= 1.0828$; PM $= 5.3°$; DM $= 0.0925$ sec. However, these margins are still small. Because the plant is "difficult to control" in nature, with right half plane poles and zeros "close to each other".