Seiberg-Witten Theory

Burak Oğuz

Middle East Technical University

oguz.burak@metu.edu.tr

June 26, 2024

Overview

Seiberg-Witten Theory is an $N = 2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$.

Its low energy description can be found exactly, and there is interesting physics in that effective action. We will focus on two interesting phenomenon

- An electric-magnetic duality in the effective action,
- Confinement of electric charge via condensation of monopoles.

Since Yang-Mills theories are asymptotically free, the low-energies are strongly coupled. So these results are obtained in a regime where we don't have many tools to answer questions.

Contents

In this presentation, we will focus on confinement and electric-magnetic duality. We will discuss the following:

- Dyons and the Montonen-Olive duality,
- $N = 2$ supersymmetric Yang-Mills theory and its low-energy action,
- Duality in the effective action,
- Monopole condensation and confinement.

The Georgi-Glashow Model

Georgi-Glashow model is an $SU(2)$ Yang-Mills Higgs theory, defined by the action

$$S = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} D_\mu \varphi^a D^\mu \varphi^a - \frac{\lambda}{4} (\varphi^a \varphi^a - v^2)^2 \right), \quad (1)$$

where $v^2 = m^2/\lambda$, and we are considering the weakly coupled regime $\lambda \ll m^2$ or $v^2 \gg 1$. The field strength and the covariant derivative are given by

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon^{abc} A^b_\mu A^c_\nu, \\ D_\mu \varphi^a &= \partial_\mu \varphi^a + e\varepsilon^{abc} A^b_\mu \varphi^c. \end{aligned} \quad (2)$$

After the Higgs field develops an expectation value v in some direction of $SU(2)$, the perturbative spectrum of this model contains two massive bosons $W^\pm$ with mass $m_W = ev$, a $U(1)$ gauge boson, and a Higgs field with mass $m_H = \sqrt{2}m$

Dyons

In this theory, there are solitonic solutions to the field equations which are monopoles. If the monopole also has an electric charge, it is called a dyon. The mass of a dyon can be calculated via

$$M = \int d^3x \left[\frac{1}{2} \left((D_0 \varphi^a)^2 + (D_i \varphi^a)^2 + (E_i^a)^2 + (B_i^a)^2 \right) + V(\varphi) \right]. \tag{3}$$

The magnetic and electric charges of the dyon are given by

$$g = \frac{1}{v} \int d^3x B_i^a (D_i \varphi)^a \quad ; \quad q = \frac{1}{v} \int d^3x E_i^a (D_i \varphi)^a. \tag{4}$$

We introduce a new parameter $\tan\theta = q/g$ and write

$$\begin{aligned} M = v(q \sin\theta + g\cos\theta) + \int d^3x \Big[\frac{1}{2} \Big(\big(D_0 \varphi^a \big)^2 + \big(E_i^a - \sin\theta D_i \varphi^a \big)^2 \\ + \big(B_i^a - \cos\theta D_i \varphi^a \big)^2 \Big) + V(\varphi) \Big]. \end{aligned} \tag{5}$$

Montonen-Olive Duality

So, we have the following bound on mass

$$M \geq v\sqrt{g^2 + q^2}, \tag{6}$$

and the states that saturate these bounds are called BPS states. Note that this bound is a universal bound in the sense that elementary excitations as well as lumps must satisfy it.

Based on the symmetry between g and q in this mass bound, Montonen and Olive conjectured that there are two descriptions of the same physics:

- The first is the electric formulation where gauge bosons are elementary excitations and monopoles are solitons,
- The second is the magnetic formulation where monopoles are elementary excitations and gauge bosons are solitons.

The two descriptions are exchanged by

$$q \to g = \frac{4\pi}{q}. \tag{7}$$

One side is a weakly coupled theory whereas the other is a strongly coupled one. We note that this is not a simple symmetry of the system but a new formulation of it in terms of a strongly coupled system.

Montonen-Olive Duality

The two descriptions are exchanged by

$$q \to g = \frac{4\pi}{q}. \tag{7}$$

One side is a weakly coupled theory whereas the other is a strongly coupled one. We note that this is not a simple symmetry of the system but a new formulation of it in terms of a strongly coupled system.

This duality is hard to maintain at the quantum level due to quantum corrections. However, as Seiberg and Witten showed, there is an electric-magnetic duality in the low-energy description of $N = 2$ supersymmetric quantum Yang-Mills theory.

$N = 2$ Yang-Mills Theory

The action of the $N = 2$ Yang-Mills theory is given by

$$\begin{aligned} S &= \int d^4x \left(\operatorname{Im}\left(\frac{\tau}{16\pi} d^2\theta W^\alpha W_\alpha \right) + \frac{1}{4g^2} \int d^2\theta d^2\bar{\theta} \mathrm{Tr}\Phi^\dagger e^{-2gV}\Phi \right) \\ &= \operatorname{Im} \int d^4x \frac{\tau}{16\pi} \mathrm{Tr} \left(\int d^2\theta W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{-2gV}\Phi \right). \end{aligned} \tag{8}$$

This action contains an $SU(2)$ valued gauge field A_μ, and its supersymmetry partners φ an adjoint valued complex scalar, and two adjoint valued spinors ψ, λ. The potential of the scalar field is given by

$$V(\varphi) \sim \mathrm{Tr}[\varphi, \varphi^\dagger]^2. \tag{9}$$

For supersymmetry to be unbroken in the theory, we must ensure that $V(\varphi) = 0$.

Thus, we need the moduli space, which means gauge inequivalent solutions of $V(\varphi) = 0$. For V to vanish, we need to choose φ so that $[\varphi, \varphi^\dagger] = 0$. After gauge fixing, the most general configuration for which this is true is given by

$$\varphi(x) = \frac{1}{2} a(x) \sigma_3, \tag{10}$$

where $a(x)$ is a complex function, and $\sigma_3 = diag(1, -1)$. There are still Weyl rotations in the $1, 2$ directions by angle π that change a to $-a$. So the parameter that labels the gauge inequivalent vacuum should be

$$\frac{1}{2} a^2 = \mathrm{Tr} \varphi^2. \tag{11}$$

This equality holds semiclassically, but after the quantum corrections, this is not the case.

The Moduli Space

Then

$$u = \langle \mathrm{Tr}\varphi^2 \rangle \tag{12}$$

labels distinct vacua so is a coordinate of the moduli space $\mathcal{M}$.

Now, observe that if $\langle \varphi \rangle \neq 0$, then the $SU(2)$ symmetry will be spontaneously broken down to $U(1)$.

The fields A_μ^3, ψ^3, λ^3, and the fluctuations of φ along the 3$^{\text{rd}}$ direction in $SU(2)$ will be massless excitations.

The Moduli Space

Then

$$u = \langle \mathrm{Tr}\varphi^2 \rangle \tag{12}$$

labels distinct vacua so is a coordinate of the moduli space $\mathcal{M}$.

Now, observe that if $\langle \varphi \rangle \neq 0$, then the $SU(2)$ symmetry will be spontaneously broken down to $U(1)$.

The fields A_μ^3, ψ^3, λ^3, and the fluctuations of φ along the 3$^{\text{rd}}$ direction in $SU(2)$ will be massless excitations. These massless fields are described by a Wilsonian low-energy effective action with $N = 2$ supersymmetry and $U(1)$ gauge group. The low-energy action is then

$$S = \frac{1}{16\pi} \mathrm{Im} \int d^4x \left[\int d^2\theta \mathcal{F}''(\Phi) W^\alpha W_\alpha + \int d^2\theta d^2\overline{\theta} \Phi^\dagger \mathcal{F}'(\Phi) \right]. \tag{13}$$

Where $\mathcal{F}(\Phi)$ is a holomorphic prepotential that depends only on Φ.

Metric of the Moduli Space

To determine the moduli space metric, we write the first and the second term of the low-energy action in terms of component fields

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x\mathcal{F}''(\varphi)\left[-\frac{1}{4}F_{\mu\nu}(F^{\mu\nu}-i*F^{\mu\nu})-i\lambda\sigma^\mu\partial_\mu\overline{\lambda}\right]+\cdots, \quad (14)$$

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x\mathcal{F}''(\varphi)\left[|\partial_\mu\varphi|^2-i\psi\sigma^\mu\partial_\mu\overline{\psi}\right]+\cdots. \quad (15)$$

Since $\mathrm{Im}\mathcal{F}''(\varphi)$ multiplies all the kinetic terms, it acts like a metric on the configuration space. Then, the moduli space metric can be written as

$$ds^2=\mathrm{Im}\mathcal{F}''(a)dad\bar{a}=\mathrm{Im}\tau(a)dad\bar{a}. \quad (16)$$

And $\tau(a)$ is the complexified coupling constant.

Metric of the Moduli Space

It turns out that this effective description is not well-defined for all vacua, i.e., for all points in the $u-$plane. We want the metric on the moduli space to be positive definite, so $\mathrm{Im}\mathcal{F}''(a) > 0$, but this cannot be the case because $\mathrm{Im}\mathcal{F}''(a)$ is a holomorphic function and holomorphic functions cannot have minima.

So, when $\mathrm{Im}\mathcal{F}''(a)$ approaches a singularity, $\mathrm{Im}\mathcal{F}''(a) \to 0$, we need a new coordinate $\tilde{a}$ for which $\mathrm{Im}\mathcal{F}''(\tilde{a}) \neq 0$ at that point. This can be achieved as long as the singularities on $\mathcal{M}$ are only coordinate singularities, which is the case here.

Duality

Let us now discuss the duality appearing in the low-energy action. If we define dual fields Φ_D and $\mathcal{F}_D(\Phi_D)$ via

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D, \tag{17}$$

which is a Legendre transformation, then one can write

$$\begin{aligned} \Phi_D &= \frac{d\mathcal{F}(\Phi)}{d\Phi}, \\ \Phi &= -\frac{d\mathcal{F}_D(\Phi_D)}{d\Phi_D}, \end{aligned} \tag{18}$$

and it can be shown that the effective action written in terms of Φ_D is equivalent to that written in terms of Φ, with the replacement

$$\tau_D(a_D) = -\frac{1}{\tau(a)}, \tag{19}$$

$$\tau_D(a_D) \equiv \mathcal{F}''_D(\Phi_D) \quad ; \quad \tau(a) = \mathcal{F}''(\Phi). \tag{20}$$

Duality

In a symmetric way, we can write the metric of $\mathcal{M}$ as

$$ds^2 = \frac{i}{2}(da\, d\bar{a}_D - da_D\, d\bar{a}). \tag{21}$$

Moreover, one can show that the effective action is invariant under

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \quad ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{22}$$

So, there is a group of descriptions which are equivalent to each other, and the duality group is $SL(2, \mathbb{Z})$.

Monopole Condensation and Confinement

In their paper, Seiberg and Witten were able to show that softly breaking $N = 2$ supersymmetry to $N = 1$ by adding a mass term $\sim m\mathrm{Tr}\Phi^2$ causes monopole condensation in the low energy theory, which induces confinement of electric charge through dual Meissner effect.

This was an achievement because confinement needs an analytic understanding. This was done in a susy theory, and the hope is to do it for non-susy theories, in particular for QCD.