Appendix A
Spinors and their properties

Our conventions are:

$$\hbar = c = 1, \quad x^\mu = (t, \mathbf{x}), \quad p^\mu = (E, \mathbf{p}), \quad g^{\mu\nu} = (1, -1, -1, -1),$$
$$x^2 = g_{\mu\nu} x^\mu x^\nu = t^2 - \mathbf{x}^2, \qquad \mathbf{p} = -\mathrm{i}\partial/\partial\mathbf{x}, \qquad p^\mu = \mathrm{i}\partial/\partial x_\mu$$

A plane wave solution to the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\mathbf{p}\rangle = E|\mathbf{p}\rangle \tag{A.1}$$

is $\langle \mathbf{x}|\mathbf{p}, t\rangle = \exp(-\mathrm{i}p \cdot x) = \Phi(t, \mathbf{x}) = \Phi(x)$. Space-time translation (change of the origin of the reference frame) $x'_\mu = x_\mu + \varepsilon_\mu$ gives $\Phi'(x') = \langle \mathbf{x}'|\mathbf{p}\rangle = \Phi(x) = \exp(-\mathrm{i}p \cdot x)$. Hence,

$$\Phi'(x) = \Phi(x - \varepsilon) = \exp[-\mathrm{i}p \cdot (x - \varepsilon)] \approx (1 + \mathrm{i}p^\mu \varepsilon_\mu)\Phi(x) \tag{A.2}$$

Thus, the translation of the state $|\mathbf{p}\rangle$ is described by the operator

$$|\mathbf{p}'\rangle = \exp(\mathrm{i}\varepsilon_\mu p^\mu)|\mathbf{p}\rangle \tag{A.3}$$

Indeed $\Phi'(x) = \langle \mathbf{x}|\mathbf{p}'\rangle \approx (1 + \mathrm{i}p^\mu \varepsilon_\mu)\Phi(x)$.

Lorentz transformations and two-dimensional representations of the group $SL(2, C)$

The Lorentz group is the group of transformations which leave invariant the square $x^2 \equiv x^\mu x_\mu \equiv g_{\mu\nu} x^\mu x^\nu$ of the four-vector x^μ. The infinitesimal form of these transformations reads

$$x^{\mu\prime} \approx x^\mu + \omega^\mu{}_\nu x^\nu \tag{A.4}$$

where $\omega_{\mu\nu} \equiv g_{\mu\kappa}\omega^{\kappa}{}_{\nu} = -\omega_{\nu\mu}$. Identifying $\omega_{\mu\nu}$ with the transformation parameters, (A.4) can be rewritten in the form

$$x^{\mu\prime} \approx x^{\mu} - \frac{\mathrm{i}}{2}\omega_{\kappa\rho}\,(M^{\kappa\rho})^{\mu}{}_{\nu}x^{\nu} \tag{A.5}$$

with the Lorentz group generators in the vector representation given by

$$(M^{\kappa\rho})^{\mu}{}_{\nu} = \mathrm{i}(g^{\kappa\mu}g^{\rho}{}_{\nu} - g^{\rho\mu}g^{\kappa}{}_{\nu}) \tag{A.6}$$

The generators $M^{\mu\nu}$ satisfy the following commutation relations

$$\left[M^{\kappa\rho}, M^{\mu\nu}\right] = \mathrm{i}(g^{\kappa\nu}M^{\rho\mu} - g^{\kappa\mu}M^{\rho\nu} - g^{\rho\nu}M^{\kappa\mu} + g^{\rho\mu}M^{\kappa\nu}) \tag{A.7}$$

In this abstract (representation-independent) form the commutation relations (A.7) define the Lie algebra of the Lorentz group. In any representation, the finite transformations $U(\omega)$ can be then written as

$$U(\omega) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\kappa\rho}M^{\kappa\rho}\right) \tag{A.8}$$

with $M^{\kappa\rho}$ in the appropriate representation.

The connection of the group $SL(2, C)$ (the group of 2×2 complex matrices M of determinant 1) to the Lorentz group is analogous to the connection of the group $SU(2)$ of the two-dimensional unitary unimodular matrices to the rotation group in three dimensions (for example, Werle (1966)). It can be established through the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.9}$$

(note that $\sigma^i\sigma^j = \delta^{ij} + \mathrm{i}\varepsilon^{ijk}\sigma^k$, $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$) in the following way. To every space-time point x^{μ} we can assign a matrix

$$\sigma \cdot x \equiv \sum_{\mu=0}^{3}\sigma^{\mu}x_{\mu} = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} \tag{A.10}$$

where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.11}$$

such that

$$\det(\sigma \cdot x) = x^2 \tag{A.12}$$

The transformation $x \to x'$ is defined by the equation

$$\sigma \cdot x' = M(\sigma \cdot x)M^{\dagger} \qquad \det M = 1 \tag{A.13}$$

Since

$$\det(\sigma \cdot x') = x'^2 = \det(\sigma \cdot x) = x^2 \tag{A.14}$$

the transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}(M)\, x^{\nu} \tag{A.15}$$

is a Lorentz transformation. The group $SL(2,C)$ is homomorphic to the restricted Lorentz group (i.e. its proper orthochronous subgroup) $\det\Lambda = +1$, $\Lambda^0{}_0 \geq 1$: $L_+^{\uparrow} = SL(2,C)/Z_2$ (because M and $-M$ generate the same Lorentz transformation).

The group $SL(2,C)$ has two inequivalent two-dimensional (fundamental) representations

$$\lambda'_{\alpha}(x') = M_{\alpha}{}^{\beta}\,\lambda_{\beta}(x) \tag{A.16}$$

$$\bar{\chi}'^{\dot{\alpha}}(x') = (M^{\dagger -1})^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}(x) \tag{A.17}$$

where λ and χ are two-component complex vectors (dots are introduced to remind us that the indices α and $\dot{\alpha}$ are the indices of the two different representations and cannot be contracted) and x' is given by (A.15). Two other possible two-dimensional representations,

$$\lambda'^{\alpha} = (M^{T-1})^{\alpha}{}_{\beta}\,\lambda^{\beta} \equiv \lambda^{\beta}(M^{-1})_{\beta}{}^{\alpha} \tag{A.18}$$

and

$$\bar{\chi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}}\,\bar{\chi}_{\dot{\beta}} \equiv \bar{\chi}_{\dot{\beta}}(M^{\dagger})^{\dot{\beta}}{}_{\dot{\alpha}} \tag{A.19}$$

are, as can be easily checked, equivalent to the representations (A.16) and (A.17), respectively, via the unitary transformations

$$\left.\begin{aligned} \lambda^{\alpha} &= \varepsilon^{\alpha\beta}\lambda_{\beta} \quad & \bar{\chi}^{\dot{\alpha}} &= \bar{\chi}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} \\ \lambda_{\alpha} &= \varepsilon_{\alpha\beta}\lambda^{\beta} \quad & \bar{\chi}_{\dot{\alpha}} &= \bar{\chi}^{\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\alpha}} \end{aligned}\right\} \tag{A.20}$$

where epsilons are defined by $\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = (\mathrm{i}\sigma_2)^{\alpha\beta}$, i.e.

$$\left.\begin{aligned} \varepsilon^{12} &= -\varepsilon^{21} = -\varepsilon_{12} = \varepsilon_{21} = 1 \\ \varepsilon^{\dot{1}\dot{2}} &= -\varepsilon^{\dot{2}\dot{1}} = -\varepsilon_{\dot{1}\dot{2}} = \varepsilon_{\dot{2}\dot{1}} = 1 \end{aligned}\right\} \tag{A.21}$$

and satisfy

$$\left.\begin{aligned} \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} &= \delta_{\alpha}^{\gamma} \\ \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\beta}\dot{\gamma}} &= \delta_{\dot{\gamma}}^{\dot{\alpha}} \end{aligned}\right\} \tag{A.22}$$

Since matrices M are unimodular, the εs are $SL(2, C)$ invariant tensors:

$$\left.\begin{aligned}
&\varepsilon_{\alpha\beta} = M_\alpha{}^\gamma M_\beta{}^\delta \varepsilon_{\gamma\delta} \qquad \varepsilon^{\alpha\beta} = \varepsilon^{\gamma\delta} M_\gamma{}^\alpha M_\delta{}^\beta \\
&\varepsilon_{\dot\alpha\dot\beta} = \varepsilon_{\dot\gamma\dot\delta} (M^{\dagger-1})^{\dot\gamma}{}_{\dot\alpha} (M^{\dagger-1})^{\dot\delta}{}_{\dot\beta} \qquad \varepsilon^{\dot\alpha\dot\beta} = (M^{\dagger-1})^{\dot\alpha}{}_{\dot\gamma} (M^{\dagger-1})^{\dot\beta}{}_{\dot\delta} \varepsilon^{\dot\gamma\dot\delta}
\end{aligned}\right\} \tag{A.23}$$

From (A.16), (A.18) and (A.17), (A.19), using anticommutativity of ϕ, λ, it follows that the combinations

$$\left.\begin{aligned}
\lambda \cdot \phi &\equiv \lambda^\alpha \phi_\alpha = \phi^\alpha \lambda_\alpha = -\lambda_\alpha \phi^\alpha = \phi \cdot \lambda \\
\bar\eta \cdot \bar\chi &\equiv \bar\eta_{\dot\beta} \bar\chi^{\dot\beta} = \bar\chi_{\dot\beta} \bar\eta^{\dot\beta} = -\bar\eta^{\dot\beta} \bar\chi_{\dot\beta} = \bar\chi \cdot \bar\psi
\end{aligned}\right\} \tag{A.24}$$

are Lorentz-invariant.

To make the connection between the $SL(2, C)$ representations ((A.16), (A.17)) and the Lorentz group more explicit recall that the Lorentz transformations (A.8) can also be parametrized in terms of

$$N_\pm^i = \tfrac{1}{2}(J^i \pm \mathrm{i} K^i) \tag{A.25}$$

where

$$J^i = \tfrac{1}{2}\varepsilon^{ijk} M^{jk}, \qquad K^i = M^{0i} \tag{A.26}$$

satisfy

$$\left.\begin{aligned}
\left[J^i, J^j\right] &= \mathrm{i}\varepsilon^{ijk} J^k \\
\left[K^i, K^j\right] &= -\mathrm{i}\varepsilon^{ijk} J^k \\
\left[J^i, K^j\right] &= \mathrm{i}\varepsilon^{ijk} K^k
\end{aligned}\right\} \tag{A.27}$$

and

$$\left.\begin{aligned}
\left[N_\pm^i, N_\pm^j\right] &= \mathrm{i}\varepsilon^{ijk} N_\pm^k \\
\left[N_+^i, N_-^j\right] &= 0
\end{aligned}\right\} \tag{A.28}$$

We see that the Lie algebra of the Lorentz group can locally be represented as a direct product of two (complexified) $SU(2)$ algebras. However, the resulting two $SL(2, C)$ groups† are not independent. This follows from the fact that their group parameters must be complex conjugate to each other:

$$\begin{aligned}
&\exp\left(-\mathrm{i}\omega_{0i} M^{0i} - \frac{\mathrm{i}}{2}\omega_{ij} M^{ij}\right) \equiv \\
&\qquad \exp\left(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i\right) = \exp\left(-\mathrm{i}\zeta_i N_+^i - \mathrm{i}\zeta_i^* N_-^i\right)
\end{aligned} \tag{A.29}$$

with $\zeta_i \equiv \eta_i - \mathrm{i}\xi_i$. The representations can be built by exploiting this decomposition. Two obvious ways in which the commutation relations (A.27) can be satisfied

† A complexified $SU(2)$ is just the $SL(2, C)$ group.

are

$$r_1(N_+^i) = \tfrac{1}{2}\sigma^i, \quad r_1(N_-^j) = 0 \tag{A.30}$$

$$r_2(N_+^i) = 0, \quad r_2(N_-^j) = \tfrac{1}{2}\sigma^j \tag{A.31}$$

It is clear that the finite-dimensional (non-unitary) representations of the Lorentz group can be labelled by the pair (m, n), where $m(m+1)$ is the eigenvalue of the $N_+^i N_+^i$ operator and $n(n+1)$ is the eigenvalue of the $N_-^i N_-^i$ operator. Thus, the representations (A.30) and (A.31) correspond to the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. Since $J^i = N_+^i + N_-^i$, we can identify the spin of the representation with $m+n$. Using (A.25), (A.30) and (A.31) we have

$$r_1\big(\exp(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i)\big) = \exp\left(-\frac{\mathrm{i}}{2}\sigma^i(\eta_i - \mathrm{i}\xi_i)\right) \tag{A.32}$$

$$r_2\big(\exp(-\mathrm{i}\xi_i K^i - \mathrm{i}\eta_i J^i)\big) = \exp\left(-\frac{\mathrm{i}}{2}\sigma^i(\eta_i + \mathrm{i}\xi_i)\right) \tag{A.33}$$

Identifying (A.32) with $M_\alpha{}^\beta$ (acting on spinors λ_β) we must identify (A.33) with $(M^{\dagger -1})^{\dot\alpha}{}_{\dot\beta}$ which acts on spinors $\bar\chi^{\dot\beta}$. Introducing two sets of matrices $(\sigma^\mu)_{\alpha\dot\beta}$ and $(\bar\sigma^\mu)^{\dot\alpha\beta}$ defined by

$$\sigma^\mu = (I, \sigma^i), \quad \bar\sigma^\mu = (I, -\sigma^i) \tag{A.34}$$

and satisfying

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu = 2g^{\mu\nu} \tag{A.35}$$

the Lorentz group generators in representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ can be written compactly as

$$\left.\begin{aligned} r_1(M^{\mu\nu}) &= \tfrac{1}{2}\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) \\ r_2(M^{\mu\nu}) &= \tfrac{1}{2}\bar\sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) \end{aligned}\right\} \tag{A.36}$$

with the obvious assignment of spinor indices: $(\sigma^{\mu\nu})_\alpha{}^\beta$ and $(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$. Here $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are two-dimensional matrices. We thus have

$$\lambda'_\alpha(x') = \{\exp[-(\mathrm{i}/4)\omega_{\mu\nu}\sigma^{\mu\nu}]\}_\alpha{}^\beta\lambda_\beta(x) \tag{A.37}$$

$$\bar\chi'^{\dot\alpha}(x') = \{\exp[-(\mathrm{i}/4)\omega_{\mu\nu}\bar\sigma^{\mu\nu}]\}^{\dot\alpha}{}_{\dot\beta}\bar\chi^{\dot\beta}(x) \tag{A.38}$$

As can be verified using (A.35), the matrices σ^μ and $\bar\sigma^\mu$ defined in (A.34) satisfy the following identities

$$\left.\begin{aligned} \tfrac{1}{2}\sigma^{\kappa\rho}\sigma^\mu - \tfrac{1}{2}\sigma^\mu\bar\sigma^{\kappa\rho} - \mathrm{i}(\sigma^\kappa g^{\rho\mu} - \sigma^\rho g^{\kappa\mu}) = 0 \\ \tfrac{1}{2}\bar\sigma^{\kappa\rho}\bar\sigma^\mu - \tfrac{1}{2}\bar\sigma^\mu\sigma^{\kappa\rho} - \mathrm{i}(\bar\sigma^\kappa g^{\mu\rho} - \bar\sigma^\rho g^{\kappa\mu}) = 0 \end{aligned}\right\} \tag{A.39}$$

Taking into account (A.16)–(A.19), (A.37), (A.38) and the explicit form of the Lorentz group generators in the vector representation (A.6) the identities (A.39) turn out to be the infinitesimal forms of the relations

$$\left.\begin{aligned} \Lambda^{\nu}{}_{\mu} M \sigma^{\mu} M^{\dagger} &= \sigma^{\nu} \\ \Lambda^{\nu}{}_{\mu} M^{\dagger -1} \bar{\sigma}^{\mu} M^{-1} &= \bar{\sigma}^{\nu} \end{aligned}\right\} \tag{A.40}$$

Thus, σ^{μ} and $\bar{\sigma}^{\mu}$ are numerically invariant tensors of the Lorentz group provided the index μ transforms according to the vector representation of $O(1,3)$. They are therefore the Clebsch–Gordan coefficients which relate the representation $(\frac{1}{2}, \frac{1}{2})$ of $SL(2, C)$ to the vector representation of $O(1,3)$.

From the completeness relation

$$(\sigma^{\mu})_{\alpha\dot{\beta}} (\bar{\sigma}_{\mu})^{\dot{\kappa}\rho} = 2\delta^{\rho}_{\alpha} \delta^{\dot{\kappa}}_{\dot{\beta}} \tag{A.41}$$

one derives the Fierz transformation for anticommuting (Grassmann) Weyl spinors†

$$(\lambda\psi)(\bar{\chi}\bar{\eta}) = \tfrac{1}{2}(\lambda\sigma^{\mu}\bar{\chi})(\psi\sigma_{\mu}\bar{\eta}) = -\tfrac{1}{2}(\lambda\sigma^{\mu}\bar{\chi})(\bar{\eta}\bar{\sigma}_{\mu}\psi) \tag{A.42}$$

Other useful relations are

$$\left.\begin{aligned} \sigma^{\mu}\bar{\sigma}^{\nu} &= g^{\mu\nu} - \mathrm{i}\sigma^{\mu\nu} \\ \bar{\sigma}^{\mu}\sigma^{\nu} &= g^{\mu\nu} - \mathrm{i}\bar{\sigma}^{\mu\nu} \end{aligned}\right\} \tag{A.43}$$

and

$$\left.\begin{aligned} \varepsilon_{\dot{\kappa}\dot{\alpha}} (\bar{\sigma}^{\mu})^{\dot{\alpha}\beta} \varepsilon_{\beta\delta} &= (\sigma^{\mu})_{\delta\dot{\kappa}} \\ \varepsilon^{\delta\beta} (\sigma^{\mu})_{\beta\dot{\alpha}} \varepsilon^{\dot{\alpha}\dot{\kappa}} &= (\bar{\sigma}^{\mu})^{\dot{\kappa}\delta} \end{aligned}\right\} \tag{A.44}$$

from which (for anticommuting spinors λ, $\bar{\chi}$) follows the identity

$$(\bar{\chi}\bar{\sigma}^{\mu}\lambda) = -(\lambda\sigma^{\mu}\bar{\chi}) \tag{A.45}$$

Finally, from the hermiticity of the σ^{μ}s we have:

$$(\sigma^{\mu}_{\alpha\dot{\beta}})^{*} = \sigma^{\mu}_{\beta\dot{\alpha}} \qquad ((\bar{\sigma}^{\mu})^{\dot{\alpha}\beta})^{*} = (\bar{\sigma}^{\mu})^{\dot{\beta}\alpha} \tag{A.46}$$

Four-component Dirac spinors are built from two Weyl spinors transforming as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of $SL(2, C)$ (and carrying the same charges under all other transformations):

$$\Psi = \begin{pmatrix} \lambda_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\Psi} = \left(\chi^{\alpha}, \bar{\lambda}_{\dot{\alpha}}\right) = \Psi^{\dagger}\gamma^{0} \tag{A.47}$$

† For commuting (c-numbers) spinors one gets + sign on the r.h.s. of (A.42).

We have, for instance, $\lambda \cdot \chi + \bar{\lambda} \cdot \bar{\chi} = \bar{\Psi}\Psi, \bar{\lambda} \cdot \bar{\chi} - \lambda \cdot \chi = \bar{\Psi}\gamma^5\Psi, \bar{\lambda}\bar{\sigma}^\mu\lambda + \chi\sigma^\mu\bar{\chi} = \bar{\Psi}\gamma^\mu\Psi$. For $\chi \equiv \lambda$ one obtains in this way the Majorana spinors. Dirac matrices have in this representation the following form:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.48}$$

This is the so-called chiral (or Weyl) representation. The matrix γ^5, defined as

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.49}$$

has in this representation the form

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{A.50}$$

The standard Dirac representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{A.51}$$

is obtained with the help of the unitary transformation:

$$\gamma^\mu_{\mathrm{Dirac}} = U^\dagger \gamma^\mu_{\mathrm{Weyl}} U, \quad \Psi_{\mathrm{Dirac}} = U^\dagger \Psi_{\mathrm{Weyl}}, \quad \text{with} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \tag{A.52}$$

It is also useful to introduce chiral Dirac spinors in the chiral representation

$$\Psi_{\mathrm{L}} = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \quad \Psi_{\mathrm{R}} = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{A.53}$$

which satisfy the relations

$$P_{\mathrm{L,R}}\Psi_{\mathrm{L,R}} \equiv \frac{1 \mp \gamma^5}{2}\Psi_{\mathrm{L,R}} = \Psi_{\mathrm{L,R}}, \quad P_{\mathrm{L,R}}\Psi_{\mathrm{R,L}} = 0 \tag{A.54}$$

These can be introduced with no restriction on the internal charges of λ and χ. In addition, every Dirac spinor can be decomposed as $\Psi = \Psi_{\mathrm{L}} + \Psi_{\mathrm{R}}$.

Under the Lorentz transformation both, Ψ_{Dirac} and $\Psi_{\mathrm{L,R}}$ transform according to

$$\Psi'(\Lambda x) = \exp\left(-\frac{\mathrm{i}}{4}\omega_{\rho\kappa}\sigma^{\rho\kappa}\right)\Psi(x) \tag{A.55}$$

with

$$\sigma^{\rho\kappa} = \frac{\mathrm{i}}{2}\left[\gamma^\rho, \gamma^\kappa\right] \tag{A.56}$$

which are four-dimensional matrices. However, the reducibility of the four-dimensional representation is manifest only in the Weyl form where

$$\sigma^{\rho\kappa}_{4\times 4} = \begin{pmatrix} \sigma^{\rho\kappa} & 0 \\ 0 & \bar{\sigma}^{\rho\kappa} \end{pmatrix} \tag{A.57}$$

Note that we use the same symbol for four- and two-dimensional matrices $\sigma^{\mu\nu}$.

Solutions of the free Weyl and Dirac equations and their properties

We begin by recalling the properties of the free particle solutions of the massless Weyl equation

$$\mathrm{i}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\partial_{\mu}\lambda_{\beta}(x) = 0 \tag{A.58}$$

Writing the positive and negative frequency solutions as

$$\left.\begin{aligned} \lambda_{\text{pos}}(x) &= \exp(-\mathrm{i}k\cdot x)\,a(\mathbf{k}) \\ \lambda_{\text{neg}}(x) &= \exp(+\mathrm{i}k\cdot x)\,b(\mathbf{k}) \end{aligned}\right\} \tag{A.59}$$

(lower case indices are understood) multiplying (A.58) by σ^0 from the left and recalling the definitions (A.34) we get (remember $E \equiv k^0 = |\mathbf{k}|$)

$$\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}a(\mathbf{k}) = -a(\mathbf{k}) \tag{A.60}$$

and the same equation for $b(\mathbf{k})$. For $\mathbf{k}/|\mathbf{k}| = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ the well-known solutions to these equations read

$$\left.\begin{aligned} a(\mathbf{k}) &= \mathcal{N}_a \exp(\mathrm{i}\varphi_a)a_-(\mathbf{k}) \\ b(\mathbf{k}) &= \mathcal{N}_b \exp(\mathrm{i}\varphi_b)a_-(\mathbf{k}) \end{aligned}\right\} \tag{A.61}$$

where $\mathcal{N}_{a,b}$ are normalization factors, and

$$a_-(\mathbf{k}) = \begin{pmatrix} -\sin(\theta/2)\exp(-\mathrm{i}\phi) \\ \cos(\theta/2) \end{pmatrix} \tag{A.62}$$

is normalized to unity.

Thus, the solution $\lambda_{\text{pos}}(x)$ describes a fermion state with the spin projection onto the direction of its momentum ($+\mathbf{k}$) equal to $-1/2$, i.e. with helicity $-1/2$. The solution $\lambda_{\text{neg}}(x)$, being an eigenstate of the three-momentum operator $\hat{\mathbf{p}} \equiv -\mathrm{i}\partial/\partial\mathbf{x}$ with the eigenvalue $-\mathbf{k}$, consequently describes a negative energy fermion with helicity $+1/2$. In the Dirac sea interpretation it represents therefore the absence of a negative energy particle with momentum $-\mathbf{k}$ and helicity $+1/2$, i.e. the presence of a positive energy antiparticle with momentum $+\mathbf{k}$ and helicity again $+1/2$ (because the absence of the spin antiparallel to $-\mathbf{k}$ is equivalent to the presence of the spin antiparallel to $\mathbf{k}$). This interpretation becomes more obvious in the second quantization language.

Similarly, for the solutions of the equation

$$\mathrm{i}(\sigma^{\mu})_{\alpha\dot{\beta}}\partial_{\mu}\bar{\chi}^{\dot{\beta}}(x)=0 \tag{A.63}$$

we get

$$\left.\begin{aligned}\bar{\chi}_{\mathrm{pos}}(x)&=\exp(-\mathrm{i}k\cdot x)a'(\mathbf{k})=\mathcal{N}_{a'}\exp(-\mathrm{i}k\cdot x)\exp(\mathrm{i}\varphi_a')a_+(\mathbf{k})\\ \bar{\chi}_{\mathrm{neg}}(x)&=\exp(+\mathrm{i}k\cdot x)b'(\mathbf{k})=\mathcal{N}_{b'}\exp(+\mathrm{i}k\cdot x)\exp(\mathrm{i}\varphi_b')a_+(\mathbf{k})\end{aligned}\right\} \tag{A.64}$$

(upper case dotted indices are understood), where

$$a_+(\mathbf{k})=\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\exp(\mathrm{i}\phi)\end{pmatrix} \tag{A.65}$$

is normalized to unity and such that we have

$$\boldsymbol{\sigma}\cdot\frac{\mathbf{k}}{|\mathbf{k}|}a'(\mathbf{k})=a'(\mathbf{k}) \tag{A.66}$$

and the same relation for $b'(\mathbf{k})$. Therefore, $\bar{\chi}_{\mathrm{pos}}(x)$ represents a $+1/2$ helicity particle, whereas $\bar{\chi}_{\mathrm{neg}}(x)$ describes a $-1/2$ helicity antiparticle. Normalization factors $\mathcal{N}$ are determined by requiring that there are $2E$ particles per unit volume, i.e. that

$$\int_{\text{unit volume}}\mathrm{d}^3x\,\bar{\lambda}\bar{\sigma}^0\lambda=\int_{\text{unit volume}}\mathrm{d}^3x\,\chi\sigma^0\bar{\chi}=2E \tag{A.67}$$

leading to $\mathcal{N}=(2E)^{1/2}$ for all $\mathcal{N}$. With this normalization we get

$$\left.\begin{aligned}a(\mathbf{k})_{\alpha}a^*(\mathbf{k})_{\dot{\beta}}&=b(\mathbf{k})_{\alpha}b^*(\mathbf{k})_{\dot{\beta}}=k_{\mu}\sigma^{\mu}_{\alpha\dot{\beta}}\\ a'(\mathbf{k})^{\dot{\alpha}}a'^*(\mathbf{k})^{\beta}&=b'(\mathbf{k})^{\dot{\alpha}}b'^*(\mathbf{k})^{\beta}=k_{\mu}(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\end{aligned}\right\} \tag{A.68}$$

which are the analogues of (A.80) for four-component spinors.

In the four-component spinor language ((A.47)–(A.54)) (A.58) and (A.63) both take the form

$$\mathrm{i}\gamma^{\mu}\partial_{\mu}\Psi_{\mathrm{L,R}}=0 \tag{A.69}$$

supplemented with different conditions $\gamma^5\Psi_{\mathrm{L}}=-\Psi_{\mathrm{L}}$ and $\gamma^5\Psi_{\mathrm{R}}=\Psi_{\mathrm{R}}$, respectively which in the Weyl representation are trivially satisfied with

$$\Psi_{\mathrm{L}}=\begin{pmatrix}\lambda_{\alpha}\\ 0\end{pmatrix},\quad \Psi_{\mathrm{R}}=\begin{pmatrix}0\\ \bar{\chi}_{\dot{\alpha}}\end{pmatrix} \tag{A.70}$$

Thus, for massless fermions chirality eigenstates are at the same time helicity eigenstates. Chiral spinors in the Dirac representation are obtained by (A.52).

The Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0 \tag{A.71}$$

has four independent free-particle solutions (two of which correspond to positive and two other to negative energies) given by

$$\left.\begin{aligned} \Psi(x)_{\mathrm{pos}} &= \exp(-ip\cdot x)\, u(\mathbf{p}, s), \quad s = 1, 2 \\ \Psi(x)_{\mathrm{neg}} &= \exp(+ip\cdot x)\, v(\mathbf{p}, s), \quad s = 1, 2 \end{aligned}\right\} \tag{A.72}$$

where $p^\mu = (E, \mathbf{p})$, $E \equiv p^0 = (\mathbf{p}^2 + m^2)^{1/2} > 0$ is the energy–momentum of the particle and the s numbers its polarizations in a chosen basis (see below). From (A.71) we get

$$(\not p - m)u(\mathbf{p}, s) = 0, \quad (\not p + m)v(\mathbf{p}, s) = 0 \tag{A.73}$$

The solutions of (A.73), valid in any representation of the Dirac matrices, read:

$$\left.\begin{aligned} u(\mathbf{p}, s) &= \frac{\not p + m}{[2m(m+E)]^{1/2}} u(\mathbf{O}, s) \\ v(\mathbf{p}, s) &= \frac{-\not p + m}{[2m(m+E)]^{1/2}} v(\mathbf{O}, s) \end{aligned}\right\} \tag{A.74}$$

In (A.74) $u(\mathbf{O}, s)$ and $v(\mathbf{O}, s)$ are four linearly independent solutions in the rest frame of the particle in which $p^\mu_{\mathrm{rest}} = (m, \mathbf{O})$. Obviously, $p^\mu = \Lambda^\mu{}_\nu p^\nu_{\mathrm{rest}}$, with $\Lambda^\mu{}_\nu$ being the appropriate Lorentz boost. In the rest frame $s = 1$ (2) numbers the solutions with spin projection onto a chosen unit spin vector $\mathbf{s} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ equal $+\frac{1}{2}$ $(-\frac{1}{2})$. In the *Dirac representation* (A.51) of the γ^μ matrices, the two linearly independent (and orthogonal) solutions with positive energy read

$$u(\mathbf{O}, 1) = \mathcal{N} \begin{pmatrix} a_+(\mathbf{s}) \\ 0 \\ 0 \end{pmatrix}, \quad u(\mathbf{O}, 2) = \mathcal{N} \begin{pmatrix} a_-(\mathbf{s}) \\ 0 \\ 0 \end{pmatrix} \tag{A.75}$$

where $\mathcal{N}$ are normalization factors and two-component spinors $a_\pm$ are defined in (A.65) and (A.62). For negative energy solutions we have

$$v(\mathbf{O}, 1) = \mathcal{N} \begin{pmatrix} 0 \\ 0 \\ -a_-(\mathbf{s}) \end{pmatrix}, \quad v(\mathbf{O}, 2) = \mathcal{N} \begin{pmatrix} 0 \\ 0 \\ a_+(\mathbf{s}) \end{pmatrix} \tag{A.76}$$

(to match the Dirac sea interpretation, the solutions with $+\frac{1}{2}$ and $-\frac{1}{2}$ spin projections are numbered in reversed order). Eqs. (A.74)–(A.76) are sufficient to explicitly construct four-component spinors for massive fermions in the Dirac representation (see, for example, Haber (1993)). Using the transformation (A.52) one can also construct them in the chiral representation. The helicity eigenstates

$$\frac{\Sigma p}{|p|}u(p,h) = \pm\frac{1}{2}u(p,h) \tag{A.77}$$

can be constructed by specifying the directions of s and p to be parallel: $\sigma p a^+(s) = a^+(s)$.

The spinors (A.74) are normalized as follows ($\bar{u} \equiv u^\dagger\gamma^0$; cf. (A.47) and (A.48)):

$$\left.\begin{aligned} \bar{u}(\mathbf{p},s)u(\mathbf{p},s') &= \mathcal{N}^2\delta_{ss'}, \quad & \bar{u}(\mathbf{p},s)v(\mathbf{p},s') = 0, \\ \bar{v}(\mathbf{p},s)v(\mathbf{p},s') &= -\mathcal{N}^2\delta_{ss'}, \quad & \bar{v}(\mathbf{p},s)u(\mathbf{p},s') = 0 \end{aligned}\right\} \tag{A.78}$$

or, equivalently,

$$u^\dagger(\mathbf{p},s)u(\mathbf{p},s') = \mathcal{N}^2\frac{E}{m}\delta_{ss'}, \quad v^\dagger(\mathbf{p},s)v(\mathbf{p},s') = \mathcal{N}^2\frac{E}{m}\delta_{ss'} \tag{A.79}$$

As for the massless chiral spinors we choose $\mathcal{N} = (2m)^{1/2}$ corresponding to $2E$ particles in a unit volume. By explicit calculation we then find

$$\left.\begin{aligned} \sum_{s=1,2} u(\mathbf{p},s)\bar{u}(\mathbf{p},s) &= \not{p} + m \\ \sum_{s=1,2} v(\mathbf{p},s)\bar{v}(\mathbf{p},s) &= \not{p} - m \end{aligned}\right\} \tag{A.80}$$

One can also define projection operators onto the states with positive and negative energy

$$\Lambda_\pm(p) \equiv \frac{\pm\not{p} + m}{2m} \tag{A.81}$$

with properties $\Lambda_\pm^2(p) = \Lambda_\pm(p)$, $\Lambda_\pm(p)\Lambda_\mp(p) = 0$, $\Lambda_+(p) + \Lambda_-(p) = 1$ and satisfying

$$\left.\begin{aligned} \Lambda_+(p)\Psi(x)_{\text{pos}} &= \Psi(x)_{\text{pos}}, \quad & \Lambda_-(p)\Psi(x)_{\text{pos}} = 0, \\ \Lambda_-(p)\Psi(x)_{\text{neg}} &= \Psi(x)_{\text{neg}}, \quad & \Lambda_+(p)\Psi(x)_{\text{neg}} = 0 \end{aligned}\right\} \tag{A.82}$$

Projection operators on the state which in the rest frame has spin parallel (antiparallel) to the direction **s** for positive (negative) energy solution reads

$$P(s) = \frac{1 + \gamma^5\not{s}}{2} \tag{A.83}$$

where the four-vector s^μ ($s^2 = -1$, $p \cdot s = 0$) is related to **s** by the appropriate Lorentz boost: $s^\mu = \Lambda^\mu{}_\nu s^\nu_{\text{rest}}$, $s^\nu_{\text{rest}} = (0, \mathbf{s})$. Because of the reverse assignment (A.76) $P(s)$ projects the positive energy solutions onto the states with spin in the rest frame parallel to **s** and negative energy solutions onto the states with spin antiparallel to $\mathbf{s}$ (which in the Dirac sea interpretation represent the antiparticle with spin parallel to $\mathbf{s}$).

A useful basis for the spin states is obtained by choosing the four-vector s^μ in the form:

$$s^\mu = \left(\frac{|\mathbf{p}|}{m}, \frac{E}{m}\frac{\mathbf{p}}{|\mathbf{p}|}\right) \tag{A.84}$$

which is obtained from the unit vector $\mathbf{s} = \mathbf{p}/|\mathbf{p}|$ by the boost relating the rest frame of the particle to the frame where it has a momentum $\mathbf{p}$. In this case $P(s)$ projects the positive energy solutions onto the states with spin parallel to $\mathbf{s}$, i.e. to the momentum of the state and, for negative energy solutions, onto the states with spin antiparallel to $\mathbf{s}$. Consequently, $P(s)$ projects onto the positive helicity states for both signs of energy since for negative energy solutions the momentum of the state (in the sense of the eigenvalue of the operator $-\mathrm{i}\partial/\partial\mathbf{x}$) is $-\mathbf{p}$, i.e. parallel to $-\mathbf{s}$. This is confirmed by noting that for s^μ given by (A.84)

$$P(s)\Lambda_\pm(p) = \frac{1}{2}\left(1 \pm \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|}\right)\Lambda_\pm(p) \tag{A.85}$$

where

$$\mathbf{\Sigma} \equiv \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{A.86}$$

is the spin operator. Thus, for the negative energy solutions the operator (A.85) projects onto the states which in the Dirac sea interpretation represent the states of the antiparticle with physical momentum $+\mathbf{p}$ and the spin parallel to $\mathbf{p}$, i.e. onto the positive helicity states of the antiparticle.

For the matrix elements of processes involving massive Majorana particles external wave-functions are provided by the solutions of the free Dirac equation.†

Parity

Transformation of the space reflection is the change of the description of the physical system from the one given in the right-handed coordinate frame to the description in the left-handed frame. Thus, $x^\mu \to x^{\mu\prime}$ with $x^{0\prime} = x^0, \mathbf{x}' = -\mathbf{x}$.

A positive (negative) energy solution (A.59) of the Weyl equation (A.58) for left-handed fields upon substituting $\mathbf{x} = -\mathbf{x}'$ becomes the positive (negative) energy solution of (A.63) (in the primed variables) for right-handed fields with $\mathbf{k}' = -\mathbf{k}$. Similarly, the solutions of the equation for right-handed fields become

† Massless Majorana fermions are indistinguishable from massless Weyl fermions (there are just two states connected by CPT).

in the reflected frame the solutions of the equation for the left-handed fields with reversed momentum. This follows from the the following relations:

$$\left.\begin{aligned}(\sigma^0)_{\alpha\dot{\beta}}a'(-\mathbf{k})^{\dot{\beta}} &= \mathrm{i}\exp[\mathrm{i}(\varphi'_a - \varphi_a)]a(\mathbf{k})_\alpha \\ (\bar{\sigma}^0)^{\dot{\alpha}\beta}a(-\mathbf{k})_\beta &= \mathrm{i}\exp[\mathrm{i}(\varphi_a - \varphi'_a)]a'(\mathbf{k})^{\dot{\alpha}}\end{aligned}\right\} \tag{A.87}$$

and similar relations for $b'(-\mathbf{k})$ and $b(-\mathbf{k})$.

For the four-component Dirac spinors, the positive (negative) energy solutions (A.72) of the Dirac equation (A.71) with momentum $\mathbf{p}$ and spin projection label s upon substituting $\mathbf{x} = -\mathbf{x}'$ and multiplying by γ^0 become the positive (negative) energy solutions of the Dirac equation in the reflected frame with momentum $-\mathbf{p}$ and the same spin label s. This follows from the relations

$$u(-\mathbf{p}, s) = \gamma^0 u(\mathbf{p}, s), \qquad v(-\mathbf{p}, s) = -\gamma^0 v(\mathbf{p}, s) \tag{A.88}$$

For helicity spinors, from (A.62) and (A.65), we get instead

$$\left.\begin{aligned}u(-p, -h) &= (-1)^{\frac{1}{2}-h}\exp(-2\mathrm{i}h\phi)\gamma^0 u(p, h) \\ v(-p, -h) &= (-1)^{\frac{1}{2}+h}\exp(2\mathrm{i}h\phi)\gamma^0 v(p, h)\end{aligned}\right\} \tag{A.89}$$

Time reversal

Transformation of the time reversal can be viewed as a change of the direction of the time axis of the reference frame. Thus, $t' = -t$, $\mathbf{x}' = \mathbf{x}$. A state with momentum $\mathbf{k}$ and helicity $-1/2$ $(+1/2)$ described originally by a solution ((A.59), (A.64)) of the Weyl equation ((A.58), (A.63)) in the new frame should be described by a solution (with the same sign of the energy) of the Weyl equation ((A.58), (A.63)) (in the primed variables) with momentum $\mathbf{k}' = -\mathbf{k}$. The solutions in the new frame $\lambda_{T\alpha}(x')$ and $\bar{\chi}_T^{\dot{\alpha}}(x')$ are obtained from the original ones by the operations

$$\left.\begin{aligned}\lambda_{T\alpha}(x') &\equiv \exp(\mathrm{i}\gamma)(\mathrm{i}\sigma^1\bar{\sigma}^3)_\alpha{}^\beta(\lambda^\star(x))_\beta \\ \bar{\chi}_T^{\dot{\alpha}}(x') &\equiv \exp(\mathrm{i}\bar{\gamma})(\mathrm{i}\bar{\sigma}^1\sigma^3)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{\chi}^\star(x))^{\dot{\beta}}\end{aligned}\right\} \tag{A.90}$$

Substituting $t = -t'$ and taking the complex conjugate brings back the exponential factors to the right form $\exp[-\mathrm{i}(Et' - \mathbf{k}'\cdot\mathbf{x}')]$ $(\exp[+\mathrm{i}(Et' - \mathbf{k}'\cdot\mathbf{x}')])$ for positive (negative) energy solutions. The correct behaviour of the spinor factors follow from the properties

$$(\mathrm{i}\sigma^1\bar{\sigma}^3)a(-\mathbf{k}) = \exp(2\mathrm{i}\varphi_a)a^\star(\mathbf{k}), \quad (\mathrm{i}\bar{\sigma}^1\sigma^3)a'(-\mathbf{k}) = -\exp(2\mathrm{i}\varphi'_a)a'^\star(\mathbf{k}) \tag{A.91}$$

$$\mathrm{i}\sigma^1\bar{\sigma}^3 = \mathrm{i}\bar{\sigma}^1\sigma^3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} \tag{A.92}$$

and similar relation for $b(\mathbf{k})$ and $b'(\mathbf{k})$.

For the four-component Dirac spinors, the positive (negative) energy solutions (A.72) of the Dirac equation (A.71) with momentum $\mathbf{p}$ and spin projection label s upon substituting $t = -t'$, complex conjugating and multiplying by $\mathrm{i}\gamma^1\gamma^3$ become the positive (negative) energy solutions of the Dirac equation in the reflected frame with momentum $-\mathbf{p}$ and opposite spin $s' = -s$. This follows from the relations

$$\left.\begin{aligned} \mathrm{i}\gamma^1\gamma^3 u(-\mathbf{p}, s) &= \mathrm{i}(-1)^s u^\star(\mathbf{p}, 3-s) \\ \mathrm{i}\gamma^1\gamma^3 v(-\mathbf{p}, s) &= -\mathrm{i}(-1)^s v^\star(\mathbf{p}, 3-s) \end{aligned}\right\} \tag{A.93}$$

Charge conjugation

Consider a charged field Φ transforming as $\Phi \rightarrow \exp(-\mathrm{i}q\Theta)\Phi$ under the $U(1)$ group of gauge transformations interacting with a classical electromagnetic potential $A_\mu(x)$. The positive energy solution† $\Phi_{\mathrm{pos}}(x)$ of the equation

$$[(\partial + \mathrm{i}qeA)^2 + m^2]\Phi_{\mathrm{pos}}(x) = 0 \tag{A.94}$$

describes the behaviour of a particle carrying charge q in the vector potential $A_\mu(x)$. The negative energy solution $\Phi_{\mathrm{neg}}(x)$ cannot by itself have such an interpretation. However, the function $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x) = \exp(\mathrm{i}\gamma)(\Phi_{\mathrm{neg}}(x))^\star$ having positive energy satisfies the equation

$$[(\partial - \mathrm{i}qeA)^2 + m^2]\Phi^{\mathrm{c}}_{\mathrm{pos}}(x) = 0 \tag{A.95}$$

and describes the behaviour of another particle (called the antiparticle) carrying charge $-q$ in the same potential. It transforms as $\Phi^{\mathrm{c}}_{\mathrm{pos}} \rightarrow \exp(+\mathrm{i}q\Theta)\Phi^{\mathrm{c}}_{\mathrm{pos}}$ under the $U(1)$ group. In the potential $A^{\mathrm{c}}_\mu(x) \equiv -A_\mu(x)$, the solution $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x)$ has obviously (up to a constant phase factor) an identical space-time form as the solution $\Phi_{\mathrm{pos}}(x)$ in the original potential $A_\mu(x)$. Thus, the simultaneous change of a particle for its antiparticle and of the sign of the potential is a symmetry of the theory because the resulting physical system behaves like the original one (their wave-functions are identical). The same applies to a set of fields Φ_i transforming as a representation R of a non-abelian gauge group G and interacting with the non-abelian potential $T^a A^a_\mu(x)$ (T^a are the generators of the group) provided we substitute $T^a A^a_\mu(x) \rightarrow T^a A^{\mathrm{c}a}_\mu(x) = -T^{a\star} A^a_\mu(x)$. $\Phi^{\mathrm{c}}_{\mathrm{pos}}(x)$ transforms as $R^\star$ under the action of G.

For left-handed Weyl fields λ^i, transforming as a representation R of the gauge group, the positive energy solution of the equation

$$\mathrm{i}\bar{\sigma} \cdot (\partial + \mathrm{i}gA^aT^a) \cdot \lambda_{\mathrm{pos}}(x) = 0 \tag{A.96}$$

describes a helicity $h = -1/2$ particle interacting with the external potential

† Strictly speaking, for energy of the solution to be well defined, the potential A_μ should not depend on time.

$T^a A_\mu^a(x)$. The negative energy solution of this equation, $\lambda^i_{\rm neg}(x)$, upon complex conjugation (and raising the spinor index) becomes the positive energy solution of the equation

$$\mathrm{i}\sigma \cdot (\partial - \mathrm{i}g A^a T^{a\star}) \cdot \bar{\lambda}_{\rm pos}(x) = 0 \tag{A.97}$$

and describes a helicity $h = +1/2$ antiparticle ($\bar{\lambda}_{\rm pos}(x)$ transforms as $R^\star$ under the action of the gauge group). This confirms the interpretation given to the negative energy solutions of (A.58) and (A.63). Clearly, there is no choice of $T^a A_\mu^{ca}(x)$ in which this antiparticle would behave like the helicity $h = -1/2$ particle in the original potential. The wave-function of a helicity $h = -1/2$ antiparticle denoted as $\lambda^{\rm c}$ transforming as $R^\star$ under the gauge transformations is a positive energy solution of the equation

$$\mathrm{i}\bar{\sigma} \cdot (\partial - \mathrm{i}g A^a T^{a\star}) \cdot \lambda^{\rm c}_{\rm pos}(x) = 0 \tag{A.98}$$

Obviously, in the potential $T^a A_\mu^{ca}(x) = -T^{a\star} A_\mu^a(x)$ the wave-function of the helicity $h = -1/2$ antiparticle is identical (up to a constant phase factor) to the one of the helicity $h = -1/2$ particle in the original potential $T^a A_\mu^a(x)$. Thus, if the physical system consists of both sets of fields, λ_i and $\lambda_i^{\rm c}$, the charge conjugation is its symmetry.

Positive energy solutions of the equation

$$\left(\mathrm{i}\partial\!\!\!/ - g A\!\!\!/^a T^a - m\right) \Psi_{\rm pos}(x) = 0 \tag{A.99}$$

for the Dirac field describe two polarization states of a massive, spin 1/2 particle and transform as representation R under the action of the gauge group. The positive energy wave-function

$$\Psi^{\rm c}_{\rm pos}(x) = \exp(\mathrm{i}\gamma) C\gamma^0 \Psi^\star_{\rm neg}(x) = \exp(\mathrm{i}\gamma) C \bar{\Psi}^{\rm T}_{\rm neg}(x) \tag{A.100}$$

where $\Psi_{\rm neg}(x)$ is a negative energy solution of (A.99) and the matrix C is chosen so that

$$(C\gamma^0)\gamma^{\mu\star}(C\gamma^0)^{-1} = -\gamma^\mu \tag{A.101}$$

satisfies the equation†

$$\left(\mathrm{i}\partial\!\!\!/ + g A\!\!\!/^a T^{a\star} - m\right) \Psi^{\rm c}_{\rm pos}(x) = 0 \tag{A.102}$$

and (having positive energy) can be interpreted as a wave-function of the corresponding antiparticle in the same potential $A^a T^a$. Obviously, $\Psi^{\rm c}_{\rm pos}(x)$ transforms as $R^\star$. In the potential $T^a A_\mu^{ca}(x) = -T^{a\star} A_\mu^a(x)$ the solution $\Psi^{\rm c}_{\rm pos}(x)$ has the same space-time form as the solution $\Psi_{\rm pos}(x)$ in the original potential and, therefore, charge conjugation is a symmetry of the physical system.

† Of course $\Psi^{\rm c}_{\rm neg}(x) = \exp(\mathrm{i}\gamma) C \bar{\Psi}^{\rm T}_{\rm pos}(x)$ satisfies (A.102) with negative energy.

The matrix C defined by (A.99) can be taken as

$$C = \mathrm{i}\gamma^0\gamma^2 = \begin{pmatrix} 0 & -\mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix} \tag{A.103}$$

which, for example, in the Weyl representation is given by (cf. (A.20) and (A.47))

$$C = \begin{pmatrix} \{\varepsilon_{\alpha\beta}\} & 0 \\ 0 & \{\varepsilon^{\dot\alpha\dot\beta}\} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.104}$$

The matrix C satisfies the following relations:

$$C^{-1} = C^{\dagger} = C^{\mathrm{T}} = -C, \quad C^2 = -I, \quad [C, \gamma^5] = 0 \tag{A.105}$$

Under the operation (A.100) a solution of (A.99) characterized by the momentum and spin four-vectors p^{μ} and s^{μ}, i.e. satisfying

$$\Psi(x) = \left(\frac{\varepsilon\not{p} + m}{2m}\right)\left(\frac{1 + \gamma^5\not{s}}{2}\right)\Psi(x) \tag{A.106}$$

transforms into

$$\begin{aligned} \Psi^{\mathrm{c}}(x) &\equiv \exp(\mathrm{i}\gamma)C\bar{\Psi}^{\mathrm{T}}(x) = \exp(\mathrm{i}\gamma)(C\gamma^0)\left(\frac{\varepsilon\not{p} + m}{2m}\right)^{\star}\left(\frac{1 + \gamma^5\not{s}}{2}\right)^{\star}\Psi^{\star}(x) \\ &= \left(\frac{-\varepsilon\not{p} + m}{2m}\right)\left(\frac{1 + \gamma^5\not{s}}{2}\right)\Psi^{\mathrm{c}}(x) \end{aligned} \tag{A.107}$$

where $\varepsilon = \pm 1$, i.e. the solution with negative energy, momentum $-\mathbf{p}$ and spin antiparallel to $\mathbf{s}$ is transformed into the solution with positive energy, momentum $\mathbf{p}$ and spin parallel to $\mathbf{s}$ etc. Therefore

$$\left.\begin{aligned} C\bar{u}^{\mathrm{T}}(\mathbf{p}, s) &= v(\mathbf{p}, s) \\ C\bar{v}^{\mathrm{T}}(\mathbf{p}, s) &= u(\mathbf{p}, s) \end{aligned}\right\} \tag{A.108}$$

The same relations are valid for the helicity eigenstates. For a field with definite chirality

$$\Psi(x) = \left(\frac{\varepsilon\not{p} + m}{2m}\right)\left(\frac{1 \pm \gamma^5}{2}\right)\Psi(x) \tag{A.109}$$

we get

$$\begin{aligned} \Psi^{\mathrm{c}}(x) &= \exp(\mathrm{i}\gamma)(C\gamma^0)\left(\frac{\varepsilon\not{p} + m}{2m}\right)^{\star}\left(\frac{1 \pm \gamma^5}{2}\right)^{\star}\Psi^{\star}(x) \\ &= \left(\frac{-\varepsilon\not{p} + m}{2m}\right)\left(\frac{1 \mp \gamma^5}{2}\right)\Psi^{\mathrm{c}}(x) \end{aligned} \tag{A.110}$$