Bootstrapping Non-Invertible Symmetries

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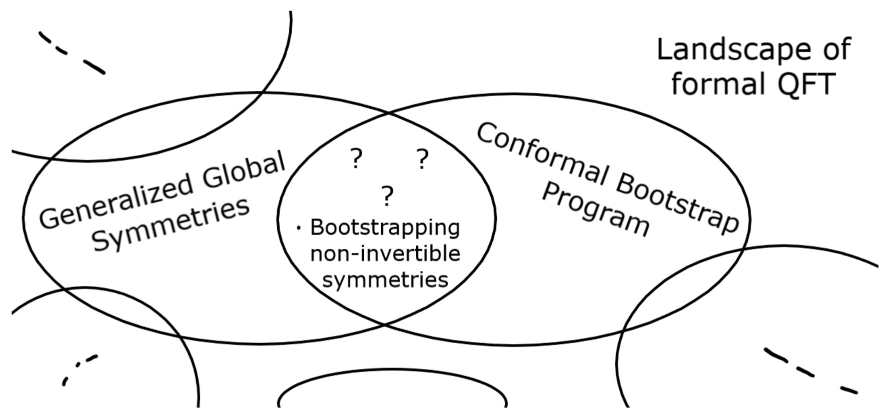
Introduction and Motivation

Quantum Field Theory (QFT) is the language used by physicists to describe phenomena from the microscopic scales to the macroscopic world. Defining QFT rigorously has been a central problem in the past several decades, driving many advancements in physics and mathematics. In our project, we explore the intersection of two up-and-coming research fields in this direction: 1) the generalized global symmetries [GKSW15] and 2) the conformal bootstrap program [SD17].



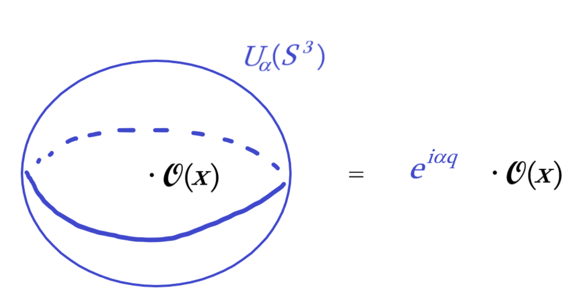
Specifically, our project approaches non-invertible symmetries from a bootstrap point of view [LS23]. The nomenclature non-invertible symmetry is very recent although there are realizations of them in the old literature on two-dimensional Conformal Field Theory (CFT) [SN24]. For example, the Kramers-Wannier duality of the 2d Ising model is a non-invertible symmetry in the modern perspective. We study the critical Ising model as a CFT admitting a categorical (non-invertible) symmetry, and using the Symmetry Topological Field Theory (SymTFT) [SN24] construction along with modularity, we will obtain bootstrap equations to constrain its spectrum [LS23].

Generalized Global Symmetries

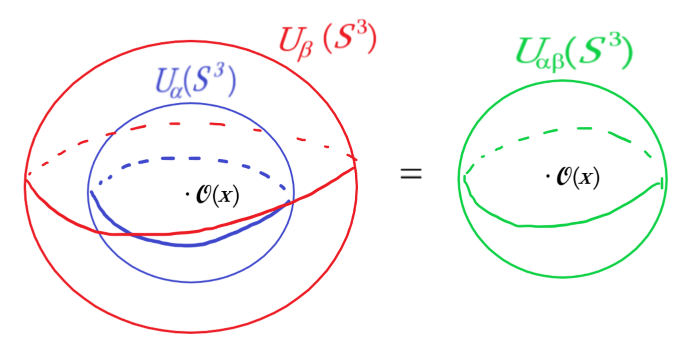
Symmetries in QFT organize the spectrum, provide selection rules and constrain the dynamics significantly. Let G be a global symmetry in a QFT, then Noether's prescription gives a conserved current j, which in turn leads to a codimension-1 topological operator U_g acting on charged local operators and obeying fusion rules. For example, in a 4d QFT with a global $U(1)$ symmetry

$$d * j = 0 \;\rightsquigarrow\; U_\alpha(S^3) = \exp\left(i\alpha \int_{S^3} *j\right), \quad \alpha \in [0, 2\pi).$$

The operators $U_\alpha(S^3)$ act on charged local operators $\mathcal{O}(x)$ via topological linking, and two $U_\alpha(S^3)$ with different parameters α, β fuse together in a way compatible with the $U(1)$ multiplication. This familiar picture is referred to as a 0-form symmetry in the language of [GKSW15].



Action of $0-$form symmetries



Fusion of $0-$form symmetries

The seminal paper [GKSW15] generalizes this picture to $p-$form symmetries. Essentially, one has a codimension$-p+1$ topological operator $U_g(S^{d-p-1})$ acting on a $p-$dimensional extended charged object $\mathcal{O}(M_p)$ via topological linking, and obeying fusion rules. The Wilson line operators $W(C)$ in *any* gauge theory (Maxwell, Yang-Mills, QCD,...), provide natural examples of 1-form symmetries. Higher form symmetries can be broken, have anomalies, and their coupling can flow under the Renormalization Group (RG). For example, the spontaneous breakdown of a 1-form symmetry, the order parameter of which being $\langle W(C)\rangle$, leads to the confining/deconfining phase transitions. Moreover, the photon in ordinary Maxwell theory arises as the Goldstone boson from the spontaneous breaking of 1-form symmetries [GKSW15]!

The Conformal Bootstrap

Conformal Field Theory (CFT) is a QFT with conformal symmetry, and is a useful tool in describing critical phenomena. Moreover, all local QFTs can be constructed from relevant deformations of CFT. Primary operators $\mathcal{O}_\Delta(x)$ of scaling dimension Δ play a central role in CFT, defined by

$$e^{-\lambda D}\mathcal{O}_\Delta(x)e^{\lambda D} = e^{-\lambda\Delta}\mathcal{O}_\Delta(\lambda x), \quad (D : x \mapsto \lambda x),$$

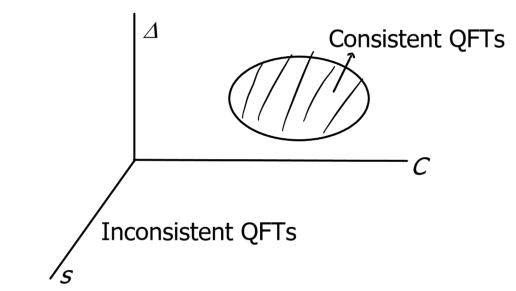
where D is the dilation operator. Knowledge of $\mathcal{O}_\Delta(x)$ gives the complete CFT spectrum via descendants, obtained by acting with the momentum operator (space-time derivative) on the primaries. Therefore, the CFT Operator Product Expansion (OPE) is a sum over all the primaries

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{k\in\{\text{Primaries}\}} C^{(\Delta,s,f)}_{ijk}(x_{12}, \partial_2)\mathcal{O}_k(x_2).$$

The coefficients $C^{(\Delta,s,f)}_{ijk}$ depend on conformal dimensions Δ_i, spins s_i, and the three-point function coefficients f_{ijk} [SD17]. Now recall that the physical information of a QFT is encoded in the $n-$point correlation functions. Using OPE recursively, one reduces an $n-$point function to:

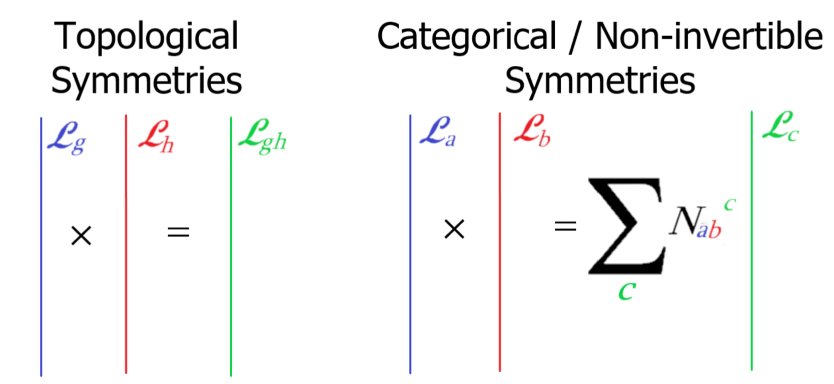
$$\langle\overbrace{\mathcal{O}_1(x_1)\mathcal{O}_2}(x_2)\cdots\mathcal{O}_n\rangle = \sum_k C_{12k}(x_{12}, \partial_2)\langle\overbrace{\mathcal{O}_k(x_2)\cdots}\mathcal{O}_n\rangle = \sum C(s, \Delta, f)\cdots\sum C(s, \Delta, f).$$

The main takeaway is that (Δ_i, s_i, f_{ijk}), called the CFT data, defines a theory. This set of data must to satisfy consistency conditions coming from physical requirements such as unitarity and crossing symmetry. The **goal** of the bootstrap program is to implement these conditions using analytical and numerical tools and carve out the *theory space* non-perturbatively.



Global Symmetry Operators in $d = 2$

In two dimensions, codimension-1 symmetry operators are topological line operators obeying fusion rules. The fusions of the lines depend on whether the global symmetry is associated to a group G or a fusion category $\mathcal{C}$, as depicted in the figure below.



For topological symmetries, the fusion of two lines give a third line, whereas for categorical symmetries, the fusion of two lines give a linear combination of all the lines in the category with coefficients $N_{ab}{}^c \in \mathbb{Z}_{\geq 0}$.

2d critical Ising model

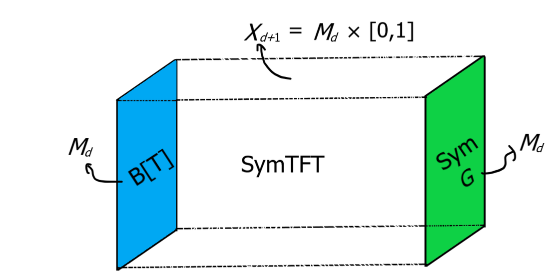
For the **2d Ising CFT**, there are three line operators $1, \eta, \mathcal{N}$ with fusion rules

$$\eta \times \eta = 1, \quad \eta \times \mathcal{N} = \mathcal{N} = \mathcal{N} \times \eta, \quad \mathcal{N} \times \mathcal{N} = 1 + \eta,$$

Where 1 is the identity line, η is a $\mathbb{Z}_2$ spin line, and $\mathcal{N}$ is a non-invertible line operator implementing the Kramers-Wannier duality relating high temperature to the low temperature.

Symmetry Topological Field Theory

Let T be a QFT on M_d with internal global symmetry G. Via the SymTFT construction, one can separate the dynamics T from the symmetry G through a Topological QFT on $X_{d+1} = M_d \times [0, 1]$ [SN24]. Upon compactifying the interval, one recovers the original theory T.



Left boundary captures the dynamics of T, and the right boundary captures the global symmetry G.

For a $p-$form symmetry $G^{(p)} = \mathbb{Z}_N$, the SymTFT is a Dijkgraaf-Witten gauge theory [SN24].

Modular Bootstrap Bounds for Ising CFT

For a 2d theory T with global categorical symmetry $\mathcal{C}$, the SymTFT is the Turaev-Viro TQFT ($\mathrm{TV}_\mathcal{C}$). The partition function of this theory in an anyon sector $\mu \in Z(\mathcal{C})$ (Drinfeld center) obeys

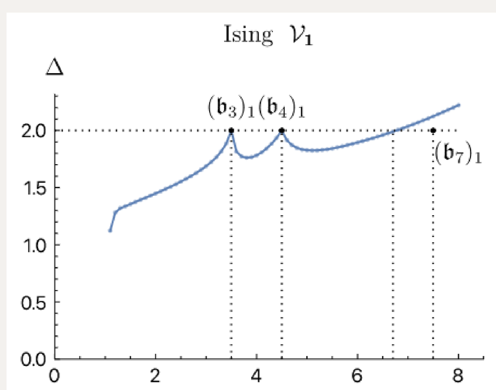
$$\text{Virasoro expansion}: Z^{3d}_\mu(\tau, \overline{\tau}) = \sum_{h,\overline{h}} n_{\mu,h\overline{h}}\, \chi_h(\tau)\overline{\chi}_{\overline{h}}(\overline{\tau}), \quad (n_{\mu,h\overline{h}} \in \mathbb{Z}_{\geq 0}),$$

$$\text{Modular covariance}: Z^{3d}_\mu(\tau, \overline{\tau}) = \sum_{\nu\in Z(\mathcal{C})} S^{-1}_{\mu\nu}\, Z^{3d}_\nu\big(S\tau, \overline{S\tau}\big), \quad (S\tau \equiv -1/\tau).$$

Combining these two, one gets the "modular bootstrap equation":

$$0 = \sum_{\nu\in Z(\mathcal{C})} \sum_{h,\overline{h}} n_{\nu,h\overline{h}} \left(\mathrm{id}_{\mu\nu}\, \chi_h(S\tau)\overline{\chi}_{\overline{h}}(\overline{S\tau}) - S_{\mu\nu}\, \chi_h(\tau)\overline{\chi}_{\overline{h}}(\overline{\tau})\right).$$

Running the numerical bootstrap for $\mathcal{C}$ = Ising, one obtains upper bounds [LS23].



Bootstrap bounds on the lightest primary operator in the trivial anyon sector [LS23].

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