Bootstrapping Non-Invertible Symmetries

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Roadmap

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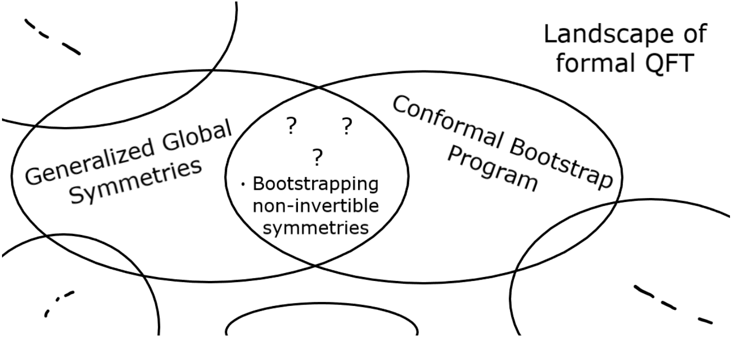
Motivation

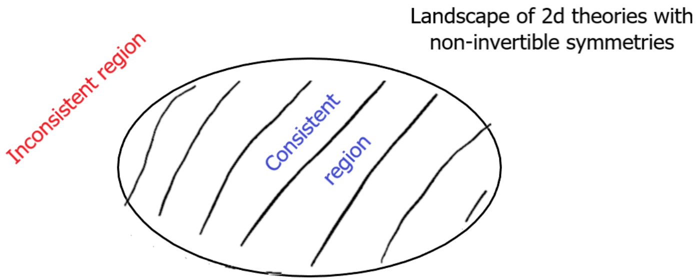
- Quantum Field Theory (QFT) is the main language for studying physics from microscopic to macroscopic scales.
- The problem of defining QFT rigorously remains unresolved, driving many advancements in physics and mathematics.
- This century has witnessed remarkable progress, and we are interested in how different approaches interact.



Statement of the Problem

- Previous decade: There are Non-invertible symmetries in QFTs.
- Status: So far best understood in 2d (Bhardwaj and Tachikawa, 2017), and it provides novel constraints.
- Following Lin and Shao (2023), we study them with bootstrap tools.
- An ambitious version of our problem can be stated as

Carve out the space of 2d CFTs with particular non-invertible symmetries.



Ingredients

Generalized Global Symmetries

- Introduced by Gaiotto, Kapustin, Seiberg, and Willett in their seminal paper (2014).
- The rough idea from today's perspective

Group Symmetries $\leftrightsquigarrow$ Topological Operators

Non-invertible Symmetries $\leftrightsquigarrow$ Categorical Operators

(Categorical operator $=$ Topological operator $\oplus$ non-invertible fusion rule)

- Far-reaching consequences for
  - phases of gauge theories,
  - 't Hooft anomalies & UV/IR mixing,
  - RG flows and emergent phenomena,
  - Ginzburg-Landau paradigm.

0-form global $U(1)$ symmetry in 4−dimensional QFTs

$$d * j = 0 \Longrightarrow U_\alpha(S^3) = \exp\left(i\alpha \int_{S^3} *j\right), \quad \alpha \in [0, 2\pi).$$

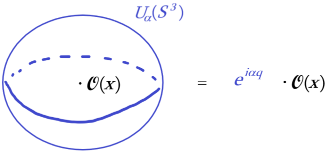


Figure: Action of 0−form symmetries

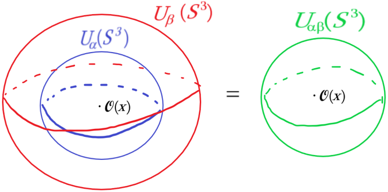


Figure: Fusion of 0−form symmetries

Gaiotto et. al. (2014) generalized this picture to p−form symmetries.

The Conformal Bootstrap

- In any QFT, physical information is encoded in correlation functions.
- For CFTs, $n-$point function $\xrightarrow{\text{OPE}} \sum F(\text{CFT data})$, where

$$\text{CFT Data } = (\Delta_i, s_i, f_{ijk})$$

$$\begin{aligned} \Delta_i &: \text{ scaling dimensions of all primary fields } \mathcal{O}_i, \\ s_i &: \text{ spins of all primaries } \mathcal{O}_i, \\ f_{ijk} &: \text{ Three-point function coefficients.} \end{aligned}$$

Crux of Conformal Bootstrap

$$(\Delta_i, s_i, f_{ijk}) \rightsquigarrow \text{CFT}.$$

Question: What are the conditions on the CFT data?
Answer: Physical requirements of unitarity, crossing symmetry, ...

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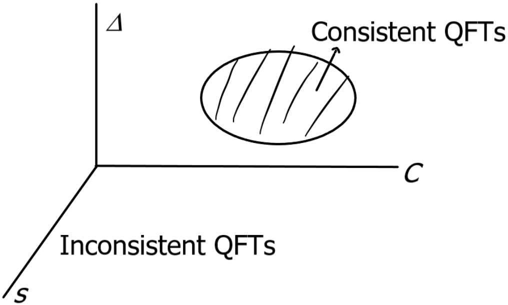
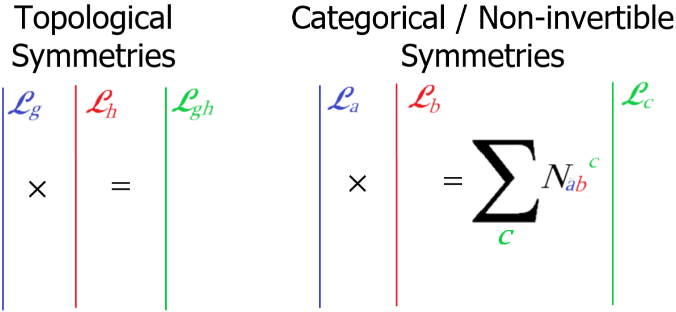
Goal: Implement consistency conditions to carve out the *theory space*.



Global Symmetry Operators in 2d

- Line operators $\oplus$ fusion rules $\rightsquigarrow$ 0-form symmetries in 2d



- **Example:** 2d critical Ising model has three lines $1, \eta, \mathcal{N}$ and fusion

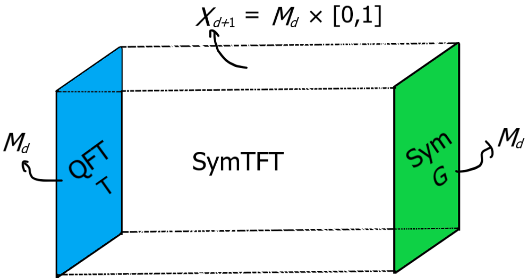
$$\eta \times \eta = 1, \quad \eta \times \mathcal{N} = \mathcal{N} = \mathcal{N} \times \eta, \quad \mathcal{N} \times \mathcal{N} = 1 + \eta,$$

$\eta = \mathbb{Z}_2$ spin line, and $\mathcal{N} =$ Kramers-Wannier duality line.

Symmetry Topological Field Theory Construction

- Let T be a QFT on M_d with internal global symmetry G.

$$\text{SymTFT}: \ \text{Dynamics T} \ \leftrightsquigarrow \ \text{Kinematics } G,$$



Example: For $G^{(p)} = \mathbb{Z}_N$, SymTFT is a Dijkgraaf-Witten gauge theory.

SymTFT and Modular Bootstrap

- For $d = 2$, $G \to \mathcal{C}$ (a fusion category),

$$\text{SymTFT} \;\to\; \text{Turaev-Viro TQFT } \text{TV}_{\mathcal{C}}.$$

- Partition function properties:

$$\text{Virasoro expansion : } Z^{3d}_{\mu}(\tau,\overline{\tau}) = \sum_{h,\overline{h}} n_{\mu,h\overline{h}}\; \chi_h(\tau)\overline{\chi}_{\overline{h}}(\overline{\tau}),$$

$$\text{Modular covariance : } Z^{3d}_{\mu}(\tau,\overline{\tau}) = \sum_{\nu} S^{-1}_{\mu\nu}\; Z^{3d}_{\nu}\big(S\tau,\overline{S\tau}\big),$$

$(S\tau \equiv -\frac{1}{\tau}, \overline{S\tau} = -\frac{1}{\overline{\tau}})$

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$$\text{Modularity} \;\oplus\; \text{Virasoro} \;\Longrightarrow\; \text{Modular Bootstrap}$$

$$0 = \sum_{\nu}\sum_{h,\overline{h}} n_{\nu,h\overline{h}}\ \ X_{\mu\nu,h\overline{h}}(\tau,\overline{\tau}),$$

$$X \sim \chi(S\tau)\ \mathrm{id}\ \overline{\chi}(\overline{S\tau}) - \chi(\tau)\ S\ \overline{\chi}(\overline{\tau}).$$

Bootstrapping the Lines of Ising CFT

- For concreteness, choose $\mathcal{C} = \text{Ising}$.
- Numerical bootstrap on $0 = \sum n \cdot X$ (Lin and Shao, 2023)

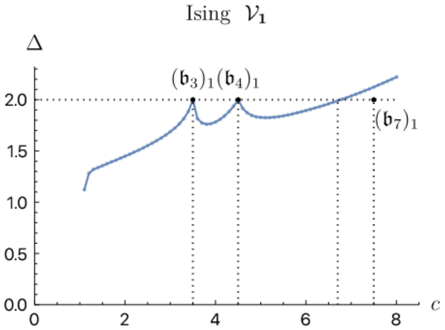

Figure: Bootstrap bounds on the lightest primary operator in the trivial anyon sector. Figure taken from Lin and Shao (2023).

Questions? If not extra slides.

Generalization to $p-$form symmetry in $d-$dimensions

$p-$**form symmetry**: Topological operators $U_g^{(p)}(S^{d-p-1})$ of codimension-$p+1$ acting on extended operators $\mathcal{O}(M_p)$ with fusion rules.

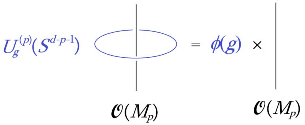

Figure: Action of $p-$form symmetries

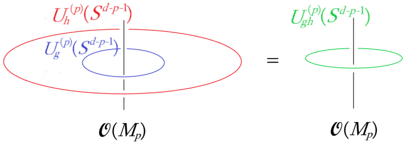

Figure: Fusion of $p-$form symmetries

Example: Wilson operators $W(C) = e^{i \oint_C A}$ are charged under 1-form symmetries for *any* gauge theory in $d \geq 2$!

CFT correlators from OPE

- Main objects: Primary operators of scaling dimension Δ

$$e^{-\lambda D}\mathcal{O}_\Delta(x)e^{\lambda D} = e^{-\lambda\Delta}\mathcal{O}_\Delta(\lambda x), \quad (D : x \mapsto \lambda x).$$

Knowledge of $\mathcal{O}_\Delta$ gives the complete spectrum via descendants

$$\mathcal{O}^{(k)}(x) \equiv \underbrace{P\cdots P}_{k \text{ times}}\mathcal{O}_\Delta(x), \quad k = 0, 1, 2, \ldots$$

Use the OPE recursively

$$\begin{aligned}
\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n\rangle &= \sum_k C_{12k}^{(s_1,s_2,\Delta_1,\Delta_2)}(x_{12},\partial_2)\langle\mathcal{O}_k(x_2)\cdots\mathcal{O}_n\rangle \\
&= \cdots \\
&= \sum C(s,\Delta,f)\cdots\sum C(s,\Delta,f).
\end{aligned}$$

2pt function to 1pt function

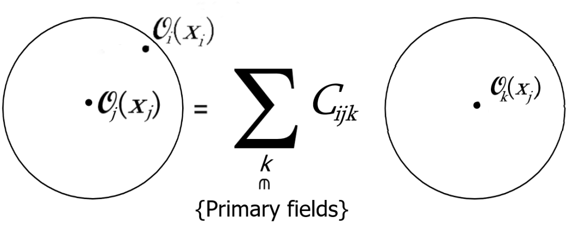

Figure: The Operator Product Expansion (OPE) in a two-point function.