

BANDIRMA ONYEDİ EYLÜL ÜNİVERSİTESİ
MÜHENDİSLİK VE DOĞA BİLİMLERİ FAKÜLTESİ



BMI3222 Signals And Systems
Course Project

**“Signal Reconstruction from
Discrete-Time Samples Using
Zero-Order and First-Order Hold
Methods in Python”**

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Abstract

This project addresses the topic of sampling and reconstruction of continuous (analog) signals, a common problem in digital signal processing (DSP) systems. The main objective of the project is to sample an analog signal at different sampling frequencies and then reconstruct it using two widely used methods—Zero-Order Hold (ZOH) and First-Order Hold (FOH)—and compare the performance of these methods.

In the implementation, three different signal models were used: sinusoidal, triangular, and exponentially decaying signals. The user interface was developed using the Python programming language and the matplotlib library, allowing interactive modification of parameters such as frequency and sampling rate. In the reconstruction processes, the ZOH method is based on holding values constant, while the FOH method applies linear interpolation. The accuracy of both methods was evaluated using the mean square error (MSE) value relative to the original signal.

The results showed that both methods may suffer from information loss at low sampling frequencies. However, the FOH method exhibited a lower error rate than the ZOH method, especially for signals with linear transitions. The project was designed as an effective learning tool to support the visual and numerical understanding of basic signal processing principles.

Introduction

Digital Signal Processing (DSP) is an interdisciplinary field involving the conversion of analog signals obtained from the real world into digital form, their processing, and, when necessary, their conversion back to analog form. Continuous-time signals such as sound, image, and temperature from the real world must first be sampled to be processed by digital systems. However, this process alone is not sufficient, as the digitally processed data often needs to be converted back into a continuous signal. This conversion process is called "reconstruction."

Reconstruction is of critical importance in many application areas, especially in digital filtering, communication systems, and audio and video processing. During the sampling process, signals are taken at specific intervals, and a continuous signal that closely resembles the original must be reconstructed from these samples. In this context, various reconstruction methods have been developed. Two of the most basic and widely used methods are the Zero-Order Hold (ZOH) and First-Order Hold (FOH) techniques.

The ZOH method creates a stepped structure in the signal by holding each sample value constant until the next one, while the FOH method aims to obtain a smoother and more continuous signal by providing linear transitions between samples. The accuracy and practical success of both methods

vary depending on the structure of the signal and the sampling frequency. Therefore, visually analyzing and comparing the effectiveness of these methods is highly valuable in DSP education and application design.

Mathematical Foundations

In digital signal processing systems, the process of reconstructing the signal after sampling is carried out using various hold techniques to obtain a representation close to the continuous-time signal. In this context, the two most commonly used basic methods are **Zero-Order Hold (ZOH)** and **First-Order Hold (FOH)**. Both methods aim to generate a continuous signal from sampled data, but they model the signal's behavior over time in different ways, producing different results.

Zero-Order Hold (ZOH)

Definition:

The Zero-Order Hold (ZOH) method is a reconstruction technique that holds each sample value constant until the next sample is taken. In other words, the signal is maintained in the form of a step function during each sampling period. Due to its hardware simplicity, this method is frequently used in digital-to-analog converters (DACs).

Mathematical Expression:

Given a sampled signal sequence $x[n]$ and a sampling period T_s the continuous-time signal reconstructed using ZOH, $x_{\text{ZOH}}(t)$ denoted as, is defined as follows:

$$x_{\text{ZOH}}(t) = x[n], \quad \text{for } nT_s \leq t < (n+1)T_s$$

Here $n \in \mathbb{Z}$, T_s , represents the sampling interval, and, $x[n]$ denotes the sampled values at time $t = nT_s$.

First-Order Hold (FOH)

Definition:

The First-Order Hold (FOH) method aims to achieve a smoother reconstruction by performing linear interpolation between two successive samples. This method typically results in lower error compared to ZOH, especially for signals that exhibit linear variation.

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The continuous-time signal reconstructed using FOH, denoted as $x_{\text{FOH}}(t)$, is expressed as a linear transition between two consecutive samples $x[n]$ and $x[n+1]$ as follows:

$$x_{\text{FOH}}(t) = x[n] + \frac{x[n+1] - x[n]}{T_s}(t - nT_s), \quad \text{for } nT_s \leq t < (n+1)T_s$$

This method estimates the value of the signal at time t by considering the slope between the current sample and the next one. The FOH (First-Order Hold) method typically produces lower reconstruction error because it better reflects the trends in the signal.

Implementation Details

In this project, three different types of continuous-time analog signals (sinusoidal, triangular, and exponentially decaying) were sampled and then reconstructed back to continuous time using two different reconstruction methods: **Zero-Order Hold (ZOH)** and **First-Order Hold (FOH)**. The code was written in Python, and scientific libraries such as **matplotlib** for visualization and **numpy** and **scipy** for mathematical operations were used.

1. Signal Definition

Three different analog signals are defined:

- **Sine Wave:**

$$x(t) = \sin(2\pi ft)$$

This signal represents the fundamental harmonic signal.

Triangle Wave:

It is a piecewise linear wave. A symmetric triangular structure is created using the *floor* and *abs* functions.

Exponential Decay:

$$x(t) = e^{-\alpha t}$$

It is a continuous signal that approaches zero over time.

2. Sampling

Analog signals were sampled according to a specific sampling frequency. The sampling process was carried out as follows:

```
# Sampling function: Sampling an analog signal
def sample_signal(signal_func, t_start, t_end, sampling_rate):
    # Sampling period: Ts = 1 / fs
    Ts = 1 / sampling_rate
    t_samples = np.arange(t_start, t_end, Ts)
    # x[n] = x(t) | t = nTs
    x_samples = signal_func(t_samples)
    return t_samples, x_samples, Ts
```

This function generates discrete samples from the continuous-time signal.

3. ZOH Reconstruction

The ZOH algorithm holds each sample value constant until the next one. In the code, this is implemented using a loop:

```
# ZOH: Zero-Order Hold Reconstruction
# Holds each x[n] value constant for Ts duration.
# Mathematically:  $x_{\text{zoh}}(t) = x[n] \quad \forall t \in [nT_s, (n+1)T_s)$ 
def zoh_reconstruct(t_samples, x_samples, t_full):
    zoh = np.zeros_like(t_full)
    for i in range(len(t_samples) - 1):
        zoh[(t_full >= t_samples[i]) & (t_full < t_samples[i+1])] = x_samples[i]
    zoh[t_full >= t_samples[-1]] = x_samples[-1] # Last sample is held for the remaining
    return zoh
```

4. FOH Reconstruction

The FOH method performs linear interpolation between two sample values. The `scipy.interpolate.interp1d` function is used for this purpose.

```
# FOH: First-Order Hold Reconstruction
# Performs linear interpolation between two consecutive samples.
#  $x_{\text{foh}}(t) = x[n] + ((x[n+1] - x[n]) / T_s) * (t - nT_s)$ 
def foh_reconstruct(t_samples, x_samples, t_full):
    foh_interp = interp1d(t_samples, x_samples, kind='linear', fill_value='extrapolate')
    return foh_interp(t_full)
```

Error Calculation (MSE)

The accuracy of the reconstructed signals is measured using the Mean Squared Error (MSE) method by comparing them to the original continuous signal.

```
# Error calculation: MSE (Mean Squared Error)
#  $MSE = (1/N) * \sum (x(t) - x_{\text{recon}}(t))^2$ 
zoh_mse = mean_squared_error(x_cont, zoh)
foh_mse = mean_squared_error(x_cont, foh)
```

Graphical User Interface (GUI)

An interactive interface was created using the `matplotlib.widgets` module. The user can observe the results by adjusting the following parameters:

- Sampling Rate

- Signal Frequency
- Signal Type

The user interface is updated with sliders and radio buttons. The `update()` function redraws the output every time a change is made.

Results and Plots

In this project, based on the sampling and reconstruction processes applied to three different types of signals (sine, triangle, exponential), the accuracy with which the ZOH and FOH methods can represent the original signal was examined through visualizations. The following graphs illustrate this process step by step.

1. Original Signal

In each simulation, a high-resolution (e.g., consisting of 1000 points) continuous-time signal was generated and accepted as the reference signal. This signal was compared with the results of the ZOH and FOH methods.

2. Sampled Signal

Samples taken from the original signal at a specific sampling frequency (indicated with red circles) were added to the graphs. These samples form the basis of the reconstruction process.

3. ZOH (Zero-Order Hold) Reconstruction

The ZOH method reconstructs the signal by holding each sample value constant until the next sample. This method creates a step-like structure in the signal and does not reflect sudden changes. The following observations were made:

- For the sine signal, significant information loss and waveform distortion were observed at low sampling rates.
- For the exponential signal, ZOH yielded better results due to its slow variation.

ZOH Graph Summary:

- Blue line: Signal reconstructed using ZOH
- Black dashed line: Original continuous-time signal
- Red dots: Sampled data points

4. FOH (First-Order Hold) Reconstruction

FOH reconstructs the signal more smoothly by performing linear transitions between samples.

Observations include:

- For periodic signals like sine and triangle waves, FOH produces results closer to the original than ZOH.
- It better tracks the trends of the signal, even at low sampling rates.

FOH Graph Summary:

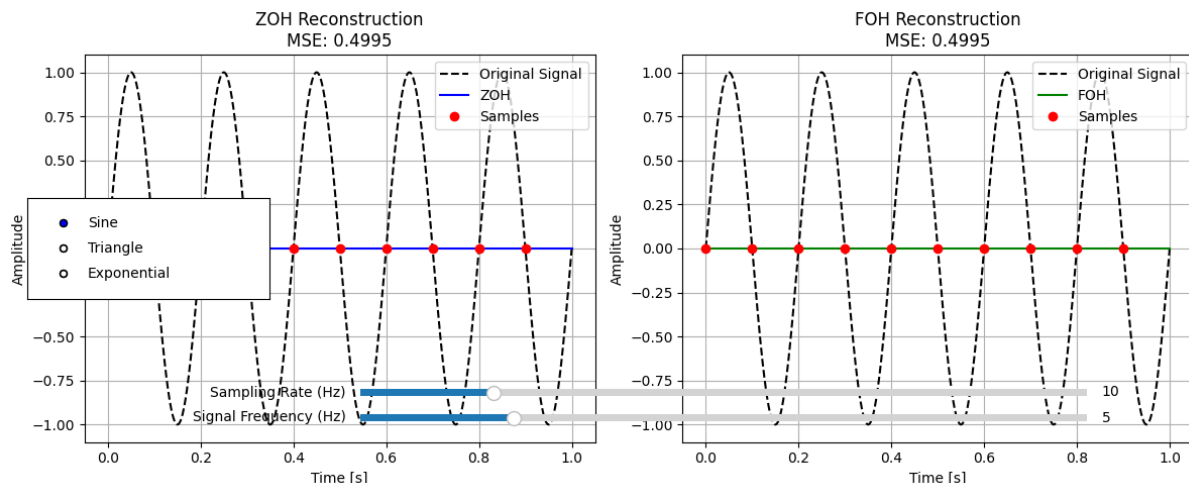
- Green line: Signal reconstructed using FOH
- Black dashed line: Original continuous-time signal
- Red dots: Sampled data points

5. Error (MSE) Comparison

On each graph, the Mean Squared Error (MSE) values of the ZOH and FOH methods were calculated and displayed in the title.

Observation:

- The MSE value of FOH is generally lower than that of ZOH.
- This indicates that the FOH method can reconstruct the sampled signal into continuous time more accurately.



Discussion

In this project, the reconstruction of digitally sampled signals was carried out using Zero-Order Hold (ZOH) and First-Order Hold (FOH) methods. While ZOH produces a step function by holding each sample value constant until the next one is taken, the FOH method provides a smoother transition by performing linear interpolation between samples. The advantages and disadvantages of both methods

were evaluated based on both visual results and the calculated Mean Squared Error (MSE) values. The ZOH method stands out particularly for its low computational cost and simplicity. However, it has limitations such as failing to capture sudden changes in the signal and producing results that are far from the original signal. This becomes more apparent in high-frequency signals. As seen in the graphs, although the signal reconstructed with ZOH aligns with the original signal, it fails to represent the waveform accurately due to its step-like structure. This indicates that the ZOH method is more suitable for low-frequency and slowly varying signals.

On the other hand, since the FOH method provides a linear transition between signal samples, the reconstructed signal presents a closer appearance to the original analog signal. Especially when the slope of the signal changes, FOH can reflect this change more accurately. This has been supported both visually and through the fact that FOH generally produces lower MSE values. Despite this accuracy advantage, it should be noted that FOH is slightly more complex computationally compared to ZOH.

Additionally, the effect of the sampling frequency on both methods was examined. According to the Nyquist criterion, in order to reconstruct the signal accurately, the sampling frequency must be at least twice the signal frequency. At sampling rates below this threshold, both methods suffer from aliasing effects, and the accuracy of the reconstructed signal significantly decreases. However, FOH still demonstrated better performance than ZOH under such conditions.

In conclusion, while ZOH stands out for its simplicity, FOH offers higher accuracy. The choice between the two methods should be based on system requirements, signal characteristics, and available processing resources.

Conclusion

In this project, the process of reconstructing analog signals from their digital samples was examined, and the performances of the ZOH and FOH methods were compared. The analysis showed that the FOH method provides more successful reconstruction, especially for high-frequency signals, while ZOH is simpler and has lower computational cost. The Mean Squared Error (MSE) values supported that FOH generally yields better results in terms of accuracy.

As a development suggestion, higher-order interpolation techniques (e.g., spline interpolation) can be explored in future studies. Additionally, evaluating scenarios such as reconstruction in the presence of different signal types and noisy environments would provide a more comprehensive analysis. Such studies could contribute to the development of more reliable and efficient signal processing algorithms for real-time DSP applications.