1) $\dim W = 2$ (infatti: $w_1 \nparallel w_2$)

$\dim U = 2$; infatti:

$$rk\begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 2 & -6 & -4 \\ -1 & 3 & -9 & -6 \end{pmatrix} = 2 \Rightarrow U = \langle u_1, u_2 \rangle$$

$\dim(U+W) = 3$; infatti:

$$rk\begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 2 & -6 & -4 \\ 1 & 1 & -1 & -1 \\ 3 & -2 & 8 & 5 \end{pmatrix} = 3$$

$$\mathcal{B}_{U+W} = \{u_1, u_2, w_1\}$$

$\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 1$.

$v \in U \cap W \iff \exists a, b, \alpha, \beta \in \mathbb{R}$ tali che $a u_1 + b u_2 = \alpha w_1 + \beta w_2$

$$\Rightarrow \begin{cases} a = \alpha + 3\beta \\ -a+b = \alpha - 2\beta \\ 3a - 3b = -\alpha + 8\beta \\ 2a - 2b = -\alpha + 5\beta \end{cases} \Rightarrow \begin{cases} \alpha = b \\ \beta = -b \\ a = -2b \end{cases} \Rightarrow v = \begin{pmatrix} -2b \\ 3b \\ -9b \\ -6b \end{pmatrix} \Rightarrow U \cap W = \left\langle \begin{pmatrix} 2 \\ -3 \\ 9 \\ 6 \end{pmatrix} \right\rangle$$

$$\mathcal{B}_{U \cap W} = \{ {}^t(2, -3, 9, 6) \}$$

2) $$D_2 = a_1^2(a_2 b_2 b_3^2 - a_3 b_2^2 b_3) - a_1 b_1 (a_2^2 b_3^2 - a_3^2 b_2^2) + b_1^2 (a_2^2 a_3 b_3 - a_2 a_3^2 b_2) = \ldots =$$

$$= (a_2 b_3 - a_3 b_2)\left[a_1^2 b_2 b_3 - a_1 b_1 (a_2 b_3 + a_3 b_2) + a_2 a_3 b_1^2\right] = \ldots =$$

$$= (a_2 b_3 - a_3 b_2)\left[a_1 b_3 (a_1 b_2 - a_2 b_1) - a_3 b_1 (a_1 b_2 - a_2 b_1)\right] =$$

$$= (a_2 b_3 - a_3 b_2)(a_1 b_2 - a_2 b_1)(a_1 b_3 - a_3 b_1) = \prod_{1 \le i < j \le 3} (a_i b_j - a_j b_i)$$

Estendo: $$D_n = \prod_{1 \le i < j \le n+1} (a_i b_j - a_j b_i)$$

Caso $a_k = b_k$ $(k = 1, 2, 3)$, $D_2 = 0$ (ha colonne uguali)

3) $(A|b) \rightarrow \begin{pmatrix} 1 & -7 & 5 & 4 \\ 0 & 9 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $\Rightarrow rkA = rk(A|b) = 2 \Rightarrow \infty^1$ soluzioni.

$\Sigma \sim \Sigma'$, dove $\Sigma': \begin{cases} x-7y+5z=4 \\ 9y-6z=-4 \end{cases}$

$\rightarrow Sol(\Sigma) = \{ {}^t(-\frac{1}{3}z+\frac{8}{9}, \frac{2}{3}z-\frac{4}{9}, z) \mid z\in\mathbb{R} \} = \{ {}^t(-\frac{z}{3}, \frac{2}{3}z, z) + (\frac{8}{9}, -\frac{4}{9}, 0) \mid z\in\mathbb{R} \}$

Dato che $Sol(\Sigma) = \xi_0 + KerA$, $\rightarrow KerA = \langle {}^t(-1,2,3) \rangle$.

$\rightarrow \mathcal{B}_{\Sigma_0} = \{ {}^t(-1,2,3) \}$

4) $b=0 \wedge c=-1$, $A = \begin{pmatrix} a & 0 & -1 \\ 0 & a & 0 \\ -1 & 0 & a \end{pmatrix}$; $P_A(t) = (a-t)\left[(a-t)^2-1\right]$

$\rightarrow \sigma(A) = \{a, a-1, a+1\}$. Tre autovalori distinti $\forall a\in\mathbb{R} \Rightarrow A$ è diagonalizzabile $\forall a\in\mathbb{R}$.

5) $|A-t\mathbb{1}| \underset{R_3+R_2}{=} \begin{vmatrix} 1-t & 2 & 2 & 1 \\ 2 & -3-t & -3 & -2 \\ 0 & -t & -t & 0 \\ -1 & 2 & 2 & 1-t \end{vmatrix} \underset{C_3-C_2}{=} \begin{vmatrix} 1-t & 2 & 0 & 1 \\ 2 & -3-t & t & -2 \\ 0 & -t & 0 & 0 \\ -1 & 2 & 0 & 1-t \end{vmatrix} = t^2(t^2-2t+2)$

$\rightarrow \sigma(A) = \{0; 1-i, 1+i\}$, $m_a(0)=2$; $m_a(1+i) = m_a(1-i) = 1$

$A - 0\cdot\mathbb{1} = A \rightarrow \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim Ker(A-0\mathbb{1}) = 1$

$\rightarrow$ 1 solo blocco elementare di Jordan di ordine 2 per $\lambda=0$: $J_2(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ sono presenti $J_1(1+i)$, $J_1(1-i)$ necessariamente.

$\Rightarrow J_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix}$.

$J_2(2)^3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^3 = \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix}$