IOP Concise Physics

Physics of the Lorentz Group

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Chapter 1

The Lorentz group and its representations

The Lorentz group starts with a group of four-by-four matrices performing Lorentz transformations on the four-dimensional Minkowski space of (t, z, x, y). The transformation leaves invariant the quantity $(t^2 - z^2 - x^2 - y^2)$. There are three generators of rotations and three boost generators. Thus, the Lorentz group is a six-parameter group.

It was Einstein who observed that this Lorentz group is also applicable to the four-dimensional energy and momentum space of (E, p_z, p_x, p_y). In this way, he was able to derive his Lorentz-covariant energy–momentum relation commonly known as $E = mc^2$. This transformation leaves $(E^2 - p_z^2 - p_x^2 - p_y^2)$ invariant. In other words, the particle mass is a Lorentz-invariant quantity.

1.1 Generators of the Lorentz group

Let us start with rotations applicable to the (z, x, y) coordinates. The four-by-four matrix for this operation is

$$Z(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{1.1}$$

which can be written as

$$Z(\phi) = \exp(-i\phi J_3), \tag{1.2}$$

with

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{1.3}$$

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The matrix J_3 is known as the generator of the rotation around the z-axis. It is not difficult to write the generators of rotations around the x- and y-axes, and they can be written as J_1 and J_2, respectively, with

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

These three rotation generators satisfy the commutation relations

$$\left[J_i, J_j\right] = \mathrm{i}\epsilon_{ijk} J_k. \tag{1.5}$$

The matrix which performs the Lorentz boost along the z-direction is

$$B(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.6}$$

with

$$B(\eta) = \exp\left(-\mathrm{i}\eta K_3\right), \tag{1.7}$$

with the generator

$$K_3 = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.8}$$

It is then possible to write the matrices for the generators K_1 and K_2, as

$$K_1 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

Then

$$\left[J_i, K_j\right] = \mathrm{i}\epsilon_{ijk} K_k \qquad \text{and} \qquad \left[K_i, K_j\right] = -\mathrm{i}\epsilon_{ijk} J_k. \tag{1.10}$$

There are six generators of the Lorentz group and they satisfy the three sets of commutation relations given in (1.5) and (1.10). It is said that the Lie algebra of the Lorentz group consists of these sets of commutation relations.

These commutation relations are invariant under Hermitian conjugation. While the rotation generators are Hermitian, the boost generators are anti-Hermitian

$$J_i^\dagger = J_i, \quad \text{while} \ K_i^\dagger = -K_i. \tag{1.11}$$

Thus, it is possible to construct two four-by-four representations of the Lorentz group, one with K_i and the other with $-K_i$. For this purpose we shall use the notation (Berestetskii 1982, Kim and Noz 1986)

$$\dot{K}_i = -K_i. \tag{1.12}$$

Since there are two representations, transformations with K_i are called the covariant transformations, while those with $\dot{K}_i$ are called contravariant transformations.

1.2 Two-by-two representation of the Lorentz group

It is possible to construct the Lie algebra of the Lorentz group from the three Pauli matrices (Dirac 1945b, Naimark 1954, Kim and Noz 1986, Başkal *et al* 2014). Let us define

$$J_i = \frac{1}{2}\sigma_i \quad \text{and} \quad K_i = \frac{\mathrm{i}}{2}\sigma_i, \tag{1.13}$$

These two-by-two matrices satisfy the Lie algebra of the Lorentz group given in (1.5) and (1.10).

These generators will lead to a two-by-two matrix of the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{1.14}$$

with four complex matrix elements, thus eight real parameters. Since its determinant is fixed and is equal to one, there are six independent parameters. This six-parameter group is commonly called $SL(2, c)$. Since the Lorentz group has six generators, this two-by-two matrix can serve as a representation of the Lorentz group. It is said in the literature that $SL(2, c)$ serves as the covering group for the Lorentz group.

For each G-matrix of $SL(2, c)$, there exists one four-by-four Lorentz transformation matrix. We can start with the Minkowskian four-vector (t, z, x, y) written as

$$X = \begin{pmatrix} t + z & x - \mathrm{i}y \\ x + \mathrm{i}y & t - z \end{pmatrix}, \tag{1.15}$$

whose determinant is

$$t^2 - z^2 - x^2 - y^2. \tag{1.16}$$

The correspondence between the two-by-two and four-by-four representations of the Lorentz group along with the generators are given in table 1.1. These representations can be used for coordinate or momentum transformations, as well as other four-vector quantities such as electromagnetic four-potentials. We can now consider the transformation

$$X' = G\ X\ G^{\dagger}, \tag{1.17}$$

Table 1.1. Two-by-two and four-by-four representations of the Lorentz group.

Generators	Two-by-two	Four-by-four
$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \exp(\mathrm{i}\phi/2) & 0 \\ 0 & \exp(-\mathrm{i}\phi/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}$
$K_3 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$	$\begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}$	$\begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & \mathrm{i}\sin(\theta/2) \\ \mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$
$K_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh\lambda & 0 & \sinh\lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\lambda & 0 & \cosh\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$J_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2 & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$K_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & -\mathrm{i}\sinh(\lambda/2) \\ \mathrm{i}\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}$

The transformation of (1.17) can be explicitly written as

$$\begin{pmatrix} t' + z' & x' - \mathrm{i}y' \\ x' + \mathrm{i}y' & t' - z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t + z & x - \mathrm{i}y \\ x + \mathrm{i}y & t - z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{1.18}$$

We can now translate this formula into

$$\begin{pmatrix} t' + z' \\ t' - z' \\ x' - \mathrm{i}y' \\ x' + \mathrm{i}y' \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha & \gamma^*\beta & \gamma^*\alpha & \alpha^*\beta \\ \beta^*\gamma & \delta^*\delta & \delta^*\gamma & \beta^*\delta \\ \beta^*\alpha & \delta^*\alpha & \beta^*\beta & \delta^*\beta \\ \alpha^*\gamma & \gamma^*\gamma & \alpha^*\delta & \gamma^*\delta \end{pmatrix} \begin{pmatrix} t + z \\ t - z \\ x - \mathrm{i}y \\ x + \mathrm{i}y \end{pmatrix}. \tag{1.19}$$

This then leads to

$$\begin{pmatrix} t' \\ z' \\ x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} t' + z' \\ t' - z' \\ x' - \mathrm{i}y' \\ x' + \mathrm{i}y' \end{pmatrix}. \tag{1.20}$$

It is important to note that the transformation of (1.17) is not a similarity transformation. In the $SL(2, c)$ regime, not all the matrices are Hermitian (Başkal *et al* 2014).

Likewise, the two-by-two matrix for the four-momentum of the particle takes the form

$$P = \begin{pmatrix} p_0 + p_z & p_x - \mathrm{i}p_y \\ p_x + \mathrm{i}p_y & p_0 - p_z \end{pmatrix} \tag{1.21}$$

with $p_0 = \sqrt{m^2 + p_z^2 + p_x^2 + p_2^2}$. The transformation of this matrix takes the same form as that for space–time given in (1.17) and (1.18). The determinant of this matrix is m^2 and remains invariant under Lorentz transformations. The explicit form of the transformation is

$$P' = G\ P\ G^{\dagger} = \begin{pmatrix} p_0' + p_z' & p_x' - \mathrm{i}p_y' \\ p_x' + \mathrm{i}p_y' & p_0' - p_z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} p_0 + p_z & p_x - \mathrm{i}p_y \\ p_x + \mathrm{i}p_y & p_0 - p_z \end{pmatrix}\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{1.22}$$

1.3 Representations based on harmonic oscillators

The matrix representations in the previous section are primarily for coordinate transformations. The question then is how can we transform functions. This problem has a stormy history. For plane waves, the form

$$\exp(\mathrm{i}p \cdot x) \tag{1.23}$$

is widely used in the literature. Since

$$p \cdot x = Et - p_x x - p_y y - p_z z$$

is a Lorentz-invariant quantity, there are no problems from the mathematical point of view.

However, for standing waves, we have to consider boundary conditions. The issue is then how to transform these conditions. One way to circumvent this difficulty is to study harmonic oscillators with built-in boundary conditions.

Indeed, Dirac (1945a, 1963), Yukawa (1953) and Feynman *et al* (1971) struggled with this problem using harmonic oscillator wave functions. Later, it was shown to be possible to construct the representation of the Poincaré group for relativistic extended particles based on harmonic oscillators (Kim *et al* 1979, Kim and Noz 1986). This representation serves a useful purpose in understanding high-speed hadrons. We shall discuss these problems systematically in chapters 5 and 6.

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