AdS/CFT Treatment of the Josephson Junction and Electric-Magnetic Duality with Compact QED_3

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ABSTRACT: The AdS/CFT treatment of superconductors is one of the beautiful applications of holography in condensed matter physics, often called the AdS/CMT correspondence. This problem is rich in physics, and it offers a better understanding of the theory of superconductors. We first discuss the general framework of holographic superconductors and the simplest gravity model that captures the effects of superconductors. Then, we focus on the Josephson junction within this description. After the holography treatment, we review a duality between the compact Quantum Electrodynamics in three dimensions and the Josephson junction. This electric-magnetic type duality has not been well studied, and our main interest is to combine this electromagnetic duality with the gravitational dual of the Josephson effect and establish a triality of descriptions of the same physics.

Contents

1 Introduction

The AdS/CFT correspondence or holography is one of the most remarkable outcomes of string theory, relating a string theory on asymptotically anti-de Sitter space-times to a Conformal Field Theory (CFT) on the boundary [1–3]. The idea of holography dates back to 't Hooft [4] and Susskind [5], from their consideration of the black hole information paradox. In [1], Maldacena showed that a string theory on $\mathrm{AdS}_5 \times S^5$ in the large N limit is dual to an $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory. From that day on, AdS/CFT correspondence has been one of the driving forces in theoretical physics research. There is a zoo of dualities between gauge theories, gravity theories, condensed matter systems, and various other strongly coupled Quantum Field Theories (QFT) [6].

Given the relation of holography to condensed matter systems, such as the gravitational description of Hall effect and Nernst effect [7, 8], it was expected that a gravitational dual to the superconductivity phenomenon, and that insight was verified in [7], leading to the literature of holographic superconductors. From 2008 to this day, there were 496 papers containing the phrase "holographic superconductor" according to INSPIRE hep. The holographic superconductor is one of the places where a consequence of holography may be seen in the lab, as superconductors can be produced in real life. Moreover, by studying certain properties of superconductors compatible with the relevant experiments, one may be able to deduce new thermodynamical aspects of black holes in the theory of gravitation.

In this paper, we are interested in using the framework of holographic superconductors for the Josephson junction (JJ). Our main interest is to leverage the electric-magnetic type duality between JJ and the compact Quantum Electrodynamics in three dimensions QED$_3$, established in [11]. The relation between QED$_3$ and the gravity dual of JJ has not been investigated, and it may lead to very interesting results for both the gauge and the gravity

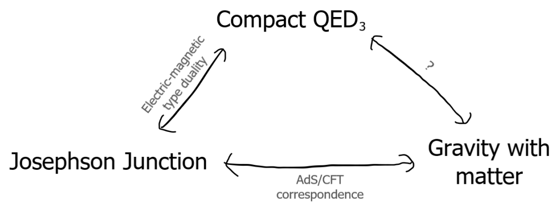

Figure 1. From [11] we know the electric-magnetic type duality between QED$_3$ and the Josephson junction, and from [12], we have a gravitational description of the Josephson junction. It is therefore natural to expect a relation between QED$_3$ and the gravitational description.

theories involved in this phenomenon. We review the results of [11] to set the stage for our problem, which is to construct a triality as in figure 1.

Additionally, this problem may also be highly relevant for the confining phases of gauge theories. In the 70's several people put forward the idea of understanding confinement via a superconductor model that is electric-magnetic dual to QCD, where the condensation of solitonic objects in the superconductor correspond to a linearly growing potential between the elementary excitations of the gauge theory. Since the main ingredients in our study are highly relevant for confinement [1], it would be wise to keep it in mind while studying this putative triality.

2 Gauge/Gravity Duality

The gauge/gravity duality establishes a relation between a gravitational quantum theory in a $d+1$ dimensional asymptotically AdS spacetime and a non-gravitational QFT in one lower dimension. The relation is between the generating functionals of the two:

$$Z_{\text{gravity}}[\phi_0 = J] = Z_{\text{QFT}}[J], \tag{2.1}$$

where the sources J in the QFT are identified with boundary conditions ϕ_0 on the fields ϕ on the conformal boundary of AdS. A remarkable feature of this correspondence is that it relates a strongly coupled theory with a weakly coupled one, referred to as a non-perturbative duality. In the QFT, taking the strongly coupled (large N for the rank of the gauge group) limit, the corresponding gravity theory is weakly coupled and the classical solution is the dominant contribution

$$Z_{\text{gravity}}[\phi_0 = J] \approx \exp\Big(S^{\text{on-shell}}_{\text{gravity}}[\phi_0 = J]\Big), \tag{2.2}$$

with the exponent being the action evaluated on the classical solution. We thus have

$$Z_{\text{QFT}}[J] \approx \exp\Big(S^{\text{on-shell}}_{\text{gravity}}[\phi_0 = J]\Big). \tag{2.3}$$

[1]Historically [19], compact QED$_3$ is the first model in which an understanding of confinement through monopole contributions and by utilizing the compact scalar description of 3d Maxwell's theory has been achieved.

Therefore, computations in a strongly coupled QFT are reduced to classical solutions of a gravity theory with appropriate boundary conditions. The correlators of QFT can be computed via

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^n S^{\text{on-shell}}_{\text{gravity}}[\phi_0 = J]}{\delta\phi_0(x_1) \cdots \delta\phi_0(x_n)}, \tag{2.4}$$

where $\mathcal{O}$ is the operator sourced by J, $\int d^d x \mathcal{O} J$, dual to ϕ on the gravity side.

3 Holographic Treatment of Superconductors

To obtain a dual description of a superconductor in a gravitational theory, the authors [7] introduce temperature by adding a black hole, and a condensate by adding a charged scalar. For the gravitational theory to reproduce the phase diagram of a superconductor, the black hole has scalar hairs at low temperatures, and no hair at high temperatures.

In [7], the authors study a 2+1 dimensional model, which captures several important unconventional superconductors, such as cuprates and organics. They take the metric to be the planar Schwarzschild AdS black hole

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2), \tag{3.1}$$

with

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r}. \tag{3.2}$$

L corresponds to the AdS radius, and M enters the Hawking temperature formula of the black hole

$$T = \frac{3M^{1/3}}{4\pi L^{4/3}}. \tag{3.3}$$

The black hole 3.1 is in 3+1 dimensions (r is not related to $x^2 + y^2$!), so the holographic dual will be in 2+1 dimensions. In this background, consider an Abelian-Higgs model with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{ab}F_{ab} - |\partial\Psi - iA\Psi|^2 - V(|\Psi|), \tag{3.4}$$

where the potential for simplicity is chosen

$$V(|\Psi|) = -\frac{2|\Psi|^2}{L^2}. \tag{3.5}$$

This is a negative mass but does not lead to instability because it is above the Breitenlohner-Freedman bound as pointed out in [7].

Taking a plane-symmetric ansatz $\Psi = \Psi(r)$, the field equations of Ψ reads

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{L^2 f}\Psi = 0, \tag{3.6}$$

where $A_t = \Phi$. With the other components of A being 0, the Maxwell equations force the phase of Ψ to a constant, so we take Ψ to be real without loss of generality. The equation for the time component of the gauge potential is the time component of the field equations of a massive vector field

$$\Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0, \tag{3.7}$$

where $2\Psi^2$ acts as an $r-$dependent mass.

Starting from these equations, and studying the asymptotic behavior of the solution the relation to superconductors is established [7]. Particularly, the solutions behave as

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \cdots\,, \tag{3.8}$$

and

$$\Phi = \mu - \frac{\rho}{r} + \cdots\,. \tag{3.9}$$

For Ψ, both fall-offs are normalizable, and with boundary conditions, one of them can be made 0. After dropping one of them, we get a one-parameter family of solutions. Properties of the dual theory can be read from this asymptotic solution. For example, from the asymptotic behavior of Φ we determine the chemical potential μ and the charge density ρ of the field theory. The condensate of the scalar operator $\mathcal{O}$ in the dual theory is given by

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(2)}, \quad i = 1, 2 \tag{3.10}$$

with the boundary condition $\varepsilon_{ij}\Psi^{(j)} = 0$. The $\sqrt{2}$ simplifies the subsequent equations and corresponds to the bulk-boundary coupling $\frac{1}{2}\int d^3x(\overline{\mathcal{O}}\Psi + \mathcal{O}\overline{\Psi})$. It is easy to see that $\mathcal{O}^{(i)}$ has dimension i. Working with units where the AdS radius is $L = 1$, we have $[T] = 1$, and $[\rho] = 2$, where $[A]$ is the mass dimension of A. We can form the dimensionless quantities $\langle \mathcal{O}_i \rangle / T$ and ρ/T^2 using these objects. Numerical solutions to the field equations of Ψ and Φ yield the curves in figure 2.

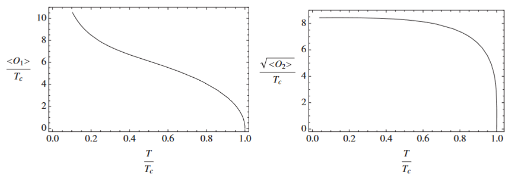

Figure 2. The condensate as a function of temperature for the two operators. Note that it goes to zero at the critical temperature $T = T_c \propto \rho^{1/2}$.

The curve in the right figure is similar to the one obtained from the BCS theory of superconductors. The left figure starts similarly, but at low temperatures, it diverges as $T^{-1/6}$. However, when the condensate becomes large the backreaction on the bulk metric cannot be ignored, so at extremely low temperatures this approximation fails.

Fitting the curves, a square root behavior is observed, which is typical of second-order phase transitions. For one boundary condition, one finds

$$\langle \mathcal{O}_1 \rangle \approx 9.3 T_c (1 - T/T_c)^{1/2}, \quad T \to T_c, \tag{3.11}$$

with a critical temperature $T_c \approx 0.226\rho^{1/2}$. For the other boundary condition,

$$\langle \mathcal{O}_2 \rangle \approx 144 T_c (1 - T/T_c)^{1/2}, \quad T \to T_c, \tag{3.12}$$

now the critical temperature is $T_c \approx 0.118\rho^{1/2}$. Considering the free energy, one can conclude that the phase transition is continuous. In 2+1 dimensions, finite temperature continuous symmetry-breaking phase transitions are only possible in the large N limit, namely the classical gravity limit of the model, in which the fluctuations are suppressed.

Hence, for $T < T_c$, a charged scalar operator $\langle \mathcal{O}_1 \rangle$ or $\langle \mathcal{O}_2 \rangle$ has condensed. Naturally, one would expect that as a result of this condensate, the associated charge will have a superconductivity in its current.

4 Gravitational Dual of the Josephson Junction

In [12], the holographic description of the Josephson junction was given. We take the following action as our model

$$S = \int d^4x \sqrt{-g} \left(R + \frac{6}{L^2} - \frac{1}{4} F_{ab} F^{ab} - |D\Psi|^2 - m^2 |\Psi|^2 \right), \tag{4.1}$$

with $D\Psi = \partial\Psi - iA\Psi$. In this model, we will restrict to the probe approximation, where we scale $\Psi = \tilde{\Psi}/q$, $A = \tilde{A}/q$ and take $q \to \infty$, keeping $\tilde{\Psi}$ and $\tilde{A}$ fixed. In this limit the fields do not have backreactions to the metric, hence we can fix the background geometry. As in section 3, we take it to be the planar planar Schwarzschild AdS black hole

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2(dx^2 + dy^2), \tag{4.2}$$

with

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r} = \frac{r^2}{L^2}\Big(1 - \frac{ML^2}{r^3}\Big). \tag{4.3}$$

The horizon of the black hole is at $r_0 = (ML^2)^{1/3}$, the temperature of which is $T = \frac{3r_0}{4\pi L^2}$.

Let us form the junction along the x-direction. The fields will now be functions of r and x. To capture the phase in the Josephson junction, we include a phase factor in the scalar field. Thus, the configurations we consider have the form

$$\tilde{\Psi} = |\Psi| e^{i\varphi}, \quad \tilde{A} = A_t dt + A_r dr + A_x dx, \tag{4.4}$$

with all the functions depending only on r and x. Note that under a gauge transformation, we have $A \to A + d\chi$ and $\varphi \to \varphi + \chi$ (the scalar field is coupled to the gauge potential with unit charge so it changes by a phase $e^{i\chi}$), so that $M = A - d\varphi$ is a gauge invariant quantity. We will focus on the gauge invariant object M in the following.

By studying the equations of motion of this theory, we obtain the asymptotic form of the scalar field as in section 3, and by fixing the boundary conditions to $\Psi^{(1)} = 0$, we have

$$\langle \mathcal{O} \rangle = \Psi^{(2)}, \tag{4.5}$$

and this operator is interpreted as the superconducting condensate [12]. Moreover, the asymptotic form of the Maxwell fields reads

$$\begin{aligned} M_t &= \mu(x) - \frac{\rho(x)}{r} + O(r^{-2}), \\ M_r &= O(r^{-3}), \\ M_x &= \nu(x) + \frac{J}{r} + O(r^{-2}). \end{aligned} \tag{4.6}$$

The functions $\mu(x), \rho(x), \nu(x)$, and J are interpreted as the chemical potential, charge density, superfluid velocity, and current in the boundary field theory, respectively. Using numerical tools, the field equations of 4.1 can be solved, and the results indicate that with appropriate choices of the chemical potential $\mu(x)$, the Josephson junction can be captured holographically.

5 Compact QED$_3$ and the Josephson Junction

This section will briefly review the electric-magnetic duality between the compact QED$_3$ and the Josephson junction. We are interested in this correspondence because a holographic description of the Josephson junction may reveal potential links between a four-dimensional Einstein gravity [12], and the three-dimensional compact QED.

In [11], a very interesting electric-magnetic type duality between the Josephson junction and the compact QED$_3$ has been worked out. Furthermore, in [18], the phase transition in the finite temperature QED$_3$, which is well studied, was used to understand the transition in the Josephson junction. We will expand upon the electric-magnetic duality. Our starting point is the compact QED$_3$, which is defined by the Euclidean path integral

$$Z = \int \mathcal{D}A \exp\Big(- \frac{1}{4g^2} \int d^3x F_{\mu\nu} F^{\mu\nu} \Big), \tag{5.1}$$

where $F_{\mu\nu}$ is the field tensor of an $O(2)$ gauge theory with potential A_μ. As is well known, one can write this path integral in terms of the field tensor F by imposing the Bianchi identity via a Lagrange multiplier

$$\mathcal{L} \propto \varepsilon^{\rho\mu\nu} \partial_\rho \sigma F_{\mu\nu}. \tag{5.2}$$

Since F is quadratic, we can integrate it out and have a description solely in terms of the scalar field σ. As shown by Polyakov, monopole contributions in the gauge theory side induce a cosine potential for the scalar field. Hence, the compact QED_3 in the low energy regime is described by the path integral

$$Z = \int \mathcal{D}\sigma \exp\Big(- \frac{g^2}{32\pi^2} \int d^3x \big((\partial\sigma)^2 + M^2 \cos\sigma \big) \Big). \tag{5.3}$$

The electromagnetic fields of the monopole solution $H_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F_{\nu\rho}$ are given by

$$H_\mu = i\frac{g^2}{4\pi}\partial_\mu \sigma(x). \tag{5.4}$$

Analytically continuing to the Lorentzian signature, we get [18]

$$\{H, E_1, E_2\} = \frac{g^2}{4\pi}\{\partial_t, -\partial_2, \partial_1\}\sigma. \tag{5.5}$$

Together with the sine-Gordon equation from the variation of σ, these are the equations of QED_3. For the Josephson junction, one has the current

$$J = J_c \sin\phi, \tag{5.6}$$

with J_c the maximum supercurrent density and ϕ is the phase difference of the Landau-Ginzburg wave-function on the two sides of the junction. The parameter ϕ is the basic parameter in the Josephson junction. Moreover, under some assumptions, the electromagnetic fields have the form

$$\{E_3, H_1, H_2\} = \frac{1}{2e(\lambda_1 + \lambda_2 + d)}\{\partial_t, -\partial_2, \partial_1\}\phi, \tag{5.7}$$

with d the thickness of the barrier separating the two sides, and λ_1, λ_2 the penetration depths of superconductors. If we look at Maxwell's equations with the current above, we find that the phase obeys a sine-Gordon equation

$$\Big(\partial_x^2 + \partial_y^2 - \frac{1}{v^2}\partial_t^2\Big)\phi = \frac{1}{\Lambda_J^2}\sin\phi, \tag{5.8}$$

where v and Λ_J are determined from the properties of the junction and the fundamental constants. Observing the similarity between the two descriptions, and noting that the electric fields and the magnetic fields of QED_3 and Josephson junction are exchanged, as expected from an electric-magnetic duality, the table of dictionary 1 was proposed in [11].

Compact QED_3	Josephson Effect
σ	ϕ
$\{H, E_1, E_2\}$	$\{E_3, H_1, H_2\}$
Instanton density	Supercurrents of Cooper pairs
Topological charge of instantons	Charge of a Cooper pair
Electric charge	Magnetic charge

Table 1. The dictionary between Compact QED_3 and the Josephson effect. The table is from [11].

6 Research Directions

- **The Josephson junction as a QED$_3$** As we discussed, there is a correspondence between the Josephson junction and the compact QED$_3$ [11]. Considering also the holographic dual of the Josephson junction [12], one naturally expects a triality between the three descriptions.

- **Confinement via Condensation in Superconductors** In the 70s, several people put forward the idea to approach the problem of confinement through the electromagnetic dual superconductor models [13–15]. The monopole condensation in superconductors leads to a linear potential in the gauge theory, hence to a confining phase. The holographic superconductors may have new insights to provide in the problem of confinement. Some discussions concerning confinement and holographic superconductors can be found in [16, 17].

- **Bootstrapping holographic superconductors?** There seems to be no approach to holographic superconductors via bootstrap tools. This may be because it is an ill-defined notion to begin with, but if it is possible, it would be interesting to involve the bootstrap tools in holographic superconductors.

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