Seiberg-Witten Theory

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Abstract

In this paper, we will review Seiberg-Witten Theory, an $N = 2$ supersymmetric Yang-Mills Theory with gauge group $SU(2)$. In their seminal paper [SW94], Seiberg and Witten studied this model and they found an exact low-energy description. Given that the coupling of Yang-Mills theories becomes stronger at low energies, this effective action is valuable to study non-perturbative effects. It was shown that this model exhibits a variety of physical phenomena. In particular, Seiberg-Witten theory exhibits confinement of electric charge via condensation of magnetic monopoles, and a version of electric-magnetic duality, which relates a weakly coupled regime with a strongly coupled one. Showing confinement analytically in ordinary QCD is an open problem, and as QCD is the theory of strong interactions, this problem has phenomenological consequences. This is one place the study of supersymmetric Yang-Mills theory can be of phenomenological value. Other physically interesting effects that Seiberg-Witten theory contains include chiral symmetry breaking and generation of mass scale from strong coupling. We will focus on the monopole condensation and confinement.

This subject requires some background, so I will write small sections on 't Hooft-Polyakov monopole and supersymmetry. Afterwards, I will obtain the low energy description of the $N = 2$ supersymmetric Yang-Mills theory, discuss the version of electric-magnetic duality that appears, and monopole condensation and confinement. Most importantly, I do not understand the details of Seiberg-Witten theory at a deep level, so this manuscript should be viewed as incomplete at the moment.

Contents

1 Introduction

Seiberg-Witten theory is an $N = 2$ supersymmetric Yang-Mills Gauge theory in 4-dimensions with gauge group $SU(2)$. It is possible to add extra matter fields, but we will deal with the case where the only matter fields appearing in the theory come from supersymmetry requirements. In their seminal paper [SW94], Seiberg and Witten studied this theory and they were able to obtain an exact low-energy description for it using supersymmetry. Moreover, they were able to show that this low-energy theory exhibits interesting physics such as the confinement of electric charge via monopole condensation, an electric magnetic duality, chiral symmetry breaking, and a generation of mass scale from strong coupling. Since Yang-Mills theories have couplings that become stronger at short distances, the results obtained in the exact effective description are valuable as they explore a regime where perturbation theory fails, and understanding theories in strong coupling has been an outstanding problem in physics. The supersymmetry in the theory allows for these exact computations, but the main goal is to see these phenomena in non-supersymmetric theories. Because although supersymmetry leads to some exact results, it is still unobserved in nature. In any case, obtaining an exact low-energy description and analyzing its physics are important achievements, making the Seiberg-Witten theory very appealing.

In this report, we will focus on the confinement of electric charge and the electric magnetic duality of the low-energy action. Confinement is observed in nature as we have never seen a single quark but only bound states of them such as the proton. In the 1970s, Kenneth Wilson came up with the formulation of lattice gauge theory [Wil74], which was a breakthrough, and this formulation shed light on quark confinement. Numerical simulations verify confinement, but understanding it through the dynamics of QCD has not yet been achieved. Some ideas have been put forward in the past, by 't Hooft, Mandelstam [Man76], Parisi [Par75], and others, which used the idea of the Meissner effect. In a superconductor, electric charges condensate (the vacuum consists of electrically charged pairs), and this expells the magnetic fields. If two probe magnetic charges are inserted here, there will form a flux tube between the probe charges and the potential between them grows linearly. This leads to the confinement of magnetic charge through the condensation of electric charges. The proposed idea was to consider the dual Meissner effect, where magnetic monopoles condensate, and this leads to the confinement of the electric charges through a linearly growing potential. In the low-energy description of $N = 2$ super-Yang-Mills theory, this was shown by softly breaking the supersymmetry to $N = 1$. We will discuss this in section 6.

The electric magnetic duality is an idea that probably goes back to Dirac, he may have been the first to talk about it. In Maxwell's theory, which is a $U(1)$ gauge theory, the source-free equations remain intact under exchanging electric fields with magnetic fields. Or, in terms of the field strength tensor, the exchange

$$F \to *F \tag{1}$$

with $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$ and $*F = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} dx^\mu \wedge dx^\nu$. This is equivalent to changing $E \to B$, $B \to -E$. To see that this is the case, recall the vacuum Maxwell's equations

$$\begin{aligned} d * F &= 0 \quad \text{(equations of motion)}, \\ dF &= 0 \quad \text{(Bianchi identity)}. \end{aligned} \tag{2}$$

We easily see that $F \to *F$ leaves the equations invariant. The duality can be maintained even when there are sources. To do that, we introduce a magnetic source J_m in addition to the usual source which we will call the electric source J_e. Then, the equations

$$\begin{aligned} d * F &= J_e, \\ dF &= J_m, \end{aligned} \tag{3}$$

still have an electric-magnetic duality provided we exchange electric sources with magnetic sources. In 1931, Dirac worked out the quantum theory in the presence of a monopole in his brilliant paper [Dir31], and showed that for the quantum theory to be consistent, one has to ensure $qg = 2\pi n$ ($n \in \mathbb{Z}$) where q is the electric charge and g is the magnetic charge. Now, the improved version of the electric magnetic duality exchanges the fundamental unit of charges ($n = 1$)

$$q \to g = \frac{2\pi}{q} \tag{4}$$

so that the equations are intact. Since q is a coupling parameter that controls the strength of interaction of matter fields to the Maxwell fields, the duality exchanges weak coupling and strong coupling, because on one side q is in the nominator, and in the other it is in the denominator.

Inspired by this, improved versions of electric-magnetic duality were conjectured in [MO77]. There were some problems with this conjecture because the main concern is the quantum theory, and the quantum effects can spoil this duality. The duality was shown to exist in $N = 4$ super-Yang-Mills theory because $N = 4$ solves the problems about quantum corrections and other issues. However, it seemed to be the case that such a duality wouldn't occur in less supersymmetric theories because it was too hard to maintain. This viewpoint changed after the work of Seiberg and Witten, where they found a low energy description for $N = 2$ super-Yang-Mills theory which contained a version of electric-magnetic duality. I will explore this in more detail in 5.

Understanding $N = 2$ supersymmetric Yang-Mills theory requires some background, so we will review some prerequisite topics. In section 2, we discuss monopoles, dyons, and the Montonen-Olive duality. In section 3, we review $N = 1$ and $N = 2$ supersymmetry and the action of supersymmetric Yang-Mills theory. In section 4, following [Bil97, SW94, Tek15], we will obtain the low-energy effective action for the $N = 2$ theory and determine the metric of the moduli space. In 5, we explore the version of electric-magnetic duality that appears in the low-energy action obtained in the previous section. In 6, we discuss the confinement of electric charge on the effective action. In 7, we will briefly comment on some recent works concerning Seiberg-Witten theory. And finally, in the discussion section, we will conclude the report.

2 't Hooft-Polyakov Monopole, the Georgi-Glashow Model and Montonen-Olive Duality

In this section, we will first investigate topological solitons (lumps) with some examples in lower dimensional models, then we will discuss the 't Hooft-Polyakov monopole, which is a lump solution of the Georgi-Glashow model in the weak coupling. There are also dyons, which is a monopole with electric charge as well, that is also a classical lump. We will also talk about the Montonen-Olive duality which exchanges the elementary excitations of the perturbative Georgi-Glashow model and the monopoles, which are solitonic solutions. The perturbative spectrum of the Georgi-Glashow model can be easily found. After the expectation value of the Higgs field breaks the $SU(2)$ symmetry down to $U(1)$, the spectrum contains two massive bosons $W^{\pm}$ with mass $m_W = ev$, a massless $U(1)$ field, and a Higgs field with mass $m_H = \sqrt{2}m$. For more detailed discussions on solitons and in particular on monopoles, we refer to [AGH97, Har96, SY23, Sre07].

2.1 Brief Discussion on Topological Solitons

Consider a classical field theory on a $d + 1$ dimensional Minkowski space-time whose vacuum configuration is not unique, and there is a manifold $\mathcal{V}$ of vacuum configurations with the same minimum energy. In some of the theories having degenerate vacua, there are special objects called topological solitons with finite energy and non-trivial topology. To understand these objects, we first look at the energy-functional of the field theory under consideration

$$\mathcal{E} = \int d^d x(T + V), \tag{5}$$

where T is the kinetic energy, which contains time derivatives of the fields in the theory, and V is the potential energy, which contains spatial derivatives and interactions of fields. Let us consider the static configurations, for which

$$\mathcal{E} = \int d^d xV. \tag{6}$$

Now, to find finite energy solutions, we need to make sure that at the spatial boundary $\partial\mathbb{R}^d \cong S^{d-1}$, fields must tend to the vacuum configurations, which are the zeros of the potential (we shift the zero point of the potential such that energy is positive definite). By hypothesis, there is a manifold of configurations $\varphi_0 \in \mathcal{V}$ that satisfy this condition. The fields can be classified as maps from the spatial boundary (a sphere) to the vacuum manifold

$$\varphi : S^{d-1} \mapsto \mathcal{V}. \tag{7}$$

It turns out that, depending on the manifold $\mathcal{V}$, such mappings can have non-trivial topological classification. Specifically, the fields, viewed as maps from S^{d-1} to $\mathcal{V}$ fall into disjoint homotopy classes, which is characterized by the $d-1$th homotopy group of the manifold $\mathcal{V}$: $\Pi_{d-1}(\mathcal{V})$. Especially interesting is the case where $\mathcal{V} = S^{d-1}$, so that $\Pi_{d-1}(S^{d-1}) = \mathbb{Z}$. This means that there are mappings classified by integers called the winding number. The winding number of the map corresponds to a topological charge for the lump, which is conserved under the dynamics of the system. To see why the topological charge would be conserved, observe that the charge is associated with the behavior of the field at the spatial infinity, and to change the topological charge we'd have to change the field at infinity. But then the derivatives of the fields would not be zero at infinity, so the energy density at infinity is not zero, hence total energy diverges. Hence, the topological charge of a classical field configuration is conserved for any finite energy process. Due to the topological conservation law, lumps are stable objects despite having large masses.

There are 4 cases of interest:

1. $\Pi_0(S^0) = \mathbb{Z}_2$ classifies kinks in $1+1$ dimensional scalar field theory with the double-well potential,
2. $\Pi_1(S^1) = \mathbb{Z}$ classifies vortices in $2+1$ dimensional Abelian Higgs model,
3. $\Pi_2(S^2) = \mathbb{Z}$ classifies monopoles in $3+1$ dimensional Georgi-Glashow model (Yang-Mills Higgs theory with gauge group $SU(2)$),
4. $\Pi_3(S^3) = \mathbb{Z}$ classifies instantons in $4+0$ dimensional Euclidean Yang-Mills gauge theory.

Our main concern will be the monopoles in the Georgi-Glashow model. But first, let us consider the simplest example to better understand solitons. We consider in $1+1$ dimensions a scalar field with a double well potential

$$S = \int d^2x \left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\lambda}{4}(\varphi^2 - v^2)^2 \right), \tag{8}$$

of which energy functional is

$$\mathcal{E} = \int dx^1 \left((\partial_0\varphi)^2 + (\partial_1\varphi)^2 + \frac{\lambda}{4}(\varphi^2 - v^2)^2 \right). \tag{9}$$

For static fields, one has

$$\mathcal{E} = \int dx^1 \left((\partial_1\varphi)^2 + \frac{\lambda}{4}(\varphi^2 - v^2)^2 \right). \tag{10}$$

The vacuum manifold consists of two points: $\mathcal{V} = \{-v, +v\}$, and the boundary of space is also two points $\partial\mathbb{R} \cong \{-\infty, \infty\} \equiv S^0$. For finite energy configurations, we must have

$$\varphi\Big|_{S^0} \in \mathcal{V} \Longrightarrow \varphi(-\infty) = \pm v \quad ; \quad \varphi(\infty) \pm v. \tag{11}$$

Thus, the space of fields is divided into different topological sectors. In the sector with boundary conditions $\varphi(-\infty) = -v$, $\varphi(\infty) = v$, the minimum energy configuration is called a kink. The equation it satisfies can be found by varying the energy functional and requiring it to be zero, and one can solve that equation to get the configuration. We will not carry that out in detail, but one can refer to [MS04, Sre07].

The topological conservation law that we talked about can be made more precise by defining a topological current:

$$J^\mu = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_\nu\varphi, \tag{12}$$

which is conserved for all field configurations: $\partial_\mu J^\mu = 0$, and the associated charge is

$$Q = \int dx^1 J^0 = \frac{1}{2v}\int dx^1 \partial_1\varphi_1 = \frac{1}{2v}\big(\varphi(x^1 = +\infty) - \varphi(x^1 = -\infty)\big). \tag{13}$$

For the kink solution, $\varphi(x^1 = +\infty) = v$ and $\varphi(x^1 = -\infty)) - v$ so the charge is

$$Q_{\text{kink}} = 1. \tag{14}$$

An anti-kink is defined as the minimum of the energy functional whose charge is -1.

The next example is vortex solutions in 2+1 dimensions. In 2+1 dimensions, one can consider an Abelian-Higgs model with potential

$$V(\varphi) = \frac{\lambda}{4}(\varphi^\dagger\varphi - v^2)^2, \tag{15}$$

so the vacuum manifold is $\mathcal{V} \cong S^1$ and the spatial boundary is also S^1. From $\Pi_1(S^1) = \mathbb{Z}$, we expect finite-energy static solutions. However, this time we need to add $U(1)$ gauge fields to the theory to have finite-energy solutions. They turn out to be vortices and their topological charge is given by the winding number of maps $\varphi : S^1 \mapsto S^1$ which has the integral representation

$$n = \frac{i}{2\pi}\int d\theta\ \varphi\frac{d}{d\theta}\varphi^{-1}. \tag{16}$$

For $3+1$ dimensions, we have the spatial boundary S^2 so we want a theory whose vacuum manifold is $\mathcal{V} \cong S^2$. This is the $SU(2)$ Yang-Mills Higgs theory with the Higgs field transforming under the adjoint representation of the gauge group and potential

$$V(\varphi) = \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2. \tag{17}$$

$SU(2)$ has 3 generators so $a = 1, 2, 3$. This potential, with the standard kinetic terms of $SU(2)$ Yang-Mills and adjoint scalar, has topological solitons. These solitons turn out to be monopoles, and a charge 1 soliton in this theory is called a 't Hooft Polyakov monopole. We will construct this in detail in the next section, but we first need to consider the winding number. We have $\Pi_2(S^2) = \mathbb{Z}$, but how can we represent the winding number?

To answer this, we write $\varphi^a = v\hat{\varphi}^a$ with φ^a an arbitrary unit vector. This unit vector has 2 degrees of freedom which are equivalent to the polar and azimuthal angles in 3-space. So $\hat{\varphi}^a$ specifies a unit sphere. We write $\mathbf{x}_\infty = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ (we take R to be very large) and so we can consider $\hat{\varphi}(\mathbf{x}_\infty)$ as a map from the spatial S^2 to vacuum S^2 as a periodic function of θ, ϕ. The winding number counts the number of times spatial S^2 winds around the vacuum S^2, and its winding number has the following integral representation:

$$n = \frac{1}{8\pi}\int d\theta d\phi \varepsilon^{abc}\varepsilon^{ij}\hat{\varphi}^a\partial_i\hat{\varphi}^b\partial_j\hat{\varphi}^c. \tag{18}$$

2.2 't Hooft Polyakov Monopole

We consider the $SU(2)$ Yang-Mills Higgs theory, defined by the action

$$S = \int d^4x \mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi D^\mu\varphi\right) - \frac{\lambda}{4}\int d^4x\Big(-2\mathrm{Tr}\varphi^2 - v^2\Big)^2. \tag{19}$$

The potential is the Higgs potential, where $v = m/\sqrt{\lambda}$. We will be concerned with the weakly coupled regime, which means $\lambda \ll m^2$.

The generators of $SU(2)$ satisfy the algebra $[t^a, t^b] = \varepsilon^{abc}t^c$, and we normalize them via Tr $t^at^b = -\frac{1}{2}\delta^{ab}$. The φ field is Lie algebra-valued, and under gauge transformations, it transforms in the adjoint representation of $SU(2)$. The field strength $F_{\mu\nu}$ is also in the adjoint representation. We can write them in terms of component fields as $F_{\mu\nu} = F^a_{\mu\nu}t^a$ and $\varphi = \varphi^a t^a$. With this, $F^a_{\mu\nu}$ and $D_\mu\varphi^a$ are given by

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon^{abc}A^b_\mu A^c_\nu, \\ D_\mu\varphi^a &= \partial_\mu\varphi^a + e\varepsilon^{abc}A^b_\mu\varphi^c, \end{aligned} \tag{20}$$

We can now write the action as

$$S = \int d^4x\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D_\mu\varphi^a D^\mu\varphi^a - \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2\right). \tag{21}$$

We will be interested in finite-energy static solutions to the equations of motion of this theory. First, let us write the equations of motion:

$$
\begin{aligned}
0 &= \frac{\delta S}{\delta A^a_\nu} = D_\mu F^{a\mu\nu} + e\varepsilon^{abc}\varphi^b D^\nu \varphi^c, \\
0 &= \frac{\delta S}{\delta \varphi^a} = (D_\mu D^\mu \varphi)^a - \lambda \varphi^a(\varphi^b\varphi^b - v^2),
\end{aligned}
\tag{22}
$$

we also have the Bianchi identity:

$$
0 = \varepsilon^{\mu\nu\rho\sigma} D_\nu F^a_{\rho\sigma}. \tag{23}
$$

Now, since we are interested in the static solutions, the vacuum configurations will be the minimum of the potential. We write the total energy as

$$
\mathcal{E} = \int d^3x \Big(E^a_i E^a_i + B^a_i B^a_i + D_0\varphi^a D_0 \varphi^a + D_i \varphi^a D_i \varphi^a + V(\varphi)\Big), \tag{24}
$$

with $E^a_i = -F^a_{0i}$, $B^a_i = \frac{1}{2}\varepsilon_{ijk}F^a_{jk}$. Clearly $\mathcal{E} \geq 0$ with equality only when $F_{\mu\nu} = 0$, and $D_\mu \varphi = 0 = V(\varphi)$. So, the vacuum configurations are given by $A_\mu = g\partial_\mu g^{-1}$, and φ^a a constant such that $\varphi^a\varphi^a = v^2$. A constant Higgs field breaks the $SU(2)$ gauge symmetry down to a $U(1)$ subgroup. Let us choose the 3-rd direction in the generator space and write $\varphi^a = v\delta^{a3}$ so that the surviving $U(1)$ gauge symmetry is along the third axis in the full gauge group $SU(2)$. With this symmetry breaking, the fields A^1_μ and A^2_μ will pick up masses. We define $W^\pm_\mu = (A_1 \pm iA_2)/\sqrt{2}$ and these complex vector fields have mass $m_W = ev$ and have $\pm e$ electric charge under the remaining $U(1)$ group. And the associated abelian field strength is $F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$. One can write this abelian field strength in a gauge invariant way as

$$
F_{\mu\nu} = \hat{\varphi}^a F^a_{\mu\nu} - \frac{1}{e}\varepsilon^{abc}\hat{\varphi}^a D_\mu \hat{\varphi}^b D_\nu \hat{\varphi}^c. \tag{25}
$$

Here $\hat{\varphi}^a = \varphi^a/\sqrt{\varphi^a\varphi^a}$. One can see that when we set $\varphi^a = v\delta^{a3}$, or $\hat{\varphi}^a = \delta^{a3}$, we have

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + e\varepsilon^{3bc}A^b_\mu A^c_\nu - \frac{1}{e}\varepsilon^{abc}\delta^{a3}(e\varepsilon^{bde}A^d_\mu \delta^{e3})(e\varepsilon^{cfg}A^f_\nu \delta^{g3}) \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + e\varepsilon^{3bc}A^b_\mu A^c_\nu - e\varepsilon^{abc}\varepsilon^{bde}\varepsilon^{cfg}\delta^{a3}\delta^{e3}\delta^{g3}A^d_\mu A^f_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + e\varepsilon^{3bc}A^b_\mu A^c_\nu + e(\delta^{ad}\delta^{ce} - \delta^{ae}\delta^{cd})\varepsilon^{cfg}\delta^{a3}\delta^{e3}\delta^{g3}A^d_\mu A^f_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + e\varepsilon^{3bc}A^b_\mu A^c_\nu - e\delta^{ae}\delta^{cd}\varepsilon^{cfg}\delta^{a3}\delta^{e3}\delta^{g3}A^d_\mu A^f_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + e\varepsilon^{3bc}A^b_\mu A^c_\nu - e\varepsilon^{df3}A^d_\mu A^f_\nu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu.
\end{aligned}
\tag{26}
$$

Moreover, using the definition of $F^a_{\mu\nu}$ and $D_\mu\varphi^a$, we can write

$$
F_{\mu\nu} = \partial_\mu(\hat{\varphi}^a A^a_\nu) - \partial_\nu(\hat{\varphi}^a A^a_\mu) - \frac{1}{e}\varepsilon^{abc}\hat{\varphi}^a \partial_\mu \hat{\varphi}^b \partial_\nu \hat{\varphi}^c. \tag{27}
$$

In particular, the magnetic field of the remaining $U(1)$ field is

$$B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk} = \varepsilon^{ijk}\partial_j(\hat{\varphi}^a A_k^a) - \frac{1}{2e}\varepsilon^{ijk}\varepsilon^{abc}\hat{\varphi}^a\partial_j\hat{\varphi}^b\partial_k\hat{\varphi}^c, \tag{28}$$

and if we calculate the flux of B through a sphere at infinity, we get

$$\Phi = \int_{S^2_\infty} dS_i B^i = \int_{S^2_\infty} dS_i \left(\varepsilon^{ijk}\partial_j(\hat{\varphi}^a A_k^a) - \frac{1}{2e}\varepsilon^{ijk}\varepsilon^{abc}\hat{\varphi}^a\partial_j\hat{\varphi}^b\partial_k\hat{\varphi}^c\right). \tag{29}$$

The first term is the surface integral of a curl, hence it is zero by the divergence theorem. The second term is related to the topological charge of the monopole, which is related to the winding number as given by

$$n = \frac{1}{8\pi}\int d^2\theta\varepsilon^{abc}\varepsilon^{ij}\hat{\varphi}^a\partial_i\hat{\varphi}^b\partial_j\hat{\varphi}^c. \tag{30}$$

The integration over the sphere can be written as

$$\begin{aligned}\Phi &= -\frac{1}{2e}\int_{S^2_\infty} dS_i\varepsilon^{ijk}\varepsilon^{abc}\hat{\varphi}^a\partial_j\hat{\varphi}^b\partial_k\hat{\varphi}^c \\ &= -\frac{1}{2e}\int_{S^2_\infty} d\theta d\phi r^2\sin\theta\hat{x}_i\varepsilon^{ijk}\varepsilon^{abc}\hat{\varphi}^a\partial_i\hat{\varphi}^b\partial_j\hat{\varphi}^c.\end{aligned} \tag{31}$$

On the sphere, dS_i is along $\hat{r}$, so $\hat{x}_i\varepsilon^{ijk} \to \varepsilon^{ij}$ with $i, j = 1, 2$ now and $\varepsilon^{12} = \varepsilon^{\theta\phi}$ $= 1$. The derivatives then become angular derivatives, and because of the ε^{ij}, there will be one θ one ϕ derivative in each term. But on spherical coordinates they come with factors: $\partial_1 = \frac{1}{r}\partial_\theta$ and $\partial_2 = \frac{1}{r\sin\theta}\partial_\phi$, so the rs and $\sin\theta$s cancel out and we are left with

$$\begin{aligned}\Phi &= \frac{-1}{2e}\int d\theta d\phi\varepsilon^{abc}\varepsilon^{ij}\hat{\varphi}^a\partial_i\hat{\varphi}^b\partial_j\hat{\varphi}^c \\ &= -\frac{4\pi n}{e}.\end{aligned} \tag{32}$$

Hence, a soliton with topological charge n acts like a magnetic monopole of charge $\Phi = -\frac{4\pi n}{e}$. Let us explicitly construct a solution for the charge 1 soliton. We look for time-independent configurations and fix the Weyl gauge $A_0 = 0$. So the energy is

$$\mathcal{E} = \int d^3x\Big(B_i^a B_i^a + D_i\varphi^a D_i\varphi^a + V(\varphi)\Big), \tag{33}$$

and the boundary condition of the scalar field is

$$\lim_{r\to\infty}\varphi^a(\mathbf{x}) = v\frac{x^a}{r}. \tag{34}$$

To find a boundary condition on A_i^a, we require $D_i\varphi^a = 0$ at the boundary. Thus

$$\begin{aligned}
0 &= \partial_i\varphi^a + e\varepsilon^{abc}A_i^b\varphi^c \\
&= \partial_i\left(\frac{x^a}{r}\right) + e\varepsilon^{abc}A_i^b\frac{x^c}{r} \\
&= \frac{1}{r}\left(\delta^{ia} - \frac{x^i x^a}{r^2}\right) + e\varepsilon^{abc}A_i^b\frac{x^c}{r}.
\end{aligned} \tag{35}$$

Multiply both sides by $rx_j\varepsilon^{jda}$ to get

$$0 = \varepsilon^{dij}x_j + e(x^d x_j A_i^j - r^2 A_i^d). \tag{36}$$

This is solved by $A_i^a = \varepsilon^{aic}\frac{x_c}{er^2}$, so the boundary conditions are

$$\begin{aligned}
\lim_{r\longrightarrow\infty}\varphi^a(\mathbf{x}) &= v\frac{x^a}{r}, \\
\lim_{r\longrightarrow\infty}A_i^a(\mathbf{x}) &= \varepsilon^{aic}\frac{x_c}{er^2}.
\end{aligned} \tag{37}$$

It is reasonable to expect that the configuration which minimizes the energy has a lot of symmetries. Based on this expectation, we make the following spherically symmetric ansatzë:

$$\begin{aligned}
\varphi^a(\mathbf{x}) &= v\frac{x^a}{r}f(r), \\
A_i^a(\mathbf{x}) &= \varepsilon^{aic}\frac{x_c}{er^2}a(r).
\end{aligned} \tag{38}$$

The boundary conditions on f and a are $f(\infty) = 1 = a(\infty)$ and $f(0) = 0 = a(0)$. The conditions at ∞ are determined by the boundary conditions of φ^a and A_i^a, and the conditions at 0 are determined by the regularity (having finite values and derivatives) conditions on φ^a and A_i^a. If we input this into the energy functional, and vary it with respect to f and a, we will get differential equations, whose solutions are stationary points of $\mathcal{E}$.

But first, it is useful to put a lower bound on $\mathcal{E}$ (since we are looking for a static configuration, we can also view this as the mass of the monopole). Observe that

$$B_i^a B_i^a + D_i\varphi^a D_i\varphi^a = \frac{1}{2}(B_i^a \pm D_i\varphi^a)^2 \mp B_i^a D_i\varphi^a. \tag{39}$$

Using some tricks, we can write $B_i^a D_i\varphi^a = \partial_i(B_i^a\varphi^a) - (D_iB_i)^a\varphi^a = \partial_i(B_i^a\varphi^a)$ ($D_iB_i^a = 0$ due to Bianchi identity). Hence

$$\int d^3x\,(B_i^a B_i^a + D_i\varphi^a D_i\varphi^a) = \frac{1}{2}\int d^3x(B_i^a \pm D_i\varphi^a)^2 \mp \int d^3x\partial_i(B_i^a\varphi^a), \tag{40}$$

and we can use Stokes' theorem to convert the divergence integral to a surface integral on a sphere at ∞:

$$\int d^3x\partial_i(B_i^a\varphi^a) = \int_{S^2_\infty} dS_i B_i^a\varphi^a = v\int_{S^2_\infty} dS_i B_i = v\Phi = -\frac{4\pi n v}{e}, \tag{41}$$

Where we used (25). So we have

$$\int d^3x \, (B_i^a B_i^a + D_i\varphi^a D_i\varphi^a) = \frac{1}{2}\int d^3x (B_i^a \pm D_i\varphi^a)^2 \pm \frac{4\pi n v}{e}. \tag{42}$$

To get rid of $\pm$, we write $|n|$ instead of n, and hence the energy (mass of the monopole) is written as

$$\mathcal{E} \equiv M = \frac{4\pi|n|v}{e} + \int d^3x \left(\frac{1}{2}\Big(B_i^a + \mathrm{sign}(n) D_i\varphi^a\Big)^2 + V(\varphi)\right). \tag{43}$$

The integral is positive definite since the integrand is the sum of a square and the potential, which itself is positive-definite. So we see that the mass of the monopole satisfies the bound $M \geq \frac{4\pi|n|v}{e}$, with equality when the integral gives 0. This bound is called a Bogomolny'i bound. For monopoles whose masses saturate the bound, which means the integral is somehow 0, we have a BPS monopole (Bogomolny'i, Prasad, Sommerfield). It is tricky how we can make the integral give 0, but there is a limit in which it is easy to saturate the bound. If the potential vanishes, $V \to 0$ (or $\lambda \to 0$), which is called the BPS limit, then the BPS monopole equations are given by

$$B_i^a = -\mathrm{sign}(n) D_i\varphi^a. \tag{44}$$

Inserting our ansatz, we can find solutions to this equation. The solution in terms of the ansatz functions $f(r)$ and $a(r)$ is

$$\begin{aligned} a(\rho) &= 1 - \frac{\rho}{\sinh\rho}, \\ f(\rho) &= \coth\rho - \frac{1}{\rho}. \end{aligned} \tag{45}$$

Where we defined $\rho = evr$.

2.3 Dyons and Montonen-Olive Duality

In the Georgi-Glashow model, if we consider a dyon (a field configuration with both electric and magnetic charges), we can express its mass via

$$M = \int d^3x \left[\frac{1}{2}\left((D_0\varphi^a)^2 + (D_i\varphi^a)^2 + (E_i^a)^2 + (B_i^a)^2\right) + V(\varphi)\right]. \tag{46}$$

Moreover, we can compute the electric and magnetic charges of the dyon with

$$g = \frac{1}{v}\int d^3x B_i^a (D_i\varphi)^a \quad ; \quad q = \frac{1}{v}\int d^3x E_i^a (D_i\varphi)^a. \tag{47}$$

To see this, we note that the electric and magnetic charges are associated with the respective fluxes of the residual $U(1)$ field. Therefore

$$g = \int_{S^2_\infty} dS_i B_i = \int_{S^2_\infty} dS_i B_i^a \frac{\varphi^a}{v} = \frac{1}{v}\int d^3x \partial_i (B_i^a \varphi^a). \tag{48}$$

Now, we will add zero into the volume integral using $D_i B_i^a = \partial_i B_i^a + \varepsilon^{abc} A_i^b B_i^c = 0$ in the following way

$$\begin{aligned} g &= \frac{1}{v} \int d^3x \Big[\partial_i (B_i^a \varphi^a) - \varphi^d D_i B_i^d \Big] \\ &= \frac{1}{v} \int d^3x \Big[B_i^a \partial_i \varphi^a - \varphi^d \varepsilon^{dbc} A_i^b B_i^c \Big] \\ &= \frac{1}{v} \int d^3x B_i^a (D_i \varphi)^a. \end{aligned} \tag{49}$$

Similarly

$$q = \int_{S_\infty^2} dS_i E_i = \frac{1}{v} \int_{S_\infty^2} dS_i E_i^a \varphi^a = \frac{1}{v} \int d^3x \partial_i (E_i^a \varphi^a). \tag{50}$$

Now, we observe that $\mathrm{Tr} \varphi D_i E_i = 0$ due to the field equations. To see this, we recall the field equations due to the variation of A_ν^a:

$$0 = \frac{\delta S}{\delta A_\nu^a} = D_\mu F^{a\mu\nu} + e\varepsilon^{abc} \varphi^b D^\nu \varphi^c, \tag{51}$$

if we choose $\nu = 0$, we get Gauss' law constraint,

$$0 = D_i F^{a\ i0} + e\varepsilon^{abc} \varphi^b D^0 \varphi^c. \tag{52}$$

Multiply each sides with φ^a to get $\varphi^c D_i E_i^c = 0$. After adding that term to the electric charge, we get

$$\begin{aligned} q &= \frac{1}{v} \int d^3x \Big[\partial_i (E_i^a \varphi^a) - \varphi^d D_i E_i^d \Big] \\ &= \frac{1}{v} \int d^3x \Big[E_i^a \partial_i \varphi^a - \varphi^d \varepsilon^{dbc} A_i^b E_i^c \Big] \\ &= \frac{1}{v} \int d^3x E_i^a (D_i \varphi)^a, \end{aligned} \tag{53}$$

as desired.

We now introduce a new parameter $\tan\theta = q/g$ and write the mass of the dyon as

$$\begin{aligned} M = &\int d^3x \left[\frac{1}{2} \Big(\big(E_i^a - \sin\theta D_i \varphi^a \big)^2 + \big(B_i^a - \cos\theta D_i \varphi^a \big)^2 + \big(D_0 \varphi^a \big)^2 \Big) + V(\varphi) \right] \\ &+ v(q \sin\theta + g \cos\theta). \end{aligned} \tag{54}$$

The integral part is positive-definite because it is a sum of squares. Then, we have the following bound on the mass

$$M \geq v\sqrt{g^2 + q^2}. \tag{55}$$

The bound is saturated for BPS (Bogomolny'i-Prasad-Sommerfield) states, for which the integral yields 0. We note that the bound above is a universal bound, in the sense that it is obeyed by both the elementary excitations in the theory and the soliton solutions of the theory. Based on this observation, and with Dirac's elegant electric-magnetic duality, Montonen and Olive made a dramatic conjecture [MO77] that there are two descriptions of the Georgi-Glashow model. One is the electric formulation with gauge bosons elementary excitations and the monopoles are extended lumps (solitons); the other is the magnetic formulation where magnetic monopoles are the elementary excitations and the familiar gauge bosons are classical lumps. The two descriptions are exchanged by

$$q \to \frac{4\pi}{q}. \tag{56}$$

This means that in the regime where the first description is a weakly coupled theory, the second must be a strongly coupled theory. One must note that this is not a simple symmetry of the model, but it is a duality in the sense that it takes a weakly coupled theory and gives a reformulation of it in terms of a strongly coupled theory. At this stage there are three immediate objections to this dramatic idea:

1. Even if this holds at the classical level, it is not clear whether this is valid at the quantum level. This is because the quantum effects will shift the Higgs potential, which will ruin the universal mass formula.
2. This duality apparently exchanges monopoles with gauge bosons, but gauge bosons are spin-1 particles. How can one assign spin-1 to a lump?
3. We are considering electric-magnetic formulations but what about dyons? Why don't we construct a theory where dyons are the elementary objects?

The first two questions are resolved by supersymmetry, and the third one will lead to S-duality.

To genuinely control quantum effects and have a precise electric-magnetic duality at the quantum level is something that could be achieved only for $N = 4$ supersymmetry. But this theory apparently is not rich enough in its phenomenology. In $N = 2$ supersymmetry, on the other hand, Seiberg and Witten showed that there is an effective electric-magnetic duality in the macroscopic description.

3 Supersymmetry in $d = 4$

In this section, we will go through a review of supersymmetry. I will first carry out some computations in $N = 1$ supersymmetry, and then talk about $N = 2$ supersymmetric Yang-Mills theory. For more details, we refer to [BL94, Har96, Sre07]. I will stick as much as possible to the conventions of [Sre07].

Supersymmetry is a symmetry that mixes bosonic and fermionic fields. In order to understand, we need to understand spinors in $d=4$. A Dirac spinor has four complex components and transforms reducibly under the Lorentz group $SO(1,3)$. We can break the 4-component Dirac spinor into two two component spinors which transform irreducibly. These spinors can be denoted χ^a and $\overline{\chi}_{\dot{a}} = (\chi_a)^\dagger$. The undotted indices represent the left-handed spinor representation $\left(\frac{1}{2},0\right)$ and the dotted indices represent the right-handed spinor representation $\left(0,\frac{1}{2}\right)$. The notation $\left(\frac{1}{2},0\right)$ means that we have a left-handed representation and similarly for $\left(0,\frac{1}{2}\right)$. Since $\overline{\chi}_{\dot{a}} = (\chi_a)^\dagger$, the right-handed spinors can be obtained from complex conjugating the left-handed ones. There is an important object $(\sigma^\mu)_{a\dot{a}}$ which carries one vector index and a pair of left- and right-handed spinor indices. When $\mu=i$, we have the Pauli matrices and $\mu=0$ is the identity matrix. We also define $(\overline{\sigma}^\mu)^{\dot{a}a}$ which has the same 0-th component as σ but its spatial components are such that $\overline{\sigma}^i$ is minus the Pauli matrices. There are also two-indexed anti-symmetric levi-civita symbols ε^{ab}. The components are

$$\varepsilon^{12} = 1 = \varepsilon_{21} \quad ; \quad \varepsilon^{21} = -1 = \varepsilon_{12}. \tag{57}$$

This symbol satisfies

$$\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c \quad ; \quad \varepsilon^{ab}\varepsilon_{bc} = \delta^a{}_c. \tag{58}$$

Clearly, we can define $\varepsilon^{\dot{a}\dot{b}}$ with the same components as ε^{ab}. We can also define the contractions

$$\psi\chi \equiv \psi^a\chi_a \quad ; \quad \overline{\psi}\overline{\chi} \equiv \overline{\psi}_{\dot{a}}\overline{\chi}^{\dot{a}} \equiv \psi^\dagger\chi^\dagger, \tag{59}$$

$$(\psi\chi)^\dagger = (\psi^a\chi_a)^\dagger = \overline{\chi}_{\dot{a}}\overline{\psi}^{\dot{a}} = \overline{\chi}\overline{\psi} = \chi^\dagger\psi^\dagger, \tag{60}$$

(we will use the bar and the dagger notation interchangeably) along with these, we have

$$\psi\sigma^\mu\chi^\dagger \equiv \psi^a\sigma^\mu_{a\dot{a}}\chi^{\dagger\dot{a}} \quad ; \quad \chi^\dagger\overline{\sigma}^\mu\psi \equiv \chi^\dagger_{\dot{a}}\overline{\sigma}^{\mu\dot{a}a}\psi_a. \tag{61}$$

Observe that

$$(\psi\sigma^\mu\chi^\dagger)^\dagger = (\psi^a\sigma^\mu_{a\dot{a}}\chi^{\dagger\dot{a}})^\dagger = \chi^a(\sigma^\mu_{\dot{a}a})^*\psi^{\dagger\dot{a}} = \chi^a\sigma^\mu_{a\dot{a}}\psi^{\dagger\dot{a}} = \chi\sigma^\mu\psi^\dagger, \tag{62}$$

where we used $(\sigma^\mu)^\dagger = \sigma^\mu$ which in components reads $(\sigma^\mu_{\dot{a}a})^* = \sigma^\mu_{a\dot{a}}$. A similar identity holds for $\overline{\sigma}$:

$$(\chi^\dagger\overline{\sigma}^\mu\psi)^\dagger = \psi^\dagger_{\dot{a}}(\overline{\sigma}^{\mu a\dot{a}})^*\chi_a = \psi^\dagger_{\dot{a}}\overline{\sigma}^{\mu\dot{a}a}\chi_a = \psi^\dagger\overline{\sigma}^\mu\chi. \tag{63}$$

3.1 $N = 1$ Supersymmetry

We will follow the discussion in [Sre07]. The $N = 1$ supersymmetry algebra is given by:

$$\begin{aligned}
[Q_a, P^\mu] &= 0, \\
[Q^\dagger_{\dot{a}}, P^\mu] &= 0, \\
[Q_a, M^{\mu\nu}] &= (S^{\mu\nu}_{\mathrm{L}})_a{}^c Q_c, \\
[Q^\dagger_{\dot{a}}, M^{\mu\nu}] &= (S^{\mu\nu}_{\mathrm{R}})_{\dot{a}}{}^{\dot{c}} Q^\dagger_{\dot{c}}, \\
\{Q_a, Q_b\} &= 0 = \{Q^\dagger_{\dot{a}}, Q^\dagger_{\dot{b}}\}, \\
\{Q_a, Q^\dagger_{\dot{b}}\} &= -2\sigma^\mu_{a\dot{a}} P_\mu.
\end{aligned} \tag{64}$$

Where Q and $Q^\dagger$ are the generators of supersymmetry. Since supersymmetry mixes bosons and fermions, it has a commutation relation with the space-time transformation generators and anti-commutation relations with the other supersymmetry generators. P^μ is the $4-$momentum operator, whose exponentiation translates the fields and $M^{\mu\nu}$ is an anti-symmetric rank-two operator valued tensor whose exponentiation generates symmetry transformations associated with the Lorentz group. The objects $(S^{\mu\nu}_{\mathrm{L}})_a{}^c$ are the generators of the Lorentz group in the representation $\left(\frac{1}{2}, 0\right)$ and $(S^{\mu\nu}_{\mathrm{R}})_{\dot{a}}{}^{\dot{c}}$ are the generators in the representation $\left(0, \frac{1}{2}\right)$.

Having laid down the algebra, we now introduce superspace and superfields. The superspace is an enhanced space that has the usual bosonic coordinates x^μ and anticommuting left-handed and its complex conjugated right-handed spinor coordinates θ_a & $\theta^*_{\dot{a}}$. The superfields are functions of these coordinates: $\Phi(x, \theta, \theta^*)$.

The 4-momentum vector operator generates the usual space-time translations:

$$[\Phi(x, \theta, \theta^*), P^\mu] = -i\partial^\mu \Phi(x, \theta, \theta^*). \tag{65}$$

We expect the supersymmetry generators to have a commutation relation with the superfield analogous to what we have above. With appropriate differential operators $\mathcal{Q}_a$ and $\mathcal{Q}^*_{\dot{a}}$, we should have

$$\begin{aligned}
[\Phi(x, \theta, \theta^*), Q_a] &= -i\mathcal{Q}_a \Phi(x, \theta, \theta^*), \\
[\Phi(x, \theta, \theta^*), Q^\dagger_{\dot{a}}] &= -i\mathcal{Q}^*_{\dot{a}} \Phi(x, \theta, \theta^*).
\end{aligned} \tag{66}$$

We want to find an expression for these differential operators, consistent with the supersymmetry algebra. To do so, we introduce

$$\begin{aligned}
\partial_a &\equiv \frac{\partial}{\partial \theta^a} \quad ; \quad \partial_a \theta^b = \delta_a{}^b, \\
\partial^*_{\dot{a}} &\equiv \frac{\partial}{\partial \theta^{\dot{a}*}} \quad ; \quad \partial^*_{\dot{a}} \theta^{\dot{b}*} = \delta_{\dot{a}}{}^{\dot{b}}.
\end{aligned} \tag{67}$$

Now, observe that

$$(\partial_a \theta^b)^* = \theta^{\dot{b}*}(\partial_a)^* = -(\partial_a)^* \theta^{\dot{b}*} \Longrightarrow (\partial_a)^* = -\partial^*_{\dot{a}}. \tag{68}$$

An ansatz for the differential operators that is consistent with the algebra is

$$\begin{aligned} \mathcal{Q}_a &= +\partial_a + i\sigma^\mu_{a\dot{b}}\theta^{\dot{b}*}\partial_\mu, \\ \mathcal{Q}^*_{\dot{a}} &= -\partial^*_{\dot{a}} - i\theta^b \sigma^\mu_{b\dot{a}}\partial_\mu. \end{aligned} \tag{69}$$

Since $\partial_a \theta^{\dot{b}*} = 0 = \partial^*_{\dot{a}}\theta^a$, we easily see that:

$$\{\mathcal{Q}_a, \mathcal{Q}_b\} = 0 = \{\mathcal{Q}^*_{\dot{a}}, \mathcal{Q}^*_{\dot{b}}\}, \tag{70}$$

and

$$\begin{aligned} \{\mathcal{Q}_a, \mathcal{Q}^*_{\dot{a}}\} &= \{+\partial_a + i\sigma^\mu_{a\dot{b}}\theta^{\dot{b}*}\partial_\mu, -\partial^*_{\dot{a}} - i\theta^b\sigma^\mu_{b\dot{a}}\partial_\mu\} \\ &= -i\{\partial_a, \theta^b\}\sigma^\mu_{b\dot{a}}\partial_\mu - i\sigma^\mu_{a\dot{b}}\{\theta^{\dot{b}*}, \partial^*_{\dot{a}}\}\partial_\mu. \end{aligned} \tag{71}$$

Observe that

$$\begin{aligned} \{\partial_a, \theta^b\}\cdot &= \partial_a(\theta^b \cdot) + \theta^b\partial_a \cdot = (\partial_a\theta^b)\cdot - \theta^b\partial_a \cdot + \theta^b\partial_a\cdot = \delta_a{}^b \cdot, \\ \{\partial^*_{\dot{a}}, \theta^{\dot{b}*}\}\cdot &= \partial^*_{\dot{a}}(\theta^{\dot{b}*}\cdot) + \theta^{\dot{b}*}\partial^*_{\dot{a}}\cdot = (\partial^*_{\dot{a}}\theta^{\dot{b}*})\cdot - \theta^{\dot{b}*}\partial^*_{\dot{a}}\cdot + \theta^{\dot{b}*}\partial^*_{\dot{a}}\cdot = \delta_{\dot{a}}{}^{\dot{b}}\cdot. \end{aligned} \tag{72}$$

Where $\cdot$ represents any object that $\{\partial_a, \theta^b\}$ or $\{\partial^*_{\dot{a}}, \theta^{\dot{b}*}\}$ acts on. Using also the fact that the anti-commutator is symmetric under the exchange of its arguments, we have:

$$\begin{aligned} \{\mathcal{Q}_a, \mathcal{Q}^*_{\dot{a}}\} &= -i\sigma^\mu_{a\dot{a}}\partial_\mu - i\sigma^\mu_{a\dot{a}}\partial_\mu, \\ &= -2i\sigma^\mu_{a\dot{a}}\partial_\mu. \end{aligned} \tag{73}$$

Compatible with the supersymmetry algebra that we wrote down above. Now, we introduce the super-covariant derivatives:

$$\begin{aligned} \mathcal{D}_a &= +\partial_a - i\sigma^\mu_{a\dot{a}}\theta^{*\dot{a}}\partial_\mu, \\ \mathcal{D}^*_{\dot{a}} &= -\partial^*_{\dot{a}} + i\theta^a\sigma^\mu_{a\dot{a}}\partial_\mu. \end{aligned} \tag{74}$$

With these, we have the following anti-commutation relations:

$$\begin{aligned} \{\mathcal{D}_a, \mathcal{D}_b\} &= 0 = \{\mathcal{D}^*_{\dot{a}}, \mathcal{D}^*_{\dot{b}}\}, \\ \{\mathcal{D}_a, \mathcal{D}^*_{\dot{a}}\} &= 2i\sigma^\mu_{a\dot{a}}\partial_\mu, \end{aligned} \tag{75}$$

$$\{\mathcal{D}_a, \mathcal{Q}_b\} = \{\mathcal{D}_a, \mathcal{Q}^*_{\dot{b}}\} = 0 = \{\mathcal{D}^*_{\dot{a}}, \mathcal{Q}_b\} = \{\mathcal{D}^*_{\dot{a}}, \mathcal{Q}^*_{\dot{b}}\}. \tag{76}$$

Observe that if we impose

$$\mathcal{D}^*_{\dot{a}}\Phi(x, \theta, \theta^*) = 0 \tag{77}$$

on a superfield, then this relation will be preserved under supersymmetry because $\mathcal{D}^*$ anti-commutes with the supercharges. We will call such a superfield a left-handed chiral superfield. Taking the Hermitean conjugate, we obtain a right-handed chiral superfield that obeys:

$$\mathcal{D}_a \Phi^\dagger(x, \theta, \theta^*) = 0. \tag{78}$$

Now, we introduce

$$y^\mu = x^\mu - i\theta\sigma^\mu\theta^*, \tag{79}$$

and observe that

$$\mathcal{D}^*_{\dot{a}} y^\mu = i\theta^a \sigma^\mu_{a\dot{a}} - \partial^*_{\dot{a}}(-i\theta\sigma^\mu\theta^*) = i\theta^a\sigma^\mu_{a\dot{a}} - i\theta^a\sigma^\mu_{a\dot{a}} = 0, \tag{80}$$

$$\mathcal{D}^*_{\dot{a}}\theta = 0. \tag{81}$$

In the first one, while taking the derivative of $\theta\sigma^\mu\theta^*$ with respect to $\partial^*_{\dot{a}}$, I anti-commuted $\partial^*_{\dot{a}}$ and θ to pick up an extra minus that resulted in the expression to be 0. Then, we can write a general left-handed superfield as a function of y^μ and θ only, so that $\mathcal{D}^*_{\dot{a}}$ annihilates it when it hits on it

$$\mathcal{D}^*_{\dot{a}}\Phi(y, \theta) = 0. \tag{82}$$

We can expand $\Phi(y, \theta)$ in powers of θ, and use the fact that θ is an anti-commuting variable to our advantage so that the expansion terminates after θ^2. Conventionally, we write the expansion as:

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{83}$$

Here, A and F are complex scalar fields, whereas ψ is a left-handed Weyl spinor field, and we recall our conventions on spinor index contractions $\theta\psi = \theta^a\psi_a = \varepsilon^{ab}\theta_b\psi_a$. We can further expand y dependence of the three fields:

$$A(y) = A(x) - i\theta\sigma^\mu\theta^*\partial_\mu A - (\theta\sigma^\mu\theta^*)(\theta\sigma^\nu\theta^*)\partial_\mu\partial_\nu A(x). \tag{84}$$

We can write the second term in a better way, using the Fierz identity

$$\begin{aligned}
(\theta\sigma^\mu\theta^*)(\theta\sigma^\nu\theta^*) &= \theta^a\sigma^\mu_{a\dot{a}}\theta^{*\dot{a}}\theta^b\sigma^\nu_{b\dot{b}}\theta^{*\dot{b}} \\
&= -\theta^a\theta^b\theta^{*\dot{a}}\theta^{*\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} \\
&= \frac{1}{4}\theta\theta^*\theta^*\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} \\
&= \frac{1}{4}\theta\theta^*\theta^*(-2g^{\mu\nu}) \\
&= -\frac{1}{2}\theta\theta^*\theta^* g^{\mu\nu},
\end{aligned} \tag{85}$$

then

$$A(y) = A(x) - i\theta\sigma^\mu\theta^*\partial_\mu A - \frac{1}{2}\theta\theta^*\theta^*\partial^2 A(x). \tag{86}$$

For the middle-term

$$\sqrt{2}\theta\psi(y) = \sqrt{2}\theta\psi(x) - \sqrt{2}\theta(i\theta\sigma^\mu\theta^*)\partial_\mu\psi + \cdots\theta\theta\partial\partial\psi, \tag{87}$$

since $\theta\theta\theta$ term will terminate, we need not worry about it. The second term needs some cleanup. Observe that

$$\theta^a\theta^b = -\frac{1}{2}\varepsilon^{ab}\theta\theta. \tag{88}$$

The easiest way to see this is by explicit computation

$$\theta^c\theta_c = \varepsilon^{cd}\theta_d\theta_c = \theta_2\theta_1 - \theta_1\theta_2 = -2\theta_1\theta_2, \tag{89}$$

so

$$\theta^a\theta^b = \varepsilon^{ab}\theta_1\theta_2. \tag{90}$$

Using

$$\theta^a = \varepsilon^{ab}\theta_b \Longrightarrow \theta^1 = \theta_2 \quad ; \quad \theta^2 = -\theta_1, \tag{91}$$

we have

$$\theta_1\theta_2 = -\theta^2\theta^1 = \theta^1\theta^2. \tag{92}$$

Hence

$$\theta^a\theta^b = \varepsilon^{ab}\theta^1\theta^2. \tag{93}$$

This is clearly correct because both sides are 0 when $a = b$ and when $a = 1, b = 2$ or $a = 2, b = 1$ we have equality on both sides. Getting back to the expansion of the fields,

$$\begin{aligned} -i\sqrt{2}\theta^a\theta^b\sigma^\mu_{b\dot{b}}\theta^{*\dot{b}}\partial_\mu\psi_a(x) &= -i\sqrt{2}\frac{-1}{2}\theta\theta\varepsilon^{ab}\varepsilon^{\dot{b}\dot{c}}\sigma^\mu_{b\dot{b}}\theta^*_{\dot{c}}\partial_\mu\psi_a(x) \\ &= \frac{i}{\sqrt{2}}\theta\theta\theta^*_{\dot{c}}\overline{\sigma}^{\mu\dot{c}a}\partial_\mu\psi_a \\ &= \frac{i}{\sqrt{2}}\theta\theta\theta^*\overline{\sigma}^\mu\partial_\mu\psi, \end{aligned} \tag{94}$$

hence

$$\sqrt{2}\theta\psi(y) = \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\theta^*\overline{\sigma}^\mu\partial_\mu\psi. \tag{95}$$

This is clean enough. Lastly, for the $\theta\theta F(y)$ term, we have

$$\theta\theta F(y) = \theta\theta F(x) - \theta\theta\theta\sigma\theta^*\partial F = \theta\theta F(x). \tag{96}$$

So, a general left-handed chiral superfield has the form

$$\begin{aligned} \Phi(x,\theta,\theta^*) = A(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) - i(\theta\sigma^\mu\theta^*)\partial_\mu A(x) \\ - \frac{i}{\sqrt{2}}\theta\theta\theta^*\overline{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\theta^*\theta^*\partial^2 A(x). \end{aligned} \tag{97}$$

Now we will investigate how a left-handed chiral superfield transforms under supersymmetry. To do so, we recall that

$$\begin{aligned} [\Phi(y,\theta), Q_a] &= -i\mathcal{Q}_a\Phi(y,\theta), \\ [\Phi(y,\theta), Q^*_{\dot{a}}] &= -i\mathcal{Q}^*_{\dot{a}}\Phi(y,\theta), \end{aligned} \tag{98}$$

and that

$$
\begin{aligned}
\mathcal{Q}_a\theta^b = \delta_a^b \quad &; \quad \mathcal{Q}_a y^\mu = 0, \\
\mathcal{Q}^*_{\dot{a}}\theta^b = 0 \quad &; \quad \mathcal{Q}^*_{\dot{a}} y^\mu = -2i\theta^c\sigma^\mu_{c\dot{a}}.
\end{aligned}
\tag{99}
$$

Therefore

$$
\mathcal{Q}_a\Phi(y,\theta) = \frac{\partial}{\partial\theta^a}\Phi(y,\theta), \tag{100}
$$

$$
\mathcal{Q}^*_{\dot{a}}\Phi(y,\theta) = -2i\theta^c\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}\Phi(y,\theta). \tag{101}
$$

Using (83),

$$
\mathcal{Q}_a\Phi(y,\theta) = \sqrt{2}\psi_a(y) + 2\theta_a F(y), \tag{102}
$$

and

$$
\begin{aligned}
\mathcal{Q}^*_{\dot{a}}\Phi(y,\theta) &= -2i\theta^c\sigma^\mu_{c\dot{a}}\left(\frac{\partial}{\partial y^\mu}A(y) + \sqrt{2}\theta^a\frac{\partial}{\partial y^\mu}\psi_a(y)\right) \\
&= -2i\theta^c\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}A(y) - 2\sqrt{2}\theta^c\theta^a\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}\psi_a(y) \\
&= -2i\theta^c\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}A(y) - 2\sqrt{2}\left(-\frac{1}{2}\theta\theta\varepsilon^{ca}\right)\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}\psi_a(y) \\
&= -2i\theta^c\sigma^\mu_{c\dot{a}}\frac{\partial}{\partial y^\mu}A(y) + \sqrt{2}\theta\theta\partial_\mu\psi^c(y)\sigma^\mu_{c\dot{a}}.
\end{aligned}
\tag{103}
$$

Thus, we have

$$
\begin{aligned}
&[\Phi(y,\theta), Q_a] = -i\mathcal{Q}_a\Phi = -i\sqrt{2}\psi_a(y) - 2i\theta_a F(y), \\
&[\Phi(y,\theta), Q^*_{\dot{a}}] = -i\mathcal{Q}^*_{\dot{a}}\Phi = -2\theta^c\sigma^\mu_{c\dot{a}}\partial_\mu A(y) - i\sqrt{2}\theta\theta\partial_\mu\psi^c(y)\sigma^\mu_{c\dot{a}}.
\end{aligned}
\tag{104}
$$

Expanding the commutator using the expansion of Φ in (83), we can match terms with the same power of θ to see how the component fields A, ψ, F transform under supersymmetry:

$$
[A(y), Q_a] = -i\sqrt{2}\psi_a(y) \quad ; \quad [A(y), Q^*_{\dot{a}}] = 0, \tag{105}
$$

$$
\sqrt{2}\theta^c\{\psi_c(y), Q_a\} = -2i\theta_a F(y) \ ; \ \sqrt{2}\theta^d\{\psi_d(y), Q^*_{\dot{a}}\} = -2\theta^c\sigma^\mu_{c\dot{a}}\partial_\mu A(y). \tag{106}
$$

To get rid of θ's, we multiply both sides of the left equation with θ_b and use $\theta_b\theta^c = \theta_b\theta_d\varepsilon^{cd} = \frac{1}{2}\varepsilon_{bd}\varepsilon^{cd}\theta\theta = -\frac{1}{2}\theta^2\delta_b^c$ to have

$$
-\sqrt{2}\frac{\theta\theta}{2}\{\psi_b, Q_a\} = -2i\frac{\theta\theta}{2}\varepsilon_{ba}F, \tag{107}
$$

which can be written as

$$
\{\psi_b(y), Q_a\} = -i\sqrt{2}\varepsilon_{ab}F(y). \tag{108}
$$

For the anti-commutator on the right, we multiply both sides by θ_e and use $\theta_e\theta^d = -\frac{1}{2}\theta^2\delta_e^d$ to get

$$
-\sqrt{2}\frac{\theta\theta}{2}\delta_e^d\{\psi_d, Q^*_{\dot{a}}\} = 2\frac{\theta\theta}{2}\delta_e^c\sigma^\mu_{c\dot{a}}\partial_\mu A, \tag{109}
$$

which is to say

$$\{\psi_c(y), Q^*_{\dot{a}}\} = -\sqrt{2}\sigma^\mu_{c\dot{a}}\partial_\mu A(y). \tag{110}$$

And, lastly,

$$[F, Q_a] = 0 \quad ; \quad [F, Q^*_{\dot{a}}] = -i\sqrt{2}\partial_\mu\psi^c(y)\sigma^\mu_{c\dot{a}}. \tag{111}$$

Altogether, the component fields have the following (anti-)commutation relations:

$$\begin{aligned} [A(y), Q_a] = -i\sqrt{2}\psi_a(y) \quad &; \quad [A(y), Q^*_{\dot{a}}] = 0, \\ \{\psi_b(y), Q_a\} = -i\sqrt{2}\varepsilon_{ab}F(y) \quad &; \quad \{\psi_c(y), Q^*_{\dot{a}}\} = -\sqrt{2}\sigma^\mu_{c\dot{a}}\partial_\mu A(y), \\ [F, Q_a] = 0 \quad &; \quad [F, Q^*_{\dot{a}}] = -i\sqrt{2}\partial_\mu\psi^c(y)\sigma^\mu_{c\dot{a}}. \end{aligned} \tag{112}$$

With these (anti-)commutation relations, we can infer how the component fields change under supersymmetry transformations.

Let us now consider more than one superfield

$$\Phi_i = \varphi_i + \sqrt{2}\theta\psi_i + \theta\theta F_i, \tag{113}$$

then

$$\Phi_i\Phi_j = \varphi_i\varphi_j + \sqrt{2}\theta(\varphi_i\psi_j + \varphi_j\psi_i) + \theta\theta(\varphi_i F_j + \varphi_j F_i - \psi_i\psi_j), \tag{114}$$

$$\begin{aligned} \Phi_i\Phi_j\Phi_k &= \varphi_i\varphi_j\varphi_k + \sqrt{2}\theta(\varphi_i\psi_j\varphi_k + \varphi_j\psi_i\varphi_k + \varphi_i\varphi_j\psi_k) \\ &\quad + 2\Big(\varphi_i(\theta\psi_j)(\theta\psi_k) + \varphi_j(\theta\psi_i)(\theta\psi_k)\Big) \\ &\quad + \theta\theta\Big(\varphi_i\varphi_j F_k + \varphi_i F_j\varphi_k + \varphi_j F_i\varphi_k - \psi_i\psi_j\varphi_k\Big) \\ &= \varphi_i\varphi_j\varphi_k + \sqrt{2}\theta\Big(\varphi_i\psi_j\varphi_k + \varphi_j\psi_i\varphi_k + \varphi_i\varphi_j\psi_k\Big) \\ &\quad + \theta\theta\Big(\varphi_i\varphi_j F_k + \varphi_i F_j\varphi_k + \varphi_j F_i\varphi_k \\ &\qquad - \varphi_k\psi_i\psi_j - \varphi_i\psi_j\psi_k - \varphi_j\psi_i\psi_k\Big). \end{aligned} \tag{115}$$

Where we've used

$$\theta\psi_i\theta\psi_k = \theta^a\psi_{ia}\theta^b\psi_{kb} = -\theta^a\theta^b\psi_{ia}\psi_{kb} = \frac{1}{2}\theta^2\varepsilon^{ab}\psi_{ia}\psi_{kb} = -\frac{1}{2}\theta^2\psi_i\psi_k. \tag{116}$$

In general, given a set of left-handed chiral superfields, we can consider a function of them, $W(\Phi)$. This function will itself be a left-handed chiral superfield, and its F term (coefficient of $\theta\theta$) can be written as

$$W(\Phi)\Big|_F = \frac{\partial W(\varphi)}{\partial\varphi_i}F_i - \frac{1}{2}\frac{\partial^2 W(\varphi)}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j. \tag{117}$$

F terms of left-handed chiral superfields are interesting because they are supersymmetry invariants. Therefore, we can construct supersymmetry-invariant potentials with *superpotentials* of the form $W(\Phi)\Big|_F$.

We will also be interested in vector superfields. It turns out the only constraint on a vector superfield is hermiticity [Sre07]

$$[V(x,\theta,\theta^*)]^\dagger = V(x,\theta,\theta^*). \tag{118}$$

Let us look at the component expansion of a generic field. As long as we have less than three of the same Grassmanian coordinates in a term, it will be nonzero. The possible coefficients in the expansion are

$$\{\theta, \theta^*, \theta\theta, \theta^*\theta^*, \theta\sigma^\mu\theta^*, \theta\theta\theta^*, \theta^*\theta^*\theta, \theta\theta\theta^*\theta^*\}. \tag{119}$$

These are all the combinations with θ and θ^* that do not vanish trivially. We need to consider each term in the expansion and choose appropriate coefficient functions for each term so that the indices are properly contracted. Thus, we have:

$$\begin{aligned} V(x,\theta,\theta^*) = C(x) &+ \theta\chi + \theta^*\xi^\dagger + \theta\theta M(x) + \theta^*\theta^* N(x) \\ &+ (\theta\sigma^\mu\theta^*)v_\mu + \theta\theta\theta^*\eta^\dagger(x) + \theta^*\theta^*\theta\lambda(x) + \frac{1}{2}\theta\theta\theta^*\theta^* D(x). \end{aligned} \tag{120}$$

Where C, D, M, N are complex scalar fields, χ, ξ, λ, η are left-handed Weyl-spinors and their hermitian conjugates are right-handed Weyl-spinors. From the Hermiticity condition, we see that

$$C^* = C, \tag{121}$$

$$\chi = \xi \quad ; \quad \chi^\dagger = \xi^\dagger, \tag{122}$$

$$M^* = N, \tag{123}$$

$$v_\mu = v_\mu^*, \tag{124}$$

$$\eta = \lambda \quad ; \quad \eta^\dagger = \lambda^\dagger, \tag{125}$$

$$D^* = D. \tag{126}$$

Thus

$$\begin{aligned} V(x,\theta,\theta^*) = C(x) &+ \theta\chi + \theta^*\chi^\dagger + \theta\theta M(x) + \theta^*\theta^* M^*(x) \\ &+ (\theta\sigma^\mu\theta^*)v_\mu + \theta\theta\theta^*\lambda^\dagger(x) + \theta^*\theta^*\theta\lambda(x) + \frac{1}{2}\theta\theta\theta^*\theta^* D(x). \end{aligned} \tag{127}$$

Here, C, D are real scalar fields, M is a complex scalar field, χ, λ are left-handed Weyl spinors, and v_μ is a real vector field. We will be interested in how the $\theta\theta\theta^*\theta^*$ component of the vector superfield changes under a supersymmetry. To compute that, we use

$$[V, Q_a] = -i\mathcal{Q}_a V, \tag{128}$$

the $\theta\theta\theta^*\theta^*$ term (henceforth we will call such terms D terms) of the left-hand-side is

$$[V, Q_a]\Big|_D = \frac{1}{2}[D, Q_a]. \tag{129}$$

What is the D term of the right-hand side? We can just use the definition of $\mathcal{Q}_a$ to get

$$-i\mathcal{Q}_a V = (-i\partial_a + (\sigma^\mu \theta^*)_a \partial_\mu)\, V. \tag{130}$$

Clearly, the ∂_a term erases a θ^a from the expansion, so when acting on V it will not give a D term. The second term has a θ^* in it, so only when it acts on the $\lambda^\dagger$ term of the expansion of V, it will give a $\theta\theta\theta^*\theta^*$. Hence, we read off that

$$\theta\theta\theta^*\theta^* \frac{1}{2}[D, Q_a] = \sigma^\mu_{a\dot{c}} \theta^{*\dot{c}} \partial_\mu (\theta\theta\theta^* \lambda^\dagger). \tag{131}$$

To get rid of θs, we first write this as

$$\theta\theta\theta^*\theta^* \frac{1}{2}[D, Q_a] = \theta\theta\sigma^\mu_{a\dot{c}} \theta^{*\dot{c}} \theta^*_{\dot{d}} \partial_\mu \lambda^{\dagger\dot{d}}, \tag{132}$$

and observe that

$$\theta^{*\dot{c}} \theta^*_{\dot{d}} \partial_\mu \lambda^{\dagger\dot{d}} = \theta^{*\dot{c}} \varepsilon_{\dot{d}\dot{e}} \theta^{*\dot{e}} \partial_\mu \lambda^{\dagger\dot{d}} = \frac{1}{2} \varepsilon^{\dot{c}\dot{e}} \theta^* \theta^* \varepsilon_{\dot{d}\dot{e}} \partial_\mu \lambda^{\dagger\dot{d}} = \frac{-1}{2} \theta^* \theta^* \partial_\mu \lambda^{\dagger\dot{c}}. \tag{133}$$

Hence

$$[D, Q_a] = -\sigma^\mu_{a\dot{c}} \partial_\mu \lambda^{\dagger\dot{c}}, \tag{134}$$

and similarly

$$[V, Q^\dagger_{\dot{a}}] = -i\mathcal{Q}^*_{\dot{a}} V. \tag{135}$$

The D term of the left-hand side is

$$[V, Q^\dagger_{\dot{a}}]\Big|_D = \frac{1}{2}[D, Q^\dagger_{\dot{a}}], \tag{136}$$

and for the right-hand-side, we have

$$-i\mathcal{Q}^*_{\dot{a}} V = (i\partial^*_{\dot{a}} - \theta^c \sigma^\mu_{c\dot{a}} \partial_\mu)\, V. \tag{137}$$

For the $\theta^2\theta^{*2}$ term, we see that

$$\begin{aligned} -\theta^c \sigma^\mu_{c\dot{a}} \theta^{*2} \theta^b \partial_\mu \lambda_b &= \theta^{*2} \sigma^\mu_{c\dot{a}} (-\theta^c \theta^b) \partial_\mu \lambda_b \\ &= \theta^{*2} \sigma^\mu_{c\dot{a}} \theta^2 \frac{1}{2} \varepsilon^{cb} \partial_\mu \lambda_b \\ &= \theta^2 \theta^{*2} \partial_\mu \lambda^c \sigma^\mu_{c\dot{a}}, \end{aligned} \tag{138}$$

thus

$$[D, Q^\dagger_{\dot{a}}] = \partial_\mu \lambda^c \sigma^\mu_{c\dot{a}}. \tag{139}$$

Together, we have

$$[D, Q_a] = -\sigma^\mu_{a\dot{c}} \partial_\mu \lambda^{\dagger\dot{c}} \quad ; \quad [D, Q^\dagger_{\dot{a}}] = \partial_\mu \lambda^c \sigma^\mu_{c\dot{a}}. \tag{140}$$

So we see that the transformation of a D term under supersymmetry gives a total divergence. Hence, a term of the form $\int d^4xD(x)$ is a supersymmetric invariant term. Now, consider a left-handed superfield Φ, its hermitian conjugate is a right-handed superfield, and we clearly have

$$(\Phi^\dagger\Phi)^\dagger = \Phi^\dagger\Phi. \tag{141}$$

So this is a vector superfield. Their component expansions give

$$\begin{aligned}\Phi(x,\theta,\theta^*) = \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) - i(\theta\sigma^\mu\theta^*)\partial_\mu\varphi(x)\\ -\frac{i}{\sqrt{2}}\theta\theta\theta^*\overline{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\theta^*\theta^*\partial^2\varphi(x),\end{aligned} \tag{142}$$

$$\begin{aligned}\Phi^\dagger(x,\theta,\theta^*) = \varphi^\dagger(x) + \sqrt{2}\theta^*\psi^\dagger(x) + \theta^*\theta^* F^\dagger(x) + i(\theta\sigma^\mu\theta^*)\partial_\mu\varphi^\dagger(x)\\ +\frac{i}{\sqrt{2}}\theta^*\theta^*\partial_\mu\psi^\dagger(x)\overline{\sigma}^\mu\theta + \frac{1}{4}\theta\theta\theta^*\theta^*\partial^2\varphi^\dagger(x).\end{aligned} \tag{143}$$

Let us find the D term of this vector superfield. In Φ, the term without θ meets with the term of $\Phi^\dagger$ with maximum θ to give

$$\theta^2\theta^{*2}\frac{1}{4}\varphi\partial^2\varphi^\dagger. \tag{144}$$

The $\sqrt{2}\theta\psi$ term meets with $\partial\psi\overline{\sigma}\theta$ term to give

$$\begin{aligned}i\theta\psi\theta^{*2}\partial_\mu\psi^\dagger\overline{\sigma}^\mu\theta &= i\theta^{*2}\psi^a\theta_a\partial_\mu\psi^\dagger_{\dot b}\overline{\sigma}^{\mu\dot b b}\theta_b\\ &= -i\theta^{*2}\psi^a\partial_\mu\psi^\dagger_{\dot b}\overline{\sigma}^{\mu\dot b b}\theta_a\theta_b\\ &= -i\theta^{*2}\psi^a\partial_\mu\psi^\dagger_{\dot b}\overline{\sigma}^{\mu\dot b b}\frac{1}{2}\varepsilon_{ab}\theta^2\\ &= i\theta^2\theta^{*2}\partial_\mu\psi^\dagger_{\dot b}\overline{\sigma}^{\mu\dot b b}\frac{1}{2}\varepsilon_{ab}\psi^a\\ &= -i\frac{1}{2}\theta^2\theta^{*2}\partial_\mu\psi^\dagger_{\dot b}\overline{\sigma}^{\mu\dot b b}\psi_b\\ &= \theta^2\theta^{*2}\frac{-i}{2}\partial_\mu\psi^\dagger\overline{\sigma}^\mu\psi.\end{aligned} \tag{145}$$

The $\theta^2 F$ term meets $\theta^{*2}F^\dagger$

$$\theta^2\theta^{*2}F^\dagger F. \tag{146}$$

The term with $-i(\theta\sigma^\mu\theta^*)\partial_\mu\varphi$ meets with $i(\theta\sigma\nu\theta^*)\partial_\nu\varphi^\dagger$ to give a D term. To simplify this we use the Fierz identity (85):

$$\theta^2\theta^{*2}\frac{-1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi^\dagger. \tag{147}$$

The term with $\theta\theta\theta^*$ in Φ and θ^* term in $\Phi^\dagger$ gives

$$\begin{aligned} -i\theta^2\theta^*\overline{\sigma}^\mu\partial_\mu\psi\theta^*\psi^\dagger &= -i\theta^2\theta^*_{\dot{a}}\overline{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a\theta^*_{\dot{c}}\psi^{\dagger\dot{c}} \\ &= i\theta^2\overline{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a\theta^*_{\dot{a}}\theta^*_{\dot{c}}\psi^{\dagger\dot{c}} \\ &= i\theta^2\overline{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a\left(\frac{-1}{2}\theta^2\varepsilon_{\dot{a}\dot{c}}\right)\psi^{\dagger\dot{c}} \\ &= \frac{-i}{2}\theta^2\theta^{*2}\overline{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a\psi^{\dagger\dot{c}} \\ &= \frac{i}{2}\theta^2\theta^{*2}\psi^\dagger_{\dot{a}}\overline{\sigma}^{\mu\dot{a}a}\partial_\mu\psi_a \\ &= \theta^2\theta^{*2}\frac{i}{2}\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi. \end{aligned} \tag{148}$$

And finally, the term with $\theta^2\theta^{*2}$ in Φ meets with $\varphi^\dagger$ term in $\Phi^\dagger$ to give

$$\theta^2\theta^{*2}\frac{1}{4}\partial^2\varphi\varphi^\dagger. \tag{149}$$

Now we combine these terms:

$$\Phi^\dagger\Phi\Big|_D = \frac{1}{4}\varphi\partial^2\varphi^\dagger - \frac{i}{2}\partial_\mu\psi^\dagger\overline{\sigma}^\mu\psi + F^\dagger F - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi^\dagger + \frac{i}{2}\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + \frac{1}{4}\partial^2\varphi\varphi^\dagger. \tag{150}$$

We can put this into a nice form if we make some integration by parts and drop total divergences:

$$\Phi^\dagger\Phi\Big|_D = -\partial_\mu\varphi^\dagger\partial^\mu\varphi + i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^\dagger F. \tag{151}$$

What we see is that the D term of $\Phi^\dagger\Phi$ produced us the canonical kinetic terms of a complex scalar field φ and a left-handed Weyl-Field ψ. The complex scalar F has no kinetic terms. We call such fields auxiliary fields. Auxiliary fields can be integrated out from the theory easily since they do not have any kinetic terms. In terms of the path integral, integrating over the field F just produces a number, and we can feed that into the normalization of the path integral.

Let us now consider a set of left-handed chiral superfields Φ_i. The following action is then supersymmetry invariant

$$S = \int d^4x\left(\Phi_i^\dagger\Phi_i\Big|_D + \left(W(\Phi)\Big|_F + \text{h.c.}\right)\right), \tag{152}$$

with

$$W(\Phi)\Big|_F = \frac{\partial W(\varphi)}{\partial\varphi_i}F_i - \frac{1}{2}\frac{\partial^2 W(\varphi)}{\partial\varphi_i\partial\varphi_j}\psi_i\psi_j. \tag{153}$$

In this action there are no derivative terms of F, we can integrate it out. In the path integral, integrating out an auxiliary field corresponds to solving the classical equations of motion of F and inserting that into the action. In the present case,

$$0 = \frac{\delta S}{\delta F_i} = F_i^\dagger + \frac{\partial W(\varphi)}{\partial \varphi_i}, \tag{154}$$

$$F_i^\dagger = -\frac{\partial W(\varphi)}{\partial \varphi_i} \quad ; \quad F_i = -\frac{\partial W^\dagger(\varphi^\dagger)}{\partial \varphi_i^\dagger}. \tag{155}$$

Putting this back in the action

$$\begin{aligned} S = \int d^4x \Bigg(& - \partial_\mu \varphi_i \partial^\mu \varphi_i^\dagger + i\psi_i^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i + \left| \frac{\partial W(\varphi)}{\partial \varphi_i} \right|^2 \\ & + \left[- \left| \frac{\partial W(\varphi)}{\partial \varphi_i} \right|^2 - \frac{1}{2} \frac{\partial^2 W(\varphi)}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \text{h.c.} \right] \Bigg) \\ = \int d^4x \Bigg(& - \partial_\mu \varphi_i \partial^\mu \varphi_i^\dagger + i\psi_i^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i - \left| \frac{\partial W(\varphi)}{\partial \varphi_i} \right|^2 \\ & + \left[- \frac{1}{2} \frac{\partial^2 W(\varphi)}{\partial \varphi_i \partial \varphi_j} \psi_i \psi_j + \text{h.c.} \right] \Bigg). \end{aligned} \tag{156}$$

Note that in the potential there are two $\left| \frac{\partial W(\varphi)}{\partial \varphi_i} \right|^2$ terms both with minus since the hermitian conjugate of that term is equal to itself, and there is one in the kinetic term so overall we have -1 that term.

We can arrive at the same results by considering the Grassmanian variables and their integration. Since they are anticommuting, the integration is a bit tricky. The properties we will require are: 1)linearity, and 2)invariance of the integral under a constant shift. Consistent with these conditions, the only non-trivial definition of a Grassmanian integration is

$$\begin{aligned} \int d\theta 1 &= 0, \\ \int d\theta \theta &= 1. \end{aligned} \tag{157}$$

Where θ is a Grassmanian number. If we have more than 1 Grassmanian number, then

$$\int d\theta^i \theta^j = \delta^{ij}. \tag{158}$$

Suppose that we have n independent Grassmanians. Then,

$$\int d\theta^n \cdots d\theta^1 \theta^1 \cdots \theta^n = n. \tag{159}$$

Note the reverse ordering in the measure. We treat the Grassmaniann differentials as anti-commuting $\{d\theta_i, d\theta_j\} = 0 = \{d\theta_i, \theta_j\}$. We can also consider complex Grassmanians. Suppose that n above is an even number, $n = 2d$. Then, define

$$\begin{aligned}\theta_a &= \theta_a + i\theta_{d+a},\\ \overline{\theta}_a &= \theta_a - i\theta_{d+a},\end{aligned} \tag{160}$$

where $a = 1, \cdots, d$. Thus, we have

$$\int d^d\theta d^d\overline{\theta}\theta^d\overline{\theta}^d = 2d. \tag{161}$$

In the superspace, we have 2 complex Grassmaniann s. In order to have an action that is supersymmetry invariant, we need to integrate over all superspace, which requires us to integrate over θ and $\overline{\theta}$s. Then, a supersymmetry invariant action would be

$$S = \int d^4xd^2\theta d^2\overline{\theta}\Phi^\dagger\Phi. \tag{162}$$

That is why in the above we concentrated on the D term, because only that term survives the $d^2\theta d^2\overline{\theta}$ integral. For supersymmetry invariant interactions, we consider

$$\int d^4x\left(\int d^2\theta W(\Phi) + \int d^2\overline{\theta W}(\Phi^\dagger)\right). \tag{163}$$

That is why we concentrated on the F term of the superpotentials constructed only out of left-handed chiral superfields.

3.2 The Vector Multiplet

We want to generate a non-abelian gauge theory from a supersymmetric action. To do so we first consider the vector multiplet, which contains a massless vector field A_μ, and its superpartner λ_α (we will use α, β for spinor indices now that we also have gauge indices which are denoted a, b, c). With an auxiliary D field, they are combined together into a superfield V via

$$V = -\theta\sigma^\mu\theta^* A_\mu + i\theta^2(\overline{\theta\lambda}) - i\overline{\theta}^2(\theta\lambda) + \frac{1}{2}\theta^2\overline{\theta}^2 D. \tag{164}$$

We will take A_μ to be a Lie algebra-valued field and $\lambda, \overline{\lambda}, D$ to be in the adjoint representation of the gauge group G. So, we can write

$$\begin{aligned}A_\mu &= A^a_\mu t^a,\\ \lambda &= \lambda^a t^a,\\ \overline{\lambda} &= \overline{\lambda}^a t^a,\\ D &= D^a t^a.\end{aligned} \tag{165}$$

Where t^a are the generators of G, normalized as $\mathrm{Tr} t^a t^b = -\frac{1}{2}\delta^{ab}$ and have the algebra

$$[t^a, t^b] = f^{abc}t^c. \tag{166}$$

From the superfield V, we define a spinorial superfield W_α as

$$W = (-i\lambda + \theta D - i\sigma^{\mu\nu}\theta F_{\mu\nu} + \theta^2\sigma^\mu D_\mu\lambda)(y), \tag{167}$$

with y the superspace coordinate, $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$, and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \\ D_\mu\lambda &= \partial_\mu\lambda - ig[A_\mu, \lambda]. \end{aligned} \tag{168}$$

Observe that we now rescaled the gauge potential so that the adjoint covariant derivative contains $-ig$. One can compactly write this as

$$W_\alpha = \frac{1}{8g^2}\overline{\mathcal{D}}^2\left(e^{2gV}\mathcal{D}_\alpha e^{-2gV}\right). \tag{169}$$

The supersymmetric Yang-Mills action then reads

$$\begin{aligned} S &= -\frac{1}{4}\int d^4x d^2\theta \ \mathrm{Tr}(W^\alpha W_\alpha) \\ &= \int d^4x \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F_{\mu\nu} * F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} + \frac{1}{2}D^2\right). \end{aligned} \tag{170}$$

In addition to the Yang-Mills term, we also got the instanton density. Ordinarily, it appears with a θ parameter. To get the θ parameter in front of the instanton term, we define

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \tag{171}$$

and observe that

$$\begin{aligned} S &= \frac{1}{16\pi}\mathrm{Im}\left(\tau\int d^4x d^2\theta \ \mathrm{Tr}(W^\alpha W_\alpha)\right) \\ &= \frac{1}{g^2}\int d^4x \mathrm{Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu D_\mu\overline{\lambda} + \frac{1}{2}D^2\right) \\ &\qquad + \frac{\theta}{32\pi^2}\int d^4x \mathrm{Tr} F_{\mu\nu} * F^{\mu\nu}. \end{aligned} \tag{172}$$

So we have produced the gauge sector with all the canonical factors.

One can also add adjoint scalars by considering

$$\begin{aligned} S &= \frac{1}{4}\int d^4x d^2\theta d^2\overline{\theta} \ \mathrm{Tr}(\Phi^\dagger e^{-2gV}\Phi) \\ &= \int d^4x \ \mathrm{Tr}\Big(|D_\mu\varphi|^2 - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + F^\dagger F - g\varphi^\dagger[D, \varphi] \\ &\qquad - \sqrt{2}ig\varphi^\dagger\{\lambda, \psi\} + \sqrt{2}ig\overline{\psi}[\overline{\lambda}, \varphi]\Big). \end{aligned} \tag{173}$$

3.3 $N = 2$ Super Yang-Mills Theory

For $N = 2$, all the fields φ, ψ and A_μ, λ gets combined into a single multiplet. That in particular means all the fields must be in the same representation of the gauge group $SU(2)$. Since A_μ is a Lie-Algebra valued field, the fields are in the adjoint representation. With a bit of thought, one may see that the action for $N = 2$ Super Yang-Mills reads

$$\begin{aligned} S &= \int d^4x \left(\mathrm{Im}\left(\frac{\tau}{16\pi} d^2\theta W^\alpha W_\alpha \right) + \frac{1}{4g^2} \int d^2\theta d^2\overline{\theta} \mathrm{Tr} \Phi^\dagger e^{-2gV} \Phi \right) \\ &= \mathrm{Im} \int d^4x \frac{\tau}{16\pi} \mathrm{Tr} \left(\int d^2\theta W^\alpha W_\alpha + \int d^2\theta d^2\overline{\theta} \Phi^\dagger e^{-2gV} \Phi \right). \end{aligned} \tag{174}$$

The auxiliary fields enter this action as

$$S_{\mathrm{aux}} = \frac{1}{g^2} \int d^4x \mathrm{Tr} \left(\frac{1}{2} D^2 - g\varphi^\dagger [D, \varphi] + F^\dagger F \right). \tag{175}$$

F can just be ignored. Varying D, one gets

$$D^a = -g\varepsilon^{abc} \varphi^{\dagger b} \varphi^c, \tag{176}$$

inserting this back into the action, one gets

$$S_{\mathrm{aux}} = -\int d^4x \frac{1}{2} \mathrm{Tr}([\varphi^\dagger, \varphi])^2. \tag{177}$$

We see that the bosonic field has a positive definite potential $V(\varphi) \sim \mathrm{Tr}[\varphi^\dagger, \varphi]^2$. To preserve supersymmetry, one needs a vacuum configuration φ_0 such that $V(\varphi_0) = 0$. This is solved for fields φ_0 such that

$$[\varphi_0^\dagger, \varphi_0] = 0. \tag{178}$$

4 Low Energy Effective Action of the $N = 2$ Theory

Following Seiberg and Witten [SW94], we will determine the low-energy effective action of $N = 2$ Super Yang-Mills with gauge group $SU(2)$. The high-energy theory describes the microscopic scales and is well known to be asymptotically free. What about the low energies? We will now tackle this issue. In this section, in addition to the original paper, I will follow the discussions in [Bil97, Tek15] which are reviews of [SW94].

4.1 Effective Actions

To obtain an effective action, one may take the generating functional $\Gamma[\varphi]$ which is the Legendre transform of $W[\varphi]$ of connected diagrams, which is related to the path integral of the QFT via

$$Z = e^{iW}. \tag{179}$$

A different way to obtain effective actions is provided by Wilsonian effective actions $S_W[\mu, \varphi]$ with μ a an infra-red cutoff. For theories with only massive particles, Wilsonian effective action is not very different from $\Gamma[\varphi]$, but when there are massless fields, such is the case in gauge theories, there are some differences.

4.2 The Moduli Space of $SU(2)$

We will be interested in obtaining the Wilsonian effective action starting from $SU(2)$ microscopic theory. This theory has a scalar potential

$$V(\varphi) = \frac{1}{2}\mathrm{Tr}([\varphi^\dagger, \varphi])^2, \tag{180}$$

and to preserve supersymmetry we need ground states such that $V = 0$. There are non-trivial configurations for which $V = 0$. Since φ is an adjoint complex scalar field, it can be written as

$$\varphi = \frac{1}{2}[a_j(x) + ib_j(x)]\sigma_j. \tag{181}$$

By a gauge transformation, we can fix the real part of φ to be in the 3-rd direction in the internal space, that is, we can set $a_1 = 0 = a_2$. Since we want the commutator of φ with $\varphi^\dagger$ to be 0, this fixes $b_1 = 0 = b_2$. Hence

$$\varphi(x) = \frac{1}{2}a(x)\sigma_3. \tag{182}$$

Where we defined a complex parameter $a = a_1 + ib_1$. Now, for the ground state, a must be a constant so that the kinetic term is 0. Moreover, rotations in the 1- or 2- directions by π can change a to $-a$, so the two are gauge equivalent. Therefore, the gauge-invariant quantity that parametrizes the inequivalent vacua can be written

$$\frac{1}{2}a^2, \tag{183}$$

or, equivalently

$$\mathrm{Tr}\varphi^2. \tag{184}$$

This equivalence holds semiclassically. When quantum effects are considered, this is no longer the case. We make the following definitions

$$u = \langle \mathrm{Tr}\varphi^2 \rangle \quad ; \quad \langle \varphi \rangle = \frac{1}{2}a\sigma_3. \tag{185}$$

The complex parameter u labels distinct vacua configurations. The gauge inequivalent vacua configurations constitute the moduli space $\mathcal{M}$ of the theory. Therefore, u is a coordinate of $\mathcal{M}$, and that makes $\mathcal{M}$ a complex plane. We will see that there are some singularities in $\mathcal{M}$, and the behavior of the theory near these singularities gives the low energy effective action.

Now, note that if $\langle\varphi\rangle \neq 0$, then the $SU(2)$ symmetry will be spontaneously broken down to $U(1)$ through the Higgs mechanism. The gauge fields A_μ^b with $b = 1, 2$ acquire mass given by $m = \sqrt{2}a$ where a is the expectation value of the Higgs field in the vacuum. Since the fields φ, λ, ψ live in the same multiplet as A, their components along the $1, 2$ directions in the internal space will also acquire the same mass as A. The fields $A_\mu^3, \lambda^3, \psi^3$ and the fluctuations of φ along the third direction remain massless. These massless fields are described by a Wilsonian low-energy effective action with $N = 2$ supersymmetry but with the gauge group $U(1)$. For a $U(1)$ gauge theory, one has

$$S = \frac{1}{16\pi}\mathrm{Im}\int d^4x \left[\int d^2\theta \mathcal{F}''(\Phi)W^\alpha W_\alpha + \int d^2\theta d^2\overline{\theta}\Phi^\dagger \mathcal{F}'(\Phi)\right]. \tag{186}$$

Where $\mathcal{F}(\Phi)$ is a holomorphic prepotential that depends only on Φ and not $\Phi^\dagger$.

4.3 Determination of the Moduli Space Metric

Let us consider the first and the second terms in the low-energy effective action and write it in terms of the component fields. Respectively, they read

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x\mathcal{F}''(\varphi)\left[-\frac{1}{4}F_{\mu\nu}(F^{\mu\nu} - i * F^{\mu\nu}) - i\lambda\sigma^\mu\partial_\mu\overline{\lambda}\right] + \cdots, \tag{187}$$

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x\mathcal{F}''(\varphi)\Big[|\partial_\mu\varphi|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi}\Big] + \cdots. \tag{188}$$

With $\cdots$ standing for non-derivative terms. If we imagine these terms as the terms of a 4-dimensional sigma model, then $\mathrm{Im}\mathcal{F}''(\varphi)$ plays the role of the metric in the field space, because it multiplies the kinetic terms. Through the same reasoning, $\mathrm{Im}\mathcal{F}''(\varphi)$ defines the metric in the space of vacuum configurations modulus gauge transformations, which is to say the metric of the moduli space.

So, we find the metric on the moduli space to be

$$ds^2 = \mathrm{Im}\tau(a)da d\overline{a}. \tag{189}$$

Where $\tau(a) = \mathcal{F}''(a)$ is the complexified coupling constant. What we did was to replace the sigma-model metric $\mathcal{F}''(\varphi)$ by its vacuum expectation value $\mathcal{F}''(a)$ corresponding to the given point on the $u-$plane or the moduli space $\mathcal{M}$.

It turns out that this effective description is not well-defined for all vacua, or for all values of u on the complex half-plane. We would like the metric on the moduli space to be positive definite, which implies $\mathrm{Im}\tau(a) > 0$. But, since $\mathcal{F}(a)$ is a holomorphic function of a (this is a requirement coming from supersymmetry [SW94]), and since $\mathrm{Im}\tau(a) = \mathrm{Im}\frac{\partial^2\mathcal{F}''(a)}{\partial a^2}$, the condition $\mathrm{Im}\tau(a) > 0$ cannot be satisfied in the plane unless it is a constant, which is the case classically but in the quantum theory this is not so. To remedy this, we need to allow for different local descriptions, that is, the coordinates a and $\mathcal{F}(a)$ are well-defined only in locally in $\mathcal{M}$. When one approaches a singularity with $\text{Im}\tau(a) = 0$, we need new coordinates in which that point is non-singular and $\text{Im}\tau(a) \neq 0$. We can do such a thing as long as the singularities in the plane are only coordinate singularities, which can be removed through coordinate choices. Apparently, that is the case (we will not show this), so this description is okay.

5 Duality

We will now discuss the version of electric-magnetic duality that appears in the effective action. For more detailed discussions, we refer to [AGH97, Bil97, Har96, SW94, Tac13, Tek15].

Let us define

$$a_D = \frac{\partial \mathcal{F}}{\partial a}, \tag{190}$$

and write the moduli space metric as

$$ds^2 = \text{Im} da_D d\overline{a} = -\frac{i}{2}(da_D d\overline{a} - da d\overline{a}_D). \tag{191}$$

This has a symmetry between a and a_D, so one can use a_D to parameterize the moduli space with a different harmonic function replacing $\text{Im}\tau$. It turns out, we have the following relation between $\tau(a)$ and $\tau_D(a_D)$:

$$-\frac{1}{\tau(a)} = \tau_D(a_D). \tag{192}$$

Then, the effective action in terms of the dual parameters reads

$$\frac{1}{16\pi}\text{Im}\int d^4x \left[\int d^2\theta \mathcal{F}_D''(\Phi_D) W_D^\alpha W_{D\alpha} + \int d^2\theta d^2\overline{\theta} \Phi_D^\dagger \mathcal{F}_D'(\Phi_D)\right]. \tag{193}$$

To see the full duality group of this action, we write it as

$$\frac{1}{16\pi}\text{Im}\int d^4x \left[\int d^2\theta \frac{d\Phi_D}{d\Phi} W^\alpha W_\alpha + \frac{1}{2i}\int d^2\theta d^2\overline{\theta}(\Phi^\dagger \Phi_D - \Phi_D^\dagger \Phi)\right], \tag{194}$$

where

$$\frac{d\Phi_D}{d\Phi} = -\frac{1}{\mathcal{F}_D''(\Phi_D)} = \mathcal{F}''(\Phi). \tag{195}$$

Then, the duality transformation above is written as

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix}. \tag{196}$$

However, this is not the only symmetry. The following transformation also leaves the action invariant

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix}. \tag{197}$$

These transformations generate the $SL(2,\mathbb{Z})$ group transformations, which is the duality group of the effective action of $N=2$ super-Yang-Mills. It is also easy to see that the metric of the moduli space is invariant under the duality transformations of $SL(2,\mathbb{Z})$.

6 Monopole Condensation and Confinement

We now discuss the confinement of electric charge in the effective action obtained in 4. We will follow [SW94, Tek15].

Confinement of electric charge is a phenomenon that needs an analytic explanation in QCD. There, we have dynamical quarks and gluon fields but experiments suggest that there are no free quarks or gluons, so they must be confined. From the calculations of lattice gauge theory, we see this to be the case, as the Wilson loops have an area law decay which means that the potential grows linearly and hence the charges are confined. Despite this, we do not have a satisfactory explanation of confinement from the dynamics of QCD. What makes the Seiberg-Witten theory phenomenologically interesting, among many other reasons to be interested in the theory, is the ability to see quark confinement in the low energy description. One downside of this is it has supersymmetry, which seems to be absent in the standard model at low energies, there is still a hope that the ideas in Seiberg-Witten theory can shed some light on the understanding of confinement in pure QCD.

An idea, put forward in the 70s by people like 't Hooft, Mandelstam, Polyakov, and others was to consider a duality to explain the confinement of electric charge. In a superconductor, it is known that magnetic charges will be confined through the Meissner effect. The idea is then to consider a dual Meissner effect, where one has a confined electric charge instead of a magnetic one. Let us review the standard Meissner effect.

6.1 Meissner Effect

We consider a $U(1)$ gauge theory of a complex scalar field with the Higgs potential:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\varphi D^\mu\varphi^\dagger - \mu^2\varphi^\dagger\varphi + \frac{\lambda}{4}(\varphi^\dagger\varphi)^2, \tag{198}$$

and the covariant derivative of φ is defined as

$$D_\mu\varphi = \partial_\mu\varphi - 2ieA_\mu\varphi. \tag{199}$$

The factor of two is put in to capture the Cooper pairs. If $\mu^2 > 0$, then through Higgs mechanism A_μ becomes massive. In the gauge $A_0 = 0$, one has for the equations of motion for B:

$$\nabla^2\mathbf{B} - m^2\mathbf{B} = 0 \quad ; \quad \nabla\times\mathbf{B} + m^2\mathbf{A} = 0. \tag{200}$$

Here $m = \sqrt{2}\mu$ is the mass of the Higgs field. The first equation tells us that the magnetic field is expelled from the superconductor except for some small region inside. This is the Meissner effect.

The vacuum of this theory consists of electrically charged pairs (condensation of electric charge) and this expels magnetic fields. If we put two magnetic charges, they will have to form a flux tube between them to conserve the flux, and this flux tube leads to a linear potential between the two magnetic charges. The two magnetic charges cannot be separated by a finite energy so they are confined.

Similarly, one can understand the confinement of electric charges. When magnetic charges condensate, two probe electric charges in this background will form a flux tube, and the potential between them will grow linearly. Hence, through the dual Meissner effect, electric charges will be confined.

In the $N = 2$ theory, one has an Abelian vector multiplet semi-classically. Break $N = 2$ to $N = 1$ by adding a mass $m\mathrm{Tr}\Phi^2$ for the scalar multiplet. Now, the near the points monopoles are massless (strong coupling, long-distance regions), $N = 1$ superpotential reads

$$W = \sqrt{2}\varphi_d M\tilde{M} + mU(\varphi_D), \tag{201}$$

with $M, \tilde{M}$ describing the monopole hyper-multiplet. When we have a nonzero mass on the scalar $m \neq 0$, the vacua corresponds to

$$M = \left(\frac{-mU'(0)}{\sqrt{2}}\right)^{1/2} = \tilde{M} \tag{202}$$

Because M is non-vanishing, monopoles condense which leads to the confinement of electric charge.

7 Some Recent Works Concerning Seiberg-Witten Theory

In this section, I will briefly comment on two recent papers concerning Seiberg-Witten theory. In the first paper, [DDN22], Seiberg-Witten theory with gauge group $SU(N)$ is analyzed. For this case, the moduli space becomes multi-dimensional, but the strong coupling regime can be investigated through some methods used for $SU(2)$.

In the second paper [GMZ24], a dictionary between the Seiberg-Witten theory and the black hole superradiance is constructed. This dictionary allows for some computations related to superradiance using methods in Seiberg-Witten theory.

8 Discussion and Conclusion

In this report, we reviewed the breakthrough paper [SW94]. This was a huge development at the time of publication, as it achieved many exact results in a regime where it is hard to answer any question. Yang-Mills gauge theories are known to be asymptotically free, which means that the coupling parameter becomes weaker at shorter distances. This means that at long distances or low energies, the theory becomes strongly coupled. This poses many problems for Quantum Field Theorists as the working tools we have for weak coupling stop working for strongly coupled systems. This brings in the challenge of describing gauge theories over long distances, which is crucially important for phenomenology. The standard model is known to be a gauge theory since the big discoveries of Gross, Wilczek, and independently Politzer in 1973. Despite the many successes of the standard model in conforming with experiments, there are still big problems. Understanding quark confinement in QCD is one of the most outstanding among these problems. The confinement has been verified in lattice gauge theory simulations, yet an analytic understanding is not present. This is why Seiberg-Witten theory was very appealing because it had an analytic description of confinement. Although this is achieved in a supersymmetric theory, this is still an important step, because we have an example to which we can compare future results.

Seiberg-Witten theory's charm comes from the exact effective action obtained, and the non-perturbative results obtained that we haven't discussed here. In addition to confinement and electric-magnetic duality, the low energy action exhibits phenomena such as chiral symmetry breaking and generating a mass scale from strong coupling. These are also important physical phenomena that we have not discussed which are present in the Seiberg-Witten theory.

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