# LOG BOOK: (DRAFT) Scattering Amplitudes

Novel new methods of calculation, using BCFW recursion

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# 1 Tues, 26 November 2019: Feynman Diagrams

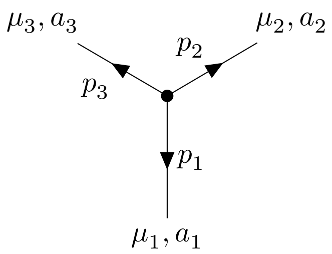



**Figure 1**

Vertex, using a cyclic rotation: The expression of the vertex is:

$$
\begin{aligned}
iV^{a_1,a_2,a_3}_{\mu_1,\mu_2,\mu_3} &= ig\left[(p_1 - p_2)_{\mu_3}\eta_{\mu_1\mu_2} + (p_2 - p_3)_{\mu_1}\eta_{\mu_2\mu_3} + (p_3 - p_1)_{\mu_2}\eta_{\mu_3\mu_1}\right] f^{a_1a_2a_3} \\
&= f^{a_1a_2a_3} V_{\mu_1\mu_2\mu_3}
\end{aligned}
$$

Amplitude, using the polarisation vector $\epsilon_{\alpha\dot{\alpha}}$:

$$
V_{\mu_1\mu_2\mu_3}\, \epsilon^{\mu_1}_{(1)}\epsilon^{\mu_2}_{(2)}\epsilon^{\mu_3}_{(3)}\Big|_{p_1^2=p_2^2=p_3^2=0} = \left.\begin{aligned}
&\left[(p_1 - p_2)\cdot\epsilon_{(3)}\right]\left(\epsilon_{(1)}\cdot\epsilon_{(2)}\right) + \\
&\left[(p_2 - p_3)\cdot\epsilon_{(1)}\right]\left(\epsilon_{(2)}\cdot\epsilon_{(3)}\right) + \\
&\left[(p_3 - p_1)\cdot\epsilon_{(2)}\right]\left(\epsilon_{(3)}\cdot\epsilon_{(1)}\right)
\end{aligned}\right\}[A]
$$

We also have to impose

- ~~Using this method, we will always have~~ momentum conservation such that $p_1 + p_2 + p_3 = 0$
- Momentum is always transverse to polarisation $p \cdot \epsilon(p) = 0$

~~Transverse:~~ Expression of the polarisation vectors - check in (1.82) of Plefka (there are two)

$$
\epsilon(p) \approx \frac{\lambda_\alpha \tilde{\eta}_{\dot{\alpha}}}{[\lambda\, \eta]} \cancel{= 0}
$$

Simplifying $[A]$: using for instance

$$
\begin{aligned}
&\cancel{p_2 = p_1 + p_3} \qquad \text{(p1-p2) . epsilon\_3} \\
&= (2p_1 + p_3)\cdot\epsilon_3 \\
&= 2(p_1\cdot\epsilon_3)
\end{aligned}
$$

Likewise, the remaining $p_i$ can be found by cyclicly permutation.

**Definition 1.0.1: Polarisation Vector, reference spinor**

$$\epsilon^{+}_{\alpha\dot\alpha} \equiv \frac{\xi_\alpha \tilde{k}_{\dot\alpha}}{\langle \xi\, k\rangle}\sqrt{2}$$
$$\epsilon^{-}_{\alpha\dot\alpha} \equiv -\frac{\tilde{\xi}_{\dot\alpha} k_\alpha}{[\tilde{\xi}\,\tilde{k}]}\sqrt{2}$$

Move them before, and then you refer to eqations numbers

We use this definition to simplify the following case: 1^+ 2^+ 3^-

$$\begin{aligned}
{\epsilon_1}_{\alpha\dot\alpha} \cdot {\epsilon_2}^{\alpha\dot\alpha} &= 2(\epsilon_1 \cdot \epsilon_2) \\
&= \left(-\sqrt{2}\right)^2 \frac{1_{\dot\alpha} k_\alpha}{\langle 1\, k\rangle} \frac{2^{\dot\alpha} k^\alpha}{\langle 2\, k\rangle} \\
&= [1\,2]\langle k\, k\rangle = 0
\end{aligned}$$

Now here we illustrate the case for MHV (Maximum Helicity Violation), where we are choosing as many as possible combinations that can be reduced to 0 on inspection without further need to calculation:

$$A(1,2,3) = 2\left[(p_i \cdot \epsilon^3)(\epsilon^1 \cdot \epsilon^2) + (p_2 \cdot \epsilon^1)(\epsilon^2 \cdot \epsilon^3)(p_3 \cdot \epsilon^1)(\epsilon^1 \cdot \epsilon^3)\right]$$

Using

$$\begin{aligned}
\epsilon_1^- &= \frac{\tilde{\xi}_{\dot\alpha} 1_\alpha}{[\xi\, 1]} \\
\epsilon_2^- &= \frac{\tilde{\xi}_{\dot\alpha} 2_\alpha}{[\xi\, 2]}
\end{aligned}$$

Looks like we take 1^- 2^- 3^+

We are now looking for $\epsilon_1^{1^-} \cdot \epsilon_2^{2^-}$

Deriving gluon three-point amplitudes from Feynman rules, colour order:

$$\begin{aligned}
2 \cdot \epsilon_2 \epsilon_3 &= {\epsilon_{(2)}}^{\dot\alpha\alpha} {\epsilon_{(3)}}_{\alpha\alpha} \\
&= \sqrt{2}\sqrt{2}\, 2^{\dot\alpha} \underbrace{{k_3}_{\dot\alpha} {k_2}^{\alpha} 3_\alpha}_{Contraction} \\
&= -2[k_3\, 2]\langle k_2\, 3\rangle
\end{aligned}$$

Using the definition of the reference spinor

Aiming to find $\perp \therefore a \cdot b = 0$, which gets rid of the interaction . This makes the first and third term vanish:

$$\epsilon_1 \cdot \epsilon_2 = 0, \qquad \epsilon_1 \cdot \epsilon_3 = 0$$

Leaving us only with the second term:

$$
\begin{aligned}
[p_2\epsilon_1^-]\,(\epsilon_2^- \cdot \epsilon_3^+) &= p_{2_{\alpha\dot\alpha}}\left(-\sqrt{2}\right)\frac{\tilde{\xi}^{\dot\alpha}1^{\alpha}}{\lambda_{2_{\dot\alpha}}} \\
2(p_2 \cdot \epsilon_1^-) &= -\sqrt{2}\frac{\langle 1\,2\rangle[2\,\tilde{\xi}]}{[\tilde{\xi}\,1]}
\end{aligned}
$$

end finally for $\epsilon_3^{(+)}$:

$$
\begin{aligned}
\epsilon_3^{(+)} &= \frac{\lambda_{1_\alpha}\tilde{\lambda}_{3_{\dot\alpha}}}{\langle 1\,3\rangle}\sqrt{2} \\
\rightarrow 2\epsilon_2^-\epsilon_3^+ &= -(\sqrt{2})(\sqrt{2})\frac{\langle 2\,1\rangle[3\,\tilde{\xi}]}{\langle 1\,3\rangle[\tilde{\xi}\,2]} \\
&= -2\frac{\langle 2\,1\rangle}{\langle 1\,3\rangle}\frac{[3\,\tilde{\xi}]}{[\tilde{\xi}\,2]}
\end{aligned}
$$

Now we simplify:

$$
\begin{aligned}
\frac{\langle 1\,2\rangle\cancel{[2\,\tilde{\xi}]}}{\langle \xi\,1\rangle}\frac{\langle 2\,1\rangle[3\,\tilde{\xi}]}{\cancel{[2\,\tilde{\xi}]}} &= -\frac{\langle 1\,2\rangle^2}{\langle 1\,3\rangle}\frac{[3\,\tilde{\xi}]}{[1\,\tilde{\xi}]} \\
&= \frac{[3\,\tilde{\xi}]}{[1\,\tilde{\xi}]}\frac{\langle 1\,2\rangle^2}{\langle 3\,1\rangle}
\end{aligned}
$$

= add intermediate step (evaluation of [3 xi]/[1 xi])

= <12>^3 / ( <23> <31>)

### 1.1 Notation

$$
\begin{aligned}
p_{\alpha\dot\alpha} &= \lambda_\alpha\lambda_{\dot\alpha} \\
p^\mu &= \#\langle\,\lambda|\,\mu\,|\tilde{\lambda}\,] \\
&= \#\lambda^\alpha\tilde{\lambda}^{\dot\alpha}\alpha\sigma^\mu{}_{\alpha\dot\alpha}
\end{aligned}
$$

where we are using the identity:

$$
P_{\alpha\dot\alpha} = p^\mu\sigma_{\mu\alpha\dot\alpha}
$$

We can use the completeness relation to expand $p^\mu$ further:

Move to formalism

$$
p^\mu = \#\lambda^\beta\tilde{\lambda}^{\dot\alpha}\beta \qquad \underbrace{\sigma^\mu{}_{\beta\dot\beta}\sigma_{\mu\alpha\dot\alpha}}_{\text{Completeness relation}}
$$

Using the Completeness Relation, our free index $\mu$ is summed over:

$$
\sigma^\mu{}_{\beta\dot\beta}\cdot\sigma_{\mu\alpha\dot\alpha} = -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}
$$

Which leads us to:

$$\epsilon^{\mu} = \frac{1}{\sqrt{2}} \frac{\langle \xi | \mu | k ]}{\langle \xi \, k \rangle}$$

## 1.2 Feynman Rule Vertex

We multiply polarisation by each vertex. Hence momentum is conserved

### 1.2.1 Steps

choose helicities (for same) Polarisation vector:

$$\begin{aligned} \epsilon_{+}^{\alpha\dot{\alpha}}(\lambda) &= -\sqrt{2}\frac{\tilde{\lambda}^{\dot{\alpha}}\alpha\mu^{\alpha}}{\langle \lambda \, \mu \rangle} \\ \epsilon_{-}^{\alpha\dot{\alpha}}(\lambda) &= \sqrt{2}\frac{\lambda^{\alpha}\tilde{\mu}^{\dot{\alpha}}}{[\, \lambda \, \mu \,]} \end{aligned}$$

From here, we choose $\epsilon$ such that $\epsilon_i \cdot \epsilon_j = 0$

Here we now have two different ways of representing the Vertex:

$$\begin{aligned} \tilde{a}_{\dot{\alpha}} b_{\alpha} \tilde{c}^{\dot{\alpha}} d^{\alpha} = [\, a \, c \,] \langle \, d \, b \, \rangle 2 a^{\mu} b_{\mu} &= 2(a \cdot b) \\ &= a^{\alpha\dot{\alpha}} b_{\alpha\dot{\alpha}} \\ &= a^{\alpha} \tilde{a}^{\dot{\alpha}} b_{\alpha} \tilde{b}_{\dot{\alpha}} \\ &= \langle \, a \, b \, \rangle [\, b \, a \,] \end{aligned}$$

Move to formalism section

The phenomenologist way of writing:

$$\begin{aligned} a^{\mu} &= \frac{1}{2} \langle \, a | \, \mu \, | a \,] \equiv \frac{1}{2} a^{\alpha} \tilde{a}^{\dot{\alpha}} \sigma_m u_{\alpha\dot{\alpha}} \\ 2(a \cdot b) &= 2\frac{1}{2}\frac{1}{2} \langle \, a | \, \mu \, | a \,] \langle \, b | \, \mu \, | b \,] \\ &= 2 \langle \, a \, b \, \rangle \cancel{[\, b \, b \,]} \quad [ba] \end{aligned}$$

Move to formalism section

# 2 ~~Tues, 21st January 2020 (general notes)~~ Lorentz group

## 2.1 Symmetries

- Spinor Helicity formalism -> Makes simplicity manifest

- Dynamics: ie Why amplitudes are simple.
- New methods are simple
- what is the dot product in terms of spinors.

---

1. Einstein Equation $E^2 = (pc)^2 + (mc^2)^2$
2. Invariant quantities in relativity. What is the set of linear transformations that make the **metric** invariant?
   - $\eta \to \overset{T}{\Lambda}\eta\Lambda = \eta$
   - Lorenz group definitions.
   - $$\overbrace{\begin{pmatrix} a & \\ & b \end{pmatrix}}^{+}\overbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}^{sp}\overbrace{\begin{pmatrix} a & \\ & b \end{pmatrix}}^{+}$$
   - Transformation of velocity $\to$ conclusion:
   $$v' = \frac{\mathrm{d}x + \beta\,\mathrm{d}x_0}{\mathrm{d}x_0 + \beta\,\mathrm{d}x} = \frac{\text{v + beta}}{\text{1 + beta v}}$$
   (Galilean transformation)
3. Lorenz transformation $SO(1,3) \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Arriving at Lorenz invariant transformation $\langle 1\,\dot{2}\rangle \equiv {\lambda_1}^{\alpha}\lambda_{2\dot{\alpha}}$

# 3 Tues, 28th January 2020 (general notes on dissertation)

## 3.1 (Looking at dissertation notes)

- Number equations. eg 1.1 (including section number)
- Relativistic $\to$ Symmetries
- Clarify notation for Hamiltonian / Lagrangian $(q, p \to x/\dot{x})$. Where $q, p$ is more general.
- Lorenz Group definition:
  - Set of transformations $G$ that have the combination of two:

$$a \otimes b \in G \qquad \begin{cases} a \in G \\ b \in G \end{cases}$$

- Associative
- $\exists \mathbb{1} : A \otimes \mathbb{1} = A$
- $\exists A^{-1} : AA^{-1} = \mathbb{1}$
- Additional groups looking at side note
- Invariance of $x^\mu x_\mu$

- Proper and improper groups
- Restricted group is the important one for our case.

# 4 Tues, 4th February 2020: BCFW recursion

(Henn and Plefka 2014, pp 35-39)

The BCFW recursion relations rely on an understanding of the behaviour of the function $A_n(z)$ in the complex $z$ plane.

The derivation proceeds in three steps.

- First, the locations of the poles of $A_n(z)$ are analyzed.]
- Then, it is shown that the residues of the poles correspond to products of lower-point tree amplitudes.
- Finally, the large $z$ behaviour of $A_n(z)$ is determined.

Using complex analysis, we want to inspect the amplitude $A_n(z)$. This is because the sum of tree-level Feynman diagrams are gauge invariant, and therefore when they are deformed by $z$, they remain unchanged. Therefore we can choose the Feynman gauge for the following discussion, without loss of generality. It is clear that $An(z)$ is a rational function of the $\lambda_i$,$\tilde{\lambda}_i$ and $z$. Moreover, $An(z = 0)$ can only have poles where the denominators of Feynman propagators become zero.

When inspecting a function using complex analysis, we try to simplify the function such that there is only one variable in which to take into the complex plane. Taking our scattering amplitude, we reduce it such that our only variable becomes the moment of a particle:

$$(p_i + p_{i+1} + p_{i+2} + ...)^2 \equiv \cancel{\delta_{ij}}$$

P_{ij}^2

Where we have the following quantities:

s (small s)
$$S = (p_1 + p_2)^2$$
$$t = (p_2 + p_3)^2$$
$$u = (p_1 + p_1)^2$$
(p1 + p3)^2
$$p_4 = -p_1 - p_2 - p_3 - p_4 \Rightarrow A(S, t, u)$$

Gauge theory, n-point amplitudes. We now deform our amplitude in such a way that:

http://inspirehep.net/search?ln=en&p=a+britto+and+cachazo+and+feng+and+witten&of=hb&action_search=Search&sf=

$$A(p_1, \dots, p_n) \rightarrow \begin{cases} p_i \rightarrow p_i(z) \\ p_j \rightarrow p_j(z) \end{cases}$$

Which leaves our amplitude in a state with only complexified momenta $p_i(z)$ and $p_i(z)$

no minus in front of p1

$$\mathscr{A}(z) = A(-p_1, p_2, \cdots, p_i(z), \cdots, p_j(z), \cdots p_n)$$
$$\mathscr{A}(0) = A(-p_1, p_2, \cdots, p_n)$$

.

This process can be particularly useful when exploring massless particles ($\sum p_i^2 = 0$); however this is not a constrain, and also works just as well with massive particles.

We are left with a transformation:

$$p_i^\mu(z) = p_i^\mu(z) + z\eta^\mu$$
$$p_j^\mu(z) = p_j^\mu(z) - z\eta^\mu$$

With $\eta =$ new complex momentum and $z =$ its respective complex variable.

$$\left.\begin{aligned} p_i^2(z) = 0 \\ p_j^2(z) = 0 \end{aligned}\right\} \quad \forall z$$

This leads to:

$$\left.\begin{aligned} p_i^2(z) = \cancelto{0}{p_i^2} + z^2\eta^2 + 2z(p_i \cdot \eta) = 0 \\ p_j^2(z) = \cancelto{0}{p_j^2} + z^2\eta^2 + 2z(p_j \cdot \eta) = 0 \end{aligned}\right\} \quad \forall z$$

This is useful for us, as we may thus choose $\eta$ to be any value we would like; so to simplify this equation, we choose $\eta = 0$, and we are left with:

$$\begin{aligned} 2(p_i \cdot \eta) = 0 \quad &\Leftrightarrow \langle i\,\eta\rangle[\eta\, i] &= 0 \\ 2(p_j \cdot \eta) = 0 \quad &\Leftrightarrow \langle j\,\eta\rangle[\eta\, j] &= 0 \end{aligned}$$

This is already a well known solution (from the 60s - find ref ), where we are keeping spacetime such that:

$$\tilde{\lambda} = \pm\lambda^*$$

Real minkowski:

$$\langle i\,\eta\rangle = 0 \implies [i\,\eta] = 0 \quad \text{and:} \quad \lambda_\eta \,/\!/\, \lambda_i \implies \tilde{\lambda}_{\dot\alpha}\eta \,/\!/\, \tilde{\lambda}_{\dot\alpha}i$$

Taking complex Minkowski:

$$\left.\begin{aligned} \underline{\langle i\,\eta\rangle}\,[\eta\, i] &\to 0 \\ \langle j\,\eta\rangle\,\underline{[\eta\, j]} &\to 0 \end{aligned}\right\} \quad \text{2 options}$$

$$\eta = \lambda_i\tilde{\lambda}_{\dot\alpha}j \qquad \text{or} \quad \eta \quad = \lambda_j\tilde{\lambda}_{\dot\alpha}i$$

Where implies that the we are left with:

$$\begin{aligned} 2(p_i \cdot \eta) = 0 \\ 2(p_j \cdot \eta) = 0 \end{aligned}$$

$$\begin{aligned} p_i \to p_i(z) &= p_i + z\eta \\ &= \lambda_i\tilde{\lambda}_{\dot\alpha}i + z\lambda_i\tilde{\lambda}_{\dot\alpha}j \\ &= \lambda_i(\tilde{\lambda}_{\dot\alpha}i + z\tilde{\lambda}_{\dot\alpha}j) \\ &\equiv \lambda_i\hat{\tilde{\lambda}}_i(z) \end{aligned} \qquad \begin{aligned} p_j \to p_j(z) &= p_j - z\eta \\ &= \lambda_j\tilde{\lambda}_{\dot\alpha}j - z\lambda_i\tilde{\lambda}_{\dot\alpha}j \\ &= (\lambda_j - z\lambda_i)\tilde{\lambda}_{\dot\alpha}j \\ &\equiv \hat{\lambda}_j\tilde{\lambda}_{\dot\alpha}j(z) \end{aligned}$$

Leaving us with the two quantities:

$$\begin{aligned} \hat{\tilde{\lambda}}_i(z) &\equiv \tilde{\lambda}_{\dot\alpha}i + z\tilde{\lambda}_{\dot\alpha}j \\ \hat{\lambda}_j(z) &\equiv \lambda_j - z\lambda_i \end{aligned}$$

indicated briefly as [ i j >

Sometimes this is given the shorthand notation:

$$\begin{aligned} \cancel{\hat{\tilde{\lambda}}_i(z)} &\equiv [i\,j\rangle \\ \cancel{\hat{\lambda}_j(z)} &\equiv \langle i\,j] \end{aligned}$$

This leads us to being able to describe amplitudes in the simple form:

$$\frac{C_1}{z - z_1} + \frac{C_2}{z - z_2} + \cdots + \frac{C_L}{z - z_L}$$

This has the simplification that there are no constant terms $(\cancelto{0}{C} + \cancelto{0}{d_1 z_1} + \cancelto{0}{d_2 z_2^2})$. This means that we only need to know pieces of information:

1. Position of poles: $(z_1, z_2, \ldots, z_L)$
2. Residues $(L_1, L_2, \ldots, L_L)$, leave us only with simple poles:

$$\frac{1}{(x - x_0)^3}$$

This is referred to as the pole to third power.

## 4.1 Feynman Diagrams

What are the singularities:

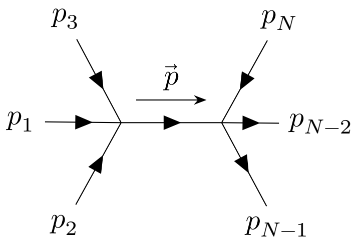



**Figure 2:** Three point amplitude Feynamn diagram

Using the above, such that $\vec{p}^2 = m^2$, and in the special massless case where $m = 0$

$$\vec{p} = -p_1 - p_2 - p_3$$

When we complexify this, we get:

$$\begin{aligned} \hat{p}(z) &= P + z\eta \\ \hat{p}^2(z) &= 0 = P^2 + 0 + 2z(P\cdot\eta) \\ ??? \qquad \frac{z}{P} &= \frac{P^2}{2(P\cdot\eta)} \end{aligned}$$

rewriting this:

$$\begin{aligned} \frac{1}{\hat{p}^2(z)} &= \frac{1}{p^2 + 2z(P\cdot z)} \\ &= \frac{1}{2(p\cdot\eta)} \cdot \frac{1}{z + \underbrace{\frac{p^2}{2(p\cdot\eta)}}_{=\frac{1}{z-z_p}}} \\ &= \sum_p \frac{C_p}{z - z_p} \end{aligned}$$

Where we have used the substitution: $z_p = \frac{p^2}{2(p\cdot\eta)}$

## 4.2 Understanding Singularities

$$A(1,2,\cdots,n) \xrightarrow{p^2\to 0} \sum_n A_L \frac{i}{p^2} A_R$$

TODO

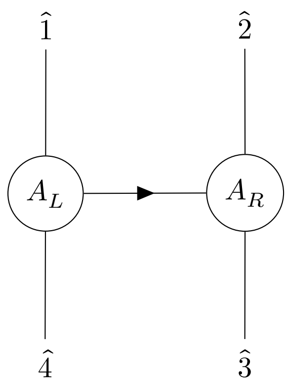



**Figure 3:** MHV amplitude

Where:

$$
\begin{aligned}
P &= p_i + p_{i+1} + \cdots + p_n + p_1 \\
\hat{P} &= P + z\eta
\end{aligned}
$$

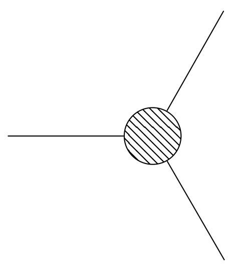
Then:

$$
\begin{aligned}
&z \to z_p \equiv \frac{-p^2}{2(p \cdot \eta)} \\
&\hat{p}^2(z) \to 0 \\
&\therefore \frac{1}{\hat{p}^2(x)} = \frac{1}{2(p \cdot \eta)[z - z_p]}
\end{aligned}
$$

$$
\begin{aligned}
C_p \lim_{z \to z_p} \mathscr{A}(z) &= \cancel{z - z_p} \sum_h A_L(\hat{1}(z), \hat{p}^h, i, i+1, \cdots) \frac{i}{2(p \cdot \eta)\cancel{(z - z_p)}} A_R(\hat{p}^h, \hat{2}(z), \cdots) \\
&= A_L(\hat{i}, \hat{p}, \cdots) A_R(-\hat{p}, \cdots) \\
&= A_L(\hat{1}(z_p), \hat{p}, \cdots) A_R(-\hat{P}, ^{-h}, 2(z_p), \cdots) \\
&= \sum_h \frac{A_L(1(z_p), \hat{p}^h) A_R(-\hat{p}^{-h}, 2(z(p)))}{2(p\eta)} \\
&\equiv C_p
\end{aligned}
$$

$$
\begin{aligned}
A(1, 2, \cdots, n) &= \sum_n A_L \frac{i}{p^2} A_R \\
\mathscr{A}(0) &= \sum_p \sum_h \frac{A_L^h(z_p) A_R^{-h}(z_p)}{s(p \cdot \eta)(\cancel{z} - z_p)} i \\
&= \sum_p \sum_h A_L^h(z_p) \frac{i}{p^2} A_R^{-h}(z_p) = A
\end{aligned}
$$

For example:



TODO

$$
\begin{aligned}
P &= p_4 + p_1, z = \frac{-1p^2}{2(p \cdot \eta)} \\
P^2 &= (p_4 + p_1)^2 = \langle 4\,1 \rangle [1\,4]
\end{aligned}
$$

$$2p\eta \quad \text{such that} \quad \eta : \begin{cases} \lambda_1 \tilde{\lambda}_{\dot{\alpha}} 2 \to 2(p \cdot \eta) & = \langle 1|p|2] \\ \lambda_2 \tilde{\lambda}_{\dot{\alpha}} 1 \to 2(p \cdot \eta) & = \langle 2|p|1] \end{cases}$$

# 5 Tues, 11th February 2020: MHV amplitudes

(Henn and Plefka 2014, pp 15)

## 5.1 General Notes:

- $\langle \quad \rangle [ \quad ]$
- $\mathbb{R}$ Minkowski implies vanishing of one implies vanishing of other

## 5.2 Shifted particles

Looking at all possible diagrams: (continued from last amplitude).

$$A_{MHV} = (1^- 2^- 3^+ 4^+ 5^+) = \frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 4\,5 \rangle \langle 5\,1 \rangle}$$

Special shifted movement.

In order to calculate this MHV amplitude, we need to find a way of simplifying this, by using the methods that we have been building up to so far. Lets try by first choosing vertexes $5$ and $1$ make a complex shift. We now have two choices:

$$\text{case } [1] \begin{cases} \hat{\lambda}_5 & = \lambda_5 + z\lambda_1 \\ \hat{\tilde{\lambda}}_1 & = \tilde{\lambda}_1 - z\tilde{\lambda}_5 \end{cases}$$

$$\text{case } [2] \begin{cases} \hat{\lambda}_1 & = \lambda_1 + z\lambda_5 \\ \hat{\tilde{\lambda}}_5 & = \tilde{\lambda}_5 - z\tilde{\lambda}_1 \end{cases}$$

### 5.2.1 Case [1]:

First shift:

$$
\begin{aligned}
\langle \hat{1}\, 2 \rangle &= \langle 1\, 2 \rangle && \text{(Same)} \\
\langle 2\, 3 \rangle &= \langle 2\, 3 \rangle && \text{(Same)} \\
\ldots & && \text{(Same)} \\
\underbrace{\langle 4\, \hat{5} \rangle}_{\text{Changed to give a pole}} &= \langle \lambda_4\, \lambda_5 + z\lambda_1 \rangle = \langle 4\, 5 \rangle + z\langle 4\, 1 \rangle \\
\langle \hat{5}\, \hat{1} \rangle &= \langle \hat{5}\, 1 \rangle = \langle 5\, 1 \rangle + z\langle 1\, 1 \rangle \\
&= \langle \lambda_5 + \lambda_1\, \lambda_1 \rangle = \langle 5\, 1 \rangle + \cancelto{0}{\langle 1\, 1 \rangle} \\
&= \langle 5\, 1 \rangle
\end{aligned}
$$

This leaves us with the new shifted amplitude:

$$\mathscr{A}(z) = \frac{\langle 1\, 2 \rangle^4}{\langle 1\, 2 \rangle \cdots \langle 3\, 4 \rangle \left( \langle 4\, 5 \rangle + z\langle 5\, 1 \rangle \right) \langle 5\, 1 \rangle}$$

And this is precisely a simple pole:

$$\frac{1}{\langle\, 4\hat{5}\, \rangle} = \frac{1}{\langle\, 45\, \rangle + z\langle\, 41\, \rangle} = \frac{1}{\langle\, 41\, \rangle} \cdot \quad \frac{1}{\underbrace{z + \frac{\langle\, 45\, \rangle}{\langle\, 41\, \rangle}}_{\text{Position of the pole } z_p}}$$

$$z_p = -\frac{\langle\, 45\, \rangle}{\langle\, 41\, \rangle}$$


**(a)** Amplitude corresponding to simple pole $z_p$

**(b)** Alternative possible arrangement, which goes to zero

**Figure 4:** Two possible MHV amplitudes for complex shifted system

#### 5.2.2 Case [2]:

Looking at the second shift:

$$
\begin{aligned}
\langle \hat{1}2 \rangle &= \langle 12 \rangle + z\langle 52 \rangle \\
...(unchanged) & \\
\langle \hat{5}\hat{1} \rangle &= \langle 5\hat{1} \rangle = \langle 51 \rangle + z\cancelto{0}{\langle 11 \rangle} \\
&= \langle 51 \rangle
\end{aligned}
$$

This leaves us with an Amplitude $\mathscr{A}(z) \approx z^3$, which at large $z : (z - z_p)^3 \approx z^3$

This example turns out to have no poles, and no physical coefficients, and we therefore cannot perform the recursion relation.

### 5.3 Helicities:


**(a)** Goes to zero, does not exist

**(b)** Goes to zero, why (next time)

**Figure 5:** MHV amplitude calculations when taking helicities into account

## 6 Tues, 18th February 2020: MHV amplitudes (Deriving Simplets MHV amplitude)

### 6.1 Deriving simplest 4-point MHV amplitude

Starting from a negative helicity amplitude, with a result of

$$A_{MHV}(1^-2^-3^+4^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle\langle 23 \rangle\langle 34 \rangle\langle 41 \rangle}$$

### 6.1.1 Special shifted:

$$Case[1]\begin{Bmatrix}\hat{\lambda}_4 = \lambda_4 - z\lambda_1 \\ \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 + z\tilde{\lambda}_4\end{Bmatrix} = \langle 4\,1\,]$$

For this, it can be easily seen that the poles come from $\langle\, 34\,\rangle$

$$\begin{aligned}\langle\,\hat{4}1\,\rangle &= \langle\,41\,\rangle - \cancelto{0}{z\langle\,11\,\rangle}\\ &= \langle\,41\,\rangle\end{aligned}$$

Of the $\overline{MHV}$ and


**(a)** $\overline{MHV}$ amplitude goes to zero, does not exist, as all LHS are + and all RHS are -

**(b)** $MHV$ amplitude

**Figure 6:** $MHV$ amplitude calculations. One of these must go to zero, as there will only be one pole

The From the two amplitudes above ( $\overline{MHV}$ fig 6a , and $MHV$ fig 6b)). We can immediately see that using the combinations of $+$ and $-$ on $A_L$ and $A_R$; only one of the objects is physically possible. We will now use this to calculate the position of the pole from the $MHV$ amplitude:

$$\begin{aligned}z_p &= (p_3 + \hat{p}_4(z_p))^2\\ &= 0\\ &= \langle\,3\hat{4}\,\rangle[\,\hat{4}3\,]\\ &= \langle\,3\hat{4}\,\rangle[\,43\,] && \text{NB no hat}\\ &= 0\\ &\implies \langle\,34\,\rangle - z\langle\,31\,\rangle = 0\\ \therefore z_p &= \frac{\langle\,34\,\rangle}{\langle\,31\,\rangle} && \text{pole}\end{aligned}$$

This is an important finding, and shows that momentum $\hat{p}$, will not generally be on-shell, but using this method, it keeps it on-shell.

#### 6.1.2 $\hat{p}$:

$$A_R = \frac{\langle 12 \rangle^3}{\langle 2\hat{p} \rangle \langle \hat{p}1 \rangle} = \frac{\langle 12 \rangle^2}{\langle 2\hat{p} \rangle \langle \hat{p}1 \rangle}$$

$$A_L = \frac{[3\hat{4}]^3}{[\hat{4}\hat{p}][\hat{p}3]} = \frac{[34]^3}{[4\hat{p}][\hat{p}3]}$$

$$\frac{i}{(p_3+p_4)^2} = \frac{\langle 12 \rangle^3}{\langle 2\hat{p} \rangle \langle \hat{p}1 \rangle} \cdot \frac{i}{\langle 34 \rangle [43]} \cdot \frac{[34]^3}{[4\hat{p}][\hat{p}3]}$$

For this, we have a possible pairing:

$$\begin{aligned} \langle \hat{p}1 \rangle [4\hat{p}] &= [4\hat{p}][\hat{p}1] \\ &= [4|\,\hat{p}\,|1 \rangle \end{aligned}$$

Here we are looking for a way to keep out particles on shell, so we use the complex momentum, and find the correct particles momentum to work allow for this calculation. In this case, we are choosing $\hat{p} = \hat{\lambda}_1 + z\lambda_2$:

$$= [4|\,\hat{1}+2\,|1 \rangle$$

$$= [4|\,1+2\,|1 \rangle = \overbrace{\cancel{[4|\,1\,|1 \rangle}}^{\text{Using } [41]\langle 11 \rangle = 0} {}^{0} + [4|\,2\,|1 \rangle$$

$$[4|\,2\,|1 \rangle = [42]\langle 21 \rangle$$

What do we need? What are the spinors $\lambda_p$ and $\tilde{\lambda}_p$. We will change to a more systematic method of computation.

Only for the value of $z_p$ should we keep the masses on shell.

$$\hat{p} = \hat{1} + 2 = \lambda_p \tilde{\lambda}_p$$

#### 6.1.3 $\hat{1}$:

$$\hat{1} + 2 = \lambda_1 ( \underbrace{\tilde{\lambda}_1 + z_p \tilde{\lambda}_4}_{\substack{\text{Need to compute this.} \\ \text{Does it look like } \lambda\tilde{\lambda}?}} ) + \lambda_2 \tilde{\lambda}_2$$

$$
\begin{aligned}
\tilde{\lambda}_1 + z_p\tilde{\lambda}_4 &= \tilde{\lambda}_1 + \frac{\langle 34\rangle}{\langle 31\rangle}\tilde{\lambda}_4 \\
&= \frac{\overbrace{\langle 31\rangle\tilde{\lambda}_1}^{3} + \overbrace{\langle 34\rangle\tilde{\lambda}_4}^{4}}{\langle 31\rangle} \\
&= \frac{\langle 3|\,(\overbrace{p_1 + p + 4}^{\text{Can look at contributions from other leg}})}{\langle 31\rangle}
\end{aligned}
$$

The momentum of each particle can also be seen as a sum from all the other contributing particles momenta. In this way we are able to write our legs in a different form: $p_1 + p_4 = -p_2 - p_3$. And so continuing with this we substitute it back into the form we arrived at above:

$$
\begin{aligned}
(\tilde{\lambda}_1 + z_p\tilde{\lambda}_4) &= -\frac{\langle 3|\,(2+3)}{\langle 31\rangle} \\
&= \frac{-\langle 32\rangle\tilde{\lambda}_2}{\langle 31\rangle}
\end{aligned}
$$

$\hat{p}$ in a factorised form:

$$
\begin{aligned}
\hat{p} &= -\frac{\langle 32\rangle}{\langle 31\rangle}\lambda_1\tilde{\lambda}_2 + \lambda_2\tilde{\lambda}_2 \\
&= -\frac{\langle 32\rangle}{\langle 31\rangle}(\lambda_1 + \lambda_2)\tilde{\lambda}_2 \\
\langle \hat{p}1\rangle &= \langle 21\rangle \\
\langle 2\hat{p}\rangle &= \underbrace{-\langle 21\rangle}_{=\langle 12\rangle}\frac{\langle 32\rangle}{\langle 31\rangle}
\end{aligned}
$$

This leaves us with the following:

$$
\begin{aligned}
[4\hat{p}] &= [42] \\
[\hat{p}3] &= [23]
\end{aligned}
$$

Now putting everything together:

$$
\frac{\overbrace{[34]^{\cancelto{2}{3}}}^{-}\overbrace{\langle 12\rangle^{\cancelto{1}{3}}}^{-}}{\langle 34\rangle\underbrace{\cancel{[43]}}_{-}[42][23]\cancel{\langle 12\rangle}\frac{\langle 32\rangle}{\langle 31\rangle}\cancel{\langle 21\rangle}} = \frac{\langle 12\rangle[34]^2\langle 31\rangle}{\langle 34\rangle[42][23]\langle 32\rangle}
$$

This can be further simplified into our shifted pole:

$$\langle 3|\, 4\, |2\,] = \langle 3|\, -1 - \cancelto{0}{2} - \cancelto{0}{3} |2\,]$$

Where we have used the face that $\cancelto{0}{-2-3}$ because: $\langle 3|\, -2\, |2\,] = 0$

$$\langle 3|\, -1\, |2\,] = -\langle 31 \rangle [\, 12\,]$$

We then substitute this into the denominator to produce:

$$\frac{\langle 12 \rangle [\, 34\,]^2 \cancel{\langle 31 \rangle}}{-\cancel{\langle 31 \rangle} [\, 12\,][\, 23\,] \langle 32 \rangle}$$

We can then group in the the terms in the denominator:

$$\begin{aligned} [\, 12\,] \langle 32 \rangle &= [\, 12\,] \langle 23 \rangle (-) \\ &= [\, 1|\, 2\, |3 \rangle (-) \\ &= [\, 1|\, -\cancel{1} - \cancel{3} - 4\, |3 \rangle (-) \\ &= +[\, 14\,] \langle 43 \rangle \end{aligned}$$

Multiplying both sides by $\frac{[\,34\,]}{[\,34\,]}$

### 6.1.4 TODO: What does this mean here??

$$\langle 32 \rangle [\, 34\,] z \text{ bottom}$$

TODO: ??? above

$$\begin{aligned} &= -\langle 23 \rangle [\, 34\,] \\ &= (-)(-) \langle 21 \rangle [\, 14\,] \\ &= \frac{\cancelto{-}{\langle 12 \rangle} [\, 34\,]^2 \cancel{\langle 34 \rangle} [\, 34\,]}{-\cancel{\langle 31 \rangle} [\, 12\,][\, 23\,] \cancel{\langle 21 \rangle} [\, 14\,]} \\ \text{Finally } &= \frac{[\, 34\,]^3}{[\, 12\,][\, 23\,][\, 41\,]} \end{aligned}$$

We can check this result in the other direction

$$\frac{[\, 34\,]^{\cancelto{2}{3}}}{\cancel{[\, 12\,]} [\, 23\,][\, 41\,]} \cdot \frac{\langle 23 \rangle \cancel{\langle 34 \rangle} \langle 41 \rangle}{\langle 12 \rangle^{\cancelto{2}{3}}} \stackrel{?}{=} \pm \mathbb{1}$$

Now we pair up the legs with the momentum:

$$
\begin{aligned}
[\,34\,]\langle\,34\,\rangle &= -2(p_3 \cdot p_4) \\
[\,12\,]\langle\,12\,\rangle &= -2(p_1 \cdot p_2) = -(p_1 + p_2)^2 \\
[\,12\,]\langle\,12\,\rangle &= [\,34\,]\langle\,34\,\rangle \\
\langle\,23\,\rangle[\,34\,] &= -\langle\,21\,\rangle[\,14\,] = -\langle\,12\,\rangle[\,41\,] \\
[\,34\,]\langle\,41\,\rangle &= -[\,32\,]\langle\,21\,\rangle = -[\,23\,]\langle\,12\,\rangle
\end{aligned}
$$

Putting all this together:

$$
= \frac{\langle\,12\,\rangle[\,41\,][\,23\,]\langle\,12\,\rangle}{\langle\,12\,\rangle^2[\,23\,][\,41\,]} = \pm\mathbb{1}
$$

### 6.2 Meeting Notes:

Slide equations:

$$
\begin{aligned}
p &= \lambda\tilde{\lambda} \\
p_\mu \to p_{\alpha\dot{\alpha}} &= p_\mu \sigma^\mu \\
\text{show } p_\mu p^\mu &= 0 \\
p_{\alpha\dot{\alpha}} &\equiv \lambda_\alpha \tilde{\lambda}_\alpha \\
\det\{(p_{\alpha\dot{\alpha}})\} &= p^2
\end{aligned}
$$

## 7 Tuesday 25th February 2020: Meeting with Andreas Brandhuber

Corrections from 19th November

## 8 Wednesday 11th March 2020: (Skype meeting) MHV amplitudes (continued)

- Derive $MHV$ amplitude
- Perform a shift for a 4-point amplitude.
- Negative helicity

$$
(MHV) \to A(1^-, 2^+, 3^-, 4^+) = \frac{\langle\,13\,\rangle^4}{\langle\,12\,\rangle\langle\,23\,\rangle\langle\,34\,\rangle\langle\,41\,\rangle}
$$

We have the choice here:

$$\begin{cases} \hat{\tilde{\lambda}}_i &= \tilde{\lambda}_i + z\lambda_j \\ \hat{\lambda}_j &= \lambda_j - z\lambda_i \end{cases}$$

Leaving us with:

$$\langle \hat{1}3 \rangle = \langle 13 \rangle + z\langle 23 \rangle$$

We will now investigate the behaviour of this object at large $z$, and begin by choosing $i = 1, j = 2$

$$\begin{aligned} \hat{\tilde{\lambda}} &= \tilde{\lambda}_1 + z\tilde{\lambda}_2 \\ \hat{\lambda}_2 &= \lambda_2 - z\lambda_1 \end{aligned}$$

Searching for the shift of $\lambda_2$

$$\begin{aligned} \langle 12 \rangle \to \langle 1\hat{2} \rangle &= \langle 1\lambda_2 - z\lambda_1 \rangle \\ &= \langle 12 \rangle - \cancelto{0}{z\langle 11 \rangle} \\ &= \langle 12 \rangle \end{aligned}$$

The other leg:

$$\langle \hat{2}3 \rangle = \langle 23 \rangle - z\langle 13 \rangle$$

This leaves us with the following amplitudes:


**(a)** cap

**(b)** cap

**Figure 7:** cap

Henn, Johannes M., and Jan C. Plefka. 2014. Scattering Amplitudes in Gauge Theories. Lecture Notes in Physics. Vol. 883. Lecture Notes in Physics. Springer Berlin Heidelberg. https://doi.org/10.1007/978-3-642-54022-6.