Bootstrapping Non-Invertible Symmetries

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Abstract

In this project, we study non-invertible (categorical) symmetries from a modular bootstrap approach. Non-invertible symmetries in Quantum Field Theory (QFT) gained much interest in recent years following their development within the framework of generalized global symmetries. So far, they have been best understood in two-dimensional Conformal Field Theories (CFT) because the old literature on Rational CFT (RCFT) provides natural examples of such categorical symmetries. We focus mostly on the 2d critical Ising model on the continuum limit, which from a modern perspective is a CFT admitting non-invertible symmetry governed by a particular fusion category called the Ising. Using the Symmetry Topological Field Theory (SymTFT) construction, we study the categorical symmetries of certain 2d theories T governed by a generic fusion category $\mathcal{C}$. The SymTFT of such a theory T is known as a Turaev-Viro Topological QFT $\mathsf{TV}_{\mathcal{C}}$ in 3d. Leveraging the modular properties of $\mathsf{TV}_{\mathcal{C}}$, we obtain modular bootstrap equations for the spectrum of T. We finally discuss recently obtained upper bounds on the spectrum of Ising CFT.

Contents

1 Introduction

Quantum Field Theory (QFT) is one of the central frameworks physicists use to describe phenomena across microscopic scales (Standard Model of particle physics, nuclear physics, ...) to the macroscopic world (cosmology, condensed-matter theory, statistical systems, ...). Despite its power to provide answers to questions we ask across these length scales, a rigorous definition of QFT has not yet been achieved. The quest to understand this language drives much progress in both physics [C+22, PR22] and pure mathematics [BFM+22]. For example, the outstanding problems of quark confinement and its dual Yang-Mills gap

are unresolved due to our inability to grasp QFTs completely. When a QFT is weakly coupled, it is possible to build a perturbation series to get some approximate answers, but not all QFTs are weakly coupled. In fact, most of the interesting phenomena, such as the confining phases of gauge theories and the superconductivity are consequences of strong coupling. Therefore, an understanding of QFT that does not rely on the couplings to be weak is needed.

There are two up-and-coming research fields with strong ideas on how to understand QFT formally: i) generalized global symmetries [GKSW15], and ii) the conformal bootstrap program [SD17, PRV19]. Our approach in this project is to investigate the intersection of these two research fields. Since both fields are very recent, their intersection has not been studied in depth. In [LS23], the bootstrap of non-invertible symmetries in the context of 2d CFTs was worked out using the Symmetry Topological Field Theory (SymTFT) construction which uses coupling a QFT to a Topological QFT (TQFT) [ABGE$^+$23, KS14, FT14, Fre14, FMT22, GK21, BS24, BSN23]. We will follow [LS23] to explore aspects of non-invertible symmetries, which are so far best understood in 2d [BT18] and general d is under development [BSN23, SN24, Sha23], from a bootstrap point of view. We will discuss how the bootstrap principle can be implemented to constrain the spectrum of the 2d CFTs admitting non-invertible symmetries, and at the same time are amenable to bootstrap tools due to their modular properties, called the modular bootstrap program. Thus, we will be concerned with 2d CFTs, 3d TQFTs, and their interesting relations, which are well understood following the seminal works in the mid-80s [BPZ84, Ver88, MS89a, MS89d, Car86, Wit89, DW90, EMSS89].

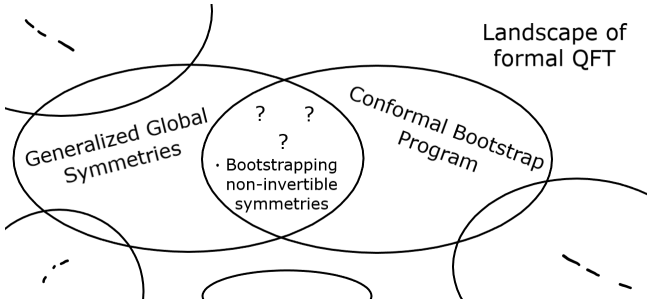


Figure 1: We aim to explore the intersections of generalized global symmetries and the bootstrap program, following [LS23] in which some bounds on the spectrum of CFTs admitting non-invertible symmetries have been found.

Let us briefly introduce the two research fields. The generalized global symmetries, initiated by the seminal paper [GKSW15], caught a lot of attention due to its richness and applications in the phases of gauge theories, 't Hooft anomalies, renormalization group (RG) flows, and the Ginzburg-Landau paradigm of phase transitions. Symmetries in QFT organize the spectrum, provide selection

rules, and overall constrain the theory significantly. [GKSW15] takes the viewpoint that a global symmetry G in QFT is associated with a topological operator, puts this simple fact at the center of the things, and provides generalizations of ordinary symmetries that leave the Lagrangian (if there is one) invariant. This generalization includes $p-$form symmetries where ordinary symmetries acting on local operators are called $0-$form. These higher-form symmetries do not act on local objects appearing in the Lagrangian but on extended operators in the spectrum via topological linking, such as Wilson loop operators in any gauge theory we know of. Following these developments, it was also realized that there are symmetry structures in this language that do not have a group-like fusion rule, dubbed the non-invertible or categorical symmetries.

The second research field, the conformal bootstrap program, gained a strong interest in this century due to the developments of numerical techniques for Conformal Field Theories (CFT) [SD17]. The bootstrap principle dates back to the 60s [Che62] where the idea of focusing only on S-matrix and the properties it should satisfy instead of the "unobservable" quarks in nuclear physics. It was clear in the 70s that the field theory approach was much more powerful, so this idea was dormant for some time. In the early 2000s, following developments in CFTs with the bootstrap principle and numerical techniques [DO01, RRTV08, SD17, PRV19], the research field quickly gained attention and took a mainstream place in formal QFT research. The reason the conformal bootstrap is very strong is that the symmetry allows one to organize the spectrum in terms of the eigenstates of the scale transformation generator, generating the evolution of Hilbert spaces in the radial quantization picture, called the primary operators. Knowing the quantum numbers of these primary operators (the scaling dimension Δ, and spins s) along with some dynamical information called the three-point function coefficients f_{ijk} turns out to capture the entire CFT! This is because in a given QFT, the physics is encoded in the $n-$point correlation functions, and using the Operator Product Expansion (OPE), one can reduce these correlators to sum over the CFT data (Δ_i, s_i, f_{ijk}). This set of data must satisfy consistency conditions coming from physical requirements such as unitarity and crossing symmetry. The **goal** of the bootstrap program is to implement these conditions using analytical and numerical tools and carve out the theory space non-perturbatively, as illustrated in figure 2. From the Wilsonian point of view, local QFTs can be obtained as relevant deformations of CFTs, so understanding all CFTs non-perturbatively is an extremely promising step toward understanding *all* QFTs non-perturbatively!

1.1 Organization of the Sections

To be somewhat self-contained, we introduce many of the ingredients from scratch, and at times while providing explicit computations. However, it is very hard to be self-contained in a subject containing two hot research topics in formal QFT, so we naturally have to rely on references. The organizations of the sections is as follows:

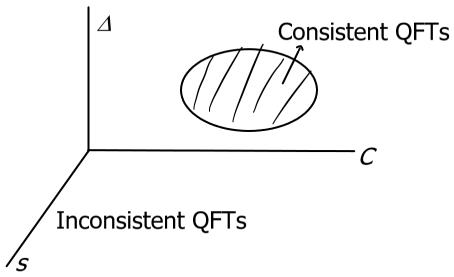


Figure 2: The theory space is the space of CFT data, in which each point corresponds to a CFT. The bootstrap program tries to find the shaded region, which contains all the consistent CFTs.

- Section 2: We give a detailed introduction to 2d CFT, following mainly the textbook [Ket95].
- Section 3: We review the seminal work of Verlinde [Ver88] on the fusion rules and modular properties of Rational CFTs (RCFT).
- Section 4: We review the 2d Ising model at the critical temperature, which is a 2d CFT with Majorana fermions. We discuss the spectrum and the fusion rules of the Ising CFT which will be connected to the non-invertible symmetries picture in 7.
- Section 5: We discuss the framework of generalized global symmetries in QFT, following [BBFT$^+$24]. We first approach ordinary symmetries as $0-$form symmetries, discuss an example of a 4d QFT with a $U(1)$ global symmetry which neatly summarizes the essential ideas, introduce $p-$form symmetries, explain how these symmetries act on charged objects, how $p-$form symmetry operators fuse together, and the notion of charges for $p-$form symmetries.
- Section 6: We discuss CFTs in dimensions higher than 2, representations of the conformal algebra, primary operators, the radial quantization picture, the Operator-State correspondence, Operator Product Expansion (OPE), and bootstrap conditions on the CFT data. Since these ideas are not directly related to the bootstrap of non-invertible symmetries which uses the modular bootstrap tools in 2d, we do not get into great detail and refer to [SD17].
- Section 7: We introduce non-invertible symmetries in 2d CFT, their actions on operators, how to obtain twisted Hilbert spaces, and discuss the explicit example of Ising CFT which has a non-invertible Kramers-Wannier-duality line. We explicitly work out the twisted Hilbert spaces of this example.

- Section 8: We briefly introduce the idea of Symmetry Topological Field Theory (SymTFT), a very recent construction that aims to obtain a broad perspective on global symmetries of QFT, be it topological (group-like) or categorical (non-invertible). We discuss the example of a $p-$form $\mathbb{Z}_N$ symmetry, which has a SymTFT with Dijkgraaf-Witten type action.

- Section 9: Using the tools built so far, we discuss the SymTFT of a 2d CFT with categorical symmetry $\mathcal{C}$, which is a Turaev-Viro TQFT $\mathsf{TV}_{\mathcal{C}}$, discuss its properties and how the Hilbert space of 2d CFT can be related with the Hilbert space of $\mathsf{TV}_{\mathcal{C}}$ on a spatial disk with certain line defect insertions. Using the modularity of the $\mathsf{TV}_{\mathcal{C}}$ partition function, we obtain the modular bootstrap equation and discuss some numerical results for the case of 2d Ising CFT.

Part I
Tools from the 20^{th} Century

2 An Introduction to 2d Conformal Field Theory

2.1 Ordinary Quantum Field Theory

A relativistic Quantum Field Theory (QFT) in $d+1$ Minkowski space-time is a theory with a set of local operators $\Phi(x)$ that transform under representations of the Poincaré group $ISO(d,1)$. This group is an extension of the group $SO(d,1)$ with d space-translations generated by the momenta P^i, 1 time translation generated by the Hamiltonian H, which combine into a $d+1$ vector $P^\mu = (H, P^i)$. Due to the locality, one can write the conserved charge, the $d+1$ momenta, as the integral of what's called the Energy-Momentum tensor $T^{\mu\nu}$, which is a symmetric tensor of rank 2. One has

$$P^\mu \equiv \int d^d x T^{0\mu}. \tag{2.1}$$

Given an action $S[\Phi]$ of the fields, one can obtain $T^{\mu\nu}$ in terms of the fields using Noether's theorem under a transformation $x^\mu \to x^\mu + \varepsilon^\mu$ with ε^μ a translation $d+1$ vector whose components are all assumed to be infinitesimal without loss of generality because one can build up arbitrary translations from the infinitesimal ones.

The group $SO(d,1)$ is the group of isometries of the Minkowski metric

$$ds^2 = -dt^2 + \sum_i^d (dx_i)^2. \tag{2.2}$$

There are $d(d-1)/2$ rotations and d Lorentz boosts that leave this metric invariant. The corresponding local currents can be written as $\mathcal{M}^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$, and whose corresponding charges that generate the isometry transformations is given by

$$M^{\mu\nu} \equiv \int d^d x \mathcal{M}^{0\mu\nu}, \tag{2.3}$$

where M^{ij} are related to the angular momentum of the fields, and the parts that contain the 0 index are related to Lorentz boosts. At the classical level, the inhomogeneous Lorentz transformations of the form $x \to x' \equiv \Lambda x + \varepsilon$, with $\Lambda \in SO(d,1)$ and $\varepsilon \in \mathbb{R}^d$, are realized on the fields by the charges $\{P^\mu, M^{\mu\nu}\}$, in the sense that the Poisson bracket of the charges with the fields will yield the transformed version of the field under the corresponding transformation. The change of the fields under these transformations is given by

$$\Phi(x) \mapsto \Phi'(x') \equiv \mathcal{D}(\Lambda) \cdot \Phi(x), \tag{2.4}$$

where $\mathcal{D}(\Lambda)$ is a representation of $SO(d,1)$ compatible with the spin of the field Φ, and $\cdot$ denotes the contraction of the spin indices that $\mathcal{D}$ and Φ carries. Note

that fields of all spins should transform the same way under translations, because spin is a characteristic property under rotations. Because of this fact, $\mathcal{D}$ does not depend on the translation parameter. One can invert the transformation $x = \Lambda^{-1} \cdot (x' - \varepsilon)$, and write the transformation law as

$$\Phi(x) \mapsto \mathcal{D}(\Lambda) \cdot \Phi(\Lambda^{-1} \cdot (x - \varepsilon)), \tag{2.5}$$

so we have knowledge about the symmetry content of the QFT based on the space-time symmetries. In certain theories, called gauge theories, there are internal symmetries in the theory that do not affect the space-time. In all honesty, these gauge symmetries are not symmetries but redundancies in the description, intentionally introduced to save the locality and Lorentz symmetry which is not present when we use the description without the redundancies. For example, in $d = 4$, a photon has 2 polarizations, hence there should be two fields associated. However, to write a Lorentz covariant Lagrangian density, one must use the redundant 1-form fields $A = A_\mu dx^\mu$, with the physical information about the electric & magnetic fields are encoded in $F = dA$, the field stress tensor. The gauge field A has 4 components, 2 more degrees of freedom than necessary. This redundancy allows a Lorentz invariant description, and there is a $U(1)$ gauge symmetry of the theory under which A changes as $A \to A + UdU^{-1}$ with U a phase $U \in U(1)$. Writing $U = e^{iq\alpha(x)}$, one has $\delta A = -iqd\alpha$ under the gauge transformation, where we introduced a coupling constant q. Since the change of A is a total derivative, and since $d^2 = 0$ identically, one has $\delta F = 0$ under gauge transformations, which is expected as it encodes physical information and physics is invariant under the redundancy transformations. There are more interesting and more involved cases where the gauge group is in general a non-abelian Lie group, for example $SU(n)$. These cases are known as Yang-Mills theories, and they are essential theories to understand many physical phenomena, and some deep mathematical fields such as the 4-manifold topology.

At the classical level, the dynamics of a field theory is governed by the field equations, obtained by imposing

$$\delta S[\Phi] = 0, \tag{2.6}$$

S being the classical action, and δS denotes the response of the action to arbitrary deformations of the fields $\delta\Phi$. This yields an Euler-Lagrange type equation that one may solve in some simple cases.

To obtain the quantum theory corresponding to the classical theory defined by the action $S[\Phi]$, one needs to construct the Hilbert space $\mathcal{H}$ of states, while promoting the fields of the theory to operators that obeys an algebra of observables. As the conserved charges $\{P^\mu, M^{\mu\nu}\}$ are constructed out of the local fields, they become operators as well, and in the Hilbert space the transformation of the quantum fields under the corresponding Lorentz transformations

reads

$$\begin{aligned}\Phi(x) &\xrightarrow{x\to x+\varepsilon} \Phi(x+\varepsilon) = e^{iP^\mu \varepsilon_\mu}\Phi(x)e^{-iP^\mu\varepsilon_\mu},\\ \Phi(x) &\xrightarrow{x\to \Lambda x} \mathcal{D}(\Lambda)\cdot\Phi(\Lambda^{-1}x) = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\Phi(x)e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}.\end{aligned} \tag{2.7}$$

Where we have written $\Lambda \sim 1+\omega$ with ω a rank-two anti-symmetric tensor. The operators $T(\varepsilon) = e^{-iP\cdot\varepsilon}$ and $U(\omega) = e^{\frac{i}{2}\omega\cdot M}$ then generate the $ISO(d,1)$ transformations in the Hilbert space. If we write these relations infinitesimally, using Baker-Campbell-Hausdorff formula, we get the following equal-time commutation relations

$$\begin{aligned}[P^\mu, \Phi(x)] &= i\partial^\mu\Phi(x),\\ [M^{\mu\nu}, \Phi(x)] &= -(\mathcal{L}^{\mu\nu}\cdot I + S^{\mu\nu})\Phi(x),\end{aligned} \tag{2.8}$$

where $\mathcal{L}^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ and we have defined $S^{\mu\nu}$ via $\mathcal{D}(\omega)\sim 1+\frac{i}{2}\omega\cdot S$, and I and S are matrices carrying the spin indices of the field Φ that we suppress. In general, constructing $\mathcal{H}$ is not an easy task. One can do it for free theories, but interacting theories proves sufficiently challenging that one has to resort to different techniques, such as building a perturbative expansion via Feynman diagrams.

There is another approach to quantization of a field theory, called the path-integral quantization. Given an action $S[\Phi]$ and some fields, one defines

$$Z \equiv \int \mathcal{D}\Phi e^{iS[\Phi]}, \tag{2.9}$$

where we assume that a measure $\mathcal{D}\Phi$ on the field space exists and is well defined. The quantum observables are defined by

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle \equiv \frac{1}{Z}\int \mathcal{D}\Phi e^{iS[\Phi]}\Phi(x_1)\cdots\Phi(x_n), \tag{2.10}$$

where we divide by Z to normalize, and in this correlation we automatically have the time-ordering of operators, that is, $x_1^0 > x_2^0 > \cdots x_n^0$. There are a lot of subtle points about the convergence of the correlation functions. It is common to analytically continue this integral by a Wick rotation $t\to -i\tau$ with τ the Wick rotated time or the Euclidean time, whose name is obvious if one considers the transformed metric $ds^2 = d\tau^2 + dx\cdot dx$ which is the metric of $\mathbb{R}^{d+1}$. In this procedure, the action changes to the Euclidean action S_E, and one defines the Euclidean path integral via

$$Z_E \equiv \int \mathcal{D}\Phi e^{-S_E[\phi]}, \tag{2.11}$$

and the quantum observables

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle_E \equiv \frac{1}{Z_E}\int \mathcal{D}\Phi e^{-S_E[\Phi]}\Phi(x_1)\cdots\Phi(x_n). \tag{2.12}$$

The physical content of a QFT is encoded in its correlation functions. Hence, solving a QFT amounts to a complete knowledge of all the possible correlation functions. The evaluation of the path integrals is a big challenge, and it becomes significantly harder when there is no small parameter in the theory to build up a perturbation series. Such is the case for Yang-Mills theories in the low-energy regimes (infrared), where the Yang-Mills coupling constant g_{YM}, the only free parameter in the theory, becoming larger and larger as the characteristic distances in the physical processes becomes larger (equivalently the energies become lower). Consequently, a complete analytical description of Yang-Mills theory or the Quantum Chromo-Dynamics (QCD) still remains out of reach, luckily numerical simulations from the lattice gauge theory ideas provides a huge help towards this end.

2.2 Conformal Field Theory

A CFT is a special type of QFT that has a larger symmetry group. Considering the metric as a dynamical variable which transforms as a rank-2 tensor, we consider a reparametrization-invariant field theory, which means that under

$$\begin{aligned} \delta x &= \varepsilon(x), \\ \delta\Phi &= L_\varepsilon \Phi, \\ \delta g_{\mu\nu}(x) &= \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \varepsilon^\rho \partial_\rho g_{\mu\nu}, \end{aligned} \tag{2.13}$$

the theory remains invariant, where L_ε is a Lie derivative appropriate to the tensorial nature of the fields Φ. Under a coordinate transformation $x \to x'$, the metric changes as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x), \tag{2.14}$$

and Lorentz invariance means $g'_{\mu\nu}(x') = g_{\mu\nu}(x)$. One can relax this condition to what is called a conformal transformation, for which

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \quad \Omega(x) = e^{\omega(x)}. \tag{2.15}$$

These transformations form a group, which is known as the conformal group. We will consider flat spaces so one can find a coordinate system such that $g_{\mu\nu} = \delta_{\mu\nu}$ globally. Moreover, the Lorentz group is a subgroup of the conformal group, with the condition $\Omega(x) = 1$. To determine all the conformal transformations, we consider the infinitesimal deformations $x \to x' = x - \varepsilon(x)$ where $g_{\mu\nu} = \delta_{\mu\nu}$. Observe that $\frac{\partial x'^\mu}{\partial x^\nu} = (\delta^\mu_\nu - \partial_\nu \varepsilon^\mu)$, the inverse of which is $\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)^{-1} = (\delta^\mu_\nu + \partial_\nu \varepsilon^\mu)$ up to first order in ε. Thus, we have

$$g'_{\mu\nu}(x') = (\delta^\rho_\mu - \partial_\mu \varepsilon^\rho)(\delta^\sigma_\nu - \partial_\nu \varepsilon^\sigma)\delta_{\rho\sigma} = \delta_{\mu\nu} + \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu. \tag{2.16}$$

For this transformation to be a conformal transformation, we must have

$$\delta_{\mu\nu} + \partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu = \Omega \delta_{\mu\nu}. \tag{2.17}$$

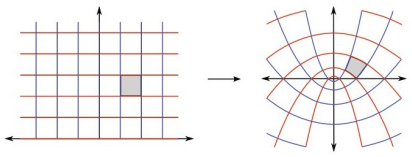


Figure 3: Demonstration of a conformal transformation. Figure taken from [BP09]

Trace both sides to get

$$\partial \cdot \varepsilon = (\Omega - 1)\frac{d}{2}, \tag{2.18}$$

putting this back above, one gets

$$\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu = \frac{2}{d}(\partial \cdot \varepsilon)\delta_{\mu\nu}. \tag{2.19}$$

One can go on to play with this equation to find constraints on ε, and finally conclude that the conformal group in d dimensional Euclidean space is the group $SO(d+1,1)$. In the Minkowski case $d = D+1$ with D the space dimensions, the conformal group reads $SO(D+1,2)$. The case of 2 dimensions (either in the Euclidean or the Minkowskian formalism) is especially interesting. Observe that the above equation on ε implies, when $d = 2$

$$\begin{aligned} &\mu = 1, \nu = 1: \quad 2\partial_1\varepsilon_1 = (\partial_0\varepsilon_0 + \partial_1\varepsilon_1) \Longrightarrow \partial_1\varepsilon_1 - \partial_0\varepsilon_0 = 0, \\ &\mu = 0, \nu = 1: \quad \partial_1\varepsilon_0 + \partial_0\varepsilon_1 = 0. \end{aligned} \tag{2.20}$$

These two equations are the Cauchy-Riemann conditions if we introduce the complex variables language

$$\begin{aligned} z = x_0 + ix_1 \quad ; \quad \varepsilon = \varepsilon_0 + i\varepsilon_1 \quad ; \quad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \\ \overline{z} = x_0 - ix_1 \quad ; \quad \overline{\varepsilon} = \varepsilon_0 - i\varepsilon_1 \quad ; \quad \overline{\partial}_{\overline{z}} = \frac{1}{2}(\partial_0 + i\partial_1), \end{aligned} \tag{2.21}$$

in terms of which the condition on ε to be a conformal transformation reads

$$\overline{\partial}\varepsilon = 0 = \partial\overline{\varepsilon}, \tag{2.22}$$

so that ε is a holomorphic function of z and $\overline{\varepsilon}$ is an anti-holomorphic function of $\overline{z}$. Hence, in the complex notation, a transformation of the form

$$z \mapsto z + \varepsilon(z) \quad ; \quad \overline{z} \mapsto \overline{z} + \overline{\varepsilon}(\overline{z}), \tag{2.23}$$

is a conformal transformation. One can expand the holomorphic ε in a Laurent series, which means it has infinitely many free parameters. This means, quite interestingly, that in 2d, there are infinitely many independent conformal transformations. This restricts the corresponding theory remarkably as we will

develop below.

Let us first write the metric in the $(z, \overline{z})$ coordinates. Since $x_0 = (z + \overline{z})/2$ and $x_1 = (z - \overline{z})/(2i)$, one finds the line element $ds^2 = dx_0^2 + dx_1^2$ as

$$ds^2 = dz d\overline{z}. \tag{2.24}$$

Then, under a holomorphic transformation $z \mapsto f(z)$ and its corresponding part in the anti-holomorphic sector $\overline{z} \mapsto \overline{f}(\overline{z})$, the change in the metric reads

$$dz d\overline{z} \to \left| \frac{df}{dz} \right|^2 dz d\overline{z}, \tag{2.25}$$

since the new metric is proportional to the old one, this can be viewed as a conformal transformation with $\Omega = \left| \frac{df}{dz} \right|^2$. In the case $f = z + \varepsilon$, we can expand ε in Laurent series

$$z \to z - \sum_{n \in \mathbb{Z}} a_n z^{n+1}, \tag{2.26}$$

and similarly for the anti-holomorphic sector. We would like to find the operators that generate the transformations corresponding to a_n. To do so, we consider the transformation of the field

$$\Phi'(z') = \Phi(z), \tag{2.27}$$

which is equivalent to

$$\Phi'(z) = \Phi(z - \varepsilon). \tag{2.28}$$

Now, we choose $\varepsilon = a_n z^{n+1}$ for any integer n, and infinitesimal a, and expand the Taylor series

$$\Phi'(z) = \Phi(z) - a_n z^{n+1} \frac{d}{dz} \Phi(z) + O(a^2). \tag{2.29}$$

Defining $\delta\Phi = \Phi'(x) - \Phi(x)$, we find that the operator that generates the conformal transformation corresponding to the parameter a_n on the fields Φ is then given by

$$l_n \equiv -z^{n+1} \partial_z \quad ; \quad \bar{l}_n \equiv -\overline{z}^{n+1} \partial_{\overline{z}}, \tag{2.30}$$

where we read off the anti-holomorphic sector easily. It is easy to find an algebra in the holomorphic sector

$$\begin{aligned} [l_n, l_m] &= z^{n+1} \partial (z^{m+1} \partial) - z^{m+1} \partial (z^{n+1} \partial) \\ &= -(n - m) z^{n+m+1} \partial \\ &= (n - m) l_{n+m}. \end{aligned} \tag{2.31}$$

It is also easy to see that l_n commutes with any $\bar{l}_m$, independent of the integers n, m. Thus, the complete algebra is

$$[l_n, l_m] = (n - m) l_{n+m} \quad ; \quad [l_n, \bar{l}_m] = 0 \quad ; \quad [\bar{l}_n, \bar{l}_m] = (n - m) \bar{l}_{n+m}. \tag{2.32}$$

This algebra is known as the 2d conformal algebra or the Witt algebra. Observe that as the operators l and $\bar{l}$ are independent, this justifies our usage of z and $\overline{z}$ as if they are independent coordinates. This corresponds to a complexification of the initial space $\mathbb{C} \to \mathbb{C}^2$, where $(x_0, x_1) \in \mathbb{C}$,$(z, \overline{z}) \in \mathbb{C}^2$. One recovers the Euclidean plane in this complexification via $\overline{z} = z^*$, where we identify the anti-holomorphic sector with the complex conjugation of the holomorphic sector.

One can also obtain a Minkwoskian signature metric via $z^* = -z$, which implies $(z, \overline{z}) = i(\tilde{\tau} + \tilde{\sigma}, \tilde{\tau} - \tilde{\sigma})$ with which the metric reads $ds^2 = -d\tilde{\tau}^2 + d\tilde{\sigma}^2$. However, we would like to associate another coordinate to the Minkowski space time. First, by a conformal transformation $z = e^{\zeta}$, $\overline{z} = e^{\overline{\zeta}}$, with $\zeta = \tau + i\sigma$, $\overline{\zeta} = \tau - i\sigma$ from the plane to a cylinder. The coordinate σ is compact because under $\sigma \to \sigma + 2\pi$, e^{ζ} remains the same, and $\tau \in (-\infty, \infty)$. Now we Wick rotate $\tau \to i\tau$ and we get $\zeta \to \zeta^+ = i(\tau + \sigma)$ and $\overline{\zeta} \to \zeta^- = i(\tau - \sigma)$, where $\zeta^{\pm}$ are the light-cone coordinates in the 2d Minkowski space $ds^2 = d\zeta^+ d\zeta^-$. Moreover, in this case, conformal transformations are reparametrizations of the light-cone coordinates: $\zeta^{\pm} \to f^{\pm}(\zeta^{\pm})$ that leave the light-cone invariant. Clearly, the Minkowski line element stays the same under these transformations up to $\Omega = f^+(\zeta^+) f^-(\zeta^-)$.

Let us get back to the Witt algebra. There are infinitely many independent generators, hence there are infinitely many symmetries in 2d CFT. Such a symmetry obviously puts severe restrictions on the dynamics of the CFT, which is the main reason 2d CFT is an integrable model. Note that although the local symmetry transformations are infinite, only a few of them are globally defined. To see this, consider the vector fields on the Riemannian sphere $S^2 = \mathbb{C} \cup \{\infty\}$ defined as

$$V(z) = -\sum_n a_n l_n = \sum_n a_n z^{n+1} \frac{d}{dz}. \tag{2.33}$$

Such a vector field will have singularities at $z = 0$ for $n < -1$, so for it to be globally defined we need to consider $n \geq -1$. To investigate the behavior around infinity, we make the conformal transformation $z \to w = -\frac{1}{z}$ so $z \to \infty$ corresponds to $w \to 0$. Using $\frac{d}{dz} = \frac{dw}{dz}\frac{d}{dw} = w^2 \frac{d}{dw}$, we write V in terms of w

$$V(w) = \sum_n a_n (-1)^{n+1} \frac{1}{w^{n+1}} w^2 \frac{d}{dw} = \sum_n a_n (-1)^{n+1} w^{-n+1} \frac{d}{dw}. \tag{2.34}$$

It is easy to see that, around $w = 0$, V will have singularities for $n > 1$. Hence, we have nailed down the globally defined transformations to those generated by $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. Let us try to understand what these generators correspond to as transformations. We start with $n = -1$

$$l_{-1} = -\frac{d}{dz} \quad ; \quad \bar{l}_{-1} = -\frac{d}{d\overline{z}}. \tag{2.35}$$

Acting with l_{-1} on Φ with an infinitesimal parameter gives $\delta_{l_{-1}}\Phi = -a_{-1}\frac{d}{dz}\Phi$. This is the transformation under a translation: $\Phi(z - a_{-1}) - \Phi(z) = -a_{-1}\frac{d}{dz}\Phi$,

so l_{-1} generates translations.

On the other hand, $l_0 + \bar{l}_0$ generates $\delta\Phi = -\lambda\left(z\frac{d}{dz} + \overline{z}\frac{d}{d\overline{z}}\right)\Phi$, which corresponds to $z \to (1-\lambda)z$ and $\overline{z} \to (1-\lambda)\overline{z}$. If we write $z = re^{i\theta}$, we see that this transformation acts as $r \to (1-\lambda)r$, which is called dilatation. The other combination, $i(l_0 - \bar{l}_0)$, generates $\delta\Phi = -i\alpha\left(z\frac{d}{dz} - \overline{z}\frac{d}{d\overline{z}}\right)\Phi$. To understand what kind of transformation this corresponds to, we change variables again to $z = re^{i\theta}$ and write $\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial\theta}{\partial z}\partial_\theta = e^{-i\theta}\partial_r + (ire^{i\theta})^{-1}\partial_\theta$, and thus $-z\partial_z = -r\partial_r + i\partial_\theta$. Similarly we find $-\overline{z}\partial_{\overline{z}} = -r\partial_r - i\partial_\theta$, so the combination $i(l_0 - \bar{l}_0) = -\partial_\theta$ generates rotations in the plane. This can also be seen from the fact that $z\partial - \overline{z}\overline{\partial}$ is the component of angular momentum $L = -i\mathbf{r} \times \nabla$ that points out of the plane, when we pass to the Euclidean sheet via $\overline{z} = z^*$.

The l_1 operators generate the special conformal transformations. All these globally defined conformal transformations can be written as the transformations of the Möbius group:

$$z \to z' = \frac{az+b}{cz+d} \quad ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\mathbb{Z}_2. \tag{2.36}$$

That is to say, a, b, c, d are complex parameters that satisfy $ad - bc = 1$. This is precisely the group $SL(2,\mathbb{C})$. We quotient out $\mathbb{Z}_2$ because if we reflect all the parameters $(a, b, c, d) \to (-a, -b, -c, -d)$, then z' will remain the same, so there is a twofold degeneracy. We note that $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$, hence $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(3,1)$, which is the conformal group of 2-dimensional space (in the Euclidean signature).

What we learned is that the local conformal algebra cannot be integrated into a globally defined group, but due to the locality in field theories, the local algebra still has consequences, which we will study. After we quantize a CFT, the physical states in the Hilbert space $\mathcal{H}$ will carry quantum numbers given by the representations of the *global* conformal algebra. Naturally, we expect a vacuum state $|0\rangle$, which has vanishing quantum numbers and transforms trivially under global transformations. The eigenstates of the operators l_0 and $\bar{l}_0$ will be labeled h and $\overline{h}$, and we call them the conformal weights of a state. Suppose we know the weights of a state. In that case, we can find their scaling dimension Δ and spin s as $\Delta = h + \overline{h}$, $s = h - \overline{h}$, because the operators that generate the corresponding transformations are given respectively by the sum and the difference of the $l_0, \bar{l}_0$ operators.

2.3 Symmetries

At the classical level, we know that, via Noether's theorem, there is an on-shell conserved local current and an associated charge for any continuous symmetry transformation that leave the action invariant, up to a boundary term. What is the corresponding symmetry statement for the quantum theory? In quantum

theory, one has off-shell field configurations so the charges that are conserved on shell need not be conserved at the quantum level. However, there are Ward identities at the correlation function levels that are the corresponding statements for the classical symmetry-conservation laws. Let us first remember Noether's theorem and the Ward identities, then we will use it in the context of 2d CFTs.

2.3.1 Noether's Theorem

Consider a field theory with a set of fields Φ_a and an action

$$S = \int d^d x \mathcal{L}(\Phi_a, \partial\Phi_a).$$

When we deform the fields

$$\Phi_a(x) \to \Phi'_a(x) = \Phi_a(x) + \varepsilon_a \delta\Phi_a(x),$$

it is called a symmetry transformation provided the action does not change under this transformation, $\delta S = 0$. In above, ε_a are constants, hence this is a global symmetry. To see that the existence of this symmetry leads to a conservation law, we localize the parameters $\varepsilon_a \to \varepsilon_a(x)$. This is no longer a symmetry transformation, hence δS is not zero. However, when we demote ε_a to global parameters, the variation of the action should be 0, so the variation should depend on the derivatives of ε_a. Then, we can write

$$\delta S = \int d^d x j^\mu_a(x) \partial_\mu \varepsilon_a(x), \tag{2.37}$$

for some coefficient functions j^μ_a. Observe that since there is a derivative, j must carry a vector index. We can integrate by parts and throw the boundary term to get

$$\delta S = -\int d^d x (\partial_\mu j^\mu_a) \varepsilon_a(x).$$

Now, recall that the equations of motion of the theory with the given action S is found by varying the fields arbitrarily, and looking at the configurations for which the variation of S is zero. The variation above with local parameters $\varepsilon_a(x)$ is such a variation, and hence for fields that satisfy the equations of motion we should have $\delta S = 0$. It follows that

$$\partial_\mu j^\mu_a = 0 \quad \text{for on shell fields.}$$

With this argument, we established the existence of conserved currents, but what about their explicit form? Can we express j^μ_a in terms of the fields in the action? We answer that question below.

We first express the variation of the action under the deformation of the fields as

$$\delta S = \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \Phi_a} \delta\Phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \partial_\mu \delta\Phi_a \right),$$

thus

$$\frac{\delta S}{\delta \Phi_a} = \frac{\partial \mathcal{L}}{\partial \Phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)}.$$

We can also write δS as

$$\begin{aligned}\delta S &= \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \Phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \right) \delta \Phi_a + \int d^d x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \right) \\ &\equiv \int d^d x \delta \mathcal{L},\end{aligned}$$

where we defined

$$\begin{aligned}\delta \mathcal{L} &= \left(\frac{\partial \mathcal{L}}{\partial \Phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \right) \delta \Phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \right) \\ &= \frac{\delta S}{\delta \Phi_a} \delta \Phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \right).\end{aligned} \tag{2.38}$$

It should be noted that $\frac{\delta S}{\delta \Phi_a}$ is an honest functional derivative since S is a functional, but $\delta \mathcal{L}$ denotes the variation of the Lagrangian under the symmetry transformation and is better not be thought of as a functional derivative since it is not a functional.

Now, a symmetry transformation is defined as a transformation such that $\delta S = 0$. This is satisfied in general if $\delta \mathcal{L} = \partial_\mu K^\mu$ so that

$$\delta S = \int_{\mathcal{M}} d^d x \delta \mathcal{L} = \int_{\partial \mathcal{M}} dS^\mu K_\mu \to 0,$$

where we've used the Stokes' theorem, and the fields are decaying fast enough that at infinity the surface integral gives 0. Now, from (2.38), we get

$$\frac{\delta S}{\delta \Phi_a} \delta \Phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a \right) = \partial_\mu K^\mu \Longrightarrow \partial_\mu j^\mu = -\frac{\delta S}{\delta \Phi_a} \delta \Phi_a,$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_a)} \delta \Phi_a - K^\mu.$$

What we see is that for field configurations that satisfy the equations of motion, which are called on-shell fields, we have a local conservation law:

$$\partial_\mu j^\mu = 0,$$

because the equations of motion following from the action principle are $\frac{\delta S}{\delta \Phi_a} = 0$. Note that we haven't specified the fields Φ_a. The index a could stand for anything: a Lorentz index, a spinor index, a gauge index, and any index that we come across in field theory. As long as under $\delta \Phi_a$ Lagrangian changes up to a boundary term, there will be a conservation law. From this conservation law, we can define a conserved charge

$$Q = \int_{X_{d-1}} d^{d-1} x j^0,$$

where X_{d-1} is a co-dimension 1 spatial manifold. If we choose $\mathcal{M} = \mathbb{R}^{1,d-1}$, then we will have $X_{d-1} = \mathbb{R}^{d-1}$. To show that this is a time-independent quantity, observe that

$$\frac{dQ}{dt} = \int_{X_{d-1}} d^{d-1}x \frac{\partial j^0}{\partial t} = -\int_{X_{d-1}} d^{d-1}x \partial_i j^i = -\int_{\partial X_{d-1}} d\Sigma_i j^i \to 0,$$

with $d\Sigma_i$ an oriented surface element on ∂X_{d-1}.

Let us study this in light of the Poisson bracket formalism. In the 1-dimensional setup, the classical observables are

$$\begin{aligned} F \; : \; \Gamma \; &\to \; \mathbb{R} \\ (q,p) &\mapsto F(q,p) \in \mathbb{R}. \end{aligned} \tag{2.39}$$

On the phase space, there is an algebraic structure, a Lie algebra, with the product defined as the Poisson bracket:

$$\{F(q,p), G(q,p)\} = \sum_i^{\dim \Gamma} \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \right).$$

There are several properties of this bracket:

- Anti-symmetry: $\{F,G\} = -\{G,F\}$,
- Bilinearity: $\{aF_1 + bF_2, G\} = a\{F_1,G\} + b\{F_2,G\}$,
- Leibniz rule: $\{FG,H\} = \{F,H\}G + F\{G,H\}$,
- Jacobi identity: $\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0$.

The first three are easy to see from the definition of the bracket, however, the Jacobi identity is not so easy to show.

The formulation of classical mechanics with this bracket is well known and has close relations to symplectic geometry. In particular, the symplectic structure on the phase space follows from the existence of a 2-form that is non-degenerate and antisymmetric: $\Omega(\cdot,\cdot)$. We have a $2n$ dimensional Phase space, where q_i and p_{i+n} $(i = (1,\cdots,n))$ are the coordinates, and one has $\{F,G\} = \Omega^{ij}\partial_i F \partial_j G$ with $\Omega(q^i,p^j) = \Omega^{ij}$ We won't deal with that here but let's make some important observations about the Poisson brackets, in particular for Noether's theorem. First we note that

$$\{q^i, p_j\} = \delta^i{}_j.$$

This relation here is very important for the quantum theory, where we impose an equation like this as a commutator equation on the canonical pairs, which are operators, now acting on the Hilbert space of the quantum theory. Recalling Hamilton's equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad ; \quad -\dot{p}_i = \frac{\partial H}{\partial q^i}.$$

If we look at the Poisson bracket of any function with the Hamiltonian,

$$\begin{aligned}\{F,H\} &= \sum_i^{\dim\Gamma}\left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}\right)\\ &= \sum_i^{\dim\Gamma}\left(\frac{\partial F}{\partial q^i}\frac{\partial q^i}{\partial t}-\frac{\partial F}{\partial p_i}(-)\frac{\partial p_i}{\partial t}\right)\\ &= \sum_i^{\dim\Gamma}\left(\frac{\partial F}{\partial q^i}\frac{\partial q^i}{\partial t}+\frac{\partial F}{\partial p_i}\frac{\partial p_i}{\partial t}\right).\end{aligned}$$

Recalling the chain rule of calculus, and taking F to be a function of the phase space and time explicitly, for the sake of generality, we observe

$$\begin{aligned}\frac{dF(q,p,t)}{dt} &= \frac{\partial F}{\partial q^i}\frac{\partial q^i}{\partial t}+\frac{\partial F}{\partial p_i}\frac{\partial p_i}{\partial t}+\frac{\partial F}{\partial t}\\ &= \{F,H\}+\frac{\partial F}{\partial t},\end{aligned}$$

so for a function F that does not explicitly depend on time but only on the phase space coordinates, the full-time derivative of the function is given by the Poisson bracket of that function with the Hamiltonian. It is in this sense Hamiltonian generates time evolution in classical mechanics.

Let us discuss Noether's theorem with this bracket. For an infinitesimal transformation

$$q^i \to q^i + \delta q^i,$$

that leaves the action invariant:

$$0 = \delta S = \int dt\left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i\right),$$

the following object is conserved

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i.$$

We can also write this as

$$Q = p_i\delta q^i,$$

with p_i the canonical momentum of Hamilton's formalism. Now, observe a very beautiful thing (we take δq^i independent of p)

$$\{Q,q^i\} = \{p_j,q^i\}\delta q^j = -\delta^i_j\delta q^j = -\delta q^i.$$

So, the Poisson bracket of the conserved charge with q^i gives back the transformation! What about the following Poisson bracket

$$\{Q,F\} = \{p_i,F\}\delta q^i = \sum_j^{\dim\Gamma}\left(\frac{\partial p_i}{\partial q^j}\frac{\partial F}{\partial p_j}-\frac{\partial p_i}{\partial p_j}\frac{\partial F}{\partial q^j}\right)\delta q^i = -\frac{\partial F}{\partial q^i}\delta q^i.$$

This is remarkable. To see what is remarkable about this, let us look at how $F(q)$ changes under the infinitesimal symmetry transformation $q \to q + \delta q$ (for the moment I neglect p, but I could just as well deform that too):

$$F(q) \to F(q + \delta q) = F(q) + \frac{\partial F}{\partial q^i} \delta q^i,$$

$$\implies \delta F = \frac{\partial F}{\partial q^i} \delta q^i.$$

Hence, the Poisson bracket of the conserved charge Q of the symmetry transformation $q \to q + \delta q$, with any general function of q gives the transformation of that function under the symmetry transformation up to a sign. For a more general deformation involving $p \to p + \delta p$, we can expect the Poisson bracket of the conserved charge Q with a phase space function $G(p, q)$ to give the variation of G under the deformation in the phase space:

$$\{Q, G(q, p)\} = -\delta_Q G,$$

where by δ_Q we mean the transformation of G under the symmetry transformation whose conserved charge is Q. This relation will be significant when we want to understand how symmetry is incorporated into the quantum mechanics in the operator formalism. The answer is quite easy: Just use the fundamental quantization condition, which is replacing observables in the phase space with hermitian operators acting on the Hilbert space $\mathcal{H}$, and replace the Poisson brackets with commutators with an addition of i so that the commutator of observables gives you an observable again. This way, we will arrive

$$[\hat{Q}, \hat{A}] = -i\delta_Q \hat{A},$$

so the conserved charge of a classical symmetry on the phase space will correspond to an operator on the Hilbert space that when commutes with observables transforms the observables as given above. One might ask whether this holds for a finite transformation. The answer is yes: In the Hilbert space, one can write the finite version of this as

$$A^Q = e^{i\alpha Q} A e^{-i\alpha Q}.$$

This result is fundamental in quantum theory, including quantum field theory with the extra complication coming from the fact that in field theory, one has ∞ many degrees of freedom and to get a charge one has to integrate a current density over all space. But the key idea is all the same: Whatever results you have in the classical theory involving the dynamical variables and the Poisson brackets, just replace the dynamical variables with hermitian operators and brackets with commutators and the result is automatically transferred to quantum theory.

We now would like to extend this formalism to field theory. There, we have infinitely many degrees of freedom, and the observables are now *functionals* of the fields. That is to say:

$$F : \Gamma \mapsto \mathbb{R}$$
$$[\varphi^i(x), \pi_i(x)] \mapsto \int d^{d-1}x \mathcal{F}[\varphi^i(x), \pi_i(x)](x).$$

Hence the natural generalization of a Poisson bracket should involve an integral and a functional derivative. Then we have

$$\{F, G\} = \int d^{d-1}x \sum_{i=1}^{N} \left(\frac{\delta F}{\delta \varphi^i(x)} \frac{\delta G}{\delta \pi_i(x)} - \frac{\delta F}{\delta \pi_i(x)} \frac{\delta G}{\delta \varphi^i(x)} \right),$$

which we can write this in terms of $\mathcal{F}$ and $\mathcal{G}$

$$\begin{aligned}
\{F, G\} &= \int d^{d-1}x d^{d-1}y d^{d-1}z \sum_{i}^{N} \left(\frac{\partial \mathcal{F}(y)}{\partial \varphi^i(x)} \delta^{d-1}(y-x) \frac{\partial \mathcal{G}(z)}{\partial \pi_i(x)} \delta^{d-1}(z-x) \right. \\
&\qquad \left. - \frac{\partial \mathcal{F}(z)}{\partial \pi_i(x)} \delta^{d-1}(z-x) \frac{\partial \mathcal{G}(y)}{\partial \varphi^i(x)} \delta^{d-1}(y-x) \right) \\
&= \int d^{d-1}y d^{d-1}z \sum_{i}^{N} \left(\frac{\partial \mathcal{F}(y)}{\partial \varphi^i(y)} \frac{\partial \mathcal{G}(z)}{\partial \pi_i(y)} \delta^{d-1}(z-y) - \frac{\partial \mathcal{F}(z)}{\partial \pi_i(y)} \delta^{d-1}(z-y) \frac{\partial \mathcal{G}(y)}{\partial \varphi^i(y)} \right) \\
&= \int d^{d-1}y \sum_{i}^{N} \left(\frac{\partial \mathcal{F}(y)}{\partial \varphi^i(y)} \frac{\partial \mathcal{G}(y)}{\partial \pi_i(y)} - \frac{\partial \mathcal{F}(y)}{\partial \pi_i(y)} \frac{\partial \mathcal{G}(y)}{\partial \varphi^i(y)} \right).
\end{aligned}$$

Observe that

$$\begin{aligned}
\{\varphi^i(x), \pi_j(y)\} &= \int d^{d-1}x' \sum_{k=1}^{N} \left(\frac{\delta \varphi^i(x)}{\delta \varphi^k(x')} \frac{\delta \pi_j(y)}{\delta \pi_k(x')} - \frac{\delta \varphi^i(x)}{\delta \pi_k(x')} \frac{\delta \pi_j(y)}{\delta \varphi^k(x')} \right) \\
&= \int d^{d-1}x' \sum_{k=1}^{N} \delta^i_k \delta^{d-1}(x-x') \delta^k_j \delta^{d-1}(y-x') \\
&= \delta^i_j \delta^{d-1}(x-y),
\end{aligned}$$

where we take $x^0 = y^0$. We recall the Noether charge

$$Q = \int d^{d-1}x \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^i)} \delta\varphi^i = \int d^{d-1}\pi_i(x) \delta\varphi^i(x),$$

which generates transformations on the phase space functionals, namely the observables

$$\{F, Q\} = \frac{\delta F}{\delta \varphi^i} \delta\varphi^i = \frac{\partial \mathcal{F}}{\partial \varphi^i} \delta\varphi^i.$$

Analogously to what happened in classical mechanics, the Poisson bracket of the Noether charge Q with a phase space observable gives the variation of that observable under the symmetry transformation whose conserved charge is Q.

2.3.2 Takahashi-Ward Identity

In a reparametrization invariant theory, which are the ones that are physically plausible, one can derive an important identity for the correlators. We first observe that under a reparametrization $x \to x' = x + \varepsilon(x)$, we have

$$\begin{aligned} \delta x &= \varepsilon(x), \\ \delta\Phi(x) &= L_\varepsilon \Phi(x), \\ \delta g_{\mu\nu}(x) &= \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu - \varepsilon^\lambda \partial_\lambda g_{\mu\nu}, \end{aligned} \tag{2.40}$$

where L_ε is the Lie derivative along the vector ε, compatible with the tensorial character of the field Φ, determined by its spin. Since $g_{\mu\nu}$ is a rank-2 tensor, we have written the explicit form of its change under the reparameterization. We observe that

$$\begin{aligned} \sum_{j=1}^{m} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_m \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \sum_{j=1}^{m} \Phi_1 \cdots \delta\Phi_j \cdots \Phi_m e^{-S[\Phi]} \\ &= \frac{1}{Z} \int \mathcal{D}\Phi \; \delta\Big(\Phi_1 \cdots \Phi_n\Big) e^{-S[\Phi]}, \end{aligned} \tag{2.41}$$

where $\Phi_j \equiv \Phi(x_j)$, and we use the Leibniz-like behaviour of δ: $\delta(AB) = (\delta A)B + A(\delta B)$. Using this again by taking e^{-S} inside the δ, we get

$$\delta\langle \Phi_1 \cdots \Phi_n \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \bigg(\delta\Big(\Phi_1 \cdots \Phi_n e^{-S[\Phi]}\Big) - \Big(\Phi_1 \cdots \Phi_n\Big) \delta e^{-S[\Phi]} \bigg). \tag{2.42}$$

The first term is a total variation, hence it is irrelevant after we integrate with $\mathcal{D}\Phi$. In the second term, we need to evaluate δe^{-S}, to do so we will use the chain rule that δ satisfies: $\delta e^{-S} = -(\delta S) e^{-S}$. We now define the energy-momentum tensor as

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.43}$$

Therefore, we have

$$\delta\langle \Phi_1 \cdots \Phi_n \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \; \Phi_1 \cdots \Phi_n \int d^2x \frac{\sqrt{g}}{2} T^{\mu\nu} \delta g_{\mu\nu}. \tag{2.44}$$

This is for a general space-time with metric g. If we work in the flat space, $g_{\mu\nu} = \delta_{\mu\nu}$, this formula reads

$$\sum_{j=1}^{n} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_n \rangle = \int d^2x \partial_\mu \varepsilon_\nu(x) \langle T^{\mu\nu}(x) \Phi_1 \cdots \Phi_n \rangle, \tag{2.45}$$

this is the consequence of the parametrization invariance at the quantum level. In particular, this identity means that in any reparameterization invariant field theory, the energy momentum tensor is the object that generates the general coordinate transformations. In flat space, $T^{\mu\nu}$ itself generates translations, and

$x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$ generates the Euclidean rotations. Both of them conserved due to $\partial_\mu T^{\mu\nu} = 0$. To see this at the quantum level, integrate by parts to put the derivative ∂_μ on $T^{\mu\nu}$ above, and using the arbitrariness of ε, we get the conservation law for points x that do not coincide with x_j $(j = 1, \cdots n)$. When the coordinate of T and one of x_j coincide, the Ward identity is singular. This tells us that, apart from the points on which an operator is inserted, the classical conservation laws hold inside a correlation function.

The Ward identities hold generically in parametrization invariant field theories. Conformal field theories have a larger class of symmetry, which contains, in addition to the translations and Lorentz rotations (metric isometries), scale transformations which is part of conformal transformations. Under a transformation $x \mapsto \lambda x$, the associated current generating this transformation is given by $x^\mu T_{\mu\nu}$. Conservation of this current means that $0 = \partial_\nu(x^\mu T_{\mu\nu}) = T^\mu_\mu$ so that the energy-momentum tensor must be traceless in a scale-invariant theory. In fact, any current of the form $f^\mu T_{\mu\nu}$ will be conserved if the symmetric part of $\partial^\nu f^\mu$ is proportional to the metric so that the divergence of the current is proportional to the trace of $T_{\mu\nu}$, which is zero in a scale-invariant theory. That is, if for any vector field f^μ satisfying $\partial^\mu f^\nu + \partial^\nu f^\mu = f(x)\eta^{\mu\nu}$ the object $f^\mu T_{\mu\nu}$ will give a conserved current.

On an Euclidean theory parametrized by the coordinates $z, \overline{z}$ with metric $ds^2 = dzd\overline{z}$, the local conservation law $\partial^\mu T_{\mu\nu}$ reads

$$\partial_{\overline{z}} T_{zz} + \partial_z T_{\overline{z}z} = 0 \quad , \quad \partial_z T_{\overline{z}\overline{z}} + \partial_{\overline{z}} T_{z\overline{z}} = 0, \tag{2.46}$$

and the tracelessness condition gives

$$T_{z\overline{z}} = 0 = T_{\overline{z}z}. \tag{2.47}$$

These two equations imply that in 2d CFT, the energy-momentum tensor can be decomposed into a holomorphic part and an anti-holomorphic part:

$$T_{zz} \equiv T(z) \quad ; \quad T_{\overline{z}\overline{z}} \equiv \overline{T}(\overline{z}), \tag{2.48}$$

and the two are related by complex conjugation in the Euclidean sheet. On the Minkowski case, they correspond to the left-moving and right-moving modes, which are independent. With the complex notation, we write the Ward identity as

$$\sum_{j=1}^{n} \langle \Phi_1 \cdots \delta\Phi_j \cdots \Phi_n \rangle = \int d^2 z \partial_{\overline{z}} \varepsilon^z(z) \Big\langle T(z)\Phi_1(z_1, \overline{z}_1) \cdots \Phi_n(z_n, \overline{z}_n) \Big\rangle, \tag{2.49}$$

where we consider only holomorphic variations, the anti-holomorphic sector follows similarly.

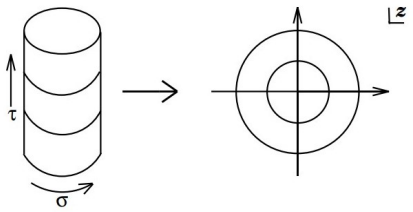


Figure 4: Conformally mapping the cylinder to the complex plane. Figure taken from [Ket95].

2.4 Quantizing CFT on the Complex Plane

We can define the generators L_n in terms of $T(z)$, since $T(z)$ is the object that generates the transformations on the fields. One has

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z), \tag{2.50}$$

and the integral is over a contour that contains the points of the fields acted on by L. We can invert this relation to

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{2.51}$$

using Cauchy's theorem. Now we would like to quantize the CFT. We parametrize the space-time $\mathbb{C}^2$ as a cylinder using the coordinates

$$z = e^{i\zeta} \quad ; \quad \zeta = \tau + i\sigma, \tag{2.52}$$

with τ and σ the coordinates on the Euclidean plane, and we have the same for the barred sector. Because $e^{2\pi i} = 1$, σ is compactified $\sigma \sim \sigma + 2\pi$. The map from ζ to z represents the mapping from the cylinder to the complex plane as in figure 4, which are conformally equivalent in the sense that the metrics of the two are proportional to each other. The infinite past and the future in the cylinder, $\tau = \mp\infty$, correspond on the plane to the origin and the point at infinity, respectively. Hence, equal radius curves on the plane correspond to equal time slices on the cylinder. Time translations $\tau \to \tau + \lambda$ corresponds to dilatations $z \to e^{\lambda} z$ on $\mathbb{C} \cup \{\infty\}$ and space translations $\sigma \to \sigma + \theta$ are rotations $z \to e^{i\theta} z$. Hence, the Hamiltonian is identified with the generator of dilatation generator, while the space of states is built out of curves of constant radius, and the energy-momentum tensors T and $\overline{T}$ become the generator of conformal transformations on the $z-$plane. To quantize this theory, we will use the radial quantization scheme, in which equal time slices are constant radii curves surrounding the origin on the plane, hence the name. Consider an operator Φ in the quantized theory. According to Heisenberg's equation, the

evolution of the operator is given by

$$\frac{d\Phi}{d\tau} = [H, \Phi], \tag{2.53}$$

where H is the Hamiltonian, and the commutator is taken at equal time. On the plane, one writes

$$\delta_\lambda \Phi = [\lambda H, \Phi], \tag{2.54}$$

and the Hamiltonian is related to the energy-momentum tensor via

$$H \equiv \oint \frac{dz}{2\pi i} z T(z) = L_0, \tag{2.55}$$

since it is the generator of dilatations. To see this, recall that for dilatations one has $z \to z + \varepsilon(z)$ with $\varepsilon(z) = \lambda z$, and moreover, for an arbitrary ε, the generator of the conformal transformation is defined as

$$T_\varepsilon \equiv \oint \frac{dz}{2\pi i} \varepsilon(z) T(z), \tag{2.56}$$

which acts on the fields as $\delta_\varepsilon \Phi = [T_\varepsilon, \Phi]$. Setting $\varepsilon = \lambda z$ we get exactly the same as what we obtained above. This is still only formal and we need to specify the fields inside the integration contour. Including transformations on the barred sector, the general version reads

$$\delta_{\varepsilon,\overline{\varepsilon}} \Phi(w, \overline{w}) = \oint \frac{dz}{2\pi i} \varepsilon(z) [T(z), \Phi(w, \overline{w})] + \text{anti-holomorphic}. \tag{2.57}$$

Due to the operator ordering issues, this equation still has problems. We will discuss how to cure these ambiguities using similar procedures used in ordinary QFT.

Before that, it is a good idea to define the notion of primary fields. We define a form

$$\Phi \equiv \Phi_{h,\overline{h}}(z, \overline{z}) dz^h d\overline{z}^{\overline{h}}, \tag{2.58}$$

and if it is a conformal invariant, then the component field $\Phi_{h,\overline{h}}$ is called a primary field of conformal weight $(h, \overline{h})$ (h and $\overline{h}$ are independent, just as z and $\overline{z}$ are viewed as independent). Since we know how dz and $d\overline{z}$ transform under conformal transformations $z \to z'$, we can infer the transformation property of the primary field as

$$\Phi'_{h,\overline{h}}(z', \overline{z}') = \left(\frac{\partial z}{\partial z'}\right)^h \left(\frac{\partial \overline{z}}{\partial \overline{z}'}\right)^{\overline{h}} \Phi_{h,\overline{h}}(z, \overline{z}). \tag{2.59}$$

The rest of the fields in CFT are called secondary. Writing $z' = z + \varepsilon(z)$, this reads at the infinitesimal level as

$$\delta_{\varepsilon,\overline{\varepsilon}} \Phi(z, \overline{z}) = \Big((h \partial \varepsilon + \varepsilon \partial) + (\overline{h} \overline{\partial} \overline{\varepsilon} + \overline{\varepsilon} \overline{\partial}) \Big) \Phi(z, \overline{z}). \tag{2.60}$$

This will be useful later on. Using the finite transformation property, we can relate the modes of Φ on the cylinder to those on the plane. We take a holomorphic field with weights $(h, 0)$ and expand it on the cylinder

$$\Phi_h(\zeta) = \sum_{n\in\mathbb{Z}} \phi_n e^{-\zeta n}, \quad \zeta = \tau + i\sigma. \tag{2.61}$$

Using the map $z = e^{\zeta}$, we map these modes to the plane. In doing so, recall that $\frac{\partial z}{\partial \zeta} = z$ so that under the map $\zeta \to z$, we have

$$\Phi_h(z) = \left(\frac{\partial z}{\partial \zeta}\right)^{-h} \Phi_h(\zeta) = z^{-h}\Phi_h(\zeta). \tag{2.62}$$

Hence, the modes on the plane read

$$\Phi_h(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h}. \tag{2.63}$$

From (2.51), we see that $T(z)$ is a primary field of conformal dimension $h = 2$ on the plane.

Now, let us consider the correlation functions and the constraints imposed by conformal symmetry. The two-point function

$$G^{(2)}(z_i, \overline{z}_i) \equiv \langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\rangle \tag{2.64}$$

is the simplest case. Under a global conformal transformation, G should be invariant, and using the transformation of the fields infinitesimally, we get the differential equation

$$\Big(\varepsilon(z_1)\partial_1 + h_1\partial_1\varepsilon(z_1) + \varepsilon(z_2)\partial_2 + h_2\partial_2\varepsilon(z_2) + \text{anti-holomorphic}\Big)G^{(2)} = 0. \tag{2.65}$$

Using the translational invariance $\varepsilon = 1 = \overline{\varepsilon}$, scale invariance $\varepsilon = z$ $\overline{\varepsilon} = \overline{z}$, and rotational plus special conformal invariance $\varepsilon = z^2$ $\overline{\varepsilon} = \overline{z}^2$ on this differential equation, we fix the two-point function up to a constant

$$G^{(2)}(z_i, \overline{z}_i) = \frac{C_{12}}{(z_1 - z_2)^{2h}(\overline{z}_1 - \overline{z}_2)^{2\overline{h}}}, \tag{2.66}$$

where $h_1 = h = h_2$, and unless $h_1 = h_2$ the correlator vanishes, by conformal invariance. Using similar arguments, the three-point function $G^{(3)} = \langle \Phi_1\Phi_2\Phi_3\rangle$ is found to be

$$G^{(3)}(z_i, \overline{z}_i) = C_{123}\frac{1}{(z_1 - z_2)^{h_1+h_2-h_3}(z_1 - z_3)^{h_1+h_3-h_2}(z_2 - z_3)^{h_2+h_3-h_1}} \times \text{anti-holomorphic}. \tag{2.67}$$

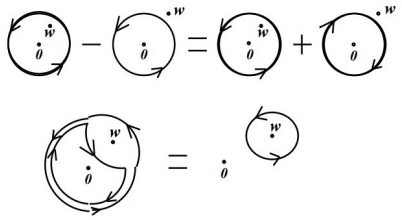


Figure 5: Deforming the contours to evaluate the commutator. Figure taken from [Ket95].

Thus, conformal invariance alone fixes the 2- and 3-point functions, without referring to a specific theory. The theory will fix the constants C. To have a geometrical understanding of why any conformal invariant theory in 2d has the same 2- and 3-point functions, is because any three points on the plane can be mapped to $\infty, 1, 0$ by a complex Möbius transformation, where

$$\lim_{z_1 \to \infty} z_1^{2h_1} \overline{z}_1^{2\overline{h}_1} G^{(3)} = C_{123}. \tag{2.68}$$

The coefficients C_{123} carry dynamical information of the CFT, and a complete knowledge of all C_{ijk} corresponds to solving the CFT.

Now we get back to the issue of operator ordering. At the quantum level, the transformation of a primary field is given as in (2.57). To define an equal time commutator, we need a radial ordering, just as we need time ordering in ordinary QFT. To do so we define

$$\begin{aligned}[T_\varepsilon, \Phi(w,\overline{w})] \equiv \lim_{|z| \to |w|} \Bigg(& \oint_{|z|>|w|} \frac{dz}{2\pi i} \varepsilon(z) R\Big(T(z)\Phi(w,\overline{w})\Big) \\ & - \oint_{|w|>|z|} \frac{dz}{2\pi i} \varepsilon(z) R\Big(T(z)\Phi(w,\overline{w})\Big) \Bigg),\end{aligned} \tag{2.69}$$

where the radial ordering operator R is defined as

$$R\Big(A(z)B(w)\Big) = \theta(|z|-|w|)A(z)B(w) \pm \theta(|w|-|z|)B(w)A(z), \tag{2.70}$$

with $\theta(x)$ defined to be 1 when $x > 0$ and 0 when $x < 0$. The $\pm$ in between takes into account the statistics, and we have $-$ only when both A and B are fermionic operators.

With these definitions, we can evaluate the integral. First observe that by playing with the contours using Cauchy's theorem, we can write the commutator as

$$[T, \Phi(w,\overline{w})] = \lim_{|z|\to|w|} \oint_{C(w)} \frac{dz}{2\pi i} \varepsilon(z) T(z)\Phi(w,\overline{w}), \tag{2.71}$$

with $C(w)$ a contour surrounding the point w. For this commutator to be non-zero, there must be a singularity in the operator product

$$\lim_{z\to w} T(z)\Phi(w,\overline{w}), \tag{2.72}$$

so that its residue contributes to the contour integral.

In general, when two operators are multiplied and the limit at which they coincide is taken, we get singularities. To make sense of such products, we use Wilson's operator product expansion

$$A(z)B(w) \sim \sum_{\Delta} C_{\Delta}(z-w)\mathcal{O}_{\Delta}(w), \tag{2.73}$$

where $\{\mathcal{O}_{\Delta}(w)\}$ is a complete set of local operators, and $C_{\Delta}(z-w)$ are numerical coefficients which are singular at $z = w$. Such equations are always understood to be valid inside a correlation function, with operator insertions away from the points z, w.

For the contour integral in (2.71) to give the correct transformation property of a primary field, we introduce the OPE between T and Φ:

$$\begin{aligned} T(z)\Phi(w,\overline{w}) =& \frac{h}{(z-w)^2}\Phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\Phi(w,\overline{w}) \\ &+ \Phi^{(-2)}(w,\overline{w}) + (z-w)\Phi^{(-3)}(w,\overline{w}) + \cdots, \end{aligned} \tag{2.74}$$

where $\cdots$ represent infinitely many non-singular terms depending on what's called the descendants or secondary fields with respect to a primary field of dimension h. To find the explicit form of a descendant field of level n, $\Phi^{(-n)}$, we use the OPE above to extract the $n-th$ descendant by multiplying both sides with $(z-w)^{-n+1}$ and using reside theorem. Namely, we have

$$\Phi^{(-n)}(w,\overline{w}) = \hat{L}_{-n}(w)\Phi(w,\overline{w}) \equiv \oint_{C(w)} \frac{dz}{2\pi i}(z-w)^{-n+1}T(z)\Phi(w,\overline{w}). \tag{2.75}$$

The operators $\hat{L}_{-n}(w)$ appear in the expansion of $T(z)$ around the point w:

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{\hat{L}_{-n}(w)}{(z-w)^{n+2}}. \tag{2.76}$$

Using the residue theorem, one obtains

$$\begin{gathered} \hat{L}_0\Phi(z,\overline{z}) = h\Phi(z,\overline{z}) \quad ; \quad \hat{L}_{-1}\Phi(z,\overline{z}) = \partial\Phi(z,\overline{z}) \\ \hat{L}_n\Phi(z,\overline{z}) = 0 \quad (n \geq 1). \end{gathered} \tag{2.77}$$

The last equality represents only the first generation of descendants, obtained from the product of Φ with a single T. We can insert more energy-momentum tensors $T\cdots T\Phi$ and consider their descendants. This way, one obtains the entire spectrum of the CFT. There exists an infinite family of descendant fields, obtained by acting $\hat{L}_{-n}$ operators on Φ several times:

$$\Phi_h^{(-\vec{k})}(z) \equiv \hat{L}_{-n_1}\cdots\hat{L}_{-n_k}\Phi_h(z) \quad ; \quad \vec{k} = (n_1, \cdots n_k). \tag{2.78}$$

These operators are classified by their eigenvalue under the $\hat{L}_0$ operator:

$$\hat{L}_0\Phi_h^{(-\vec{k})}(z) = (h + |\vec{k}|)\Phi_h^{(-k)} \quad ; \quad |\vec{k}| = \sum_{j=1}^{k} n_j. \tag{2.79}$$

Note that the descendants do not necessarily transform properly under conformal transformations, unlike primaries. This can be seen from the fact that $\hat{L}_n\Phi \neq 0$ $(n > 0)$ for a descendant field.

It is convenient to introduce a short-hand for the OPE notation, written as

$$T(z)\Phi(w) \sim \frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w), \tag{2.80}$$

and similarly for the barred sector. When considering the correlation function, only the terms that we wrote above give a contribution, the descendants giving 0 hence they are not written. Given a set of primary fields $\{\Phi_i\}$, one conveniently normalizes them so that their 2-point function reads

$$\langle\Phi_i(z,\overline{z})\Phi_j(w,\overline{w})\rangle = \delta_{ij}\frac{1}{(z-w)^{2h_i}} \times \text{anti-holomorphic}. \tag{2.81}$$

And the OPE between them has the following form

$$\Phi_i(z,\overline{z})\Phi_j(z,\overline{z}) \sim \sum_k C_{ij}{}^k(z-w)^{h_k-h_i-h_j}(\overline{z}-\overline{w})^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\Phi_k(w,\overline{w}), \tag{2.82}$$

which can be obtained from the coincidence limit of the 3-point function [DFMS97].

The OPEs between T and Φ and Φ between Φ contain all the information about a CFT, and determining them solves the theory. From the OPE $T\Phi$, we can find the commutation relation between the modes of T -which are the L_n

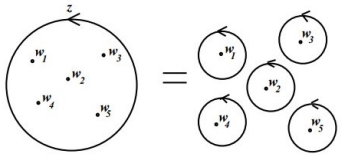


Figure 6: The deformation of the contour appearing in the conformal Ward identity. Figure taken from [Ket95]

operators, and those of Φ -which are denoted ϕ_n. We have

$$\begin{aligned}
[L_n, \phi_m] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+h-1} T(z)\Phi(w) \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{h}{(z-w)^2}\Phi + \frac{1}{z-w}\partial_w \Phi(w) \right) \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \left(h \frac{d}{dz}(z^{m+1})\big|_{z=w} \Phi(w) + w^{m+1}\partial_w \Phi(w) \right) \\
&= \oint \frac{dw}{2\pi i} \left(h(m+1) w^{m+n+h-1}\Phi(w) + w^{m+n+h}\partial_w \Phi(w) \right) \\
&= \oint \frac{dw}{2\pi i} \Big(h(m+1) w^{m+n+h-1} - (m+n+h) w^{m+n+h-1} \Big) \Phi(w) \\
&= h(m+1)\phi_{m+n} - (m+n+h)\phi_{m+n} \\
&= (hm - m - n)\phi_{m+n},
\end{aligned} \tag{2.83}$$

where we used the contour integral formula

$$\oint \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^{n+1}} = \frac{1}{n!} \left(\frac{d^n f(z)}{dz^n} \right) \Big|_{z=w}, \tag{2.84}$$

and the fact that $\oint dw w^{m+n+h-1} \sim \delta_{m+n+h,0}$ hence

$$\begin{aligned}
\oint \frac{dw}{2\pi i} w^{m+n+h-1}\Phi(w) &= \sum_{k\in\mathbb{Z}} \oint \frac{dw}{2\pi i} w^{m+n-k-1}\phi_k \\
&= \sum_{k\in\mathbb{Z}} \delta_{m+n-k,0}\phi_k \\
&= \phi_{n+m}.
\end{aligned} \tag{2.85}$$

One can also calculate the correlator of T with an arbitrary number of primaries

Φ. To do so we integrate $\varepsilon(z)T(z)$ on a contour as in figure 6, we get

$$\begin{aligned}
&\left\langle \oint_{C(0)} \frac{dz}{2\pi i}\varepsilon(z)T(z)\Phi_1(w_1)\cdots\Phi_m(w_m)\right\rangle \\
&\quad=\sum_{j=1}^{m}\left\langle \Phi_1(w_1)\cdots\left(\oint_{C(w_j)}\frac{dz}{2\pi i}\varepsilon(z)T(z)\Phi_j(w_j)\right)\cdots\Phi_m(w_m)\right\rangle \\
&\quad=\sum_{j=1}^{m}\left\langle \cdots\left(\oint_{C(w_j)}\frac{dz}{2\pi i}\varepsilon(z)\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_j\right]\right)\Phi_j(w_j)\cdots\right\rangle,
\end{aligned} \tag{2.86}$$

and taking out the integral over z and the parameter $\varepsilon(z)$ since the equality holds for arbitrary ε and for all z, we get the conformal Ward identity

$$\langle T(z)\Phi_1(w_1)\cdots\rangle=\sum_{j=1}^{m}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right]\langle\Phi_1(w_1)\cdots\rangle. \tag{2.87}$$

rom this equation, we read that correlation functions are also meromorphic functions with singularities at operator insertions, whose residues are determined by the conformal transformation properties of the operators. Moreover, this is the local version of the conformal Ward identity, which is a fundamental equation in CFT and in String Theory. The OPE's for products of primary fields with the stress tensor is equivalent to the Ward identity in CFT.

Finding the correlators for primaries is not enough, as they are not the complete basis in the space of fields. To get all the correlators, we need to account for the descendants, which is a relatively easy task. This is because the dynamics of descendants are determined from the dynamics of the primaries, hence a knowledge of the correlators of Φs with T is precisely a solution to that problem. Specifically, consider a correlator with all primaries and one descendant Φ^{-n}, which has the form

$$\begin{aligned}
\langle\cdots\hat{L}_{-n}\Phi_j(w_j)\cdots\rangle&=\oint_{C(w_j)}\frac{dz}{2\pi i}(z-w_j)^{-n+1}\langle\cdots T(z)\Phi_j(w_j)\cdots\rangle \\
&=\oint_{C(w_j)}\frac{dz}{2\pi i}(z-w_j)^{-n+1}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right]\langle\cdots\Phi_j(w_j)\cdots\rangle.
\end{aligned} \tag{2.88}$$

For a correlator with several descendants, we will have several such integrals, one for each descendant. But, in principle, all the correlators can be found via this procedure, just from the knowledge of correlators from primaries.

2.5 Virasoro Algebra

From dimensional reasons and analyticity, the OPE TT has the general form

$$T(z)T(w)=\frac{c/2}{(z-w)^4}+\frac{2}{(z-w)^2}T(w)+\frac{1}{z-w}\partial_w T(w)+\cdots, \tag{2.89}$$

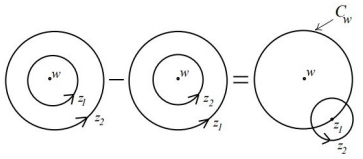


Figure 7: The contour for the difference of $\hat{L}_n\hat{L}_m$ and $\hat{L}_m\hat{L}_n$ acting on $\Phi(w)$. Figure taken from [Ket95]

with $\cdots$ representing the descendants, which are obtained by acting $\hat{L}_{-n}$ on $T(w)$, multiplied by a factor of $(z-w)^{n-2}$. The constant c is called the central charge of the theory, and the easiest way to isolate the coefficient is by looking at the two-point function

$$\langle T(z)T(0)\rangle = \frac{c/2}{z^4}. \tag{2.90}$$

This can be justified from the general form of the two-point function. The constant c depends on the CFT on which T is computed, and is related to the conformal anomaly. We write the OPE between two Ts as

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w). \tag{2.91}$$

We read off from this that T is almost a primary field of conformal dimension $h=2$. If $c=0$, then T is honestly a primary field, which is the classical case. That $c \neq 0$ is a purely quantum effect.

Given the OPE, we can find an algebra between the modes $\hat{L}$ by considering

$$[\hat{L}_n, \hat{L}_m]\Phi(w), \tag{2.92}$$

to evaluate this, we separate the commutator with choice of contour as in figure 7, and evaluate

$$\hat{L}_n\hat{L}_m\Phi(w) = \oint_{C_2(w)} \frac{dz_2}{2\pi i}(z_2-w)^{n+1} \oint_{C_1(w)} \frac{dz_1}{2\pi i}(z_1-w)^{m+1}T(z_1)T(z_2)\Phi(w), \tag{2.93}$$

where $|C_1(w)| < |C_2(w)|$, which is why $z_1 - w$ goes with the power of $m+1$, because $\hat{L}_m$ is the first mode to act on Φ, which corresponds to the contour having smaller radius. For the second term of the commutator, we have

$$\hat{L}_m\hat{L}_n\Phi(w) = \oint_{C_1(w)} \frac{dz_1}{2\pi i}(z_1-w)^{m+1} \oint_{C_2(w)} \frac{dz_2}{2\pi i}(z_2-w)^{n+1}T(z_2)T(z_1)\Phi(w), \tag{2.94}$$

and here $|C_2(w)| < |C_1(w)|$. Taking the difference, we obtain

$$[\hat{L}_n, \hat{L}_m]\Phi(w) = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1-w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2-w)^{n+1}T(z_2)T(z_1)\Phi(w). \tag{2.95}$$

Inserting the OPE TT into this, we get

$$\begin{aligned}[\hat{L}_n, \hat{L}_m] = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \\ \times \left(\frac{c/2}{(z_2 - z_1)^4} + \frac{2}{(z_2 - z_1)^2} T(z_1) + \frac{1}{z_2 - z_1} \partial_{z_1} T(z_1) \right).\end{aligned} \tag{2.96}$$

Now, to take the dz_2 integral, we will go term by term inside the OPE. The first term is of the form

$$\begin{aligned}\oint_{C(z_1)} \frac{dz_2}{2\pi i} \frac{f(z_2; w)}{(z_1 - z_2)^4} &= \frac{1}{3!} \Big(\frac{d^3}{dz_2^3} f(z_2; w) \Big) \Big|_{z_2 = z_1} \\ &= \frac{c}{12}(n+1)n(n-1)(z_1 - w)^{n-2},\end{aligned} \tag{2.97}$$

where $f(z_2; w) = \frac{c}{2}(z_2 - w)^{n+1}$. Now, we take the integral of dz_1 over the contour $C(w)$, which is of the form

$$\frac{c}{12}(n^3 - n) \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n-1} = \frac{c}{12}(n^3 - n)\delta_{m+n,0}. \tag{2.98}$$

For the second term in the large parenthesis, we have

$$\begin{aligned}&\oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \frac{2T(z_1)}{(z_2 - z_1)^2} \\ &\quad = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1}(n+1)(z_1 - w)^n 2T(z_1) \\ &\quad = 2(n+1) \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n+1} T(z_1) \\ &\quad = 2(n+1)\hat{L}_{n+m},\end{aligned} \tag{2.99}$$

and finally, for the last term

$$\begin{aligned}&\oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+1} \oint_{C(z_1)} \frac{dz_2}{2\pi i}(z_2 - w)^{n+1} \frac{1}{z_2 - z_1} \partial_{z_1} T(z_1) \\ &\quad = \oint_{C(w)} \frac{dz_1}{2\pi i}(z_1 - w)^{m+n+2} \partial_{z_1} T(z_1) \\ &\quad = -\oint_{C(w)} \frac{dz_1}{2\pi i}(m+n+2)(z_1 - w)^{m+n+1} T(z_1) + \text{boundary term} \\ &\quad = -(m+n+2)\hat{L}_{m+n},\end{aligned} \tag{2.100}$$

where we integrated by parts. Combining all three parts, we finally obtain the Virasoro algebra

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.101}$$

The Virasoro algebra is the central extension of the Witt algebra. The operators with a hat means that they are expanded around some point, in the above case w. If we set w to be the origin, we recover the L_n operators: $\hat{L}_n(0) = L_n$, and these operators satisfy the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.102}$$

One can show that this is the unique central extension of the Witt algebra, up to additive constant redefinitions. To see this, consider an arbitrary extension

$$[L_n, L_m] = (n-m)L_{n+m} + c_{n,m}, \tag{2.103}$$

with $c_{n,m}$ coefficients anti-symmetric under exchange of n and m. For this algebra to satisfy the Jacobi identity, we must have

$$[L_n, [L_m, L_k]] + [L_k, [L_n, L_m]] + [L_m, [L_k, L_n]] = 0. \tag{2.104}$$

Explicitly, the terms are of the form

$$\begin{aligned}
[L_n, [L_m, L_k]] &= [L_n, (m-k)L_{m+k} + c_{m,k}] \\
&= (m-k)(n-m-k)L_{n+m+k} + (m-k)c_{n,m+k}, \\
[L_k, [L_n, L_m]] &= (n-m)(k-n-m)L_{n+m+k} + (n-m)c_{k,n+m}, \\
[L_m, [L_k, L_n]] &= (k-n)(m-k-n)L_{n+m+k} + (k-n)c_{m,n+k}.
\end{aligned} \tag{2.105}$$

When we sum the three terms, the coefficient of L_{n+m+k} will be identically 0, which is no surprise as the Witt algebra has a Lie algebra structure. For the central extension to satisfy the Jacobi identity, the objects $c_{n,m}$ must satisfy

$$(m-k)c_{n,m+k} + (n-m)c_{k,n+m} + (k-n)c_{m,n+k} = 0. \tag{2.106}$$

It is not immediately obvious what is the solution to this equation, but inserting

$$c_{n,m} = \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{2.107}$$

one sees that the Jacobi identity is satisfied identically.

Observe that for $n = -1, 0, 1$, the central charge term does not appear in the algebra because they are the roots of $n^3 - n = 0$. This reflects the fact that the globally defined transformations, generated by $\{L_{-1}, L_0, L_1\}$, reflects the invariance of the ground state under $SL(2, \mathbb{C})$, since $SL(2, \mathbb{C})$ remains the exact global symmetry of the quantized CFT. In particular, $T(z)$ is still a primary field of dimension $h = 2$ with respect to the global conformal group.

Out of the OPE between two Ts, we can extract the infinitesimal transfor-

mation of T under conformal transformations

$$\begin{aligned}\delta_\varepsilon T &= [T_\varepsilon, T] \\ &= \oint_{C(w)} dz\varepsilon(z)T(z)T(w) \\ &= \oint_{C(w)} dz\varepsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)\right) \\ &= \frac{c}{12}\partial_w^3\varepsilon(w) + 2\partial_w\varepsilon(w)T(w) + \varepsilon(w)\partial_w T(w) \\ &= \Big(\varepsilon(w)\partial_w + 2\partial_w\varepsilon(w)\Big)T(w) + \frac{c}{12}\partial_w^3\varepsilon(w).\end{aligned} \tag{2.108}$$

We can integrate this to a finite transformation of $T(w)$ from w to $z = f(w)$

$$T(w) \to T'(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}S[f;w], \tag{2.109}$$

where the Schwartzian derivative is defined as

$$S[f;w] = \Big(\partial_w f\partial_w^3 f - \frac{3}{2}(\partial_w^2 f)^2\Big)/(\partial_w f)^2. \tag{2.110}$$

In particular, for the map from the cylinder w to the plane $z = e^w$, we have

$$\begin{aligned}T_{\text{cylinder}}(w) \to T_{\text{plane}}(z) &= \left(\frac{dw}{dz}\right)^2 T_{\text{cylinder}}(w) + \frac{c}{12}S[z;w] \\ &= z^{-2}T_{\text{cylinder}}(w) - \frac{c}{24}.\end{aligned} \tag{2.111}$$

Hence, L_0 on the plane and L_0 on the cylinder are related by $\frac{c}{24}$.

2.6 The Free Boson

The free bosonic CFT has the following action

$$S = \frac{1}{4\pi}\int d^2z\partial\Phi\overline{\partial}\Phi. \tag{2.112}$$

The field Φ must have 0 conformal weight for the action to be invariant. The variation of the action gives the equations of motion

$$\partial\overline{\partial}\Phi = 0. \tag{2.113}$$

From this we can find the two-point function as

$$\langle\Phi(z,\overline{z})\Phi(w,\overline{w})\rangle = -\ln|z-w|^2, \tag{2.114}$$

so that $\partial\overline{\partial}$ gives a result proportional to $\delta^2(z-w)$. To see this, we use $\partial\ln|z-w|^2 = \frac{1}{z-w}$ and the identity $\overline{\partial}_z\frac{1}{z-w} = \delta^2(z-w)$.

From the equations of motion, one can define (anti-)holomorphic currents

$$j = \partial\Phi \quad ; \quad \overline{j} = \overline{\partial}\Phi, \tag{2.115}$$

and by the equations of motion $\overline{\partial} j = 0 = \partial\overline{j}$. One can think of j as a primary field of conformal dimension 1, since Φ has conformal dimension 0 and ∂ brings a conformal dimension 1. One other way to see this is to consider the expansion

$$\Phi = \sum_{n\in\mathbb{Z}} \phi_n z^{-n}, \tag{2.116}$$

the derivative of which gives

$$j = \partial\Phi = \sum_{n\in\mathbb{Z}} (-n)\phi_n z^{-n-1}, \tag{2.117}$$

and this is precisely the expansion of a primary field of dimension $h = 1$. A similar statement for $\overline{j}$ holds.

Now, observe that if we split Φ into ϕ and $\overline{\phi}$, we can write the two-point function as

$$\langle\phi(z)\phi(w)\rangle = \ln(z-w) \quad ; \quad \langle\overline{\phi}(\overline{z})\overline{\phi}(\overline{w})\rangle = \ln(\overline{z}-\overline{w}). \tag{2.118}$$

With this, we can also obtain the correlator between j with itself by

$$\langle j(z)j(w)\rangle = \langle\partial\phi(z)\partial\phi(w)\rangle = -\frac{1}{(z-w)^2}, \tag{2.119}$$

and the OPE

$$j(z)j(w) = -\frac{1}{(z-w)^2} + \cdots, \tag{2.120}$$

where $\cdots$ represents non-singular terms that do not contribute to the 2-point function.

From the action, we find the holomorphic part of the energy-momentum tensor to be

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) :\equiv -\frac{1}{2}\lim_{w\to z}\left(\partial\phi(z)\partial\phi(w) + \frac{1}{(z-w)^2}\right), \tag{2.121}$$

where $: A :$ denotes the normal ordering used to remove divergences, and the reason for $1/(z-w)^2$ in the limit is because the divergence of $\langle jj\rangle$ goes like $-1/(z-w)^2$ and the normal ordering's purpose is to remove that divergence at the coincidence limit.

Given that j has conformal dimension 1 and is a primary field, we see that it has the following OPE with T:

$$T(z)j(w) \sim \frac{1}{(z-w)^2}j(w) + \frac{1}{z-w}\partial_w j(w), \tag{2.122}$$

or, in terms of ϕ:

$$T(z)\partial\phi(w) \sim \frac{1}{(z-w)^2}\partial_w\phi(w) + \frac{1}{z-w}\partial_w^2\phi(w). \tag{2.123}$$

One can define Wick rules for contractions when there is a normal ordered product $: A :: B :$. Using that, one could find that for the scalar field CFT the central charge is 1 because of the following OPE

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \tag{2.124}$$

2.7 Representations of the Virasoro Algebra

In the previous section, we saw that the entire spectrum of CFT can be obtained from the primary fields $\{\phi_p(z,\overline{z})\}$ and their descendants $\{\phi_p^{(\vec{k},\vec{k})}(z,\overline{z})\}$ which are defined as the actions of certain set of $\hat{L}_{-n}$ operators on the primaries. The index p is over all the primary fields in the theory $p = \{h, \overline{h}\}$.

The vacuum $|0\rangle$ is defined to be $SL(2,\mathbb{C})$ invariant, and it can be viewed as an insertion of the identity operator at the infinite past (the origin on the plane). Moreover, since L_n ($n \geq -1$) have no poles at $z = 0$, upon acting on $|0\rangle$ they will give zero

$$L_n|0\rangle = 0, \quad n \geq -1. \tag{2.125}$$

On the other hand, acting L_{-n} with $n > 2$ on the vacuum, we get non-trivial states.

In the radial quantization scheme, we naturally define asymptotic in-states that act on the vacuum at the origin of the plane as follows

$$|\Phi_{\text{in}}\rangle = \lim_{z,\overline{z}\to 0}\Phi(z,\overline{z})|0\rangle. \tag{2.126}$$

To define an out-state, we make the transformation $z = 1/w$ and take the limit $w \to \infty$:

$$\langle\Phi_{\text{out}}| = \lim_{w,\overline{w}\to 0}\langle 0|\Phi'(w,\overline{w}). \tag{2.127}$$

We need to relate $\Phi(z,\overline{z})$ with $\Phi'(w,\overline{w})$. For primary fields, the transformation law of the fields under $w \to z = 1/w$ reads

$$\Phi'(w,\overline{w}) = (-w^{-2})^h(-\overline{w}^{-2})^{\overline{h}}\Phi(1/w,1/\overline{w}) = z^{2h}\overline{z}^{2\overline{h}}\Phi(z,\overline{z}), \tag{2.128}$$

so we define the out-states as

$$\langle\Phi_{\text{out}}| = \lim_{z,\overline{z}\to\infty}\langle 0|\Phi(z,\overline{z})z^{2h}\overline{z}^{2\overline{h}}. \tag{2.129}$$

This motivates us to define the adjoint of a field as

$$\Phi^\dagger(z,\overline{z}) \equiv \Phi\left(\frac{1}{z},\frac{1}{\overline{z}}\right)z^{-2h}\overline{z}^{-2\overline{h}}, \tag{2.130}$$

so that we have the relation

$$\langle \Phi_{\rm out}| = \lim_{w,\overline{w}\to 0}\langle 0|\Phi(1/z,1/\overline{z})z^{-2h}\overline{z}^{-2\overline{h}} = \lim_{z,\overline{z}\to 0}\langle 0|\Phi^\dagger(z,\overline{z}) = |\Phi_{\rm in}\rangle^\dagger. \quad (2.131)$$

Let us consider the $T(z)$ operator. In terms of the modes one has

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad (2.132)$$

and upon conjugation

$$T^\dagger(z) = z^{-4}T(1/z) = z^{-4}\sum_{m\in\mathbb{Z}} L_m^\dagger z^{m+2} = \sum_{m\in\mathbb{Z}} L_m^\dagger z^{m-2}. \quad (2.133)$$

Compared with the expansion of $T(z)$, we define the conjugation of the modes of the energy-momentum tensor as $L_n^\dagger = L_{-n}$. A consequence of this definition is that c is a positive number for a Hilbert space of positive-definite inner product. To see this, observe that due to the positive-definiteness of the inner product,

$$\langle 0|[L_2,L_{-2}]|0\rangle = \langle 0|L_2L_{-2} - \underbrace{L_{-2}L_2|0\rangle}_{=0} = \langle 0|L_2L_2^\dagger|0\rangle = ||L_2^\dagger|0\rangle||^2 \geq 0. \quad (2.134)$$

On the other hand, we know that $[L_2,L_{-2}] = 4L_0 + \frac{c}{2}$ and since $L_0|0\rangle = 0$, we have

$$\frac{c}{2} = \langle 0|[L_2,L_{-2}]|0\rangle \geq 0. \quad (2.135)$$

We now define the primary states. These are simply the states obtained by acting a primary field on the vacuum at the origin of the plane. This is the idea of operator-state correspondence in CFT, there is a one-to-one correspondence between primary fields and primary states. Using the L_{-n} $(n>1)$ operators, we can generate the rest of the fields, the descendants. One denotes the primary states as $|h,\overline{h}\rangle$ which is given by

$$|h,\overline{h}\rangle = \lim_{z,\overline{z}\to 0}\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle. \quad (2.136)$$

Using the OPE between T and a primary field Φ, we can compute the following commutator

$$\begin{aligned}
[L_n,\Phi_{h,\overline{h}}(z,\overline{z})] &= \oint_{C(z)}\frac{dw}{2\pi i}w^{n+1}T(w)\Phi_{h,\overline{h}}(z,\overline{z}) \\
&= \oint_{C(z)}\frac{dw}{2\pi i}w^{n+1}\left(\frac{h}{(w-z)^2}\Phi_{h,\overline{h}}(z,\overline{z}) + \frac{1}{w-z}\partial_z\Phi_{h,\overline{h}}(z,\overline{z})\right) \\
&= \Big((n+1)w^n h\Phi_{h,\overline{h}}(z,\overline{z})\Big)\Big|_{z=w} + \Big(w^{n+1}\partial_z\Phi_{h,\overline{h}}(z,\overline{z})\Big)\Big|_{z=0} \\
&= \left(z^{n+1}\frac{d}{dz} + (n+1)z^n h\right)\Phi_{h,\overline{h}}(z,\overline{z}).
\end{aligned} \quad (2.137)$$

Now, using the fact that $L_n|0\rangle = 0$ for $n > 0$, we obtain

$$\begin{aligned} L_n|h,\overline{h}\rangle &= \lim_{z,\overline{z}\to 0}[L_n, \Phi_{h,\overline{h}}(z,\overline{z})]|0\rangle \\ &= \lim_{z,\overline{z}\to 0}\left(z^{n+1}\frac{d}{dz} + (n+1)z^n h\right)\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle = 0, \quad n > 0, \end{aligned} \tag{2.138}$$

and for the $n = 0$ case

$$\begin{aligned} L_0|h,\overline{h}\rangle = \lim_{z,\overline{z}\to 0}[L_0, \Phi_{h,\overline{h}}(z,\overline{z})]|0\rangle &= \lim_{z,\overline{z}\to 0}\left(z\frac{d}{dz} + h\right)\Phi_{h,\overline{h}}(z,\overline{z})|0\rangle \\ &= h|h,\overline{h}\rangle, \end{aligned} \tag{2.139}$$

and similarly for $\overline{L}_n$ operators. We will call the states $|h,\overline{h}\rangle$ the highest-weight states. The reason for this naming will be clear shortly. Here we note that the condition $L_n|h\rangle = 0$ for $n > 0$ follows for all n if we ensure $L_1|h\rangle = 0 = L_2|h\rangle$, since the rest of L_ns can be generated by the Virasoro algebra.

To obtain the Hilbert space corresponding to the CFT, we need to look at the representations of the Virasoro algebra, the symmetry algebra of the theory. This is in parallel with the case of quantum mechanics of a particle with a spherically symmetric Hamiltonian, where the Hilbert space is associated with the irreducible representations of the symmetry group $\mathfrak{su}(2)$ (we ignore the part of the Hilbert space associated to the motion of the particle, which in simple cases is the space of integrable smooth functions like $L^2(\mathbb{R})$ if the particle moves on the real line). The particle's spin labels the irreps of this group, and within the spin j sector, there are $2j+1$ states. To see this we recall that $\mathfrak{su}(2)$ has 3 generators and by defining $J_0 \equiv J_3$, $J_\pm \equiv J_x \pm iJ_y$ we get the algebra

$$[J_0, J_\pm] = \pm J_\pm \quad ; \quad [J_+, J_-] = 2J_0, \tag{2.140}$$

with the operators $J_\pm$ called raising-lowering operators. We assume that in a given spin sector, there is a maximum eigenvalue m of J_0 such that

$$J_0|j\rangle = j|j\rangle \quad ; \quad J_+|j\rangle = 0. \tag{2.141}$$

To get the other eigenvalues of J_0 in that spin sector, we act on the highest-weight state with the lowering operator

$$|m\rangle = (J_-)^{j-m}|j\rangle. \tag{2.142}$$

The inner products of these states can be found using

$$\begin{aligned} \langle m-1|m-1\rangle = \langle m|J_-^\dagger J_-|m\rangle &= \langle m|J_+J_-|m\rangle \\ &= \langle m|J_x^2 - i[J_x,J_y] + J_y^2|m\rangle \\ &= \langle m|J_x^2 + J_y^2 + J_z^2 - J_z^2 + J_z|m\rangle \\ &= \langle m|J^2 - J_z(J_z-1)|m\rangle \\ &= \Big(j(j+1) - m(m-1)\Big)\langle m|m\rangle, \end{aligned} \tag{2.143}$$

where we used the fact that $J^2 = J_x^2 + J_y^2 + J_z^2$ is a Casimir operator, namely an operator that satisfies $[J_i, J^2] = 0$, and the spin j of an irreps is related to the eigenvalue of J^2 as $j(j+1)$. What we see from the above computation is that when m is small enough, the inner product $\langle m-1|m-1\rangle$ becomes negative, rendering the representation non-unitary. This situation is remedied only if j is either an integer or a half-integer. In that case, the state $|-j-1\rangle$ has norm $j(j+1) - (-j)(-j-1) = j(j+1) - j(j+1) = 0$, and hence all the other operators obtained by acting J_- on $|-j-1\rangle$ has zero norm as well. These singular vectors are said to decouple from the first $(2j+1)$ states because any operator A constructed from J_is will have vanishing matrix element $\langle m|A|m'\rangle$ when m' is a singular vector. This is because when A is expressed in terms of $J_\pm$ and J_0, the matrix element will be proportional to $\langle m|\rangle m'$, which is zero since the two are orthogonal. Hence, the representation space has dimension $2j+1$, and the vector space corresponding to spin j is spanned by the vectors in the orthogonal set $\{|m\rangle \,|-j \leq m \leq j\}$, which gives a unitary finite-dimensional representation of the Lie algebra $\mathfrak{su}(2)$.

After this brief review, we get back to the CFT case. Observe that

$$[L_0, L_{-m}] = mL_{-m}, \quad m > 0 \tag{2.144}$$

mimics the commutation of the raising operator in analogy with $\mathfrak{su}(2)$ case and

$$[L_0, L_m] = -mL_m, \quad L_m|h\rangle = 0, \quad m > 0 \tag{2.145}$$

mimics the properties of the lowering operator with $|h\rangle$ a highest-weight state. The analogy is made even further by noting that $L_n^\dagger = L_{-n}$. Then, we can think of the descendant operators as operators obtained by acting with the raising operators, of which there are infinitely many which is the difference of this situation to $\mathfrak{su}(2)$ since there is only one independent raising operator in $\mathfrak{su}(2)$. The descendant operators

$$\Phi_p^{(-\vec{k},-\vec{\bar{k}})} \equiv L_{-k_1} \cdots L_{-k_m} \overline{L}_{-\bar{k}_1} \cdots \overline{L}_{-\bar{k}_{m'}} \Phi_p(0,0) \tag{2.146}$$

give corresponding descendant states, which have different eigenvalue under L_0, just as $|m\rangle$ states have different J_z eigenvalue for different m. A simple descendant state has the form $L_{-n}|h\rangle = L_{-n}(\Phi(0)|0\rangle) = (\hat{L}_{-n}\Phi)(0)|0\rangle = \Phi^{(-n)}(0)|0\rangle$.

A highest-weight representation of the Virasoro algebra gives us a Verma module. As there are infinitely many ladder operators, the representation is infinite-dimensional and is completely characterized by its central charge and the dimension of the highest-weight state. The appearance of the dimensionality is present in the $\mathfrak{su}(2)$ case and the representation is completely characterized by the spin, but in the present context, we also need the central charge. So, the construction of the Hilbert space (the Verma module) in CFT corresponds to finding unitary representations of the Virasoro algebra, and due to the conformal symmetry, one can organize the states neatly starting from the primaries and algorithmically obtaining the descendants.

Algebra	$\mathfrak{su}(2)$	Vir
Grading Operator	J_0	L_0
Ladder Operator(s)	$J_\pm$	$L_{\mp n}$
Conjugation	$J_\pm^\dagger = J_\mp$	$L_{\mp n}^\dagger = L_{\pm n}$
Highest-weight state	$\|j\rangle$	$\|h\rangle$

Table 1: The analogies between the irreps of $\mathfrak{su}(2)$ and that of Vir.

From the requirement that the Verma module has a positive-definite inner product, which is the requirement for it to be a unitary representation, we can constrain the values of h and c. We consider

$$0 < \langle h|L_{-n}^\dagger L_{-n}|h\rangle = \langle h|[L_n, L_{-n}]|h\rangle = \left(2nh + \frac{c}{12}(n^3 - n)\right)\langle h|h\rangle. \quad (2.147)$$

Setting $n = 1$, we get the condition $h > 0$. To see why c must be positive if the Verma module is to admit a positive-definite inner product, we consider the case where n is large. Then, the $2nh$ term and $-\frac{c}{12}n$ terms are negligible compared to $\frac{c}{12}n^3$ term, hence we have

$$0 < \langle h|L_{-N}^\dagger L_{-N}|h\rangle \sim \frac{c}{12}N^3\langle h|h\rangle, \quad N \gg 1. \quad (2.148)$$

If c is a negative number, then the inner product of $L_{-N}|h\rangle$ in the module will be negative, and if it is 0 then a non-zero vector will have zero norm which is against the condition of having a positive-definite inner product, where only the zero vectors have zero norm.

Observe that one has $h = 0$ only if $L_{-1}|h\rangle = 0$, that is, if $|h\rangle = |0\rangle$. And from (2.138), we see that $[L_{-1}, \Phi] = \partial\Phi \quad ; \quad [\overline{L}_{-1}, \Phi] = \overline{\partial}\Phi$. In particular, a primary field of dimension $(h, 0)$ has dependence only on z and is independent of $\overline{z}$.

2.8 The Correlation Functions of CFT

Given the correlators of primaries, we can find those of descendants, as we discussed before. We write the complete OPE between two primary fields as

$$\begin{aligned}\Phi_n(z,\overline{z})\Phi_m(w,\overline{w}) = \sum_p \sum_{\vec{k},-\vec{\overline{k}}} C_{nm}^{p(-\vec{k},-\vec{\overline{k}})} z^{h_p-h_n-h_m+|\vec{k}|} \\ \times \overline{z}^{\overline{h}_p-\overline{h}_n-\overline{h}_m+|\vec{\overline{k}}|} \Phi_p^{(-\vec{k},-\vec{\overline{k}})}(w,w),\end{aligned} \tag{2.149}$$

where $C_{nm}^{p(-\vec{k},-\vec{\overline{k}})}$ are some coefficients. This OPE means that there is no new operator beyond the primaries and their descendants in a CFT. For convenience, since the holomorphic and anti-holomorphic sectors always separate, we will write only the holomorphic sector and we will write $\phi(z)$ to mean the holomorphic part of $\Phi(z,\overline{z})$. With this convention, the OPE reads

$$\phi_n(z)\phi_m(w) = \sum_p \sum_{\vec{k}} C_{nm}^{p(-\vec{k})} z^{h_p-h_n-h_m+|\vec{k}|} \phi_p^{(-\vec{k})}(w). \tag{2.150}$$

We can compute the conformal dimension of a descendant state by

$$\begin{aligned}L_0\phi_h^{(-\vec{k})}(z) &= L_0 L_{-k_1}\cdots L_{-k_n}\phi_h(z) \\ &= ([L_0,L_{-k_1}] + L_{-k_1}L_0)L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &= (k_1 L_{-k_1} + L_{-k_1}L_0)L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &= k_1\phi_h^{(-\vec{k})}(z) + L_{-k_1}L_0 L_{-k_2}\cdots L_{-k_n}\phi_h(z) \\ &\quad\vdots \\ &= (k_1+\cdots k_{n-1})\phi_h^{(-\vec{k})}(z) + L_{-k_1}\cdots L_{-k_{n-1}}L_0 L_{-k_n}\phi_h(z) \\ &= (k_1+\cdots k_n)\phi_h^{(-\vec{k})}(z) + L_{-k_1}\cdots L_{-k_n}L_0\phi_h(z) \\ &= (|\vec{k}|+h)\phi_h^{(-\vec{k})}(z),\end{aligned} \tag{2.151}$$

where $|\vec{k}| = k_1+\cdots k_n$. Moreover, the number of descendants at level n is $P(n)$, where $P(n)$ is the number of partitions of n into sums of positive integers. For example, at level 1, there is only one descendant, which is $L_{-1}\phi_h(z)$. At level 2, there are two descendants, given by $L_{-2}\phi_h(z)$ and $L_{-1}L_{-1}\phi_h(z)$, and the story goes on similarly for higher n.

The energy-momentum tensor T itself is a descendant of level two in the family of the identity operator

$$I^{(-2)}(w) = (\hat{L}_{-2}I)(w) = \oint_{C(w)} \frac{dz}{2\pi i}(z-w)^{-2+1}T(z)I = T(w). \tag{2.152}$$

The OPE coefficients of the descendant fields can be determined once the coefficients for those of primaries are known. For example, one has

$$\langle\Phi_1(w_1)\cdots(\hat{L}_{-n}\Phi_m)(w_m)\rangle = \mathcal{L}_{-n}\langle\Phi_1(w_1)\cdots\Phi_m(w_m)\rangle, \tag{2.153}$$

where we defined the differential operator $n \geq 2$

$$\mathcal{L}_{-n} = -\sum_{j=1}^{m-1} \left(\frac{(1-n)h_j}{(w_j - w_m)^n} + \frac{1}{(w_j - w_m)^{n-1}} \frac{\partial}{\partial w_j} \right). \tag{2.154}$$

What we did above is to move $\hat{L}_{-n}$ operator through all Φ_j $(j < m)$ for $\hat{L}_{-n}$ to act on the vacuum bra $\langle 0|$ and while doing so we picked up the contributions of the OPEs.

Due to a theorem by Belavin, Polyakov, and Zamolodchikov [BPZ84], the coefficients $C_{mn}^{p(-\vec{k},-\vec{\bar{k}})}$ can be decomposed in terms of the coefficients of primaries times some functions of the parameters (h_m, h_n, h_p, c). Explicitly, one has

$$C_{mn}^{p(-\vec{k},-\vec{\bar{k}})} = C_{mnp} \beta_{mn}^{p(-\vec{k})} \overline{\beta}_{mn}^{p(-\vec{\bar{k}})}, \tag{2.155}$$

where C_{mnp} are the coefficients appearing in the OPE of primary fields, $\beta_{mn}^{p(-\vec{k})}$ ($\overline{\beta}_{mn}^{p(-\vec{\bar{k}})}$) are functions of h_m, h_n, h_p, c ($\overline{h}_m, \overline{h}_n, \overline{h}_p, \overline{c}$). One may in principle compute these coefficient functions, but the procedure becomes increasingly more involved.

2.9 Null States

The irreps of Virasoro algebra are constructed in analogy with that of $\mathfrak{su}(2)$, but Virasoro algebra has complications that $\mathfrak{su}(2)$ does not have. Namely, in the irreps of Virasoro algebra, there are null states, states that have a 0 inner product with all vectors, and hence these states spoil the positive-definiteness of the irreps. But, if the inner product is not positive definite, then unitarity is broken so the quantum theory is physically ill-defined. To remedy this, we must detect the null states and quotient them out. That is to say, if $|\chi\rangle$ is a null state, then we identify $|\psi\rangle + |\chi\rangle$ with $|\psi\rangle$, in effect setting $|\chi\rangle$ to zero, so that it having zero inner product is not a problem. In this section we will discuss how to deal with this issue.

In constructing the Hilbert space of the CFT, we need to ensure that the inner product is positive-definite. The only relevant reps which can contain unitary irreps of the Virasoro algebra are the highest-weight ones, because those are the reps that have a lowest-energy state. Starting from a highest-weight state $|h\rangle$, we can construct the corresponding Verma module. One must keep in mind though that this module need not have a positive definite inner product right away. Whether we have such a product depends on the values of h and c.

A descendant state χ that satisfies the equations

$$L_0|\chi\rangle = (h+N)|\chi\rangle, \quad L_n|\chi\rangle = 0 \quad (n > 0), \tag{2.156}$$

is called a null state. It is a primary and descendant state (descendant because its eigenvalue under L_0 is $h+N$), so it is a descendant at level N; on the other hand, it is also a highest-weight state because L_n ($n>0$) annihilates it, so it is primary. To obtain a non-degenerate reps, we need to mod out the submodules generated by the null states. Schematically, if $\mathrm{Verm}(h,c)$ is the full Verma module and $\mathrm{Null}(h,c)$ is the submodule of null states, then the physically plausible Hilbert space is defined as the quotient $\mathcal{H}(h,c)=\mathrm{Verm}(h,c)/\mathrm{Null}(h,c)$. In other words, we identify any state $|\psi\rangle\in\mathrm{Verm}(h,c)$ with any state of the form $|\psi\rangle+|\chi\rangle$ where $|\chi\rangle\in\mathrm{Null}(h,c)$. In some sense, we set all the null states to zero, which renders the module physically viable since after setting all null states to zero only the 0 vectors will have 0 norm in the Verma module, hence it has a positive-definite inner product.

Now let us try to understand null states at low levels. For $N=1$, the only possible null state is $|\chi\rangle=L_{-1}|h\rangle$, or equivalently $\hat{L}_{-1}\phi_h(z)=\partial_z\phi_h(z)$. Setting this to 0 corresponds to the identity operator and $|h\rangle=|0\rangle$, and hence is a trivial representation. For $N=2$ case, we have the possibility

$$|\chi\rangle=L_{-2}|h\rangle+aL_{-1}^2|h\rangle=0, \tag{2.157}$$

for some value of the parameter a. Because all L_n operators with $n>0$ can be generated by L_1 and L_2 using the Virasoro algebra, it is enough to act with those two on the above null state and get the conditions for $|\chi\rangle$ to be a null state. We act on the level 2 χ with L_1 to get

$$\begin{aligned}
0&=L_1\Big(L_{-2}|h\rangle+aL_{-1}^2|h\rangle\Big)\\
&=\Big([L_1,L_{-2}]+L_{-2}L_1+a\big([L_1,L_{-1}]L_{-1}+L_{-1}L_1L_{-1}\big)\Big)|h\rangle\\
&=\Big(3L_{-1}+2aL_0L_{-1}+aL_{-1}[L_1,L_{-1}]-aL_{-1}^2L_1\Big)|h\rangle\\
&=\Big(3L_{-1}+2aL_0L_{-1}+2aL_{-1}L_0\Big)|h\rangle\\
&=\Big(3L_{-1}+2a[L_0,L_{-1}]+4aL_{-1}L_0\Big)|h\rangle\\
&=\Big(3L_{-1}+2aL_{-1}+4ahL_{-1}\Big)|h\rangle\\
&=(3+2a(2h+1))L_{-1}|h\rangle.
\end{aligned} \tag{2.158}$$

This is one equation, we also act with L_2 to get a second equation

$$\begin{aligned}
0 &= L_2\Big(L_{-2} + aL_{-1}^2\Big)|h\rangle \\
&= \Big([L_2, L_{-2}] + L_{-2}L_2 + a[L_2, L_{-1}]L_{-1} + aL_{-1}L_2L_{-1}\Big)|h\rangle \\
&= \Big(\Big(4L_0 + \frac{c}{2}\Big) + a\left(3L_1L_{-1} + L_{-1}[L_2, L_{-1}] + L_{-1}^2L_2\right)\Big)|h\rangle \\
&= \Big(4h + \frac{c}{2} + a(3(\underbrace{2L_0}_{[L_1, L_{-1}]} + 3L_{-1}L_1)\Big)|h\rangle \\
&= \Big(4h + \frac{c}{2} + 6ah\Big)|h\rangle.
\end{aligned} \tag{2.159}$$

Solving these two equations, we find

$$a = -\frac{3}{2(2h+1)} \quad ; \quad c = \frac{2h(5-8h)}{2h+1}. \tag{2.160}$$

Hence, we write

$$|\chi_h\rangle = \Big(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\Big)|h\rangle, \tag{2.161}$$

where h can be written in terms of c as

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right). \tag{2.162}$$

Via the operator-state correspondence, the null state equation can be written as a null operator equation

$$\chi_h(z) \equiv \hat{L}_{-2}\phi_h(z) - \frac{3}{2(2h+1)}\hat{L}_{-1}^2\phi_h(z). \tag{2.163}$$

This equation has the consequence that if we take a correlator with $\phi_h(z)$, then it will be annihilated by the differential operator $\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2$. To see this first recall (2.154). Since

$$\hat{L}_{-1}\phi_h(z) = \partial\phi_h(z), \tag{2.164}$$

and via the definition (2.154), one has

$$\begin{aligned}
0 &= \langle\phi_1(z_1)\cdots\chi_h(z)\rangle \\
&= \left[\frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2} - \sum_{j=1}^{m}\left(\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\right)\right]\langle\phi_1(z_1)\cdots\phi_h(z)\rangle.
\end{aligned} \tag{2.165}$$

This is a second order partial differential equation that the CFT correlators must satisfy, and whose solutions can be expressed in terms of hypergeometric functions [DFMS97, Ket95]. The procedure of modding out the null states can

be continued level by level. The states of a Verma module $\mathcal{V}(c,h)$ at level N is a linear space spanned by

$$\Big\{|h,(-\vec{k})\rangle = L_{-n_1}\cdots L_{-n_k}|h\rangle \;\Big|\; \sum n_i = N\Big\}. \tag{2.166}$$

And the null state at level N can be determined by solving the equation

$$|\chi^{(N)}\rangle = \sum b_{-\vec{k}}|h,(-\vec{k})\rangle = 0, \tag{2.167}$$

which is a natural generalization of the equation that we wrote for the $N=2$ case (2.157).

Within the subspace generated by $|h,(-\vec{k})\rangle$, we can form the inner products of the states, and all possible inner products can be arranged into a matrix of inner products, whose components are defined as

$$\hat{M}^{(N)}_{(\vec{k}_n)(\vec{k}_m)} \equiv \langle h|L_{n_k}\cdots L_{n_1}L_{-m_1}\cdots L_{-m_{k'}}|h\rangle, \quad \sum n_i = N = \sum m_j. \tag{2.168}$$

All the matrix elements can be computed by commuting the $L^\dagger_{-n_i} = L_{n_i}$ operators through L_{-m_j} operators, until L_{n_i} operators act on $|h\rangle$ to give 0. The importance of this matrix, called the Gram matrix, lies in its determinant, or the zero modes. When a vector in this subspace is annihilated by $M(c,h)$, it is precisely a null state. So the zero eigenvalues of $M^{(N)}(c,h)$ determines the values (c,h) at which the $N-$level Verma module contains a null state. The determinant of this matrix at arbitrary level N was computed by Kac, which is called the Kac determinant and the formula is given by

$$\det \hat{M}^{(N)}(c,h) = \prod_{k=1}^{N}\prod_{mn=k}(h-h(m,n))^{P(N-k)}, \tag{2.169}$$

with $m,n\in\mathbb{Z}^+$, and

$$\begin{aligned} h(m,n) = \frac{1}{48}\Big((13-c)(m^2+n^2) - 24mn \\ - 2(1-c) + \sqrt{(1-c)(25-c)}(m^2-n^2)\Big). \end{aligned} \tag{2.170}$$

It can be shown that for a Verma module $\mathcal{V}(c,h)$ of central charge c, there is a null vector at level $N=m\times n$ iff

$$h = h(m,n) = -\alpha_0^2 + \frac{1}{4}(n\alpha_+ + m\alpha_-)^2, \tag{2.171}$$

and we defined

$$c = c(\alpha_+,\alpha_-) = 1-24\alpha_0^2, \quad 2\alpha_0 = \alpha_+ + \alpha_-, \quad \alpha_+\alpha_- = -1, \tag{2.172}$$

or

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1} = \frac{\sqrt{1-c}\pm\sqrt{25-c}}{\sqrt{24}}. \tag{2.173}$$

Given a central charge value $0 < c < 1$, the solution for $h = h(n, m)$ is characterized by two integers $n, m \geq 1$, and the table of integers is known as the Kac table. The discrete set of dimensions means the discrete set of primaries $\{\phi_{(n,m)}, h_{(n,m)}\}$ in the CFT.

If $c > 25$, one can see that both $\alpha_\pm$ are imaginary numbers. As n, m increases, one gets $h(n, m) < 0$, in which case the representation is non-unitary. When $1 < c < 25$, the conformal weights are complex numbers. The dimensions are real and positive only for the case $c \leq 1$.

Now we discuss a remarkable observation. When α_-/α_+ is a rational number $\mathbb{Q}$, each module $\mathcal{V}(c, h)$ has infinitely many null highest-weight states. When these null states are modded out, the algebra of the operators consists of a finite set [BPZ84], namely, there are finitely many primary fields in the CFT. Such theories are called Rational Conformal Field Theories (RCFT). The minimal models introduced in [BPZ84], called the BPZ models, are particular examples of RCFTs in which $c < 1$.

The minimal models of BPZ are characterized by

$$\begin{aligned} \alpha_-/\alpha_+ &= -p/q, \quad c = 1 - \frac{6(p-q)^2}{pq}, \\ h(n, m) &= \frac{1}{4pq}\Big((nq - mp)^2 - (p - q)^2\Big), \end{aligned} \tag{2.174}$$

where p, q are positive integers which are relatively primes. Observe that if we change $n \to (p - n)$ and $m \to (q - m)$, we have the same $h(n, m)$:

$$h(n, m) = h(p - n, q - m). \tag{2.175}$$

Moreover, $h(n, m) = h(-n, -m)$. Using these two, we can infer

$$h(n + p, m + q) = h(-n, -m) = h(n, m), \tag{2.176}$$

so that we can restrict (n, m) to the rectangle

$$0 < n < p, \quad 0 < m < q, \quad p < q. \tag{2.177}$$

Depending on the magnitude of c and h, one can study the Kac determinant to see whether an irrep of Virasoro algebra, the Verma module, is a unitary representation. If the determinant is negative, that means there are negative inner products in the module, violating unitarity. Even if $detM^{(N)}(h, c) \geq 0$ for all N, it still does not guarantee a unitarity, one needs further checks.

3 The Rational Conformal Field Theory

RCFT is a special CFT which was studied in great detail in the late 1980s [BPZ84, Ver88, MS89a, MS89c, MS88, MS89b, MS89d, Car86]. In particular,

Figure 8: The Riemann sphere on the left, and the torus on the right.

[Ver88] discusses the fusion rules in RCFT, which is a rule to determine the OPE of two operators $[\phi_i]$, $[\phi_j]$ inside a conformal family, which takes the form

$$[\phi_i] \times [\phi_j] = \sum_k {N_{ij}}^k [\phi_k] \tag{3.1}$$

for some coefficients ${N_{ij}}^k$. The main result of [Ver88] is that certain modular transformations of $SL(2,\mathbb{Z})$ (the diffeomorphism group of the torus T^2) diagonalize the matrices $N_i \equiv (N_i)_j{}^k$. This has many deep connections to Topological Quantum Field Theories [Wit89, DW90], and in the modern literature the non-invertible symmetries are heavily studied in relation to these constructions coming from RCFT [Sha23, SN24, LOST23]. In this section we shall give a brief review of [Ver88].

3.1 Fusion Rules in RCFT

A CFT defined over a Riemann surface has certain analyticity structures on its partition function and its correlators. Due to the conformal invariance, the CFT depends only on the conformal class of the Riemann surface, or equivalently the moduli space of complex structures. For example, for the Riemann sphere, there is no moduli, it has a unique conformal class. On the other hand, the torus has a moduli $\tau \in \mathbb{C}^+/SL(2,\mathbb{Z})$, where $\mathbb{C}^+$ is the complex upper plane and $SL(2,\mathbb{Z})$ is the group of large diffeomorphisms of the torus [1] under which it is invariant. These surfaces are given in figure 8. The $SL(2,\mathbb{Z})$ invariance of the torus implies modular invariance on the CFT $n-$point functions on the torus. This modular invariance plays a special role for CFTs on generic Riemann surfaces, because tori can be sewn together to generate higher genus, as demonstrated in figure 9.

Consider a correlator of n primary fields on a genus g surface, which we call G. The $n-$point function depends on the coordinates of the operator insertions,

[1] There are two generators denoted T and S. These act on the modular parameter as $T : \tau \mapsto \tau + 1$, $S : \tau \mapsto -\frac{1}{\tau}$.

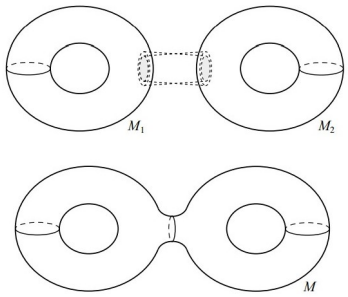


Figure 9: Sewing two tori to get a genus 2 surface. Figure taken from [Pol07]

the punctures, $z = (z_1, \cdots, z_n)$, and it also depends on the moduli parameters of the Riemann surface $m = (m_1, \cdots, m_{3g-3})$. The set (z, m) comprises analytic coordinates on the moduli space of $n-$punctured $g-$genus Riemann surface $\mathscr{M}_{g,n}$. The correlator G has a nice analyticity structure, in the sense that it can be decomposed as follows

$$G(z, m, \overline{z}, \overline{m}) = \sum_{\overline{I}, J} \overline{\mathscr{F}}_{\overline{I}}(\overline{z}, \overline{m}) h_{\overline{I}J} \mathscr{F}_J(z, m) \tag{3.2}$$

where $\mathscr{F}_J$ ($\overline{\mathscr{F}}_{\overline{I}}$) are the (anti-)holomorphic blocks of the correlator and h is a Hermitian metric. When $n = 0$, we have the partition function of a CFT over the Riemann surface with g-punctures. Further, when $g = 1$, which is the torus T^2, we have the partition function over the torus decomposed into (anti-)holomorphic parts and expressed in terms of the characters. The correlator G and hence the blocks $\mathscr{F}$ depend on the representations of the primaries at the punctures.

The blocks $\mathscr{F}$ may have non-trivial monodromy and modular properties, but at the end of the day G should be modular invariant, which puts restrictions on the metric h. In particular, for the torus, one finds restrictions on the operator spectrum of the CFT. As is standard in CFT, we will drop the anti-holomorphic part and will recover it from the holomorphic part in a straight-forward way.

For a rational CFT, one can understand $\mathscr{F}$ better. For an RCFT with N primary operators ϕ_i $(i = 0, \cdots N-1)$,[2] corresponding to irreps $[\phi_i]$ of a chiral algebra. We pick a direction, say left, and don't put indices for the other direction. Our convention is such that $[\phi_0]$ corresponds to the representation containing the identity [1], and a multiplet of primaries in the same $[\phi]$ reps will be denoted as ϕ_i. In this context, we only consider conformal weights that are

[2] Q1: what exactly is the definition of a rational CFT? If we only check if the ratio of $\alpha_1 2$ and α_2 is a rational number, then that completely determines a rational CFT via the value of its central charge. Q2: Do all RCFTs have a finitely many primaries?

integer.

As we discussed in the previous section, there are null states within the representations $[\phi_i]$, whose existence constraints the correlators such that they obey some partial differential equation. The solutions to these PDE's form a vector space holomorphic solutions, whose basis is precisely the blocks $\mathscr{F}_I$.

From an RCFT point there are only certain natural choices for the basis $\mathscr{F}_I$. First we note that any punctured Riemann surface can be obtained by sewing 3−punctured spheres. Then, the conformal blocks $\mathscr{F}_I$ can be constructed out of the three-point functions. It is possible to interpret this sewing procedure by a $g-$loop Feynman diagram for a φ^3 theory, in which the propagators represents a sum over all states in the representation $[\phi_i]$. The vertex factors can be seen as the fusion of three representations $[\phi_i]$, $[\phi_j]$, $[\phi_k]$, which is done through a three-point function $\langle\phi_i\phi_j\phi_k\rangle$. Associated to a diagram, we have a basis $\mathscr{F}_I$, which is unique up to phase transformations. Since the diagrams are in an equivalence class under phases, operations on the φ^3 diagram that do not change it should be represented by phases. For example, upon a modular transformation, under which the diagram should be unchanged, the change in the diagram should be a phase.

We note that this entire construction can be formulated as a vector bundle $V_{g,n}$ over the moduli space $\mathscr{M}_{g,n}$, where the fibres $V_{g,n}$ are uniquely determined by the requirement that its holomorphic sections satisfy the PDEs coming from the null state equations. For RCFT, the fibre spaces $V_{g,n}$ are finite dimensional, which follows from the fact that RCFT has finitely many primary fields.

Now we introduce the fusion rule. Let us consider a Riemann sphere (zero genus), with three punctures; namely the three-point function on the Riemann sphere. Considering the components $V_{0,ijk}$ of the vector bundle $V_{0,3}$, where at the punctures we insert the fields ϕ_i, ϕ_j, ϕ_k. Let $N_{ijk} \equiv \dim V_{0,ijk}$. The fusion rules is then defined as the formal sum

$$\phi_i \times \phi_j = \sum_k N_{ij}{}^k \phi_k. \tag{3.3}$$

One often uses diagrams to represent these fusions, as in figure 10. In raising the k index, we use the conjugation matrix $C_{ij} = N_{ij0}$ as a metric. We interpret the coefficients $N_{ij}{}^k$ as counting the number of independent fusion paths from ϕ_i, ϕ_j to ϕ_k. To determine the rules, one would have to investigate the three-point function $\langle\phi_i\phi_j\phi_k\rangle$, or equivalently the OPE between two primaries ϕ_i and ϕ_j.

Getting back to $\mathscr{F}_I$ for an arbitrary surface, the number of $\mathscr{F}_I$ $(= \dim V_{g,n})$ can be determined using the φ^3 diagram and counting the different ways of fusing together representations. These are like Feynman rules for the φ^3 diagrams, with vertex factors N_{ijk}, and the indices are contracted as indicated by the

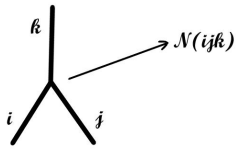


Figure 10: Representing the fusion as a φ^3 diagram in analogy with Feynman diagrams.

"propagators". The result of the diagram should be independent of the way the spheres are sewn together, that is, how the diagram is drawn. From figure 11, we read off the condition

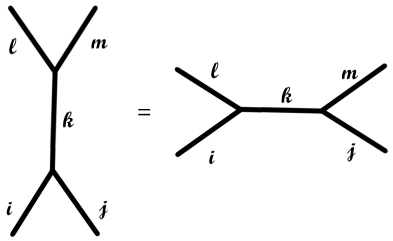


Figure 11: The independence of how the spheres are sewn together puts constraints on the fusion rule coefficients.

$$\sum_k N_{ij}{}^k N_{klm} = \sum_k N_{il}{}^k N_{kjm}. \tag{3.4}$$

Algebraically, we can derive this by using the symmetry of $N_{ij}{}^k$ and the associativity of the fusion rules as follows:

$$(\phi_i \times \phi_j) \times \phi_l = \sum_k N_{ij}{}^k \phi_k \phi_l = \sum_{k,m} N_{ij}{}^k N_{kl}{}^m \phi_m \tag{3.5}$$

$$\phi_i \times (\phi_j \times \phi_l) = \sum_k \phi_i N_{jl}{}^k \phi_k = \sum_{k,m} N_{jl}{}^k N_{ik}{}^m \phi_m, \tag{3.6}$$

equating both sides gives the same condition as (3.4). This condition further implies that the matrices $(N_i)_j{}^k$ furnishes a representation of the fusion algebra. To see this, observe that

$$\begin{aligned} (N_i \cdot N_j)_l{}^n = \sum_k N_{il}{}^k N_{jk}{}^n &= \sum_k N_{il}{}^k N_{kj}{}^n \\ &= \sum_k N_{ij}{}^k N_{kl}{}^n \\ &= \sum_k N_{ij}{}^k (N_k)_l{}^n, \end{aligned} \tag{3.7}$$

or in matrix notation

$$N_i \cdot N_j = \sum_k N_{ij}{}^k N_k. \tag{3.8}$$

Since N_{ijk} is totally symmetric in ijk, one proves another important property

$$N_i \cdot N_j = (N_i)_k{}^l (N_j)_l{}^m = (N_i)^l{}_k (N_j)^m{}_l = (N_j)^m{}_l (N_i)^l{}_k = N_j \cdot N_i, \tag{3.9}$$

that is to say, N_i commutes with themselves, so they can be simultaneously diagonalized, by a standard theorem of linear algebra. Their eigenvalues form the N one-dimensional representations of the fusion rules. This statement plays an important role in the following.

3.2 Relation with Modular Transformations

Let us consider the characters of the chiral algebra. We will choose two oriented cycles on the torus such that the a-cycle consists of equal time points, and the $b-$cycle is along the time evolution direction, generated by L_0 operator. The character χ_i is defined as the trace of the exponential of L_0, the evolution operator, over the representation $[\phi_i]$:

$$\chi_i \equiv \mathrm{Tr}_{[\phi_i]}\big(q^{L_0+\varepsilon}\big), \tag{3.10}$$

with $q = e^{2\pi i\tau}$, τ is the modular parameter of the torus, and $\varepsilon = -\frac{1}{24}c$.

One can obtain the moduli space of genus 1 surfaces $\mathscr{M}_1$ as the upper half-plane divided by the modular transformations generated by $T : \tau \mapsto \tau + 1$, and $S : \tau \mapsto -\frac{1}{\tau}$. For chiral algebras whose operators have integer conformal weight, the characters transform in a finite-dimensional representation of the modular group $SL(2,\mathbb{Z})$. For other algebras, things are more involved and one has to specify boundary conditions.

The transformation of the character under T is easy to read. Due to the $+1$, we get $\mathrm{Tr}_{[\phi_i]} e^{2\pi i(L_0+\varepsilon)} = e^{2\pi i(h_i+\varepsilon)}$, so

$$T : \chi_i \mapsto e^{2\pi i(h_i+\varepsilon)} \chi_i. \tag{3.11}$$

Under the S-transformations, the characters transform in a unitary representation as

$$S : \chi_i \mapsto \sum_j S_i{}^j \chi_j. \tag{3.12}$$

Observe that under $\tau \mapsto -\frac{1}{\tau}$, the cycles a, b are mapped to $-b, a$. Hence, S^2 maps a, b to $-a, -b$, in particular, it reverses the time direction so under S^2 the character χ_i becomes that of the conjugate representation χ^i. So in general we have $S^2 = C$, and only when the representation of the character is self-conjugate do we have $S^2 = 1$.

Now the crucial point is the following. There is a relation between the matrix S and the fusion rules of the primary fields. The basic idea used in [Ver88] is using the primaries to play with the characters and compare them with the transformation under the modular transformations. First consider χ_0, the character of the identity sector. We can obtain the other characters χ_i by inserting into the trace in χ_0 an identity, and by writing the identity as the OPE between a primary field ϕ_i and its conjugate. After this insertion, we move the ϕ_i field along the $b-$cycle (the direction of time evolution), and after it has gone around a complete cycle to get back to its starting point, we annihilate the ϕ_i and its conjugate. As a result of this "wrapping of ϕ_i", the trace is no longer in the identity sector [1] but in the $[\phi_i]$ sector. To see this, map the torus on an annular region $|q|^{1/2} \leq z < |q|^{-1/2}$, that is, if $\tau = \tau_1 + i\tau_2$, then $|q| = e^{-2\pi\tau_2}$ hence $e^{-\pi\tau_2} \leq z < e^{\pi\tau_2}$. If we send $q \to 0$, then the $a-$ cycle will be squished, causing ϕ_i to be moved from ∞ to the origin and thus changing the representation (because the in-state is moved from [1] to $[\phi_i]$). Hence, this operation, denoted $\phi_i(b)$ acts on the characters as

$$\phi_i(b)\chi_0 = \chi_i. \tag{3.13}$$

What happens when $\phi_i(b)$ acts on a character with a $j-$twist χ_j? In that case, the ϕ_i that moves into the origin will meet with ϕ_j, and from their OPE we will get several fusion paths, so the action of $\phi_i(b)$ on χ_j includes several other characters, with some coefficients that will be related to the coefficients of the fusion rules ${N_{ij}}^k$:

$$\phi_i(b)\chi_j = \sum_k {A_{ij}}^k \chi_k. \tag{3.14}$$

It is easy to see that if the three-point function $\langle\phi_i\phi_j\phi_k\rangle$ vanishes (namely, if ${N_{ij}}^k$ vanishes), then ${A_{ij}}^k$ vanishes as well.

Because the OPE between the primaries is symmetric, the operations $\phi_i(b)$ commute as well, as a consequence of which we have

$${A_{ij}}^k = {A_{ji}}^k, \tag{3.15}$$

$$\sum_k {A_{ij}}^k {A_{kl}}^m = \sum_k {A_{il}}^k {A_{kj}}^m. \tag{3.16}$$

The second "crossing symmetry" equation is related to the fact that the operators $\phi_i(b)$ form an associative algebra

$$\phi_i(b)\phi_j(b) = \sum_k {A_{ij}}^k \phi_k(b). \tag{3.17}$$

These properties are all completely the same ones that ${N_{ij}}^k$ obey, so it is reasonable to suspect they are the same. Indeed, it was shown in [Ver88] that this is the case.

The operation of moving ϕ_i along the $b-$cycle can equivalently be done for the $a-$cycle. This time, however, the "twisting" operation does not change the representation of the character, because the $a-$cycle consists of equal time points so there are no insertions of ϕ_i on the origin. Since under the operation, $\phi_i(a)$ the representation should be the same, the action of $\phi_i(a)$ on χ_j is easy to find: χ_j must be eigenstates of the operators $\phi_i(a)$ with some eigenvalues. Hence, we have

$$\phi_i(a)\chi_j = \lambda_i^{(j)}\chi_j. \tag{3.18}$$

The eigenvalues are in general complex (or real) numbers, not necessarily phases. The operators $\phi_i(a)$ satisfies the same algebra as $\phi_i(b)$ does. Using this, we find the following

$$\phi_i(a)\phi_j(a)\chi_n = \lambda_i^{(n)}\lambda_j^{(n)}\chi_n, \tag{3.19}$$

$$\phi_i(a)\phi_j(a)\chi_n = \sum_k A_{ij}{}^k\phi_k(a)\chi_n = \sum_k A_{ij}{}^k\lambda_k^{(n)}\chi_n, \tag{3.20}$$

consequently,

$$\lambda_i^{(n)}\lambda_j^{(n)} = \sum_k A_{ij}{}^k\lambda_k^{(n)}. \tag{3.21}$$

Now comes the key point: The modular transformations generated by S change the cycles, hence they change $\phi_i(a)$ operators to $\phi_i(b)$ operators and vice-versa. In particular, the transformed characters in (3.12) become eigenstates of $\phi_i(b)$.

Let us explicitly show this. We act on both sides of (3.18) with the modular transformation S and use (3.12:

$$S : \phi_i(b)\chi_j \mapsto \phi_i(a)\sum_l S_j{}^l\chi_l = \sum_l S_j{}^l\lambda_i^{(l)}\chi_l, \tag{3.22}$$

$$S : \sum_k A_{ij}{}^k\chi_k \mapsto \sum_{k,l} A_{ij}{}^k S_k{}^l\chi_l. \tag{3.23}$$

Equating the two we get

$$S_j{}^l\lambda_i^{(l)} = \sum_k A_{ij}{}^k S_k{}^l. \tag{3.24}$$

Moreover, using the fact that $S_j{}^l$ furnishes a unitary representation, we get

$$A_{ij}{}^k = \sum_l S_j{}^l\lambda_i^{(l)}S_l^{\dagger k}, \tag{3.25}$$

which means that the matrices $S_j{}^l$ diagonalizes the $(A_i)_j{}^k$ matrices, whose eigenvalues are $\lambda_i^{(l)}$. In addition to this, if we use the fact that $A_{i0}{}^k = \delta_i^k$ (a fusion of $[\phi_i]$ with [1] will give something inside $[\phi_i]$), we get

$$\lambda_i^{(n)} = \frac{S_i{}^n}{S_0{}^n}. \tag{3.26}$$

Hence, if we know the matrix $S_i{}^j$, we can find all the eigenvalues $\lambda_i^{(n)}$ and hence all the coefficients $A_{ij}{}^k$. Since it is expected that $A_{ij}{}^k = N_{ij}{}^k$ for many RCFTs, we have thus solved the fusion rules of the RCFT. Indeed, in many explicit computations, $A_{ij}{}^k = N_{ij}{}^k$ was verified [Ver88]. Since it is an extremely important result, following [Ver88], we will state it below

$$\text{The modular transformations } S : \tau \mapsto -\frac{1}{\tau} \text{ diagonalizes the fusion rules!} \tag{3.27}$$

3.3 Derivation with Diagrammatic Computation

We will derive the result above using a diagrammatic approach, which is a standard tool in the modern literature of non-invertible symmetries, for example in [LOST23].

We represent a torus as a rectangular with $b-$cycle representing the time direction (vertical) and $a-$cycle representing the space direction (horizontal). We define the Virasoro characters χ_j in the $[\phi_j]$ representation as

$$\chi_j = \mathrm{Tr}_{[\phi_j]}\left(q^{L_0-\varepsilon}\right) \equiv \boxed{\quad j \uparrow}. \tag{3.28}$$

($q = e^{2\pi i \tau}$) Under modular transformations generated by $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -\frac{1}{\tau}$, χ_j changes as

$$\begin{aligned} T &: \boxed{\quad j \uparrow} \ \mapsto\ e^{2\pi i (h_j - \varepsilon)} \ \boxed{\quad j \uparrow}, \\ S &: \boxed{\quad j \uparrow} \ \mapsto\ \sum_k S_j{}^k \ \ \boxed{\quad k \uparrow}, \end{aligned} \tag{3.29}$$

with $S_j{}^k$ a unitary matrix. These characters can be twisted with ϕ_i operators along the $a-$cycle and the $b-$cycle, which we represent as follows

$$\phi_i(a)\chi_j \equiv \boxed{\underset{i}{\rightarrow}\ \ j \uparrow}, \tag{3.30}$$

$$\phi_i(b)\chi_j \equiv \boxed{i \uparrow \quad j \uparrow}. \tag{3.31}$$

These correspond to an ϕ_i insertion in the trace and winding it once along the respective cycle. The twist along a-cycle should be proportional to χ_j as explained in the previous subsection, and the twist along the $b-$cycle introduces a fusion $[\phi_i]$ with $[\phi_j]$ inside the trace of Virasoro character. Based on this picture we have

$$\boxed{\underset{i}{\rightarrow}\ \ j \uparrow} = \lambda_i^{(j)} \ \boxed{\quad j \uparrow}, \tag{3.32}$$

$$\boxed{i\uparrow \quad j\uparrow} = \sum_k A_{ij}{}^k \boxed{\qquad k\uparrow}, \tag{3.33}$$

since there is a fusion involved in the second line, the coefficients $A_{ij}{}^k$ are related with $\mathcal{N}(ijk)$, in fact they are generically equal to each other. Now, the key idea was to observe that under the modular transformation $S : \tau \mapsto -\frac{1}{\tau}$, the $a-$cycle and the $b-$cycle are exchanged. So under the exchange, the diagrams behave like

$$S : \boxed{i\uparrow \quad j\uparrow} \mapsto \sum_m S_j{}^m \boxed{\underset{i}{\rightarrow} \quad m\uparrow} = \sum_m S_j{}^m \lambda_i^{(m)} \boxed{\qquad m\uparrow} \tag{3.34}$$

$$S : \sum_k A_{ij}{}^k \boxed{\qquad k\uparrow} \mapsto \sum_k A_{ij}{}^k \sum_m S_k^m \boxed{\qquad m\uparrow}, \tag{3.35}$$

using the fact that S_k^m is unitary, we find

$$A_{ij}{}^k = \sum_m S_j{}^m \lambda_i^{(m)} S_m^{\dagger k}. \tag{3.36}$$

This is the result of [Ver88] reviewed in the previous section from a more algebraic point of view.

4 2d Critical Ising Model and RCFT

An Ising model at 2d is a two-dimensional square lattice system with a spin σ_i at each site, having values either spin up or spin down. The Ising model in general dimensions is very interesting, and in the case of $d = 2$, it is exactly solvable. One important feature is the Kramers-Wannier duality, mapping the high-temperature regime partition function with the low-temperature one. On one side, one uses the order parameter σ, the spin, and on the dual lattice one uses the disorder parameter μ.

At the critical temperature T_c, the Ising model, like all statistical systems, is scale invariant. In fact, the scale invariance in this case implies the stronger conformal invariance. Hence, the continuum limit of the theory should be described by a 2d CFT. The corresponding continuum CFT turns out to be the free Majorana fermion with the Lagrangian [BPZ84]

$$S = \int d^2z \left(\frac{1}{2}\psi\overline{\partial}\psi + \frac{1}{2}\overline{\psi}\partial\overline{\psi} + m\overline{\psi}\psi \right), \tag{4.1}$$

where $m \propto (T - T_c)$. Thus, one deals with massless fermions at the critical temperature. The equations of motion for $T = T_c$ then read

$$\overline{\partial}\psi = 0, \quad \partial\overline{\psi} = 0 \Longrightarrow \psi = \psi(z), \quad \overline{\psi} = \overline{\psi}(\overline{z}). \tag{4.2}$$

We would like to find the stress-energy tensor. To do so, we make the transformation $z \to z + \varepsilon(z)$, so that $\psi \to \psi + \varepsilon\partial\psi$, under which the action changes as

$$\begin{aligned} S &\to \int d^2z \frac{1}{2}(\psi + \varepsilon\partial\psi)\overline{\partial}(\psi + \varepsilon\partial\psi) + \text{c.c.} \\ &= \int d^2z \frac{1}{2}\Big(\psi\overline{\partial}\psi + \varepsilon(\psi\overline{\partial}\partial\psi + \partial\psi\overline{\partial}\psi + O(\varepsilon^2)\Big) + \text{c.c.} \\ &= S + \int d^2z \frac{1}{2}\varepsilon\Big(\overline{\partial}(\psi\partial\psi) - \overline{\partial}\psi\partial\psi + \partial\psi\overline{\partial}\psi\Big) + \text{c.c.}, \end{aligned} \tag{4.3}$$

hence, we have

$$\delta S = \int d^2z \overline{\partial}\varepsilon \left(-\frac{1}{2}\psi\partial\psi\right) + \text{c.c.}, \tag{4.4}$$

from which we can infer the stress-energy tensors

$$T = -\frac{1}{2}\psi(z)\partial\psi(z), \quad \overline{T} = -\frac{1}{2}\overline{\psi}(\overline{z})\overline{\partial\psi}(\overline{z}). \tag{4.5}$$

We can also find the two-point function by solving

$$\overline{\partial}\langle\psi(z)\psi(w)\rangle \sim \delta^2(z-w). \tag{4.6}$$

Recall the identity

$$\overline{\partial}_z\partial_z \ln|z-w| = \overline{\partial}_z\frac{1}{z-w} = \pi\delta^2(z-w), \tag{4.7}$$

from which we read off the correlation function and the $\psi\psi$ OPE

$$\begin{aligned} \psi(z)\psi(w) &\sim \frac{1}{z-w}, \\ \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) &\sim \frac{1}{\overline{z}-\overline{w}}. \end{aligned} \tag{4.8}$$

We know that for a general primary of scaling dimension $(h, \overline{h})$, the two-point function reads

$$\langle\Phi_{h,\overline{h}}(z,\overline{z})\Phi_{h,\overline{h}}(w,\overline{w})\rangle = \frac{C}{(z-w)^{2h}(\overline{z}-\overline{w})^{2\overline{h}}}, \tag{4.9}$$

which suggests that the Majorana fermions have the following dimensions

$$\begin{aligned} \big(h_\psi, \overline{h}_\psi\big) &= \Big(\frac{1}{2}, 0\Big), \\ \big(h_{\overline{\psi}}, \overline{h}_{\overline{\psi}}\big) &= \Big(0, \frac{1}{2}\Big). \end{aligned} \tag{4.10}$$

We can also compute the $T\psi$ OPE. The first three terms read [BPZ84]

$$T(z)\psi(w) = \frac{1/2}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w\psi(w) + \frac{3}{4}\partial_w^2\psi(w). \tag{4.11}$$

Recall the Ward identity for the correlation function with a T insertion and primaries Φ:

$$\langle T(z)\Phi_1(w_1)\cdots\rangle = \sum_{j=1}^{m}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right]\langle\Phi_1(w_1)\cdots\rangle. \tag{4.12}$$

Inserting an additional $\psi(w)$, we get

$$\begin{aligned}&\langle T(z)\psi(w)\cdots\rangle\\ &=\left[\frac{1/2}{(z-w)^2}+\frac{1}{z-w}\partial_w+\sum_{j=1}^{m}\left(\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right)\right]\langle\psi(w)\cdots\rangle.\end{aligned} \tag{4.13}$$

But, from the $T\psi$ OPE we find that this is equal to

$$\langle T(z)\psi(w)\cdots\rangle = \left[\frac{1/2}{(z-w)^2}+\frac{1}{z-w}\partial_w+\frac{3}{4}\partial_w^2\right]\langle\psi(w)\cdots\rangle. \tag{4.14}$$

Subtracting the two, we obtain

$$0=\left[\frac{3}{4}\partial_w^2-\sum_{j=1}^{m}\left(\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\partial_{w_j}\right)\right]\langle\psi(w)\cdots\rangle \tag{4.15}$$

In the continuum limit of the Ising model, we have the spin label σ as a continuous field $\sigma(z,\overline{z})$, and through Kramers-Wannier duality we also have $\mu(z,\overline{z})$. These turn out to be primary operators with scaling dimension [BPZ84]

$$\Delta_\sigma=\Delta_\mu=\frac{1}{16}=\overline{\Delta}_\mu=\overline{\Delta}_\sigma, \tag{4.16}$$

so that their two-point functions go like $\langle\sigma\sigma\rangle\sim z^{-1/8}$ and similarly for μ.

In addition to spin operator σ, we have the identity operator I as a primary operator with dimension $\Delta_I=(0,0)$, and the energy operator ε with dimension $\Delta_\varepsilon=(\frac{1}{2},\frac{1}{2})$. These are all scalar operators in the sense that $\Delta=\overline{\Delta}$, so their spin is $s=\Delta-\overline{\Delta}=0$. The operator spectrum of the critical Ising model consists of this, so these are all the primaries at hand. The table of primaries thus reads

Name	Primary Operator	Dimension
Identity	I	$\Delta=(0,0)$
Spin	σ	$\Delta=(\frac{1}{16},\frac{1}{16})$
Energy	ε	$\Delta=(\frac{1}{2},\frac{1}{2})$

And, the fusion algebra between these primaries is as follows [DFMS97]

$$\sigma\times\sigma=I+\varepsilon,\quad \sigma\times\varepsilon=\sigma,\quad \varepsilon\times\varepsilon=I. \tag{4.17}$$

In the modern language, the Ising model is a CFT with a categorical (non-invertible) symmetry, and one gets different spin selection rules for the defect Hilbert spaces in the language of RCFT [LS23].

Part II
Tools from the 21st Century

5 Generalized Symmetries

The notion of symmetry is extremely important in the study of Quantum Field Theories. The insight of the paper [GKSW15] is to realize the deep connection between symmetries and topology. This new perspective allows for higher-form symmetries, in which 0-form symmetries are the ordinary symmetries we know of. We will give a review this perspective, mostly following [BBFT$^+$24].

5.1 0-form Symmetries

Consider a field theory with some action $S[\Phi]$ of the fields Φ. An infinitesimal global transformation $\Phi \to \Phi + \varepsilon\delta\Phi$ is a symmetry transformation if

$$\delta S = 0. \tag{5.1}$$

How does this lead to symmetry? To see this connection, we upgrade the infinitesimal global transformation to an infinitesimal local one $\Phi \to \Phi + \varepsilon(x)\delta\Phi$ such that when the space-time dependency of ε is killed, the transformation is a symmetry transformation. Since the transformation with $\varepsilon(x)$ is not a symmetry transformation, the change of the action will be nonzero. However, since the deformation is still infinitesimal, the change of the action can be at most first order in $\varepsilon(x)$. Moreover, since space-time independent ε must be a symmetry transformation, δS must be related only to the derivative of $\varepsilon(x)$:

$$\delta S = \int dx \partial_\mu \varepsilon(x) j^\mu, \tag{5.2}$$

where j^μ is an object introduced to contract the μ index in the derivative, and it is constructed out of the fields Φ. With an integration by parts, and omitting the boundary term, we are left with

$$\delta S = -\int dx \varepsilon(x) \partial_\mu j^\mu. \tag{5.3}$$

Now comes the crucial part: The action principle states that the fields follow the trajectory in the configuration space that minimizes the action, in the sense that small arbitrary deformations of the path in the configuration space have the same value for the action as the on-shell configuration. Since $\Phi \to \Phi + \varepsilon(x)\delta\Phi$ is such a deformation, the fields solving the equations of motion must have $\delta S = 0$. This imposes, for on-shell fields, the following continuity equation

$$\partial_\mu j^\mu = 0. \tag{5.4}$$

So far is the familiar argument showing the existence of conserved currents following from the existence of symmetries. The form of j^μ can be found using Noether's prescription that we discussed in chapter 2. The crucial insight of the paper [GKSW15] starts from the continuity equation above. Let us write that in the differential form language

$$d * j = 0, \tag{5.5}$$

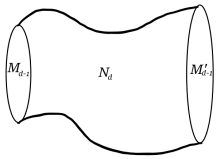


Figure 12: Deforming M_{d-1} to M'_{d-1}

where j is a $1-$form on the spacetime, and $*$ is the Hodge star operator. Now, recall that in the ordinary setup, we integrate j^0 over all space to get a charge that is independent of time up to surface terms:

$$Q = \int dx j^0; \quad \frac{dQ}{dt} = 0. \tag{5.6}$$

In the perspective of generalized symmetries, what happens is that we define a closed $(d-1)$-form on spacetime

$$J = *j, \tag{5.7}$$

and integrate this over a codimension-1 submanifold M_{d-1} to define a charge

$$Q(M_{d-1}) = \int_{M_{d-1}} J. \tag{5.8}$$

This charge is then topological in the sense that

$$Q(M_{d-1}) = Q(M'_{d-1}), \tag{5.9}$$

where M'_{d-1} is a manifold obtained by the topological deformation of M_{d-1} such that (as in figure 12)

$$\partial N_d = M'_{d-1} - M_{d-1}, \tag{5.10}$$

with ∂N_d the boundary of a $d-$manifold inside spacetime. We can see this as follows. Schematically we write $M' = M + \partial N$, and observe that

$$\begin{aligned} Q(M_{d-1} + \partial N_d) = \int_{M_{d-1}+\partial N_d} J &= \int_{M_{d-1}} J + \int_{\partial N_d} J \\ &= Q(M_{d-1}) + \int_{N_d} dJ \\ &= Q(M_{d-1}), \end{aligned} \tag{5.11}$$

where we used Stokes' theorem and the fact that J is a closed form. This is the statement that $Q(M_{d-1})$ is a topological charge, and choosing M_{d-1} to be purely spatial, we recover the familiar statement that Q is time-independent. Hence, this perspective connects the symmetries present in the classical description to

topological operators in the quantum theory, which generates the corresponding symmetry transformations. So far, we have not discussed any sources in space-time. It will turn out that in the quantum theory, the symmetry operator is topological as long as the deformation does not cross a local operator, which corresponds to inserting a local field inside a correlation function. We will get back to this point.

This is not the only outcome of this perspective. Since we now regard charges as being defined (or inserted) over a co-dimension 1 manifold, we can generalize this picture to a charge operator that is inserted on a co-dimension $q+1$ manifold, which we will call a $q-$form symmetry. The ordinary symmetries we know of are $0-$form symmetries in this case, and their charges are inserted on a codimension-1 manifold. In this language, a $0-$form symmetry has charge $Q(M_{d-1})$, which acts on local operators, the quantum fields $\Phi(x)$, to transform it under certain representations of the relevant symmetry group. How the charge acts on the local field $\Phi(x)$ supported on a point is given by a topological connection between the co-dimension-1 manifold M_{d-1} and the point x. For generalized $q-$form symmetries, this extends to the topological linking between the co-dimension$-q+1$ manifold M_{d-q-1} on which the charge lives, and the $q-$dimensional manifold C_q on which a charged extended operator lives.

Given an ordinary $0-$form symmetry at the classical level, there is a corresponding charge $Q = \int_{\Sigma_{t=t_0}} d^{d-1}xj^0$, where $\Sigma_{t=t_0}$ is the spatial slice at time t_0. At the quantum level, we implement this symmetry by a unitary operator

$$U(t), \tag{5.12}$$

which is the exponentiation of the quantized charge. This is time-independent because the charge Q is time-independent:

$$U(t) = U(t'). \tag{5.13}$$

This operator acts on all points of the space, and its action on local fields is by conjugation

$$U(t)\mathcal{O}(\mathbf{x},t)U^{-1}(t) = \mathcal{O}'(\mathbf{x},t), \tag{5.14}$$

which transforms a local operator into another local one. As discussed in the beginning of this section, the perspective of [GKSW15] is that U is a $0-$form symmetry operator living on a codimension-1 manifold and is invertible.

Insertion Along General Submanifolds The operator U can be inserted on any codimension-1 manifold M_{d-1} inside the $d-$dimensional space-time, which we denote as

$$U(M_{d-1}). \tag{5.15}$$

The topological nature of this operator tells us that

$$U(M_{d-1}) = U(M'_{d-1}). \tag{5.16}$$

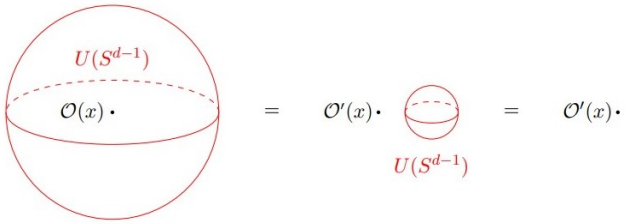


Figure 13: Action of U on a local operator $\mathcal{O}$ via linking. From the second to the third picture, we are using the topological nature of U to contract the sphere that does not enclose any operator insertions.

Moreover, if we choose M_{d-1} and M'_{d-1} to be spatial slices at times t and t', we recover $U(t) = U(t')$.

We can view the action of U on $\mathcal{O}$ as

$$U(M_{d-1})\mathcal{O}(x) = \mathcal{O}'(x)U(M'_{d-1}), \tag{5.17}$$

which is valid inside a correlation function. Here, what we mean is that when we deform the submanifold M_{d-1} across an operator insertion, the operator is transformed, and the manifold M'_{d-1} does not contain the point x anymore. Choosing $M_{d-1} = \Sigma_t$, $M'_{d-1})\Sigma_{t'}$, then in the limit $\Delta t = t' - t \to 0$, we recover

$$U(t)\mathcal{O}(x) = \mathcal{O}'(x)U(t), \tag{5.18}$$

which is again conjugation. There is an equivalent way to see this by deforming the submanifolds M_{d-1} and M'_{d-1} to a sphere enclosing the point x, which we discuss below.

Action via Enclosing We can describe 5.17 as

$$U(S^{d-1})\mathcal{O}(x) = \mathcal{O}'(x), \tag{5.19}$$

where on the left-hand-side, we insert an operator U on a sphere S^{d-1} that is enclosing the point x, as in figure 13.

0-form Symmetry Group The symmetry transformations can be combined to get other symmetry transformations, and these form a group, which can in general be non-abelian. We will denote this group of transformations associated with a $0-$form symmetry as

$$G^{(0)}. \tag{5.20}$$

Corresponding to each element in the group $g \in G^{(0)}$, we have a topological operator implementing that transformation on a local field inside the correlation function

$$\forall g \in G^{(0)} \hookrightarrow U_g. \tag{5.21}$$

Inserting two operators corresponding to two group elements on the same manifold M_{d-1}), we get a multiplication of the two operators. Since the topological operators U_g are expected to act according to the group structure $G^{(0)}$, that is, since it is a homomorphism, we have the relation

$$U_g(M_{d-1})U_{g'}(M_{d-1}) = U_{gg'}(M_{d-1}), \tag{5.22}$$

where gg' is the multiplication of the group, and we often write this in shorthand as a fusion rule

$$U_g \otimes U_{g'} = U_{gg'}. \tag{5.23}$$

Question: Can we allow for a phase in this fusion rule? That is, instead of a faithful representation, can we have a projective representation

$$U_g \otimes U_{g'} = \phi(g, g')U_{gg'}, \tag{5.24}$$

where $\phi(g, g') \in U(1)$ is a phase that is independent of which state or operator U_g is acting on? In ordinary discussions, one usually centrally extends the group to lift the projective to a faithful representation, would there be a similar thing here for the higher-form symmetry counterparts?

Coming back to the fusion rule, if we go to the canonical quantization picture where $M_{d-1} = \Sigma_t$, then we have

$$U_g(t)U_{g'}(t) = U_{gg'}(t), \tag{5.25}$$

which implies that the Hilbert space forms a representation of the group $G^{(0)}$. Furthermore, it should be a unitary representation as required by the probability conservation.

Moreover, the space of local operators at a point x should also form a representation of the group $G^{(0)}$. To see this, consider

$$U_g(S^{d-1})\mathcal{O}(x) = g \cdot \mathcal{O}(x), \tag{5.26}$$

where $\mathcal{O}'(x) = g \cdot \mathcal{O}(x)$ for some g acting on the operators. From the fusion rule, we must have

$$U_g(S^{d-1})U_{g'}(S^{d-1})\mathcal{O}(x) = U_{gg'}(S^{d-1})\mathcal{O}(x). \tag{5.27}$$

Let us unpack the left-hand-side

$$U_g(S^{d-1})U_{g'}(S^{d-1})\mathcal{O}(x) = U_g(S^{d-1})\Big(g' \cdot \mathcal{O}(x)\Big) = g \cdot \Big(g' \cdot \mathcal{O}(x)\Big), \tag{5.28}$$

and the right-hand-side

$$U_{gg'}(S^{d-1})\mathcal{O}(x) = (gg') \cdot \mathcal{O}(x). \tag{5.29}$$

Thus, we have

$$g \cdot \Big(g' \cdot \mathcal{O}(x)\Big) = (gg') \cdot \mathcal{O}(x), \tag{5.30}$$

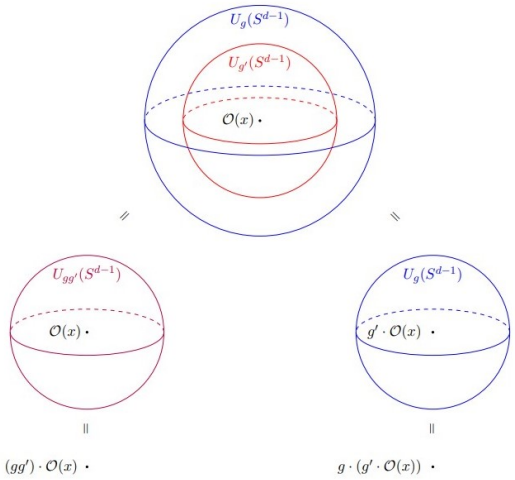


Figure 14: Diagrammatic representation of fusion rules. One can also read off how $\mathcal{O}(x)$ transforms under representations of $G^{(0)}$. Figure taken from [BBFT$^+$24].

meaning that $\mathcal{O}$ transforms under a representation of $G^{(0)}$. There is a diagrammatic way to see this by using spheres enclosing a point and using topological equivalence, as in figure 14.

5.2 A 4d QFT with 0-form Global $U(1)$ Symmetry

Let us discuss a 4d theory admitting a global $G^{(0)} = U(1)$ symmetry, which captures all of what we discussed in a simple way. There is a conserved current j^μ as a result of $U(1)$ invariance, and the corresponding transformations are implemented on the quantum theory by the topological operators

$$U_\alpha(M_{d-1}) = \exp\Big(i\alpha \int_{M_{d-1}} J\Big), \quad U(1) \ni g = \exp(i\alpha), \tag{5.31}$$

where $J = *j$ and due to the conservation $d * j = 0$, J is a closed $d-1$-form, hence $U_\alpha(M_{d-1}) = U_\alpha(M'_{d-1})$ for any two co-dimension-1 manifolds enclosing the same local operators. Let us choose the codimension-1 ($=4-1=3$) manifold to be a sphere S^3 for simplicity. If we insert a local operator $\mathcal{O}_q(x)$ of charge $q \in \mathbb{Z}$ under $U(1)$, where x is enclosed inside S^3 as in figure 15 then we will have the following Ward identity with the contact term

$$\langle \partial_\mu j^\mu(x') \mathcal{O}_q(x) \cdots \rangle = q\delta(x - x') \langle \mathcal{O}_q(x) \cdots \rangle. \tag{5.32}$$

We define a Poincaré dual to the delta function $\delta^4(x)$, which is a 4 form that gives, upon integration, one if the integration manifold M_4 contains the point

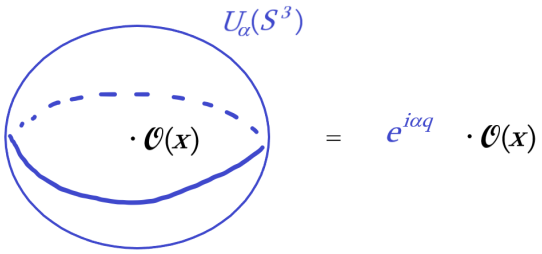


Figure 15: Action via linking for a 4d QFT with $U(1)$ global symmetry.

x, and 0 otherwise. With this, we write

$$\langle dJ(x')\mathcal{O}_q(x)\cdots\rangle = q\delta^4(x-x')\langle\mathcal{O}_q(x)\cdots\rangle. \tag{5.33}$$

Now integrate x' over a disk D^4 with a boundary as a sphere $\partial D^4 = S^3$ containing the point x

$$\Big\langle \int_{D^4} dJ(x')\mathcal{O}_q(x)\cdots \Big\rangle = q\int_{D^4}\delta^d(x-x')\langle\mathcal{O}_q(x)\cdots\rangle. \tag{5.34}$$

The integral on the right-hand-side is 1 by definition, and the left-hand-side becomes upon using Stoke's theorem

$$\Big\langle \int_{\partial D^4} J(x')\mathcal{O}_q(x)\cdots \Big\rangle = \Big\langle \int_{S^3} J(x')\mathcal{O}_q(x)\cdots \Big\rangle = \langle Q(S^3)\mathcal{O}_q(x)\cdots\rangle. \tag{5.35}$$

Thus, we have the infinitesimal transformation law

$$\langle Q(S^3)\mathcal{O}_q(x)\cdots\rangle = q\langle\mathcal{O}_q(x)\cdots\rangle. \tag{5.36}$$

At the exponential level, we get the finite transformation law

$$\langle U_g(S^3)\mathcal{O}_q(x)\cdots\rangle = \exp(iq\alpha)\langle\mathcal{O}_q(x)\cdots\rangle. \tag{5.37}$$

Moreover, the topological operators $U_\alpha(S^3)$ with different parameters can be fused together as in figure 16 to get a third operator, compatible with the $U(1)$ multiplication. Generalization of all these discussions to d-dimensional QFT with a global symmetry G is straightforward.

5.3 Higher-Form Symmetries

We get back to the point of our discussion where the action of a topological operator acts on a charged object via topological linking. Namely, we have

$$\Big\langle U_g(M_{d-1}) = e^{i\alpha Q(M_{d-1})} \quad \curvearrowright \quad \mathcal{O}(x)\Big\rangle = \#(M_{d-1},x)\Big\langle \mathcal{O}'(x)\Big\rangle. \tag{5.38}$$

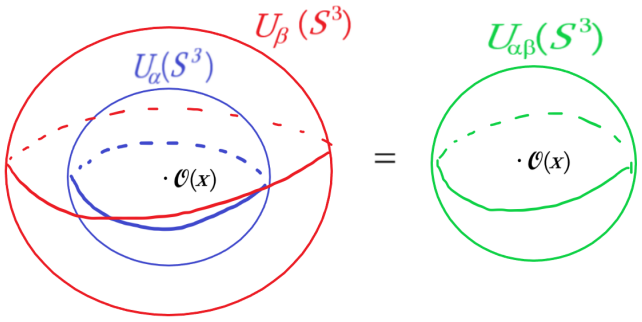


Figure 16: Fusing two symmetry operators to get a third operator compatible with the group multiplication in $U(1)$.

In the case when we act on the operator by enclosing with a sphere, $\#(S^{d-1}, x)$ measures whether the point x is within the space enclosed by the sphere, in which case $\# = 1$, and if x is outside sphere, $\# = 0$. Given this, we make the observation that if we increase the *co-dimension* of U and the *dimension of the support* of $\mathcal{O}$, then we can still have a topological action, which suggests that there are extended objects charged under symmetry operators that is distinct from the $0-$form symmetries we know of. These are called $p-$form symmetries, and the definition is as follows:

$p-$form Symmetry: A $p-$form symmetry is a co-dimension $p+1$ topological operator which is topological, invertible, and acts on an operator with support on a p-dimensional submanifold.

On a co-dimension $p+1$-manifold, we insert a $p-$form topological operator

$$U(M_{d-p-1}), \tag{5.39}$$

which is i) topological

$$U(M_{d-p-1}) = U(M'_{d-p-1}) \tag{5.40}$$

where $M'_{d-p-1} - M_{d-p-1} = \partial M_{d-p}$ analogous to the case of a $0-$form symmetry, and ii) invertible

$$U(M_{d-p-1})U^{-1}(M_{d-p-1}) = 1 = U^{-1}(M_{d-p-1})U(M_{d-p-1}). \tag{5.41}$$

$p-$form symmetry group: Just as in the case of $0-$form symmetries, the $p-$form symmetry transformations form a group, which we denote to be

$$G^{(p)}. \tag{5.42}$$

Then, for any element $g \in G^{(p)}$, we have a corresponding topological operator of codimension-$p+1$ denoted U_g. The group multiplication induces a composition

of these operators

$$U_g(M_{d-p-1})U_{g'}(M_{d-p-1}) = U_{gg'}(M_{d-p-1}), \tag{5.43}$$

which we denote as a fusion rule

$$U_g \otimes U_{g'} = U_{gg'}. \tag{5.44}$$

Given a $0-$form symmetry, we can have any gauge group -mathematically speaking- G acting on the theory, and sometimes positive-definiteness of energy requires the group to be compact, for example in Yang-Mills theory [Sre07]. What about $p-$form symmetry groups $G^{(p)}$? Can they be any group -mathematically speaking-, or do they have extra constraints due to the new complications coming from the topological action? Let us discuss the composition of higher-form symmetry operators in an easy example, then we will discuss its action. In three dimensions, a $1-$form symmetry will have a topological operator of codimension$(3-1-1=1)$ inserted in M_1. Take two operators $U_g(M_1)$ and $U_{g'}(N_1)$ where N_1 is obtained from M_1 by an infinitesimal shift as depicted in figure 17. Along the axis of M_1, we can rotate N_1 by π degrees to take it from one side to the other, and this is a smooth deformation, so does not change the topology.

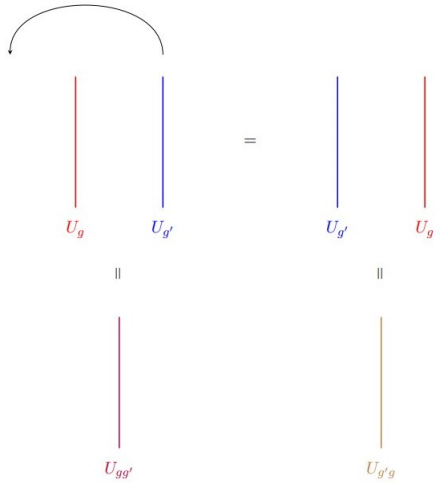


Figure 17: Moving $U_{g'}$ past U_g by a smooth deformation that does not change topology. In both sides, taking the limit where M_1 and N_1 are coincident, we infer that $U_{gg'} = U_{g'g}$. Figure taken from [BBFT$^+$24].

Now, fusing the two topological operators means inserting them on the same manifold M_1. To achieve this in the above figure, we smoothly shift the manifold N_1 onto M_1. In the left-hand-side of 17, N_1 is to the right of M_1, so if the

composed operator acts on an extended object on its right, $\lim_{N_1 \to M_1} U_{g'}(N_1)$ will act on it first, so the limit produces $U_{gg'}(M_1)$. On the right-hand-side, we have the converse. If there is an extended object on the right of the topological operators, then in the limit N_1 is incident to M_1, $U_g(M_1)$ will be the first one to act, so the coincident limit produces $U_{g'g}(M_1)$. Using the fact that these two configurations can be produced via topological deformation, we conclude these are equal to each other, that is

$$U_{gg'} = U_{g'g}. \tag{5.45}$$

Since the fusion of topological operators is induced from a group multiplication on $G^{(p)}$, equation 5.45 implies that multiplication on $G^{(p)}$ must obey

$$g \cdot g' = g' \cdot g. \tag{5.46}$$

Since 5.45 holds for two arbitrary elements $g, g' \in G^{(p)}$, this implies that $G^{(p)}$ is an **Abelian group**.

5.4 1-form Symmetries of Maxwell's Theory

Before moving on to the action of $p-$form symmetries, let us discuss an example from the $U(1)$ gauge theory in $d-$dimensions. A is a $U(1)$ valued gauge field and its field strength is given by $F = dA$, which is a closed $2-$form on spacetime, in the sense that $dF = 0$, also known as Bianchi identity. Recall that when we have a closed $d-p-1$-form, we have a $p-$form symmetry. Therefore, F being a $2-$form, we have a $d-p-1 = 2 \Longrightarrow p = d-3$ form symmetry with charge

$$Q(M_2) = \int_{M_2} F. \tag{5.47}$$

Via exponentiation, we obtain the topological operators of the $(d-3)-$form symmetry

$$U_g^{(m)}(M_2) = \exp\Big(i\alpha \int_{M_2} F\Big), \qquad U(1) \ni g = e^{i\alpha}. \tag{5.48}$$

The superscript m stands for magnetic symmetry, as the charge obtained is closely related to magnetic flux through the surface M_2. In addition to the Bianchi identity $dF = 0$, we have the source-free equations of motion in Maxwell theory, which reads $d * F = 0$. This implies that the $d-2$-form $*F$ is a closed form in spacetime, therefore by integrating it on a codimension$-2(= 1+1)$ manifold, we obtain the charge of a $1-$form symmetry:

$$Q(M_{d-2}) = \int_{M_{d-2}} *F, \tag{5.49}$$

and via exponentiation we obtain the topological operator of what's called the electric $1-$form symmetry

$$U_g^{(e)}(M_{d-2}) = \exp\Big(i\alpha \int_{M_{d-2}} *F\Big), \qquad U(1) \ni g = e^{i\alpha}. \tag{5.50}$$

Thus, in the simplest gauge theory we know of, there are two different higher-form symmetries

$$\begin{aligned} G^{(d-3)} &= U(1) \quad \text{magnetic } (d-3)-\text{form symmetry}, \\ G^{(1)} &= U(1) \quad \text{electric} \quad 1 \quad -\text{form symmetry}. \end{aligned} \tag{5.51}$$

5.5 Action of $p-$form Symmetries

Let us now discuss the topological action of $p-$form symmetries with an Abelian group $G^{(p)}$, where one should remember that the groups associated to $p-$form symmetries with $p \geq 1$ are always Abelian. The symmetries are implemented via $p-$form symmetry operators $U_g^{(p)}(M_{d-p-1})$ of codimension-$p+1$, acting on extended charged operators $\mathcal{O}(M_p)$ living on a manifold of dimension p (otherwise the codimension$-p+1$ manifold M_{d-p-1} cannot enclose the operator $\mathcal{O}(M_p)$ on $d-$dimensional spacetime). We can choose for simplicity M_{d-p-1} to be a codimension-$p+1$ sphere S^{d-p-1}, and if the sphere is topologically tangled with the manifold M_p on which the extended object $\mathcal{O}(M_p)$, it can act on the extended object to transform it as illustrated in figure 18. At the level of the correlation functions, we could write

$$\langle U_g^{(p)}(S^{d-p-1})\mathcal{O}(M_p)\rangle = \phi(g)\langle \mathcal{O}(M_p)\rangle, \tag{5.52}$$

where the linking of S^{d-p-1} and M_p is taken to be 1. Here, the factor $\phi(g)$ is a homomorphism $\phi : G^{(p)} \to U(1)$, which means again that charged operators transform under representations of the group $G^{(p)}$.

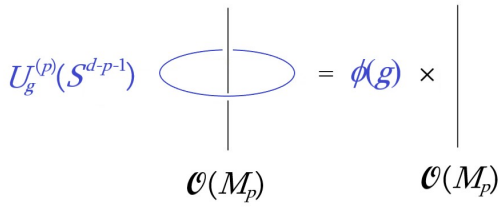


Figure 18: Action of a codimension-$p+1$ topological operator on an extended $p-$operator $\mathcal{O}(M_p)$ is given by topological linking, as in the case of 0-form symmetries.

Of course, given symmetry operators $U_g^{(p)}(S^{d-p-1})$ with different group elements, we can fuse them as illustrated in figure 19. At the level of correlation

functions, one writes

$$\langle U_g^{(p)}(S^{d-p-1})U_h^{(p)}(S^{d-p-1})\cdots\rangle = \langle U_{gh}^{(p)}(S^{d-p-1})\cdots\rangle. \tag{5.53}$$

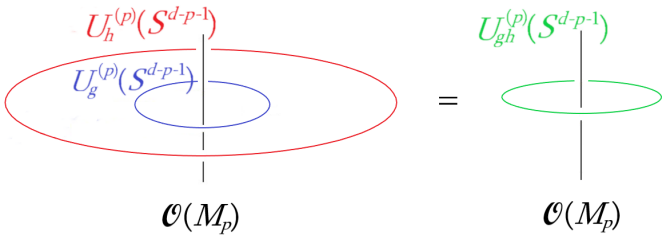


Figure 19: Fusion of higher form symmetry operators.

5.6 Charges of Higher-Form Symmetries

It is a good question to ask, how do we study the irreps of Abelian groups, because understanding irreps would allow us to characterize higher-form charges. The language to understand the irreps of Abelian groups is somewhat straightforward to understand. Given an Abelian group $G^{(p)}$, its irreps are equivalently homomorphisms

$$\phi : G^{(p)} \to U(1), \tag{5.54}$$

called the characters. The set of characters of an Abelian group forms a group itself, with the group product being

$$(\phi\phi')(g) = \phi(g) \times \phi'(g) \in U(1). \tag{5.55}$$

The resulting group is called the Pontryagin dual group of $G^{(p)}$, denoted $\widehat{G}^{(p)}$ [BBFT+24], which is of the form

$$\widehat{G}^{(p)} = \{\text{Homomorphisms } \phi : G^{(p)} \to U(1)\}. \tag{5.56}$$

Note that an element in the Pontryagin group characterizes an irrep, and that charged operators under a $p-$form symmetry transform under some irrep, therefore the charge of the extended object is valued in the group $\widehat{G}^{(p)}$. Let us briefly discuss some prominent examples.

1. When $G^{(p)} = U(1)$, the elements can be parameterized by $g = e^{i\alpha}$ where $\alpha \in [0, 2\pi)$. Possible homomorphisms are some powers of g:

$$\phi(g) = g^q = e^{iq\alpha}. \tag{5.57}$$

From the compactness $\alpha \sim \alpha + 2\pi$, we get the condition $e^{2\pi i q} = 1$, restricting $q \in \mathbb{Z}$. Therefore, the set of homomorphisms are in one-to-one correspondence with the number of integers, so $\widehat{U}(1) = \mathbb{Z}$ for any $p-$form symmetry.

2. For $G^{(p)} = \mathbb{Z}_N$, we parameterize elements by $g = e^{\frac{2\pi i \alpha}{N}}$. To see that this gives the desired group just shift the integer parameter α as $\alpha \to \alpha + N$, and observe that under this shift we end up with the same element, so $\alpha \in \mathbb{Z}_N$. Possible homomorphisms are again powers of g

$$\phi(g) = g^q = e^{\frac{2\pi i q \alpha}{N}}. \tag{5.58}$$

Because α is an integer, shifting $q \to q + N$ does not change the homomorphism, so q is likewise an integer modulo N, $q \in \mathbb{Z}_N$. The possible homomorphisms are therefore isomorphic to $\mathbb{Z}_N$, therefore $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$.

3. Generalizing the case of $\mathbb{Z}_N$, we can say that a finite Abelian group $G^{(p)}$ is self-dual under the Pontryagin dual operation

$$\widehat{G}^{(p)} = G^{(p)}. \tag{5.59}$$

To see this, we can decompose the finite group in terms of "primaries" as

$$G^{(p)} \simeq \prod_{i=1}^{n} \mathbb{Z}_{N_i}, N_i \in \mathbb{N}, \tag{5.60}$$

and since the groups $\mathbb{Z}_N$ are self-dual, we have

$$\widehat{G}^{(p)} = \prod_{i}^{n} \widehat{\mathbb{Z}}_{N_i} = \prod_{i}^{n} \mathbb{Z}_{N_i} = G^{(p)}. \tag{5.61}$$

These three cases are the most prominent higher-form symmetries in the literature (even the third one has not yet made a big appearance). Usually, one has either a $U(1)$ symmetry or a $\mathbb{Z}_N$ symmetry with N usually small numbers like $2, 3$ and maybe 6.

6 Conformal Bootstrap Program

The bootstrap principle dates back to the 1960s with Geoffrey Chew's idea [Che62] of considering only the S-matrix instead of using the machinery of field theory, since we do not observe the constituent quarks and only bound states such as protons and neutrons. For a decade or so this was a mainstream approach, but the developments due to Gerardus 't Hooft on the renormalizability of non-Abelian gauge theories, and the seminal result of David Gross, Frank Wilczek, and independently by David Politzer on the asymptotic freedom of QCD made people realize the usefulness of field theory formalism. The following years witnessed many remarkable non-perturbative developments in QFT,

due to seminal works by Alexander Polyakov, Stanley Mandelstam, and again 't Hooft starting with the introduction of topological solitons and instantons in non-Abelian gauge theories. After that, the bootstrap tools were somewhat shadowed by those of QFT and maybe String Theory. However, in the early 2000s, the bootstrap principle was realized to be extremely useful in Conformal Field Theories (CFT) [DO01, RRTV08], and is now one of the mainstream research fields in formal QFT [C$^+$22, PR22, SD17, PRV19] with the developments of many numerical tools. In this chapter, we will briefly go through the important ideas without proof, but one can consult [SD17] for details.

6.1 Conformal Algebra and Primary Operators

The conformal algebra in a d dimensional Euclidean QFT is given by

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \\
[M_{\mu\nu}, K_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu, \\
[D, P_\mu] &= P_\mu, \\
[D, K_\mu] &= -K_\mu, \\
[K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2M_{\mu\nu},
\end{aligned} \tag{6.1}$$

where $M_{\mu\nu}$ generate rotations, P_μ generate translations, D generate scale transformations, and K_μ generate special conformal transformations which are related to translations via inversion $K \sim I \cdot P \cdot I$. One can show that the part of this algebra without momentum operators is isomorphic to $\mathfrak{so}(d+1,1)$. In fact, one can use the embedding space formalism, where one studies the action of the conformal algebra on $\mathbb{R}^{d+1,1}$ instead of $\mathbb{R}^d$, and one can use this formalism to find the kinematic form of the correlation functions.

In a conformally invariant theory, we clearly have the Lorentz group as a subgroup, so the objects in the spectrum are organized by their behavior under rotational transformations, namely their spins. Given an operator with a representation index a under the rotation group $SO(d)$, we have

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (\mathcal{S}_{\mu\nu})^a{}_b \mathcal{O}^b(0), \tag{6.2}$$

where $(S_{\mu\nu})^a{}_b$ obey the $SO(d)$ algebra and the indices a, b run depending on the dimensionality of the representation space. We can also suppress the representation indices to write $\mathcal{S}_{\mu\nu} \cdot \mathcal{O}(0)$ to have more clarity in notation. So far, this is the standard discussion in a relativistic QFT. For a CFT (more generally in a scale-invariant theory), the main objects of interest are eigenstates of the dilation operator, satisfying

$$[D, \mathcal{O}_\Delta(0)] = \Delta \mathcal{O}_\Delta(0), \tag{6.3}$$

where Δ is the eigenvalue of $\mathcal{O}_\Delta(0)$ under the (adjoint) action of D. Using Baker-Campbell-Hausdorff expansion we can write this as a finite transformation

$$e^{-\lambda D} \mathcal{O}_\Delta(x) e^{\lambda D} = e^{-\lambda\Delta} \mathcal{O}_\Delta(\lambda x), \quad (D : x \mapsto \lambda x). \tag{6.4}$$

Observe that using the conformal algebra, we can see that the momentum operator P and the special conformal transformation generator K act like raising-lowering operators on the conformal dimension of a primary operator. We use the short-hand notation $[Q, \mathcal{O}] \equiv Q\mathcal{O}$, and write

$$DK_\mu \mathcal{O}_\Delta(0) = ([D, K_\mu] + K_\mu D)\mathcal{O}_\Delta = (-K_\mu + K_\mu \Delta)\mathcal{O}_\Delta = (\Delta - 1)K_\mu \mathcal{O}_\Delta(0). \tag{6.5}$$

Similarly for the momentum operator

$$DP_\mu \mathcal{O}_\Delta(0) = ([D, P_\mu] + P_\mu D)\mathcal{O}_\Delta = (P_\mu + P_\mu \Delta)\mathcal{O}_\Delta = (\Delta + 1)P_\mu \mathcal{O}_\Delta(0). \tag{6.6}$$

Now, we note that in a CFT, if we quantize the theory by radial slicing of spacetime where at each constant R foliation, we have a Hilbert space $\mathcal{H}_R$, the unitary operator that will generate "time evolution" from one $\mathcal{H}_R$ to another will be the dilation operator D, hence it acts like the time evolution operator, that is, the Hamiltonian. Since we expect the operator generating time evolutions to be bounded below, there should be a primary operator having the lowest Δ possible, a lowest weight state. We define these primaries via

$$[K_\mu, \mathcal{O}_\Delta(0)] = 0. \tag{6.7}$$

Given a primary, we can act with P_μ to get operators with higher conformal dimensions, called the descendants

$$\begin{aligned} \mathcal{O}_\Delta(0) &\mapsto P_{\mu_1} \cdots P_{\mu_n} \mathcal{O}_\Delta(0), \\ \Delta &\mapsto \Delta + n. \end{aligned} \tag{6.8}$$

In summary, a primary operator along with its quantum numbers of spin and scaling dimension is defined via the relations

$$\begin{aligned} [D, \mathcal{O}_\Delta(0)] &= \Delta \mathcal{O}(0), \\ [M_{\mu\nu}, \mathcal{O}_\Delta(0)] &= \mathcal{S}_{\mu\nu} \cdot \mathcal{O}(0), \\ [K_\mu, \mathcal{O}_\Delta(0)] &= 0. \end{aligned} \tag{6.9}$$

Starting from this data, we can construct a representation of the conformal algebra out of primaries and their descendants, which gives us the spectrum.

Using the conformal symmetry, we can also get some of the correlation functions of the CFT. For example, it can be shown that the 2-point function of scalars is fixed as

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle = \frac{C_{ij} \delta_{\Delta_i \Delta_j}}{x^{2\Delta_i}}. \tag{6.10}$$

Conformal symmetry also fixes the 3-point functions of scalars to be

$$\langle \mathcal{O}_i(x_i) \mathcal{O}_j(x_j) \mathcal{O}_k(x_k) \rangle = \frac{f_{ijk}}{x_{ij}^{\Delta_i + \Delta_j - \Delta_k} x_{jk}^{\Delta_j + \Delta_k - \Delta_i} x_{ki}^{\Delta_k + \Delta_i - \Delta_j}}, \tag{6.11}$$

where f_{ijk} are some constants called the 3-point function coefficients, and $x_{ij} = x_i - x_j$. One can obtain the form of 4-point functions and those of the spinning correlators (insertions of operators with nonzero spin) using the embedding space formalism up to some dynamical constants such as f_{ijk} above.

6.2 Radial Quantization and the Operator-State Correspondence

Given a CFT on $\mathbb{R}^d$, we can slice the space as constant radius spheres, and the canonical quantization will give a Hilbert space $\mathcal{H}$ at each radius, this is the radial quantization picture. To act on $\mathcal{H}$, we need to insert an operator on the sphere S^{d-1}. For example, to implement a symmetry transformation U, we need to insert the operator into a sphere $U(S^{d-1})$. Since a Hilbert space at a given radius r_1 is related to another Hilbert space at radius $r_2 > r_1$ by a scaling transformation, the "Hamiltonian" operator evolving the Hilbert spaces are the dilation operators D now, with eigenstates $\mathcal{O}_\Delta(0)$ of "energy" Δ. A small note is that for this theory to be well-defined, we need D to be a trace-class operator, because for the partition function

$$Z = \mathrm{Tr} e^{-\beta D} \tag{6.12}$$

to be finite, we need D to be trace-class. Hence, given a CFT with trace class operator D, we can understand the entire spectrum via the "energy eigenstates" $\mathcal{O}_\Delta$. We should make this claim more precise. In the path integral formalism, given an operator, we can always create a state and vice-versa. In CFT, the remarkable thing is that there is an *isomorphism* between operators and states so that the path-integral assignment between an arbitrary operator $\mathcal{O}$ with a state $|\mathcal{O}\rangle$ is unique. Although this is not directly relevant to our goal of bootstrapping non-invertible symmetries, the construction of an extended defect in QFT is closely related to the idea of operator-state correspondence and cutting open a manifold to insert states via boundary conditions.

6.3 Operator Product Expansion and CFT Data

Inserting two operators in a ball and performing the path integral, we produce a state on the boundary. Since any state in the spectrum can be expanded over the primaries and descendants, we can decompose the produced state as

$$\mathcal{O}_i(x)\mathcal{O}_j(0)|0\rangle = \sum_{k\in\text{Primaries}} C_{ijk}(x,P)\mathcal{O}_k(0)|0\rangle, \tag{6.13}$$

where $C_{ijk}(x,P)$ are differential operators organize primaries and descendants in the conformal family of $\mathcal{O}_k$. In the path integral, if all the other insertions are outside of the sphere with radius $|x|$, then this is an exact relation from the operator-state correspondence, therefore

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k\in\text{Primaries}} C_{ijk}(x,P)\mathcal{O}_k(0) \quad \text{(OPE)}. \tag{6.14}$$

It is possible to carry out a different radial quantization so that x and 0 points change, so the differential operator $C_{ijk}(x,P)$ changes, but at the end of the day it is more sensible to choose the simpler case while knowing that the generic case makes sense. In general, whenever we can enclose two operators with a sphere

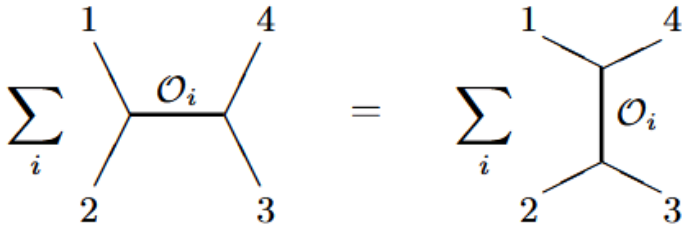


Figure 20: The crossing symmetry.

that has no other operators, we can perform the OPE. Moreover, the insertions can have spin quantum numbers, in which case we have the OPE modified as

$$\mathcal{O}_i^a(x_1)\mathcal{O}_j^b(x_2) = \sum_k (C_{ijk})^{ab}{}_c \mathcal{O}_k^c(x_2), \tag{6.15}$$

where a, b, c are representation indices of the rotation group $SO(d)$, but they do not have to belong to the same representation space, namely they can have different spins. So, by taking OPE of some vectors, we may produce operators of various spins in the OPE.

The main virtue of the OPE is that we can use it to reduce $n-$point correlation functions to sums over certain quantities such as the quantum numbers of $\mathcal{O}_i^a$. In a QFT, all the physical information is encoded in the correlators, so if we know how to compute these correlators we know the QFT. For the more symmetric case of CFT, we in principle have a way to compute these correlators from a set of data, called the CFT data. This is because an $n-$point function with spin indices suppressed

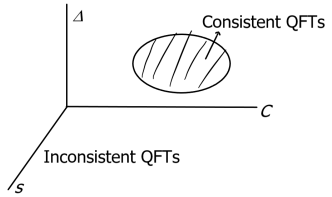
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle \tag{6.16}$$

can be reduced iteratively by utilizing OPE. Schematically we have

$$\begin{aligned}\langle \overbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\cdots\mathcal{O}_n(x_n)\rangle &= \sum_k C_{12k}(x_{12},\partial_2)\langle \overbrace{\mathcal{O}_k(x_2)\cdots}\mathcal{O}_n(x_n)\rangle \\ &= \sum F(s,\Delta,f)\cdots\sum F(s,\Delta,f),\end{aligned} \tag{6.17}$$

where Δ are the conformal dimensions, s are spins, and f_{ijk} are the 3-point function coefficients, and F is some function of these quantities.

Because an arbitrary $n-$point function can be expressed in terms of the data (Δ_i, s_i, f_{ijk}), this set can be viewed as a CFT. Now comes the crux of the bootstrap principle: An arbitrary set of data (Δ_i, s_i, f_{ijk}) need not define a consistent QFT. In fact, there are conditions on this data from requirements



of unitarity, crossing symmetry, and other conditions. For example, unitarity imposes $\Delta \geq \frac{d-2}{2}$ for a scalar operator in a $d \geq 3$ dimensional CFT. On the other hand, the crossing symmetry is a condition we met in RCFT. It states that a 4-point function can be reduced in different contractions, which should give the same result

$$\langle \overbrace{\mathcal{O}_1 \mathcal{O}_2} \overbrace{\mathcal{O}_3 \mathcal{O}_4} \rangle = \langle \overbrace{\mathcal{O}_1 \mathcal{O}_4} \overbrace{\mathcal{O}_2 \mathcal{O}_3} \rangle, \tag{6.18}$$

or diagrammatically as in 20.

The main takeaway is that the CFT data must satisfy some consistency conditions, meaning that if we were to draw a theory space with an axis for each of the data as in figure 6.3, in which each point is a QFT, some of the points would be inconsistent and some of them will be consistent. We obviously want consistent QFTs, and the modern bootstrap program is very good at implementing various conditions numerically to carve out the theory space and find consistent CFTs non-perturbatively. This in turn would lead us to consistent QFTs since from a Wilsonian perspective local QFTs are relevant deformations of CFTs.

7 Non-Invertible Topological Lines

In the language of generalized symmetries, global symmetries are implemented in the quantum theory by topological defects. In the case of 1+1 dimensions, an ordinary (0-form) symmetry G, that can be continuous or discrete, is implemented by line operators in spacetime. Let the space be the circle S^1. A topological line labeled by a group element $g \in G$, $\mathcal{L}_g$ that is wrapped around the spatial circle leads to an action on the Hilbert space $\mathcal{H}$ as in figure 21. We can wrap two lines $\mathcal{L}_{g_1}$ and $\mathcal{L}_{g_2}$ on the circle, and by moving them on top of each other, we get the fusion of lines

$$\mathcal{L}_{g_1} \times \mathcal{L}_{g_2} = \mathcal{L}_{g_1 g_2}, \tag{7.1}$$

as discussed in section 5. Since these elements are associated to a group G, we can find their inverses

$$\mathcal{L}_g^{-1} = \mathcal{L}_{g^{-1}}. \tag{7.2}$$

Given two lines, one can also take their direct sum $\mathcal{L}_{12} \equiv \mathcal{L}_1 + \mathcal{L}_2$. This is defined at the correlation function level as

$$\langle \mathcal{L}_{12} \cdots \rangle = \langle \mathcal{L}_1 \cdots \rangle + \langle \mathcal{L}_2 \cdots \rangle. \tag{7.3}$$

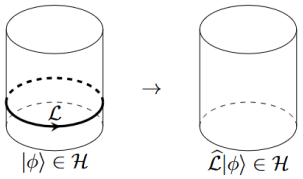


Figure 21: A defect $\mathcal{L}$ wrapped on the spatial circle leads to an action on $\mathcal{H}$. Figure taken from [LS23].

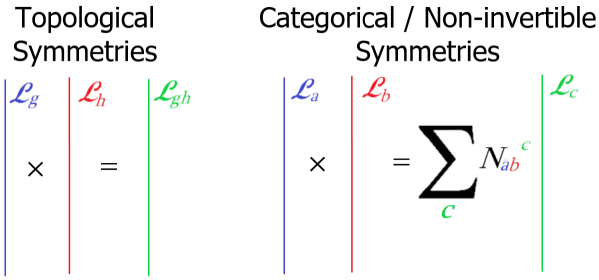


Figure 22: The difference between topological (group-like) symmetries and categorical (non-invertible) symmetries in 2-dimensions, where codimension-1 topological operators are line operators.

We can go further to define linear combinations of lines with non-negative integer coefficients. If a line is not the combination of different lines, or in other words if the combination is such that only one coefficient is 1 and the rest is 0, then we will call that line a simple line.

Having discussed invertible lines, let us divert towards the non-invertible lines. In 1+1d CFTs in general, there are non-invertible lines not associated with a global symmetry. The fusion of such lines is of the form

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \mathcal{L}_c, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}, \tag{7.4}$$

as also demonstrated in figure 22 emphasizing the difference between topological and non-invertible symmetries. this is not a group-like composition, and in particular, $\mathcal{L}$ may not have inverses. In a unitary 1+1d QFT, a finite set of topological line operators form a unitary fusion category $\mathcal{C}$, which is a generalization of the concept of a finite group.

For every simple line $\mathcal{L}$, there is an orientation reversed line $\overline{\mathcal{L}}$, which is the dual object in the fusion category $\mathcal{C}$, such that $\mathcal{L} \times \overline{\mathcal{L}} = \overline{\mathcal{L}} \times \mathcal{L} = 1 + \cdots$, where

1 is the identity object in $\mathcal{C}$. The coefficient $N_{ab}^c \in \mathbb{Z}_{\geq 0}$ in the fusion rule is the dimension of the Hilbert space of topological operators living at the vertex of the operators $\mathcal{L}_a, \mathcal{L}_b, \overline{\mathcal{L}}_c$.

Given a line $\mathcal{L}$, we can define an action $\widehat{\mathcal{L}} : \mathcal{H} \to \mathcal{H}$ on the Hilbert space of local operators by wrapping the line on the spatial circle as in figure 21. For a global symmetry, the action of the topological line gives the symmetry transformation of the field $\widehat{\mathcal{L}}_g : \phi \mapsto g \cdot \phi$. For a line in the category $\mathcal{C}$, the corresponding action $\widehat{\mathcal{L}}$ might not be an invertible action. Namely, it can map a non-trivial operator to the 0 operator.

There is another similar notion of the topological line acting on the local operator ϕ. One can insert a topological line near a local operator, and deform the line such that it moves through the point on which the local operator lives. A characteristic of non-invertible symmetries is to change the local operator into an extended one during this deformation. For example, in the Kramers-Wannier duality of Ising model, the local order operator σ becomes the non-local disorder operator μ upon passing through a duality line, as we will discuss.

A local operator is said to commute with the line if the action of the line on the local operator is trivial, in the sense that if we move the line through the point, the correlation function is unchanged as depicted in figure 23. This is equivalent to the Hilbert space statement $\widehat{\mathcal{L}}|\phi\rangle = \langle\mathcal{L}\rangle|\phi\rangle$.

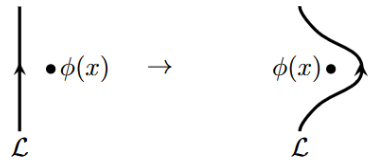


Figure 23: Passing the line operator $\mathcal{L}$ through the local operator $\phi(x)$. If ϕ commutes with the line, then a correlation function does not change as one moves the line through the point x. Figure taken from [LS23].

These topological lines are closely related to the Verlinde operators that we discussed in section 3. One may then ask about the twisted Hilbert spaces with a Verline operator insertions, or in the category language a line insertion. Suppose we insert a line defect extending along the time axis of the cylinder, and intersecting with the spatial circle at a point. This twisted quantization gives $\mathcal{H}_{\mathcal{L}}$, the twisted Hilbert space. Using the conformal equivalece between the cylinder and the complex plane, we can map a state $|\psi\rangle \in \mathcal{H}_{\mathcal{L}}$ to a point operator[3] $\psi(x)$ living at the end of a topological line $\mathcal{L}$ on the plane, as in figure 24. This is a generalization of the idea of state-operator correspondence

[3]Be careful that we are not calling these local operators but point operators, because they live at the end of a line operator.

in ordinary CFT.

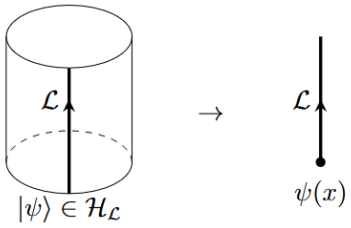


Figure 24: The operator state correspondence in the presence of a line defect extending along the time direction. Figure taken from [LS23].

In this twisted scheme, the spins $s = h - \overline{h}$ of the operators get constraints due to the extended defects. For example, in a bosonic CFT, $s \in \mathbb{Z}$, by the mutual locality (?). By contrast, for a point operator $\psi(x)$, the spin need not be integer, and they are constrained by the category $\mathcal{C}$. This constraint is called the spin selection rule for $\mathcal{H}_{\mathcal{L}}$.

7.1 Example of Non-Invertible Symmetry in Ising CFT

We discussed in 4 the relation between the Ising model in the continuum limit and a Majorana CFT as a minimal model. There, we had the fusion algebra

$$\varepsilon \times \varepsilon = I, \quad \sigma \times \varepsilon = \sigma = \varepsilon \times \sigma, \quad \sigma \times \sigma = I + \varepsilon. \tag{7.5}$$

This is actually a non-invertible fusion rule, because σ line fused with itself does not give a single element, but a combination of two elements. This is an example of a theory with a symmetry that is not captured by a group, but by a category, called the Ising category. This category has the same lines as the $\mathfrak{su}(2)_2$ category, and both Ising and $\mathfrak{su}(2)_2$ categories have the fusion algebra

$$\eta \times \eta = 1, \quad \eta \times \mathcal{N} = \mathcal{N} = \mathcal{N} \times \eta, \quad \mathcal{N} \times \mathcal{N} = 1 + \eta, \tag{7.6}$$

where η is a $\mathbb{Z}_2$ line, and $\mathcal{N}$ is called the Kramers-Wannier duality line, related to the Kramers-Wannier duality of the 2d Ising model relating high temperature to low temperature. Although the Ising and $\mathfrak{su}(2)_2$ categories appear identical in terms of their fusions, they have some differences in the crossing relations [LS23], or mathematically their Frobenius-Schur indicators have opposite signs.

Recall that by inserting the line operators along the time axis, we can get twisted Hilbert spaces. For the case of Ising, we have three Hilbert spaces and corresponding primary operators. For the untwisted case, we have

$$\mathcal{H} : \underbrace{|0,0\rangle}_{1\times 1}, \quad \underbrace{|\frac{1}{2},\frac{1}{2}\rangle}_{1\times \eta}, \quad \underbrace{|\frac{1}{16},\frac{1}{16}\rangle}_{1\times \mathcal{N}}. \tag{7.7}$$

Note that these states correspond respectively to 1, ε, and σ in the Ising CFT we discussed.

For a Hilbert space twisted with $\mathcal{N}$, we need to recall the fusion algebra. $\mathcal{N}$ when fused with 1 gives $\mathcal{N}$ again, and we can insert it for either chirality (left-handed holomorphic or the right-handed anti-holomorphic sector). We can also fuse with η to get $\mathcal{N}$ again in either chirality to get two more states. Note that if we fuse with $\mathcal{N}$, we get a combination of 1 and η, but that will not be linearly independent of the 4 states we counted above. So, we have

$$\mathcal{H}_{\mathcal{N}} : \underbrace{|\frac{1}{16},0\rangle}_{\mathcal{N}\times 1 \text{ (l)}}, \quad \underbrace{|0,\frac{1}{16}\rangle}_{\mathcal{N}\times 1 \text{ (r)}}, \quad \underbrace{|\frac{1}{16},\frac{1}{2}\rangle}_{\mathcal{N}\times \eta \text{ (l)}}, \quad \underbrace{|\frac{1}{2},\frac{1}{16}\rangle}_{\mathcal{N}\times \eta \text{ (r)}}, \tag{7.8}$$

where (l) and (r) denotes which chiral sector we insert the defect. Note that if we look at $\mathcal{N}$ twist to the left sector, we will obtain a state of the form $|0,\frac{1}{16}\rangle + |\frac{1}{2},\frac{1}{16}\rangle$ from the fusion rule, but this is a combination of the states above. The same thing goes for an $\mathcal{N}$ twist in the right sector.

We laslty look at the η twisted Hilbert space. Inserting an η for the left sector makes the $|0,0\rangle$ state into a $|\frac{1}{2},0\rangle$ state and since $\eta\times\eta = 1$ the state $|\frac{1}{2},\frac{1}{2}\rangle$ will turn into $|0,\frac{1}{2}\rangle$. The insertion into the right sector produces $|0,\frac{1}{2}\rangle$ state for $|0,0\rangle$ and $|\frac{1}{2},0\rangle$ state for $|\frac{1}{2},\frac{1}{2}\rangle$, which do not give any independent states. We lastly need to check $\mathcal{N}$. From the fusion, $\eta\times\mathcal{N} = \mathcal{N}$, so the state $|\frac{1}{16},\frac{1}{16}\rangle$ does not see the insertion of η. Therefore, we have

$$\mathcal{H}_{\eta} : \quad \underbrace{|0,\frac{1}{2}\rangle}_{\eta\times 1 \text{ (l) or } \eta\times\eta \text{ (r)}} \quad , \quad \underbrace{|\frac{1}{2},0\rangle}_{\eta\times 1 \text{ (r) or } \eta\times\eta \text{ (l)}} \quad , \quad \underbrace{|\frac{1}{16},\frac{1}{16}\rangle}_{\eta\times\mathcal{N}=\mathcal{N}}. \tag{7.9}$$

The main takeaway is that these are new sectors with new spin selection rules.

8 Symmetry Topological Field Theory Construction

In this section, we will briefly introduce the idea of Symmetry Topological Field Theory (SymTFT) construction. In SymTFT, one tries to understand the generalized global symmetries of a d-dimensional QFT T on M_d with global symmetry G by coupling T to a Topological Quantum Field Theory (TQFT) on $X_{d+1} = M_d \times [0,1]$ such that the boundaries of X_{d+1} separate the dynamics of T and the global symmetry G via the sandwich construction as illustrated in figure 25. Some of the relevant works on this sandwich construction are [FT14, Fre14, KS14, GK21, ABGE$^+$23, FMT22, SN24, BSN23, BS24].

Although this construction is still in progress, there are some cases where it is well understood. For example, for a QFT with a $p-$form global symmetry

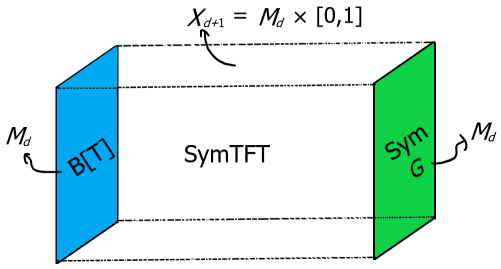


Figure 25: The SymTFT construction is a TQFT on $X_{d+1} = M_d \times [0,1]$ with some boundary conditions at the boundaries, the left (blue) one capturing the dynamics of T, and the right (green) one capturing the global symmetry G of T.

of gauge group $G^{(p)} = \mathbb{Z}_N$, the corresponding SymTFT is the Dijkgraaf-Witten gauge theory [DW90] with action

$$S_{DW} = \frac{N}{2\pi} \int_{X_{d+1}} a_p \wedge db_{d-p}, \tag{8.1}$$

where a_p is a $p-$form $U(1)$ gauge field, and b_{d-p} is a $(d-p)-$form $U(1)$ gauge field. The equations of motion give $da_p = 0$, $db_{d-p} = 0$, and we can define topological objects with these flat gauge fields

$$W_n(M_p) \equiv e^{in \int_{M_p} a_p}, \quad T_m(M_{d-p}) \equiv e^{im \int_{M_{d-p}} b_{d-p}}, \quad n, m \in \mathbb{Z}_N \tag{8.2}$$

and these obey the algebra

$$W_n(M_p) T_m(M_{d-p}) = e^{2\pi i \frac{nm}{N} \times \text{Link}(M_p, M_{d-p})} T_m(M_{d-p}) W_n(M_p). \tag{8.3}$$

We will not discuss the procedure by which to understand how this theory captures the symmetries of the QFT T on M_d, but details can be found at [BS24]. We note that what Witten did in his seminal paper [Wit89] while understanding Chern-Simons theory with the use of Verlinde's fusion rules in 2d RCFT is a prototypical example of this idea. On passing we also point out that since $G^{(p)}$ is a discrete group, the SymTFT can be described as a discrete gauge theory in the cochain formalism. Roughly, one triangulates the spacetime, and assigns a phase at each simplex, with the action

$$S \propto \int_{X_{d+1}} a_p \cup \delta b_{d-p}, \tag{8.4}$$

where a_p is a $G^{(p)}$ valued $p-$cochain (so it assigns an element of $G^{(p)}$ to each simplex), b_{d-p} is a $\widehat{G}^{(p)}$ valued $(d-p)-$cochain, $\cup$ is a cup product such that $G^{(p)}$ valued connection a_p and $\widehat{G}^{(p)}$ valued b_{d-p} combine to give a phase $\mathbb{R}/\mathbb{Z}$, and δ is a differential mapping that maps a $q-$cochain to a $q+1$-cochain. For details one can refer to [BBFT$^+$24].

9 Modular Bootstrap of 2d Non-Invertible Symmetries from the SymTFT

In addition to the $G^{(p)} = \mathbb{Z}_N$ case, the SymTFT for a 2d CFT T with a non-invertible symmetry governed by the fusion category $\mathcal{C}$ is also well understood. The relevant TQFT here is the Turaev-Viro TQFT $\mathsf{TV}_{\mathcal{C}}$, worked out in 90s [TV92].

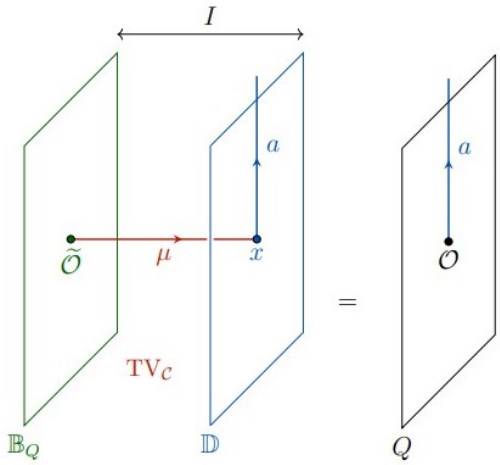


Figure 26: The slab construction for a 2d theory T with categorical global symmetry $\mathcal{C}$. The boundary $\mathbb{B}_{\mathsf{T}}$ has the objects of T, and $\mathbb{D}$ has the categorical lines, connected by the anyon lines $\mu \in Z(\mathcal{C})$ valued in the Drinfeld center.

The $\mathsf{TV}_{\mathcal{C}}$ captures the non-invertible defects of T through the slab construction in 3d with two boundaries as in figure 26. On the right boundary, $\mathbb{D}$, we have the topological Dirichlet boundary condition, and on the right boundary, $\mathbb{B}_{\mathsf{T}}$, we have the 2d CFT as a boundary condition on the TQFT. In this correspondence, a local operator $\widetilde{\mathcal{O}}$ -living in the CFT denoted by T- becomes a point-like operator on the boundary $\mathbb{B}_{\mathsf{T}}$, which is attached to a line operator in the bulk TQFT. These line operators are called anyons, and they are valued in the Drinfeld center of the category $Z(\mathcal{C})$. [4] This construction is such that all the symmetry operations happen on the boundary $\mathbb{D}$, the anyons label the representation of the relevant symmetry algebra, which is called the tube algebra $\mathrm{Tube}(\mathcal{C})$.

Now, for each anyon insertion μ, we can consider the Hilbert space $\mathcal{V}_\mu$ from quantizing $\mathsf{TV}_{\mathcal{C}}$ on a spatial disk with boundary condition $\mathbb{B}_{\mathsf{T}}$ where the anyon

[4] This is reminiscent of the center symmetry of $SU(N)$ Yang-Mills theory where Wilson operators transform under the $\mathbb{Z}_N$ 1-form symmetry, which comes from the center of $SU(N)$. Things are much different here, but there are some similarities.

defect inserted at the origin, as illustrated in figure 27. It is possible to relate the twisted Hilbert spaces $\mathcal{H}_{\mathcal{L}_a} \equiv \mathcal{H}_a$ of T, the 2d CFT with $\mathcal{C}$ global symmetry, and the Hilbert spaces of the $\mathsf{TV}_{\mathcal{C}}$ with an anyon insertion through the formula

$$\mathcal{H}_a = \bigoplus_{\mu} W_a^{\mu} \otimes \mathcal{V}_{\mu}, \tag{9.1}$$

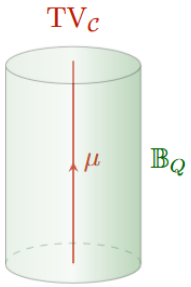


Figure 27: Quantizing the $\mathsf{TV}_{\mathcal{C}}$ on a spatial disk with an anyon defect insertion at the origin. Figure taken from [LS23].

where $W_{\mu}^{a} = \mathrm{Hom}_{\mathcal{C}}(F(\mu), a)$ [5] is the vector space of $\mathbb{D}$-boundary defects x at the intersection of the anyon μ and the object a. Particularly, what this means is that the untwisted Hilbert space $\mathcal{H}$ of the 2d CFT T is given by $\mathcal{V}_1$, the TQFT Hilbert space with trivial anyon insertion.

Let $Z_{\mu}^{3d}(\tau, \overline{\tau})$ denote the partition function for the Hilbert space $\mathcal{V}_{\mu}$. So, it is the partition function on a solid torus of modular parameter τ, boundary condition $\mathbb{B}_{\mathsf{T}}$, and an anyon μ wrapped around the non-contractible cycle of the solid torus. Since the action of $SL(2,\mathbb{Z})$ on the torus leaves it the same, the partition function must behave properly under these modular transformations, namely it should be an irrep of the group $SL(2,\mathbb{Z})$. The group $SL(2,\mathbb{Z})$ has two transformations $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto S\tau \equiv -\frac{1}{\tau}$, so the partition function should obey

$$Z_{\mu}^{3d}(\tau+1, \overline{\tau}+1) = T_{\mu\mu} Z_{\mu}^{3d}(\tau, \overline{\tau}), \quad Z_{\mu}^{3d}(S\tau, \overline{S\tau}) = \sum_{\nu \in Z(\mathcal{C})} S_{\mu\nu} Z_{\nu}^{3d}(\tau, \overline{\tau}), \tag{9.2}$$

where $T_{\mu\mu}$ and $S_{\mu\nu}$ are representation matrices. Note that these could be seen equivalently from a 2d CFT perspective very well. Using the modularity property of the $\mathsf{TV}_{\mathcal{C}}$ partition function, we will now obtain modular bootstrap equations for the spectrum of 2d CFTs with categorical symmetry $\mathcal{C}$.

[5] With $F(\mu)$ a forgetful map $F : Z(\mathcal{C}) \to \mathcal{C}$ defined by transporting parallelly a bulk anyon μ to the boundary and producing a boundary topological line $F(\mu)$.

In addition to modular covariance, we will also need one more property of Z_μ^{3d}, the Virasoro character expansion. Using the $SL(2,\mathbb{Z})$ duality present, we can represent the partition function kinematically as sums over the characters, with some dynamical coefficients

$$Z_\mu^{3d}(\tau,\overline{\tau}) = \sum_{(h,\overline{h})\in\mathcal{H}_\mu} n_{\mu,h\overline{h}}\chi_h(\tau)\overline{\chi_{\overline{h}}}(\overline{\tau}), \tag{9.3}$$

where $n_{\mu,h\overline{h}} \in \mathbb{Z}_{\geq 0}$ are non-negative integers by unitarity. Assuming $c > 1$ so that the characters are not degenerate, we have

$$\chi_h(\tau) = \frac{q^{(h-c/24)}}{\eta(\tau)}, \quad (h \neq 0) \tag{9.4}$$

and for the vacuum character $h = 0$

$$\chi_0(\tau) = \frac{q^{h-c/24}}{\eta(\tau)}(1-q), \tag{9.5}$$

where we defined the parameter $q = e^{2\pi i\tau}$ as familiar from 2d CFT. Now we write the modular covariance and modular expansion together

$$\begin{aligned} Z_\mu^{3d}(\tau,\overline{\tau}) &= \sum_\nu S^{-1}_{\mu\nu} Z_\nu^{3d}(S\tau,\overline{S\tau}) = \sum_\nu \sum_{h,\overline{h}} n_{\nu,h\overline{h}}\ S^{-1}_{\mu\nu}\chi_h(S\tau)\overline{\chi_{\overline{h}}}(\overline{S\tau}), \\ Z_\mu^{3d}(\tau,\overline{\tau}) &= \sum_{h,\overline{h}} n_{\mu,h\overline{h}}\chi_h(\tau)\overline{\chi_{\overline{h}}}(\overline{\tau}) = \sum_\nu \sum_{h,\overline{h}} n_{\nu,h\overline{h}}\ \mathrm{id}_{\mu\nu}\chi_h(\tau)\overline{\chi_{\overline{h}}}(\overline{\tau}), \end{aligned} \tag{9.6}$$

and subtract the two equations, yielding the modular bootstrap equation

$$0 = \sum_\nu \sum_{h,\overline{h}} n_{\nu,h\overline{h}} X_{\mu\nu,h\overline{h}}(S\tau,\overline{S\tau}), \quad \forall \mu \in Z(\mathcal{C}), \tag{9.7}$$

with the matrix X defined as

$$X_{\mu\nu,h\overline{h}}(S\tau,\overline{S\tau}) = \mathrm{id}_{\mu\nu}\chi_h(\tau)\overline{\chi_{\overline{h}}}(\overline{\tau}) - S^{-1}_{\mu\nu}\chi_h(S\tau)\overline{\chi_{\overline{h}}}(S\overline{\tau}). \tag{9.8}$$

Equivalently, we can write this modular bootstrap equation exactly as in [LS23] by multiplying both sides with $-S_{\mu\nu}$ which changes X as

$$X_{\mu\nu,h\overline{h}}(S\tau,\overline{S\tau}) = \mathrm{id}_{\mu\nu}\chi_h(S\tau)\overline{\chi_{\overline{h}}}(\overline{S\tau}) - S_{\mu\nu}\chi_h(\tau)\overline{\chi_{\overline{h}}}(\overline{\tau}). \tag{9.9}$$

One can run the numerical bootstrap on this equation now as was done in [LS23] using the linear functional method. The idea is to act on the equation $0 = (\sum n \cdot X)_\mu$ with a functional

$$\alpha_\mu = \sum_{m=0,n=0}^{\Lambda} \alpha_\mu^{m,n}\partial_\tau^m\partial_{\overline{\tau}}^n\Big|_{\tau=i=-\overline{\tau}}, \quad \alpha_\mu^{m,n} \in \mathbb{R}, \tag{9.10}$$

yielding

$$0 = \sum_{\mu,\nu} \sum_{h,\overline{h}} n_{\nu,h\overline{h}} \alpha_\mu[X_{\mu\nu,h\overline{h}}(S\tau, \overline{S\tau})]. \tag{9.11}$$

The way to run the numerics on this equation is to recall that n are non-negative integers, so in order for the above equation to hold, we must have $\alpha_\mu[X]$ negative at some points. One selects some putative support $\mathcal{P}$ for $(h, \overline{h})$ and checks whether this spectrum satisfies the above equation, if not it is ruled out. Implementing this idea into an algorithm produces various constraints for the spectrum of 2d CFTs with categorical symmetries.

In [LS23], the numerical bootstrap is done for three categories, the Fibonacci category, the Ising category, and the $\mathfrak{su}(2)_2$ category. We are interested in the bounds for the Ising CFT, which are given in 28. In the figures, the bounds on the spectrum with various anyon insertions and defect Hilbert spaces are presented for scalar primary operators.

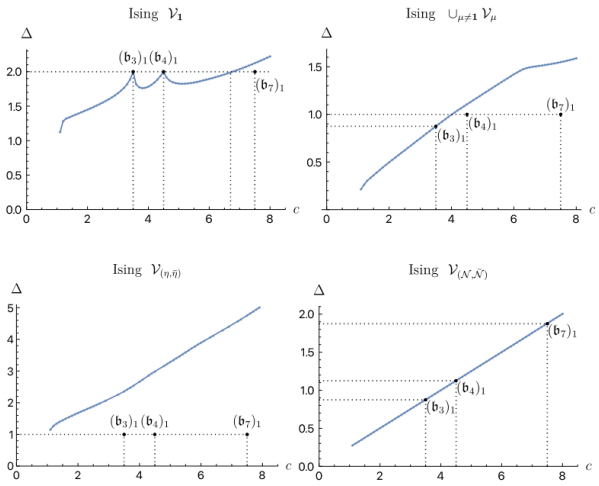


Figure 28: The modular bootstrap bounds on the scalar primaries of the Ising CFT with i) No anyon insertions in the untwisted Hilbert space, ii) Anyon insertions in the untwisted sectors, iii) Twisted Hilbert space with η line insertion, iv) Twisted Hilbert space with $\mathcal{N}$ line insertion.

10 Summary & Future Prospects

In this project, we aimed to investigate the intersections of two research fields, generalized symmetries and the bootstrap. Since both of these research fields are recent, and they are highly involved in various aspects of Quantum Field

Theories, Conformal Field Theories, and Topological Quantum Field Theories, introducing the necessary ingredients resulted in a lengthy report. In the first part, we have reviewed in detail the subjects of 2d CFTs, RCFTs, and the critical Ising model as a minimal model, which set up the stage for the bootstrap of non-invertible symmetries. The second part introduced modern ideas from generalized global symmetries, the conformal bootstrap program, non-invertible (categorical) symmetries in 2d CFTs, and the Symmetry Topological Field Theory (SymTFT) construction. All of these topics are very important for formal QFT and are under intense development, so our discussion was by default incomplete and probably lacking in conceptual clarity.

Despite these shortcomings, we were able to introduce the ideas as they appear in the up-to-date literature, and hopefully, the idea investigated in the paper [LS23], namely how does modular bootstrap constrain non-invertible symmetries, is conveyed in this report. Given that non-invertible symmetries caught interest only recently [BT18, Sha23, SN24], and that as far as we can tell the first paper ever to investigate these non-invertible symmetries from a modular bootstrap point of view is [LS23], it is reasonable to expect that this idea may receive a lot more attention in the next few years following some resolved problems in understanding non-invertible symmetries in generic QFTs, particularly higher dimensions. In 2d, there are many contexts tractable under the modular bootstrap tools, particularly the modular invariant WZW models, admitting non-invertible symmetries. It may be interesting to work these out, although whether the computing power spent on these theories that are already well understood with analytical tools is a valid question, with a likely negative answer. However, following the developments of non-invertible symmetries in QFTs for generic spacetime dimension d, the bootstrap principle may be a useful tool for understanding these theories. It is an amusing question to ponder whether the bootstrap equations can always be obtained from the SymTFT construction as easily as we did in our context, by just subtracting two equations the TQFT partition function satisfies to get an equation appropriate for the bootstrap. With these possible research directions and questions, we conclude our project with the hope that these tools may shed light on our understanding of the concept of symmetry in QFT, the most powerful tool so far, and the concept of theory space, which the bootstrap program aims to carve out with a procedure similar to what was done the previous section.

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