LOG BOOK: (DRAFT) Scattering Amplitudes

Novel new methods of calculation, using BCFW recursion

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Contents

1 Introduction

- A definition of Scattering amplitudes
- Why are SAs used?
-

In Quantum Field Theory (QFT), a scattering amplitude is the mechanism for measuring and modelling the properties of fundamental particles and how they interact.

In order to begin understanding a scattering amplitude, we must first understand the rules by which we can describe an individual particles motion and then build up a langauge for how these motions interact in a system of particles. We open here by exploring current methods available to us, in the forms of mathematical formalisms, and how we might use certain properties to expand on this knowledge, to

certain properties when joining one or more individual things, in order to represent systems, or groups of particles, and how they interact.

We begin by mapping the energy and momentum associated with each entering particle to exiting particles while preserving Einsteins energy-momentum relationship of:

$$E^2 = (\vec{p}c)^2 + (mc^2)^2 \tag{1.1}$$

currently available to us for exploring interactions from their fundamental rules in physical systems and building up the language we need in order to model how they interact. From this, we explore further formalisms which

form of relativistic interactions, which themselves have been bootstrapped from classical mechanics.

setting out the notation for which we will use throughout the duration of this paper, in order to be able to concisely explain

In order to take into account relativistic effects, we begin here by working through the formalism of the Lorentz Group to arrive at the relativistic formalisms for

the a special group called the Lorentz Group,

Let us first take a look at the Lagrangian equation, and how they differ from their relativistic to non-relativistic formulations.

- in order to account for relativistic physics, we need to introduce some changes. These changes are explored in the following:

Traditionally, they have been computed using Feynman diagrams, which set the boundary conditions to the summed values of all entering particles and those exiting within the same interaction, such that they conserve this E-M relation between the entering and exiting particles at the macro level.

The Feynman method makes use of virtual particles, so called because they are gauge variant or **Off-Shell** (or off the mass shell) to keep tally of the energy and momenta of all the paths the particles and interactions make, so that an individual interaction may temporarily break the energy momentum relations creating such a virtual particle with negative mass, or energy (see Hodges 2013; Henn and

Plefka 2014). Although it allows for broken symmetries, each virtual particle must cancel out by the final exiting interaction, ensuring that the system as a whole is conservative and invariant and we are only left with **On-Shell** particles at the boundaries.

This method is flexible as it is gauge variant; a property which is useful for simple particle interactions, however it becomes problematic for computing interactions involving numerous particles as the complexity of computation quickly grows Bern, Dixon, and Kosower (2012). (For example a quark-quark interaction might produce more than one gluon or involve more than one virtual-particle loop, or both. The calculations quickly become unmanageable.) The scattering amplitude on the other-hand is gauge invariant and only knows about on shell degrees of freedom.

Our aim will be to search for new symmetries, using the analytic structure found in the texts to probe tree level amplitudes in gauge theory.

The proposed strategy will explore scattering amplitudes between massless particles using restricted on-shell formalism where interactions and particles must be gauge invariant throughout, which is known as the BCFW recursion relation (see Britto et al. 2005, @Britto:2005aa).

These methods build on using a colour decomposition of the gauge theory amplitudes and on expressing them in a spinor helicity basis particularly suited for massless particles (as outlined in Henn and Plefka (2014) and other texts mentioned)

2 Elements of the Lorentz group

Here we will introduce some basic concepts: Groups, Algebras and Representations of the Lorenz Group

2.1 Moving Beyond Galilean Relativity

In order for us to begin understanding interactions between free particles, we will first inspect their classical form and the systems by which such interactions have been be described; using reference systems or co-ordinates. This fist step will allow us to set a time and spatial dimensions of the system, and allow us to compare two systems in relation to one another, and measure the change in this relation as one of the systems evolves over time in relation to the other.

2.1.1 The changes in the Lagrangian L and Hamiltonian H

$$\mathsf{L}_{\mathsf{NR}} = \frac{1}{2}m\dot{\vec{x}}^2 \tag{2.1}$$

$$\rightarrow \mathsf{L}_{\mathsf{Rel}} = -m\sqrt{1-\dot{\vec{x}}^2} \tag{2.2}$$

$$H_{\mathsf{NR}} = \frac{\vec{p}^2}{2m} + V(x) \tag{2.3}$$

$$\rightarrow H_{Rel} = (something) \tag{2.4}$$

To calculate this relation, we inspect the Lagrangian relation between its two component measurables (in an unspecific basis, these can be any two variables $\vec{p}, \vec{q}$, but to relate them to the classical equations involving momentum and position, we will be calling them $\vec{p}, \vec{x}$)

$$p \equiv \frac{\partial \mathsf{L}}{\partial \dot{\vec{x}}} = \frac{m\dot{\vec{x}}}{\sqrt{1-\dot{\vec{x}}^2}} \tag{2.5}$$

2.1.2 The time evolution of the system

Where

$$H = H(\vec{q}, \vec{p}, t) \tag{2.6}$$

$$\dot{\vec{p}} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \qquad = -\frac{\partial H}{\partial \vec{q}} \tag{2.7}$$

$$\dot{\vec{q}} = \frac{\mathrm{d}\vec{q}}{\mathrm{d}t} \qquad = +\frac{\partial H}{\partial \vec{p}} \tag{2.8}$$

Thus leaving us with the following set of equations:

$$H = T + V \qquad = \sum_i \dot{\vec{q}}^i \frac{\partial L}{\partial \dot{\vec{q}}^i} - L = \sum_i \dot{\vec{q}}^i p_i - L \tag{2.9}$$

where

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}^i} \tag{2.10}$$

leading to

$$\hat{H} = m\dot{\vec{x}}^2 - (-m\sqrt{1-\dot{\vec{x}}^2}) \tag{2.11}$$
$$= \frac{m\dot{\vec{x}}}{\sqrt{1-\dot{\vec{x}}^2}}\dot{\vec{x}} + m\sqrt{1-\dot{\vec{x}}^2} \tag{2.12}$$
$$= \frac{\cancel{m\dot{\vec{x}}^2} + m - \cancel{m\dot{\vec{x}}^2}}{\sqrt{1-\dot{\vec{x}}^2}} \tag{2.13}$$
$$= \frac{m}{\sqrt{1-\dot{\vec{x}}^2}} \tag{2.14}$$

Using the approximation we showed above, we can show that

$$H \to \frac{\vec{p}^2}{2m} = \frac{m\dot{\vec{x}}^2}{2} \tag{2.15}$$

For small velocities, where $\vec{x} \ll 1$, we can say:

$$\sqrt{1-\dot{\vec{x}}^2} \approx 1 - \frac{\dot{\vec{x}}^2}{2} + ... [\quad \text{Limit} \quad] \tag{2.16}$$

2.1.3 Einstein Equation

Where we arrive at Einsteins formula, when setting c = 1

$$H = E = \frac{m}{\sqrt{1-\dot{\vec{x}}^2}} \tag{2.17}$$

This explains the principle of relatively according to Einsteins equations, and allows us to begin setting up the formalism for representing interactions which include relativistic properties.

2.1.4 Principle of relativity

(Classical theory of fields, Landau and Lifshits (1975)) Inertial reference frame in which a body moves in respect to and without any other force enacting on it, proceeds at constant velocity.

2.1.5 Reference frames:

Two inertial reference frames are related by:

$$\Delta x = x - x' \tag{2.18}$$

where $\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \to x$ and x' have same inertial property.

2.1.6 Intervals:

Event: described by the place where it occurred and times

$$e_1 = [t, x, y, z] \tag{2.19}$$

$$\text{worldline } = [t \to t', f(x,y,z) - f'(x,y,z)] \tag{2.20}$$

2.2 Transformations:

2.2.1 Lorenz Transformations

Why $x_{\dot{\alpha}\alpha}$ What is this object?

For the Lorenz transformation:

Definition 2.2.1: Lorenz Transformations

$$P^\mu \to \Lambda^\mu{}_\nu p^\nu \equiv p'^\nu \tag{2.21}$$

Expanding on this, the can arrive at the following explanation:

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \lambda_{\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^2 \end{pmatrix} \tag{2.22}$$

$$P_{\alpha\dot{\alpha}} = \rho^\mu \sigma_{\mu\alpha\dot{\alpha}} \to MPM^\dagger \tag{2.23}$$

Using $P^\mu \rho_\mu = \det p$, the following can be obtained:

$$\det(MpM^\dagger) = \det(p) \tag{2.24}$$
$$\det(M)\det(p)\det(M^\dagger) = \det(p) \tag{2.25}$$
$$|\det(M)| = 1 \tag{2.26}$$

$$\underbrace{SO(1,3)}_{\text{group}} \rightarrow \underbrace{SL(2,\mathbb{C}) \times SL(2,\mathbb{C})}_{\text{group} \quad \mathbb{Z}_2} \tag{2.27}$$

The Lorenz Group $SO(1,3)$ is isomorephic to $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

A transformation that is a representation of this group is are known as the **Spinor Transofrmations**:

Definition 2.2.2: Spinor Transformations

The spinors transform as follows:

α

$$\Psi'_\alpha = M_\alpha{}^\beta \Psi_\beta \tag{2.28}$$
$$\bar{\Psi}'_{\dot{\alpha}} = M_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\Psi}_\beta \tag{2.29}$$
$$\tag{2.30}$$

Such that $M_\alpha{}^\beta$ belongs to $SL(2,\mathbb{C})$, i.e. with $\det(M) = \mathbb{1}$

To build invariant spinors, we first need invariant tensors. The objects we will be working with are tensors of $\text{rank}\,2$, so having an invariant tensor would give us a way of simplifying our equations, by giving us a principal invariant which equal to the coefficient of the characteristic polynomial (see definition: 2.31)

The principal invariants do not change under rotations of the coordinate system and satisfy the principle of material frame-indifference and any function of the principal invariants is also objective.

Definition 2.2.3: Characteristic polynomials

For A, which is a tensor of $\text{rank}\,2$, the characteristic polynomial equation p:

$$p(\lambda) = \det(A - \lambda\mathbb{1}) \tag{2.31}$$

where $\mathbb{1}$ is the identity matrix and $\lambda_i \in \mathbb{C}$ represent the polynomial's eigenvalues.

When working with determinants, a common method of calculation and simplification is the Levi-Civita symbol.

Definition 2.2.4: Determinants

The determinant is a the reduction of a $\text{rank}\,n$ tensor to a $\text{rank}\,0$ or scalar value. It is computed from each of the tensors elements, only if each of the number of elements are equal. For example, for an $n = 2$ ranked tensor, the corresponding matrix A_{ij} where there are $i = j$ elements. The quantity of the determinant is used in understanding certain properties of the linear transformations.

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{2.32}$$

Using the Levi-Civita symbol, we can calculate a determinant as such:

Definition 2.2.5: Levi Civita symbol

The Levi-Civita symbol reduces a tensor of $\text{rank}\,n$ to one of $\text{rank}\,0$ (a scalar). It is denoted by $\epsilon_{a_1,a_2,a_3,\cdots,a_n}$, where ϵ has n subscripts, each of which identifies one of the objects.

Depending on the reference order of its objects, the Levi-Civita symbol is defined to be: - $+1$ if $a_1, a_2, a_3, \cdots, a_n$ represents an even permutation. - -1 if $a_1, a_2, a_3, \cdots, a_n$ represents an odd permutation. - 0 if $a_1, a_2, a_3, \cdots, a_n$ does not represent a permutation (i.e., contains duplicate of any of its entries).

$$\epsilon_{a_1,a_2,a_3,\cdots,a_n} = \begin{cases} +1 & = & \text{if} & a_1, a_2, a_3, \cdots, a_n & \text{even permutation} \\ -1 & = & \text{if} & a_1, a_2, a_3, \cdots, a_n & \text{odd permutation} \\ 0 & = & \text{otherwise} \end{cases} \tag{2.33}$$

For a tensor or $\mathrm{rank}\, 2$, which is our most common use case, this can be reprenented by matrix as such:

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.34}$$

How does this transform with M

$$\epsilon_{\alpha\beta} \to M_{\alpha}{}^{\alpha'} M_{\beta}{}^{\beta'} \epsilon_{\alpha'\beta'} = \det M \epsilon_{\alpha\beta} \tag{2.35}$$

And this is precisely $\epsilon_{\alpha\beta} \det(M) = \epsilon_{\alpha\beta}$ and therefore invariant if $\det M) = 1$

Definition 2.2.6: Spinors

Raising and lowering spinor indices

$$\lambda_{\alpha} = \epsilon_{\alpha\beta}\lambda^{\beta} \qquad \lambda^{\alpha} \equiv \epsilon^{\alpha\beta}\lambda_{\beta}$$
$$\tilde{\lambda}_{\dot{\alpha}}\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\alpha}}\beta \qquad \tilde{\lambda}^{\dot{\alpha}}\alpha \equiv \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha}}\dot{\beta}$$

Proof:

$$\lambda^{\alpha\prime} = \epsilon^{\alpha\beta} M_{\beta}{}^{\gamma} \lambda_{\gamma} \tag{2.36}$$
$$= \underbrace{\epsilon^{\alpha\beta} M_{\beta}{}^{\gamma}}_{\left[(M^T)^{-1}\right]^{\alpha}{}_{\delta}\lambda^{\delta}} \epsilon_{\gamma\delta}\lambda^{\delta} \tag{2.37}$$
$$\to \lambda^{\alpha'} = \lambda^{\delta}\left(M^{-1}\right)_{\delta}{}^{\alpha} \tag{2.38}$$
$$\epsilon^{\alpha\beta} M_{\beta}{}^{\gamma} \epsilon_{\gamma\delta} \stackrel{?}{=} \left(M^{-1}\right)_{\delta}{}^{\alpha} \tag{2.39}$$

Side note:

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \tag{2.40}$$

$$-\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \quad i\sigma_2 \tag{2.41}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.42}$$

Calculating a Spinor, we have a consistency condition:

$$\epsilon^{\alpha\beta}\epsilon_{\beta\rho} \equiv \delta^{\alpha}{}_{\rho} \tag{2.43}$$

2.2.2 Galilean to Poincaré

A velocity transformation

$$\frac{\mathrm{d}}{\mathrm{d}x'^0}(x') = v \tag{2.44}$$

$$\implies \mathrm{d}x' = \gamma(\mathrm{d}x + \beta \mathrm{d}x^0) \tag{2.45}$$

$$\implies \mathrm{d}x'^0 = \gamma(\mathrm{d}x^0 + \beta \mathrm{d}x) \tag{2.46}$$

For small velocities:

$$S = -mc^2 \int \mathrm{d}t \sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}} \tag{2.47}$$

$$\simeq \frac{1 - \dot{\vec{x}}^2}{\left(2c^2\right)^2} \tag{2.48}$$

Using $p^2 = m^2$:

$$E(\dot{\vec{x}}) = \frac{m}{\sqrt{1 - \dot{\vec{x}}^2}} \to m + m\frac{\dot{\vec{x}}^2}{2} \tag{2.49}$$

Claim:

$$(E, \vec{p}) = \quad \text{4-momentum} \tag{2.50}$$
$$= \quad \text{4-velocity} \tag{2.51}$$

There exists these two values

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu \tag{2.52}$$
$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \tag{2.53}$$

To prove that this quantity os a 4-vector:

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \to \frac{\mathrm{d}x^\mu}{\mathrm{d}x} \tag{2.54}$$
$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.55}$$

Lagrangian using the following Taylor expansion:

$$\sqrt{1+\epsilon} \sim 1 + \frac{\epsilon}{2} \tag{2.56}$$
$$\frac{1}{1+\epsilon} \sim 1 - \epsilon \tag{2.57}$$

Combined:

$$\frac{1}{\sqrt{1-\epsilon}} = 1 + \frac{\epsilon}{2} \tag{2.58}$$

(Fundamentally we are searching for a massless particles, where $p^2 = m^2 = 0$)

2.2.3 Pauli Matrices

Definition 2.2.7: Pauli Matrices

$$p^2 = p_\mu p^m u = 0, \quad \text{and} \quad p_0^2 - \vec{p}^2 = 0 \tag{2.59}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.60}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.61}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.62}$$

Squaring each of the pauli matrices produces the identity matrix:

$$\left(\sigma^i\right)^2 = \mathbb{1} \tag{2.63}$$

Proof:

$$\left(\sigma^1\right)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.64}$$

Multiplication $\rightarrow$ identity matrix: there is a recursive relation, when multiplying the i^{th} matrix by the j^{th}:

$$\sigma^i \sigma^j = \delta^{ij}\mathbb{1} + i\epsilon^{ijk}\sigma k \tag{2.65}$$

commutation and anti commutation: They have the following anti/commutator relations

$$\left[\sigma^i, \sigma^j\right] = i\epsilon^{ijk}\sigma^k = \left\{\sigma^j, \sigma^i\right\} \tag{2.66}$$

Trace of multiplication: The trace of two pauli matrices multiplied together

$$\mathrm{Tr}\left(\sigma_i \sigma_j\right) = 2\delta^{ij} \tag{2.67}$$

where

$$\sigma_i \sigma_j = \delta^{ij}\mathbb{1}_2 + i\epsilon^{ijk}\sigma^k \tag{2.68}$$

2.3 Spinor Helicity formalism and Pauli Matrices properties:

2.4 Spinor helicity formalism $\vec{\rho} \to \vec{\rho} \cdot \vec{\sigma}$:

The spinor helicity formalism is a very useful description of scattering amplitudes of massless particles, enabling us to use the on-shell degrees of freedom with respect to the particles momentum and polarisation to measure the scattering amplitude of massless particles of all helicities (gluons, fermions, scalars).

This method also simplifies the scattering amplitude, by rendering the analytic expressions in a form which does which is much more compact then the standard four-vector notation (Henn and Plefka 2014, pp 15)

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{2.69}$$

$$\rho^\mu = 2 \times 2 \quad \text{matrix} \tag{2.70}$$

$$\sigma^0 = \mathbb{1}_2 \tag{2.71}$$

$$\sigma^i = \quad \text{Pauli spin matrices} \tag{2.72}$$

$$p^\mu \sigma_\mu \equiv \rho^0 \sigma_0 + \vec{\rho} \cdot \vec{\sigma} \tag{2.73}$$

where:

$$\det\big(p^\mu \sigma_\mu\big) \equiv \vec{p}^2 \tag{2.74}$$

$$\text{Where, massless particles} \quad p^2 = 0 \tag{2.75}$$

$$(\rho^0)^2 - (\vec{(p)})^2 = 0 \tag{2.76}$$

These relations here set the stage for being able to work with massless particles.

$$(p^\mu \sigma_\mu)_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{2.77}$$

2.4.1 Facts about σ_μ:

When using the Minkowski spacetime metric, we represent as following:

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{1}, \vec{\sigma}) \tag{2.78}$$
$$\bar{\sigma}^{\mu\ \dot{\alpha}\beta} = (\mathbb{1}, -\vec{\sigma}) \tag{2.79}$$
$$\tag{2.80}$$

With the Euclidian metric, we represent as following:

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{1}, i\vec{\sigma}) \tag{2.81}$$
$$\bar{\sigma}^{\mu\ \dot{\alpha}\beta} = (\mathbb{1}, -i\vec{\sigma}) \tag{2.82}$$
$$\tag{2.83}$$

So that:

$$P_{\alpha\dot{\alpha}} = p^\mu \sigma_{\mu\ \alpha\dot{\alpha}} \tag{2.84}$$
$$\bar{P}^{\dot{\alpha}\alpha} = p^\mu \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \tag{2.85}$$

Then $\epsilon_{\alpha\beta}$ similar to Levi-Cevita.

$$\epsilon_{12} = -\epsilon_{21} \tag{2.86}$$

Introducing the baracket notation: $\langle \quad \rangle$ and $[\quad]$

$$\rho^\mu \sigma_\mu = \rho^0 \sigma_0 + \vec{p} \cdot \vec{\sigma} \tag{2.87}$$
$$\rho_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \tag{2.88}$$

2.5 Angular Momentum properties:

$$\vec{J} = \vec{x} \times \vec{p} \tag{2.89}$$
$$= xp_y - yp_x \tag{2.90}$$

Using the a commutator:

$$[J^1, J^2] = [x^2p^3 - x^3p^2, x^3p^1 - x^1p^3] \tag{2.91}$$
$$= (x^2p^3 - x^3p^2)(x^3p^1 - x^1p^3) - (x^3p^1 - x^1p^3)(x^2p^3 - x^3p^2) \tag{2.92}$$

(Only commutator with same index are non-zero)

$$[J_i, J_i] = 0 \tag{2.93}$$
$$\left[J_i, J_j\right] \neq 0 \tag{2.94}$$
$$\tag{2.95}$$

This leads to the formal relation of angular momentum:

$$\left[J^i, J^j\right] = i\hbar J^{ijk}\sigma^k J^k \tag{2.96}$$

This related transformation and sum to Noethers theorem

2.6 Noethers Theorem

$$I = \sum_r \frac{\partial \mathcal{L}}{\partial \dot{q}_r} \delta q_r \tag{2.97}$$

2.7 SU Group:

2.7.1 $SU(2)$

$$SU(2) \to U(N) = N \times Nmatrix \tag{2.98}$$
$$= U^\dagger U = \mathbb{1} \quad \text{Unitary} \tag{2.99}$$

\begin{(SU(N)\end{(} are a special subset of unitary matrices ($\det U = \pm 1$)

Taylor expansion of U:

$$U = e^{iA} \tag{2.100}$$
$$e^x = 1 + \frac{x^2}{2} + \frac{x^3}{3} \dots \tag{2.101}$$

Impose the following conditions:

$$\det\left\{e^{iA}\right\} = \mathbb{1} \implies \det\{M\} = e^{\mathrm{Tr}(\log M)} \tag{2.102}$$
$$= e^{\mathrm{Tr}(\log(\exp(iA)))} \tag{2.103}$$
$$= e^{\mathrm{Tr}(iA)} \tag{2.104}$$
$$= e^{i\,\mathrm{Tr}(A)} \implies \mathrm{Tr}(A) = 0 \tag{2.105}$$

Condition: $U^\dagger U = \mathbb{1}$ Special unitarity: add $\det A = \pm 1 \to SU(N)$

$$U = e^{iA} \tag{2.106}$$
$$A^\dagger = A \quad \text{Unitary, hermitian} \tag{2.107}$$
$$\mathrm{Tr}\, A = 0 \tag{2.108}$$

This implies we can write:

$$U = e^{i\alpha^a \frac{\sigma^a}{2}}, \quad a = (1, 2, 3) \tag{2.109}$$

where:

$$e^{i\alpha^a \frac{\sigma^a}{2}} = \exp\left(\sum_{a=1}^{3} i\alpha^a \frac{\sigma^a}{2}\right) \tag{2.110}$$

2.7.2 Proofs:

$$U^d agger U = \mathbb{1} \tag{2.111}$$

$$(U_1 U_2)^\dagger (U_1 U_2) = \mathbb{1} \tag{2.112}$$

$$U_2^\dagger U_1^\dagger U_1 U_2 = \mathbb{1} \tag{2.113}$$

3 Four vecotrs

Definition 3.0.1: Invariance

Take the following:

$$\lambda^\alpha \mu_\alpha = \langle \lambda\, \mu \rangle = \lambda^\alpha \mu^\beta \epsilon_{\alpha\beta} \tag{3.1}$$

$$\tilde{\lambda}_{\dot{\alpha}} \dot{\alpha} \tilde{\mu}^{\dot{\alpha}} = [\tilde{\lambda}\, \tilde{\mu}] \tag{3.2}$$

Both of these are antisymmetric:

$$[\tilde{\lambda}\, \tilde{\mu}] = \tilde{\mu}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} \alpha = \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \lambda^{\dot{\alpha}} \tag{3.3}$$

$$= \tilde{\mu}^{\dot{\beta}} (-\epsilon_{\dot{\beta}\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}}) \tag{3.4}$$

$$= -\tilde{\lambda}_{\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \tag{3.5}$$

Where $\lambda, \tilde{\lambda}$ are two types of spinors:

$$\text{Undotted:} \quad \lambda^\alpha, \lambda_\alpha \tag{3.6}$$

$$\text{Dotted:} \quad \mu_{\dot{\alpha}}, \mu^{\dot{\alpha}} \tag{3.7}$$

3.1 Dot product:

Definition 3.1.1: Dot Product

In order to do a dot product using this convention:

$$p_1{}^{\dot{\alpha}\alpha} p_{2\,\dot{\alpha}\alpha} = \tilde{\lambda}_1{}^{\dot{\alpha}} \lambda_1{}^{\alpha} \lambda_{2\,\alpha} \tilde{\lambda}_{2\,\dot{\alpha}} \tag{3.8}$$

$$= \langle 1\,2 \rangle [2\,1] = 2(p_1 p_2) \tag{3.9}$$

and

$$\langle 1\,2 \rangle \equiv \lambda_1{}^{\alpha} \lambda_{2\,\alpha} \qquad \lambda^{\alpha} = \epsilon^{\alpha\beta} \lambda_{\beta}$$

$$[1\,2] \equiv \lambda_{1\,\dot{\alpha}} \lambda_2{}^{\dot{\alpha}} \qquad \lambda_{\alpha} = \epsilon_{\alpha\beta} \lambda^{\beta}$$

Hence we define:

$$\lambda^{\alpha} = \epsilon^{\alpha\beta} \lambda_{\beta} \tag{3.10}$$

$$\lambda_{\alpha} = \epsilon_{\alpha\beta} \lambda^{\beta} \tag{3.11}$$

Similar to dotted spinors:

$$\sigma_{\mu\,\alpha\dot{\alpha}} \equiv (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\alpha}} \tag{3.12}$$

$$\sigma_{\mu}{}^{\dot{\alpha}\alpha} \equiv (\mathbb{1}, \vec{\sigma})^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \sigma_{\mu\,\beta\dot{\beta}} \tag{3.13}$$

$$\overline{\sigma}^{\mu\,\dot{\alpha}\alpha} \equiv (\mathbb{1}, -\vec{\sigma})^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\mu\,\beta\dot{\beta}} \tag{3.14}$$

3.2 Polarisation vectors for massless particles

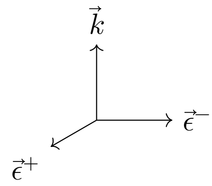
We now examine the way to calculate the polarisation vector for gluons. This process can also be used in the calculation of any particle with similar properties. They have the follwing properties:

- Massless
- Spin 1
- Two states of polarisation: helicity $h = \pm 1$

Definition 3.2.1: Polarisation vector $\vec{\epsilon}$

A polarisation vector $\vec{\epsilon}$ may be found from a momentum vector $\vec{k} =$ such that they are orthogonal as in:

$$\vec{\epsilon}(\vec{k}) \cdot \vec{k} = 0 \tag{3.15}$$



$$\vec{k} = \left|\vec{k}\right|(0,0,1) \tag{3.16}$$
$$\epsilon_L = \frac{1}{\sqrt{2}}(1,i,0) \tag{3.17}$$
$$\epsilon_L = \frac{1}{\sqrt{2}}(1,-i,0) \tag{3.18}$$

In 4D:

$$\epsilon^\mu(\vec{k}^\pm) \tag{3.19}$$

We will use covariant 4-vector notation with μ index to represent in $\alpha\dot{\alpha}$ form:

$$\epsilon^{(+)}_{\alpha\dot{\alpha}}(k) = \sqrt{2}(\text{something } B) \tag{3.20}$$
$$\epsilon^{(-)}_{\alpha\dot{\alpha}}(k) = \sqrt{2}(\text{something } A) \tag{3.21}$$

Here we need to introduce the reference spinor $(\mu, \tilde{\mu})$, such that the corresponding λ or $\tilde{\lambda}$ are not parralell:

$$\tilde{\mu} \nparallel \tilde{\lambda}: \qquad (A) = \frac{\lambda_\alpha \tilde{\mu}_{\dot{\alpha}}}{[\tilde{\lambda}\,\tilde{\mu}]} \tag{3.22}$$

$$\mu \nparallel \lambda: \qquad (B) = \frac{\tilde{\lambda}_{\dot{\alpha}} \dot{\alpha} \mu_\alpha}{\langle \lambda\,\mu \rangle} \tag{3.23}$$

We don't want denominator to vanish such that $\frac{\#}{\langle \lambda\,\lambda \rangle \to 0}$, as this would cause it to divide by zero. So we will use:

$$\mu_\alpha \to \mu^{'} = a\mu_\alpha + b\lambda_\alpha \tag{3.24}$$

Therefore we choose $\epsilon^{(+)}_{\alpha\dot{\alpha}}$ leaving us with:

$$\epsilon^{(+)}_{\alpha\dot{\alpha}} - \frac{\tilde{\lambda}_{\dot{\alpha}}(a\mu_\alpha + b\lambda_\alpha)}{a\langle \lambda\,\mu \rangle + \cancel{b\langle \lambda\,\lambda \rangle}} \tag{3.25}$$

Where this is a gauge freedom:

$$\epsilon^{(+)}_{\alpha\dot{\alpha}} + \#\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} = \epsilon \tag{3.26}$$

3.3 Invariance using Maxwell's Equations of electrodynamics

We can now explore this gauge invariance using Maxwell equations; taking us from a classical form to our new form, while continuing to be gauge invariant. Starting from the classical form of the Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{3.27}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.28}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.29}$$

$$\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t} \tag{3.30}$$

Taking a closer look at equation ~3.28, we can set see that this suggests that there is a vector potential

$$\vec{B} = \left(\vec{\nabla} \times \vec{A}\right) \tag{3.31}$$

The $\vec{B}$ field remains unchanged when the gradient of an arbitrary scalar field $\left(\vec{\nabla}\Lambda\right)$ is added:

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \tag{3.32}$$

Similarly, with eq ~3.29 it follows that:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.33}$$

Which suggests we could have a scalar potential ϕ, which would satisfy:

$$-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} = \vec{E} \tag{3.34}$$

And we can therefore add any arbitrary scalar field's Λ time derivative to the scalar potential ϕ, while keeping the $\vec{E}$ field unchanged.

$$\phi \rightarrow \phi - \frac{\partial \Lambda}{\partial t} \tag{3.35}$$

alternatively, this can be written in a more compact form as:

$$\vec{\nabla} \times \left(E + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0} \tag{3.36}$$

Maxwell's equations can be recast as

$$A^{\mu} = (\phi, \vec{A}) \tag{3.37}$$

Where, using the covariant notation, there is a symmetry in gauge invariance with the field-strength tensor:

$$\partial_{\mu} F^{\mu\nu} = J^{\nu} \tag{3.38}$$

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{3.39}$$

Therefore if we set the following, it will lead to is a symmetry in gauge invariance.

Using

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{3.40}$$

$$(\vec{a} \times \vec{b})^i = \epsilon^{ijk} a^j b^k \tag{3.41}$$

Where ϵ^{ijk} is Levi Civita symbol we define in equation ~2.33.

Combing the above mentioned expansions, we will now find a different way to write the Maxwell equations:

3.3.1 A new method of deriving J, ρ and F

$$\vec{\nabla}(\underbrace{\vec{\nabla} \times A}_{=\vec{B}}) = J\frac{\partial}{\partial t}\left(\underbrace{-\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t}}_{=E}\right) \tag{3.42}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \tag{3.43}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} \tag{3.44}$$

$$= \left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{A} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial A^0}{\partial t}\right) = \vec{J} \tag{3.45}$$

$$\vec{\nabla} \cdot \left(-\vec{\nabla} \cdot \vec{A} - \frac{\partial A}{\partial t}\right) = \rho \tag{3.46}$$

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)A^0 - \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A} + \frac{\partial A^0}{\partial t}\right) = \rho \tag{3.47}$$

This leads us to our new Box operator '$\Box$' which we defined in equation ~4.4 for a four-vector, defined as:

$$\Box A^\mu = \partial^\mu \left(\partial_\nu A^\nu\right) = J^\mu \tag{3.48}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.49}$$

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{3.50}$$

Definition 3.3.1: Gauge redundancy

$$A^\mu \to A^\mu + \partial^\mu \Lambda \tag{3.51}$$

$$F^{\mu\nu} \to F_{\mu\nu} \tag{3.52}$$

4 Spinor helicity formalism (null vectors)

4.1 Contraction using [ ] and ⟨ ⟩ notation

In order to move back and forth between our

$$\langle \lambda\, \chi \rangle \equiv \lambda^\alpha \chi^\beta \epsilon_{\alpha\beta} = \lambda^\alpha \chi_\alpha \tag{4.1}$$

$$\tilde{\lambda}_{\dot\alpha} \tilde{\chi}^{\dot\alpha} = \epsilon_{\dot\alpha\beta} \tilde{\lambda}^{\dot\alpha} \beta \tilde{\lambda}^{\dot\alpha} \alpha \equiv [\tilde{\lambda}\, \tilde{\chi}] \tag{4.2}$$

$$2(p_i \cdot p_j) \equiv {p_i}_{\alpha\dot\alpha} {p_i}^{\dot\alpha\alpha} = \langle i\, j \rangle [j\, i] \tag{4.3}$$

Definition 4.1.1: Box operator

$$\Box \equiv \partial_\mu \partial^\mu \equiv \partial_0 \partial^0 + \partial_i \partial^i = \partial_j^2 - \nabla^2 \tag{4.4}$$

It has the following properties:

- For massless particles: $(\Box + m^2)\, \phi = 0$
- Transform for ∂_μ : $\partial_\mu e^{ikx}$ for massless particles (i.e. where $k^2 = m^2$):

4.1.1 Weyl Spinors and equations

Definition 4.1.2: Weyl Equations

$$i\partial_\mu \bar{\sigma}^{\dot\alpha\alpha} \Psi_\alpha = 0 \tag{4.5}$$

$$i\partial_\mu \sigma_{\alpha\dot\alpha} \tilde{\Psi}_\alpha = 0 \tag{4.6}$$

Where $\Psi_\alpha, \tilde{\Psi}_\alpha$ are Weyl spinors

What do we need to solve $\Psi_\alpha = \xi_\alpha e^{ikx}$:

$$i\partial^{\dot{\alpha}\alpha}\left(\xi_\alpha e^{ikx}\right) = i^2 k^{\dot{\alpha}\alpha}\xi_\alpha e^{ikx} \tag{4.7}$$
$$= -\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}\alpha\xi_\alpha e^{ikx} \tag{4.8}$$

Definition 4.1.3: Spin operator $\hat{h}$

$$\hat{h} = -\frac{1}{2}\lambda\frac{\partial}{\partial\lambda} + \frac{1}{2}\frac{\partial}{\partial\tilde{\lambda}}\tilde{\lambda} \tag{4.9}$$

4.2 Creating Amplitudes for a three particle interaction:

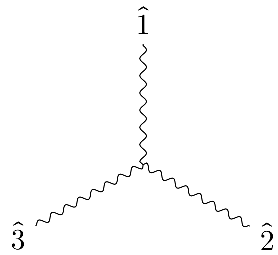


Figure 1: Feynman diagram of a three particle interaction

$$A(1,2,3) \approx \langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle \tag{4.10}$$

Mixing brackets for three particles doesn't work, for example a three particle "Scattering":

$$\langle 1\,2\rangle[2\,3]\langle 3\,1\rangle \to p_1 + p_2 + p_3 = 0 \tag{4.11}$$
$$\to p_1 = -(p_2 + p_3) \tag{4.12}$$

As we are looking at massless particles; here our equation goes:

$$
\begin{aligned}
p_1^2 &= 0 && (4.13)\\
&= p_2^2 + p_3^2 + 2(p_2 \cdot p_3) && (4.14)\\
&= 2(p_2 \cdot p_3) && (4.15)\\
\implies p_1 \cdot p_2 = p_2 \cdot p_3 &= p_3 \cdot p_1 = 0 && (4.16)
\end{aligned}
$$

And complex momenta:

$$
\begin{aligned}
2p_1 \cdot p_2 &= 2p_2 \cdot p_3 = 2p_3 \cdot p_1 && (4.17)\\
\implies \langle 1\,2\rangle[2\,1] &= 0 && (4.18)\\
\langle 2\,3\rangle[3\,1] &= 0 && (4.19)\\
\langle 3\,1\rangle[1\,3] &= 0 && (4.20)\\
& && (4.21)
\end{aligned}
$$

(Henn and Plefka 2014, pp 17, section 1.11 Vanishing Tree Amplitudes)

We will use this and the following to explore a complex momenta:

$$
\begin{aligned}
\text{When} \quad \langle 1\,2\rangle &= 0\lambda_1 && \parallel \lambda_2 && (4.22)\\
\lambda_{1_\alpha} &= a\lambda_{2_\alpha} && && (4.23)\\
A(1,2,3) &= \begin{cases} \overbrace{\langle 1\,2\rangle}^{a_1}\, \overbrace{\langle 2\,3\rangle}^{a_2}\, \overbrace{\langle 3\,1\rangle}^{a_3} \\ \underbrace{[1\,2]}_{\tilde{a_1}}\, \underbrace{[2\,3]}_{\tilde{a_3}}\, \underbrace{[3\,1]}_{\tilde{a_3}} \end{cases} && && (4.24)
\end{aligned}
$$

4.3 Examples

4.3.1 Simple using $[\quad]$:

$$\frac{[1\,2]^3}{[2\,3][3\,1]} = \qquad 1 : +\frac{1}{2}(3-1) = +1 \tag{4.25}$$
$$2 : +\frac{1}{2}(3-1) = +1 \tag{4.26}$$
$$3 : +\frac{1}{2}(0-2) = -1 \tag{4.27}$$
$$\tag{4.28}$$

4.3.2 Simple using $\langle \quad \rangle$:

$$\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,1\rangle} = \qquad 1 : -\frac{1}{2}(3-1) = -1 \tag{4.29}$$
$$2 : -\frac{1}{2}(3-1) = -1 \tag{4.30}$$
$$3 : -\frac{1}{2}(0-2) = +1 \tag{4.31}$$
$$\tag{4.32}$$

Note:

$$\langle i\,j\rangle = -\frac{1}{2}(\quad U_i \quad - \quad D_j \quad) \tag{4.33}$$
$$[i\,j] = +\frac{1}{2}(\quad U_i \quad - \quad D_j \quad) \tag{4.34}$$

5 Spinors and transformations

5.1 Symmetries

- Spinor Helicity formalism -> Makes simplicity manifest
- Dynamics: ie Why amplitudes are simple.
- New methods are simple
- what is the dot product in terms of spinors.

---

1. Einstein Equation $E^2 = (pc)^2 + (mc^2)^2$
2. Invariant quantities in relativity. What is the set of linear transformations that make the **metric** invariant?

   - $\eta \to \Lambda\eta\Lambda = \eta$
   - Lorenz group definitions.

$$\overbrace{\begin{pmatrix} a & \\ & b \end{pmatrix}}^{+}\overbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}^{sp}\overbrace{\begin{pmatrix} a & \\ & b \end{pmatrix}}^{+} \tag{5.1}$$

   - Transformation of velocity $\to$ conclusion:

$$v' = \frac{\mathrm{d}x + \beta\,\mathrm{d}x_0}{\mathrm{d}x_0 + \beta\,\mathrm{d}x} \tag{5.2}$$

   (Galilean transformation)

3. Lorenz transformation $SO(1,3) \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Arriving at Lorenz invariant transformation $\langle 1\,\dot{2}\rangle \equiv {\lambda_1}^{\alpha}\lambda_{2\dot{\alpha}}$

6 Little group and weights

7 Determination of three-point amplitudes of massless particles

7.1 Determining three point amplitudes

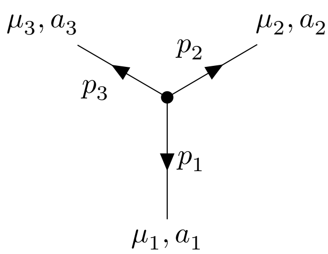


Figure 2

Vertex, using a cyclic rotation:

$$iV^{a_1,a_2,a_3}_{\mu_1,\mu_2,\mu_3} = ig\left[(p_1-p_2)_{\mu_3}\eta_{\mu_1\mu_2} + (p_2-p_3)_{\mu_1}\eta_{\mu_2\mu_3} + (p_3-p_1)_{\mu_2}\eta_{\mu_3\mu_1}\right] f^{a_1a_2a_3} \tag{7.1}$$
$$= f^{a_1a_2a_3}V_{\mu_1\mu_2\mu_3} \tag{7.2}$$

Amplitude, using the polarisation vector $\epsilon_{\alpha\dot{\alpha}}$:

$$V_{\mu_1\mu_2\mu_3} = \epsilon^{\mu_1}_{(1)}\epsilon^{\mu_2}_{(2)}\epsilon^{\mu_3}_{(3)}\Big|_{p_1^2=p_2^2=p_3^2=0} = \left.\begin{aligned} &\left[(p_1-p_2)\cdot\epsilon_{(3)}\right]\left(\epsilon_{(1)}\cdot\epsilon_{(2)}\right) + \\ &\left[(p_2-p_3)\cdot\epsilon_{(1)}\right]\left(\epsilon_{(2)}\cdot\epsilon_{(3)}\right) + \\ &\left[(p_3-p_1)\cdot\epsilon_{(2)}\right]\left(\epsilon_{(3)}\cdot\epsilon_{(1)}\right)\end{aligned}\right\}[A] \tag{7.3}$$

- Using this method, we will always have momentum conservation such that $p_1+p_2+p_3=0$
- Momentum is always transverse to polarisation $p\cdot\epsilon(p)=0$

Transverse:

$$\epsilon(p) \approx \frac{\lambda_\alpha\tilde{\eta}_{\dot{\alpha}}}{[\lambda\,\eta]} = 0 \tag{7.4}$$

Simplifying $[A]$:

$$-p_2 = p_1 + p_3 \tag{7.5}$$
$$= (2p_1+p_3)\cdot\epsilon_3 \tag{7.6}$$
$$= 2(p_1\cdot\epsilon_3) \tag{7.7}$$

Likewise, the remaining p_i can be found by cyclicly permutation.

Definition 7.1.1: Polarisation Vector, reference spinor

$$\epsilon^+_{\alpha\dot{\alpha}} \equiv \frac{\xi_\alpha\tilde{k}_{\dot{\alpha}}}{\langle\xi\,k\rangle}\sqrt{2} \tag{7.8}$$
$$\epsilon^-_{\alpha\dot{\alpha}} \equiv -\frac{\tilde{\xi}_{\dot{\alpha}}k_\alpha}{[\tilde{\xi}\,\tilde{k}]}\sqrt{2} \tag{7.9}$$
$$\tag{7.10}$$

We use this definition to simplify the following case:

$$\epsilon_{1\,\alpha\dot{\alpha}} \cdot \epsilon_2{}^{\alpha\dot{\alpha}} = 2(\epsilon_1 \cdot \epsilon_2) \tag{7.11}$$
$$= \left(-\sqrt{2}\right)^2 \frac{1_{\dot{\alpha}} k_\alpha}{\langle 1\, k\rangle} \frac{2^{\dot{\alpha}} k^\alpha}{\langle 2\, k\rangle} \tag{7.12}$$
$$= [1\, 2]\langle k\, k\rangle = 0 \tag{7.13}$$

Now here we illustrate the case for MHV (Maximum Helicity Violation), where we are choosing as many as possible combinations that can be reduced to 0 on inspection without further need to calculation:

$$A(1,2,3) = 2\left[(p_i \cdot \epsilon^3)(\epsilon^1 \cdot \epsilon^2) + (p_2 \cdot \epsilon^1)(\epsilon^2 \cdot \epsilon^3)(p_3 \cdot \epsilon^1)(\epsilon^1 \cdot \epsilon^3)\right] \tag{7.14}$$

Using

$$\epsilon_1^- = \frac{\tilde{\xi}_{\dot{\alpha}} 1_\alpha}{[\xi\, 1]} \tag{7.15}$$
$$\epsilon_2^- = \frac{\tilde{\xi}_{\dot{\alpha}} 2_\alpha}{[\xi\, 2]} \tag{7.16}$$
$$\tag{7.17}$$

We are now looking for $\epsilon_1^{1^-} \cdot \epsilon_2^{2^-}$

Deriving gluon three-point amplitudes from Feynman rules, colour order:

$$2 \cdot \epsilon_2 \epsilon_3 = \epsilon_{(2)}{}^{\dot{\alpha}\alpha} \epsilon_{(3)}{}_{\alpha\alpha} \tag{7.18}$$
$$= \sqrt{2}\sqrt{2}\, 2^{\dot{\alpha}}\, \underbrace{k_{3\,\dot{\alpha}} k_2{}^\alpha 3_\alpha}_{Contraction} \tag{7.19}$$
$$= -2[k_3\, 2]\langle k_2\, 3\rangle \tag{7.20}$$

Using the definition of the reference spinor

Aiming to find $\perp \therefore \quad a \cdot b = 0$, which gets rid of the interaction . This makes the first and third term vanish:

$$\epsilon_1 \cdot \epsilon_2 = 0, \qquad \epsilon_1 \cdot \epsilon_3 = 0 \tag{7.21}$$

Leaving us only with the second term:

$$[p_2 \epsilon_1^-] \, (\epsilon_2^- \cdot \epsilon_3^+) = p_{2_{\alpha\dot{\alpha}}} \left(-\sqrt{2}\right) \frac{\tilde{\xi}^{\dot{\alpha}} 1^{\alpha}}{\lambda_{2_{\dot{\alpha}}}} \tag{7.22}$$

$$2(p_2 \cdot \epsilon_1^-) = -\sqrt{2} \frac{\langle 1\, 2\rangle [2\, \tilde{\xi}]}{[\tilde{\xi}\, 1]} \tag{7.23}$$

end finally for $\epsilon_3^{(+)}$:

$$\epsilon_3^{(+)} = \frac{\lambda_{1_\alpha} \tilde{\lambda}_{3_{\dot{\alpha}}}}{\langle 1\, 3\rangle} \sqrt{2} \tag{7.24}$$

$$\rightarrow 2\epsilon_2^- \epsilon_3^+ = -(\sqrt{2})(\sqrt{2}) \frac{\langle 2\, 1\rangle [3\, \tilde{\xi}]}{\langle 1\, 3\rangle [\tilde{\xi}\, 2]} \tag{7.25}$$

$$= -2 \frac{\langle 2\, 1\rangle}{\langle 1\, 3\rangle} \frac{[3\, \tilde{\xi}]}{[\tilde{\xi}\, 2]} \tag{7.26}$$

Now we simplify:

$$\frac{\langle 1\, 2\rangle \cancel{[2\, \tilde{\xi}]}}{\langle \xi\, 1\rangle} \frac{\langle 2\, 1\rangle [3\, \tilde{\xi}]}{\cancel{[2\, \tilde{\xi}]}} = -\frac{\langle 1\, 2\rangle^2}{\langle 1\, 3\rangle} \frac{[3\, \tilde{\xi}]}{[1\, \tilde{\xi}]} \tag{7.27}$$

$$= \frac{[3\, \tilde{\xi}]}{[1\, \tilde{\xi}]} \frac{\langle 1\, 2\rangle^2}{\langle 3\, 1\rangle} \tag{7.28}$$

7.2 Notation

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \lambda_{\dot{\alpha}} \tag{7.29}$$

$$p^\mu = \#\langle \lambda | \, \mu \, | \tilde{\lambda} \,] \tag{7.30}$$

$$= \# \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \tag{7.31}$$

where we are using the identity:

$$P_{\alpha\dot{\alpha}} = p^{\mu}\sigma_{\mu\alpha\dot{\alpha}} \tag{7.32}$$

We can use the completeness relation to expand p^{μ} further:

$$p^{\mu} = \#\lambda^{\beta}\tilde{\lambda}^{\dot{\alpha}}\beta \qquad \underbrace{\sigma^{\mu}{}_{\beta\dot{\beta}}\sigma_{\mu\alpha\dot{\alpha}}}_{\text{Completeness relation}} \tag{7.33}$$

Using the Completeness Relation, our free index μ is summed over:

$$\sigma^{\mu}{}_{\beta\dot{\beta}} \cdot \sigma_{\mu\alpha\dot{\alpha}} = -2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{7.34}$$

Which leads us to:

$$\epsilon^{\mu} = \frac{1}{\sqrt{2}}\frac{\langle \xi |\, \mu \,| k]}{\langle \xi \, k \rangle} \tag{7.35}$$

7.3 Feynman Rule Vertex

We multiply polarisation by each vertex. Hence momentum is conserved

7.3.1 Steps

choose helicities (for same) Polarisation vector:

$$\epsilon_{+}^{\alpha\dot{\alpha}}(\lambda) = -\sqrt{2}\frac{\tilde{\lambda}^{\dot{\alpha}}\alpha\mu^{\alpha}}{\langle \lambda \, \mu \rangle} \tag{7.36}$$

$$\epsilon_{-}^{\alpha\dot{\alpha}}(\lambda) = \sqrt{2}\frac{\lambda^{\alpha}\tilde{\mu}^{\dot{\alpha}}}{[\lambda \, \mu]} \tag{7.37}$$

From here, we choose ϵ such that $\epsilon_i \cdot \epsilon_j = 0$

Here we now have two different ways of representing the Vertex:

$$\tilde{a}_{\dot{\alpha}} b_{\alpha} \tilde{c}^{\dot{\alpha}} d^{\alpha} = [\, a\, c\,] \langle\, d\, b\,\rangle 2a^{\mu} b_{\mu} = 2(a \cdot b) \tag{7.38}$$

$$= a^{\alpha\dot{\alpha}} b_{\alpha\dot{\alpha}} \tag{7.39}$$

$$= a^{\alpha} \tilde{a}^{\dot{\alpha}} b_{\alpha} \tilde{b}_{\dot{\alpha}} \tag{7.40}$$

$$= \langle\, a\, b\,\rangle [\, b\, a\,] \tag{7.41}$$

The phenomenologist way of writing:

$$a^{\mu} = \frac{1}{2} \langle\, a |\, \mu\, | a\,] \equiv \frac{1}{2} a^{\alpha} \tilde{a}^{\dot{\alpha}} \sigma_m u_{\alpha\dot{\alpha}} \tag{7.42}$$

$$2(a \cdot b) = 2 \frac{1}{2} \frac{1}{2} \langle\, a |\, \mu\, | a\,] \langle\, b |\, \mu\, | b\,] \tag{7.43}$$

$$= 2 \langle\, a\, b\,\rangle [\, b\, b\,] \tag{7.44}$$

8 Introduction to Feynman diagrams - reproducing amplitudes for Yang-Mills theory

(Henn and Plefka 2014, pp 35-39)

The BCFW recursion relations rely on an understanding of the behaviour of the function $A_n(z)$ in the complex z plane.

The derivation proceeds in three steps.

- First, the locations of the poles of $A_n(z)$ are analyzed.]
- Then, it is shown that the residues of the poles correspond to products of lower-point tree amplitudes.
- Finally, the large z behaviour of $A_n(z)$ is determined.

Using complex analysis, we want to inspect the amplitude $A_n(z)$. This is because the sum of tree-level Feynman diagrams are gauge invariant, and therefore when they are deformed by z, they remain unchanged. Therefore we can choose the Feynman gauge for the following discussion, without loss of generality. It is clear that $An(z)$ is a rational function of the λ_i,$\tilde{\lambda}_i$ and z. Moreover, $An(z = 0)$ can only have poles where the denominators of Feynman propagators become zero.

When inspecting a function using complex analysis, we try to simplify the function such that there is only one variable in which to take into the complex plane. Taking our scattering amplitude, we reduce it such that our only variable becomes the moment of a particle:

$$(p_i + p_{i+1} + p_{i+2} + \ldots)^2 \equiv \ \delta_{ij} \tag{8.1}$$

Where we have the following quantities:

$$S = (p_1 + p_2)^2 \tag{8.2}$$
$$t = (p_2 + p_3)^2 \tag{8.3}$$
$$u = (p_1 + p_1)^2 \tag{8.4}$$
$$p_4 = -p_1 - p_2 - p_3 - p_4 \Rightarrow A(S, t, u) \tag{8.5}$$

Gauge theory, n-point amplitudes. We now deform our amplitude in such a way that:

$$A(p_1, \ldots, p_n) \rightarrow \begin{cases} p_i \rightarrow p_i(z) \\ p_j \rightarrow p_j(z) \end{cases} \tag{8.6}$$

Which leaves our amplitude in a state with only complexified momenta $p_i(z)$ and $p_i(z)$

$$\mathscr{A}(z) = A(-p_1, p_2, \cdots, p_i(z), \cdots, p_j(z), \cdots p_n) \tag{8.7}$$
$$\mathscr{A}(0) = A(-p_1, p_2, \cdots, p_n) \tag{8.8}$$

.

This process can be particularly useful when exploring massless particles $(\sum p_i^2 = 0)$; however this is not a constrain, and also works just as well with massive particles.

We are left with a transformation:

$$p_i^\mu(z) = p_i^\mu(z) + z\eta^\mu \tag{8.9}$$
$$p_j^\mu(z) = p_j^\mu(z) - z\eta^\mu \tag{8.10}$$

With $\eta =$ new complex momentum and $z =$ its respective complex variable.

$$\left.\begin{aligned} p_i^2(z) = 0 \\ p_j^2(z) = 0 \end{aligned}\right\} \qquad \forall z \tag{8.11}$$

This leads to:

$$\left.\begin{aligned} p_i^2(z) = &\ \cancelto{0}{p_i^2} + z^2\eta^2 + 2z(p_i \cdot \eta) = 0 \\ p_j^2(z) = &\ \cancelto{0}{p_j^2} + z^2\eta^2 + 2z(p_j \cdot \eta) = 0 \end{aligned}\right\} \qquad \forall z \tag{8.12}$$

This is useful for us, as we may thus choose η to be any value we would like; so to simplify this equation, we choose $\eta = 0$, and we are left with:

$$2(p_i \cdot \eta) = 0 \qquad \Leftrightarrow \langle i\,\eta\rangle[\eta\, i] \qquad = 0 \tag{8.13}$$

$$2(p_j \cdot \eta) = 0 \qquad \Leftrightarrow \langle j\,\eta\rangle[\eta\, j] \qquad = 0 \tag{8.14}$$

This is already a well known solution (from the 60s - find ref), where we are keeping spacetime such that:

$$\tilde{\lambda} = \pm\lambda^* \tag{8.15}$$

Real minkowski:

$$\langle i\,\eta\rangle = 0 \implies [i\,\eta] = 0 \quad \text{and:} \quad \lambda_\eta \,/\!/\, \lambda_i \implies \tilde{\lambda}_{\dot\alpha}\eta \,/\!/\, \tilde{\lambda}_{\dot\alpha} i \tag{8.16}$$

Taking complex Minkowski:

$$\left.\begin{aligned} \underline{\langle i\,\eta\rangle}\,[\eta\, i] \to 0 \\ \langle j\,\eta\rangle\,\underline{[\eta\, j]} \to 0 \end{aligned}\right\} \quad \text{2 options} \tag{8.17}$$

$$\eta = \lambda_i \tilde{\lambda}_{\dot\alpha} j \quad \text{or} \quad \eta = \lambda_j \tilde{\lambda}_{\dot\alpha} i \tag{8.18}$$

Where implies that the we are left with:

$$2(p_i \cdot \eta) = 0 \tag{8.19}$$

$$2(p_j \cdot \eta) = 0 \tag{8.20}$$

$$
\begin{aligned}
p_i \rightarrow p_i(z) &= p_i + z\eta & \qquad p_j \rightarrow p_j(z) &= p_j - z\eta \\
&= \lambda_i \tilde{\lambda}_{\dot{\alpha}} i + z\lambda_i \tilde{\lambda}_{\dot{\alpha}} j & &= \lambda_j \tilde{\lambda}_{\dot{\alpha}} j - z\lambda_i \tilde{\lambda}_{\dot{\alpha}} j \\
&= \lambda_i (\tilde{\lambda}_{\dot{\alpha}} i + z\tilde{\lambda}_{\dot{\alpha}} j) & &= (\lambda_j - z\lambda_i)\tilde{\lambda}_{\dot{\alpha}} j \\
&\equiv \lambda_i \hat{\tilde{\lambda}}_i(z) & &\equiv \hat{\lambda}_j \tilde{\lambda}_{\dot{\alpha}} j(z)
\end{aligned} \tag{8.21}
$$

Leaving us with the two quantities:

$$
\hat{\tilde{\lambda}}_i(z) \equiv \tilde{\lambda}_{\dot{\alpha}} i + z\tilde{\lambda}_{\dot{\alpha}} j \tag{8.22}
$$

$$
\hat{\lambda}_j(z) \equiv \lambda_j - z\lambda_i \tag{8.23}
$$

Sometimes this is given the shorthand notation:

$$
\hat{\tilde{\lambda}}_i(z) \equiv [i\, j\rangle \tag{8.24}
$$

$$
\hat{\lambda}_j(z) \equiv \langle i\, j] \tag{8.25}
$$

This leads us to being able to describe amplitudes in the simple form:

$$
\frac{C_1}{z - z_1} + \frac{C_2}{z - z_2} + \cdots + \frac{C_L}{z - z_L} \tag{8.26}
$$

This has the simplification that there are no constant terms $(\cancelto{0}{C} + \cancelto{0}{d_1 z_1} + \cancelto{0}{d_2 z_2^2})$. This means that we only need to know pieces of information:

1. Position of poles: $(z_1, z_2, \dots, z_L)$
2. Residues $(L_1, L_2, \dots, L_L)$, leave us only with simple poles:

$$
\frac{1}{(x - x_0)^3} \tag{8.27}
$$

This is referred to as the pole to third power.

8.1 Feynman Diagrams

What are the singularities:

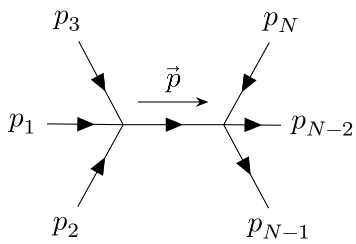


Figure 3: Three point amplitude Feynamn diagram

Using the above, such that $\vec{p}^2 = m^2$, and in the special massless case where $m = 0$

$$\vec{p} = -p_1 - p_2 - p_3 \tag{8.28}$$

When we complexify this, we get:

$$\widehat{p}(z) = P + z\eta \tag{8.29}$$

$$\widehat{p}^2(z) = 0 = P^2 + 0 + 2z(P \cdot \eta) \tag{8.30}$$

$$??? \qquad \frac{z}{P} = \frac{P^2}{2(P \cdot \eta)} \tag{8.31}$$

rewriting this:

$$\frac{1}{\widehat{p}^2(z)} = \frac{1}{p^2 + 2z(P \cdot z)} \tag{8.32}$$

$$= \frac{1}{2(p \cdot \eta)} \cdot \frac{1}{z + \underbrace{\frac{p^2}{2(p \cdot \eta)}}_{=\frac{1}{z - z_p}}} \tag{8.33}$$

$$= \sum_p \frac{C_p}{z - z_p} \tag{8.34}$$

Where we have used the substitution: $z_p = \frac{p^2}{2(p \cdot \eta)}$

8.2 Understanding Singularities

$$A(1,2,\cdots,n) \xrightarrow{p^2\to 0} \sum_n A_L \frac{i}{p^2} A_R \tag{8.35}$$

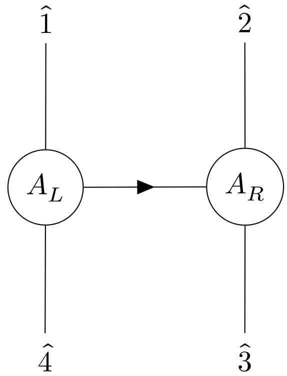


Figure 4: MHV amplitude

Where:

$$P = p_i + p_{i+1} + \cdots + p_n + p_1 \tag{8.36}$$
$$\hat{P} = P + z\eta \tag{8.37}$$

Then:

$$z \to z_p \equiv \frac{-p^2}{2(p\cdot\eta)} \tag{8.38}$$
$$\hat{p}^2(z) \to 0 \tag{8.39}$$
$$\therefore \frac{1}{\hat{p}^2(x)} = \frac{1}{2(p\cdot\eta)[z - z_p]} \tag{8.40}$$
$$\tag{8.41}$$

$$C_p \lim_{z\to z_p} \mathscr{A}(z) = \cancel{z - z_p} \sum_h A_L(\hat{1}(z), \hat{p}^h, i, i+1, \cdots) \frac{i}{2(p\cdot\eta)\cancel{(z-z_p)}} A_R(\hat{p}^h, \hat{2}(z), \cdots) \tag{8.42}$$

$$= A_L(\hat{i}, \hat{p}, \cdots) A_R(-\hat{p}, \cdots) \tag{8.43}$$

$$= A_L(\hat{1}(z_p), \hat{p}, \cdots) A_R(-\hat{P}, ^{-h}, 2(z_p), \cdots) \tag{8.44}$$

$$= \sum_h \frac{A_L(1(z_p), \hat{p}^h) A_R(-\hat{p}^{-h}, 2(z(p)))}{2(p\eta)} \tag{8.45}$$

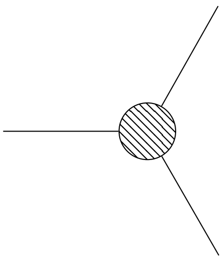
$$\equiv C_p \tag{8.46}$$

$$A(1, 2, \cdots, n) = \sum_n A_L \frac{i}{p^2} A_R \tag{8.47}$$

$$\mathscr{A}(0) = \sum_p \sum_h \frac{A_L^h(z_p) A_R^{-h}(z_p)}{s(p\cdot\eta)(\cancel{z} - z_p)} i \tag{8.48}$$

$$= \sum_p \sum_h A_L^h(z_p) \frac{i}{p^2} A_R^{-h}(z_p) = A \tag{8.49}$$

For example:



$$P = p_4 + p_1, z = \frac{-1p^2}{2(p\cdot\eta)} \tag{8.50}$$

$$P^2 = (p_4 + p_1)^2 = \langle 4\,1\rangle[1\,4] \tag{8.51}$$

$$2p\eta \quad \text{such that} \quad \eta : \begin{cases} \lambda_1 \tilde{\lambda}_{\dot{\alpha}} 2 \to 2(p\cdot\eta) & = \langle 1|p|2] \\ \lambda_2 \tilde{\lambda}_{\dot{\alpha}} 1 \to 2(p\cdot\eta) & = \langle 2|p|1] \end{cases} \tag{8.52}$$

9 Three-point amplitudes and factorisation

(Henn and Plefka 2014, pp 15)

9.1 General Notes:

- $\langle \quad \rangle [\quad]$
- $\mathbb{R}$ Minkowski implies vanishing of one implies vanishing of other

9.2 Shifted particles

Looking at all possible diagrams: (continued from last amplitude).

$$A_{MHV} = (1^- 2^- 3^+ 4^+ 5^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 4\rangle\langle 4\, 5\rangle\langle 5\, 1\rangle} \tag{9.1}$$

Special shifted movement.

In order to calculate this MHV amplitude, we need to find a way of simplifying this, by using the methods that we have been building up to so far. Lets try by first choosing vertexes 5 and 1 make a complex shift. We now have two choices:

$$\text{case } [1] \begin{cases} \hat{\lambda}_5 & = \lambda_5 + z\lambda_1 \\ \hat{\tilde{\lambda}}_1 & = \tilde{\lambda}_1 - z\tilde{\lambda}_5 \end{cases} \tag{9.2}$$

$$\text{case } [2] \begin{cases} \hat{\lambda}_1 & = \lambda_1 + z\lambda_5 \\ \hat{\tilde{\lambda}}_5 & = \tilde{\lambda}_5 - z\tilde{\lambda}_1 \end{cases} \tag{9.3}$$

9.2.1 Case [1]:

First shift:

$$
\begin{aligned}
\langle \hat{1}\, 2 \rangle &= \langle 1\, 2 \rangle && \text{(Same)} \quad (9.4)\\
\langle 2\, 3 \rangle &= \langle 2\, 3 \rangle && \text{(Same)} \quad (9.5)\\
\ldots & && \text{(Same)} \quad (9.6)\\
\underbrace{\langle 4\, \hat{5} \rangle}_{\text{Changed to give a pole}} &= \langle \lambda_4\, \lambda_5 + z\lambda_1 \rangle = \langle 4\, 5 \rangle + z \langle 4\, 1 \rangle && (9.7)\\
\langle \hat{5}\, \hat{1} \rangle &= \langle \hat{5}\, 1 \rangle = \langle 5\, 1 \rangle + z \langle 1\, 1 \rangle && (9.8)\\
&= \langle \lambda_5 + \lambda_1\, \lambda_1 \rangle = \langle 5\, 1 \rangle + \cancelto{0}{\langle 1\, 1 \rangle} && (9.9)\\
&= \langle 5\, 1 \rangle && (9.10)
\end{aligned}
$$

This leaves us with the new shifted amplitude:

$$
\mathscr{A}(z) = \frac{\langle 1\, 2 \rangle^4}{\langle 1\, 2 \rangle \cdots \langle 3\, 4 \rangle \left(\langle 4\, 5 \rangle + z \langle 5\, 1 \rangle \right) \langle 5\, 1 \rangle} \tag{9.11}
$$

And this is precisely a simple pole:

$$
\frac{1}{\langle\, 4\hat{5}\, \rangle} = \frac{1}{\langle\, 45\, \rangle + z \langle\, 41\, \rangle} = \frac{1}{\langle\, 41\, \rangle} \cdot \quad \underbrace{\frac{1}{z + \frac{\langle\, 45\, \rangle}{\langle\, 41\, \rangle}}}_{\text{Position of the pole } z_p} \tag{9.12}
$$

$$
z_p = -\frac{\langle\, 45\, \rangle}{\langle\, 41\, \rangle} \tag{9.13}
$$

(a) Amplitude corresponding to simple pole z_p

(b) Alternative possible arrangement, which goes to zero

Figure 5: Two possible MHV amplitudes for complex shifted system

9.2.2 Case [2]:

Looking at the second shift:

$$\langle \hat{1}2 \rangle = \langle 12 \rangle + z\langle 52 \rangle \tag{9.14}$$

$$...(unchanged) \tag{9.15}$$

$$\langle \hat{5}\hat{1} \rangle = \langle 5\hat{1} \rangle = \langle 51 \rangle + z\cancelto{0}{\langle 11 \rangle} \tag{9.16}$$

$$= \langle 51 \rangle \tag{9.17}$$

This leaves us with an Amplitude $\mathscr{A}(z) \approx z^3$, which at large $z : (z - z_p)^3 \approx z^3$

This example turns out to have no poles, and no physical coefficients, and we therefore cannot perform the recursion relation.

9.3 Helicities:

(a) Goes to zero, does not exist

(b) Goes to zero, why (next time)

Figure 6: MHV amplitude calculations when taking helicities into account

10 BCFW recursion relations in Yang-Mills and Gravity

(Henn and Plefka 2014, pp 35-39)

The BCFW recursion relations rely on an understanding of the behaviour of the function $A_n(z)$ in the complex z plane.

The derivation proceeds in three steps.

- First, the locations of the poles of $A_n(z)$ are analyzed.

- Then, it is shown that the residues of the poles correspond to products of lower-point tree amplitudes.
- Finally, the large z behaviour of $A_n(z)$ is determined.

Using complex analysis, we want to inspect the amplitude $A_n(z)$. This is because the sum of tree-level Feynman diagrams are gauge invariant, and therefore when they are deformed by z, they remain unchanged. Therefore we can choose the Feynman gauge for the following discussion, without loss of generality. It is clear that $An(z)$ is a rational function of the λ_i,$\tilde{\lambda}_i$ and z. Moreover, $An(z = 0)$ can only have poles where the denominators of Feynman propagators become zero.

When inspecting a function using complex analysis, we try to simplify the function such that there is only one variable in which to take into the complex plane. Taking our scattering amplitude, we reduce it such that our only variable becomes the moment of a particle:

$$(p_i + p_{i+1} + p_{i+2} + \dots)^2 \equiv \ \delta_{ij} \tag{10.1}$$

Where we have the following quantities:

$$S = (p_1 + p_2)^2 \tag{10.2}$$
$$t = (p_2 + p_3)^2 \tag{10.3}$$
$$u = (p_1 + p_1)^2 \tag{10.4}$$
$$p_4 = -p_1 - p_2 - p_3 - p_4 \Rightarrow A(S, t, u) \tag{10.5}$$

Gauge theory, n-point amplitudes. We now deform our amplitude in such a way that:

$$A(p_1, \dots, p_n) \rightarrow \begin{cases} p_i \rightarrow p_i(z) \\ p_j \rightarrow p_j(z) \end{cases} \tag{10.6}$$

Which leaves our amplitude in a state with only complexified momenta $p_i(z)$ and $p_i(z)$

$$\mathscr{A}(z) = A(-p_1, p_2, \cdots, p_i(z), \cdots, p_j(z), \cdots p_n) \tag{10.7}$$
$$\mathscr{A}(0) = A(-p_1, p_2, \cdots, p_n) \tag{10.8}$$

.

This process can be particularly useful when exploring massless particles $(\sum p_i^2 = 0)$; however this is not a constrain, and also works just as well with massive particles.

We are left with a transformation:

$$p_i^\mu(z) = p_i^\mu(z) + z\eta^\mu \tag{10.9}$$
$$p_j^\mu(z) = p_j^\mu(z) - z\eta^\mu \tag{10.10}$$

With $\eta =$ new complex momentum and $z =$ its respective complex variable.

$$\left.\begin{matrix} p_i^2(z) = 0 \\ p_j^2(z) = 0 \end{matrix}\right\} \qquad \forall z \tag{10.11}$$

This leads to:

$$\left.\begin{matrix} p_i^2(z) = & \cancelto{0}{p_i^2} + z^2\eta^2 + 2z(p_i \cdot \eta) = 0 \\ p_j^2(z) = & \cancelto{0}{p_j^2} + z^2\eta^2 + 2z(p_j \cdot \eta) = 0 \end{matrix}\right\} \qquad \forall z \tag{10.12}$$

This is useful for us, as we may thus choose η to be any value we would like; so to simplify this equation, we choose $\eta = 0$, and we are left with:

$$2(p_i \cdot \eta) = 0 \Leftrightarrow \langle i\,\eta\rangle[\eta\,i] = 0 \tag{10.13}$$
$$2(p_j \cdot \eta) = 0 \Leftrightarrow \langle j\,\eta\rangle[\eta\,j] = 0 \tag{10.14}$$

This is already a well known solution (from the 60s - find ref), where we are keeping spacetime such that:

$$\tilde{\lambda} = \pm\lambda^* \tag{10.15}$$

Real minkowski:

$$\langle i\,\eta\rangle = 0 \implies [i\,\eta] = 0 \quad \text{and:} \quad \lambda_\eta \mathbin{/\!/} \lambda_i \implies \tilde{\lambda}_{\dot{\alpha}}\eta \mathbin{/\!/} \tilde{\lambda}_{\dot{\alpha}} i \tag{10.16}$$

Taking complex Minkowski:

$$\left.\begin{aligned} \underline{\langle i\,\eta\rangle}\,[\eta\, i] &\to 0 \\ \langle j\,\eta\rangle\,\underline{[\eta\, j]} &\to 0 \end{aligned}\right\} \quad \text{2 options} \tag{10.17}$$

$$\text{either} \quad \eta = \lambda_i \tilde{\lambda}_{\dot{\alpha}} j \tag{10.18}$$

$$\text{or} \quad \eta = \lambda_j \tilde{\lambda}_{\dot{\alpha}} i \tag{10.19}$$

Where implies that the we are left with:

$$2(p_i \cdot \eta) = 0 \tag{10.20}$$

$$2(p_j \cdot \eta) = 0 \tag{10.21}$$

$$\begin{aligned} p_i \to p_i(z) &= p_i + z\eta \\ &= \lambda_i \tilde{\lambda}_{\dot{\alpha}} i + z\lambda_i \tilde{\lambda}_{\dot{\alpha}} j \\ &= \lambda_i (\tilde{\lambda}_{\dot{\alpha}} i + z\tilde{\lambda}_{\dot{\alpha}} j) \\ &\equiv \lambda_i \hat{\tilde{\lambda}}_i(z) \end{aligned} \qquad \begin{aligned} p_j \to p_j(z) &= p_j - z\eta \\ &= \lambda_j \tilde{\lambda}_{\dot{\alpha}} j - z\lambda_i \tilde{\lambda}_{\dot{\alpha}} j \\ &= (\lambda_j - z\lambda_i)\tilde{\lambda}_{\dot{\alpha}} j \\ &\equiv \hat{\lambda}_j \tilde{\lambda}_{\dot{\alpha}} j(z) \end{aligned} \tag{10.22}$$

Leaving us with the two quantities:

$$\hat{\tilde{\lambda}}_i(z) \equiv \tilde{\lambda}_{\dot{\alpha}} i + z\tilde{\lambda}_{\dot{\alpha}} j \tag{10.23}$$

$$\hat{\lambda}_j(z) \equiv \lambda_j - z\lambda_i \tag{10.24}$$

Sometimes this is given the shorthand notation:

$$\hat{\tilde{\lambda}}_i(z) \equiv [i\, j\rangle \tag{10.25}$$

$$\hat{\lambda}_j(z) \equiv \langle i\, j] \tag{10.26}$$

This leads us to being able to describe amplitudes in the simple form:

$$\frac{C_1}{z - z_1} + \frac{C_2}{z - z_2} + \cdots + \frac{C_L}{z - z_L} \tag{10.27}$$

This has the simplification that there are no constant terms $(\cancelto{0}{c} + \cancelto{0}{d_1 z_1} + \cancelto{0}{d_2 z_2^2})$. This means that we only need to know pieces of information:

1. Position of poles: $(z_1, z_2, \dots, z_L)$
2. Residues $(L_1, L_2, \dots, L_L)$, leave us only with simple poles:

$$\frac{1}{(x - x_0)^3} \tag{10.28}$$

This is referred to as the pole to third power.

10.1 Feynman Diagrams

What are the singularities:

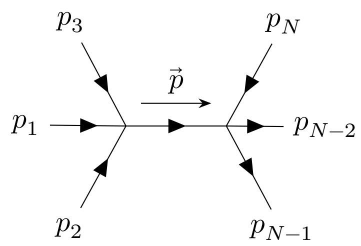


Figure 7: Three point amplitude Feynamn diagram

Using the above, such that $\vec{p}^2 = m^2$, and in the special massless case where $m = 0$

$$\vec{p} = -p_1 - p_2 - p_3 \tag{10.29}$$

When we complexify this, we get:

$$\hat{p}(z) = P + z\eta \tag{10.30}$$
$$\hat{p}^2(z) = 0 = P^2 + 0 + 2z(P \cdot \eta) \tag{10.31}$$
$$??? \qquad \frac{z}{P} = \frac{P^2}{2(P \cdot \eta)} \tag{10.32}$$

rewriting this:

$$
\begin{aligned}
\frac{1}{\hat{p}^2(z)} &= \frac{1}{p^2 + 2z(P \cdot z)} && (10.33) \\
&= \frac{1}{2(p \cdot \eta)} \cdot \frac{1}{z + \underbrace{\frac{p^2}{2(p\cdot\eta)}}_{=\frac{1}{z-z_p}}} && (10.34) \\
&= \sum_p \frac{C_p}{z - z_p} && (10.35)
\end{aligned}
$$

Where we have used the substitution:

$$
z_p = \frac{p^2}{2(p \cdot \eta)} \tag{10.36}
$$

10.2 Understanding Singularities

$$
A(1, 2, \cdots, n) \xrightarrow{p^2 \to 0} \sum_n A_L \frac{i}{p^2} A_R \tag{10.37}
$$

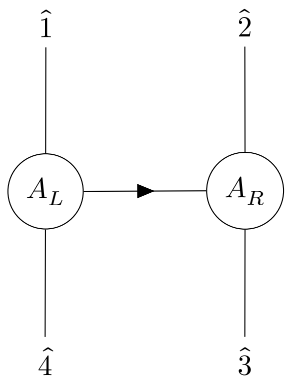


Figure 8: MHV amplitude

Where:

$$P = p_i + p_{i+1} + \cdots + p_n + p_1 \tag{10.38}$$
$$\hat{P} = P + z\eta \tag{10.39}$$

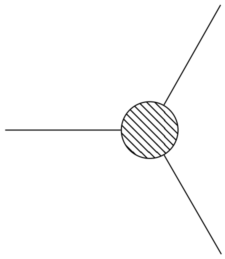
Then:

$$z \to z_p \equiv \frac{-p^2}{2(p \cdot \eta)} \tag{10.40}$$
$$\hat{p}^2(z) \to 0 \tag{10.41}$$
$$\therefore \frac{1}{\hat{p}^2(x)} = \frac{1}{2(p \cdot \eta)[z - z_p]} \tag{10.42}$$

$$\begin{aligned}
C_p \lim_{z \to z_p} \mathscr{A}(z) &= \cancel{z - z_p} \sum_h A_L(\hat{1}(z), \hat{p}^h, i, i+1, \cdots) \frac{i}{2(p \cdot \eta)\cancel{(z - z_p)}} A_R(\hat{p}^h, \hat{2}(z), \cdots) && (10.43)\\
&= A_L(\hat{i}, \hat{p}, \cdots) A_R(-\hat{p}, \cdots) && (10.44)\\
&= A_L(\hat{1}(z_p), \hat{p}, \cdots) A_R(-\hat{P}, ^{-h}, 2(z_p), \cdots) && (10.45)\\
&= \sum_h \frac{A_L(1(z_p), \hat{p}^h) A_R(-\hat{p}^{-h}, 2(z(p)))}{2(p\eta)} && (10.46)\\
&\equiv C_p && (10.47)
\end{aligned}$$

$$\begin{aligned}
A(1, 2, \cdots, n) &= \sum_n A_L \frac{i}{p^2} A_R && (10.48)\\
\mathscr{A}(0) &= \sum_p \sum_h \frac{A_L^h(z_p) A_R^{-h}(z_p)}{s(p \cdot \eta)(\cancel{z} - z_p)} i && (10.49)\\
&= \sum_p \sum_h A_L^h(z_p) \frac{i}{p^2} A_R^{-h}(z_p) = A && (10.50)
\end{aligned}$$

For example:



$$P = p_4 + p_1, z = \frac{-1p^2}{2(p \cdot \eta)} \tag{10.51}$$

$$P^2 = (p_4 + p_1)^2 = \langle 4\,1\rangle [1\,4] \tag{10.52}$$

$$2p\eta \quad \text{such that} \quad \eta : \begin{cases} \lambda_1 \tilde{\lambda}_{\dot{\alpha}} 2 \to 2(p \cdot \eta) & = \langle 1|p|2] \\ \lambda_2 \tilde{\lambda}_{\dot{\alpha}} 1 \to 2(p \cdot \eta) & = \langle 2|p|1] \end{cases} \tag{10.53}$$

10.3 Deriving simplest 4-point MHV amplitude

Starting from a negative helicity amplitude, with a result of

$$A_{MHV}(1^- 2^- 3^+ 4^+) = \frac{\langle\, 12\,\rangle^4}{\langle\, 12\,\rangle\langle\, 23\,\rangle\langle\, 34\,\rangle\langle\, 41\,\rangle} \tag{10.54}$$

10.3.1 Special shifted:

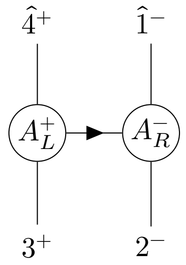
$$Case[1] \begin{Bmatrix} \hat{\lambda}_4 = \lambda_4 - z\lambda_1 \\ \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 + z\tilde{\lambda}_4 \end{Bmatrix} = \langle\, 4\,1\,] \tag{10.55}$$

For this, it can be easily seen that the poles come from $\langle\, 34\,\rangle$

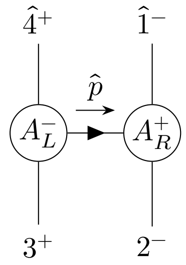
$$\langle\, \hat{4}1\,\rangle = \langle\, 41\,\rangle - \cancelto{0}{z\langle\, 11\,\rangle} \tag{10.56}$$

$$= \langle\, 41\,\rangle \tag{10.57}$$

Of the $\overline{MHV}$ and



(a) $\overline{MHV}$ amplitude goes to zero, does not exist, as all LHS are + and all RHS are -



(b) MHV amplitude

Figure 9: MHV amplitude calculations. One of these must go to zero, as there will only be one pole

The From the two amplitudes above ($\overline{MHV}$ fig 9a , and MHV fig 9b)). We can immediately see that using the combinations of $+$ and $-$ on A_L and A_R; only one of the objects is physically possible. We will now use this to calculate the position of the pole from the MHV amplitude:

$$
\begin{aligned}
z_p &= (p_3 + \hat{p}_4(z_p))^2 && (10.58)\\
&= 0 && (10.59)\\
&= \langle 3\hat{4} \rangle [\hat{4}3] && (10.60)\\
&= \langle 3\hat{4} \rangle [43] && \text{NB no hat} \quad (10.61)\\
&= 0 && (10.62)\\
&\implies \langle 34 \rangle - z\langle 31 \rangle = 0 && (10.63)\\
\therefore z_p &= \frac{\langle 34 \rangle}{\langle 31 \rangle} && \text{pole} \quad (10.64)
\end{aligned}
$$

This is an important finding, and shows that momentum $\hat{p}$, will not generally be on-shell, but using this method, it keeps it on-shell.

10.3.2 $\hat{p}$:

$$
\begin{aligned}
A_R &= \frac{\langle 12 \rangle^3}{\langle 2\hat{p} \rangle \langle \hat{p}1 \rangle} && = \frac{\langle 12 \rangle^2}{\langle 2\hat{p} \rangle \langle \hat{p}1 \rangle} && (10.65)\\
A_L &= \frac{[3\hat{4}]^3}{[\hat{4}\hat{p}][\hat{p}3]} && = \frac{[34]^3}{[4\hat{p}][\hat{p}3]} && (10.66)
\end{aligned}
$$

$$\frac{i}{(p_3+p_4)^2} = \frac{\langle 12\rangle^3}{\langle 2\hat{p}\rangle\langle \hat{p}1\rangle} \cdot \frac{i}{\langle 34\rangle[43]} \cdot \frac{[34]^3}{[4\hat{p}][\hat{p}3]} \tag{10.67}$$

For this, we have a possible pairing:

$$\langle \hat{p}1\rangle[4\hat{p}] = [4\hat{p}][\hat{p}1] \tag{10.68}$$

$$= [4|\hat{p}|1\rangle \tag{10.69}$$

$$\tag{10.70}$$

Here we are looking for a way to keep out particles on shell, so we use the complex momentum, and find the correct particles momentum to work allow for this calculation. In this case, we are choosing $\hat{p} = \hat{\lambda}_1 + z\lambda_2$:

$$= [4|\hat{1}+2|1\rangle \tag{10.71}$$

$$= [4|1+2|1\rangle = \overbrace{\cancel{[4|1|1\rangle}}^{\text{Using } [41]\langle 11\rangle=0} \to 0 \quad +[4|2|1\rangle \tag{10.72}$$

$$[4|2|1\rangle = [42]\langle 21\rangle \tag{10.73}$$

What do we need? What are the spinors λ_p and $\tilde{\lambda}_p$. We will change to a more systematic method of computation.

Only for the value of z_p should we keep the masses on shell.

$$\hat{p} = \hat{1}+2 = \lambda_p\tilde{\lambda}_p \tag{10.74}$$

10.3.3 $\hat{1}$:

$$\hat{1}+2 = \lambda_1(\underbrace{\tilde{\lambda}_1 + z_p\tilde{\lambda}_4}_{\substack{\text{Need to compute this.}\\ \text{Does it look like } \lambda\tilde{\lambda}?}}) + \lambda_2\tilde{\lambda}_2 \tag{10.75}$$

$$\tilde{\lambda}_1 + z_p\tilde{\lambda}_4 = \tilde{\lambda}_1 + \frac{\langle\,34\,\rangle}{\langle\,31\,\rangle}\tilde{\lambda}_4 \tag{10.76}$$

$$= \frac{\overbrace{\langle\,31\,\rangle\tilde{\lambda}_1}^{3} + \overbrace{\langle\,34\,\rangle\tilde{\lambda}_4}^{4}}{\langle\,31\,\rangle} \tag{10.77}$$

$$= \frac{\langle 3|\,(\ \overbrace{p_1 + p + 4}^{\text{Can look at contributions from other leg}}\ )}{\langle\,31\,\rangle} \tag{10.78}$$

The momentum of each particle can also be seen as a sum from all the other contributing particles momenta. In this way we are able to write our legs in a different form: $p_1 + p_4 = -p_2 - p_3$. And so continuing with this we substitute it back into the form we arrived at above:

$$(\tilde{\lambda}_1 + z_p\tilde{\lambda}_4) = -\frac{\langle 3|\,(2+3)}{\langle\,31\,\rangle} \tag{10.79}$$

$$= \frac{-\langle\,32\,\rangle\tilde{\lambda}_2}{\langle\,31\,\rangle} \tag{10.80}$$

$\hat{p}$ in a factorised form:

$$\hat{p} = -\frac{\langle\,32\,\rangle}{\langle\,31\,\rangle}\lambda_1\tilde{\lambda}_2 + \lambda_2\tilde{\lambda}_2 \tag{10.81}$$

$$= -\frac{\langle\,32\,\rangle}{\langle\,31\,\rangle}(\lambda_1 + \lambda_2)\tilde{\lambda}_2 \tag{10.82}$$

$$\langle\,\hat{p}1\,\rangle = \langle\,21\,\rangle \tag{10.83}$$

$$\langle\,2\hat{p}\,\rangle = \underbrace{-\langle\,21\,\rangle}_{=\langle\,12\,\rangle}\frac{\langle\,32\,\rangle}{\langle\,31\,\rangle} \tag{10.84}$$

This leaves us with the following:

$$[\,4\hat{p}\,] = [\,42\,] \tag{10.85}$$

$$[\,\hat{p}3\,] = [\,23\,] \tag{10.86}$$

Now putting everything together:

$$\frac{\overbrace{[\,34\,]}^{-}\cancelto{2}{3}\overbrace{\langle\,12\,\rangle}^{-}\cancelto{1}{3}}{\langle\,34\,\rangle\underbrace{\cancel{[\,43\,]}}_{-}[\,42\,][\,23\,]\cancel{\langle\,12\,\rangle}\frac{\langle 32\rangle}{\langle 31\rangle}\cancel{\langle\,21\,\rangle}} = \frac{\langle\,12\,\rangle[\,34\,]^2\langle\,31\,\rangle}{\langle\,34\,\rangle[\,42\,][\,23\,]\langle\,32\,\rangle} \tag{10.87}$$

This can be further simplified into our shifted pole:

$$\langle\,3|\,4\,|2\,] = \langle\,3|\,-1-\cancelto{0}{2}-\cancelto{0}{3}|2\,] \tag{10.88}$$

Where we have used the face that $\cancelto{0}{-2-3}$ because: $\langle\,3|\,-2\,|2\,]=0$

$$\langle\,3|\,-1\,|2\,] = -\langle\,31\,\rangle[\,12\,] \tag{10.89}$$

We then substitute this into the denominator to produce:

$$\frac{\langle\,12\,\rangle[\,34\,]^2\cancel{\langle\,31\,\rangle}}{-\cancel{\langle\,31\,\rangle}[\,12\,][\,23\,]\langle\,32\,\rangle} \tag{10.90}$$

We can then group in the the terms in the denominator:

$$[\,12\,]\langle\,32\,\rangle = [\,12\,]\langle\,23\,\rangle(-) \tag{10.91}$$
$$= [\,1|\,2\,|3\,\rangle(-) \tag{10.92}$$
$$= [\,1|\,-\cancel{1}-\cancel{3}-4\,|3\,\rangle(-) \tag{10.93}$$
$$= +[\,14\,]\langle\,43\,\rangle \tag{10.94}$$

Multiplying both sides by $\frac{[\,34\,]}{[\,34\,]}$

$$\langle\,32\,\rangle[\,34\,]z\text{ bottom} \tag{10.95}$$

$$= -\langle 23 \rangle [34] \tag{10.96}$$

$$= (-)(-)\langle 21 \rangle [14] \tag{10.97}$$

$$= \frac{\cancelto{-}{\langle 12 \rangle}[34]^2 \cancel{\langle 34 \rangle}[34]}{-\cancel{\langle 31 \rangle}[12][23]\cancel{\langle 21 \rangle}[14]} \tag{10.98}$$

$$\text{Finally } = \frac{[34]^3}{[12][23][41]} \tag{10.99}$$

We can check this result in the other direction

$$\frac{[34]^{\cancelto{2}{3}}}{\cancel{[12]}[23][41]} \cdot \frac{\langle 23 \rangle \cancel{\langle 34 \rangle} \langle 41 \rangle}{\langle 12 \rangle^{\cancelto{2}{3}}} \stackrel{?}{=} \pm \mathbb{1} \tag{10.100}$$

Now we pair up the legs with the momentum:

$$[34]\langle 34 \rangle = -2(p_3 \cdot p_4) \tag{10.101}$$

$$[12]\langle 12 \rangle = -2(p_1 \cdot p_2) = -(p_1 + p_2)^2 \tag{10.102}$$

$$[12]\langle 12 \rangle = [34]\langle 34 \rangle \tag{10.103}$$

$$\langle 23 \rangle [34] = -\langle 21 \rangle [14] = -\langle 12 \rangle [41] \tag{10.104}$$

$$[34]\langle 41 \rangle = -[32]\langle 21 \rangle = -[23]\langle 12 \rangle \tag{10.105}$$

Putting all this together:

$$= \frac{\langle 12 \rangle [41][23]\langle 12 \rangle}{\langle 12 \rangle^2 [23][41]} = \pm \mathbb{1} \tag{10.106}$$

10.4 Meeting Notes:

Slide equations:

$$p = \lambda\tilde{\lambda} \tag{10.107}$$

$$p_\mu \to p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu \tag{10.108}$$

$$\text{show } p_\mu p^\mu = 0 \tag{10.109}$$

$$p_{\alpha\dot{\alpha}} \equiv \lambda_\alpha \tilde{\lambda}_\alpha \tag{10.110}$$

$$\det\{(p_{\alpha\dot{\alpha}})\} = p^2 \tag{10.111}$$

10.5 Deriving 3 point amplitudes

- Derive MHV amplitude
- Perform a shift for a 4-point amplitude.
- Negative helicity

$$(\mathrm{MHV}) \to \mathrm{A}(1^-2^+3^-4^+) = \frac{\langle\, 13\,\rangle^4}{\langle\, 12\,\rangle\langle\, 23\,\rangle\langle\, 34\,\rangle\langle\, 41\,\rangle} \tag{10.112}$$

We have the choice here:

$$\begin{cases} \hat{\tilde{\lambda}}_i & = \tilde{\lambda}_i + z\lambda_j \\ \hat{\lambda}_j & = \lambda_j - z\lambda_i \end{cases} \tag{10.113}$$

Leaving us with:

$$\langle\, \hat{1}3\,\rangle = \langle\, 13\,\rangle + z\langle\, 23\,\rangle \tag{10.114}$$

We will now investigate the behaviour of this object at large z, and begin by choosing $i = 1, j = 2$

$$\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 + z\tilde{\lambda}_2 \tag{10.115}$$

$$\hat{\lambda}_2 = \lambda_2 - z\lambda_1 \tag{10.116}$$

Searching for the shift of λ_2

$$\langle 12 \rangle \to \langle 1\hat{2} \rangle = \langle 1\lambda_2 - z\lambda_1 \rangle \tag{10.117}$$
$$= \langle 12 \rangle - \cancelto{0}{z\langle 11 \rangle} \tag{10.118}$$
$$= \langle 12 \rangle \tag{10.119}$$

The other leg:

$$\langle \hat{2}3 \rangle = \langle 23 \rangle - z\langle 13 \rangle \tag{10.120}$$

This leaves us with the following amplitudes:

(a) **(b)**

Figure 10: Amplitude example

- In real Minkowski, three point doesn't exist
- Only angle or square brackets can be zero, but not at the same time.
- Two particle invariance (2 half + 3 squares)

refering to ~fig 10a, we can see that it is null at the polling z, and therefore the product of the brackets must go to zero:

$$\underbrace{\langle \hat{2}3 \rangle}_{\text{Must equal } 0} [32] = 0 = (\hat{p}_2 + p_3)^2 \tag{10.121}$$

The angle brakets ($\langle \hat{2}3 \rangle$) must be equal to zero because of the line between the two shells. The condition for blobs to be on-shell are zero

This means that the right hand leg in ~fig 10a must go to zero:

$$\frac{\langle 3\hat{p} \rangle^3}{\langle \hat{2}3 \rangle \langle \hat{p}2 \rangle} = \frac{0^3}{0^2} \tag{10.122}$$

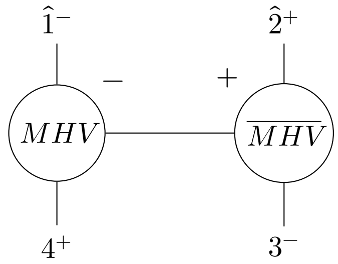


Figure 11: Non vanishing Diagram

This shows us that 10a is not physically possible, as the amplitude is only made of square brackets $(+-+)$. This could be zero if square is 0, but square brackets can't be zero. Because we are shifting the particle λ_2. $\mathrm{MHV}\ 2^-1^+$ means angle brackets. One of these will always go to zero.

10.6 Example ~Fig. 10b

Try to write down what the MHV diagram with the brackets. Start the brackets.

To writ this amplitude:

$$A_L = \frac{\langle \hat{1}\hat{p} \rangle^3}{\langle \hat{p}4 \rangle \langle 4\hat{1} \rangle} \underbrace{= \frac{\langle 1\hat{p} \rangle^3}{\langle \hat{p}4 \rangle \langle 41 \rangle}}_{\text{because } \lambda_1 \text{ is not shifted}} \tag{10.123}$$

And the amplitude on the right is $\overline{\mathrm{MHV}}$:

$$A_R = \frac{[\hat{p}\hat{2}]^3}{[3\hat{p}][\hat{2}p]} = \frac{[\hat{p}2]}{[3\hat{p}][23]} \tag{10.124}$$

$$A(1^-2^+3^-4^+) = \frac{\langle 1\hat{p} \rangle^3}{\langle \hat{p}3 \rangle \langle 41 \rangle} \frac{i}{p_{23}^2} \frac{[\hat{p}2]}{[3\hat{p}][23]} \tag{10.125}$$

Where p_{23} is the momentum between A_L and A_R

$$p_{23}^2 = (p_2 + p_3)^2 = \langle 23 \rangle [32] \tag{10.126}$$

Putting this all together, all we need to understand is $\hat{p}$

We have to find the spinors of $\hat{p}$ at the positions of the poles:

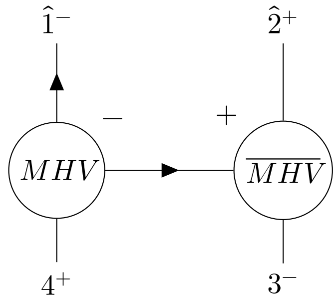


Figure 12: Non vanishing Diagram

$$\hat{p} = \lambda_{\hat{p}}\tilde{\lambda}_{\hat{p}} \tag{10.127}$$

The position of the pole in z is found by requiring $\hat{p}$ to be on-shell.

Where we have two equivalent conditions:

$$\langle \hat{2}3 \rangle [32] = 0 \tag{10.128}$$
$$\langle 41 \rangle [\hat{1}4] = 0 \tag{10.129}$$

From these two equations, we have two possibilities:

$$\langle \hat{2}3 \rangle = 0 \implies \qquad \langle 23 \rangle - z\langle 13 \rangle = 0 \tag{10.130}$$
$$[\hat{1}4] = 0 \implies \qquad [14] + z[s4] = 0 \tag{10.131}$$

And solving these two equations simultaneously:

$$z = \frac{\langle 23 \rangle}{\langle 13 \rangle} \qquad z = \frac{-[14]}{[24]} \tag{10.132}$$

Here the two z are the same, so we have found two equations that are equal. When two mathematical objects are equal, we can divide one by the other and their ratio must equal one:

$$-\frac{\langle 23 \rangle}{\langle 13 \rangle} \cdot \frac{[24]}{[14]} \stackrel{?}{=} 1 \tag{10.133}$$

Let us interrogate the ratio a little further, and we can replace with the momenta p

$$\frac{(-)[4|2|3\rangle}{[4|\underbrace{1}_{p_1}|3\rangle} = -\frac{[4|2|3\rangle}{(-)[4|2+3+4|3\rangle} \tag{10.134}$$

$$p_1 = -p_2 - p_3 - p_4 \qquad (-)[4|2+\cancel{3}+\cancel{4}|3\rangle \tag{10.135}$$

Summary: We have two values of z, which look different, but are actually the same. We now have to find $\langle\ \rangle[\ ]$

$$\hat{p} = \hat{p}_2 + p_3 = \lambda_2\tilde{\lambda}_2 - z\lambda_1\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 \tag{10.136}$$

$$= (\lambda_2 - z\lambda_1)\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 \tag{10.137}$$

Using in our $z = -\frac{[14]}{[24]}$, and $\hat{\lambda}_2 = \lambda_2 + \frac{[14]}{[24]}\lambda_1$

$$\hat{\lambda}_2 = \lambda_2 + \frac{[14]}{[24]}\lambda_1 \tag{10.138}$$

$$= \frac{\lambda_2[24] + \lambda_1[14]}{[24]} \tag{10.139}$$

$$\text{which is the same as } = \frac{(p_2+p_1)|4]}{[24]} \tag{10.140}$$

$$\text{with momentum conservation } = \frac{-(p_3+p_4)|4]}{[24]} \quad (p_4 \text{ next to 4 gives 0}) \tag{10.141}$$

$$\text{becoming } = -\lambda_3\frac{[34]}{[24]} \tag{10.142}$$

Finally:

$$\hat{\lambda}_2 = \lambda_3\frac{[34]}{[24]} \tag{10.143}$$

Plugging this back into the original expression, this becomes:

$$\hat{p} = \underbrace{\lambda_3}_{\lambda_{\hat{p}}} \underbrace{\left[-\frac{[34]}{[24]}\tilde{\lambda}_2 + \tilde{\lambda}_3\right]}_{\tilde{\lambda}_{\hat{p}}} \tag{10.144}$$

$$\lambda_{\hat{p}} = \lambda_3 \tag{10.145}$$

$$\tilde{\lambda}_{\hat{p}} = \tilde{\lambda}_3 - \frac{[34]}{[24]}\tilde{\lambda}_2 \tag{10.146}$$

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