Topological Phases and Quantum Lattice Systems Review

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Abstract

This is a review of the paper arXiv:2201.01327 [KS22]. I took these notes from a lecture series given by A. Kapustin which can be found here. The main purpose here is to understand what is the phase of matter in a quantum lattice system, and some topological notions that characterize it.

Contents

Introduction

In condensed matter systems, there are what's called the gapless phases and gapped phases that characterize a system. When a system is gapless, it basically means that there are long-range excitations in the system, so that the creation of the corresponding mode is continuously connected to the ground state. Photon would be one such example, where we could naively view the ground state as a photon at the limit where its wavelength goes to infinity. In such systems, no matter the length scale, there are always excitations that one must consider, like the soft photons. In contrast, when the system is gapped, the excitations above the ground state cost certain energy Δ, which can be viewed as a mass, so that at energies below that gap, we can effectively integrate out the corresponding fields, which are called short-range modes. A field whose modes have a mass m is such an example. The gapped phases sometimes include the word "topological", whose meaning is not crystal clear as far as I understand.

When one deals with systems of ∞ volume and $T \geq 0$, the study of the phases of matter requires the technology of C^*-algebras and the operator formalism in quantum mechanics, for which [BR79] is a good reference.

Also, the problem of the phases of matter is related to Topological Quantum Field Theories (TQFTs), whose study involves dealing with topology and category theory deeply. In particular, the ground states of the quantum matter systems are approximated by TQFTs. The relation between the gapped (topological) phases of matter and the study of TQFT is provided by the homotopy theory

$$\text{Gapped (topological) phases} \xleftarrow{\text{Homotopy}} \text{Topological Quantum Field Theory.}$$

Now we ask the following question: What is a phase of matter? The answer to this question is that systems that qualitatively behave the same way are in the same phase. This is made more precise by the mathematical notion of an equivalence class, and systems that are in the same class will have the same behavior under this formalism. We note that we are considering the case of ∞ volume systems. We will give the precise formulation of what we mean by a system in the next section, which will require the theory of C^* algebras. I will mainly follow the video lectures, but will include some stuff from [BR79] as well.

1 Lecture 1: Phases of Matter

1.1 The Notion of Physical (Quantum) System

We start with a formal construction of a physical system. Since we are doing quantum theory, the usual construction involves the Hilbert space $\mathcal{H}$ and operators acting on it. However, it turns out the C^*-algebra formalism is better suited here. Of course, the two formalisms are not independent. We can go from one to the other easily. In particular, by the Gelfand-Naimark-Segal (GNS) construction, if we know the algebra $\mathcal{A}$ and the states $\omega : \mathcal{A} \to \mathbb{C}$, we can construct the corresponding Hilbert space.

On the other direction, we have an algebra of observables $\mathcal{A}$, and view the physical state ω as a linear functional from the algebra $\mathcal{A}$ to the number field $\mathbb{C}$ as $\omega : \mathcal{A} \to \mathbb{C}$, and the evolution of the system will be governed by some Hamiltonian H. To connect this with the Hilbert space formalism, we can just view a state $\psi \in \mathcal{H}$ in the density matrix formulation $\rho = |\psi\rangle\langle\psi|$ and identify $\omega : \mathsf{a} \mapsto \omega(\mathsf{a}) \in \mathbb{C}$ as $\mathrm{Tr}\big(\mathsf{a}\rho\big) \in \mathbb{C}$, where $\mathsf{a} \in \mathcal{A}$ and $\mathsf{a}\rho$ is the matrix multiplication on $\mathcal{H}$ and Tr is the trace operation on $\mathcal{H}$.

Of course, we need to make sure that $\mathcal{A}$ is a well-defined algebra, in the sense that it is an algebra containing bounded operators. With this, the physical system consists of the following data:

$$\mathsf{S} : \Big(\mathcal{A},\ \omega : \mathcal{A} \to \mathbb{C},\ H\Big), \tag{1}$$

where $\mathcal{A}$ is a uniformly hyperfinite (UHF) C^* algebra. This is basically an infinite product of matrix algebras in the sense of a norm $||\ ||$. The state ω is a linear complex-valued function as we discussed above. Its linearity is easy to see from the connection with the Hilbert space: $\omega(\cdot) \sim \mathrm{Tr}(\rho_\omega \cdot)$, where ρ_ω is a density matrix acting on $\mathcal{H}$ that corresponds to the state ω. The Hamiltonian H generates the time evolution of the system via its one-parameter automorphisms. Since H is an observable as well, it acts on the Hilbert space $\mathcal{H}$ associated to the pair $(\mathcal{A},\ \omega)$ by the GNS construction.

1.2 The Notion of Gapped System

What does it mean for a system to be gapped? A system is said to be gapped if there is a unique ground state satisfying

$$H|0\rangle = 0, \tag{2}$$

and the spectrum of H must be such that

$$\text{spec } H \subset \{0\} \cup [\Delta, \infty), \quad \Delta > 0. \tag{3}$$

That is to say, there is a unique state of zero energy, and any excitation costs at least Δ energy so that H is a positive-semidefinite operator; or positive definite

on the subspace $\mathcal{H}\backslash\{\lambda|0\rangle \ |\lambda \in \mathbb{C}^\times\}$, with $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.

A simple example of a system is the trivial system. For the trivial system, we have

$$\mathcal{A} = \bigotimes_{j\in\mathbb{Z}^d} \mathcal{A}_j, \quad \mathcal{A}_j = B(\mathcal{H}_j), \tag{4}$$

$$\omega = \bigotimes_{j\in\mathbb{Z}^d} \omega_j, \tag{5}$$

where $B(\mathcal{H}_j)$ is a set of operators acting on a finite-dimensional Hilbert space and ω_j is a pure state of the individual algebras $\mathcal{A}_j$. This system is trivial since the states and the observables completely decompose as tensor products. ("pattern of entanglement")

How can one identify nontrivial patterns of entanglement? Let us discuss some examples of non-trivial gapped phases. We will start from the well-known example of the Hall effect, which are examples of gapped phases.

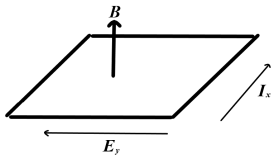


Figure 1: The setup of the Hall effect. There is an $xy-$plane and a constant magnetic field along the $z-$axis. As shown in the figure, in this setup there is no flow along the electric field E_y, but along the diagonal direction, there is a current I_x.

Integer Quantum Hall Phases (IQHE)

The main setup of the Integer Hall Effect is as in Figure 1. At 0 temperature, the material is an insulator along the direction of the electric field E_y, but there is a current I_x which is given by

$$I_x = \sigma_H E_y, \quad 2\pi\sigma_H \in \mathbb{Z}. \tag{6}$$

The parameter σ_H is called the Hall conductance, and that it is an integer comes from topological effects. In fact, σ_H is an invariant of the gapped phase in the IQHE setup, which characterizes the phase

Fractional Quantum Hall Phases (FQHE)

The setup is the same as that of IQHE. Here, we are on finite temperature, and the difference with IQHE is that

$$2\pi\sigma_H \in \mathbb{Q} \tag{7}$$

where $\mathbb{Q}$ is the set of rational numbers. Typically, σ_H is such that its denominator is small. Again, σ_H can be seen as an invariant of a phase.

One of the goals here is to understand why σ_H is robust and why it is $\mathbb{Q}$ valued.

1.3 Topological Quantum Field Theory

Now we discuss TQFT. These are special types of Quantum Field Theories that do not depend on the geometric structure of the spacetime M over which they are defined. Typically, a QFT is sensitive to the geometry (and things like the spin structure) of the manifold M they are defined over, but TQFTs have the property that they depend only on the topology of M, and the $n-$point functions -the observables- of the theory are topological invariants of M (unless there is an anomaly). In TQFT, the energy-momentum tensor is proportional to the identity operator

$$T^{\mu\nu} \sim \frac{\delta S}{\delta g_{\mu\nu}} \sim 1 \cdot f(g), \tag{8}$$

where $f(g)$ is some function of the metric. There is a categorical description of TQFT, given by Atiyah-Segal axioms, which is as follows.

1.3.1 Atiyah-Segal Picture

A TQFT in d-spacetime dimensions is a functor between two categories. We will first discuss the notion of category. It is a generalization of structures in mathematics. In a category $\mathcal{C}$, we have objects $\mathsf{a}, \mathsf{b} \cdots \in \mathsf{Ob}(\mathcal{C})$, and morphisms $f, g, \cdots \in \mathsf{Hom}(\mathcal{C})$, which maps objects to other objects, for example, $f : \mathsf{a} \mapsto \mathsf{b} \in \mathsf{Hom}(\mathsf{a}, \mathsf{b})$ is a morphism from a to b. A morphism between two objects need not be unique, as we will see in examples below. These morphisms have a natural composition $\circ$ such that if $f : \mathsf{a} \mapsto \mathsf{b}$ and $g : \mathsf{b} \mapsto \mathsf{c}$, then $g \circ f : \mathsf{a} \mapsto \mathsf{c}$. This composition satisfies two natural axioms:

1. Associativity: For any three morphisms $f \in \mathsf{Hom}(\mathsf{a}, \mathsf{b})$, $g \in \mathsf{Hom}(\mathsf{b}, \mathsf{c})$, and $h \in \mathsf{Hom}(\mathsf{b}, \mathsf{c})$, one has

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{9}$$

2. Identity Morphism: For any object $\mathsf{a} \in \mathsf{Ob}(\mathcal{C})$, there exists an identity morphism $\mathrm{id}_{\mathsf{a}} : \mathsf{a} \mapsto \mathsf{a}$, such that when composed with a morphism $f \in$

$\mathsf{Hom}(\mathsf{a},\mathsf{b})$ from both sides, one has

$$f \circ \mathrm{id}_{\mathsf{a}} = f = \mathrm{id}_{\mathsf{b}} \circ f. \tag{10}$$

Let us give some straightforward examples for categories.

$\mathcal{C} = \mathsf{Set}$

The category of sets consists of the following data:

1. The objects $\mathsf{Ob}(\mathsf{Set})$ consists of all the sets,
2. The morphisms $\mathsf{Hom}(\mathsf{Set})$ are all maps (functions) from any set A to any set B: $f \in \mathsf{Hom}(A,B)$, $f : A \mapsto B$. The composition of morphisms is the familiar composition of functions: $g \circ f = g(f(\cdot))$ such that $g(f(\cdot)) : A \mapsto g(f(A)) = g(B) = C$. It is easy to see that this composition rule is associative and there is the identity morphism $\mathrm{id}_A : A \mapsto A$.

$\mathcal{C} = \mathsf{Top}$

The category of topological spaces (spaces for which the transition functions defined on the intersections of the charts are always continuous functions) consists of the following data:

1. The objects $\mathsf{Ob}(\mathsf{Top})$ consists of all continuous manifolds (in the sense of the above parenthesis),
2. The morphisms $\mathsf{Hom}(\mathsf{Top})$ are homeomorphisms $\varphi : M \mapsto N$ between topological spaces. The homeomorphisms have a natural composition since their composition is again a homeomorphism. Let $\psi : N \mapsto L$, then $\psi \circ \varphi : M \mapsto L$, which is clearly associative, and there is an identity morphism $\mathrm{id}_M : M \mapsto M$ from M to itself that is a homeomorphism.

$\mathcal{C} = \mathsf{Man}^\infty$

The category of manifolds that are infinitely differentiable (that is, manifolds M for which the transition functions defined on the intersections of the charts are always in $C^\infty(M)$) consists of the following data:

1. The objects $\mathsf{Ob}(\mathsf{Man}^\infty)$ consists of all smooths manifolds (in the sense of the above parenthesis),
2. The morphisms $\mathsf{Hom}(\mathsf{Man}^\infty)$ are diffeomorphisms $\varphi : M \mapsto N$ between smooth manifolds. The diffeomorphisms have a natural composition since two diffeomorphism composed gives a diffeomorphism. Let $\psi : N \mapsto L$ be another diffeomorphism, then $\psi \circ \varphi : M \mapsto L$, which is clearly associative. There is an identity isomorphism which is a diffeomorphism of M to itself: $\mathrm{id}_M : M \mapsto M$.

Note that Man^∞ is just an improvement of Top where we replace continuity and homomorphism with differentiability (smoothness) and diffeomorphism, respectively.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, there exists a functor that maps the objects and morphisms of $\mathcal{C}$ to those of $\mathcal{D}$. Covariant functors are structure-preserving maps which consist of the following:

- F is a functor from $\mathcal{C}$ to $\mathcal{D}$, which is written as $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, such that for each $\mathsf{c} \in \mathcal{C}$ and $\mathsf{d} \in \mathcal{D}$, one has

$$\mathsf{F} : \mathsf{c} \mapsto \mathsf{d} = \mathsf{F}(\mathsf{c}), \tag{11}$$

  and moreover, for each morphism $f_c \in \mathsf{Hom}(\mathsf{c}, \mathsf{c}')$ and $g_d \in \mathsf{Hom}(\mathsf{d}, \mathsf{d}')$, one has

$$\mathsf{F} : \Big(f_c : \mathsf{c} \mapsto \mathsf{c}'\Big) \mapsto \Big(g_d : \mathsf{d} \mapsto \mathsf{d}'\Big). \tag{12}$$

- The functor F satisfies two natural conditions, which are the associativity

$$\mathsf{F} \circ \Big(\mathsf{G} \circ \mathsf{H}\Big) = \Big(\mathsf{F} \circ \mathsf{G}\Big) \circ \mathsf{H}, \tag{13}$$

  where $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, $\mathsf{G} : \mathcal{D} \to \mathcal{E}$, and $\mathsf{H} : \mathcal{E} \mapsto \mathcal{F}$; and there is an identity functor $\mathsf{F}(1_\mathsf{c})$, corresponding to each object $\mathsf{c} \in \mathcal{C}$, such that

$$\mathsf{F}(1_\mathsf{c}) : \mathrm{id}_\mathsf{c} \mapsto \mathrm{id}_\mathsf{c}. \tag{14}$$

A contravariant functor is similar to a covariant functor, except a contravariant functor changes the directions of the arrows in the morphisms, whereas covariant functors preserve the direction. In particular, every morphism $f_\mathsf{c} : \mathsf{c} \mapsto \mathsf{c}'$ in $\mathcal{C}$ is assigned to a morphism $\mathsf{F} : \Big(f_c : \mathsf{c} \mapsto \mathsf{c}'\Big) \mapsto \Big(g_d : \mathsf{d}' \mapsto \mathsf{d}\Big)$ where $\mathsf{d} = \mathsf{F}(\mathsf{c})$

and $\mathsf{d}' = \mathsf{F}(\mathsf{c}')$. Hence, a contravariant functor acts as a covariant functor from the opposite (dual) category $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. It is straightforward to see that $\mathcal{C}^{\mathrm{op}}$ is obtained from $\mathcal{C}$ by simply reversing the directions of the arrows, and that one has the usual property of a duality $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

We are almost ready to define TQFT in a categorical language. We need to define two categories.

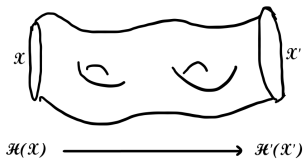


Figure 2: TQFT as a functor from the category Bord_n to the category $\mathsf{Vect}_{\mathbb{C}}$. In the figure, there are two boundaries $\mathcal{X}$ and $\mathcal{X}'$, to which there are associated Hilbert spaces $\mathcal{H}(\mathcal{X})$ and $\mathcal{H}'(\mathcal{X}')$, respectively. The partition function is a map between these Hilbert spaces, and this construction can be seen as a functor as there is an enclosing cobordism between the two manifolds $\mathcal{X}$ and $\mathcal{X}'$.

$\mathcal{C} = \mathsf{Bord}_d$

The category of bordisms Bord_d, whose objects are closed $d-$manifolds and whose morphisms are the cobordisms between two $d-$manifolds $\mathcal{X}$ and $\mathcal{X}'$ as shown in Figure 2.

$\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$

The category of vector spaces $\mathsf{Vect}_{\mathbb{C}}$, whose objects are vector spaces of finite dimensions over the field of complex number $\mathbb{C}$, and whose morphisms are linear maps between the vector spaces.

Now we define TQFT à la Atiyah-Segal. To the boundaries, we associate complex vector spaces $\mathcal{H}$, which live in the category of vector spaces, and between the boundaries, there is a cobordism, living in the category of $n-$bordisms. In this setup, we have

$$\mathsf{Hom}(\mathcal{X}, \mathcal{X}') \ni \mathscr{M} : \mathcal{X} \mapsto \mathcal{X}', \tag{15}$$

$$\mathsf{Hom}(\mathcal{H}(\mathcal{X}), \mathcal{H}'(\mathcal{X}')) \ni Z : \mathcal{H}(\mathcal{X}) \mapsto \mathcal{H}'(\mathcal{X}'), \tag{16}$$

and TQFT is a functor $\mathcal{F}$ between the two

$$\mathcal{F} : \mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}. \tag{17}$$

Figure 3: A $d-$bordism is a higher bordism between two points. The enclosing manifold is of dimension d.

This definition is not very useful, because it depends on global data and there is no notion of locality, which is one of the main features of QFTs.

Apparently, there is a better definition of TQFT using the notion of higher categories. A higher category has a more detailed structure than an ordinary category, where there are higher morphisms between the ordinary morphisms defined above. For example, a 2-morphism $f^{(2)} \in \mathsf{Hom}^{(2)}$ is a higher morphism between the morphisms $f \in \mathsf{Hom}$ and $g \in \mathsf{Hom}$, in the sense that $f^{(2)} : f \mapsto g$. This way one can go to ad infinitum, which are called ∞ categories. These notions contain higher structures, and each step is a refinement. It is possible to view these higher morphisms as homotopies between homotopies in the following sense: Two loops γ_1 and γ_2 in some space are homotopic if there exists a continuous map F_t such that $F_0 = \gamma_1$ and $F_1 = \gamma_2$. The F_t is a homotopy between the two curves γ_1 and γ_2, but is not unique. There are infinitely many different ways to deform one curve to another, which is analogous to a $2-$morphism. Suppose that there are two deformations $\gamma^{(1)}_{1\to 2}$ and $\gamma^{(2)}_{1\to 2}$, then one can find a homotopy between the two which is like a $3-$morphism. The story goes on and on.

Using this technology of higher categories, one defines a TQFT as follows. We consider the $d-$category $\widetilde{\mathsf{Bord}}_d$, whose objects are points, and morphisms are $d-$bordisms between the two points as shown in Figure 3. Although this is an improvement, there are still problems with this definition. One may replace $\widetilde{\mathsf{Bord}}_d$ with a $d-$infinity category, but then the question is what will replace $\mathsf{Vect}_{\mathbb{C}}$ as an ∞ category?

These are involved attempts at constructing a TQFT in a mathematical way. One can also define a TQFT from a physicist's approach, namely by defining an action and quantizing the given classical theory. The archetypal example of such a TQFT is the Chern-Simons gauge theory in 3-dimensions [Wit89, Wit88, DW90]. This theory will turn out to be very relevant for the description of the IQHE and FQHE [Ton16].

1.4 Integer Quantum Hall Effect and the Chern-Simons TQFT

We define the Chern-Simons theory as a $U(1)$ gauge theory on a $3-$manifold $M = M_2 \times \mathbb{R}$ with the action

$$S = \frac{k}{4\pi} \int_{M_2 \times \mathbb{R}} A \wedge dA, \tag{18}$$

where we have a Principle $U(1)$-bundle E over the spacetime $M_2 \times \mathbb{R}$: $U(1) \to E \xrightarrow{\pi} M$. If the bundle is trivial, that is if $E = M \times U(1)$ globally, then A is a $1-$form gauge field. It is possible to consider non-trivial bundles E, which are trivial locally $E = M \times U(1)$, but globally it cannot be decomposed, in which case A is a connection 1-form of the bundle defined locally like a section.

At the moment, S seems like a gauge non-invariant action, which does not make sense. However, if $k \in \mathbb{Z}$, then the action will be invariant up to $2\pi\mathbb{Z}$, so that the path integral weight e^{iS} will be gauge invariant. We have a current

$$j = \frac{k}{2\pi} * dA, \tag{19}$$

which is a topological current because the charge is given by $\int *j = \frac{k}{2\pi} \int dA$ (we use $*^2 = 1$). From this, one can make a connection with the Hall conductance. Specifically, looking at j_x:

$$j_x = \frac{k}{2\pi} \varepsilon_{x\nu\rho} \partial_\nu A_\rho = \frac{k}{2\pi}(-\partial_0 A_y + \partial_y A_0) \sim \frac{k}{2\pi} E_y, \tag{20}$$

we see that it resembles the equation $I_x = \sigma_H E_y$, from which we make the identification $k = 2\pi\sigma_H$.

There is another way to show that k must be integer for the gauge invariance of the Chern-Simons action. We extend the $3-$manifold $M_2 \times \mathbb{R}$ to a $4-$manifold X_4 and use Stoke's theorem

$$\frac{k}{4\pi} \int_{M_2 \times \mathbb{R}} A \wedge dA = \frac{k}{4\pi} \int_{X_4} F \wedge F, \tag{21}$$

where $F = dA$ is the curvature, which is gauge invariant for $U(1)$ case (in the non-abelian cases, F transforms in the adjoint representation of the gauge group, and one needs to take an inner product in the Lie algebra for gauge invariance in the $4d$ action). One can argue, by looking at the spin structure of X_4 and other geometric structures, that k should be an integer, which is a $4-$dimensional point of view in which gauge invariance is more manifest.

For the case of FQHE, one has $2\pi\sigma_H \in \mathbb{Q}$, for example $2\pi\sigma_H = \frac{1}{3}$. This corresponds, for example, to electrons of charge $e/3$. There are other curious

properties of gapped phases, which cannot be captured entirely by the field theory perspective (fractons etc.).

A big and important problem is finding lattice models behaving like TQFTs in the continuum limit. On the other hand, given a TQFT, it's not immediately obvious how to construct the corresponding lattice models. This is important because there is a general expectation that gapped phases have a relation with TQFTs. In the video, Kapustin says that he does not believe in this idea anymore.

1.5 Kitaev's Conjecture on Invertible Systems

To the idea that gapped (topological) phases are related to TQFTs, there came an improvement by Kitaev in a talk at 2013. His improvement focuses on special gapped-phases and special TQFTs, for which one can construct a correspondence.

The idea is the following: "invertible" systems are quite special, and there is a good chance to connect these systems to TQFTs using the homotopy theory. What is an invertible field theory? The mathematical definition is in some sense a monoid. One can tensor field theories together with a symmetric operation, and a similar thing can be done with the phases. If we have two systems $(\mathcal{A}, \omega)$ and $(\mathcal{A}', \omega')$, the symmetric operation reads

$$(\mathcal{A}, \omega) \circ (\mathcal{A}', \omega') = (\mathcal{A} \otimes \mathcal{A}', \omega \otimes \omega'), \tag{22}$$

where the composition $\circ$ above is stacking the systems together such that the two does not interact. The system is called invertible if there is an inverse, in the sense of the following. $(\mathcal{A}', \omega')$ is inverse to $(\mathcal{A}, \omega)$ if

$$(\mathcal{A} \otimes \mathcal{A}', \omega \otimes \omega') \sim (\mathcal{A} \otimes \mathcal{A}', \omega_0), \tag{23}$$

where ω_0 is a trivial state (unentangled spins), and $\sim$ means that the state $\omega \otimes \omega'$ can homotopically be deformed to a trivial state ω_0. Such a $(\mathcal{A}, \omega)$ is called an invertible system. In this construction, the phases of invertible systems forms a group (apparently, this has close relations to K-theory). Moreover, these invertible systems also has relations to topological insulators, again apparently[1].

Kitaev's argument is that all invertible systems (which are gapped by assumption) form a space X_d, and moreover he argued that $\Omega X_d \sim X_{d-1}$, where ΩX_d is the loop space, which consists of all maps $f : S^1 \to X_d$, and can be topologized. Note that the set of path components of ΩX_d is the fundamental group $\pi_1(X_d)$. So the spaces of all such spaces form an Ω spectrum. Hence, the

[1] When I use the word apparently, I mean I do not know where does this come from but I am interested in it.

set of phases in $d-$dimensions is equivalent to $\pi_0(X_d)$, which measures whether X_d is a connected manifold. And to each spectrum, there corresponds a generalized cohomology theory, $\pi_0(X_d) = hX^d(pt)$ where $hX^d(pt)$ is the generalized cohomology group of rank d. So according to Kitaev, if one wants to classify all invertible phases, one just needs to find the generalized cohomology group.

Kapustin has an approach to understanding the topology of X_d using the cohomology theory called the cobordism. It's apparently some dual of the cobordism, and one uses the relation (which holds under certain circumstances)

$$hX^d(M) = \mathrm{Hom}(\Omega_d^{\mathrm{SO}}(M), U(1)). \tag{24}$$

Apparently, Freed and Hopkins arXiv:1604.06527 [FH21] showed that invertible TQFTs are classified, up to homotpy, by $hY^d(pt)$, with Y the Anderson dual of oriented Thom spectrum. Based on this, the Kitaev's conjecture was refined to

$$X_d \sim Y_d, \tag{25}$$

where $\sim$ denotes homotopical equivalence. Apparently, Freed and Hopkins were working with functors from some ∞-categories to not-∞-categories, and they had to find a target category which they guessed correctly for the case of invertible TQFTs, and from that point on they were able to carry out the classification involving Y_d.

One interesting observation is that when $\Omega_d^{SO}(M)$ has torsion, $hX^d(M) = hY^d(pt)$. I do not know why this is interesting, but it may be closely related to the recent paper by Cordova and Freed arXiv:2408.15148 [CFT24].

The takeaway is that invertible TQFTs seems to be related with invertible gapped phases, which is the current hypothesis at the time of the talk. What are the consequences of this refined conjecture? This will be the next discussion, and the construction in the following also works for general gapped phases, not exclusive for invertible gapped phases.

1.6 Predictions on the Homotopy Type of X_d

Let us give some predictions for X_d spaces.

d	0	1	2	3
X_d	$K(\mathbb{Z}, 2)$	$K(\mathbb{Z}, 3)$	$K(\mathbb{Z}, 4) \times \mathbb{Z}$	$K(\mathbb{Z}, 5) \times S^1$

Table 1: Predictions for the X_d space in dimensions 0 to 3.

The $K(G, n)$ is called the Eilenberg-Maclane space that is defined by

$$\pi_m\big(K(G, n)\big) \simeq \delta_{nm} G \quad (n \text{ is fixed}), \tag{26}$$

that is, $K(G,n)$ has only $n-$th homotopy group non-trivial, and the rest of the homotopy groups are trivial. For certain G and n, we know what they are in terms of familiar manifolds. The space $K(\mathbb{Z},2)$ is $\mathbb{C}P^\infty$, which can be viewed as the lines in the Hilbert space. That X_0 is a connected space is the reflection of the fact that there is only one phase in $0-$dimensions, since not much happens in 0-dimensions. In any case, $K(\mathbb{Z},2)$ has interesting topology despite being simply connected.

For $1-$dimensions, there are again no interesting phases since $K(\mathbb{Z},3)$ is connected (π_0 vanishes), which means that there are no non-trivial phases. However, the topology of $K(\mathbb{Z},3)$ is more interesting than that of $K(\mathbb{Z},2)$. One can show that the space $K(\mathbb{Z},1)$ is the circle S^1, and $K(\mathbb{Z},2)$ is the complex projective space of infinite dimension $\mathbb{C}P^\infty = \lim_{n\to\infty}\mathbb{C}P^n$. It is interesting that for the $n=0$ case, one has $\pi_0(K(G,0)) = G$, so the phase is classified by a group. Does this make sense, is there any interesting appearances of the space $K(G,0)$ for certain choices of G like $\mathbb{Z}_2$? For dimension 2 and higher, it is not known (?) how to construct a system that gives the characteristic classes appearing in the table 1.

One can also give predictions for systems with $U(1)$ symmetry, which are related to Symmetry Protected Topological (SPT) phases. The table is as below

d	0	1	2	3
$\pi_0\Big(X_d^{U(1)}\Big)$	$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0

Table 2: Predictions on π_0 of $X_d^{U(1)}$.

For $d=2$, the one of the integers is related to the IQHE. And, one has

$$\pi_0\Big(X_d^{U(1)}\Big) = hX^d\Big(BU(1)\Big), \tag{27}$$

where $BU(1)$ is the classifying space of $U(1)$, which is equal to $K(\mathbb{Z},2)$. The classifying space is the space defined for the following purpose. Given a manifold M, we can construct a $G-$bundle E such that

$$\begin{array}{c} G\to E \\ \downarrow\pi \\ M, \end{array} \tag{28}$$

and one can ask how many different G-bundles are there over M? The answer to this question is provided by BG, the classifying space. We first construct a G-bundle $\xi_P(G)$ over BG

$$\begin{array}{c} G\to \xi_P(G) \\ \downarrow \\ BG. \end{array} \tag{29}$$

Now, there is a map f from M to BG $f : M \to BG$, and this induces a map from $\xi_P(G)$ to E

$$\begin{array}{ccccc} G & \to & E & \xleftarrow{f^*} & \xi_P(G) \leftarrow G \\ & & \downarrow & & \downarrow \\ & & M & \xrightarrow{f} & BG. \end{array} \tag{30}$$

Then, the following theorem holds [HH99]:

The Classification Theorem: There is a natural 1-1 correspondence between the set of equivalence classes of Principal $G-$bundles over M, and the homotopy classes of mappings $[M, BG]$ from M to BG of a universal bundle $\xi_P(G)$.

1.7 Lattice Systems and C^*-Algebras

In a lattice system, we have the algebra of observables $\mathcal{A}$, which is a Uniformly Hyperfinite (UHF) C^*-algebra

$$\mathcal{A} = \overline{\bigotimes_{j \in \Lambda} \mathcal{A}_j}, \tag{31}$$

where $\Lambda \subset \mathbb{R}$, $\mathcal{A}_j = B(\mathcal{H}_j)$ is a matrix algebra of finite dimension. There is a norm $||\ ||$ in this algebra, and $\overline{\otimes \mathcal{A}}$ denotes the completion of the algebra with respect to the norm. A C^*-algebra has the following properties.

- $\mathcal{A}$ is a Banach space,
- $||x \cdot y|| \leq ||x|| \cdot ||y||$,
- $||x^*|| = ||x||$,
- $||x^* x|| = ||x||^2$.

In a C^*-algebra, one can define a positive element $\mathsf{a} \in \mathcal{A}$ with two conditions

$$\begin{array}{c} \mathsf{a}^* = \mathsf{a}, \\ \mathrm{Spec}(\mathsf{a}) \subset [0, \infty), \end{array} \tag{32}$$

and the space of all such elements forms a subspace $\mathcal{A}^+$ of positive elements. We can also have a notion of potisive function $\omega : \mathcal{A} \to \mathbb{C}$ via

$$\begin{array}{l} \omega(\mathsf{a}) \geq 0, \quad \forall \mathsf{a} \in \mathcal{A}^+ \\ \omega(1) = 1. \end{array} \tag{33}$$

Given the algebra and the states, we get, by GNS construction, a representation on Hilbert space

$$\begin{array}{ccc} (\mathcal{A}, \omega) & \rightsquigarrow & \text{Representation} \\ & & \mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathcal{H}). \end{array} \tag{34}$$

However, we can just work with the C^*-algebras, without any reference to $\mathcal{H}$, which turns out to be useful.

In this picture, what gives the time evolution? First, we need a Hamiltonian H, and with that H the time evolution is going to be the one parameter -strongly continuous- group of automorphisms such that

$$\alpha_t : \mathcal{A} \to \mathcal{A}, \quad t \in \mathbb{R}; \tag{35}$$

- $\alpha_t(\mathsf{a}^*) = \Big(\alpha_t(\mathsf{a})\Big)^*$,
- $\alpha_t \cdot \alpha_s = \alpha_{t+s}$,
- $\alpha_t(\mathsf{a})$ is a continuous function of t.

Now we will write a condition on the state ω, without leaving the C^*-algebra side, such that it is a gapped ground state of the corresponding Hamiltonian. We first define a derivation map

$$\delta : \mathcal{D}(\delta) \to \mathcal{A}, \tag{36}$$

$$\delta(\mathsf{a}) = \lim_{t\to 0} \frac{\alpha_t(\mathsf{a}) - \alpha_0(\mathsf{a})}{t}, \tag{37}$$

where the domain of δ is a dense subset $\mathcal{D}(\delta) \subset \mathcal{A}$, and it satisfies a Leibniz rule

$$\delta(\mathsf{ab}) = \delta(\mathsf{a})\mathsf{b} + \mathsf{a}\delta(\mathsf{b}). \tag{38}$$

The derivations are related to the symmetry generators, such as the Hamiltonian. And, physically interesting derivations can be written as

$$\delta(\mathsf{a}) = \sum_j [\mathsf{h}_j, \mathsf{a}], \quad \mathsf{h}_j \in \mathcal{A}, \tag{39}$$

where we can view h_j as the "energy density" for each site on the lattice. There are issues about the domains, in particular whether they are dense in $\mathcal{A}$ or not. These issues are solved by the Lieb-Robinson bounds, which means that when we have a nice enough H, we can solve problems associated to whether the domains are dense in the algebra or not.

How does one define the ground state in this setting? We call ω a ground state for δ, if

$$-i\omega\Big(\mathsf{a}^*\delta(\mathsf{a})\Big) \geq 0, \quad \forall\ \mathcal{D}(\delta). \tag{40}$$

Roughly, this condition states that ω has 0 energy and all other states have non-negative energy.

The above only tells us if ω is a ground state, but it does not tell us whether it is a gapped state. To determine that we need the following condition: We call ω a gapped state with gap $\Delta > 0$ if

$$-i\omega\Big(\mathsf{a}^*\delta(\mathsf{a})\Big) \geq \Delta\Big(\omega(\mathsf{a}^*) - |\omega(\mathsf{a})|^2\Big), \tag{41}$$

which means that ω is the unique state of 0 energy -the ground state-, and all the other states have at least Δ energy. Observe that this condition is stronger than the above, and $\Delta \to 0$ recovers the first condition.

Nota bene that if ω is a gapped ground state for δ_1 and δ_2 with respective gaps Δ_1 and Δ_2, then it is also a gapped state with respect to the derivation

$$\delta_t \equiv t\delta_1 + (1-t)\delta_2, \quad t \in [0,1], \tag{42}$$

of which the gap satisfies $\Delta_t \geq \min(\Delta_1, \Delta_2)$. Therefore, for a given state ω, the space of Hamiltonians for which ω is a gapped ground state is contractbile, as one can define a homotopy δ_t between any two such Hamiltonians. Hence, when we consider the homotopy theory of states, the Hamiltonian becomes irrelevant, that is, it is automatically trivial.

Some notes and questions from lecture 1:

- Apparently, starting from a vertex operator algebra, one can reverse engineer a lattice model. But, in this construction, each site in the lattice one has ∞ dimensional space of vertex operators. But we would like a finite space, which is a problem.
- Regarding the above issue, what if we construct a lattice model starting from an RCFT? Due to the simplified structure of RCFT, there are finitely many families, so one may naively expect that the corresponding lattice system has a finite dimensional space of vertex operators at each site. For example, considering the compact scalar CFT, one has a $\mathbb{Z}_k$ algebra of vertex operators V_n $n \in \mathbb{Z}_k$, where n of V_n can be thought of as a momentum mode. Would the corresponding lattice system from the algebra of V_n have a finite dimensional space of vertex operators at each site? To learn about this one needs to learn how the reverse engineering construction works and apply the RCFT idea to it.

2 Lecture 2: Homology and all that

Getting back to the lattice systems, we have a C^*-algebra $\mathcal{A}$, and a completion $\mathcal{A}_\ell \subset \mathcal{A}$ which contains the local observables, and we have

$$\mathcal{A}_l = \bigotimes_{j \in \mathbb{Z}^d} \mathcal{A}_j. \tag{43}$$

And we would like to extract invariants for states $\omega : \mathcal{A} \to \mathbb{C}$. Physically, we never consider the totality of $\mathcal{A}$, the interesting observables are either local $\in \mathcal{A}_l$, or almost local $\in \mathcal{A}_{al}$, where almost local observables contains local observables: $\mathcal{A}_l \subset \mathcal{A}_{al} \subset \mathcal{A}$. To give a measure of how local is an observable $\mathsf{a} \in \mathcal{A}$, we define a function

$$f_j(\mathsf{a}, r) = \inf_{\mathsf{b} \in \mathcal{A}_{B_j(r)}} ||\mathsf{a} - \mathsf{b}||, \tag{44}$$

where $j \in \mathbb{Z}^d$, $B_j(r)$ is a ball of radius r centered at the $j-$th site, and inf is the infimum of the set of all such norms $||\mathsf{a} - \mathsf{b}||$. We thus define the algebra of almost local observables as

$$\mathcal{A}_{al} = \Big\{ \mathsf{a} \in \mathcal{A} \;\Big|\; \underbrace{\sup_r (1+r)^\alpha f_j(\mathsf{a}, r)}_{= ||\mathsf{a}||_{j,\alpha}} < \infty; \; \forall \alpha \in \mathbb{N} \Big\}, \tag{45}$$

and this does not depend on the site j, because the number of points inside a ball grows polynomially, whereas we require that the accuracy of the approximation is faster than polynomial growth, hence it does not matter what j we choose, it all gives the same subset.

Instead of this, we can view $\mathcal{A}_{al}$ as a completion of $\mathcal{A}_l$ with respect to a complete family of norms

$$||\mathsf{a}||, \quad ||\mathsf{a}||_{j,\alpha}, \quad \alpha = 1, 2, \cdots, \tag{46}$$

where we define the norm as

$$||\mathsf{a}||_{j,\alpha} = \sup_r (1+r)^\alpha f_j(\mathsf{a}, r). \tag{47}$$

Due to this completion, $\mathcal{A}_{al}$ has the structure of a Fréchet algebra. We define the subset $\mathfrak{d}_{al} \subset \mathcal{A}_{al}$ as the subspace of anti-self-adjoint $\mathsf{a} = \mathsf{a}^*$ and traceless observables. The subalgebra $\mathfrak{d}_{al}$ is a Lie algebra, and we will mainly use this subalgebra, as it is physically relevant. Given this, we define the derivations

Definition: A Uniformly Almost Local (UAL) derivation is a derivation of the form

$$\delta_{\mathsf{F}}(\mathsf{a}) = \sum_{j \in \mathbb{Z}^d} [\mathsf{f}_j, \mathsf{a}], \tag{48}$$

where $\mathsf{f}_j \in \mathfrak{d}_{al}$ is h-confined at j, in the sense of the following. An observable a is h-confined if

$$f_j(\mathsf{a}, r) \leq h(r) = O(r^{-\infty}), \tag{49}$$

for some positive function $h(r)$. Also, the UAL derivation satisfies

$$\delta_{\mathsf{F}}(\mathsf{a}^*) = \delta_{\mathsf{F}}(\mathsf{a})^*. \tag{50}$$

We denote the space of UAL derivations (of $\mathcal{A}_{al}$) as $\mathfrak{D}_{al}$, and this is a Fréchet Lie-algebra.

2.1 Ambiguities in Densities & Currents in QFT

In a local QFT, one has symmetries which leads to conserved local currents, written as

$$\frac{\partial \rho(x)}{\partial t} = -\nabla \cdot \mathbf{j}, \tag{51}$$

which is called the continuity equation. This can be written in the form language as

$$\begin{aligned} \frac{\partial \rho}{\partial t} = -dj; \quad & \rho \in \Omega^D(M \times \mathbb{R}), \\ & j \in \Omega^{D-1}(M \times \mathbb{R}), \end{aligned} \tag{52}$$

where the spacetime is of the form $M \times \mathbb{R}$ with $\dim M = D$, and $\Omega^k(M \times \mathbb{R})$ is the space of k forms on space-time. Associated to the local currents, there are conserved charges defined as

$$Q = \int d^d x \rho(x). \tag{53}$$

There are ambiguities in the definitions of the local density and the current here.

- First ambiguity: Under $\rho \mapsto \rho + dk$, Q changes up to a surface term,
- Second ambiguity: Under $j \mapsto j + dm$, dj does not change at all.

Hence, the physics of the field theory is independent under certain choices of ρ and j. There is a corresponding ambiguity for the lattice systems and their derivations which we discuss now.

Lattice Ambiguities: As an analogue of Heisenberg's equation on a lattice, we have

$$\delta_Q(\mathsf{a}) = \sum_{j \in \mathbb{Z}^d} [\mathsf{q}_j, \mathsf{a}], \tag{54}$$

on a lattice. The first ambiguity is

$$\mathsf{q}_j \mapsto \mathsf{q}_j + \sum_{i \in \mathbb{Z}^d} K_{ij}, \tag{55}$$

where $K_{ij} \in \mathfrak{d}_{al}$ satisfies $K_{ij} = -K_{ij}$ and is f-confined at both i and j. If i and j are far apart, then K_{ij} must be small because it localized both at i and j, and the only way to preserve that when the two are far away is to ensure K_{ij} is small.

The conservation equation on the lattice reads

$$\begin{aligned} \frac{d\mathsf{q}_j}{dt} = -\sum_i J_{ij}, \quad & J_{ij} \in \mathfrak{d}_{al} \\ & J_{ij} = -J_{ji}, \end{aligned} \tag{56}$$

and J_{ij} is $f-$confined both at i and j. Physically, J_{ij} is the flow of charge from point j to i. According to Kapustin, this equation is not discussed in condensed

matter texts very much, but it simplifies a lot of things. It is discussed in appendix D of [Kit06].

If we set $\mathsf{q}_j = \mathsf{h}_j$, with h_j the energy density at the lattice site, we write

$$H = \text{"}\sum \mathsf{h}_j\text{"}. \tag{57}$$

With this, the conservation of energy can be written, using the analog of the Heisenberg equation, as

$$\frac{d\mathsf{h}_j}{dt} = [H, \mathsf{h}_j] = \sum_i [\mathsf{h}_i, \mathsf{h}_j], \tag{58}$$

and we define

$$J^E_{ij} \equiv -[\mathsf{h}_i, \mathsf{h}_j], \tag{59}$$

so that

$$\frac{d\mathsf{h}_j}{dt} = -\sum_i J^E_{ij}, \tag{60}$$

where we remember that J^E_{ij} is the energy flux from site j to site i.

In the field theory case, the two ambiguities on ρ and j are the only ambiguities, due to Stoke's theorem and Poincaré lemma. On the lattice, the situation will be discussed presently.

Definition: An $n-$chain is a function ($\Lambda = \mathbb{Z}^d$)

$$\begin{aligned} &\overbrace{\Lambda \times \cdots \times \Lambda}^{n-\text{times}} \to \mathfrak{d}_{al} \\ &(j_1, \cdots, j_n) \mapsto \mathsf{f}_{j_1 \cdots j_n}, \end{aligned} \tag{61}$$

such that

- $\mathsf{f}_{j_1 \cdots j_n}$ is skew-symmetric,
- $\mathsf{f}_{j_1 \cdots j_n}$ is $h-$confined at $j_1, \cdots, j_n$.

We denote the space of $n-$chains with C_n, where $n = 1, 2, \cdots$. Moreover, for convenience, we denote $\mathfrak{D}_{al} \equiv C_0$. With this, we get a chain complex

$$\cdots C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0, \tag{62}$$

where the boundary map is defined as

$$(\partial \mathsf{f})_{j_1 \cdots j_{n-1}} = \sum_k \mathsf{f}_{k j_1 \cdots j_{n-1}} \quad (\mathsf{f} \in C_n, n > 1). \tag{63}$$

Clearly, ∂ is nilpotent

$$(\partial\partial\mathsf{f})_{j_1\cdots j_{n-2}} = \sum_k \sum_l \mathsf{f}_{klj_1\cdots j_{n-1}} = 0, \tag{64}$$

since f is skew-symmetric. Hence $\partial^2 = 0$. The case of $n = 1$ is a bit different. Taking $\mathsf{f} \in C_1$, we define

$$\partial\mathsf{f}(\mathsf{a}) = \sum_j [\mathsf{f}_j, \mathsf{a}]. \tag{65}$$

Now we get back to the ambiguities on the lattice. Observe that q_j is ambiguous up to

$$\sum_i K_{ij}, \quad K_{ij} = -K_{ij}, \tag{66}$$

for a $K_{ij} \in \mathfrak{d}_{al}$. By definition of boundary operator, we must have a $2-$chain whose boundary gives $\sum_i K_{ij}$

$$(\partial\mathsf{k})_j \equiv \sum_i K_{ij}, \tag{67}$$

hence, we will say that q_j is ambiguous up to a $\partial-$exact 1-chain, or equivalently, the ambiguity is up to the image of ∂. Having understood this ambiguity (the gauge redundancy), what are the other ambiguities?

Theorem: The chain complex (62) is acyclic.

Apparently, this means that there is no homology, hence no other obvious ambiguities here. $\Big($I am a bit confused here, in above, we discussed q_j lives in some sort of homology because we "mod out" $\partial-$exact currents, but then we say there is no homology. What does this really mean? $\Big)$

$C_0 = \mathfrak{D}_{al}$ is a Lie algebra, so the bracket of C_0 can be lifted to C_k. Actually, this lifting is possible because C_0 is a graded differential Lie algebra. This is apparently in sharp contrast to QFT, where we can't lift the Lie algebra to higher C_k, because for higher chains the bracket may not satisfy the Jacobi identity.

Claim: (C_0, ∂) is a differential graded Lie algebra (DGLA). In fact, it is a Fréchet Lie algebra.

$$[\mathsf{f}, \mathsf{f}']_{i_1\cdots i_n j_1\cdots j_m} \sim [\mathsf{f}_{i_1\cdots i_n}, \mathsf{f}_{j_1\cdots j_m}] + \text{signed permutations}, \tag{68}$$

where $\mathsf{f} \in C_n$, $\mathsf{f}' \in C_m$. Suppose $n = 1 = m$, then one has

$$[\mathsf{f}, \mathsf{f}']_{ij} = [\mathsf{f}_i, \mathsf{f}'_j] - [\mathsf{f}_j, \mathsf{f}'_i], \tag{69}$$

and

$$\partial[\mathsf{f}, \mathsf{f}'] = [\partial\mathsf{f}, \mathsf{f}'] + (-1)^{|\mathsf{f}|}[\mathsf{f}, \partial\mathsf{f}'], \tag{70}$$

which is the graded Leibniz rule. In this chain notation, the energy current J^E_{ij} defined above is a $2-$chain

$$J^E_{ij} = -[\mathsf{h}_i, \mathsf{h}_j], \tag{71}$$

or equivalently

$$J^E = -\frac{1}{2}[\mathsf{h}, \mathsf{h}]. \tag{72}$$

For the case of electric charge, we define

$$Q = \sum_j \mathsf{q}_j, \tag{73}$$

and we have the current

$$J^{\text{electric}} = -[\mathsf{h}, \mathsf{q}], \tag{74}$$

from which we can write the Heisenberg equation

$$\frac{d\mathsf{q}}{dt} \sim -[\mathsf{h}, \mathsf{q}] \sim -J^{\text{electric}}. \tag{75}$$

2.2 Symmetries of a State:

Suppose $\omega : \mathcal{A} \to \mathbb{C}$ is a state. We want to define symmetries for this state without referring to a Hamiltonian, as it is irrelevant for the discussion of phases. By a symmetry of the state, we mean a transformation that does not excite it.

Definition: $\mathsf{F} \in \mathfrak{D}_{al}$ does not excite the state ω if

$$\omega\Big(\mathsf{F}(\mathsf{a})\Big) = 0 \quad \forall \mathsf{a} \in \mathcal{A}_{al}. \tag{76}$$

In the Hilbert space side, this corresponds to (under GNS construction)

$$\langle 0|[\hat{\mathsf{F}}, \hat{\mathsf{a}}]|0\rangle = 0, \tag{77}$$

where $\hat{\mathsf{F}}$ is some charge. This actually means that there is no spontaneous symmetry breaking, since $|0\rangle$ is an eigenstate of the charge. In the C^*-algebra formalism, we have the following definition for the chains:

Definition: $\mathsf{f} \in C_n$ does not excite the state ω if

$$\omega\Big([\mathsf{f}_{j_1\cdots j_n}, \mathsf{a}]\Big) = 0 \quad \forall j_1, \cdots j_n, \text{ and } \forall \mathsf{a} \in \mathcal{A}_{al}. \tag{78}$$

Let us give an example. If $n = 1$, then

$$\omega\Big([\mathsf{q}_j, \mathsf{a}]\Big) = 0 \tag{79}$$

is the condition for $\mathsf{q}_j \in C_1$ to not excite the state ω. In the Hilbert space side, this means that every single local charge has ground state as its eigenstate.

Now suppose that

$$Q = \sum_j \mathsf{q}_j, \tag{80}$$

that acts on a local observable as

$$Q(\mathsf{a}) = \sum_j [\mathsf{q}_j, \mathsf{a}], \tag{81}$$

and which does not excite ω. Can we choose q_j, using the gauge redundancy from $Q = \partial \mathsf{q}$, such that $\mathsf{q}_j \in C_1$ does not excite ω? To do so we will be led to the idea of homology. We first define

Definition: As a subspace of all chains, we define $C_n^\omega \subset C_n$ to be the space of $n-$chains that do not excite ω.

We note that $(C_\bullet^\omega, \partial)$ is a sub-complex of (C^ω, ∂). In fact, it is a sub DGLA (Differential Graded Lie Algebra), which is a direct consequence of the Jacobi identity.

Question: Can we lift a charge $Q \ni C_0^\omega$ to C_1^ω? What is the homology of this complex, and are there any objects that cannot be lifted to C_1^ω? That is to say, are there objects that are cycles but are not boundaries in this complex? Physically, it is not generally plausible for a $1-$chain to not excite a state (why?), so at first sight it seems like this lifting is not possible.

More specifically, taking ω to be pure and the unique ground state of a gapped H, is it the case that the complex $C_1^\omega \to C_0^\omega \subset C_0$ is exact for such ω?

Theorem: $(C_\bullet^\omega, \partial)$ is acyclic if ω is a unique gapped ground state of some UAL Hamiltonian. (Apparently, the proof of this theorem has some non-trivial steps.)

Kitaev showed a special case of this, where the derivation was H itself. Of course, by definition of a ground state, we have

$$\omega\Big(H(\mathsf{a})\Big) = 0, \tag{82}$$

that is to say, the state is not excited by H. Now we ask the question whether we can write H as

$$H = \sum_j \tilde{\mathsf{h}}_j, \tag{83}$$

such that

$$\omega\Big([\tilde{\mathsf{h}}_j, \mathsf{a}]\Big) = 0, \tag{84}$$

for all j. Kitaev showed this to be the case. Moreover, the argument can be generalized to derivations other than H that do not excite ω; and moreover can

be generalized for higher chains using the theorem about the acyclicity of the complex $(C_\bullet, \partial)$. Apparently, this is an important fact, and this is where the condition of being gapped enters into the theory.(!!!)

2.3 Invariants of States:

Now, using the technology developed up to here, we will define invariants of states. By invariants, we mean stuff that does not change under an equivalence class. And these invariants can be seen as numbers assigned to the phases.

Kapustin says that they didn't find any new non-trivial invariants which can be assigned to a state. The prediction is that such states on which new non-trivial invariants can be assigned would appear first in 2d systems. Recall that according to Kitaev's conjecture, at least for invertible states, one expects some $\mathbb{Z}$ valued classification in $2d$ and up. In $1d$, there's nothing in the sense that the space of states is connected, despite having some interesting topology. For $2d$ one can consider a marginal gapped state (invertible), in which case the disconnected pieces are high in number. But, according to Kapustin, it is not obvious whether this technology helps in zeroing in on any one of those disconnected pieces.

We will now assign numerical invariants to states with symmetries, where symmetry is in the sense of th above.

2.3.1 Numerical Invariants of Gapped Ground States with Symmetries:

Suppose ω is a unique gapped ground state of some UAL derivation

$$H = \sum_j \mathsf{h}_j, \tag{85}$$

and in addition a Lie group G acts on $\mathcal{A}_j$, the local algebra of observables. Moreover, suppose the action of G commutes with each h_j and preserves ω.

Actually, the last part, that G-action preserves ω, is not ver crucial. It will follow from the Goldstone theorem because if there is a gap then the ground state must be invariant under the action of G anyway (otherwise there is a spontaneous breaking of symmetry). Also, the condition that h_j commutes with the action of G is not very crucial. Suppose G is compact, which we take to be. In that case, even if the local densities are not separately invariant, one can always average over the action of G and define a new H that is invariant under G. But, since we are looking at the homotopy of the space of states, and since H is immaterial for that, imposing this condition will not be a separate constraint. Hence, all h_j commuting with G is merely a convenient assumption, whereas H being invariant under G is a constraint. Apparently, this is related to the fact

that there are no anomalies in the lattice (!!!).

Let us discuss an example where $G = U(1)$. This is actually a very relevant example, because this is the case where Hall conductivity comes from. We have

$$\omega \rightsquigarrow C_n^{\omega,U(1)}, \tag{86}$$

$$Q = \sum_j \mathfrak{q}_j, \tag{87}$$

where Q denotes the $U(1)$ charge. We will again assume, for convenience, that all $\mathfrak{q}_j$ are invariant under $U(1)$ action. The space $C_n^{\omega,U(1)}$ is the set of $n-$chains that does not excite ω and also being invariant under $U(1)$. From the acyclicity of (C_n^{ω}, ∂), it follows that $(C_n^{\omega,U(1)}, \partial)$ is also acyclic (again, by averaging over $U(1)$ we can make sure this is the case). We note that $Q \in C_0^{\omega,U(1)}$, and it can be written as a ∂ exact $0-$chain due to (87)

$$Q = \partial\tilde{\mathfrak{q}}, \quad \tilde{\mathfrak{q}} \in C_1^{\omega,U(1)}, \tag{88}$$

and the existence of such a $\tilde{\mathfrak{q}}$ follows from acyclicity. Let us try to compute the following commutator

$$[\tilde{\mathfrak{q}}_i, \tilde{\mathfrak{q}}_j] =?. \tag{89}$$

Note that we can always choose $[\mathfrak{q}_i, \mathfrak{q}_j] = 0$ since the action of $U(1)$ at different sites is separate. Since $\tilde{\mathfrak{q}}$ is a "fermionic" object, it may not commute with itself. Let us act on $[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}]$ with the boundary operator

$$\begin{aligned} \partial[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}] &= [Q, \tilde{\mathfrak{q}}] + (-1)^1[\tilde{\mathfrak{q}}, Q] \\ &= 2[Q, \tilde{\mathfrak{q}}] \\ &= 0, \end{aligned} \tag{90}$$

since $\tilde{\mathfrak{q}}$ is by assumption $U(1)$ invariant, hence it commutes with Q. Therefore, $[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}]$ is ∂ exact

$$[\tilde{\mathfrak{q}}, \tilde{\mathfrak{q}}] = \partial M; \quad M \in C_3^{\omega,U(1)}, \tag{91}$$

this is in some sense an abstraction of making $\tilde{\mathfrak{q}}_i$ on site. Note that $\tilde{\mathfrak{q}}$ are only approximately localized (confined to some i), whereas $\mathfrak{q}$ are localized.

Note that these chains under consideration are analogous objects to forms in field theory. Then, it would be interesting to define the notion of integration of these chains. Here the dimensionality of the system comes into play. $(d+1)-$chains can be integrated

$$\int : C_{d+1} \to \mathfrak{d}_{al}, \tag{92}$$

but this map is not unique, since there are co-cylces (kernel of $\int$ in some sense). Let us consider the $d = 1$ case. Here we can integrate a $2-$chain, called a

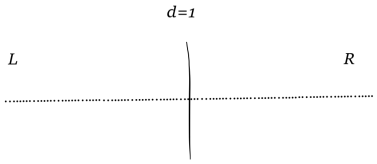


Figure 4: A $1d$-lattice divided into left and right.

current. In a $1d$-chain, we can define integration as counting the flux from some left to right on the lattice as in Figure 4:

$$\int_{L,R} J = \sum_{\substack{i\in L\\ j\in R}} J_{ij}. \tag{93}$$

For $d = 2$ case, we divide the lattice into three, and integrate a $3-$cycle as in figure 5

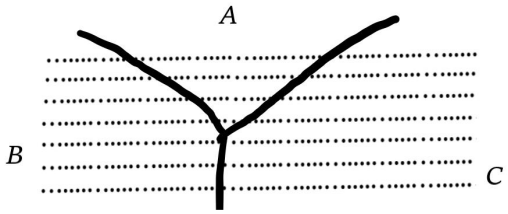


Figure 5: A $2d$-lattice divided into three regions A, B, C.

$$\int_{A,B,C} M = \sum_{\substack{i\in A\\ j\in B\\ k\in C}} M_{ijk}. \tag{94}$$

These integrals are convergent since J_{ij} and M_{ijk} are decaying quickly as the sites are far from each other. Hence, the sum localizes around the centers of the divisions, as illustrated in Figure 6 Note that there is an ambiguity in the choice

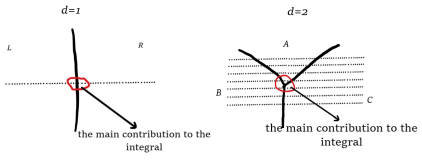


Figure 6: The integral of J in $1d$ lattice and that of M in $2d$ lattice localizes to the centers of the divisions.

of how to separate the regions. Kapustin says he doesn't know how to choose the most general integration region (!!!). He also says that it is sufficient to just choose any region in which the ∞ looks like some ordinary conical regions, they are all non-overlapping and the union of the three covers the entire lattice. Moreover, since the integral is localized to the centers, how wiggly the lines are is not crucially important, as long as the behavior at ∞ is well enough. An example of a reasonable choice is given in Figure 7. Let us now discuss some

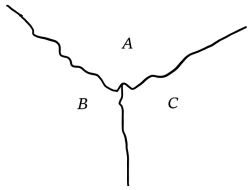


Figure 7: A reasonable choice of region in $2d$ lattice.

important properties. Let $\mathsf{f} \in C_{d+1}$, such that $\partial \mathsf{f} = 0$. Then, the following holds

$$I = \int_{A_1,\cdots,A_{d+1}} \mathsf{f}; \quad I \text{ does not depend on the choice of } A_1, \cdots A_{d+1}. \tag{95}$$

Physically, being closed means that the charge does not change, and hence, the net current along any region should be the same. That is to say, because no charge is accumulated anywhere, any integration region will give the same answer. For example, on a $1d$-lattice, if we take $J = \partial \mathsf{q}$, then we can choose the point that separates left and right any way we would like, and the integral of J will remain the same.

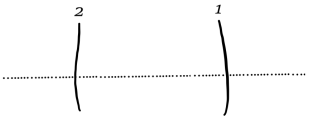


Figure 8: Two different divisions will give the same integral over J when it is closed: $J = \partial \mathsf{q}$.

$$\int_{①} J = \int_{②} J. \tag{96}$$

We can argue along similar lines for higher dimensional lattices. If $\partial \mathsf{f} = 0$, then $\mathsf{f} = \partial \mathsf{g}$, and hence the integral is actually trivial due to Stokes theorem

$$\int_{A_1,\cdots,A_{d+1}} \mathsf{f} = \int_{A_1,\cdots,A_{d+1}} \partial \mathsf{g} = 0. \tag{97}$$

One might ask then what is the point of all this? When discussing numerical chains, the integral $\int_A \mathsf{f}$ for $\partial \mathsf{f} = 0$ being independent of A will be more interesting because there, $\partial \mathsf{f} = 0$ will not mean ∂-exactness, that is, there may be closed f that cannot be written as $\mathsf{f} = \partial \mathsf{g}$.

We get back to the invariants of states. We look at

$$\omega\Big(\int_{A,B,C} M\Big) = \sum_{\substack{i\in A\\ j\in B\\ k\in C}} \omega\Big(M_{ijk}\Big) \in i\mathbb{R}, \tag{98}$$

this is purely imaginary because the chains were defined to be anti-self-adjoint. If we defined them self-adjoint, we would have it purely real.

Claim: $\omega\Big(\int M\Big)$ does not depend on the choice of a region, but it depends on an orientation. This is because $\partial\omega(M) = \omega(\partial M) = \omega([\tilde{\mathsf{q}}, \tilde{\mathsf{q}}]) = 0$, where $\omega([\tilde{\mathsf{q}}, \tilde{\mathsf{q}}]) = 0$ because $\tilde{\mathsf{q}}_i$ does not excite the ground state, for all i, hence

$$\omega([\tilde{\mathsf{q}}_i, \tilde{\mathsf{q}}_j]) = 0 \quad \forall i, j. \tag{99}$$

Note that changing M or $\tilde{\mathsf{q}}$ will not change anything since $\partial\omega(M) = 0$. Hence, $\omega\Big(\int_{A,B,C} M\Big)$ is genuinely an invariant of ω.

Important note: An invariant constructed in this way can only be done for $d \geq 2$, because we need a $3-$chain. The reason we need a $3-$chain is that the current is naturally a $2-$chain, and it is in the image of ∂ when it acts on some $3-$chain. In $d = 1$, we do not have any such object, so this definition of an invariant does not make sense there.

How does this construction work for higher dimensions? Say $d = 3$, then we need a $4-$chain to integrate over. One immediate guess would be

$$[\tilde{\mathsf{q}}, \tilde{M}], \tag{100}$$

since it is a $4-$chain. But this does not turn out to be the correct $4-$chain. We check

$$\begin{aligned} \partial[\tilde{\mathsf{q}}, \tilde{M}] &= [Q, \tilde{M}] + (-1)^1[\tilde{\mathsf{q}}, \partial M] \\ &= 0 + [\tilde{\mathsf{q}}, \partial\partial\cdots] \\ &= 0, \end{aligned} \tag{101}$$

where $[Q, \tilde{M}]$ is 0 because of $U(1)$ invariance, and the second term is zero because we have ∂^2. Hence

$$[\tilde{\mathsf{q}}, \tilde{M}] = \partial N, \quad N \in C_5^{\omega, U(1)}. \tag{102}$$

Since N is a $5-$chain, we can use that in $4d$ constructions. Iterating this procedure, we see that one can get a numerical invariant for every even dimension. Comparing this with physical expectations, this is apparently where one would

expect for Hall conductance to appear. In the field theory approach, one can measure the response of a system to some external field, and topological responses are given by Chern-Simons actions. And, CS actions (of gauge group $U(1)$) exist only on $d = D + 1$ dimensions, where D is even. For $3d$ CS of gauge group $U(1)$, there is only one invariant, which is identified with σ_H, and higher dimensional CS have generalizations of σ_H.

We can carry out similar constructions for general gauge groups G, and one apparently finds out that similar constructions do not give numbers but invariant polynomials on the Lie algebra $\mathfrak{g}$. And they again occur only for even spatial dimensions, which precisely allows CS. All invariants expected from the TQFT grounds are obtained in this formalism. Now we can see that the Hall conductance is an obstruction to promoting the symmetry to local symmetry in such a way that different sites still have commuting symmetries. This is exactly what M was an obstruction to.

We note that so far, what we did could have been done for any collections of points, not necessarily a lattice. In the lattice, the difference is that there is a notion of minimum distance between sites. The takeaway here is that for states with symmetries, we can associate invariants. Another thing we can do is define invariants of families of states, without caring about symmetry, which is what we will do next.

2.4 Invariants of Smooth Families of Gapped States:

Let ω_m be a gapped ground state of H_m (UAL derivation), where $m \in M$. Also, suppose that H_m depends smoothly on the parameter space M of the theory. ω_m here denotes a family of states $m \in M$. Because derivations form a Fréchet algebra, the notion of derivatives and smoothness makes sense. There is a more convenient viewpoint for this. One can show that if we have a family ω_m in the above sense then we have another interesting property that different states in the family are related by some automorphisms. This is referred to as a quasi-adiabatic evolution. The curve γ joining two points is a map

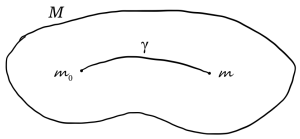


Figure 9: The parameter space as a manifold M. The two points are joined by a curve γ.

$$\gamma : [0, 1] \to M, \tag{103}$$

such that $\gamma(0) = m_0$ and $\gamma(1) = m_1$. Under these circumstances, there exists a map G such that

$$G : [0,1] \to \mathfrak{D}_{al}, \tag{104}$$

where $\mathfrak{D}_{al}$ is a smooth family of "Hamiltonians" but not the ones for which ω_{m_0} and ω_{m_1} are the ground states. From the G above, we define

$$\alpha_G(t) = P \exp\Big(\int^t ds G(s) \Big), \tag{105}$$

which we will call the evolution operator. Using this evolution operator, we can write

$$\omega_{\gamma(t)} = \omega_{\gamma(0)} \circ \alpha_G(t). \tag{106}$$

That is to say, there is a smooth family of automorphisms which maps state m_0 to state m_1, by applying some automorphism. In this perspective, $G(t)$ is the connection on a trivial bundle. Thus, a nice definition of smooth families is then the following: WE have a 1-form on M, valued on $\mathfrak{D}_{al}$, such that the family of states is sort of parallel with respect to this Hamiltonian. So we define

Definition: $\omega : M \times \mathcal{A} \to \mathbb{C}$ is smooth if there is a $G \in \Omega^1(M, \mathfrak{D}_{al})$ such that

$$d\omega(\mathsf{a}) = \omega\Big(G(\mathsf{a})\Big) \quad \forall \mathsf{a} \in \mathcal{A}_{al}, \tag{107}$$

which is like the equation of parallel transport, provided with the connection G. Hence, this is like a trivial bundle over a special subgroup (sub Lie-algebra) $\mathfrak{D}_{al}$.

Equivalently, we let $\mathsf{a} \in C^\infty(M, \mathcal{A}_{al})$. Then

$$d(\omega(\mathsf{a})) = \omega(D\mathsf{a}), \quad D\mathsf{a} = d\mathsf{a} + G(\mathsf{a}), \tag{108}$$

which is the condition for ω to be a local section of a dual bundle to the trivial bundle of observables.

Now that we have a smooth family, we want to attach some invariants to it. In ordinary finite-dimensional quantum mechanics, there's something called a Berry class. In that case, ω is just a family of projectors with $1-$dimensional, image in some Hilbert space $\mathcal{H}$, so one gets a line bundle out of the parameter space. And the Chern class of the line bundle is the Berry class.

Claim: To any smooth family, one can attach a class in $H^{d+2}(M)$, with H the de-Rham cohomology.

For $d = 0$, it is the usual Berry class. The question is, can we quantize this, that is, can we maybe lift it up to an integral cohomology? Kapustin says they don't know how to do it in general.

Idea: This sort of arguments are used a lot in [DW90]. Maybe those TQFTs discussed there are useful in understanding higher Berry classes.

For invertible theories, the expectation is that they be quantized, but it is not expected for general cases. Hence, the expectation is to prove that $H^{d+2}(M)$ is an integral class for invertible systems (!!!). Kapustin says that they proved it for $d = 1$ case for invertible systems.

We consider forms in $\Omega^\bullet(M, C^{-\bullet}) \curvearrowleft d + \partial \equiv \mathcal{D}$. Define

$$\mathcal{G} = G + \sum_{n=1}^{\infty} g^{(n)}, \tag{109}$$

where $g^{(n)} \in \Omega^{n+1}(M, C_n^\omega)$. Then, $\mathcal{G}$ is like a connection of degree 1. And we have the Maurer-Cartan equation

$$\mathcal{D}\mathcal{G} + \frac{1}{2}[\mathcal{G}, \mathcal{G}] = 0, \tag{110}$$

which says that the curvature of the generalized connection $\mathcal{G}$, which has forms of arbitrary degree and chains of arbitrary degree, is 0. G by itself has nonzero curvature in general, but by adding higher-form guys we make it to a flat connection. We will have $g^{(d+1)} \in \Omega^{d+1}(M, C_{d+1}^\omega)$ in the sum. We define

$$\omega\Big(\int_{A_1,\cdots,A_{d+1}} g^{(d+1)}\Big) \in \Omega^{d+2}_{\text{closed}}(M, i\mathbb{R}). \tag{111}$$

Moreover, we will have a cohomology class that does not depend on details like how we solve the Maurer-Cartan equation. It just depends on the family of the system we started with. This procedure gives the higher Berry class.

3 Lecture 3: Higher Berry Classes

A Brief Summary of Lecture 2:

We have a UHF C^*-algebra $\mathcal{A}$, and a Fréchet subalgebra $\mathcal{A}_{al}$ of almost local observables. The subalgebra $\mathcal{A}_{al}$ carries all the information about the metric structure of the space. From this, we construct

$$\Big(\bigoplus_{n=0}^{\infty} C_n(\mathfrak{d}_{al}), \partial, [\ ,\ ]\Big) = C_\bullet(\mathfrak{d}_{al}), \tag{112}$$

where $C_\bullet(\mathfrak{d}_{al})$ is the subspace of $\mathcal{A}_{al}$ consisting of anti-self-adjoint traceless observables. We have $C_0 = \mathfrak{D}_{al}$. And we note that apparently, for any differential graded Lie algebra (DGLA) $0-$th in the chain is always a Lie algebra.

C_1 can be seen as a density of a Hamiltonian, and

$$\mathfrak{D}_{al} \ni \mathsf{F} = \sum_{j\in\Lambda} \mathsf{f}_j. \tag{113}$$

Suppose we add a state $\omega : \mathcal{A} \to \mathbb{C}$. For any ω, we can define a sub-differential graded Lie algebra of elements that does not excite ω (symmetries of ω) $C^\omega_\bullet \subset C_\bullet$. And, that ω is a gapped phase is implies $C^\omega_\bullet$ has zero homology

$$\omega \text{ is gapped} \implies C^\omega_\bullet \text{ has trivial homology.} \tag{114}$$

Version 1: G acts on $\mathcal{A}_{\text{local}}$ while preserving ω.

$$Q \in \text{Hom}(\mathfrak{g}, \mathfrak{D}_{al}) \tag{115}$$

For $G = U(1)$, $\mathsf{q} = \bigoplus_{n=0}^\infty \mathsf{q}^{(2n+1)}$, $\mathsf{q}^{(p)} \in C_p(\mathfrak{d}_{al})$. And we have $[\mathsf{q}^{(1)}, Q] = 0$. We have the following

$$Q = \partial\mathsf{q}^{(1)}, \quad [\mathsf{q}^{(1)}, \mathsf{q}^{(1)}] = \partial\mathsf{q}^{(3)}, \quad \frac{1}{2}[\mathsf{q}^{(1)}, \mathsf{q}^{(3)}] + \frac{1}{2}[\mathsf{q}^{(3)}, \mathsf{q}^{(1)}] = \partial\mathsf{q}^{(5)}. \tag{116}$$

Moreover, $\omega(\mathsf{q}^{(d+1)}) \in C_{d+1}(\mathbb{R})$, $\partial\omega = 0$. We now define numerical chains.

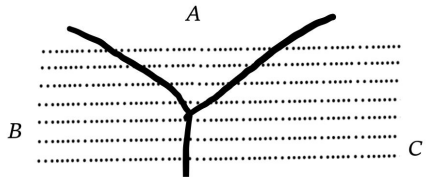
Definition: A numerical $n-$chain is a function

$$\mathsf{f} : (i_1, \cdots, i_n) \mapsto \mathsf{f}_{i_1,\cdots,i_n} \in \mathbb{R}, \tag{117}$$

such that

- f is skew-symmetric,
- f decays faster than $|i_l - i_k|^{-a}$ for all $a \in \mathbb{N}$.

Since $\partial\omega = 0$, we get an element of $H_{d+1}(C_\bullet(\mathbb{R}), \partial)$. Suppose $d = 2$, then we have $H_3(C_\bullet(\mathbb{R}), \partial)$, a $3-$chain. We then evaluate



$$\int \omega(\mathsf{q}^{(3)}) = \sum_{\substack{i\in A\\ j\in B\\ k\in C}} \omega(\mathsf{q}^{(3)})_{ijk}. \tag{118}$$

In $2d$ and at 0 temperature, this turns out to be precisely the Hall conductance, up to some numerical factor. In $d = 4$, one gets a generalization of the Hall

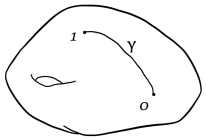


Figure 10: The two states connected by the map α.

conductance, so a coefficient in the CS term.

Version 2: Instead of a group action, consider a family of states

$$\psi : M \times \mathcal{A} \to \mathbb{C}, \tag{119}$$

where M is the parameter space. There exists a $G \in \Omega^1(M, \mathfrak{D}_{al})$ such that

$$d\psi(\mathsf{a}) = \psi(G(\mathsf{a})), \quad \forall \mathsf{a} \in \mathcal{A}_{al}. \tag{120}$$

The 1-form G can be used to connect different states on M via the automorphisms

$$\gamma : [0,1] \to M, \tag{121}$$

$$\psi_{\gamma(1)} = \psi_{\gamma(0)} \circ \alpha, \quad \alpha = P \exp\Big(\int_0^1 \gamma^* G \Big), \tag{122}$$

where γ^* is the pullback of G on the interval. The curvature of G is defined as

$$F = dG + \frac{1}{2}[G, G] \in \Omega^2(M, \mathfrak{D}_{al}), \tag{123}$$

or equivalently $F = (d + G)^2$. Write

$$g = g^{(1)} + g^{(2)} + g^{(3)} + \cdots \in \Omega^\bullet(M, C^\omega_\bullet(\mathfrak{d}_{al})). \tag{124}$$

$F = \partial \mathsf{q}^{(1)}$, and we write

$$\mathcal{D} g^{(1)} = \partial g^{(1)}, \tag{125}$$

$$[g^{(1)}, g^{(1)}] + \mathcal{D}^{(2)} = \partial g^{(3)}, \cdots \tag{126}$$

Moreover,

$$\omega(g^{(d+1)}) \in \Omega^{d+2}(M, C_{d+1}(\mathbb{R}), \tag{127}$$

$$(d + \partial)\omega(g^{(d+1)}) = 0. \tag{128}$$

To get a closed thing on M, we just integrate $\omega(g^{(d+1)}$

$$\int_{\text{cones}} \omega(g^{(d+1)}) \in \Omega^{d+2}(M), \tag{129}$$

which we can write as Ψ such that

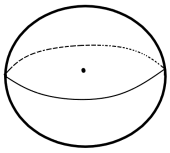
$$\int_{\text{cones}} \omega(g^{(d+1)}) = \Psi, \quad d\Psi = 0. \tag{130}$$

What are the applications of these constructions? Apparently, they provide constraints on the phase diagram of a many-body system.

3.1 Higher Berry Class

An Old Story of von Neumann-Wigner Crossings (1930s)

Apparently, generic level crossings in a system occur in codimension 3, so one needs to tune 3−parameters of H. Let the parameter space be S^2 embedded into the Euclidean space $S^2 \subset \mathbb{R}^3$. Can one deform S^2 to a bigger parameter



space in an interesting way? Apparently, not much happens because the point cannot be removed. The reason for this comes from topology, which was realized by Berry some decades later. Namely, the curvate of the Berry connection on this S^2 is nonzero, hence one cannot remove the level crossing there. Hence, when $\int_{S^2} F \neq 0$, then there must be a level crossing inside the sphere.

The many body version is as follows: A phase diagram will have 3−parameters. So one can imagine $S^{d+2} \subset \mathbb{R}^{d+3}$, and on that S^{d+2}, the Hihher Berry curvature integrates to non-zero. Then, there must be a point inside the sphere on which the gap closes. Because if it didn't close then the class would be on the ball but the ball has no cohomology. In this case we can view crossing of the excited states with the ground state by this Higher Berry curvature. This is a more straightforward interpretation of the higher Berry curvature. Let $G \in C^\infty([0,1], \mathfrak{D}_{al})$. The following theorem holds

Theorem: The ODE below has a unique solution

$$\frac{d\alpha(t)(\mathsf{a})}{dt} = \alpha(t)\Big(G(\mathsf{a})\Big), \quad \forall \mathsf{a} \in \mathcal{A}_{al}, \tag{131}$$

such that

$$\alpha(t)(\mathsf{a}) \in \mathcal{A}_{al}, \quad \alpha(0) = \mathrm{id}. \tag{132}$$

This theorem follows from Lieb-Robinson bounds. The interpretation of the theorem is that exponentiation of the initial state with a Hamiltonian preserves the locality of the initial state.

Definition: $\alpha : \mathcal{A}_{al} \to \mathcal{A}_{al}$ is a locally generated automorphism if $\alpha = \alpha(1)$ for some $G = C^\infty([0,1], \mathfrak{D}_{al})$.

Theorem: Locally generated automorphisms (LGA) form a group. (In fact a diffeological Lie group), whose Lie algebra is $\mathfrak{D}_{al}$.

Definition: Let ψ, ψ' be pure states on $\mathcal{A}$. We call $\psi \approx \psi'$ (LGA equivalent) if

$$\psi' = \psi \circ \alpha, \tag{133}$$

for some LGA α. If ψ is the ground state for a gapped H, then $\psi' = \psi \circ \alpha$ will be the ground state of another gapped H'. And, the two states are in the same phase $\mathcal{A} \ni \psi \sim \psi' \in \mathcal{A}'$ if there exists (purely factorized) ψ_0, ψ_0' such that

$$\psi \otimes \psi_0 \approx \psi' \otimes \psi_0'. \tag{134}$$

This is the definition of a phase of matter. Namely, when ψ_0 is a factorized state, we have

$$\psi \sim \psi_0 \Longleftrightarrow \psi \text{ is in a trivial phase.} \tag{135}$$

This means that if we tensor ψ with another pure factorized state in another algebra, then one can reduce ψ into ψ_0 by one of LGAs. That is to say, after tensoring, the states can be disentangled.

$$\begin{aligned} \psi \approx \psi_0 &\Longrightarrow \psi \text{ can be disentangled using an LGA} \\ &\Longleftrightarrow \psi \text{ is short range entangled.} \end{aligned} \tag{136}$$

ψ is invertible if there is a ψ' on $\mathcal{A}'$ such that

$$\begin{aligned} &\psi \otimes \psi' \text{ is short range entangled,} \\ (\psi \otimes \psi' \approx \psi_0) \rightarrow\ &\text{using an LGA on } \mathcal{A} \otimes \mathcal{A}', \end{aligned} \tag{137}$$

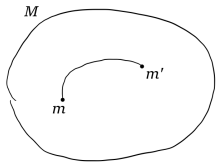
and such a ψ' is called the inverse of ψ. Apparently, invertible states have interesting properties. For example, if you stack different systems, the phases form a semi-group (a monoid). So the trivial element is like the trivial state. But, invertible states actually form an abelia ngroup since there is an inverse here.

Recall Kitaev's conjecture. If we look at the space of invertible states properly defined, should become an Ω-spectrum, as discussed in subsection 1.5. For that to make sense, we need to take in some sense a limit. Here, we have a fixed algebra $\mathcal{A}$, but to get an $\Omega-$spectru one needs to take a limit over all algebras, make a finer and finer lattice and put some extra DOF. Kapustin says he's not trying to construct the mystical ∞-dimensional space of states; rather, his purpose is to extract invariants out of it, by restricting to subspaces and by probing the topology via homology classes.

3.1.1 Another perspective on what the higher Berry class is an obstruction to

Given the parameter space M, we ask the question does there exist some $G \in \Omega^{0,1}(M \times [0,1], \mathfrak{D}_{al})$ such that

$$\alpha_m = P \exp\Big(\int G(m) \Big), \tag{138}$$



satisfying

$$\psi_m = \psi \circ \alpha_m. \tag{139}$$

In other words, the question is that can one globally generate all the states $m' \in M$, using LGAs, which depends smoothly on the initial point m?. Independent of what the manifold M is, we can always do this locally, because on a local chart the manifold is diffeomorphic to $\mathbb{R}^n$. However, when it is global there are some problems, depending on the structure of M. In particular, we can cover M with some charts U_α, and generate all the states within the chart provided U_α is a contractible chart. However, the condition of it being global is a stronger condition, and depends on whether the higher Berry class is trivial or not, namely whether the topology of M is non-trivial.

Question: Given the involvement of topology and connections, the following is a natural question to ask. Can we reformulate the above problem as a question of whether a $G-$bundle over M is trivial or not, in such a way that the Chern number of the bundle is related to the higher Berry class? In this construction, if it is viable, what do we choose G as?

Indeed, this perspective is in similar vein to the usual Berry class. Let us discuss the ordinary Berry class and a simple example.

3.1.2 The Ordinary Berry Phase

I will follow the discussion in [Nak03]. Let us consider a quantum theory with a Hamiltonian $H(m)$, depending on some parameter space $m \in M$. Suppose that m changes adiabatically with time $m = m(t)$. The time evolution of any state is governed by

$$H(m(t))|\psi(t)\rangle = i\frac{d}{dt}|\psi(t)\rangle. \tag{140}$$

Assume that $|\psi(0)\rangle = |n, m(0)\rangle$ where

$$H(m(0))|n, m(0)\rangle = E_n(m(0))|n, m(0)\rangle. \tag{141}$$

What is the evolved state $\psi(t)$? When $M = pt$, and H is time-independent the energy eigenstate is a stationary state, and through time it picks up a harmless phase $e^{-iE_n t}|n\rangle$ and remains an energy eigenstate. Here, things are a bit complicated since both $H(m(t))$ and $|n, m(t)\rangle$ has time dependence. When H is time dependent, the energy eigenstate evolves with the phase $e^{-i\int_0^t dsE(s)}$,

which comes from infinitesimally evolving the initial state and each time picking the phase $e^{-iE_n(\delta t)}$ and adding them up. So the evolved state will be of the form $e^{-i\int_0^t dsE(m(s))}|n, m(t)\rangle$, where $|n, m(t)\rangle$ is defined as

$$H(m(t))|n, m(t)\rangle = E_n(m(t))|n, m(t)\rangle. \tag{142}$$

Since the state $|n, m(t)\rangle$ is time dependent as well, the phase $e^{-i\int_0^t dsE(m(s))}$ is not enough to solve the equation. Thus, we make the ansatz

$$|\psi(t)\rangle = e^{i\eta_n(t)-i\int_0^t dsE(m(s))}|n, m(t)\rangle. \tag{143}$$

For convenience we will denote the phase as $e^{i\alpha(t)}$. Inserting into the Schrödinger equation, we get

$$e^{i\alpha(t)}E_n(m(t))|n, m(t)\rangle = e^{i\alpha(t)}\left(-\frac{d\eta_n}{dt} + E_n(m(t)) + i\frac{d}{dt}\right)|n, m(t)\rangle. \tag{144}$$

The left hand side cancels with the middle term of the parenthesis, and upon multiplying the rest of the equation with $\langle n, m(t)|$, we are left with

$$0 = -\frac{d\eta_n}{dt} + \langle n, m(t)|i\frac{d}{dt}|n, m(t)\rangle. \tag{145}$$

Solving for $\eta_n(t)$

$$\begin{aligned}\eta_n(t) &= i\int_0^t ds\langle n, m(s)|\frac{d}{ds}|n, m(s)\rangle \\ &= i\int_{m(0)}^{m(t)} dm\langle n, m|\nabla_m|n, m\rangle,\end{aligned} \tag{146}$$

where ∇_m is a gradient in M. It can be shown that $\eta_n(t)$ is real so that in the evolution the state only picks up a phase given by $\alpha(t) = \eta_n(t) - \int_0^t dsE(m(s))$. If we draw a closed loop in M in a time T, then η_n will be a loop integral

$$\eta_n(T) = i\int_{m(0)}^{m(T)} dm\langle n, m|\nabla_m|n, m\rangle. \tag{147}$$

Because it is a loop, $m(0) = m(T)$, so one might expect η_n to be zero, as it looks like the loop integral of a gradient. However, depending on the topology of M, this integral may or may not vanish. The integrand is like a connection A, called the Berry connection, and the loop integral can be written as $\int_D F$, where F is the Berry curvature $F = dA$, and D is a surface enclosing the closed loop. It was Simons who realized the connection between the Berry phase and the holonomy in the parameter space, which gives it a very nice geometrical interpretation.

The Berry connection is defined as

$$A = \langle m|d|m\rangle, \tag{148}$$

namely, a $1-$form on the parameter space. This is a connection in the following sense. In the quantum theory, $|m\rangle$ is identified with $e^{i\theta}|m\rangle$, so the physical states can be seen as living inside the equivalence class $[|m\rangle]$ under the projection operator π of a $U(1)$ bundle. Let us take two overlapping charts U_i and U_j

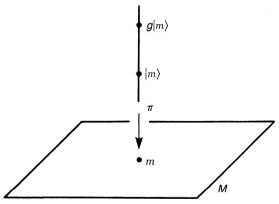


Figure 11: The states seen as a $U(1)$ bundle over M. Figure taken from [Nak03].

of M, with the respective local sections $\sigma_i(m) = |m\rangle_i$ and $\sigma_j(m) = |m\rangle_j$. The two sections on the overlap $U_i \cup U_j$ are related by the transition functions $\sigma_j(m) = \sigma_i(m)t_{ij}(m)$. Let us see how the two connections are related:

$$\begin{aligned} A_j = \ & {}_j\langle m|d|m\rangle_j = t_{ij}(m)^{-1}\ {}_i\langle m|d\Big(|m\rangle_i t_{ij}(m)\Big) \\ & = A_i + t_{ij}(m)^{-1}dt_{ij}(m), \end{aligned} \tag{149}$$

which is indeed the way a $U(1)$ connection transforms under a gauge transformation. From this connection, one defines a curvature $2-$form

$$F = dA. \tag{150}$$

To understand these structures better, we discuss a simple example where a Dirac monopole makes an intriguing appearance.

Example: Spin in a Magnetic Field
Let us discuss an electron living in $\mathbb{R}^3$ with some magnetic field $\mathbf{B}$, the Hamiltonian of which is

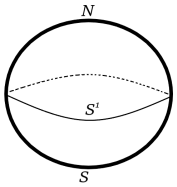
$$H(\mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\sigma}, \tag{151}$$

where $\boldsymbol{\sigma}$ is the Pauli matrices, associated to the spin-1/2 representation of the relevant rotation group, $SU(2)$. Fixing the length of $\mathbf{B}$ to some B, the parameter space is the 2-sphere S^2, that is, the direction of $\mathbf{B}$. We will shift H for convenience and work with

$$H = \mathbf{B} \cdot \boldsymbol{\sigma} - B. \tag{152}$$

In the polar coordinates (θ, ϕ) where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, the Hamiltonian reads

$$H = B \begin{pmatrix} \cos\theta - 1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - 1 \end{pmatrix}, \tag{153}$$



and the following is its ground state

$$|m\rangle_N = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}. \tag{154}$$

This state is singular when $\theta = \pi$, because at the north pole ϕ is not well defined. On the other hand, if we define

$$|m\rangle_S = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}, \tag{155}$$

this will be well defined at $\theta = \pi$ because $\cos \pi/2 = 0$, but this time it is singular at $\theta = 0$. This reflects the fact that a sphere can be covered with two charts, the north hemisphere and the south hemisphere, with a circle intersection at the equator, as in Figure 3.1.2. In these two different charts we have the connections

$$A_N = {}_N\langle m|d|m\rangle_N = -i\sin^2(\theta/2)d\phi = \frac{i}{2}(\cos\theta - 1)d\phi, \tag{156}$$

$$A_S = {}_S\langle m|d|m\rangle_S = i\cos^2(\theta/2)d\phi = \frac{i}{2}(\cos\theta + 1)d\phi. \tag{157}$$

On the equator S^1, the two can be related with a gauge transformation, or a transition function:

$$A_N = A_S - id\phi = A_S + e^{i\phi}de^{-i\phi}. \tag{158}$$

Since the two are related by a gauge transformation, they give rise to the same curvature:

$$F = dA_N = dA_S = \frac{i}{2}d(\cos\theta)d\phi = -\frac{i}{2}\sin\theta d\theta d\phi. \tag{159}$$

The above expressions can be identified with that of a Dirac monopole of charge $1/2$. Indeed, the Dirac monopole is also a $U(1)$ bundle over S^2, that is the reason for the similarity with here. The Berry phase η_n evaluated on a loop is identified with the flux of the monopole of charge $1/2$ through the enclosing surface.

Getting back to the Higher Berry class, let us discuss the case where $M = pt$, and the symmetry group H is not trivial. In this case, the Higher Berry class takes values in

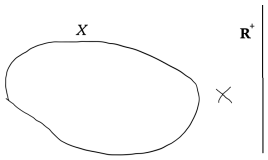
$$H_H^{(d+2)}(pt, \mathbb{R}) = \mathrm{Inv}^{\frac{d+2}{2}}(\mathfrak{h}). \tag{160}$$

What does this mean? Suppose we are given a state, and the question is can we, with a symmetry, disentangle it using only evolutions which preserve the symmetry. The Higher Berry class is an obstruction to that. So the non-vanishing of this class prevents us from disentangling a state with the symmetry, while preserving the symmetry, even after stacking with anchillas (?).

3.2 Gromow Asymptotic Dimension?

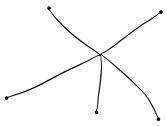
Remark: Apparently, all these constructions depend on the fact that volumes are growing with some power of radius, but not more than some power.

There is something called asymptotic dimension, and also something called the Gromow asymptotic dimension, and there's some spaces which have finite Gromow asymptotic dimensions. Kapustin says he is curious whether the constructions we did above can be done for such spaces, spaces that have a finite Gromow asymptotic dimension. For example, take a Riemannian manifold X, and consider $X \times \mathbb{R}^+$ with metric



$$ds^2 = g_X r^2 + dr^2, \quad r \in \mathbb{R}^+. \tag{161}$$

On X, choose some lattice, some discrete set of points, which uniformly fill the entire space. This is an example of a space described above. If we choose X to be a finite number of points, then the total space looks like and we would like



to consider lattice systems on it. One immediate question is what would be the invariants here? It is not obvious whether this is physically interesting, but it may in principle be useful.

3.3 Quantization of Higher Berry Class

Recall Kitaev's conjecture. We have a space of invertible theories and there are some predictions on the homotopy of the space of invertible theories. We

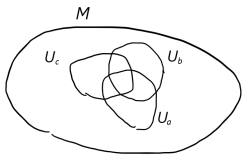


Figure 12: The parameter space divided into local patches.

list the space of invertible theories for the bosonic case, which is the same as Table 1: where $K(G,n)$ is the Eilenberg-Maclane space that we discussed before.

d	0	1	2	3
X_d	$\mathbb{C}P^\infty$	$K(\mathbb{Z},3)$	$K(\mathbb{Z},4)\times\mathbb{Z}$	$K(\mathbb{Z},5)\times S^1$

Table 3: Predictions for the X_d space in dimensions 0 to 3.

Consider a map from M to X_1

$$f : M \to X_1, \tag{162}$$

there should be a class $\Omega \in H^3(X_1,\mathbb{Z})$ and its pullback lies in $f^*\Omega \in H^3(M,\mathbb{Z})$. This is the first non-trivial Higher Berry class. Note that it is not expected that the cohomology class to be quantized in general, but it is expected to be quantized for invertible systems. For example, in IQHE, any invertible state will have an integer Hall conductance. Apparently, people expect the cohomology class for a general system to be a rational number, the reason for which is the FQHE.

3.3.1 Quantization of Higher Berry Class for $d = 1$

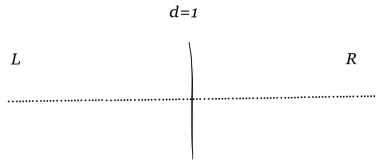
It helps to think of Higher Berry class as an obstruction to constructing a family of globally defined LGAs which means an obstruction to constructing all the states in the family starting from a particular one $m \in M$. Looking at 3 intersecting (contractible) charts of M, as demonstrated in Figure 12, we have the following conditions

$$U_{ab} = U_a \cap U_b \text{ is contractible,} \tag{163}$$

$$U_{abc} = U_a \cap U_b \cap U_c \text{ is contractible.} \tag{164}$$

On each chart U_a, we can produce $\psi = \psi_0 \circ \tilde{\alpha}_a$ by some automorphism $\tilde{\alpha}_a$. Here, ψ_0 is assumed to be purely factorazed, and as a consequence all $\psi = \psi_0 \circ \tilde{\alpha}_a$ are in the trivial phase, by definition of the trivial phase. Also, it is a short-range entangled state. Once we have the quantization for a family of short range entangled states, we also get it automatically for invertible states. This is because

we can tensor the whole thing if there exists some family which is invertible at every point, because it is tensored with an ancillary state and disentangle. Hence, via this trick, we can replace the family of invertible states with a family of short range entangled states, which does not change the Berry class. We can split this system into 2 (remember we are working on $d = 1$).



Now, let α_a be the restriction of $\tilde{\alpha}_a$ to L (so α_a decays very rapidly on R). And evolve ψ_0 with only the L side using the α_a automorphism. On the intersection U_{ab}, we define

$$\alpha_{ab} = \alpha_a \circ \alpha_b^{-1}, \tag{165}$$

and with this automorphism, $\psi \circ \alpha_{ab}$ is very close to ψ far from the origin, because on the left α_a and α_b both become $\tilde{\alpha}$, whereas on the right both are rapidly decaying so they are close to identity. Therefore, if we have two states agreeing at ∞, then they must lie in the same Hilbert space. Can we obtain one from the other by acting with an almost local unitary operator? The easiest way to prove this is by choosing one state to be pure and the other mixed. Apparently, the dimensionality is not relevant here. So, we have one factorized and one not factorized states in $d-$dimensions, but they are both pure and agree at ∞. Then, one can be obtained from the other by some almost local unitary observable. The proof of this fact uses some quantum information tools, and it is a useful fact. If we take it to be true, we can write

$$\psi \circ \alpha_{ab} = \psi \circ \mathrm{Ad}_{V_{ab}}, \tag{166}$$

where

$$\begin{aligned} \mathrm{Ad}_V : \mathcal{A}_{al} &\to \mathcal{A}_{al} \\ \mathsf{a} &\mapsto V \mathsf{a} V^{-1}, \end{aligned} \tag{167}$$

and V is a unitary element in $\mathcal{A}_{al}$. We can choose V to depend smoothly on M, the parameter space $V_{ab} \in C^\infty(U_{ab}, \mathcal{A}_{al})$, however, proving this is a bit non-trivial. Once we believe that such a unitary V exists, we can consider

$$W_{abc} := V_{ac}^{-1} \alpha_{ab}(V_{bc}) V_{ab} \in C^\infty(U_{abc}, \mathcal{A}_{al}). \tag{168}$$

This is clearly unitary

$$\begin{aligned} W_{abc} W_{abc}^\dagger &= \left(V_{ac}^{-1} \alpha_{ab}(V_{bc}) V_{ab} \right) \left(V_{ac}^\dagger [\alpha_{ab}(V_{bc}^\dagger)] V_{ab} \right) \\ &= V_{ac}^{-1} \alpha_{ab}(V_{bc}) \alpha_{ab}(V_{bc}^{-1}) V_{ac} \\ &= V_{ac}^{-1} V_{ac} \\ &= \mathrm{id}, \end{aligned} \tag{169}$$

where we used $V^\dagger = V^{-1}$ and $\alpha(V^{-1}) = [\alpha(V)]^{-1}$. We can also check that

$$\psi \circ \mathrm{Ad}_{W_{abc}} = \psi, \tag{170}$$

so the adjoint action of W_{abc} defined on the 3−intersection U_{abc} preserves the state ψ. This implies that there is a corresponding operator $\hat{W}_{abc}$ such that it acts on the ground state to give a phase

$$\hat{W}_{abc}|0\rangle = e^{i\alpha_{abc}}|0\rangle. \tag{171}$$

This can be shown using von Neumann's bicommutant theorem. We define the average of W_{abc} over ψ as $h_{abc} \equiv \psi(W_{abc}) \in C^\infty(U_{abc}, U(1))$, where the $U(1)$ part comes from the $e^{i\alpha_{abc}}$ phase above. This average satisfies the cocycle condition

$$h_{acd}h_{abc} = h_{abd}h_{bcd}, \tag{172}$$

on quadrupole overlaps U_{abcd}. Due to the cocycle condition, these $U(1)$ valued functions define a class in $H^2(M, U(1)_M) \approx H^3(M, \mathbb{Z})$ and with some extra work to show that the image of this is the usual de Rham cohomology $H^3(M, \mathbb{R})$.

Some Questions:

- The cocycle condition on h_{abc} looks somewhat like the crossing symmetry condition that the fusion algebra in RCFT [Ver88] satisfies. I wonder if there is a relationship.
- In the isomorphism $H^2(M, U(1)_M) \approx H^3(M, \mathbb{Z})$, was it a coincidence that $U(1)$ became its Pontryagin dual $\hat{U}(1) \approx \mathbb{Z}$?

It can be shown that the class obtained above is the same one constructed by different means for the one that used the Maurer-Cartan equation. Hence, $H^3(M, \mathbb{Z})$ is a refinement of the higher Berry class to an integer class. The other construction, the one using Maurer-Cartan, has the advantage that it works in arbitrary dimensions; this construction, although apparently more straightforward, is valid only for $d = 1$. Kapustin says he does not know how to generalize this construction to higher d.

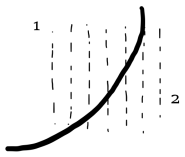
Also, he says that there is a similar definition in $2d$, but if you want to show that there is a class of degree 4, then you need to deal with a lot of indices on $\psi(W_{abc})$ due to more overlaps. So the 3-cocycle condition for that case is too messy. And Kapustin says they didn't work out whether that class is the same as the de Rham cohomology class, because apparently there is no systematic way to do so. In $d = 2$, one can construct a degree 4 class, because of the $K(\mathbb{Z}, 4)$ factor on X_2, and it is quantized. Kapustin says they did not check whether it is the same class as above.

Let us discuss a few cases in which the quantization is known.

3.4 Some Interesting Cases:

Case 1: G, the symmetry group, is a compact semi-simple group, and the dimension is 2. Any such system gives rise to a family of quasi $1d-$systems parameterized by G. How do we get the quasi $1d$-systems?

Start from a $2d$ lattice, and cut the system into two. We do the $G-$transformations



(constant one) for one half of the system (say 2). The system is invariant under such transformations. Indeed, if we did a transformation with G constant on the whole plane, the system wouldn't change. So when we do the transformations on one half, the system does not know that region 1 has different transformations than region 2. The only change occurs on the neighborhood the boundary cutting the plane, and this defines a quasi $1d$-system with some family of states transforming under G. And for any compact and semi-simple G, there exists a natural class of rank 3, obtained by taking any pseudo-subgroup that defines a $3-$class.

Hence, we ask the question whether we can find the higher Berry class on the $1d$ quasi system, and then just integrate over the degree 3 cycle of a $3-$sphere embedded into the group $S^3 \subset G$. Upon the integration what would we get? We would get a number which is quantized by the above considerations, generalized to the quasi $1d$ situation. On the other hand, we can show that the quasi system on the line is an equivariant Berry class or something (?)

$$H^3(G, \mathbb{Z}) \simeq \mathbb{Z}. \tag{173}$$

This is the non-Abelian version of Hall conductance.

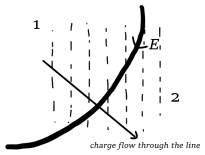
Case 2: Fix $G = U(1)$ in $d = 1$. We take M to be the parameter space. In this case we expect the invariants to live in

$$H^3_{U(1)}(M) = H^3(M) \oplus H^1(M), \tag{174}$$

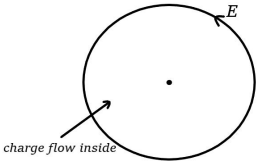
where $H^3(M)$ is quantized, and $H^1(M)$ is known as the Thouless pump, which is also quantized. Apparently, this is a rigorous version of Laughlin's flux insertion argument, which is the following.

What happens when we apply a $U(1)$ gauge transformations to region 2? Basically, it turns on a $U(1)$ electric field, and the Hall conductance drives

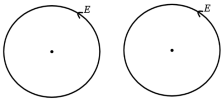
the charges along the boundary. The Thouless pump measures the charge flow through the line separating the two regions, as demonstrated in the figure below. The context in which Laughlin argued is the following. Consider flux insertion



at a point, and an electric field as in the figure. In the above situation, where we



divide the lattice into two with a line, the point is mapped to ∞, this is how the two situations are connected with each other. Actually, having the point inside the plane has its advantages. For example, in that situation, one can consider the statistics of two such points as in the figure below. These cases are all the



cases in which the quantization is known.

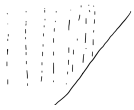
3.5 The Big Open Problem:

According to Kapustin, this is the most vexing and the most physical open problem related to these constructions. The refined Kitaev conjecture tells us that

$$X_2 \sim K(\mathbb{Z}, 4) \times \mathbb{Z}, \tag{175}$$

what is the $\mathbb{Z}$ part, and what does it label? It is called the chiral central charge, but it does not have a rigorous definition for it. There are some descriptions of it:

#1: Given a $2d$ system, how do we know it is in a non-trivial phase? We introduce a physical boundary, so the system is modified and the lower part can be made trivial. If it is a non-trivial phase, then one necessarily has a gapless



system. Kapustin says he does not know which system one gets. In the bulk, the system hardly changed, but near the boundary there should be essentially a field theory. Apparently it is not clear what kind of field theory is this, but it could be for example a $(1+1)-d$ CFT. In which case, the belief is that the invariant from $\mathbb{Z}$ is simply the difference between the central charge of the left-moving modes and that of the right-moving modes $c_L - c_R$ (so we have a left Virasoro and a right Virasoro algebra). There are some problems with this perspective:

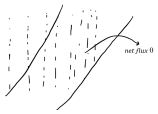
- Why would the edge of a lattice system be described by a field theory at all?
- Why the field theory would be Lorentz invariant?
- Why would it be conformally invariant?

So, there are a lot of assumptions here.

#2: Let us insert the system in Figure 3.5 into a lattice at some finite temperature $T > 0$. In a CFT at $T > 0$, whenever we have $c_L - c_R \neq 0$, there is a net energy flux along the boundary line

$$\langle T_1^0 \rangle = \kappa T^2, \tag{176}$$

where T_1^0 is the component of the energy momentum tensor, T is the temperature, and κ is related to the central charge. This equation makes sense even if the field theory is not Lorentz (or conformally) invariant. The takeaway is that when we insert the system at $T > 0$, there will be a flux along the boundary, and apparently this flux is independent of how we choose the boundary. This is because if we take a strip of the system, the net flux should be 0.

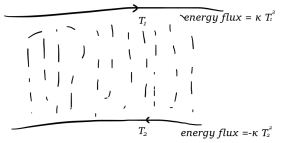


κ is the Thermal Hall conductance, and the reason for this name is the following. Imagine a $2d$ system with two boundaries, one at T_1 temperature and the other at T_2 temperature. The total flux in this system is given by

$$J^E = \kappa(T_1^2 - T_2^2). \tag{177}$$

When the temperature difference is small,

$$J^E \simeq 2\kappa \frac{T_1 + T_2}{2}(T_1 - T_2), \tag{178}$$



that is, the energy flux is proportional to the temperature difference and to the central charge κ. This is called the thermal Hall effect because the temperature gradient is along $\uparrow$ whereas the heat flow is along $\rightarrow$. It is expected that 2κ is integer or fractional. Nobody knows why because we do not have any analog of the constructions above for finite temperature systems. These constructions are relevant for gapped systems at zero temperature.

So, this is the outstanding problem, and it is believed that this quantity makes sense for general gapped systems. And for gapped systems it is still called the central charge, but it is not necessarily an integer, but is expected to be rational. Alos, it is not known how to construct that invariant, let alone show that it is quantized.

Apparently, there is some indication that this invariant has to do with a hidden rotational symmetry of the problem. When dealing with lattice systems, and there is a $U(1)$ symmetry, one defines the Hall conductance. When there is no $U(1)$ symmetry, there is the thermal conductance, which is similar to the electric situation (electric conductance), so the question is what is the analogous quantity here that is the counterpart of Hall conductance in $U(1)$ case. And apparently, the answer seems to be that it is a rotational symmetry. But, the tricky part is that this is not a field theory, it is a lattice system, and in the lattice, there is no rotational symmetry which is the main problem: How does one see a hidden rotational symmetry in a lattice setup?

What are some systems that has no $U(1)$ symmetry but shows this effect? Apparently, the thermal Hall effect is very general. Take any insulator that has no interesting quantization at all, and put it in a magnetic field. After turning on the B, you are going to get a thermal Hall effect.

Kapustin says that they are interested in cases where there is some contributions from just electrons. Integer Quantum Hall states exhibit thermal Hall effect as well as the usual Hall effect. And in this case, the two corresponding invariants are proportional. In the case of the Fractional Hall effect, there is no reason for them to be proportional since the Fractional Hall effect comes up in interacting systems. In the few cases where a measurement has been made, they turn out to be proportionally the same coefficient as is the case for IQHE. But, this has not been measured in many systems. This is a mysterious proportionality, and Kapustin says that they have been working on this problem for years without any progress.

4 Summary and Questions:

The main purpose in all these constructions is to understand the phases of matter from the quantum lattice systems approach. In doing so, a significant simplification occurs when we restrict to the subspace of invertible systems. Due to Kitaev's conjecture, invertible systems are related to invertible TQFTs, and the topology of the space of invertible systems X_d gives us invariants classifying the phases of matter. There are some predictions on X_d for low dimensions, and the approach of Kapustin, as far as I understand, is to study systems which realize the effects coming from the predictions, and construct the predicted invariants. In doing so, some advanced tools in mathematics are used:

- Topology,
- the theory of operator algebras, in particular C^*-algebras,
- homology & cohomology chains,
- the bundle technology,
- differential geometry,
- Lie algebras,
- some category theory (due to the relation with TQFT).

Below are some things I would be interested in working out

1. Construct a lattice system whose continuum limit is a TQFT. From that, try to understand the main ingredients to develop a framework in which TQFTs can be obtained as continuum limits quantum lattice systems. Which TQFTs can be obtained from lattice systems? Due to Kitaev's conjecture, it seems plausible to think that invertible TFTs can be related with lattice systems describing invertible systems, first working out the lattice formulation of invertible TFTs would be a good starting point.

   • One proposal I have for the above is the following. It is possible to consider supersymmetric gauge theories on a lattice (I am not familiar with the literature so am unable to give detailed references, but [Sch23] is a recent review). And, in ordinary supersymmetric gauge theories, there is the notion of topological twist introduced by Witten, which makes the supersymmetric theory a topological theory. Making analogy with this topological twist, can we define a topological twisting of a supersymmetric gauge theory formulated on a lattice in such a way that the continuum limit gives us the topologically twisted theory on the continuum? That would be a fantastic construction, copying the topological twist of ordinary supersymmetric gauge theories to lattice supersymmetric gauge theories. Also, if this works out, it would be a very interesting question to ask whether this formulation of topological twisting on the lattice sheds some

light on 4-manifold topology, just as "continuum" susy gauge theories are very powerful for 4-mainfold topology.

• To take things even further, would it be possible to discretize the $6d$ $\mathcal{N} = (2, 0)$ SCFT in the sense of the above bullet? If so, what would this teach us about the nature of electric magnetic dualities and the Geometric Langlands?

2. Starting from a chiral vertex algebra, one can reverse engineer a lattice system (again, I do not know the literature well, but [Sop24, SS24] are some recent papers related to this). The problem with this is that there are ∞ dimensional space of vertex operators at each site in the lattice. If we start from the chiral vertex algebra of an RCFT, would we get a lattice system that does not have the above problem, since RCFT has finitely many primary fields? If so, this would be a nice improvement.

• The above idea, if successful, can shed some light on the problem discussed in subsection 3.5, where the lattice system is described by a field theory on the physical boundary. It is not clear why the boundary of a lattice system would be described by a field theory, but if we have a construction starting from RCFT and obtaining a lattice system, this phenomenon can be understood as the inverse of that. Schematically, using the "inverse" of the construction starting from the vertex algebra to the lattice system, we may understand why the boundary is a field theory, and why it would be a conformally invariant field theory.

$$\text{Chiral vertex algebra} \underset{?}{\overset{\rightarrow}{\leftarrow}} \text{Lattice system} \tag{179}$$

3. As far as I understand, Kapustin's purpose is to test the predictions of the homotopy types of X_d, by explicitly constructing lattice systems that realize these effects. The above two items could be such constructions, and even if they fail, there are many other interesting models to work out, for example, the spin chain systems are very rich and it would be very nice to learn about them. It would be fruitful to study various models and try to construct the invariants using the mathematical tools I typed above, on each of which I have some level of familiarity, and it would be very useful to learn them in greater detail. It would teach a lot of math and physics, and the problem is well motivated.

4. The Eilenberg-Maclane space is relevant for the study of phases of matter as it appears in X_d. There are also what's called Moore spaces, which is defined similarly to Eilenberg-Maclane spaces, but we use homology instead of homotopy. Given a Lie group G, an a fixed integer $n \geq 1$, a Moore space X is defined to satisfy

$$H_m(X) \simeq \delta_{nm} G, \tag{180}$$

roughly, X has only the $n-th$ homology non-vanishing, and the other homologies are all trivial. For example, S^n is a Moore space of $\mathbb{Z}$ for $n \geq 1$. And $\mathbb{RP}^2$ is a Moore space of $\mathbb{Z}_2$ for $n = 1$.

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