Rational Conformal Field Theory and Verlinde Operators

Burak Oğuz

September 1, 2024

Abstract

We study Conformal Field Theory (CFT) in 2 dimensions. This is a well-studied example of a solvable Quantum Field Theory (QFT), due to the rich symmetry structure special to 2d. CFT also has important appearances in pure mathematics, especially in the Geometric Langlands program and the representation theory of affine Lie algebras. We first give a review of the standard construction of these theories from a physics perspective and define the Rational Conformal Field Theories (RCFTs), which are special models. In RCFT we discuss the fusion rules and Verlinde operators, which are insightful tools to understand the fusion algebra. We discuss how modular invariance on the torus diagonalizes and solves the fusion rules, following Verlinde's seminal paper. [1]

Contents

[1] I thank Batuhan Acar for helping me understand some of the detailed parts of this manuscript.

1 Introduction

Conformal Field Theory (CFT) in 2 dimensions is a well studied subject appearing in many places in physics and mathematics, due to its very strong symmetry structure which makes CFT interesting to both physicists and mathematicians. Some of the places CFT plays an important role are the following:

- In String Theory [Pol07], a string propagating in spacetime draws a worldsheet, since it is an extended line. Many of the properties of the theory is encoded in a conformal field theory living on the worldsheet. This was one of the first places where physicists got interested in CFT.

- Independent of String Theory, conformal invariance have been known to physicists from statistical systems [Pol70, Kad66]. When a statistical system goes through a phase transition, the theory is scale invariant at the critical points[2] and hence can be described by a CFT.

- Due to the rich symmetry structure, 2d CFT is an integrable system [BPZ84, PBZ84, Neg16]. Integrability is the phenomenon where the system has enough symmetries that one can fix everything from the symmetries. 2d CFT contains infinite dimensional algebras that give infinitely many symmetries, hence it is integrable.

- In the AdS/CFT correspondence, $d-$dimensional CFT plays an important role as the dual of a gravitational theory on an $d+1$ dimensional AdS space. In the $\mathrm{AdS}_3/\mathrm{CFT}_2$ case, many interesting results about gravity and CFT were studied. In particular, because $3d$ gravity is a topological theory [Wit88], interesting results between Topological Field Theories (TFTs) and CFTs have been studied in [Wit89, DW90, EMSS89, MS89b, MS89d] and in many other papers building on these works.

- On the mathematics side, the infinite dimensional algebra present in CFT has attracted a lot of attention from mathematicians studying the Geometric Langlands program, the representation theory of Affine Lie algebras, and the quantum algebras. We are by no means knowledgeable in these areas, but one may consult [Fre07] for more information regarding this.

[2]For example, if one heats up water to the critical temperature $T_c \approx 374°C$, it has a phase transition and thus can be described by a conformal theory called the Ising CFT [SD17].

There are other places where CFT is crucial not mentioned above. But it is clear that this remarkable theory is worth studying both from physics perspective and mathematics perspective. We will focus on Rational Conformal Field Theories (RCFT), which are even more restricted CFTs. Our purpose is to give a detailed review of the standard construction of CFT from a physicist point of view, following [Ket95] and getting help from [DFMS97, Pol07, BP09]. After reviewing the standards, we will study Verlinde lines in RCFT on a Riemann surface. We reproduce the results of the seminal paper [Ver88] in section 3. In subsection 3.3, we use a diagrammatic approach to get the same results. Diagrammatic tools are very useful in the modern literature of generalized symmetries [GKSW15, Sha23, SN24, BBFT$^+$24]. The Verlinde lines studied here are related to non-invertible defects in 2d CFTs with a modular fusion category $\mathcal{M}$ [Sha23].

2 Conformal Field Theory in 2 Dimensions

We will carry out the standard construction of CFTs in 2 dimensions, following the presentation of [Ket95] and sometimes drawing results from [DFMS97]. We start from ordinary QFT, and then move on to the conformally invariant field theories and study their quantization.

The section is organized as follows: First, we will briefly review postulates of the quantum theory. Then we discuss the dynamics and symmetries of relativistic QFT. In subsection 2.3, we start discussing CFTs, which are invariant under a larger group of transformations, the conformal group, than ordinary QFT. In we derive the Ward-Takahashi identities which are quantum manifestations of symmetries on the correlation functions, the objects of interest in quantum theories.

2.1 The Quantum Theory

During the first half of twentieth century, there were great discoveries that changed our perspective on nature drastically. These developments can broadly be categorized into two frameworks: the Quantum Theory and the General Theory of Relativity. We will mainly be concerned with quantum theory here.

The quantum theory is a framework used to describe microscopic systems, and has had great success in doing so. As much as it was a success, physicists had -and still have- a hard time understanding it. Our current understanding of the quantum theory starts from of a set of postulates about a quantum system, and after which the rest is built up. In this subsection we follow [Wei05].

The postulates are as follows:

(i) The starting point of the quantum theory is the state vector $|\Psi\rangle$, which captures a physical system. This vector $|\Psi\rangle$ lives in the Hilbert space of states $\mathcal{H}$, which is a complex vector space with hermitian inner product. Also, we physically identify the state $e^{i\theta}|\Psi\rangle$ with $|\Psi\rangle$, where θ is a real number. Hence, the states are rays in the Hilbert space. The properties of the inner product can be summarized as

$$\begin{aligned} \langle\Phi|\Psi\rangle &= \langle\Psi|\Phi\rangle^*, \\ \langle\Phi|x_1\Psi_1 + x_2\Psi_2\rangle &= x_1\langle\Phi|\Psi_1\rangle + x_2\langle\Phi|\Psi_2\rangle, \\ \langle y_1\Phi_1 + y_2\Phi_2|\Psi\rangle &= y_1^*\langle\Phi_1|\Psi\rangle + y_2^*\langle\Phi_2|\Psi\rangle, \end{aligned} \tag{1}$$

where $x_i, y_i \in \mathbb{C}$ and y_i^* is the complex conjugate. The norm additionally has a positivity condition $\langle\Psi|\Psi\rangle \geq 0$ with equality only when $|\Psi\rangle$ is the zero vector.

(ii) The physical observables are represented by Hermitian operators acting on the Hilbert space:

$$\begin{aligned} \mathcal{A} : \mathcal{H} &\to \mathcal{H} \\ |\Psi\rangle &\mapsto \mathcal{A}\cdot|\Psi\rangle, \end{aligned} \tag{2}$$

which is linear in the sense of

$$\mathcal{A}\Big(x_1|\Psi_1\rangle + x_2|\Psi_2\rangle\Big) = x_1\mathcal{A}|\Psi_1\rangle + x_2\mathcal{A}|\Psi_2\rangle. \tag{3}$$

The reality condition is $\mathcal{A}^\dagger = \mathcal{A}$, where $\mathcal{A}^\dagger$ is defined via

$$\langle\Phi|\mathcal{A}^\dagger\Psi\rangle = \langle\Phi\mathcal{A}|\Psi\rangle = \langle\Psi|\mathcal{A}\Phi\rangle. \tag{4}$$

There are technical assumptions about boundedness and smoothness of $\mathcal{A}|\Psi\rangle$. A state $|\mathsf{a}\rangle$ has definite value $\alpha(\mathsf{a})$ under the measurement of the observable $\mathcal{A}$ if

$$\mathcal{A}|\mathsf{a}\rangle = \alpha(\mathsf{a})|\mathsf{a}\rangle, \tag{5}$$

where $\alpha(\mathsf{a})$ is the result of the measurement. Since $\mathcal{A}^\dagger = \mathcal{A}$, an elementary theorem in linear algebra states that $\alpha(\mathsf{a})$ are real numbers and their associated eigenstates are orthonormal in the sense of the inner product: $\langle\mathsf{a}'|\mathsf{a}\rangle = 0$ if $\mathsf{a}' \neq \mathsf{a}$ as physical states.

(iii) If a system is in a generic state $|\Psi\rangle$, a measurement on this system for the observable $\mathcal{A}$ disturbs the system and projects the vector $|\Psi\rangle$ into one of the eigenstates of $\mathcal{A}$:

$$|\Psi\rangle \xrightarrow{\mathcal{A}} |\mathsf{a}\rangle, \tag{6}$$

but to which eigenstate $|\mathsf{a}\rangle$ will $|\Psi\rangle$ collapse is not known with certainty. Measuring the value $\alpha(\mathsf{a})$ from the state $|\Psi\rangle$ has a probability $P(\Psi \to \mathsf{a})$ given by

$$P(\Psi \to \mathsf{a}) = |\langle\mathsf{a}|\Psi\rangle|^2, \tag{7}$$

and P satisfies

$$\sum_{\{\mathsf{a}\}} P(\Psi \to \mathsf{a}) = 1, \tag{8}$$

where $\{\mathsf{a}\}$ is a complete set of eigenstates of $\mathcal{A}$. This condition is the conservation of probability, all probabilities of possible outcomes from a process must add up to 1.

In quantum theory, the systems associated to $\mathcal{H}$ has certain symmetries, transformations under which the physical information remains the same. Accounting the symmetries in a quantum system constraints the dynamics and is a powerful tool. We will briefly discuss the incorporation of symmetry into quantum theory.

Suppose there is an observer O who sees a system in state $|\Psi\rangle$, and another equivalent observer O' sees it in state $|\Phi\rangle$. The two observers sees the system in different states, but if the observers are related with a symmetry transformation, then they must observe the same probability:

$$P(\Psi \to \mathsf{a}_\Psi) = P(\Phi \to \mathsf{a}_\Phi). \tag{9}$$

According to a theorem by Wigner, for a symmetry transformation $|\Psi\rangle \to |\Phi\rangle$, we can find a unitary operator U such that $|\Psi\rangle = U|\Phi\rangle$ satisfying

$$\begin{aligned} \langle U\Psi_1|U\Psi_2\rangle &= \langle \Psi_1|\Psi_2\rangle, \\ U\Big(x|\Psi_1\rangle + y|\Psi_2\rangle\Big) &= xU|\Psi_1\rangle + yU|\Psi_2\rangle. \end{aligned} \tag{10}$$

(There is another possibility where U is anti-unitary and anti-linear, instead of unitary and linear, but we usually have U unitary and linear.) The condition of unitarity can be found to be $U^\dagger U = 1 = UU^\dagger$, which implies $U^\dagger = U^{-1}$. In the next subsection we will see some explicit examples of unitary operators and observables for relativistic QFTs.

In the quantum theory, one is interested in calculating probabilities $P(i \to f)$ for arbitrary initial and final states. For this one needs a time evolution operator, which is closely related to the notion of energy. In a time translation invariant system, probabilities must be conserved through translations of time, hence must be implemented through unitary transformations on the Hilbert space. In quantum theory, the observables have an additional role to play here. The one-parameter automorphisms of hermitian operators give the unitary transformations. For the case of time evolution, we have

$$U(t) = \exp\Big(-i\hat{H}t\Big), \tag{11}$$

where $\hat{H}$ is the Hamiltonian operator, whose eigenvalues are related to energy states. Then, the probability amplitude for a system starting in an initial state i, and evolving for some time t, to be measured in the final state f is given by

$$A(i \to f, t) = \Big\langle f \Big| \exp\Big(-i\hat{H}t\Big) \Big| i \Big\rangle. \tag{12}$$

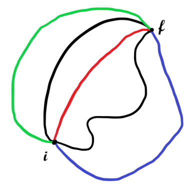

Figure 1: The demonstration of sum over all configurations from the initial state to the final state.

This can be calculated using the sum over histories approach. That is, by summing over the amplitudes of all possible paths in from i to f and weighting the sum by the respective amplitudes, as demonstrated in figure 1. This process corresponds to a trace over the Hilbert space with appropriate operator insertions (the explicit form of β is not specified yet, we will leave it ambiguous)

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \equiv \mathrm{Tr}_{\mathcal{H}} \Big(e^{-\beta \hat{H}} \mathcal{O}_1 \cdots \mathcal{O}_n \Big). \tag{13}$$

Hence, the problem of calculating all amplitudes from arbitrary i to f is solved if one computes all the $n-$point functions. These objects are the main object of interest in Quantum Theory. The $0-$point function is traditionally called the partition function

$$Z = \mathrm{Tr}_{\mathcal{H}} e^{-\beta \hat{H}}. \tag{14}$$

2.1.1 Atiyah-Segal Picture of QFT

It turns out that the notions of QFT are (partially) captured by Atiyah-Segal axioms. In this framework, a QFT is viewed as a functor between two categories, and the partition function is the object that maps Hilbert spaces at given times to Hilbert spaces at other times as demonstrated in figure 2.

Specifically, QFT is a symmetric monoidal functor $\mathcal{Z}$, from the category of bordisms $\mathsf{Bord}_{(n,n-1)}$, to the category of vector spaces Vect:

$$\mathcal{Z} : \mathsf{Bord}_{(n,n-1)} \to \mathsf{Vect} \tag{15}$$

2.2 Ordinary Quantum Field Theory

Now we discuss relativistic Quantum Field Theory (QFT) in $d+1$ Minkowski space-time. In this theory, we have a set of local operators $\Phi(x)$ that transform under representations of the Poincaré group $ISO(d,1)$. This group is an extension of the group $SO(d,1)$ with d space-translations generated by the momenta P^i, 1 time translation generated by the Hamiltonian H, which combine into a $d+1$ vector $P^\mu = (H, P^i)$. Due to the locality, one can write

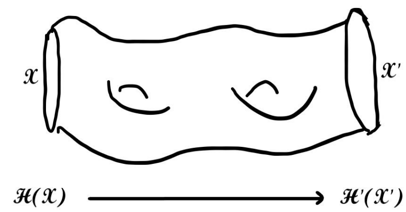

Figure 2: The partition function is a map from the Hilbert space on the left boundary $\mathcal{X}$ to the right boundary $\mathcal{X}'$.

the conserved charge, the $d+1$ momenta, as the integral of what's called the Energy-Momentum tensor $T^{\mu\nu}$, which is a symmetric tensor of rank 2. One has

$$P^{\mu} \equiv \int d^d x T^{0\mu}. \tag{16}$$

Given an action $S[\Phi]$ of the fields, one can obtain $T^{\mu\nu}$ in terms of the fields using Noether's theorem under a transformation $x^\mu \to x^\mu + \varepsilon^\mu$ with ε^μ a translation $d+1$ vector whose components are all assumed to be infinitesimal without loss of generality because one can build up arbitrary translations from the infinitesimal ones.

The group $SO(d,1)$ is the group of isometries of the Minkowski metric

$$ds^2 = -dt^2 + \sum_i^d (dx_i)^2. \tag{17}$$

There are $d(d-1)/2$ rotations and d Lorentz boosts that leave this metric invariant. The corresponding local currents can be written as $\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$, and whose corresponding charges that generate the isometry transformations is given by

$$M^{\mu\nu} \equiv \int d^d x \mathcal{M}^{0\mu\nu}, \tag{18}$$

where M^{ij} are related to the angular momentum of the fields, and the parts that contain the 0 index are related to Lorentz boosts. At the classical level, the inhomogeneous Lorentz transformations of the form $x \to x' \equiv \Lambda x + \varepsilon$, with $\Lambda \in SO(d,1)$ and $\varepsilon \in \mathbb{R}^d$, are realized on the fields by the charges $\{P^\mu, M^{\mu\nu}\}$, in the sense that the Poisson bracket of the charges with the fields will yield the transformed version of the field under the corresponding transformation. The change of the fields under these transformations is given by

$$\Phi(x) \mapsto \Phi'(x') \equiv \mathcal{D}(\Lambda) \cdot \Phi(x), \tag{19}$$

where $\mathcal{D}(\Lambda)$ is a representation of $SO(d,1)$ compatible with the spin of the field Φ, and $\cdot$ denotes the contraction of the spin indices that $\mathcal{D}$ and Φ carries. Note that fields of all spins should transform the same way under translations, because spin is a characteristic property under rotations. Because of this fact, $\mathcal{D}$ does not depend on the translation parameter. One can invert the transformation $x = \Lambda^{-1} \cdot (x' - \varepsilon)$, and write the transformation law as

$$\Phi(x) \mapsto \mathcal{D}(\Lambda) \cdot \Phi(\Lambda^{-1} \cdot (x - \varepsilon)), \tag{20}$$

so we have knowledge about the symmetry content of the QFT based on the space-time symmetries. In certain theories, called gauge theories, there are internal symmetries in the theory that do not affect the space-time. In all honesty, these gauge symmetries are not symmetries but redundancies in the description, intentionally introduced to save the locality and Lorentz symmetry which is not present when we use the description without the redundancies. For example, in $d = 4$, a photon has 2 polarizations, hence there should be two fields associated. However, to write a Lorentz covariant Lagrangian density, one must use the redundant 1-form fields $A = A_\mu dx^\mu$, with the physical information about the electric & magnetic fields are encoded in $F = dA$, the field stress tensor. The gauge field A has 4 components, 2 more degrees of freedom than necessary. This redundancy allows a Lorentz invariant description, and there is a $U(1)$ gauge symmetry of the theory under which A changes as $A \to A + UdU^{-1}$ with U a phase $U \in U(1)$. Writing $U = e^{iq\alpha(x)}$, one has $\delta A = -iqd\alpha$ under the gauge transformation, where we introduced a coupling constant q. Since the change of A is a total derivative, and since $d^2 = 0$ identically, one has $\delta F = 0$ under gauge transformations, which is expected as it encodes physical information and physics is invariant under the redundancy transformations. Mathematically, what we are doing is introduce an equivalence class under the nilpotent operator $d^2 = 0$. The equivalence class $[A]$ is such that we have the identification $A \sim A + d\alpha$ for $\alpha \in \Omega^0(M)$.

There are more interesting and more involved cases where the gauge group is in general a non-abelian Lie group, for example, $SU(N)$. These cases are known as Yang-Mills theories, and they are essential theories to understanding many physical phenomena and some deep mathematical fields such as the 4-manifold topology in relation to Donaldson theory [FU84] and Seiberg-Witten theory [Iga02].

At the classical level, the dynamics of a field theory is governed by the field equations, obtained by imposing

$$\delta S[\Phi] = 0, \tag{21}$$

S being the classical action, and δS denotes the response of the action to arbitrary deformations of the fields $\delta\Phi$. This yields an Euler-Lagrange type equation that one may solve in some simple cases.

To obtain the quantum theory corresponding to the classical theory defined by the action $S[\Phi]$, one needs to construct the Hilbert space $\mathcal{H}$ of states while promoting the fields of the theory to operators $\hat{\Phi} \in \mathrm{End}(\mathcal{H})$ [3] acting on $\mathcal{H}$, which obeys an algebra of observables. As the conserved charges $\{P^\mu, M^{\mu\nu}\}$ are constructed out of the local fields, they become operators as well, and in the Hilbert space the transformation of the quantum fields under the corresponding Lorentz transformations reads

$$\begin{aligned}
\Phi(x) &\xrightarrow{x\to x+\varepsilon} \Phi(x+\varepsilon) = e^{iP^\mu\varepsilon_\mu}\Phi(x)e^{-iP^\mu\varepsilon_\mu}, \\
\Phi(x) &\xrightarrow{x\to\Lambda x} \mathcal{D}(\Lambda)\cdot\Phi(\Lambda^{-1}x) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\Phi(x)e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}.
\end{aligned} \tag{22}$$

Where we have written $\Lambda \sim 1+\omega$ with ω a rank-two anti-symmetric tensor. The operators $T(\varepsilon) = e^{-iP\cdot\varepsilon}$ and $U(\omega) = e^{\frac{i}{2}\omega\cdot M}$ then generate the $ISO(d,1)$ transformations in the Hilbert space. If we write these relations infinitesimally, using Baker-Campbell-Hausdorff formula, we get the following equal-time commutation relations

$$\begin{aligned}
[P^\mu, \Phi(x)] &= i\partial^\mu\Phi(x), \\
[M^{\mu\nu}, \Phi(x)] &= -(\mathcal{L}^{\mu\nu}\cdot I + S^{\mu\nu})\Phi(x),
\end{aligned} \tag{23}$$

where $\mathcal{L}^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ and we have defined $S^{\mu\nu}$ via $\mathcal{D}(\omega) \sim 1 + \frac{i}{2}\omega\cdot S$, and I and S are matrices carrying the spin indices of the field Φ that we suppress. In general, constructing $\mathcal{H}$ is not an easy task. One can do it for free theories, but interacting theories proves sufficiently challenging that one has to resort to different techniques, such as building a perturbative expansion via Feynman diagrams.

There is another approach to quantization of a field theory, called the path-integral quantization. Given an action $S[\Phi]$ and some fields, one defines

$$Z \equiv \int \mathcal{D}\Phi e^{iS[\Phi]}, \tag{24}$$

where we assume that a measure $\mathcal{D}\Phi$ on the field space exists and is well defined. The quantum observables are defined by

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle \equiv \frac{1}{Z}\int \mathcal{D}\Phi e^{iS[\Phi]}\Phi(x_1)\cdots\Phi(x_n), \tag{25}$$

where we divide by Z to normalize, and in this correlation we automatically have the time-ordering of operators, that is, $x_1^0 > x_2^0 > \cdots x_n^0$. There are a lot of subtle points about the convergence of the correlation functions. It is common to analytically continue this integral by a Wick rotation $t \to -i\tau$ with τ the Wick rotated time or the Euclidean time, whose name is obvious if one considers the transformed metric $ds^2 = d\tau^2 + dx\cdot dx$ which is the metric of $\mathbb{R}^{d+1}$. In this procedure, the action changes to the Euclidean action S_E, and one defines the Euclidean path integral via

$$Z_E \equiv \int \mathcal{D}\Phi e^{-S_E[\phi]}, \tag{26}$$

[3] $\mathrm{End}(\mathcal{H})$ denotes the endomorphisms of the Hilbert space, namely, maps from $\mathcal{H}$ to itself $\hat{\Phi}: \mathcal{H} \to \mathcal{H}$, $|\psi\rangle \mapsto \hat{\Phi}\cdot|\psi\rangle$.

and the quantum observables

$$\langle \Phi(x_1)\cdots\Phi(x_n)\rangle_E \equiv \frac{1}{Z_E}\int \mathcal{D}\Phi e^{-S_E[\Phi]}\Phi(x_1)\cdots\Phi(x_n). \tag{27}$$

The physical content of a QFT is encoded in its correlation functions. Hence, solving a QFT amounts to a complete knowledge of all the possible correlation functions. The evaluation of the path integrals is a big challenge, and it becomes significantly harder when there is no small parameter in the theory to build up a perturbation series. Such is the case for Yang-Mills theories in the low-energy regimes (infrared), where the Yang-Mills coupling constant g_{YM}, the only free parameter in the theory, becoming larger and larger as the characteristic distances in the physical processes becomes larger (equivalently the energies become lower). Consequently, a complete analytical description of Yang-Mills theory or the Quantum Chromo-Dynamics (QCD) still remains out of reach, luckily numerical simulations from the lattice gauge theory ideas provides a huge help towards this end.

2.3 Conformal Field Theory

A CFT is a special type of QFT that has a larger symmetry group. Considering the metric as a dynamical variable which transforms as a rank-2 tensor, we consider a reparametrization-invariant field theory, which means that under

$$\begin{aligned} \delta x &= \varepsilon(x), \\ \delta\Phi &= L_\varepsilon \Phi, \\ \delta g_{\mu\nu}(x) &= \partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu - \varepsilon^\rho\partial_\rho g_{\mu\nu}, \end{aligned} \tag{28}$$

the theory remains invariant, where L_ε is a Lie derivative appropriate to the tensorial nature of the fields Φ. Under a coordinate transformation $x \to x'$, the metric changes as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x), \tag{29}$$

and Lorentz invariance means $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$. One can relax this condition to what is called a conformal transformation, for which

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \quad \Omega(x) = e^{\omega(x)}. \tag{30}$$

These transformations form a group, which is known as the conformal group. We will consider flat spaces so one can find a coordinate system such that $g_{\mu\nu} = \delta_{\mu\nu}$ globally. Moreover, the Lorentz group is a subgroup of the conformal group, with the condition $\Omega(x) = 1$. To determine all the conformal transformations, we consider the infinitesimal deformations $x \to x' = x - \varepsilon(x)$ where $g_{\mu\nu} = \delta_{\mu\nu}$. Observe that $\frac{\partial x'^\mu}{\partial x^\nu} = (\delta^\mu_\nu - \partial_\nu\varepsilon^\mu)$, the inverse of which is $\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)^{-1} = (\delta^\mu_\nu + \partial_\nu\varepsilon^\mu)$ up to first order in ε. Thus, we have

$$g'_{\mu\nu}(x') = (\delta^\rho_\mu - \partial_\mu\varepsilon^\rho)(\delta^\sigma_\nu - \partial_\nu\varepsilon^\sigma)\delta_{\rho\sigma} = \delta_{\mu\nu} + \partial_\nu\varepsilon_\mu + \partial_\mu\varepsilon_\nu. \tag{31}$$

For this transformation to be a conformal transformation, we must have

$$\delta_{\mu\nu} + \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu = \Omega \delta_{\mu\nu}. \tag{32}$$

Trace both sides to get

$$\partial \cdot \varepsilon = (\Omega - 1)\frac{d}{2}, \tag{33}$$

putting this back above, one gets

$$\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu = \frac{2}{d}(\partial \cdot \varepsilon)\delta_{\mu\nu}. \tag{34}$$

One can go on to play with this equation to find constraints on ε, and finally conclude that the conformal group in d dimensional Euclidean space is the group $SO(d+1,1)$. In the Minkowski case $d = D+1$ with D the space dimensions, the conformal group reads $SO(D+1,2)$. The case of 2 dimensions (either in the Euclidean or the Minkowskian formalism) is especially interesting. Observe that the above equation on ε implies, when $d = 2$

$$\begin{aligned} &\mu = 1, \nu = 1: \quad 2\partial_1 \varepsilon_1 = (\partial_0 \varepsilon_0 + \partial_1 \varepsilon_1) \Longrightarrow \partial_1 \varepsilon_1 - \partial_0 \varepsilon_0 = 0, \\ &\mu = 0, \nu = 1: \quad \partial_1 \varepsilon_0 + \partial_0 \varepsilon_1 = 0. \end{aligned} \tag{35}$$

These two equations are the Cauchy-Riemann conditions if we introduce the complex variables language

$$\begin{aligned} z = x_0 + ix_1 \quad &; \quad \varepsilon = \varepsilon_0 + i\varepsilon_1 \quad ; \quad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \\ \overline{z} = x_0 - ix_1 \quad &; \quad \overline{\varepsilon} = \varepsilon_0 - i\varepsilon_1 \quad ; \quad \overline{\partial}_{\overline{z}} = \frac{1}{2}(\partial_0 + i\partial_1), \end{aligned} \tag{36}$$

in terms of which the condition on ε (ε) to be a conformal transformation reads

$$\overline{\partial}\varepsilon = 0 = \partial\overline{\varepsilon}, \tag{37}$$

so that ε is a holomorphic function of z and $\overline{\varepsilon}$ is an anti-holomorphic function of $\overline{z}$. Hence, in the complex notation, a transformation of the form

$$z \mapsto z + \varepsilon(z) \quad ; \quad \overline{z} \mapsto \overline{z} + \overline{\varepsilon}(\overline{z}), \tag{38}$$

is a conformal transformation. A conformal transformation on the complex plane is demonstrated in figure 3. One can expand the holomorphic ε in a Laurent series, which means it has infinitely many free parameters. This means, quite interestingly, that in 2d, there are infinitely many independent conformal transformations. This restricts the corresponding theory remarkably as we will develop below.

Let us first write the metric in the $(z, \overline{z})$ coordinates. Since $x_0 = (z + \overline{z})/2$ and $x_1 = (z - \overline{z})/(2i)$, one finds the line element $ds^2 = dx_0^2 + dx_1^2$ as

$$ds^2 = dzd\overline{z}. \tag{39}$$

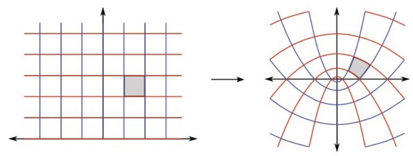

Figure 3: Demonstration of a conformal transformation. Figure taken from [BP09]

Then, under a holomorphic transformation $z \mapsto f(z)$ and its corresponding part in the anti-holomorphic sector $\overline{z} \mapsto \overline{f}(\overline{z})$, the change in the metric reads

$$dz d\overline{z} \to \left| \frac{df}{dz} \right|^2 dz d\overline{z}, \tag{40}$$

since the new metric is proportional to the old one, this can be viewed as a conformal transformation with $\Omega = \left| \frac{df}{dz} \right|^2$. In the case $f = z + \varepsilon$, we can expand ε in Laurent series

$$z \to z - \sum_{n \in \mathbb{Z}} a_n z^{n+1}, \tag{41}$$

and similarly for the anti-holomorphic sector. We would like to find the operators that generate the transformations corresponding to a_n. To do so, we consider the transformation of the field

$$\Phi'(z') = \Phi(z), \tag{42}$$

which is equivalent to

$$\Phi'(z) = \Phi(z - \varepsilon). \tag{43}$$

Now, we choose $\varepsilon = a_n z^{n+1}$ for any integer n, and infinitesimal a, and expand the Taylor series

$$\Phi'(z) = \Phi(z) - a_n z^{n+1} \frac{d}{dz} \Phi(z) + O(a^2). \tag{44}$$

Defining $\delta\Phi = \Phi'(x) - \Phi(x)$, we find that the operator that generates the conformal transformation corresponding to the parameter a_n on the fields Φ is then given by

$$l_n \equiv -z^{n+1} \partial_z \quad ; \quad \bar{l}_n \equiv -\overline{z}^{n+1} \partial_{\overline{z}}, \tag{45}$$

where we read off the anti-holomorphic sector easily. It is easy to find an algebra in the holomorphic sector

$$\begin{aligned} [l_n, l_m] &= z^{n+1} \partial (z^{m+1} \partial) - z^{m+1} \partial (z^{n+1} \partial) \\ &= -(n-m) z^{n+m+1} \partial \\ &= (n-m) l_{n+m}. \end{aligned} \tag{46}$$

It is also easy to see that l_n commutes with any $\bar{l}_m$, independent of the integers n, m. Thus, the complete algebra is

$$[l_n, l_m] = (n-m)l_{n+m} \quad ; \quad [l_n, \bar{l}_m] = 0 \quad ; \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}. \tag{47}$$

This algebra is known as the 2d conformal algebra or the Witt algebra. Observe that as the operators l and $\bar{l}$ are independent, this justifies our usage of z and $\overline{z}$ as if they are independent coordinates. This corresponds to a complexification of the initial space $\mathbb{C} \to \mathbb{C}^2$, where $(x_0, x_1) \in \mathbb{C}$,$(z, \overline{z}) \in \mathbb{C}^2$. One recovers the Euclidean plane in this complexification via $\overline{z} = z^*$, where we identify the anti-holomorphic sector with the complex conjugation of the holomorphic sector.

One can also obtain a Minkwoskian signature metric via $z^* = -z$, which implies $(z, \overline{z}) = i(\tilde{\tau} + \tilde{\sigma}, \tilde{\tau} - \tilde{\sigma})$ with which the metric reads $ds^2 = -d\tilde{\tau}^2 + d\tilde{\sigma}^2$. However, we would like to associate another coordinate to the Minkowski space time. First, by a conformal transformation $z = e^{\zeta}$, $\overline{z} = e^{\overline{\zeta}}$, with $\zeta = \tau + i\sigma$, $\overline{\zeta} = \tau - i\sigma$ from the plane to a cylinder. The coordinate σ is compact because under $\sigma \to \sigma + 2\pi$, e^{ζ} remains the same, and $\tau \in (-\infty, \infty)$. Now we Wick rotate $\tau \to i\tau$ and we get $\zeta \to \zeta^+ = i(\tau + \sigma)$ and $\overline{\zeta} \to \zeta^- = i(\tau - \sigma)$, where $\zeta^\pm$ are the light-cone coordinates in the 2d Minkowski space $ds^2 = d\zeta^+ d\zeta^-$. Moreover, in this case, conformal transformations are reparametrizations of the light-cone coordinates: $\zeta^\pm \to f^\pm(\zeta^\pm)$ that leave the light-cone invariant. Clearly, the Minkowski line element stays the same under these transformations up to $\Omega = f^+(\zeta^+) f^-(\zeta^-)$.

Let us get back to the Witt algebra. There are infinitely many independent generators, hence there are infinitely many symmetries in 2d CFT. Such a symmetry obviously puts severe restrictions on the dynamics of the CFT, which is the main reason 2d CFT is an integrable model. Note that although the local symmetry transformations are infinite, only a few of them are globally defined. To see this, consider the vector fields on the Riemannian sphere $S^2 = \mathbb{C} \cup \{\infty\}$ defined as

$$V(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1} \frac{d}{dz}. \tag{48}$$

Such a vector field will have singularities at $z = 0$ for $n < -1$, so for it to be globally defined we need to consider $n \geq -1$. To investigate the behavior around infinity, we make the conformal transformation $z \to w = -\frac{1}{z}$ so $z \to \infty$ corresponds to $w \to 0$. Using $\frac{d}{dz} = \frac{dw}{dz}\frac{d}{dw} = w^2 \frac{d}{dw}$, we write V in terms of w

$$V(w) = \sum_n a_n (-1)^{n+1} \frac{1}{w^{n+1}} w^2 \frac{d}{dw} = \sum_n a_n (-1)^{n+1} w^{-n+1} \frac{d}{dw}. \tag{49}$$

It is easy to see that, around $w = 0$, V will have singularities for $n > 1$. Hence, we have nailed down the globally defined transformations to those generated by $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. Let us try to understand what these generators correspond to as transformations. We start with $n = -1$

$$l_{-1} = -\frac{d}{dz} \quad ; \quad \bar{l}_{-1} = -\frac{d}{d\overline{z}}. \tag{50}$$

Acting with l_{-1} on Φ with an infinitesimal parameter gives $\delta_{l_{-1}}\Phi = -a_{-1}\frac{d}{dz}\Phi$. This is the transformation under a translation: $\Phi(z - a_{-1}) - \Phi(z) = -a_{-1}\frac{d}{dz}\Phi$, so l_{-1} generates translations.

On the other hand, $l_0 + \bar{l}_0$ generates $\delta\Phi = -\lambda\left(z\frac{d}{dz} + \overline{z}\frac{d}{d\overline{z}}\right)\Phi$, which corresponds to $z \to (1-\lambda)z$ and $\overline{z} \to (1-\lambda)\overline{z}$. If we write $z = re^{i\theta}$, we see that this transformation acts as $r \to (1-\lambda)r$, which is called dilatation. The other combination, $i(l_0 - \bar{l}_0)$, generates $\delta\Phi = -i\alpha\left(z\frac{d}{dz} - \overline{z}\frac{d}{d\overline{z}}\right)\Phi$. To understand what kind of transformation this corresponds to, we change variables again to $z = re^{i\theta}$ and write $\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial\theta}{\partial z}\partial_\theta = e^{-i\theta}\partial_r + (ire^{i\theta})^{-1}\partial_\theta$, and thus $-z\partial_z = -r\partial_r + i\partial_\theta$. Similarly we find $-\overline{z}\partial_{\overline{z}} = -r\partial_r - i\partial_\theta$, so the combination $i(l_0 - \bar{l}_0) = -\partial_\theta$ generates rotations in the plane. This can also be seen from the fact that $z\partial - \overline{z}\overline{\partial}$ is the component of angular momentum $L = -i\mathbf{r}\times\nabla$ that points out of the plane, when we pass to the Euclidean sheet via $\overline{z} = z^*$.

The l_1 operators generate the special conformal transformations. All these globally defined conformal transformations can be written as the transformations of the Möbius group:

$$z \to z' = \frac{az+b}{cz+d} \quad ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{51}$$

That is to say, a, b, c, d are complex parameters that satisfy $ad - bc = 1$. This is precisely the group $SL(2,\mathbb{C})$. We quotient out $\mathbb{Z}_2$ because if we reflect all the parameters $(a, b, c, d) \to (-a, -b, -c, -d)$, then z' will remain the same, so there is a twofold degeneracy. We note that $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$, hence $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(3,1)$, which is the conformal group of 2-dimensional space (in the Euclidean signature).

What we learned is that the local conformal algebra cannot be integrated into a globally defined group, but due to the locality in field theories, the local algebra still has consequences, which we will study. After we quantize a CFT, the physical states in the Hilbert space $\mathcal{H}$ will carry quantum numbers given by the representations of the global conformal algebra. Naturally, we expect a vacuum state $|0\rangle$, which has vanishing quantum numbers and transforms trivially under global transformations. The eigenstates of the operators l_0 and $\bar{l}_0$ will be labeled h and $\overline{h}$, and we call them the conformal weights of a state. Suppose we know the weights of a state. In that case, we can find their scaling dimension Δ and spin s as $\Delta = h + \overline{h}$, $s = h - \overline{h}$, because the operators that generate the corresponding transformations are given respectively by the sum and the difference of the $l_0, \bar{l}_0$ operators.

2.4 The Ward-Takahashi Identity

In a reparametrization invariant theory, which are the ones that are plausible physically, one can derive an important identity for the correlators. These are the quantum manifestations of the symmetries of the classical theory that is quantized. We first recall that under a reparametrization $x \to x' = x + \varepsilon(x)$, we have

$$\begin{aligned} \delta x &= \varepsilon(x), \\ \delta\Phi(x) &= L_\varepsilon \Phi(x), \\ \delta g_{\mu\nu}(x) &= \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \varepsilon^\lambda \partial_\lambda g_{\mu\nu}, \end{aligned} \tag{52}$$

where L_ε is the Lie derivative along the vector ε, compatible with the tensorial character of the field Φ, determined by its spin. Since $g_{\mu\nu}$ is a rank-2 tensor, we have written the explicit form of its change under the reparameterization. We observe that

$$\begin{aligned} \sum_{j=1}^{m} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_m \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \sum_{j=1}^{m} \Phi_1 \cdots \delta\Phi_j \cdots \Phi_m e^{-S[\Phi]} \\ &= \frac{1}{Z} \int \mathcal{D}\Phi \ \delta\Big(\Phi_1 \cdots \Phi_n\Big) e^{-S[\Phi]}, \end{aligned} \tag{53}$$

where $\Phi_j \equiv \Phi(x_j)$, and we use the Leibniz-like behaviour of δ: $\delta(AB) = (\delta A)B + A(\delta B)$. Using this again by taking e^{-S} inside the δ, we get

$$\delta\langle \Phi_1 \cdots \Phi_n \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \bigg(\delta\Big(\Phi_1 \cdots \Phi_n e^{-S[\Phi]}\Big) - \Big(\Phi_1 \cdots \Phi_n\Big) \delta e^{-S[\Phi]} \bigg). \tag{54}$$

The first term is a total variation, hence it is irrelevant after we integrate with $\mathcal{D}\Phi$. In the second term, we need to evaluate δe^{-S}, to do so we will use the chain rule that δ satisfies: $\delta e^{-S} = -(\delta S) e^{-S}$. We now define the energy-momentum tensor as

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{55}$$

Therefore, we have

$$\delta\langle \Phi_1 \cdots \Phi_n \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \ \Phi_1 \cdots \Phi_n \int d^2x \frac{\sqrt{g}}{2} T^{\mu\nu} \delta g_{\mu\nu}. \tag{56}$$

This is for a general space-time with metric g. If we work in the flat space, $g_{\mu\nu} = \delta_{\mu\nu}$, this formula reads

$$\sum_{j=1}^{n} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_n \rangle = \int d^2x \partial_\mu \varepsilon_\nu(x) \langle T^{\mu\nu}(x) \Phi_1 \cdots \Phi_n \rangle, \tag{57}$$

this is the consequence of the parametrization invariance at the quantum level. In particular, this identity means that in any reparameterization invariant field theory, the energy momentum tensor is the object that generates the general coordinate transformations. In flat space, $T^{\mu\nu}$ itself generates translations, and $x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$ generates the Euclidean rotations. Both of them conserved due to $\partial_\mu T^{\mu\nu} = 0$. To see this at the quantum level, integrate by parts to put the derivative ∂_μ on $T^{\mu\nu}$ above, and using the arbitrariness of ε, we get the conservation law for points x that do not coincide with x_j $(j = 1, \cdots n)$. When the coordinate of T and one of x_j coincide, the Ward identity is singular. This tells us that, apart from the points on which an operator is inserted, the classical conservation laws hold inside a correlation function.

The Ward identities hold generically in parametrization invariant field theories. Conformal field theories have a larger class of symmetry, which contains, in addition to the translations and Lorentz rotations (metric isometries), scale transformations which is part of conformal transformations. Under a transformation $x \mapsto \lambda x$, the associated current generating this transformation is given by $x^\mu T_{\mu\nu}$. Conservation of this current means that $0 = \partial_\nu(x^\mu T_{\mu\nu}) = T^\mu_\mu$ so that the energy-momentum tensor must be traceless in a scale-invariant theory. In fact, any current of the form $f^\mu T_{\mu\nu}$ will be conserved if the symmetric part of $\partial^\nu f^\mu$ is proportional to the metric so that the divergence of the current is proportional to the trace of $T_{\mu\nu}$, which is zero in a scale-invariant theory. That is, if for any vector field f^μ satisfying $\partial^\mu f^\nu + \partial^\nu f^\mu = f(x)\eta^{\mu\nu}$ the object $f^\mu T_{\mu\nu}$ will give a conserved current.

On an Euclidean theory parametrized by the coordinates $z, \overline{z}$ with metric $ds^2 = dzd\overline{z}$, the local conservation law $\partial^\mu T_{\mu\nu}$ reads

$$\partial_{\overline{z}} T_{zz} + \partial_z T_{\overline{z}z} = 0 \quad , \quad \partial_z T_{\overline{z}\overline{z}} + \partial_{\overline{z}} T_{z\overline{z}} = 0, \tag{58}$$

and the tracelessness condition gives

$$T_{z\overline{z}} = 0 = T_{\overline{z}z}. \tag{59}$$

These two equations imply that in 2d CFT, the energy-momentum tensor can be decomposed into a holomorphic part and an anti-holomorphic part:

$$T_{zz} \equiv T(z) \quad ; \quad T_{\overline{z}\overline{z}} \equiv \overline{T}(\overline{z}), \tag{60}$$

and the two are related by complex conjugation in the Euclidean sheet. On the Minkowski case, they correspond to the left-moving and right-moving modes, which are independent. With the complex notation, we write the Ward identity as

$$\sum_{j=1}^{n} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_n \rangle = \int d^2 z \partial_{\overline{z}} \varepsilon^z(z) \Big\langle T(z)\Phi_1(z_1, \overline{z}_1) \cdots \Phi_n(z_n, \overline{z}_n) \Big\rangle, \tag{61}$$

where we consider only holomorphic variations, the anti-holomorphic sector follows similarly.

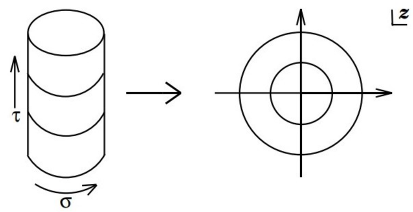

Figure 4: Conformally mapping the cylinder to the complex plane. Figure taken from [Ket95].

2.5 Quantizing CFT on the Complex Plane

We can define the generators L_n in terms of $T(z)$, since $T(z)$ is the object that generates the transformations on the fields. One has

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \tag{62}$$

and the integral is over a contour that contains the points of the fields acted on by L. We can invert this relation to

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{63}$$

using Cauchy's theorem. Now we would like to quantize the CFT. We parametrize the space-time $\mathbb{C}^2$ as a cylinder using the coordinates

$$z = e^{i\zeta} \quad ; \quad \zeta = \tau + i\sigma, \tag{64}$$

with τ and σ the coordinates on the Euclidean plane, and we have the same for the barred sector. Because $e^{2\pi i} = 1$, σ is compactified $\sigma \sim \sigma + 2\pi$. The map from ζ to z represents the mapping from the cylinder to the complex plane as in figure 4, which are conformally equivalent in the sense that the metrics of the two are proportional to each other. The infinite past and the future in the cylinder, $\tau = \mp\infty$, correspond on the plane to the origin and the point at infinity, respectively. Hence, equal radius curves on the plane correspond to equal time slices on the cylinder. Time translations $\tau \to \tau + \lambda$ corresponds to dilatations $z \to e^{\lambda} z$ on $\mathbb{C} \cup \{\infty\}$ and space translations $\sigma \to \sigma + \theta$ are rotations $z \to e^{i\theta} z$. Hence, the Hamiltonian is identified with the generator of dilatation generator, while the space of states is built out of curves of constant radius, and the energy-momentum tensors T and $\overline{T}$ become the generator of conformal transformations on the $z-$plane. To quantize this theory, we will use the radial quantization scheme, in which equal time slices are constant radii curves surrounding the origin on the plane, hence the name. Consider an operator Φ in the quantized theory. According to Heisenberg's equation, the evolution of the operator is given by

$$\frac{d\Phi}{d\tau} = [H, \Phi], \tag{65}$$

where H is the Hamiltonian, and the commutator is taken at equal time. On the plane, one writes

$$\delta_\lambda \Phi = [\lambda H, \Phi], \tag{66}$$

and the Hamiltonian is related to the energy-momentum tensor via

$$H \equiv \oint \frac{dz}{2\pi i} z T(z) = L_0, \tag{67}$$

since it is the generator of dilatations. To see this, recall that for dilatations one has $z \to z + \varepsilon(z)$ with $\varepsilon(z) = \lambda z$, and moreover, for an arbitrary ε, the generator of the conformal transformation is defined as

$$T_\varepsilon \equiv \oint \frac{dz}{2\pi i} \varepsilon(z) T(z), \tag{68}$$

which acts on the fields as $\delta_\varepsilon \Phi = [T_\varepsilon, \Phi]$. Setting $\varepsilon = \lambda z$ we get exactly the same as what we obtained above. This is still only formal and we need to specify the fields inside the integration contour. Including transformations on the barred sector, the general version reads

$$\delta_{\varepsilon,\overline{\varepsilon}} \Phi(w, \overline{w}) = \oint \frac{dz}{2\pi i} \varepsilon(z) [T(z), \Phi(w, \overline{w})] + \text{anti-holomorphic}. \tag{69}$$

Due to the operator ordering issues, this equation still has problems. We will discuss how to cure these ambiguities using similar procedures used in ordinary QFT.

Before that, it is a good idea to define the notion of primary fields. We define a form

$$\Phi \equiv \Phi_{h,\overline{h}}(z, \overline{z}) dz^h d\overline{z}^{\overline{h}}, \tag{70}$$

and if it is a conformal invariant, then the component field $\Phi_{h,\overline{h}}$ is called a primary field of conformal weight $(h, \overline{h})$ (h and $\overline{h}$ are independent, just as z and $\overline{z}$ are viewed as independent). Since we know how dz and $d\overline{z}$ transform under conformal transformations $z \to z'$, we can infer the transformation property of the primary field as

$$\Phi'_{h,\overline{h}}(z', \overline{z}') = \left(\frac{\partial z}{\partial z'}\right)^h \left(\frac{\partial \overline{z}}{\partial \overline{z}'}\right)^{\overline{h}} \Phi_{h,\overline{h}}(z, \overline{z}). \tag{71}$$

The rest of the fields in CFT are called secondary. Writing $z' = z + \varepsilon(z)$, this reads at the infinitesimal level as

$$\delta_{\varepsilon,\overline{\varepsilon}} \Phi(z, \overline{z}) = \Big((h\partial\varepsilon + \varepsilon\partial) + (\overline{h\partial\varepsilon} + \overline{\varepsilon\partial})\Big)\Phi(z, \overline{z}). \tag{72}$$

This will be useful later on. Using the finite transformation property, we can relate the modes of Φ on the cylinder to those on the plane. We take a holomorphic field with weights $(h, 0)$ and expand it on the cylinder

$$\Phi_h(\zeta) = \sum_{n\in\mathbb{Z}} \phi_n e^{-\zeta n}, \quad \zeta = \tau + i\sigma. \tag{73}$$

Using the map $z = e^{\zeta}$, we map these modes to the plane. In doing so, recall that $\frac{\partial z}{\partial \zeta} = z$ so that under the map $\zeta \to z$, we have

$$\Phi_h(z) = \left(\frac{\partial z}{\partial \zeta}\right)^{-h} \Phi_h(\zeta) = z^{-h}\Phi_h(\zeta). \tag{74}$$

Hence, the modes on the plane read

$$\Phi_h(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h}. \tag{75}$$

From (63), we see that $T(z)$ is a primary field of conformal dimension $h = 2$ on the plane.

Now, let us consider the correlation functions and the constraints imposed by conformal symmetry. The two-point function

$$G^{(2)}(z_i, \overline{z}_i) \equiv \langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\rangle \tag{76}$$

is the simplest case. Under a global conformal transformation, G should be invariant, and using the transformation of the fields infinitesimally, we get the differential equation

$$\Big(\varepsilon(z_1)\partial_1 + h_1\partial_1\varepsilon(z_1) + \varepsilon(z_2)\partial_2 + h_2\partial_2\varepsilon(z_2) + \text{anti-holomorphic}\Big)G^{(2)} = 0. \tag{77}$$

Using the translational invariance $\varepsilon = 1 = \overline{\varepsilon}$, scale invariance $\varepsilon = z\ \overline{\varepsilon} = \overline{z}$, and rotational plus special conformal invariance $\varepsilon = z^2\ \overline{\varepsilon} = \overline{z}^2$ on this differential equation, we fix the two-point function up to a constant

$$G^{(2)}(z_i, \overline{z}_i) = \frac{C_{12}}{(z_1 - z_2)^{2h}(\overline{z}_1 - \overline{z}_2)^{2\overline{h}}}, \tag{78}$$

where $h_1 = h = h_2$, and unless $h_1 = h_2$ the correlator vanishes, by conformal invariance. Using similar arguments, the three-point function $G^{(3)} = \langle\Phi_1\Phi_2\Phi_3\rangle$ is found to be

$$G^{(3)}(z_i, \overline{z}_i) = C_{123}\frac{1}{(z_1 - z_2)^{h_1+h_2-h_3}(z_1 - z_3)^{h_1+h_3-h_2}(z_2 - z_3)^{h_2+h_3-h_1}} \times \text{anti-holomorphic}. \tag{79}$$

Thus, conformal invariance alone fixes the 2- and 3-point functions, without referring to a specific theory. The theory will fix the constants C. To have a

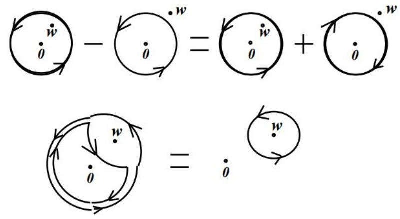

Figure 5: Deforming the contours to evaluate the commutator. Figure taken from [Ket95].

geometrical understanding of why any conformal invariant theory in 2d has the same 2- and 3-point functions, is because any three points on the plane can be mapped to $\infty, 1, 0$ by a complex Möbius transformation, where

$$\lim_{z_1 \to \infty} z_1^{2h_1} \overline{z}_1^{2\overline{h}_1} G^{(3)} = C_{123}. \tag{80}$$

The coefficients C_{123} carry dynamical information of the CFT, and a complete knowledge of all C_{ijk} corresponds to solving the CFT.

Now we get back to the issue of operator ordering. At the quantum level, the transformation of a primary field is given as in (69). To define an equal time commutator, we need a radial ordering, just as we need time ordering in ordinary QFT. To do so we define

$$\begin{aligned}[T_\varepsilon, \Phi(w, \overline{w})] \equiv \lim_{|z| \to |w|} \Bigg(&\oint_{|z|>|w|} \frac{dz}{2\pi i} \varepsilon(z) R\Big(T(z)\Phi(w, \overline{w})\Big) \\ &- \oint_{|w|>|z|} \frac{dz}{2\pi i} \varepsilon(z) R\Big(T(z)\Phi(w, \overline{w})\Big)\Bigg),\end{aligned} \tag{81}$$

where the radial ordering operator R is defined as

$$R\Big(A(z)B(w)\Big) = \theta(|z| - |w|)A(z)B(w) \pm \theta(|w| - |z|)B(w)A(z), \tag{82}$$

with $\theta(x)$ defined to be 1 when $x > 0$ and 0 when $x < 0$. The $\pm$ in between takes into account the statistics, and we have $-$ only when both A and B are fermionic operators.

With these definitions, we can evaluate the integral. First observe that by playing with the contours using Cauchy's theorem, we can write the commutator as

$$[T, \Phi(w, \overline{w})] = \lim_{|z| \to |w|} \oint_{C(w)} \frac{dz}{2\pi i} \varepsilon(z) T(z) \Phi(w, \overline{w}), \tag{83}$$

with $C(w)$ a contour surrounding the point w. For this commutator to be non-zero, there must be a singularity in the operator product

$$\lim_{z \to w} T(z)\Phi(w, \overline{w}), \tag{84}$$

so that its residue contributes to the contour integral.

In general, when two operators are multiplied and the limit at which they coincide is taken, we get singularities. To make sense of such products, we use Wilson's operator product expansion

$$A(z)B(w) \sim \sum_{\Delta} C_{\Delta}(z-w)\mathcal{O}_{\Delta}(w), \tag{85}$$

where $\{\mathcal{O}_{\Delta}(w)\}$ is a complete set of local operators, and $C_{\Delta}(z-w)$ are numerical coefficients which are singular at $z = w$. Such equations are always understood to be valid inside a correlation function, with operator insertions away from the points z, w.

For the contour integral in (83) to give the correct transformation property of a primary field, we introduce the OPE between T and Φ:

$$\begin{aligned} T(z)\Phi(w, \overline{w}) =& \frac{h}{(z-w)^2}\Phi(w, \overline{w}) + \frac{1}{z-w}\partial_w \Phi(w, \overline{w}) \\ &+ \Phi^{(-2)}(w, \overline{w}) + (z-w)\Phi^{(-3)}(w, \overline{w}) + \cdots , \end{aligned} \tag{86}$$

where $\cdots$ represent infinitely many non-singular terms depending on what's called the descendants or secondary fields with respect to a primary field of dimension h. To find the explicit form of a descendant field of level n, $\Phi^{(-n)}$, we use the OPE above to extract the $n-th$ descendant by multiplying both sides with $(z-w)^{-n+1}$ and using reside theorem. Namely, we have

$$\Phi^{(-n)}(w, \overline{w}) = \hat{L}_{-n}(w)\Phi(w, \overline{w}) \equiv \oint_{C(w)} \frac{dz}{2\pi i}(z-w)^{-n+1}T(z)\Phi(w, \overline{w}). \tag{87}$$

The operators $\hat{L}_{-n}(w)$ appear in the expansion of $T(z)$ around the point w:

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{\hat{L}_{-n}(w)}{(z-w)^{n+2}}. \tag{88}$$

Using the residue theorem, one obtains

$$\begin{gathered} \hat{L}_0\Phi(z, \overline{z}) = h\Phi(z, \overline{z}) \quad ; \quad \hat{L}_{-1}\Phi(z, \overline{z}) = \partial\Phi(z, \overline{z}) \\ \hat{L}_n\Phi(z, \overline{z}) = 0 \quad (n \geq 1). \end{gathered} \tag{89}$$

The last equality represents only the first generation of descendants, obtained from the product of Φ with a single T. We can insert more energy-momentum tensors $T \cdots T\Phi$ and consider their descendants. This way, one obtains the entire spectrum of the CFT. There exists an infinite family of descendant fields, obtained by acting $\hat{L}_{-n}$ operators on Φ several times:

$$\Phi_h^{(-\vec{k})}(z) \equiv \hat{L}_{-n_1} \cdots \hat{L}_{-n_k} \Phi_h(z) \quad ; \quad \vec{k} = (n_1, \cdots n_k). \tag{90}$$

These operators are classified by their eigenvalue under the $\hat{L}_0$ operator:

$$\hat{L}_0 \Phi_h^{(-\vec{k})}(z) = (h + |\vec{k}|)\Phi_h^{(-k)} \quad ; \quad |\vec{k}| = \sum_{j=1}^{k} n_j. \tag{91}$$

Note that the descendants do not necessarily transform properly under conformal transformations, unlike primaries. This can be seen from the fact that $\hat{L}_n \Phi \neq 0$ $(n > 0)$ for a descendant field.

It is convenient to introduce a short-hand for the OPE notation, written as

$$T(z)\Phi(w) \sim \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w), \tag{92}$$

and similarly for the barred sector. When considering the correlation function, only the terms that we wrote above give a contribution, the descendants giving 0 hence they are not written. Given a set of primary fields $\{\Phi_i\}$, one conveniently normalizes them so that their 2-point function reads

$$\langle \Phi_i(z,\overline{z})\Phi_j(w,\overline{w})\rangle = \delta_{ij}\frac{1}{(z-w)^{2h_i}} \times \text{anti-holomorphic}. \tag{93}$$

And the OPE between them has the following form

$$\Phi_i(z,\overline{z})\Phi_j(z,\overline{z}) \sim \sum_k C_{ij}{}^k (z-w)^{h_k - h_i - h_j} (\overline{z} - \overline{w})^{\overline{h}_k - \overline{h}_i - \overline{h}_j} \Phi_k(w,\overline{w}), \tag{94}$$

which can be obtained from the coincidence limit of the 3-point function [DFMS97].

The OPEs between T and Φ and Φ between Φ contain all the information about a CFT, and determining them solves the theory. From the OPE $T\Phi$, we can find the commutation relation between the modes of T -which are the L_n

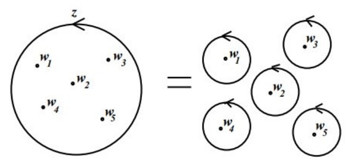

Figure 6: The deformation of the contour appearing in the conformal Ward identity. Figure taken from [Ket95]

operators, and those of Φ -which are denoted ϕ_n. We have

$$\begin{aligned}
[L_n, \phi_m] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+h-1} T(z)\Phi(w) \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{h}{(z-w)^2}\Phi + \frac{1}{z-w}\partial_w \Phi(w) \right) \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \left(h \frac{d}{dz}(z^{m+1})\big|_{z=w} \Phi(w) + w^{m+1}\partial_w \Phi(w) \right) \\
&= \oint \frac{dw}{2\pi i} \big(h(m+1) w^{m+n+h-1}\Phi(w) + w^{m+n+h}\partial_w \Phi(w) \big) \\
&= \oint \frac{dw}{2\pi i} \Big(h(m+1) w^{m+n+h-1} - (m+n+h) w^{m+n+h-1} \Big) \Phi(w) \\
&= h(m+1)\phi_{m+n} - (m+n+h)\phi_{m+n} \\
&= (hm - m - n)\phi_{m+n},
\end{aligned} \tag{95}$$

where we used the contour integral formula

$$\oint \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^{n+1}} = \frac{1}{n!} \left(\frac{d^n f(z)}{dz^n} \right) \Big|_{z=w}, \tag{96}$$

and the fact that $\oint dw w^{m+n+h-1} \sim \delta_{m+n+h,0}$ hence

$$\begin{aligned}
\oint \frac{dw}{2\pi i} w^{m+n+h-1}\Phi(w) &= \sum_{k\in\mathbb{Z}} \oint \frac{dw}{2\pi i} w^{m+n-k-1}\phi_k \\
&= \sum_{k\in\mathbb{Z}} \delta_{m+n-k,0}\phi_k \\
&= \phi_{n+m}.
\end{aligned} \tag{97}$$

One can also calculate the correlator of T with an arbitrary number of primaries

Φ. To do so we integrate $\varepsilon(z)T(z)$ on a contour as in figure 6, we get

$$\begin{aligned}
&\left\langle \oint_{C(0)} \frac{dz}{2\pi i}\varepsilon(z)T(z)\Phi_1(w_1)\cdots\Phi_m(w_m)\right\rangle \\
&\quad = \sum_{j=1}^{m}\left\langle \Phi_1(w_1)\cdots\left(\oint_{C(w_j)}\frac{dz}{2\pi i}\varepsilon(z)T(z)\Phi_j(w_j)\right)\cdots\Phi_m(w_m)\right\rangle \\
&\quad = \sum_{j=1}^{m}\left\langle \cdots\left(\oint_{C(w_j)}\frac{dz}{2\pi i}\varepsilon(z)\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_j\right]\right)\Phi_j(w_j)\cdots\right\rangle ,
\end{aligned} \tag{98}$$

and taking out the integral over z and the parameter $\varepsilon(z)$ since the equality holds for arbitrary ε and for all z, we get the conformal Ward identity

$$\langle T(z)\Phi_1(w_1)\cdots\rangle = \sum_{j=1}^{m}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right]\langle\Phi_1(w_1)\cdots\rangle. \tag{99}$$

From this equation, we read that correlation functions are also meromorphic functions with singularities at operator insertions, whose residues are determined by the conformal transformation properties of the operators. Moreover, this is the local version of the conformal Ward identity, which is a fundamental equation in CFT and in String Theory. The OPE's for products of primary fields with the stress tensor is equivalent to the Ward identity in CFT.

Finding the correlators for primaries is not enough, as they are not the complete basis in the space of fields. To get all the correlators, we need to account for the descendants, which is a relatively easy task. This is because the dynamics of descendants are determined from the dynamics of the primaries, hence a knowledge of the correlators of Φs with T is precisely a solution to that problem. Specifically, consider a correlator with all primaries and one descendant Φ^{-n}, which has the form

$$\begin{aligned}
&\langle\cdots\hat{L}_{-n}\Phi_j(w_j)\cdots\rangle = \oint_{C(w_j)}\frac{dz}{2\pi i}(z-w_j)^{-n+1}\langle\cdots T(z)\Phi_j(w_j)\cdots\rangle \\
&\quad = \oint_{C(w_j)}\frac{dz}{2\pi i}(z-w_j)^{-n+1}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right]\langle\cdots\Phi_j(w_j)\cdots\rangle.
\end{aligned} \tag{100}$$

For a correlator with several descendants, we will have several such integrals, one for each descendant. But, in principle, all the correlators can be found via this procedure, just from the knowledge of correlators from primaries.

2.6 The Virasoro Algebra

From dimensional reasons and analyticity, the OPE TT has the general form

$$T(z)T(w) = \frac{c/2}{(z-w)^4}+\frac{2}{(z-w)^2}T(w)+\frac{1}{z-w}\partial_w T(w)+\cdots , \tag{101}$$

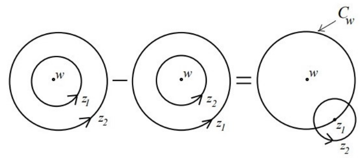

Figure 7: The contour for the difference of $\hat{L}_n\hat{L}_m$ and $\hat{L}_m\hat{L}_n$ acting on $\Phi(w)$. Figure taken from [Ket95]

with $\cdots$ representing the descendants, which are obtained by applying $\hat{L}_{-n}$ on $T(W)$, multiplied by a factor of $(z-w)^{n-2}$. The constant c is called the central charge of the theory, and the easiest way to isolate the coefficient is by looking at the two-point function

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}. \tag{102}$$

This can be justified from the general form of the two-point function. The constant c depends on the CFT on which T is computed, and is related to the conformal anomaly. We write the OPE between two Ts as

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w). \tag{103}$$

We read off from this that T is almost a primary field of conformal dimension $h=2$. If $c=0$, then T is honestly a primary field, which is the classical case. That $c \neq 0$ is a purely quantum effect.

Given the OPE, we can find an algebra between the modes $\hat{L}$ by considering

$$[\hat{L}_n, \hat{L}_m]\Phi(w), \tag{104}$$

to evaluate this, we separate the commutator with choice of contour as in figure 7, and evaluate

$$\hat{L}_n\hat{L}_m\Phi(w) = \oint_{C_2(w)} \frac{dz_2}{2\pi i}(z_2-w)^{n+1}\oint_{C_1(w)}\frac{dz_1}{2\pi i}(z_1-w)^{m+1}T(z_1)T(z_2)\Phi(w), \tag{105}$$

where $|C_1(w)| < |C_2(w)|$, which is why $z_1 - w$ goes with the power of $m+1$, because $\hat{L}_m$ is the first mode to act on Φ, which corresponds to the contour having smaller radius. For the second term of the commutator, we have

$$\hat{L}_m\hat{L}_n\Phi(w) = \oint_{C_1(w)} \frac{dz_1}{2\pi i}(z_1-w)^{m+1}\oint_{C_2(w)}\frac{dz_2}{2\pi i}(z_2-w)^{n+1}T(z_2)T(z_1)\Phi(w), \tag{106}$$

and here $|C_2(w)| < |C_1(w)|$. Taking the difference, we obtain

$$[\hat{L}_n, \hat{L}_m]\Phi(w) = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1-w)^{m+1}\oint_{C(z_1)}\frac{dz_2}{2\pi i}(z_2-w)^{n+1}T(z_2)T(z_1)\Phi(w). \tag{107}$$

Inserting the OPE TT into this, we get

$$[\hat{L}_n, \hat{L}_m] = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \\ \times \left(\frac{c/2}{(z_2 - z_1)^4} + \frac{2}{(z_2 - z_1)^2} T(z_1) + \frac{1}{z_2 - z_1} \partial_{z_1} T(z_1) \right). \tag{108}$$

Now, to take the dz_2 integral, we will go term by term inside the OPE. The first term is of the form

$$\begin{aligned} \oint_{C(z_1)} \frac{dz_2}{2\pi i} \frac{f(z_2; w)}{(z_1 - z_2)^4} &= \frac{1}{3!} \left(\frac{d^3}{dz_2^3} f(z_2; w) \right) \Big|_{z_2 = z_1} \\ &= \frac{c}{12}(n+1)n(n-1)(z_1 - w)^{n-2}, \end{aligned} \tag{109}$$

where $f(z_2; w) = \frac{c}{2}(z_2 - w)^{n+1}$. Now, we take the integral of dz_1 over the contour $C(w)$, which is of the form

$$\frac{c}{12}(n^3 - n) \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n-1} = \frac{c}{12}(n^3 - n)\delta_{m+n,0}. \tag{110}$$

For the second term in the large parenthesis, we have

$$\begin{aligned} &\oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \frac{2T(z_1)}{(z_2 - z_1)^2} \\ &= \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1}(n+1)(z_1 - w)^n 2T(z_1) \\ &= 2(n+1) \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n+1} T(z_1) \\ &= 2(n+1)\hat{L}_{n+m}, \end{aligned} \tag{111}$$

and finally, for the last term

$$\begin{aligned} &\oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \frac{1}{z_2 - z_1} \partial_{z_1} T(z_1) \\ &= \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n+2} \partial_{z_1} T(z_1) \\ &= -\oint_{C(w)} \frac{dz_1}{2\pi i}(m+n+2)(z_1 - w)^{m+n+1} T(z_1) + \text{boundary term} \\ &= -(m+n+2)\hat{L}_{m+n}, \end{aligned} \tag{112}$$

where we integrated by parts. Combining all three parts, we finally obtain the Virasoro algebra

$$[\hat{L}_n, \hat{L}_m] = (n - m)\hat{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{113}$$

The Virasoro algebra is the central extension of the Witt algebra. The operators with a hat means that they are expanded around some point, in the above case w. If we set w to be the origin, we recover the L_n operators: $\hat{L}_n(0) = L_n$, and these operators satisfy the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{114}$$

One can show that this is the unique central extension of the Witt algebra, up to additive constant redefinitions. To see this, consider an arbitrary extension

$$[L_n, L_m] = (n-m)L_{n+m} + c_{n,m}, \tag{115}$$

with $c_{n,m}$ coefficients anti-symmetric under exchange of n and m. For this algebra to satisfy the Jacobi identity, we must have

$$[L_n, [L_m, L_k]] + [L_k, [L_n, L_m]] + [L_m, [L_k, L_n]] = 0. \tag{116}$$

Explicitly, the terms are of the form

$$\begin{aligned} [L_n, [L_m, L_k]] &= [L_n, (m-k)L_{m+k} + c_{m,k}] \\ &= (m-k)(n-m-k)L_{n+m+k} + (m-k)c_{n,m+k}, \\ [L_k, [L_n, L_m]] &= (n-m)(k-n-m)L_{n+m+k} + (n-m)c_{k,n+m}, \\ [L_m, [L_k, L_n]] &= (k-n)(m-k-n)L_{n+m+k} + (k-n)c_{m,n+k}. \end{aligned} \tag{117}$$

When we sum the three terms, the coefficient of L_{n+m+k} will be identically 0, which is no surprise as the Witt algebra has a Lie algebra structure. For the central extension to satisfy the Jacobi identity, the objects $c_{n,m}$ must satisfy

$$(m-k)c_{n,m+k} + (n-m)c_{k,n+m} + (k-n)c_{m,n+k} = 0. \tag{118}$$

It is not immediately obvious what is the solution to this equation, but inserting

$$c_{n,m} = \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{119}$$

one sees that the Jacobi identity is satisfied identically.

Observe that for $n = -1, 0, 1$, the central charge term does not appear in the algebra because they are the roots of $n^3 - n = 0$. This reflects the fact that the globally defined transformations, generated by $\{L_{-1}, L_0, L_1\}$, reflects the invariance of the ground state under $SL(2, \mathbb{C})$, since $SL(2, \mathbb{C})$ remains the exact global symmetry of the quantized CFT. In particular, $T(z)$ is still a primary field of dimension $h = 2$ with respect to the global conformal group.

Out of the OPE between two Ts, we can extract the infinitesimal transformation of T under conformal transformations

$$\begin{aligned}
\delta_\varepsilon T &= [T_\varepsilon, T] \\
&= \oint_{C(w)} dz \varepsilon(z) T(z) T(w) \\
&= \oint_{C(w)} dz \varepsilon(z) \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) \right) \\
&= \frac{c}{12} \partial_w^3 \varepsilon(w) + 2\partial_w \varepsilon(w) T(w) + \varepsilon(w) \partial_w T(w) \\
&= \Big(\varepsilon(w) \partial_w + 2\partial_w \varepsilon(w) \Big) T(w) + \frac{c}{12} \partial_w^3 \varepsilon(w).
\end{aligned} \tag{120}$$

We can integrate this to a finite transformation of $T(w)$ from w to $z = f(w)$

$$T(w) \to T'(z) = \left(\frac{dw}{dz} \right)^2 T(w) + \frac{c}{12} S[f; w], \tag{121}$$

where the Schwartzian derivative is defined as

$$S[f; w] = \Big(\partial_w f \partial_w^3 f - \frac{3}{2} (\partial_w^2 f)^2 \Big) / (\partial_w f)^2. \tag{122}$$

In particular, for the map from the cylinder w to the plane $z = e^w$, we have

$$\begin{aligned}
T_{\text{cylinder}}(w) \to T_{\text{plane}}(z) &= \left(\frac{dw}{dz} \right)^2 T_{\text{cylinder}}(w) + \frac{c}{12} S[z; w] \\
&= z^{-2} T_{\text{cylinder}}(w) - \frac{c}{24}.
\end{aligned} \tag{123}$$

Hence, L_0 on the plane and L_0 on the cylinder are related by $\frac{c}{24}$.

2.7 The Free Boson

The free bosonic CFT has the following action

$$S = \frac{1}{4\pi} \int d^2 z \partial \Phi \overline{\partial} \Phi. \tag{124}$$

The field Φ must have 0 conformal weight for the action to be invariant. The variation of the action gives the equations of motion

$$\partial \overline{\partial} \Phi = 0. \tag{125}$$

From this we can find the two-point function as

$$\langle \Phi(z, \overline{z}) \Phi(w, \overline{w}) \rangle = -\ln |z - w|^2, \tag{126}$$

so that $\partial \overline{\partial}$ gives a result proportional to $\delta^2(z - w)$. To see this, we use $\partial \ln |z - w|^2 = \frac{1}{z-w}$ and the identity $\overline{\partial}_z \frac{1}{z-w} = \delta^2(z - w)$.

From the equations of motion, one can define (anti-)holomorphic currents

$$j = \partial\Phi \quad ; \quad \overline{j} = \overline{\partial}\Phi, \tag{127}$$

and by the equations of motion $\overline{\partial}j = 0 = \partial\overline{j}$. One can think of j as a primary field of conformal dimension 1, since Φ has conformal dimension 0 and ∂ brings a conformal dimension 1. One other way to see this is to consider the expansion

$$\Phi = \sum_{n\in\mathbb{Z}} \phi_n z^{-n}, \tag{128}$$

the derivative of which gives

$$j = \partial\Phi = \sum_{n\in\mathbb{Z}} (-n)\phi_n z^{-n-1}, \tag{129}$$

and this is precisely the expansion of a primary field of dimension $h = 1$. A similar statement for $\overline{j}$ holds.

Now, observe that if we split Φ into ϕ and $\overline{\phi}$, we can write the two-point function as

$$\langle\phi(z)\phi(w)\rangle = \ln(z-w) \quad ; \quad \langle\overline{\phi}(\overline{z})\overline{\phi}(\overline{w})\rangle = \ln(\overline{z}-\overline{w}). \tag{130}$$

With this, we can also obtain the correlator between j with itself by

$$\langle j(z)j(w)\rangle = \langle\partial\phi(z)\partial\phi(w)\rangle = -\frac{1}{(z-w)^2}, \tag{131}$$

and the OPE

$$j(z)j(w) = -\frac{1}{(z-w)^2} + \cdots, \tag{132}$$

where $\cdots$ represents non-singular terms that do not contribute to the 2-point function.

From the action, we find the holomorphic part of the energy-momentum tensor to be

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) :\equiv -\frac{1}{2}\lim_{w\to z}\left(\partial\phi(z)\partial\phi(w) + \frac{1}{(z-w)^2}\right), \tag{133}$$

where $: A :$ denotes the normal ordering used to remove divergences, and the reason for $1/(z-w)^2$ in the limit is because the divergence of $\langle jj\rangle$ goes like $-1/(z-w)^2$ and the normal ordering's purpose is to remove that divergence at the coincidence limit.

Given that j has conformal dimension 1 and is a primary field, we see that it has the following OPE with T:

$$T(z)j(w) \sim \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}\partial_w j(w), \tag{134}$$

or, in terms of ϕ:

$$T(z)\partial\phi(w) \sim \frac{1}{(z-w)^2}\partial_w\phi(w) + \frac{1}{z-w}\partial_w^2\phi(w). \tag{135}$$

One can define Wick rules for contractions when there is a normal ordered product $:A::B:$. Using that, one could find that for the scalar field CFT the central charge is 1 because of the following OPE

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w). \tag{136}$$

2.8 Representations of the Virasoro Algebra

In the previous section, we saw that the entire spectrum of CFT can be obtained from the primary fields $\{\phi_p(z,\overline{z})\}$ and their descendants $\{\phi_p^{(\vec{k},\vec{\overline{k}})}(z,\overline{z})\}$ which are defined as the actions of certain set of $\hat{L}_{-n}$ operators on the primaries. The index p is over all the primary fields in the theory $p = \{h, \overline{h}\}$.

The vacuum $|0\rangle$ is defined to be $SL(2,\mathbb{C})$ invariant, and it can be viewed as an insertion of the identity operator at the infinite past (the origin on the plane). Moreover, since L_n ($n \geq -1$) have no poles at $z = 0$, upon acting on $|0\rangle$ they will give zero

$$L_n|0\rangle = 0, \quad n \geq -1. \tag{137}$$

On the other hand, acting L_{-n} with $n > 2$ on the vacuum, we get non-trivial states.

In the radial quantization scheme, we naturally define asymptotic in-states that act on the vacuum at the origin of the plane as follows

$$|\Phi_{\text{in}}\rangle = \lim_{z,\overline{z}\to 0} \Phi(z,\overline{z})|0\rangle. \tag{138}$$

To define an out-state, we make the transformation $z = 1/w$ and take the limit $w \to \infty$:

$$\langle\Phi_{\text{out}}| = \lim_{w,\overline{w}\to 0} \langle 0|\Phi'(w,\overline{w}). \tag{139}$$

We need to relate $\Phi(z,\overline{z})$ with $\Phi'(w,\overline{w})$. For primary fields, the transformation law of the fields under $w \to z = 1/w$ reads

$$\Phi'(w,\overline{w}) = (-w^{-2})^h(-\overline{w}^{-2})^{\overline{h}}\Phi(1/w,1/\overline{w}) = z^{2h}\overline{z}^{2\overline{h}}\Phi(z,\overline{z}), \tag{140}$$

so we define the out-states as

$$\langle\Phi_{\text{out}}| = \lim_{z,\overline{z}\to\infty} \langle 0|\Phi(z,\overline{z})z^{2h}\overline{z}^{2\overline{h}}. \tag{141}$$

This motivates us to define the adjoint of a field as

$$\Phi^\dagger(z,\overline{z}) \equiv \Phi\left(\frac{1}{z},\frac{1}{\overline{z}}\right) z^{-2h}\overline{z}^{-2\overline{h}}, \tag{142}$$

so that we have the relation

$$\langle\Phi_{\text{out}}| = \lim_{w,\overline{w}\to 0}\langle 0|\Phi(1/z,1/\overline{z})z^{-2h}\overline{z}^{-2\overline{h}} = \lim_{z,\overline{z}\to 0}\langle 0|\Phi^\dagger(z,\overline{z}) = |\Phi_{\text{in}}\rangle^\dagger. \tag{143}$$

Let us consider the $T(z)$ operator. In terms of the modes one has

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{144}$$

and upon conjugation

$$T^\dagger(z) = z^{-4}T(1/z) = z^{-4}\sum_{m\in\mathbb{Z}} L_m^\dagger z^{m+2} = \sum_{m\in\mathbb{Z}} L_m^\dagger z^{m-2}. \tag{145}$$

Compared with the expansion of $T(z)$, we define the conjugation of the modes of the energy-momentum tensor as $L_n^\dagger = L_{-n}$. A consequence of this definition is that c is a positive number for a Hilbert space of positive-definite inner product. To see this, observe that due to the positive-definiteness of the inner product,

$$\langle 0|[L_2,L_{-2}]|0\rangle = \langle 0|L_2L_{-2} - \underbrace{L_{-2}L_2|0\rangle}_{=0} = \langle 0|L_2L_2^\dagger|0\rangle = ||L_2^\dagger|0\rangle||^2 \geq 0. \tag{146}$$

On the other hand, we know that $[L_2,L_{-2}] = 4L_0 + \frac{c}{2}$ and since $L_0|0\rangle = 0$, we have

$$\frac{c}{2} = \langle 0|[L_2,L_{-2}]|0\rangle \geq 0. \tag{147}$$

We now define the primary states. These are simply the states obtained by acting a primary field on the vacuum at the origin of the plane. This is the idea of operator-state correspondence in CFT, there is a one-to-one correspondence between primary fields and primary states. Using the L_{-n} ($n > 1$) operators, we can generate the rest of the fields, the descendants. One denotes the primary states as $|h,\overline{h}\rangle$ which is given by

$$|h,\overline{h}\rangle = \lim_{z,\overline{z}\to 0}\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle. \tag{148}$$

Using the OPE between T and a primary field Φ, we can compute the following commutator

$$\begin{aligned}
[L_n,\Phi_{h,\overline{h}}(z,\overline{z})] &= \oint_{C(z)}\frac{dw}{2\pi i}w^{n+1}T(w)\Phi_{h,\overline{h}}(z,\overline{z}) \\
&= \oint_{C(z)}\frac{dw}{2\pi i}w^{n+1}\left(\frac{h}{(w-z)^2}\Phi_{h,\overline{h}}(z,\overline{z}) + \frac{1}{w-z}\partial_z\Phi_{h,\overline{h}}(z,\overline{z})\right) \\
&= \left((n+1)w^n h\Phi_{h,\overline{h}}(z,\overline{z})\right)\Big|_{z=w} + \left(w^{n+1}\partial_z\Phi_{h,\overline{h}}(z,\overline{z})\right)\Big|_{z=0} \\
&= \left(z^{n+1}\frac{d}{dz} + (n+1)z^n h\right)\Phi_{h,\overline{h}}(z,\overline{z}).
\end{aligned} \tag{149}$$

Now, using the fact that $L_n|0\rangle = 0$ for $n \geq -1$, we obtain

$$\begin{aligned} L_n|h,\overline{h}\rangle &= \lim_{z,\overline{z}\to 0}[L_n, \Phi_{h,\overline{h}}(z,\overline{z})]|0\rangle \\ &= \lim_{z,\overline{z}\to 0}\left(z^{n+1}\frac{d}{dz} + (n+1)z^n h\right)\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle = 0, \quad n \geq -1, \end{aligned} \tag{150}$$

in particular, for the $n = 0$ case

$$\begin{aligned} L_0|h,\overline{h}\rangle = \lim_{z,\overline{z}\to 0}[L_0, \Phi_{h,\overline{h}}(z,\overline{z})]|0\rangle &= \lim_{z,\overline{z}\to 0}\left(z\frac{d}{dz} + h\right)\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle \\ &= h|h,\overline{h}\rangle, \end{aligned} \tag{151}$$

and similarly for $\overline{L}_n$ operators. We will call the states $|h,\overline{h}\rangle$ the highest-weight states. The reason for this naming will be clear shortly. Here we note that the condition $L_n|h\rangle = 0$ for $n > 0$ follows for all n if we ensure $L_1|h\rangle = 0 = L_2|h\rangle$, since the rest of L_ns can be generated by the Virasoro algebra.

The relevant algebra of the quantum CFT is the Virasoro algebra. Hence, the Hilbert space can be constructed as the irreps of Virasoro algebra, which we write as

$$\mathcal{H} = \sum_{h,\overline{h}} \mathsf{Vir}(h,c) \otimes \overline{\mathsf{Vir}}(h,c), \tag{152}$$

where Vir is called the Verma modules, h is the conformal weight of primary fields and c is the central extension. The Verma modules depend only on these two numbers, which must be positive as we will discuss.

Thus, to obtain the Hilbert space corresponding to the CFT, we need to look at the representations of the Virasoro algebra, the symmetry algebra of the theory. This is in parallel with the case of quantum mechanics of a particle with a spherically symmetric Hamiltonian, where the Hilbert space is associated with the irreducible representations of the symmetry group $\mathfrak{su}(2)$ (we ignore the part of the Hilbert space associated to the motion of the particle, which in simple cases is the space of integrable smooth functions like $L^2(\mathbb{R})$ if the particle moves on the real line). The particle's spin labels the irreps of this group, and within the spin j sector, there are $2j+1$ states. To see this we recall that $\mathfrak{su}(2)$ has 3 generators and by defining $J_0 \equiv J_3$, $J_\pm \equiv J_x \pm iJ_y$ we get the algebra

$$[J_0, J_\pm] = \pm J_\pm \quad ; \quad [J_+, J_-] = 2J_0, \tag{153}$$

with the operators $J_\pm$ called raising-lowering operators. We assume that in a given spin sector, there is a maximum eigenvalue m of J_0 such that

$$J_0|j\rangle = j|j\rangle \quad ; \quad J_+|j\rangle = 0. \tag{154}$$

To get the other eigenvalues of J_0 in that spin sector, we act on the highest-weight state with the lowering operator

$$|m\rangle = (J_-)^{j-m}|j\rangle. \tag{155}$$

The inner products of these states can be found using

$$\begin{aligned}
\langle m-1|m-1\rangle = \langle m|J_-^\dagger J_-|m\rangle &= \langle m|J_+J_-|m\rangle \\
&= \langle m|J_x^2 - i[J_x,J_y] + J_y^2|m\rangle \\
&= \langle m|J_x^2 + J_y^2 + J_z^2 - J_z^2 + J_z|m\rangle \\
&= \langle m|J^2 - J_z(J_z-1)|m\rangle \\
&= \Big(j(j+1) - m(m-1)\Big)\langle m|m\rangle,
\end{aligned} \tag{156}$$

where we used the fact that $J^2 = J_x^2 + J_y^2 + J_z^2$ is a Casimir operator, namely an operator that satisfies $[J_i, J^2] = 0$, and the spin j of an irreps is related to the eigenvalue of J^2 as $j(j+1)$. What we see from the above computation is that when m is small enough, the inner product $\langle m-1|m-1\rangle$ becomes negative, rendering the representation non-unitary. This situation is remedied only if j is either an integer or a half-integer. In that case, the state $|-j-1\rangle$ has norm $j(j+1)-(-j)(-j-1) = j(j+1)-j(j+1) = 0$, and hence all the other operators obtained by acting J_- on $|-j-1\rangle$ has zero norm as well. These singular vectors are said to decouple from the first $(2j+1)$ states because any operator A constructed from J_is will have vanishing matrix element $\langle m|A|m'\rangle$ when m' is a singular vector. This is because when A is expressed in terms of $J_\pm$ and J_0, the matrix element will be proportional to $\langle m|\rangle m'$, which is zero since the two are orthogonal. Hence, the representation space has dimension $2j+1$, and the vector space corresponding to spin j is spanned by the vectors in the orthogonal set $\{|m\rangle \,|-j \leq m \leq j\}$, which gives a unitary finite-dimensional representation of the Lie algebra $\mathfrak{su}(2)$.

After this brief review, we get back to the CFT case. Observe that

$$[L_0, L_{-m}] = mL_{-m}, \quad m > 0 \tag{157}$$

mimics the commutation of the raising operator in analogy with $\mathfrak{su}(2)$ case and

$$[L_0, L_m] = -mL_m, \quad L_m|h\rangle = 0, \quad m > 0 \tag{158}$$

mimics the properties of the lowering operator with $|h\rangle$ a highest-weight state. The analogy is made even further by noting that $L_n^\dagger = L_{-n}$. Then, we can think of the descendant operators as operators obtained by acting with the raising operators, of which there are infinitely many which is the difference of this situation to $\mathfrak{su}(2)$ since there is only one independent raising operator in $\mathfrak{su}(2)$. The descendant operators

$$\Phi_p^{(-\vec{k},-\vec{\bar{k}})} \equiv L_{-k_1}\cdots L_{-k_m}\overline{L}_{-\bar{k}_1}\cdots\overline{L}_{-\bar{k}_{m'}}\Phi_p(0,0) \tag{159}$$

give corresponding descendant states, which have different eigenvalue under L_0, just as $|m\rangle$ states have different J_z eigenvalue for different m. A simple descendant state has the form $L_{-n}|h\rangle = L_{-n}(\Phi(0)|0\rangle) = (\hat{L}_{-n}\Phi)(0)|0\rangle = \Phi^{(-n)}(0)|0\rangle$.

A highest-weight representation of the Virasoro algebra gives us a Verma module. As there are infinitely many ladder operators, the representation is infinite-dimensional and is completely characterized by its central charge and the dimension of the highest-weight state. The appearance of the dimensionality is present in the $\mathfrak{su}(2)$ case and the representation is completely characterized by the spin, but in the present context, we also need the central charge. So, the construction of the Hilbert space (the Verma module) in CFT corresponds to finding unitary representations of the Virasoro algebra, and due to the conformal symmetry, one can organize the states neatly starting from the primaries and algorithmically obtaining the descendants.

Algebra	$\mathfrak{su}(2)$	Vir
Grading Operator	J_0	L_0
Ladder Operator(s)	$J_{\pm}$	$L_{\mp n}$
Conjugation	$J_{\pm}^{\dagger} = J_{\mp}$	$L_{\mp n}^{\dagger} = L_{\pm n}$
Highest-weight state	$\lvert j\rangle$	$\lvert h\rangle$

Table 1: The analogies between the irreps of $\mathfrak{su}(2)$ and that of Vir.

From the requirement that the Verma module has a positive-definite inner product, which is the requirement for it to be a unitary representation, we can constrain the values of h and c. We consider

$$0 < \langle h|L_{-n}^{\dagger}L_{-n}|h\rangle = \langle h|[L_n, L_{-n}]|h\rangle = \Big(2nh + \frac{c}{12}(n^3 - n)\Big)\langle h|h\rangle. \tag{160}$$

Setting $n = 1$, we get the condition $h > 0$. To see why c must be positive if the Verma module is to admit a positive-definite inner product, we consider the case where n is large. Then, the $2nh$ term and $-\frac{c}{12}n$ terms are negligible compared to $\frac{c}{12}n^3$ term, hence we have

$$0 < \langle h|L_{-N}^{\dagger}L_{-N}|h\rangle \sim \frac{c}{12}N^3\langle h|h\rangle, \quad N \gg 1. \tag{161}$$

If c is a negative number, then the inner product of $L_{-N}|h\rangle$ in the module will be negative, and if it is 0 then a non-zero vector will have zero norm which is against the condition of having a positive-definite inner product, where only the zero vectors have zero norm.

Observe that one has $h = 0$ only if $L_{-1}|h\rangle = 0$, that is, if $|h\rangle = |0\rangle$. And from (150), we see that $[L_{-1}, \Phi] = \partial\Phi \quad ; \quad [\overline{L}_{-1}, \Phi] = \overline{\partial}\Phi$. In particular, a primary field of dimension $(h, 0)$ has dependence only on z and is independent of $\overline{z}$.

2.9 The Correlation Functions of CFT

Given the correlators of primaries, we can find those of descendants, as we discussed before. We write the complete OPE between two primary fields as

$$\begin{aligned}\Phi_n(z,\overline{z})\Phi_m(w,\overline{w}) = \sum_p \sum_{\vec{k},\vec{\overline{k}}} & C_{nm}^{p(-\vec{k},-\vec{\overline{k}})} z^{h_p-h_n-h_m+|\vec{k}|} \\ & \times \overline{z}^{\overline{h}_p-\overline{h}_n-\overline{h}_m+|\vec{\overline{k}}|} \Phi_p^{(-\vec{k},-\vec{\overline{k}})}(w,w),\end{aligned} \tag{162}$$

where $C_{nm}^{p(-\vec{k},-\vec{\overline{k}})}$ are some coefficients. This OPE means that there is no new operator beyond the primaries and their descendants in a CFT. For convenience, since the holomorphic and anti-holomorphic sectors always separate, we will write only the holomorphic sector and we will write $\phi(z)$ to mean the holomorphic part of $\Phi(z,\overline{z})$. With this convention, the OPE reads

$$\phi_n(z)\phi_m(w) = \sum_p \sum_{\vec{k}} C_{nm}^{p(-\vec{k})} z^{h_p-h_n-h_m+|\vec{k}|} \phi_p^{(-\vec{k})}(w). \tag{163}$$

We can compute the conformal dimension of a descendant state by

$$\begin{aligned}L_0\phi_h^{(-\vec{k})}(z) &= L_0 L_{-k_1}\cdots L_{-k_n}\phi_h(z) \\ &= ([L_0, L_{-k_1}] + L_{-k_1}L_0)L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &= (k_1 L_{-k_1} + L_{-k_1}L_0)L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &= k_1\phi_h^{(-\vec{k})}(z) + L_{-k_1}L_0L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &\qquad\vdots \\ &= (k_1 + \cdots k_{n-1})\phi_h^{(-\vec{k})}(z) + L_{-k_1}\cdots L_{-k_{n-1}}L_0L_{-k_n}\phi_h(z) \\ &= (k_1 + \cdots k_n)\phi_h^{(-\vec{k})}(z) + L_{-k_1}\cdots L_{-k_n}L_0\phi_h(z) \\ &= (|\vec{k}| + h)\phi_h^{(-\vec{k})}(z),\end{aligned} \tag{164}$$

where $|\vec{k}| = k_1 + \cdots k_n$. Moreover, the number of descendants at level n is $P(n)$, where $P(n)$ is the number of partitions of n into sums of positive integers. For example, at level 1, there is only one descendant, which is $L_{-1}\phi_h(z)$. At level 2, there are two descendants, given by $L_{-2}\phi_h(z)$ and $L_{-1}L_{-1}\phi_h(z)$, and the story goes on similarly for higher n.

The energy-momentum tensor T itself is a descendant of level two in the family of the identity operator

$$I^{(-2)}(w) = (\hat{L}_{-2}I)(w) = \oint_{C(w)} \frac{dz}{2\pi i}(z-w)^{-2+1}T(z)I = T(w). \tag{165}$$

The OPE coefficients of the descendant fields can be determined once the coefficients for those of primaries are known. For example, one has

$$\langle \Phi_1(w_1)\cdots(\hat{L}_{-n}\Phi_m)(w_m)\rangle = \mathcal{L}_{-n}\langle \Phi_1(w_1)\cdots\Phi_m(w_m)\rangle, \tag{166}$$

where we defined the differential operator $n \geq 2$

$$\mathcal{L}_{-n} = -\sum_{j=1}^{m-1}\left(\frac{(1-n)h_j}{(w_j-w_m)^n} + \frac{1}{(w_j-w_m)^{n-1}}\frac{\partial}{\partial w_j}\right). \tag{167}$$

What we did above is to move $\hat{L}_{-n}$ operator through all Φ_j $(j<m)$ for $\hat{L}_{-n}$ to act on the vacuum bra $\langle 0|$ and while doing so we picked up the contributions of the OPEs.

Due to a theorem by Belavin, Polyakov, and Zamolodchikov [BPZ84], the coefficients $C_{mn}^{p(-\vec{k},-\vec{\bar{k}})}$ can be decomposed in terms of the coefficients of primaries times some functions of the parameters (h_m, h_n, h_p, c). Explicitly, one has

$$C_{mn}^{p(-\vec{k},-\vec{\bar{k}})} = C_{mnp}\beta_{mn}^{p(-\vec{k})}\overline{\beta}_{mn}^{p(-\vec{\bar{k}})}, \tag{168}$$

where C_{mnp} are the coefficients appearing in the OPE of primary fields, $\beta_{mn}^{p(-\vec{k})}$ ($\overline{\beta}_{mn}^{p(-\vec{\bar{k}})}$) are functions of h_m, h_n, h_p, c ($\overline{h}_m, \overline{h}_n, \overline{h}_p, \overline{c}$). One may in principle compute these coefficient functions, but the procedure becomes increasingly more involved.

2.10 Null States

The irreps of Virasoro algebra are constructed in analogy with that of $\mathfrak{su}(2)$, but Virasoro algebra has complications that $\mathfrak{su}(2)$ does not have. Namely, in the irreps of Virasoro algebra, there are null states, states that have a 0 inner product with all vectors, and hence these states spoil the positive-definiteness of the irreps. But, if the inner-product is not positive definite, then unitarity is broken so the quantum theory is physically ill defined. To remedy this, we must detect the null states and quotient them out. That is to say, if $|\chi\rangle$ is a null state, then we identify $|\psi\rangle + |\chi\rangle$ with $|\psi\rangle$, in effect setting $|\chi\rangle$ to zero, so that it having zero inner product is not a problem. In this section we will discuss how to deal with this issue.

In constructing the Hilbert space of the CFT, we need to ensure that the inner product is positive-definite. The only relevant reps which can contain unitary irreps of the Virasoro algebra are the highest-weight ones, because those are the reps that have a lowest-energy state. Starting from a highest-weight state $|h\rangle$, we can construct the corresponding Verma module. One must keep in mind though that this module need not have a positive definite inner product right away. Whether we have such a product depends on the values of h and c.

A descendant state χ that satisfies the equations

$$L_0|\chi\rangle = (h+N)|\chi\rangle, \quad L_n|\chi\rangle = 0 \quad (n>0), \tag{169}$$

is called a null state. It is a primary and descendant state (descendant because its eigenvalue under L_0 is $h+N$), so it is a descendant at level N; on the other hand, it is also a highest-weight state because L_n $(n>0)$ annihilates it, so it is primary. To obtain a non-degenerate reps, we need to mod out the submodules generated by the null states. Schematically, if $\mathrm{Verm}(h,c)$ is the full Verma module and $\mathrm{Null}(h,c)$ is the submodule of null states, then the physically plausible Hilbert space is defined as the quotient $\mathcal{H}(h,c) = \mathrm{Verm}(h,c)/\mathrm{Null}(h,c)$. In other words, we identify any state $|\psi\rangle \in \mathrm{Verm}(h,c)$ with any state of the form $|\psi\rangle + |\chi\rangle$ where $|\chi\rangle \in \mathrm{Null}(h,c)$. In some sense, we set all the null states to zero, which renders the module physically viable since after setting all null states to zero only the 0 vectors will have 0 norm in the Verma module, hence it has a positive-definite inner product.

Now let us try to understand null states at low levels. For $N=1$, the only possible null state is $|\chi\rangle = L_{-1}|h\rangle$, or equivalently $\hat{L}_{-1}\phi_h(z) = \partial_z\phi_h(z)$. Setting this to 0 corresponds to the identity operator and $|h\rangle = |0\rangle$, and hence is a trivial representation. For $N=2$ case, we have the possibility

$$|\chi\rangle = L_{-2}|h\rangle + aL_{-1}^2|h\rangle = 0, \tag{170}$$

for some value of the parameter a. Because all L_n operators with $n>0$ can be generated by L_1 and L_2 using the Virasoro algebra, it is enough to act with those two on the above null state and get the conditions for $|\chi\rangle$ to be a null state. We act on the level 2 χ with L_1 to get

$$\begin{aligned}
0 &= L_1\Big(L_{-2}|h\rangle + aL_{-1}^2|h\rangle\Big) \\
&= \Big([L_1, L_{-2}] + L_{-2}L_1 + a\big([L_1, L_{-1}]L_{-1} + L_{-1}L_1L_{-1}\big)\Big)|h\rangle \\
&= \Big(3L_{-1} + 2aL_0L_{-1} + aL_{-1}[L_1, L_{-1}] - aL_{-1}^2L_1\Big)|h\rangle \\
&= \Big(3L_{-1} + 2aL_0L_{-1} + 2aL_{-1}L_0\Big)|h\rangle \\
&= \Big(3L_{-1} + 2a[L_0, L_{-1}] + 4aL_{-1}L_0\Big)|h\rangle \\
&= \Big(3L_{-1} + 2aL_{-1} + 4ahL_{-1}\Big)|h\rangle \\
&= (3 + 2a(2h+1))L_{-1}|h\rangle.
\end{aligned} \tag{171}$$

This is one equation, we also act with L_2 to get a second equation

$$
\begin{aligned}
0 &= L_2\Big(L_{-2} + aL_{-1}^2\Big)|h\rangle \\
&= \Big([L_2, L_{-2}] + L_{-2}L_2 + a[L_2, L_{-1}]L_{-1} + aL_{-1}L_2L_{-1}\Big)|h\rangle \\
&= \Big(\Big(4L_0 + \frac{c}{2}\Big) + a\left(3L_1L_{-1} + L_{-1}[L_2, L_{-1}] + L_{-1}^2L_2\right)\Big)|h\rangle \\
&= \Big(4h + \frac{c}{2} + a(3(\underbrace{2L_0}_{[L_1,L_{-1}]} + 3L_{-1}L_1)\Big)|h\rangle \\
&= \Big(4h + \frac{c}{2} + 6ah\Big)|h\rangle.
\end{aligned}
\tag{172}
$$

Solving these two equations, we find

$$
a = -\frac{3}{2(2h+1)} \quad ; \quad c = \frac{2h(5-8h)}{2h+1}. \tag{173}
$$

Hence, we write

$$
|\chi_h\rangle = \Big(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\Big)|h\rangle, \tag{174}
$$

where h can be written in terms of c as

$$
h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{175}
$$

Via the operator-state correspondence, the null state equation can be written as a null operator equation

$$
\chi_h(z) \equiv \hat{L}_{-2}\phi_h(z) - \frac{3}{2(2h+1)}\hat{L}_{-1}^2\phi_h(z). \tag{176}
$$

This equation has the consequence that if we take a correlator with $\phi_h(z)$, then it will be annihilated by the differential operator $\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2$. To see this first recall (167). Since

$$
\hat{L}_{-1}\phi_h(z) = \partial\phi_h(z), \tag{177}
$$

and via the definition (167), one has

$$
\begin{aligned}
0 &= \langle\phi_1(z_1)\cdots\chi_h(z)\rangle \\
&= \left[\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2} - \sum_{j=1}^{m}\left(\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right)\right]\langle\phi_1(z_1)\cdots\phi_h(z)\rangle.
\end{aligned}
\tag{178}
$$

This is a second order partial differential equation that the CFT correlators must satisfy, and whose solutions can be expressed in terms of hypergeometric functions [DFMS97, Ket95]. The procedure of modding out the null states can be continued level by level. The states of a Verma module $\mathcal{V}(c, h)$ at level N is a linear space spanned by

$$
\Big\{|h, (-\vec{k})\rangle = L_{-n_1}\cdots L_{-n_k}|h\rangle \;\Big|\; \sum n_i = N\Big\}. \tag{179}
$$

And the null state at level N can be determined by solving the equation

$$|\chi^{(N)}\rangle = \sum b_{-\vec{k}}|h,(-\vec{k})\rangle = 0, \tag{180}$$

which is a natural generalization of the equation that we wrote for the $N = 2$ case (170).

Within the subspace generated by $|h,(-\vec{k})\rangle$, we can form the inner products of the states, and all possible inner products can be arranged into a matrix of inner products, whose components are defined as

$$\hat{M}^{(N)}_{(\vec{k}_n)(\vec{k}_m)} \equiv \langle h|L_{n_k}\cdots L_{n_1}L_{-m_1}\cdots L_{-m_{k'}}|h\rangle, \quad \sum n_i = N = \sum m_j. \tag{181}$$

All the matrix elements can be computed by commuting the $L^\dagger_{-n_i} = L_{n_i}$ operators through L_{-m_j} operators, until L_{n_i} operators act on $|h\rangle$ to give 0. The importance of this matrix, called the Gram matrix, lies in its determinant, or the zero modes. When a vector in this subspace is annihilated by $M(c,h)$, it is precisely a null state. So the zero eigenvalues of $M^{(N)}(c,h)$ determines the values (c,h) at which the $N-$level Verma module contains a null state. The determinant of this matrix at arbitrary level N was computed by Kac, which is called the Kac determinant and the formula is given by

$$\det \hat{M}^{(N)}(c,h) = \prod_{k=1}^{N}\prod_{mn=k}(h-h(m,n))^{P(N-k)}, \tag{182}$$

with $m,n \in \mathbb{Z}^+$, and

$$\begin{aligned} h(m,n) = \frac{1}{48}\Big((13-c)(m^2+n^2) - 24mn \\ - 2(1-c) + \sqrt{(1-c)(25-c)}(m^2-n^2)\Big). \end{aligned} \tag{183}$$

It can be shown that for a Verma module $\mathcal{V}(c,h)$ of central charge c, there is a null vector at level $N = m \times n$ iff

$$h = h(m,n) = -\alpha_0^2 + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2, \tag{184}$$

and we defined

$$c = c(\alpha_+,\alpha_-) = 1 - 24\alpha_0^2, \quad 2\alpha_0 = \alpha_+ + \alpha_-, \quad \alpha_+\alpha_- = -1. \tag{185}$$

For a central charge $0 < c < 1$, this solution is characterized by two integers $n, m \geq 1$, and the table of integers is known as the Kac table.

If $c > 25$, one can see that both $\alpha_\pm$ are imaginary numbers. As n, m increase, one gets $h(n,m) < 0$, in which case the representation is non-unitary. When $1 < c < 25$, the conformal weights are complex numbers. The dimensions are real and positive only for the case $c \leq 1$.

Now we discuss a remarkable observation. When α_-/α_+ is a rational number $\mathbb{Q}$, each module $\mathcal{V}(c,h)$ has infinitely many independent null states. When these null states are modded out, the algebra of the operators is consists of a finite set [BPZ84], namely, there are finitely many primary fields in the CFT. Such theories are called Rational Conformal Field Theories (RCFT). The minimal models introduced in [BPZ84], called the BPZ models, are particular examples of RCFTs in which $c < 1$.

3 The Rational Conformal Field Theory

RCFT is a special CFT which was studied in great detail in the late 1980s [BPZ84, Ver88, MS89a, MS89c, MS88, MS89b, MS89d, Car86]. In particular, [Ver88] discusses the fusion rules in RCFT, which is a rule to determine the OPE of two operators $[\phi_i]$, $[\phi_j]$ inside a conformal family, which takes the form

$$[\phi_i] \times [\phi_j] = \sum_k {N_{ij}}^k [\phi_k] \tag{186}$$

for some coefficients ${N_{ij}}^k$. The main result of [Ver88] is that certain modular transformations of $SL(2,\mathbb{Z})$ (the diffeomorphism group of the torus T^2) diagonalize the matrices $N_i \equiv {(N_i)_j}^k$. This has many deep connections to Topological Quantum Field Theories [Wit89, DW90], and in the modern literature the non-invertible symmetries are heavily studied in relation to these constructions coming from RCFT [Sha23, SN24, LOST23]. In this section we shall give a brief review of [Ver88].

3.1 Fusion Rules in RCFT

A CFT defined over a Riemann surface has certain analyticity structures on its partition function and its correlators. Due to the conformal invariance, the CFT depends only on the conformal class of the Riemann surface, or equivalently the moduli space of complex structures. For example, for the Riemann sphere, there is no moduli, it has a unique conformal class. On the other hand, the torus has a moduli $\tau \in \mathbb{C}^+ \; SL(2,\mathbb{Z})$, where $\mathbb{C}^+$ is the complex upper plane and $SL(2,\mathbb{Z})$ is the group of large diffeomorphisms of the torus [4] under which it is invariant. These surfaces are given in figure 8. The $SL(2,\mathbb{Z})$ invariance of the torus implies modular invariance on the CFT $n-$point functions on the torus. This modular invariance plays a special role for CFTs on generic Riemann surfaces, because tori can be sewn together to generate higher genus, as demonstrated in figure 9.

Consider a correlator of n primary fields on a genus g surface, which we call G. The $n-$point function depends on the coordinates of the operator insertions,

[4] There are two generators denoted T and S. These act on the modular parameter as $T : \tau \mapsto \tau + 1$, $S : \tau \mapsto -\frac{1}{\tau}$.

Figure 8: The Riemann sphere on the left, and the torus on the right.

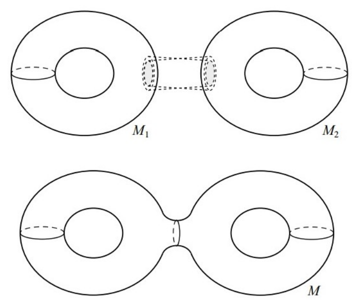

Figure 9: Sewing two tori to get a genus 2 surface. Figure taken from [Pol07]

the punctures, $z = (z_1, \cdots, z_n)$, and it also depends on the moduli parameters of the Riemann surface $m = (m_1, \cdots, m_{3g-3})$. The set (z, m) comprises analytic coordinates on the moduli space of $n-$punctured $g-$genus Riemann surface $\mathscr{M}_{g,n}$. The correlator G has a nice analyticity structure, in the sense that it can be decomposed as follows

$$G(z, m, \overline{z}, \overline{m}) = \sum_{\overline{I}, J} \overline{\mathscr{F}}_{\overline{I}}(\overline{z}, \overline{m}) h_{\overline{I}J} \mathscr{F}_J(z, m) \tag{187}$$

where $\mathscr{F}_J$ ($\overline{\mathscr{F}}_{\overline{I}}$) are the (anti-)holomorphic blocks of the correlator and h is a Hermitian metric. When $n = 0$, we have the partition function of a CFT over the Riemann surface with g-punctures. Further, when $g = 1$, which is the torus T^2, we have the partition function over the torus decomposed into (anti-)holomorphic parts and expressed in terms of the characters. The correlator G and hence the blocks $\mathscr{F}$ depend on the representations of the primaries at the punctures.

The blocks $\mathscr{F}$ may have non-trivial monodromy and modular properties, but at the end of the day G should be modular invariant, which puts restrictions on the metric h. In particular, for the torus, one finds restrictions on the operator spectrum of the CFT. As is standard in CFT, we will drop the anti-holomorphic part and will recover it from the holomorphic part in a straight-forward way.

For a rational CFT, one can understand $\mathscr{F}$ better. For an RCFT with N primary operators ϕ_i $(i = 0, \cdots N-1)$, corresponding to irreps $[\phi_i]$ of a chiral algebra. We pick a direction, say left, and don't put indices for the other direction. Our convention is such that $[\phi_0]$ corresponds to the representation containing the identity $[1]$, and a multiplet of primaries in the same $[\phi]$ reps will be denoted as ϕ_i. In this context, we only consider conformal weights that are integer.

As we discussed in the previous section, there are null states within the representations $[\phi_i]$, whose existence constraints the correlators such that they obey some partial differential equation. The solutions to these PDE's form a vector space holomorphic solutions, whose basis is precisely the blocks $\mathscr{F}_I$.

From an RCFT point there are only certain natural choices for the basis $\mathscr{F}_I$. First we note that any punctured Riemann surface can be obtained by sewing $3-$punctured spheres. Then, the conformal blocks $\mathscr{F}_I$ can be constructed out of the three-point functions. It is possible to interpret this sewing procedure by a $g-$loop Feynman diagram for a φ^3 theory, in which the propagators represents a sum over all states in the representation $[\phi_i]$. The vertex factors can be seen as the fusion of three representations $[\phi_i]$, $[\phi_j]$, $[\phi_k]$, which is done through a three-point function $\langle \phi_i \phi_j \phi_k \rangle$. Associated to a diagram, we have a basis $\mathscr{F}_I$, which is unique up to phase transformations. Since the diagrams are in an equivalence class under phases, operations on the φ^3 diagram that do not change it should be represented by phases. For example, upon a modular transformation, under which the diagram should be unchanged, the change in the diagram should be a phase.

We note that this entire construction can be formulated as a vector bundle $V_{g,n}$ over the moduli space $\mathscr{M}_{g,n}$, where the fibres $V_{g,n}$ are uniquely determined by the requirement that its holomorphic sections satisfy the PDEs coming from the null state equations. For RCFT, the fibre spaces $V_{g,n}$ are finite dimensional, which follows from the fact that RCFT has finitely many primary fields.

Now we introduce the fusion rule. Let us consider a Riemann sphere (zero genus), with three punctures; namely the three-point function on the Riemann sphere. Considering the components $V_{0,ijk}$ of the vector bundle $V_{0,3}$, where at the punctures we insert the fields ϕ_i, ϕ_j, ϕ_k. Let $N_{ijk} \equiv \dim V_{0,ijk}$. The fusion rules is then defined as the formal sum

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{188}$$

One often uses diagrams to represent these fusions, as in figure 10. In raising

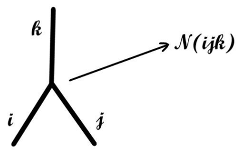

Figure 10: Representing the fusion as a φ^3 diagram in analogy with Feynman diagrams.

the k index, we use the conjugation matrix $C_{ij} = N_{ij0}$ as a metric. We interpret the coefficients ${N_{ij}}^k$ as counting the number of independent fusion paths from ϕ_i, ϕ_j to ϕ_k. To determine the rules, one would have to investigate the three-point function $\langle \phi_i \phi_j \phi_k \rangle$, or equivalently the OPE between two primaries ϕ_i and ϕ_j.

Getting back to $\mathscr{F}_I$ for an arbitrary surface, the number of $\mathscr{F}_I$ ($= \dim V_{g,n}$) can be determined using the φ^3 diagram and counting the different ways of fusing together representations. These are like Feynman rules for the φ^3 diagrams, with vertex factors N_{ijk}, and the indices are contracted as indicated by the "propagators". The result of the diagram should be independent of the way the spheres are sewn together, that is, how the diagram is drawn. From figure 11, we read off the condition

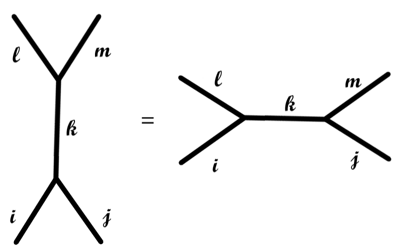

Figure 11: The independence of how the spheres are sewn together puts constraints on the fusion rule coefficients.

$$\sum_k {N_{ij}}^k N_{klm} = \sum_k {N_{il}}^k N_{kjm}. \tag{189}$$

Algebraically, we can derive this by using the symmetry of ${N_{ij}}^k$ and the associativity of the fusion rules as follows:

$$(\phi_i \times \phi_j) \times \phi_l = \sum_k {N_{ij}}^k \phi_k \phi_l = \sum_{k,m} {N_{ij}}^k {N_{kl}}^m \phi_m \tag{190}$$

$$\phi_i \times (\phi_j \times \phi_l) = \sum_k \phi_i {N_{jl}}^k \phi_k = \sum_{k,m} {N_{jl}}^k {N_{ik}}^m \phi_m, \tag{191}$$

equating both sides gives the same condition as (189). This condition further implies that the matrices $(N_i)_j{}^k$ furnishes a representation of the fusion algebra. To see this, observe that

$$\begin{aligned}(N_i \cdot N_j)_l{}^n = \sum_k N_{il}{}^k N_{jk}{}^n &= \sum_k N_{il}{}^k N_{kj}{}^n \\ &= \sum_k N_{ij}{}^k N_{kl}{}^n \\ &= \sum_k N_{ij}{}^k (N_k)_l{}^n,\end{aligned} \tag{192}$$

or in matrix notation

$$N_i \cdot N_j = \sum_k N_{ij}{}^k N_k. \tag{193}$$

Since N_{ijk} is totally symmetric in ijk, one proves another important property

$$N_i \cdot N_j = (N_i)_k{}^l (N_j)_l{}^m = (N_i)^l{}_k (N_j)^m{}_l = (N_j)^m{}_l (N_i)^l{}_k = N_j \cdot N_i, \tag{194}$$

that is to say, N_i commutes with themselves, so they can be simultaneously diagonalized, by a standard theorem of linear algebra. Their eigenvalues form the N one-dimensional representations of the fusion rules. This statement plays an important role in the following.

3.2 Relation with Modular Transformations

Let us consider the characters of the chiral algebra. We will choose two oriented cycles on the torus such that the a-cycle consists of equal time points, and the $b-$cycle is along the time evolution direction, generated by L_0 operator. The character χ_i is defined as the trace of the exponential of L_0, the evolution operator, over the representation $[\phi_i]$:

$$\chi_i \equiv \mathrm{Tr}_{[\phi_i]}\big(q^{L_0+\varepsilon}\big), \tag{195}$$

with $q = e^{2\pi i\tau}$, τ is the modular parameter of the torus, and $\varepsilon = -\frac{1}{24}c$.

One can obtain the moduli space of genus 1 surfaces $\mathscr{M}_1$ as the upper half-plane divided by the modular transformations generated by $T : \tau \mapsto \tau + 1$, and $S : \tau \mapsto -\frac{1}{\tau}$. For chiral algebras whose operators have integer conformal weight, the characters transform in a finite-dimensional representation of the modular group $SL(2, \mathbb{Z})$. For other algebras, things are more involved and one has to specify boundary conditions.

The transformation of the character under T is easy to read. Due to the $+1$, we get $\mathrm{Tr}_{[\phi_i]} e^{2\pi i(L_0+\varepsilon)} = e^{2\pi i(h_i+\varepsilon)}$, so

$$T : \chi_i \mapsto e^{2\pi i(h_i+\varepsilon)}\chi_i. \tag{196}$$

Under the S-transformations, the characters transform in a unitary representation as

$$S : \chi_i \mapsto \sum_j S_i^{\ j} \chi_j. \tag{197}$$

Observe that under $\tau \mapsto -\frac{1}{\tau}$, the cycles a, b are mapped to $-b, a$. Hence, S^2 maps a, b to $-a, -b$, in particular, it reverses the time direction so under S^2 the character χ_i becomes that of the conjugate representation χ^i. So in general we have $S^2 = C$, and only when the representation of the character is self-conjugate do we have $S^2 = 1$.

Now the crucial point is the following. There is a relation between the matrix S and the fusion rules of the primary fields. The basic idea used in [Ver88] is using the primaries to play with the characters and compare them with the transformation under the modular transformations. First consider χ_0, the character of the identity sector. We can obtain the other characters χ_i by inserting into the trace in χ_0 an identity, and by writing the identity as the OPE between a primary field ϕ_i and its conjugate. After this insertion, we move the ϕ_i field along the $b-$cycle (the direction of time evolution), and after it has gone around a complete cycle to get back to its starting point, we annihilate the ϕ_i and its conjugate. As a result of this "wrapping of ϕ_i", the trace is no longer in the identity sector [1] but in the $[\phi_i]$ sector. To see this, map the torus on an annular region $|q|^{1/2} \leq z < |q|^{-1/2}$, that is, if $\tau = \tau_1 + i\tau_2$, then $|q| = e^{-2\pi\tau_2}$ hence $e^{-\pi\tau_2} \leq z < e^{\pi\tau_2}$. If we send $q \to 0$, then the $a-$ cycle will be squished, causing ϕ_i to be moved from ∞ to the origin and thus changing the representation (because the in-state is moved from [1] to $[\phi_i]$). Hence, this operation, denoted $\phi_i(b)$ acts on the characters as

$$\phi_i(b)\chi_0 = \chi_i. \tag{198}$$

What happens when $\phi_i(b)$ acts on a character with a $j-$twist χ_j? In that case, the ϕ_i that moves into the origin will meet with ϕ_j, and from their OPE we will get several fusion paths, so the action of $\phi_i(b)$ on χ_j includes several other characters, with some coefficients that will be related to the coefficients of the fusion rules ${N_{ij}}^k$:

$$\phi_i(b)\chi_j = \sum_k {A_{ij}}^k \chi_k. \tag{199}$$

It is easy to see that if the three-point function $\langle \phi_i \phi_j \phi_k \rangle$ vanishes (namely, if ${N_{ij}}^k$ vanishes), then ${A_{ij}}^k$ vanishes as well.

Because the OPE between the primaries is symmetric, the operations $\phi_i(b)$ commute as well, as a consequence of which we have

$${A_{ij}}^k = {A_{ji}}^k, \tag{200}$$

$$\sum_k {A_{ij}}^k {A_{kl}}^m = \sum_k {A_{il}}^k {A_{kj}}^m. \tag{201}$$

The second "crossing symmetry" equation is related to the fact that the operators $\phi_i(b)$ form an associative algebra

$$\phi_i(b)\phi_j(b) = \sum_k A_{ij}{}^k \phi_k(b). \tag{202}$$

These properties are all completely the same ones that $N_{ij}{}^k$ obey, so it is reasonable to suspect they are the same. Indeed, it was shown in [Ver88] that this is the case.

The operation of moving ϕ_i along the $b-$cycle can equivalently be done for the $a-$cycle. This time, however, the "twisting" operation does not change the representation of the character, because the $a-$cycle consists of equal time points so there are no insertions of ϕ_i on the origin. Since under the operation, $\phi_i(a)$ the representation should be the same, the action of $\phi_i(a)$ on χ_j is easy to find: χ_j must be eigenstates of the operators $\phi_i(a)$ with some eigenvalues. Hence, we have

$$\phi_i(a)\chi_j = \lambda_i^{(j)}\chi_j. \tag{203}$$

The eigenvalues are in general complex (or real) numbers, not necessarily phases. The operators $\phi_i(a)$ satisfies the same algebra as $\phi_i(b)$ does. Using this, we find the following

$$\phi_i(a)\phi_j(a)\chi_n = \lambda_i^{(n)}\lambda_j^{(n)}\chi_n, \tag{204}$$

$$\phi_i(a)\phi_j(a)\chi_n = \sum_k A_{ij}{}^k \phi_k(a)\chi_n = \sum_k A_{ij}{}^k \lambda_k^{(n)}\chi_n, \tag{205}$$

consequently,

$$\lambda_i^{(n)}\lambda_j^{(n)} = \sum_k A_{ij}{}^k \lambda_k^{(n)}. \tag{206}$$

Now comes the key point: The modular transformations generated by S change the cycles, hence they change $\phi_i(a)$ operators to $\phi_i(b)$ operators and vice-versa. In particular, the transformed characters in (197) become eigenstates of $\phi_i(b)$.

Let us explicitly show this. We act on both sides of (203) with the modular transformation S and use (197:

$$S : \phi_i(b)\chi_j \mapsto \phi_i(a)\sum_l S_j{}^l \chi_l = \sum_l S_j{}^l \lambda_i^{(l)}\chi_l, \tag{207}$$

$$S : \sum_k A_{ij}{}^k \chi_k \mapsto \sum_{k,l} A_{ij}{}^k S_k{}^l \chi_l. \tag{208}$$

Equating the two we get

$$S_j{}^l \lambda_i^{(l)} = \sum_k A_{ij}{}^k S_k{}^l. \tag{209}$$

Moreover, using the fact that ${S_j}^{l}$ furnishes a unitary representation, we get

$$
{A_{ij}}^{k} = \sum_l {S_j}^{l} \lambda_i^{(l)} S_l^{\dagger k}, \tag{210}
$$

which means that the matrices ${S_j}^{l}$ diagonalizes the ${(A_i)_j}^{k}$ matrices, whose eigenvalues are $\lambda_i^{(l)}$. In addition to this, if we use the fact that ${A_{i0}}^{k} = \delta_i^k$ (a fusion of $[\phi_i]$ with [1] will give something inside $[\phi_i]$), we get

$$
\lambda_i^{(n)} = \frac{{S_i}^{n}}{{S_0}^{n}}. \tag{211}
$$

Hence, if we know the matrix ${S_i}^{j}$, we can find all the eigenvalues $\lambda_i^{(n)}$ and hence all the coefficients ${A_{ij}}^{k}$. Since it is expected that ${A_{ij}}^{k} = {N_{ij}}^{k}$ for many RCFTs, we have thus solved the fusion rules of the RCFT. Indeed, in many explicit computations, ${A_{ij}}^{k} = {N_{ij}}^{k}$ was verified [Ver88]. Since it is an extremely important result, following [Ver88], we will state it below

$$
\text{The modular transformations } S : \tau \mapsto -\frac{1}{\tau} \text{ diagonalizes the fusion rules!} \tag{212}
$$

3.3 Derivation with Diagrammatic Computation

We will derive the result above using a diagrammatic approach, which is a standard tool in the modern literature of non-invertible symmetries, for example in [LOST23].

We represent a torus as a rectangular with $b-$cycle representing the time direction (vertical) and $a-$cycle representing the space direction (horizontal). We define the Virasoro characters χ_j in the $[\phi_j]$ representation as

$$
\chi_j = \mathrm{Tr}_{[\phi_j]}\left(q^{L_0-\varepsilon}\right) \equiv \boxed{\quad\ j\uparrow}. \tag{213}
$$

($q = e^{2\pi i\tau}$) Under modular transformations generated by $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -\frac{1}{\tau}$, χ_j changes as

$$
\begin{aligned}
T :& \boxed{\quad\ j\uparrow} \mapsto e^{2\pi i(h_j+\varepsilon)} \boxed{\quad\ j\uparrow}, \\
S :& \boxed{\quad\ j\uparrow} \mapsto \sum_k {S_j}^{k} \boxed{\quad\ k\uparrow},
\end{aligned} \tag{214}
$$

with ${S_j}^{k}$ a unitary matrix. These characters can be twisted with ϕ_i operators along the $a-$cycle and the $b-$cycle, which we represent as follows

$$
\phi_i(a)\chi_j \equiv \boxed{\underset{i}{\rightarrow}\ \ j\uparrow}, \tag{215}
$$

$$\phi_i(b)\chi_j \equiv \boxed{i\uparrow \quad j\uparrow}. \tag{216}$$

These correspond to an ϕ_i insertion in the trace and winding it once along the respective cycle. The twist along a-cycle should be proportional to χ_j as explained in the previous subsection, and the twist along the $b-$cycle introduces a fusion $[\phi_i]$ with $[\phi_j]$ inside the trace of Virasoro character. Based on this picture we have

$$\boxed{\underset{i}{\rightarrow}\ \ j\uparrow} = \lambda_i^{(j)} \boxed{\quad\quad j\uparrow}, \tag{217}$$

$$\boxed{i\uparrow \quad j\uparrow} = \sum_k {A_{ij}}^k \boxed{\quad\quad k\uparrow}, \tag{218}$$

since there is a fusion involved in the second line, the coefficients ${A_{ij}}^k$ are related with $\mathcal{N}(ijk)$, in fact they are generically equal to each other. Now, the key idea was to observe that under the modular transformation $S:\tau\mapsto -\frac{1}{\tau}$, the $a-$cycle and the $b-$cycle are exchanged. So under the exchange, the diagrams behave like

$$S: \boxed{i\uparrow \quad j\uparrow} \mapsto \sum_m {S_j}^m \boxed{\underset{i}{\rightarrow}\ \ m\uparrow} = \sum_m {S_j}^m \lambda_i^{(m)} \boxed{\quad\quad m\uparrow} \tag{219}$$

$$S: \sum_k {A_{ij}}^k \boxed{\quad\quad k\uparrow} \mapsto \sum_k {A_{ij}}^k \sum_m {S_k}^m \boxed{\quad\quad m\uparrow}, \tag{220}$$

using the fact that ${S_k}^m$ is unitary, we find

$${A_{ij}}^k = \sum_m {S_j}^m \lambda_i^{(m)} S_m^{\dagger k}. \tag{221}$$

This is the result of [Ver88] reviewed in the previous section from a more algebraic point of view.

4 Conclusions & Discussions:

We have given a detailed review of CFT and studied the fusion rules in RCFT. As can be seen from the manuscript, there are many mathematical tools needed to understand CFTs and the rich structures appearing here have many connections to different areas of physics and mathematics. Conformal Field Theory is still a field teeming with activity. What we studied in section 3 is closely related to the modern study of symmetries [GKSW15, Sha23, SN24, BBFT$^+$24]. Our purpose was to review the paper [LOST23] deriving the asymptotic density of states in the presence of non-invertible line-defects, which we did to some extent. However, we did not discuss that in this manuscript because our understanding of it is not complete. In any case, we hope that we have been able to present a coherent picture on the usefulness of CFT and the result of [Ver88].

References

[BBFT+24] Lakshya Bhardwaj, Lea E. Bottini, Ludovic Fraser-Taliente, Liam Gladden, Dewi S. W. Gould, Arthur Platschorre, and Hannah Tillim. Lectures on generalized symmetries. *Phys. Rept.*, 1051:1–87, 2024.

[BP09] Ralph Blumenhagen and Erik Plauschinn. *Introduction to conformal field theory: with applications to String theory*, volume 779. 2009.

[BPZ84] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984.

[Car86] John L. Cardy. Operator Content of Two-Dimensional Conformally Invariant Theories. *Nucl. Phys. B*, 270:186–204, 1986.

[DFMS97] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[DW90] Robbert Dijkgraaf and Edward Witten. Topological Gauge Theories and Group Cohomology. *Commun. Math. Phys.*, 129:393, 1990.

[EMSS89] Shmuel Elitzur, Gregory W. Moore, Adam Schwimmer, and Nathan Seiberg. Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory. *Nucl. Phys. B*, 326:108–134, 1989.

[Fre07] Edward Frenkel. Lectures on the Langlands program and conformal field theory. In *Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry*, pages 387–533, 2007.

[FU84] D. S. Freed and K. K. Uhlenbeck. *INSTANTONS AND FOUR - MANIFOLDS*. 1984.

[GKSW15] Davide Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett. Generalized Global Symmetries. *JHEP*, 02:172, 2015.

[Iga02] Kevin Iga. What do topologists want from Seiberg-Witten theory? *Int. J. Mod. Phys. A*, 17:4463–4514, 2002.

[Kad66] L. P. Kadanoff. Scaling laws for Ising models near T(c). *Physics Physique Fizika*, 2:263–272, 1966.

[Ket95] S. V. Ketov. *Conformal field theory*. 1995.

[LOST23] Ying-Hsuan Lin, Masaki Okada, Sahand Seifnashri, and Yuji Tachikawa. Asymptotic density of states in 2d CFTs with non-invertible symmetries. *JHEP*, 03:094, 2023.

[MS88] Gregory W. Moore and Nathan Seiberg. Polynomial Equations for Rational Conformal Field Theories. *Phys. Lett. B*, 212:451–460, 1988.

[MS89a] Gregory W. Moore and Nathan Seiberg. Classical and Quantum Conformal Field Theory. *Commun. Math. Phys.*, 123:177, 1989.

[MS89b] Gregory W. Moore and Nathan Seiberg. LECTURES ON RCFT. In *1989 Banff NATO ASI: Physics, Geometry and Topology*, 9 1989.

[MS89c] Gregory W. Moore and Nathan Seiberg. Naturality in Conformal Field Theory. *Nucl. Phys. B*, 313:16–40, 1989.

[MS89d] Gregory W. Moore and Nathan Seiberg. Taming the Conformal Zoo. *Phys. Lett. B*, 220:422–430, 1989.

[Neg16] Stefano Negro. Integrable structures in quantum field theory. *J. Phys. A*, 49(32):323006, 2016.

[PBZ84] Alexander M. Polyakov, A. A. Belavin, and A. B. Zamolodchikov. Infinite Conformal Symmetry of Critical Fluctuations in Two-Dimensions. *J. Statist. Phys.*, 34:763, 1984.

[Pol70] Alexander M. Polyakov. Conformal symmetry of critical fluctuations. *JETP Lett.*, 12:381–383, 1970.

[Pol07] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007.

[SD17] David Simmons-Duffin. The Conformal Bootstrap. In *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pages 1–74, 2017.

[Sha23] Shu-Heng Shao. What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetries. 8 2023.

[SN24] Sakura Schafer-Nameki. ICTP lectures on (non-)invertible generalized symmetries. *Phys. Rept.*, 1063:1–55, 2024.

[Ver88] Erik P. Verlinde. Fusion Rules and Modular Transformations in 2D Conformal Field Theory. *Nucl. Phys. B*, 300:360–376, 1988.

[Wei05] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6 2005.

[Wit88] Edward Witten. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys. B*, 311:46, 1988.

[Wit89] Edward Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.