1) $\dim \mathbb{R}_2[x] = 3$;

$\{p_1(x), p_2(x), p_3(x)\}$ sono linearm. indip., in quanto $(a_1p_1 + a_2p_2 + a_3p_3)(x) = 0$

$\Rightarrow a_1 = a_2 = a_3 = 0$

$$q_1(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x) \Rightarrow \begin{cases} c_1 + c_3 = 0 \\ c_2 + 2c_3 = 2 \\ c_1 + c_2 = -3 \end{cases} \longrightarrow \begin{cases} c_1 = -5/3 \\ c_2 = -4/3 \\ c_3 = 5/3 \end{cases}$$

$$q_2(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x) \Rightarrow \begin{cases} c_1 + c_3 = 2 \\ c_2 + 2c_3 = 0 \\ c_1 + c_2 = -1 \end{cases} \longrightarrow \begin{cases} c_1 = 1 \\ c_2 = -2 \\ c_3 = 1 \end{cases}.$$

2) $\det A = 3k - 1 \neq 0 \rightarrow k \neq 1/3$. $\Rightarrow$ A e' invertibile se e solo se $k \neq 1/3$.

$rkA = 4$ se $k \neq 1/3$; $rkA = 3$ se $k = 1/3$ perche', ad esempio, il minore

$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2$ e' non nullo $\forall k \in \mathbb{R}$.

3) $\operatorname{rk} A = 3$ (ad esempio, $\begin{vmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 0 & 1 & 0 \end{vmatrix} = -1$) e, ovviamente, $\operatorname{rk}(A|b)=3$.

$\rightarrow$ il sistema è compatibile e possiede ∞^1 soluzioni:

$$\begin{cases} x_1+2x_2-x_3-x_4=-10 \\ x_2+x_4=6 \\ 2x_1-2x_2-x_3=3 \end{cases} \rightarrow \begin{cases} x_1+2x_2-x_3-x_4=-10 \\ x_2+x_4=6 \\ -6x_2+x_3+2x_4=23 \end{cases} \rightarrow \begin{cases} x_1+2x_2-x_3-x_4=-10 \\ x_2+x_4=6 \\ x_3+8x_4=59 \end{cases}$$

$$\Rightarrow \quad Sol(\Sigma) = \{ {}^t(-5x_4+37; -x_4+6; -8x_4+59; x_4) \mid x_4 \in \mathbb{R} \}$$

4) $p_A(t) = (1-t)[(t-k)^2-1] = 0 \rightarrow t_1 = 1, \quad t_2 = k-1, \quad t_3 = k+1$

Se $k-1 \neq 1 \wedge k+1 \neq 1$, t_1, t_2, t_3 sono a 2 a 2 distinti, e quindi ϕ è diagonalizzabile. $k-1\neq 1 \wedge k+1 \neq 1 \Rightarrow k \neq 0 \wedge k \neq 2$.

$\underline{\lambda_1 = 1}$ $\quad (A-I)x=0 \rightarrow \begin{cases} (k-1)x_1+x_2+2x_3=0 \\ x_1+(k-1)x_2+kx_3=0 \end{cases}$

$\rightarrow V_1 = \{ {}^t(-\frac{1}{k}x_3, -\frac{k+1}{k}x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle {}^t(1, k+1, -k) \rangle$

$\underline{\lambda_2 = k-1}$, $\quad (A-(k-1)I)x=0 \rightarrow \begin{cases} x_1+x_2+2x_3=0 \\ x_1+x_2+kx_3=0 \\ (2-k)x_3=0 \end{cases}$

$\rightarrow V_{k-1} = \{ {}^t(x_1, -x_1, 0) \mid x_1 \in \mathbb{R} \} = \langle {}^t(1,-1,0) \rangle$

$\underline{\lambda_3 = k+1}$, $\quad (A-(k+1)I)x=0 \rightarrow \begin{cases} -x_1+x_2+2x_3=0 \\ x_1-x_2+kx_3=0 \\ -kx_3=0 \end{cases}$

$\rightarrow V_{k+1} = \{ {}^t(x_1, x_1, 0) \mid x_1 \in \mathbb{R} \} = \langle {}^t(1,1,0) \rangle$

Posto $v_1 = \begin{pmatrix} 1 \\ k+1 \\ -k \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathcal{B} = \{v_1, v_2, v_3\}$ è una base di autovettori di ϕ.

$$P = \begin{pmatrix} 1 & 1 & 1 \\ k+1 & -1 & 1 \\ -k & 0 & 0 \end{pmatrix}, \quad P^{-1}AP = D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & k+1 \end{pmatrix}.$$

$\boxed{k=2}$ $\lambda_1=\lambda_2=1$, $\lambda_3=3$; $m_a(1)=2$, $m_a(3)=1$.

$(A-I)x=0 \rightarrow x_1+x_2+2x_3=0 \rightarrow V_1=\{{}^t(-x_2-2x_3; x_2; x_3) \mid x_2,x_3\in\mathbb{R}\} = \left\langle \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix} \right\rangle$

$\Rightarrow m_g(1)=2$

$(A-3I)x=0 \rightarrow \begin{cases} -x_1+x_2+2x_3=0 \\ x_1-x_2+2x_3=0 \\ -2x_3=0 \end{cases} \rightarrow V_3=\{{}^t(x_1,x_1,0) \mid x_1\in\mathbb{R}\} = \langle {}^t(1,1,0) \rangle$

$\Rightarrow m_g(3)=1.$

ϕ è diagonalizzabile. Nella base $\mathcal{B}_2=\left\{ \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$, la matrice di ϕ è

$D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&3\end{pmatrix}$, e $P^{-1}AP=D$, con $P=\begin{pmatrix}-1&-2&1\\1&0&1\\0&1&0\end{pmatrix}$.

$\boxed{k=0}$ $\lambda_1=\lambda_3=1$, $\lambda_2=-1$, $m_a(1)=2$, $m_a(-1)=1$

$(A-I)x=0 \rightarrow \begin{cases} -x_1+x_2+2x_3=0 \\ x_1-x_2=0 \end{cases} \rightarrow V_1=\{{}^t(x_1,x_1,0) \mid x_1\in\mathbb{R}\} = \langle {}^t(1,1,0) \rangle$

$\Rightarrow m_g(1)=1 \neq m_a(1)$ $\rightarrow$ ϕ non è diagonalizzabile.

5) $P_A(t) = (t-1)^4$; $\sigma(A) = \{1\}$, $m_a(1) = 4$.

$$A - I = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -3 & 1 & 1 \\ 0 & 3 & -2 & -2 \end{pmatrix} \xrightarrow{\text{GAUSS}} \dots \to \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & -3 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $rk(A-I) = 2$ $\Rightarrow$ $\dim Ker(A-I) = 2$. $\Rightarrow$ 2 blocchi di Jordan :

$J_1(1), J_3(1)$ $\vee$ $J_2(1), J_2(1)$. $N_1(1) = 2$

$$(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $rk(A-I)^2 = 1$ $\Rightarrow$ $\dim Ker(A-I)^2 = 3$

$N_1(1) + N_2(1) = 3$ $\longrightarrow$ $N_2(1) = 3 - 2 = 1$ $\Rightarrow$ 1 solo blocco di ordine almeno 2.

$\Rightarrow$ $A \cong diag(J_1(1), J_3(1))$, cioè $\exists P \in GL_4(\mathbb{R})$: $P^{-1}AP = diag(J_1(1), J_3(1))$

dove $diag(J_1(1), J_3(1)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$