Machine Learning from Data - IDC
HW6 – Theory

Instructions: This assignment <u>must be submitted individually</u>, submitting in pairs <u>is not allowed</u>.

1 VC-Dimension

Compute the VC-dimension of the following hypothesis classes:

1. (10 pt.) Assume the instance space $\mathcal{X}$ satisfies $|\mathcal{X}| = \infty$. The space of binary hypotheses, which given a training set, returns the target y of $\mathbf{x}$ if the pair $(\mathbf{x}, y)$ was observed in the training set, and $+1$ otherwise. Formally, compute $VC(\mathcal{H})$ of

$$\mathcal{H} = \{h : \mathcal{X} \to \{-1, +1\} : h \text{ equals } -1 \text{ on a finite subset of } \mathcal{X} \text{ and } +1 \text{ elsewhere}\}.$$

2. (15 pt.) n-Interval classifiers of length ≥ 2. Let $\mathcal{X} = \mathbb{R}$,

$$\mathcal{H} = \{x \mapsto +1 \iff x \in [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n] : a_1 + 2 \leq b_1, \ldots, a_n + 2 \leq b_n\}.$$

3. (20 pt.) Linear classifiers in the plain. Let $\mathcal{X} = \mathbb{R}^2$,

$$\mathcal{H} = \left\{ (x_1, x_2) \mapsto \begin{cases} +1 & w_1x_1 + w_2x_2 + b > 0 \\ -1 & w_1x_1 + w_2x_2 + b \leq 0 \end{cases} : w_1, w_2, b \in \mathbb{R} \right\}.$$

Show that $\text{VCdim}(\mathcal{H}) = 3$:

(a) Find a set of size 3 that $\mathcal{H}$ shatters.

(b) Show that no set of size 4, $A = (\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4)$, $\mathbf{z}_i \in \mathbb{R}^2$ can be shattered by $\mathcal{H}$.
Guidance: First prove the following lemma:

Lemma 1. *Suppose a linear classifier h obtains prediction $y \in \{-1, +1\}$ on a set of points $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^2$ $(h(\mathbf{z}) = h(\mathbf{z}') = y)$. Then it also obtains the same prediction on any intermediate point. Namely,*

$$\forall \alpha \in [0, 1] \quad h((1 - \alpha)\mathbf{z} + \alpha\mathbf{z}') = y.$$

And use it in each of the following 3 possible cases:

- The convex hull of A forms a line.
- The convex hull of A forms a triangle.
- The convex hull of A forms a quadrilateral.

2 Learning Conjunctions of Literals

(30 pt.) Let $\mathcal{X} = \{0,1\}^n$ (all boolean strings of length n), let $C = \mathcal{H} =$ the set of all conjunctions on $\mathcal{X}$ (e.g. $x_1 \wedge \neg x_3 \wedge x_n$ is in C and $\mathcal{H}$). Define an algorithm L so that C is PAC-learnable by L using $\mathcal{H}$. Prove all your steps.

3 (Almost) PAC-learnability

(25 pt.) Let C denote the class of all possible target concepts defined over a set of instances $\mathcal{X}$. Suppose that $\mathcal{H}$ is a space of binary hypotheses containing the constant concept c_1 defined by $c_1(x) = +1$ for all $x \in \mathcal{X}$, and having the property that $C \setminus \{c_1\}$ is PAC-learnable by an algorithm L using $\mathcal{H}$ with sample complexity $m(\delta, \epsilon)$.

Provide a learning algorithm L' that uses L, so that C (including c_1) is PAC-learnable by L' using $\mathcal{H}$ with sample complexity $\max\{m(\delta,\epsilon), \lceil \frac{\log(1/\delta)}{\epsilon} \rceil\}$. Prove all your steps.