Mineshaft Report

Introduction

As the geological makeup of the Earth differs as you dig further into its crust, it is in the interest of any mining company to be able to easily test the mining depth, so that it may know what elements and minerals that it is likely to find within a certain mineshaft, and most importantly, whether further digging is required.

In this report, we will discuss and analyze several approaches to measuring the depth of a mine shaft we have dug. All measurements center around taking a test mass calibrated to $1\ kg$, dropping it from the surface of the Earth, and measuring the time it takes for the mass to drop down the length of the shaft (four kilometers).

We will first discuss the ideal case, then consider variables such as gravity, drag, and Coriolis forces. Eventually, we will discuss an infinitely deep mine (going through the entire Earth), to understand the feasibility of increasing the depth of the mine.

Calculation of Fall Time

Ideal Case

Neglecting drag, Coriolis forces, and assuming constant gravity means that the equation of motion for our mass is,

$$\frac{d^2y}{dt} = \frac{dv}{dt} = -g \tag{1}$$

Where g, the gravitational acceleration, is 9.81 meters per second squared, and $v = \frac{dy}{dt}$ is the velocity of the test mass. When we solve (1) analytically, we find that the time to reach the bottom of the mineshaft (henceforth called “fall time”) is 28.6 seconds. The fall time, when calculated numerically using the event detection capabilities of the scipy.integrate module from scipy, a python library, was found to be 28.6 seconds.

Variable Gravity

In practice, gravity varies over such distances, and we should include this variance in gravity in our calculations. For a homogeneously distributed earth, the gravitational acceleration is,

$$g(y) = g_o\left(\frac{y}{R_\oplus}\right) \tag{2}$$

Where y is the distance from Earth's center, g_o is the gravitational acceleration at the Earth's surface, and $R_\oplus$ is the Earth's radius. This case gives us a fall time of 28.6 seconds as well. ~~(~~With more precision, however, we see that with variable gravity, the mass drops in 28.558 seconds, whereas with constant gravity, we have 28.557 seconds.~~)~~ That said, although negligible, the fall time is larger.

Variable Gravity and Drag

Due to the speeds the mass achieves, drag also plays an important factor. We find that the fall time, with variable gravity and drag included in the calculations, is 83.5 seconds. The calculations of the drag parameters are given in the attached ipynb file. Therefore, including drag in these calculations increases the fall time dramatically (over 2x faster).

Feasibility of Depth Measurement Approach

On top of drag and varying gravity, we also ought to include the forces on the mass, due to the rotation of the Earth. This last force is called a Coriolis force. Solving the system of differential equations by including the effects of linearly decreasing gravity, drag, and the Coriolis force gives the trajectory shown in Figure 1. Both with and without drag, the mass hits the wall before the bottom of the mineshaft. With drag, this happens at 1848.1 meters below Earth's surface and 40.5 seconds, and without drag, this happens at 3736.5 meters and 27.6 seconds.

As the walls of the mineshaft are quite jagged, and it will be considerably difficult to ensure elastic collisions (such that speed is not lost), it is not feasible to continue with this test; there are too many uncontrollable variables at play for this test to reliably measure the mineshaft's depth.

Calculation of Crossing Times

The remainder of this report will be somewhat hypothetical. We will suppose a pole-to-pole mineshaft (to neglect Coriolis forces) and calculate crossing times and velocities that the test mass could achieve. We will do this for the Earth and the Moon, for possible future endeavors in lunar mining.

Pole-To-Pole Mineshaft on Earth

We will assume Earth's density is spherically symmetric and of the form

$$\rho(r) = \rho_n \left(1 - \frac{r^2}{R_\oplus^2}\right)^n \tag{5}$$

Where ρ_n is a normalization constant chosen such that the density integrates to Earth's mass, and n is the index of the density. We will show trajectories for the indices $n = 0, 1, 2, 9$. If we replace $g(y)$ from earlier with the gravitational field generated by these density functions, the solutions to the systems give the graphs shown in Fig. 2 below.

The crossing time for the n= 0, n=1, n=2, and n=9 are 1267.2 seconds, 1096.9 seconds, 1035.1 seconds, and 943.9 seconds, respectively. Clearly, as density increases, fall time decreases. In the attached ipynb file, the entire calculation is worked out, and it is shown that $t_{fall} \propto 1/\sqrt{\rho}$. Curiously, for a uniform Earth, the time to cross the Earth through the tunnel is almost the same as the time it takes to go to that same point through orbit (2534.7 seconds)

Discussion and Future Work

In the future, several more aspects should be added to the calculations. When looking at Coriolis forces, we neglected drag in the transverse direction. This is of interest for a more realistic simulation. The ellipsoid nature of the Earth must be considered, as it will affect Coriolis forces. Further, more analysis must be done on how the Coriolis force affects the trajectory of the test mass when the mine is not at the equator or the poles. Further, a more realistic density function should be considered. In Graphic 1, Earth's density is not modeled well by a differentiable function. Therefore, a new density function ought to be used in the attached python file to more accurately model the gravitational forces at play. Finally, when modeling the pole-to-pole mineshaft, we must consider using different drag coefficients. As the fluids which make up the earth are at different densities and temperatures, they will interfere with the test mass's motion in different ways, resulting in different drag forces and different velocity dependences.

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Figures & Graphics

Figure 1

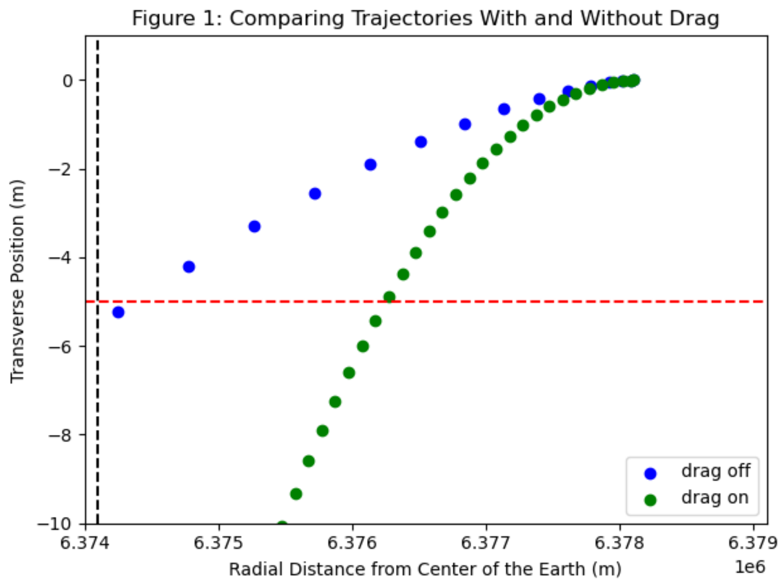


Figure 1: The trajectory of the test mass with variable gravity and Coriolis forces. The case with neglected drag is shown in blue and the case with included drag is shown in green. The horizontal red dashed line represents the wall of the mineshaft. The black dashed line represents the bottom of the mineshaft.

Figure 2

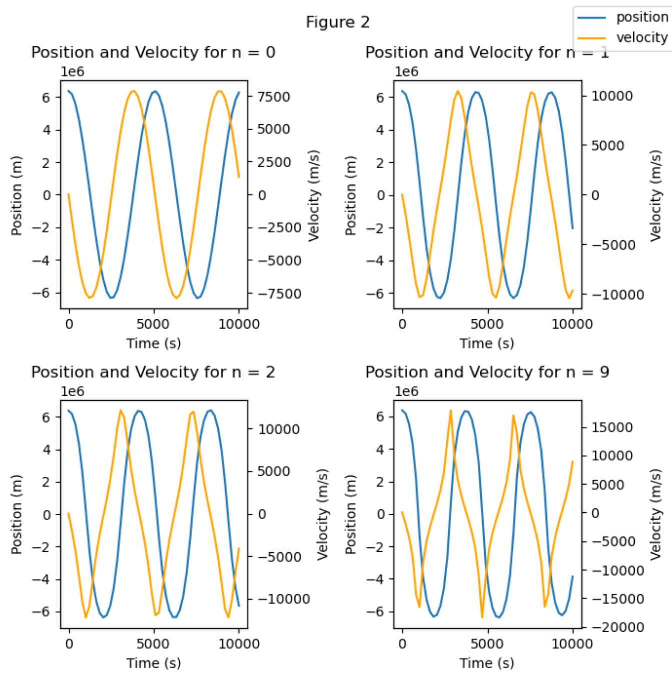
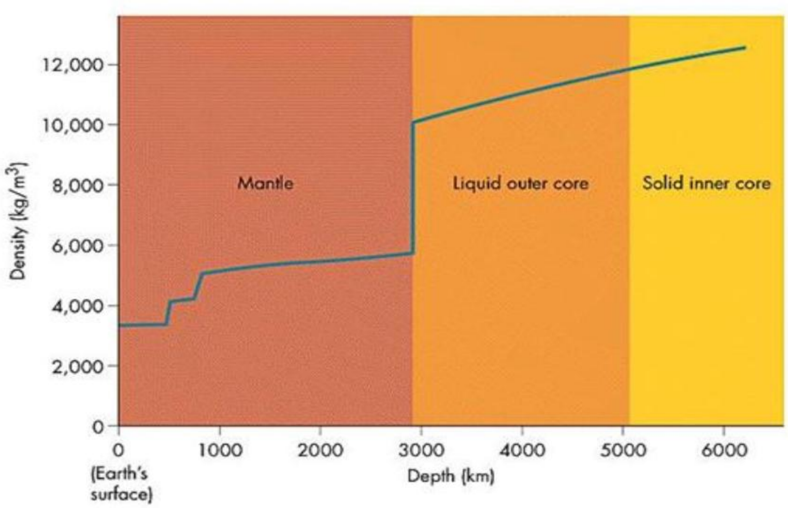


Figure 2: Position and velocity graphed against time for n=0, n=1, n=2, n=9 in (5). Derived from solving for position from the gravitational force for the mass distribution corresponding to each index. The blue lines correspond to position, and the yellow lines correspond to velocity.

Graphic 1



Graphic 1: A qualitative density profile of the Earth.