Rational Conformal Field Theory and Verlinde Operators

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Motivation

Conformal Field Theory in 2-dimensions has wide applications in both physics and mathematics. Here are some subset of places where it plays an important role:

- String Theory (worldsheet CFT),
- Integrable systems,
- Statistical systems (Ising model, ...),
- The Geometric Langlands program,
- Infinite dimensional (quantum) algebras,
- Representation theory of Affine Lie algebras,
- etc.

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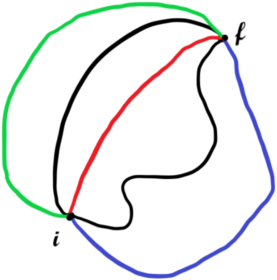
$$\mathcal{A}|a\rangle = \alpha(a)|a\rangle. \tag{1}$$
3. For a generic state $|\psi\rangle$, the probability of finding it in the a state is given by
$$P(\psi \to a) = |\langle a|\psi\rangle|^2, \tag{2}$$
with the constraint
$$\sum_a P(\psi \to a) = 1. \tag{3}$$

What is QFT?

We would like understand the time evolution of the physical states, that is, to calculate $P(i \to f)$ for all i and f. This is calculated by summing up the probability amplitude of all possible intermediate processes that evolve $i \to f$.



This corresponds to taking a trace over all states in $\mathcal{H}$ with appropriate operator insertions

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \equiv \mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H} \mathcal{O}_1 \cdots \mathcal{O}_n \right). \tag{4}$$

What is QFT?

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ is called the $n-$point function. The physical content of a QFT is encoded in the $n-$point functions. The 0$-$point is called the partition function, which captures the time evolution

$$Z = \mathrm{Tr}_{\mathcal{H}} e^{-\beta H}. \tag{5}$$

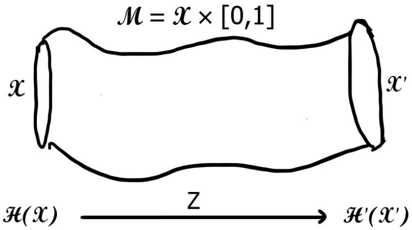


Figure: Partition function as a map from the Hilbert space at the left boundary to the right boundary, which can be interpreted as time evolution map, with $\mathcal{X}$ the space at time $t = 0$ and $\mathcal{X}'$ the space at $t = 1$.

The 2d Conformal Field Theory in flat space is a 2d QFT that is invariant under conformal transformations $x \to x'$ such that:

$$g'_{\mu\nu}(x') = e^{-\omega(x)}\delta_{\mu\nu}. \tag{6}$$

For an infinitesimal transformation $x' = x + \varepsilon(x)$, this reads

$$\partial_0\varepsilon_0 = \partial_1\varepsilon_1, \quad \partial_0\varepsilon_1 = -\partial_1\varepsilon_0, \tag{7}$$

and these are the Cauchy-Riemann equations with complex coordinates $z = x^0 + ix^1$ and $\overline{z} = x^0 - ix^1$

$$\overline{\partial}\varepsilon(z) = 0 = \partial\overline{\varepsilon}(\overline{z}). \tag{8}$$

Thus, $z \mapsto z + \varepsilon(z)$ and $\overline{z} \mapsto \overline{z} + \overline{\varepsilon}(\overline{z})$ are conformal transformations, where we view z and $\overline{z}$ to be independent.

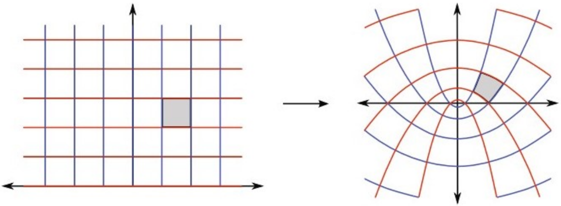


Figure: Demonstration of a 2d conformal transformation. Taken from Blumenhagen & Plauschinn (2009).

Expanding the holomorphic $\varepsilon(z)$ in a Laurent series, we get

$$\varepsilon(z) = \sum_{n \in \mathbb{Z}} a_n z^{n+1}, \tag{9}$$

which has ∞ many free parameters. Hence, a CFT in 2d has ∞ many local symmetries!

Generators of the Conformal Transformations

The relevant symmetry algebra of the quantum CFT is the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{10}$$

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \tag{11}$$

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where L_n & $\overline{L}_n$ are the generators of conformal transformations acting on the Hilbert space.

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The Hilbert space can thus be decomposed as direct sums of the irreps of the Virasoro algebra,

$$\mathcal{H} = \bigoplus_{h,\overline{h}} \mathrm{Vir}(h,c) \otimes \overline{\mathrm{Vir}}(\overline{h},c), \tag{13}$$

where $\mathrm{Vir}(h,c)$ is called a Verma module. Constructing the irreps of Virasoro algebra is analogous to that of $\mathfrak{su}(2)$.

Analogy: Irreps of $\mathfrak{su}(2)$

In $\mathfrak{su}(2)$, one has three generators $J_0, J_\pm \equiv J_1 \pm iJ_2$ satisfying the algebra

$$[J_0, J_\pm] = \pm J_\pm \quad ; \quad [J_+, J_-] = 2J_0. \tag{14}$$

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4. each $|j\rangle$ and its descendants comprises a unitary irrep,

Irreps of Virasoro

Analogous to $|j\rangle$, whose descendants form each of the irreps, there exists highest-weight states $|h\rangle$ in the reps of Virasoro algebra, defined by

$$L_n|h\rangle = 0 \ (n > 0), \quad L_0|h\rangle = h|h\rangle. \tag{15}$$

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Then, similarly, we can generate the spectrum from the descendants:

$$\left|h, (-\vec{k})\right\rangle \equiv L_{-k_1} \cdots L_{-k_n}|h\rangle. \tag{16}$$

Operator State Correspondence

In CFT, there is something called the operator state correspondence, which means that to each $|h\rangle$, there corresponds a primary operator ϕ_h of conformal dimension h. We will use this correspondence.

Analogy With $\mathfrak{su}(2)$

Algebra	$\mathfrak{su}(2)$	Vir
Grading Operator	J_0	L_0
Ladder Operator(s)	$J_\pm$	$L_{\mp n}$
Conjugation	$J_\pm^\dagger = J_\mp$	$L_{\mp n}^\dagger = L_{\pm n}$
Highest-weight state	$\lvert j \rangle$	$\lvert h \rangle$

Difference With $\mathfrak{su}(2)$

Null states at level $N = 2$

As opposed to $\mathfrak{su}(2)$, in Vir we have null states such as:

$$\chi_h(z) \equiv \hat{L}_{-2}\phi_h(z) - \frac{3}{2(2h+1)}\hat{L}_{-1}^2\phi_h(z). \tag{17}$$

that satisfy $\langle\chi|\chi\rangle = 0 = \langle\psi|\chi\rangle$ for any $|\psi\rangle$.

We have to mod out all the null states for unitarity!

Rational Conformal Field Theory

For certain values of h and c, there are ∞ many null states. After modding out the corresponding submodule, we end up with finitely many primary fields. These are called rational CFTs:

RCFT

An RCFT is a CFT with finitely many primary fields.

CFT on Riemann Surfaces

A CFT depends on the moduli space of complex structures of the Riemann surface over which it is defined.

- For the Riemann sphere, the conformal class is trivial.
- For the torus, there is a moduli $\tau \in \mathbb{C}^+/SL(2,\mathbb{Z})$.

The Riemann sphere
(no moduli)

The Torus
(moduli parameter: τ)

CFT on Riemann Surfaces

What about Riemann surfaces of higher genus? One can sew the tori together to get higher genus, so the case of torus and modular invariance plays a special role.

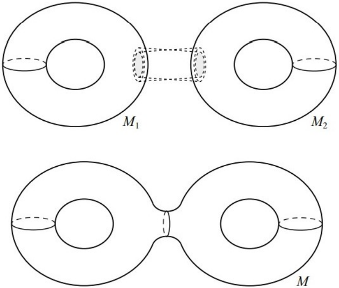
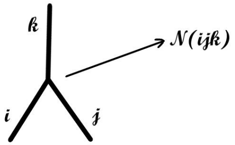


Figure: Figure taken from Polchinski (1998).

RCFT on the Torus

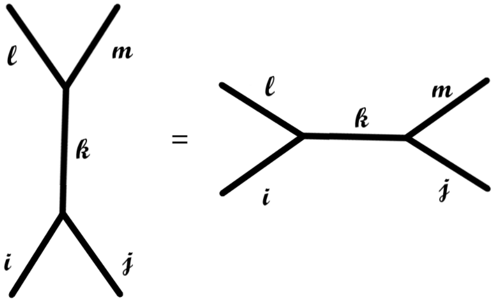
We focus on RCFT on the torus now. There are finitely many families $[\phi_i]$, and these representations can be fused together to produce another representation, which we represent with the following diagram:



$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}(ijk)[\phi_k] \tag{18}$$

The Fusion Diagrams

These diagrams satisfy the following, which gives the fusion rules the structure of an associative algebra



Moreover, one has $(N_i)_j{}^k = \mathcal{N}(ijk)$ as the representations of the algebra

$$N_i N_j = \sum_k \mathcal{N}(ijk) N_k \tag{19}$$

Virasoro Characters

In a given representation $[\phi_j]$, we define the Virasoro characters

$$\chi_j = \mathrm{Tr}_{[\phi_j]}\left(q^{L_0-\varepsilon}\right) \equiv \boxed{\quad j\uparrow}. \tag{20}$$

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$(q = e^{2\pi i\tau})$ Under modular transformations generated by $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -\frac{1}{\tau}$, χ_j changes as

$$\begin{aligned} T : \boxed{\quad j\uparrow} &\mapsto e^{2\pi i(h_j+\varepsilon)} \boxed{\quad j\uparrow}, \\ S \ : \boxed{\quad j\uparrow} &\mapsto \sum_k S_j{}^k \ \boxed{\quad k\uparrow}, \end{aligned} \tag{21}$$

with $S_j{}^k$ a unitary matrix.

Verlinde Lines

We define twist operations of the characters with ϕ_i operators along the $a-$cycle and the $b-$cycle, which we represent as follows

$$\phi_i(a)\chi_j \equiv \boxed{\underset{i}{\rightarrow}\quad j\uparrow}, \tag{22}$$

$$\phi_i(b)\chi_j \equiv \boxed{i\uparrow \quad j\uparrow}. \tag{23}$$

These correspond to winding a ϕ_i operator inside the trace once. One can see that

$$\boxed{\underset{i}{\rightarrow}\quad j\uparrow} = \lambda_i^{(j)} \boxed{\quad\quad j\uparrow}, \tag{24}$$

$$\boxed{i\uparrow \quad j\uparrow} = \sum_k A_{ij}{}^k \boxed{\quad\quad k\uparrow}, \tag{25}$$

where $A_{ij}{}^k$ are related to $\mathcal{N}(ijk)$.

"Twisting Operations" and Modular Invariance

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using the fact that $S_k{}^m$ is unitary, we find

$$A_{ij}{}^k = \sum_m S_j{}^m \lambda_i^{(m)} S_m^{\dagger k}. \tag{28}$$

Verlinde Rules

This is a remarkable result. The matrices $(N_i)_j{}^k \equiv \mathcal{N}(ijk)$, are diagonalized by the S-matrices of modular transformations. This is the main result of the seminal paper by E. Verlinde (1988).

Verlinde's Result

The modular transformations S for an RCFT on the torus diagonalize the fusion algebra, hence solves the fusion rules purely in terms of the unitary matrices $S_i{}^j$.

The Verlinde lines play an important role in 3d Topological Quantum Field Theory (TQFT), and in Knot Theory through Witten's Knot invariants defined via Chern-Simons TQFT.

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- The Verlinde lines studied here are closely related to the fusion lines $a, b, \cdots \in \mathcal{M}$ for theories incorporating non-invertible symmetries .
- We also studied extensions of results about the asymptotic density of states in the presence of $\mathcal{M}$, but we do not have the time time dwell on that.