Sphere Packing in Dimension 8

Undergraduate Seminar Work

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The Sphere Packing Problem

Sphere packing is a geometric problem dealing with a seemingly simple question:

What is the most efficient way to arrange spheres in space?

1 Lattices and Packings

Preliminary Definitions

We denote by $B_r^n(x)$ a closed ball of radius r centered at x.

Sphere packing

A non-empty subset of $\mathbb{R}^n$ of equal-radius spheres whose interiors do not overlap.

Upper density of any packing $\mathcal{P}$ is defined as:

$$\limsup_{r\to\infty} \frac{\mathrm{vol}(B_r^n(0)\cap\mathcal{P})}{\mathrm{vol}(B_r^n(0))}$$

Here we use the standard Euclidean space and vol is the Lebesgue measure.

The sphere packing density in $\mathbb{R}^n$, denoted by $\Delta_{\mathbb{R}^n}$, is the supremum of all upper densities.[1]

[1]This definition does not guarantee that this density is in fact achieved by some packing, but it can be proven (Groemer, 1963).

Lattices and Periodic Packings

Lattice

A rank-n subgroup consisting of a basis of $\mathbb{R}^n$, and all integer linear combinations of members of the base

If B is a basis of $\mathbb{R}^n$, then the lattice Λ can be represented as:

$$\Lambda := \{a_1 v_1 + \ldots + a_n v_n \mid a_i \in \mathbb{Z}, v_i \in B\}$$

Periodic packing

A sphere packing for which there exists a lattice Λ so that it is invariant under translation by any lattice element

$$\mathcal{P} = \mathcal{P} + v, \forall v \in \Lambda$$

Lattice packing

A periodic packing in which all sphere centers lie on a lattice, up to translation

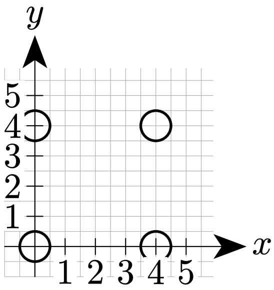
$$\mathcal{P} = \Lambda + x_0, x_0 \in \mathbb{R}^n$$

We can use the basis $B = \{(0,4),(4,0)\}$ to construct $\Lambda = \{(4n,4m) \mid n,m \in \mathbb{Z}\}$, and define:

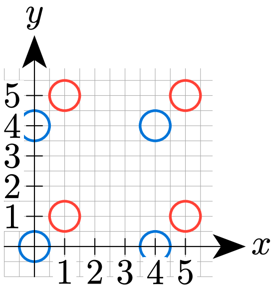
$$\mathcal{P}_1 = \left\{ B^2_{1/2}(x) \mid x \in \Lambda \right\}$$

$$\mathcal{P}_2 = \left\{ B^2_{1/2}(x) \mid x \in \Lambda \cup (\Lambda + (1,1)) \right\}$$

Then $\mathcal{P}_1$ is a lattice packing, while $\mathcal{P}_2$ is a periodic packing that is not a lattice packing.



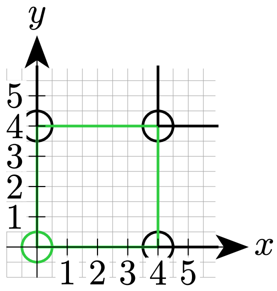
$\mathcal{P}_1$



$\mathcal{P}_2$

Lattice Packing Density

We can think of the lattice as a tiling of space by parallelepipeds.



The fundamental cell of a sphere packing based on a lattice Λ with basis B is given by:

$$C := \{x_1 v_1 + \ldots + x_n v_n \mid 0 \leq x_i \leq 1, v_i \in B\}$$

Translations of the fundamental cell by elements of Λ tile the entire space, and a lattice packing places a sphere at each edge.

Because there is one sphere per copy of the cell, the density is given by $\operatorname{vol}(B_r^n) / \operatorname{vol}(C)$. We can use the fact that $\operatorname{vol}(C) = \operatorname{vol}(\mathbb{R}^n/\Lambda)$, so the lattice packing density is:

$$\frac{\operatorname{vol}(B_r^n)}{\operatorname{vol}(\mathbb{R}^n/\Lambda)}$$

Definition and Basic Properties of E_8

E_8 is a sphere packing in $\mathbb{R}^8$. We present the lattice on which it is based, Λ_8:

$$\Lambda_8 = \left\{ (x_1, \ldots, x_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 \,\middle|\, \sum_{i=1}^{8} x_i \equiv 0 (\mathrm{mod}\ 2) \right\}$$

Λ_8 is an *integral lattice*, i.e., all inner products between its basis vectors are integers.

Λ_8 is an *even lattice*, i.e., the squared length of every vector in it is an even integer.

In particular, the distance between any two points in Λ_8 is of the form $\sqrt{2k}$, and between two closest points it is $\sqrt{2}$.

We choose a lattice packing with $r = \frac{\sqrt{2}}{2}$, denoted by E_8. Its resulting density is:

$$\mathrm{vol}(\mathbb{R}^8 / \Lambda_8) = 1, \qquad \frac{\mathrm{vol}\left(B^8_{\sqrt{2}/2}\right)}{\mathrm{vol}(\mathbb{R}^8/\Lambda_8)} = \frac{\pi^4}{384} = 0.2538\ldots$$

The Duality of Λ_8

We need one more property of Λ_8: it is it's own *dual lattice.*

For a lattice Λ with a given basis $\{v_1, \ldots, v_n\}$ define:

***Dual lattice* of Λ**

The lattice with basis $\{v_1^*, \ldots, v_n^*\}$ such that

$$\langle v_i, v_j^* \rangle = \begin{cases} 1 \text{ if } i = j, \text{and} \\ 0 \text{ otherwise} \end{cases}$$

We denote the dual lattice by Λ^*. If $\Lambda = \Lambda^*$ then Λ is its own dual lattice.

2 Using Harmonic Analysis to Find Density Bounds

If we can prove that the highest possible sphere packing density in $\mathbb{R}^8$ is equal to the density of E_8, we will in fact prove that it is an optimal packing in this dimension.

This problem was addressed by Cohn and Elkies, who showed that one can use auxiliary functions with certain properties (to be detailed shortly) to bound the maximal possible density in a given dimension.

The main insight is in the relationship between a function and its Fourier transform, through the Poisson summation formula.

Schwartz Functions

We consider functions from $\mathbb{R}^n$ to $\mathbb{R}$ (and sometimes complex-valued functions of a real variable).

Schwartz functions - $\mathcal{S}(\mathbb{R}^n)$

All infinitely differentiable functions for which

$$\sup_{x \in \mathbb{R}^n} \left| x^\beta \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right| < \infty$$

for all $\alpha, \beta \in \mathbb{N}$, where $\left(\frac{\partial}{\partial x}\right)^\alpha$ is a series of partial derivatives in some order.

In other words, these are functions whose values and all derivatives decay "very rapidly", or faster than any polynomial.

The Fourier Transform in $\mathbb{R}^n$ for Schwartz Functions

Define the Fourier transform of f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x, \text{ for } \xi \in \mathbb{R}^n$$

We'll use the notation $f(x) \to g(\xi)$ to mean that g is the Fourier transform of f.

For Schwartz functions, the inverse Fourier transform also holds:

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi$$

A *radial function*

A function that depends only on the value of $|x|$

A key property of radial functions is $\hat{\hat{f}} = f$:

$$\begin{aligned}\hat{\hat{f}}(y) &= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-2\pi i \xi \cdot y} \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot (-y)} \,\mathrm{d}\xi \\ &= f(-y) \\ &= f_0(|-y|) \\ &= f(y)\end{aligned}$$

Poisson Summation Formula

To apply tools from analysis to the density problem, we use the Poisson summation formula.

We start by proving for the one-dimensional case:

For $f \in \mathcal{S}(\mathbb{R})$,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

We know two functions are identical if their Fourier series expansions match.

We define two 1-periodic functions and compare their Fourier coefficients to show equality.

Let:

$$F_1(x) := \sum_{n=-\infty}^{\infty} f(x+n)$$

F_1 is 1-periodic, and since f decays very rapidly, the sum converges absolutely and F_1 is continuous.

We can define another 1-periodic function:

$$F_2(x) := \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

We can think of F_2 as a discrete version of the inverse Fourier transform of f, by replacing the integration with a sum in $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, \mathrm{d}\xi$.

For $F_2(x)$, the m-th Fourier coefficient is $\hat{f}(m)$. For $F_1(x)$:

$$
\begin{aligned}
a_m &= \int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) \right) e^{-2\pi i m x} \, \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m x} \, \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(y) e^{-2\pi i m y} \, \mathrm{d}y \\
&= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y} \, \mathrm{d}y \\
&= \hat{f}(m)
\end{aligned}
$$

Note that swapping integration and summation is allowed (f is $\mathcal{S}$).

So have $F_1 = F_2$, and setting $x = 0$ gives:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

This generalizes to $\mathbb{R}^n$:

$$\sum_{x \in \mathbb{Z}^n} f(x) = \sum_{x \in \mathbb{Z}^n} \hat{f}(x)$$

To extend the result to general lattices, let $M : Z^n \to \mathbb{R}^n$ be the linear transformation with respect to the basis $B = \{v_1, ..., v_n\}$ of Λ. Then M is a matrix whose columns are the basis vectors, and $\Lambda = M\mathbb{Z}^n$.

Thus,

$$\sum_{x \in \Lambda} f(x) = \sum_{x \in \mathbb{Z}^n} f(Mx)$$

Let $g(x) = f(Mx)$ and from the formula we obtain that $\hat{g}(\xi) = \int_{\mathbb{R}^n} f(Mx) e^{-2\pi i \xi \cdot x} \,\mathrm{d}x$.

Through a bit of algebraic manipulation, we obtain that

$$\hat{g}(\xi) = \frac{1}{\mathrm{vol}(\mathbb{R}^n / \Lambda)} \int_{\mathbb{R}^n} f(u) e^{-2\pi i \left(M^{-1}\right)^T \xi \cdot u} \,\mathrm{d}u = \frac{1}{\mathrm{vol}(\mathbb{R}^n / \Lambda)} \hat{f}\left(\left(M^{-1}\right)^T \xi\right)$$

We substitute back and obtain that

$$\sum_{x\in\Lambda} f(x) = \sum_{x\in\mathbb{Z}^n} f(Mx) = \sum_{\xi\in\mathbb{Z}^n} \hat{g}(\xi) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{\xi\in\mathbb{Z}^n} \hat{f}\left(\left(M^{-1}\right)^T \xi\right)$$

Let D be the basis of the dual lattice Λ^*. Directly from the definition, we obtain that $M^T D = I$, so $D = \left(M^T\right)^{-1}$ and therefore $\Lambda^* = D\mathbb{Z}^n = \left(M^T\right)^{-1}\mathbb{Z}^n$. That is:

$$\sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{y\in\Lambda^*} \hat{f}(y)$$

And this is Poisson summation formula for lattices in $\mathbb{R}^n$.

Linear Programming Bounds

The main theorem of Cohn and Elkies:

Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < r \in \mathbb{R}$, such that:

1. $f(0) = \hat{f}(0) > 0$
2. $\hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^n$
3. $f(x) \leq 0$ for all $|x| \geq r$

Then The density in a lattice packing in $\mathbb{R}^n$ is bounded above by $\mathrm{vol}\left(B^n_{r/2}\right)$.

We start by proving for lattice packings, then extend to general packings.

Let Λ be a lattice in $\mathbb{R}^n$.

Without loss of generality, assume Λ has minimal vector length r[1].

Then there's a lattice packing with spheres of radius $\frac{r}{2}$ and density

$$\frac{\mathrm{vol}\left(B^n_{r/2}\right)}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

We need to show $\mathrm{vol}(\mathbb{R}^n/\Lambda) \geq 1$.

[1]Scaling the lattice (and thus the sphere radius) does not change the packing density.

By Poisson summation, $\sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{y\in\Lambda^*} \hat{f}(y)$.

Expressing the left sum:

$$\sum_{x\in\Lambda} f(x) = f(0) + \sum_{x\in\Lambda, x\neq 0} f(x)$$

We have $f(0) > 0$ and $f(x) \leq 0$ for $|x| \geq r$ (and for $x \in \Lambda, x \neq 0$). Thus:

$$\sum_{x\in\Lambda} f(x) \leq f(0)$$

For the right sum:

$$\hat{f}(0) \leq \sum_{y\in\Lambda^*} \hat{f}(y)$$

So

$$\frac{\hat{f}(0)}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \leq \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{y\in\Lambda^*} \hat{f}(y)$$

Combining:

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \hat{f}(y) \geq \frac{\hat{f}(0)}{\operatorname{vol}(\mathbb{R}^n/\Lambda)}$$

So

$$\frac{\hat{f}(0)}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \leq f(0)$$

But we required $f(0) = \hat{f}(0)$, thus

$$\operatorname{vol}(\mathbb{R}^n/\Lambda) \geq 1$$

Extension to periodic and general packings

Our proof only cover lattice packings. What about general packings?

Every general packing can be approximated arbitrarily closely by a periodic packing. If we repeat a large enough finite arrangement periodically, the density loss is negligible.

For a periodic packing constructed from Λ and N translations $t_1, \ldots, t_N$, we again assume minimal vector length r, so radius $\frac{r}{2}$ and density $N \cdot \frac{\mathrm{vol}\left(B^n_{r/2}\right)}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$.

We need $\mathrm{vol}(\mathbb{R}^n/\Lambda) \geq N$, proven similarly using:

$$\sum_{j,k=1}^{N} \sum_{x \in \Lambda} f\left(t_j - t_k + x\right)$$

The zeros of the function and further conclusions

The density of a lattice packing of spheres of radius $\frac{r}{2}$ is:

$$\frac{\mathrm{vol}\left(B^n_{r/2}\right)}{\mathrm{vol}(\mathbb{R}^n/\Lambda)}$$

So if we have such a function f and r, the density equals the bound when $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$.

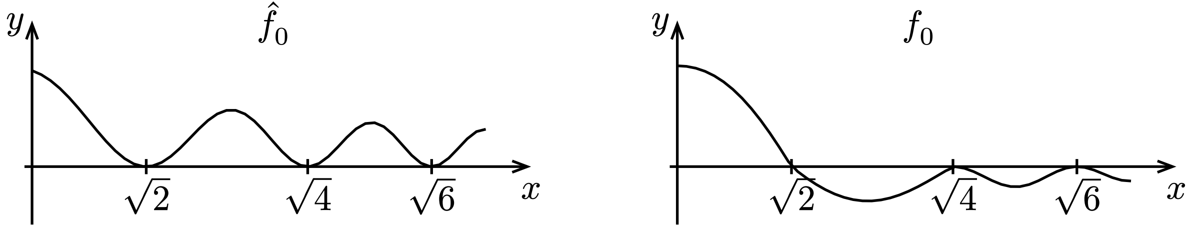
Plugging into Poisson summation:

$$f(0) + \sum_{x\in\Lambda, x\neq 0} f(x) = \hat{f}(0) + \sum_{y\in\Lambda^*, y\neq 0} \hat{f}(y)$$

From the conditions, both infinite sums vanish. So f must vanish on the lattice (except at 0), and $\hat{f}$ must vanish on the dual lattice (except at 0).

All the conditions on the desired function are linear and remain invariant under rotation, and therefore we can think of f as a *radial* function $f_0(x), x \geq 0$.

With respect to the lattice Λ_8, we see it must vanish at every point $\sqrt{2k}$ and can also deduce the parity of the order of each zero, and schematically depict the expected shape of the function and its Fourier transform:



3 Modular Forms

It seems we made a lot of progress: we have a “recipe” for the required function: Find a function with zeros at the E_8 lattice points and whose Fourier transform also has zeros there.

But constructing such a function is very hard.

While we can build a function with given zeros, and one whose Fourier transform has given zeros, controlling both simultaneously is not straightforward.

We have seen that for radial functions, $\hat{\hat{\hat{f}}} = f$. So such f, we can have:

$$g_{+1} = \frac{f + \hat{f}}{2}, \qquad g_{-1} = \frac{f - \hat{f}}{2}$$

Under Fourier transform: $\widehat{g_{+1}} = \left(\hat{f} + f\right)/2 = g_{+1}, \widehat{g_{-1}} = \left(\hat{f} - f\right)/2 = -g_{-1}$.

Eigenfunction of the Fourier transform

A function satisfying for some scalar $\alpha \in \mathbb{R}$ somewhere $\hat{f} = \alpha f$.

So g_{+1} is an eigenfunction with $\alpha = 1$, and g_{-1} is an eigenfunction with $\alpha = -1$. They receive the required roots as f, and we can obtain via them back the f since $f = g_{+1} + g_{-1}$. The two new functions are independent of each other, and therefore we can construct each of them separately.

The Laplace Transform

To find such eigenfunctions, we turn to the *Laplace Transform* of *Gaussian functions*.

Laplace Transform

For a function g, we define its Laplace transform using:

$$f(x) = \int_0^\infty e^{-t\pi|x|^2} g(t)\,\mathrm{d}t$$

As long as g is simple, we can obtain the Fourier transform of f by a simple substitution:

$$\hat{f}(y) = \int_0^\infty t^{-\frac{n}{2}} e^{-\pi|y|^2/t} g(t)\,\mathrm{d}t = \int_0^\infty e^{-\pi|y|^2 t} t^{\frac{n}{2}-2} g(1/t)\,\mathrm{d}t$$

If it holds that $g(1/t) = \varepsilon t^{2-\frac{n}{2}} g(t)$ then $\hat{f} = \varepsilon f$, and so we are looking for such functions.

Initial Definitions

Modular forms are functions that satisfy certain functional equations.

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ (upper half-plane).

Weakly modular form of weight k

If $F : \mathfrak{h} \to \mathbb{C}$ and $k \in \mathbb{Z}$, F is a weakly modular form of weight k if:

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

If F is holomorphic in $\mathfrak{h}$ and bounded as $\operatorname{Im} z \to \infty$, then F is a modular form.

Eisenstein Series - E_k

The Eisenstein series is defined[1]

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

ζ is the Riemann zeta function ($\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$). For even $k > 2$:

$$E_k(z+1) = E_k(z),\ \ E_k\left(-\frac{1}{z}\right) = z^k E_{k(z)}$$

So E_k are modular forms of weight k.

[1] Although there's a notation overlap between the packing E_8 and E_k when $k = 8$, we'll only use $k \in \{2, 4, 6\}$.

For $k = 2$, the series converges conditionally, not absolutely. Define the summation order:

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{n \neq 0} \frac{1}{n^k} + \frac{1}{2\zeta(k)} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^k}$$

For $k = 2$, $E_2(z+1) = E_2(z)$, but

$$E_2(-1/z) = z^2 E_2(z) - 6iz/\pi$$

E_2 is a *quasimodular form* of weight 2.

Sketch of the proof: For every $\varepsilon > 0$, we can denote

$$G_{2,\varepsilon}(z) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^2 \; |mz+n|^{2\varepsilon}}$$

and obtain a series that converges absolutely. The new function satisfies the equation

$$G_{2,\varepsilon}\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 \; |cz+d|^{2\varepsilon} \; G_{2,\varepsilon}(z)$$

The limit $G_2^*(z) := \lim_{\varepsilon\to 0} G_{2,\varepsilon}(z)$ exists and is equal to $\zeta(2)E_2(z) - \pi/(2\,\mathrm{Im}(z))$.

G_2^* behaves similarly to a modular form of weight 2, and substituting it into the functional equation gives the desired result.

Fourier expansion of E_k

A 1-periodic function has the following Fourier expansion: $g(z) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}$.

From here on, we denote $q = e^{2\pi i z}$. To find the Fourier expansion for E_k we use:

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \frac{\pi}{\tan(\pi z)} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right), \quad z \in \mathbb{C}/\mathbb{Z}$$

and by using k term-by-term derivatives, we can get an expansion for $\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k}$. In total, we get that:

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m$$

where $\sigma_{k-1}(m)$ returns the sum of the divisors of m, each is raised to the power $k-1$.

Theta Function of a Lattice - $\Theta_\Lambda(z)$

For every lattice Λ we can define a function, called the theta function, as follows

$$\Theta_\Lambda(z) = \sum_{x \in \Lambda} e^{\pi i |x|^2 z}$$

This function converges when $\operatorname{Im} z > 0$ and defines an analytic function on $\mathfrak{h}$.

This function has an important property: For every lattice Λ in $\mathbb{R}^n$, it holds that

$$\Theta_\Lambda(z) = \frac{1}{\operatorname{vol}(\mathbb{R}^n/\Lambda)} \left(\frac{i}{z}\right)^{n/2} \Theta_{\Lambda^*}(-1/z)$$

for every $z \in \mathfrak{h}$.

The proof is by using Poisson's summation formula on the inner function, $e^{-t\pi|x|^2}$.

We will use the following theta functions, defined on the lattice $\mathbb{Z}$:

$$\theta_{00}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

$$\theta_{01}(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z}$$

$$\theta_{10}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i \left(n+\frac{1}{2}\right)^2 z}$$

$$z^{-2}\theta_{00}^4(-1/z) = -\theta_{00}^4(z) \qquad \theta_{00}^4(z+1) = \theta_{01}^4(z)$$

$$z^{-2}\theta_{01}^4(-1/z) = -\theta_{10}^4(z) \qquad \theta_{01}^4(z+1) = \theta_{00}^4(z)$$

$$z^{-2}\theta_{10}^4(-1/z) = -\theta_{01}^4(z) \qquad \theta_{10}^4(z+1) = -\theta_{10}^4(z)$$

4 Viazovska’s function

Constructing a Function with the Appropriate Zeros

Modular forms, combined with the Laplace transform, are well-suited for our search for Fourier-transform eigenfunctions.

But how can we build a function with the required zeros? One very direct way is by multiplying by a simple function that already has similar zeros. This is exactly the approach Viazovska used to prove the existence of the necessary functions.

The function $\sin^2(\pi|x|^2/2)$ has second-order zeros at every $x = \sqrt{2k}$, $k = 1, 2, \ldots$.

Combined with the Laplace transform, we get a template suitable for both eigenfunctions we are looking for:

$$\sin^2(\pi|x|^2/2) \int_0^\infty e^{-t\pi|x|^2} f(t)\,\mathrm{d}t$$

Conditions for Suitable Eigenfunctions

We begin by searching for the eigenfunction with eigenvalue 1. Suppose we have a function:

$$a(x) = \sin^2(\pi|x|^2/2) \int_0^\infty e^{-t\pi|x|^2} g_0(t)\,\mathrm{d}t$$

We want to know when $\hat{a} = a$.

We seek non-necessary conditions. We do not need tools to characterize all functions that could serve as a proof of the bound, just an example of a single suitable function.

We will try making educated guesses to narrow down the "search area."

As a start, let us assume that $|x| > \sqrt{2}$, because that is where the function's behavior should be simpler.

We can also think of a as a radial function defined[1] for $r \in \mathbb{R}_{\geq 0}$.

We saw that the Fourier transform of the Laplace transform takes the following form:

$$\int_0^\infty e^{-t\pi|x|^2} g(t)\,\mathrm{d}t \to \int_0^\infty e^{-\pi|y|^2 t} t^2 g\left(\frac{1}{t}\right) \mathrm{d}t$$

So if we assume a is an eigenfunction (i.e. $\hat{a} = a$), we can move toward the desired form if we choose $g_0(t) = h_0\left(\frac{1}{t}\right)t^2$, meaning

$$a(r) = \sin^2(\pi r^2/2) \int_0^\infty e^{-\pi r^2 t} t^2 h_0\left(\frac{1}{t}\right) \mathrm{d}t$$

[1] When we discuss the Fourier transform, we continue denoting $a(x) : \mathbb{R}^n \to \mathbb{R}$ and its transform by $\hat{a}(y)$, replacing r^2 with the square of the norm of the vector x.

We want to work in $\mathfrak{h}$, the upper half-plane in order to use modular forms, and we can simplify even further if we look for a function $\phi_0 : i\mathbb{R} \to i\mathbb{R}$. We can denote:

$$i\phi_0\left(\frac{1}{it}\right) = h_0\left(\frac{1}{t}\right)$$

We can now change the integration domain to $(0, i\infty)$ by changing variables $z = it \Rightarrow t = -iz$:

$$\begin{aligned} a(r) &= \sin^2(\pi r^2/2) \int_0^\infty e^{-\pi r^2 t} t^2 i\phi_0\left(\frac{1}{it}\right) \mathrm{d}t \\ &= -\sin^2(\pi r^2/2) \int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{z\pi i r^2} \, \mathrm{d}z \end{aligned}$$

What if we try to get even closer to the form of a Laplace transform? That is, we want $e^{z\pi i r^2}$ to be one of the factors inside the integral, avoiding factors outside it.

We can do this using the identity

$$\sin^2(\pi r^2/2) = -\frac{1}{4}\left(e^{i\pi r^2} - 2 + e^{-i\pi r^2}\right)$$

So we define:

$$a(r) := -4\sin^2(\pi r^2/2)\int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z}\,\mathrm{d}z$$

From now on, we will no longer change a, and we will focus on the inner function, ϕ_0.

We will use the identity and obtain

$$a(r) = \int_0^{i\infty} \left[\phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 (z+1)} - 2\phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z} + \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 (z-1)}\right] \mathrm{d}z$$

We will deal with the exponents $z + 1, z - 1$ in the factors $e^{\pi i r^2 (z\pm1)}$, which prevent us from obtaining a Laplace transform.

We can perform a change of variables in each of the terms, and get:

$$\int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 (z+1)} \,\mathrm{d}z = \int_1^{i\infty+1} \phi_0\left(-\frac{1}{z-1}\right) (z-1)^2 e^{\pi i r^2 z} \,\mathrm{d}z$$

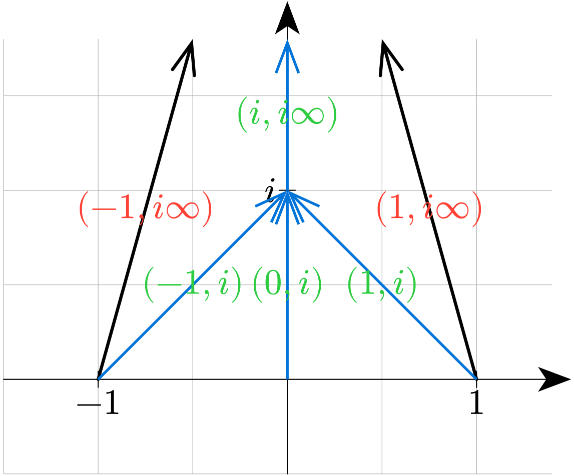
$$\int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 (z-1)} \,\mathrm{d}z = \int_{-1}^{i\infty-1} \phi_0\left(-\frac{1}{z+1}\right) (z+1)^2 e^{\pi i r^2 z} \,\mathrm{d}z$$

Let us assume it is possible to go from $\phi_0(-\frac{1}{z})z^2$ to $\phi_0(z)$. Changing the integration domain is problematic for us:

$$\int_{ia}^{ib} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i |x|^2 z}\, \mathrm{d}z \to$$

$$\int_{i\frac{1}{a}}^{i\frac{1}{b}} \phi_0(w) e^{\pi i |y|^2 w}\, \mathrm{d}w,\ \ a, b \in [-\infty, \infty]$$

Divide the segment $(0, i\infty)$ into $(0, i)$ and $(i, i\infty)$, and find they mirror each other. We will change the integration along the segments $(\pm 1, i\infty)$ to the segments $(\pm 1, i) \to (i, i\infty)$.



$$
\begin{aligned}
a(r) = \int_{-1}^{i} & \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
& -2\int_{0}^{i} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
& +\int_{1}^{i} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
& +\int_{i}^{i\infty} \Bigg[\phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z} \\
& \qquad\qquad -2\phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z} + \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\Bigg]\,\mathrm{d}z
\end{aligned}
$$

We will look at the Fourier transform of the integral over the segment $(-1, i)$ under the substitution $w = -\frac{1}{z}$, and we get:

$$\int_{-1}^{i} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{\pi i |x|^2 z}\, \mathrm{d}z \to \int_{1}^{i} \phi_0\left(-1 - \frac{1}{w-1}\right)(w-1)^2 e^{\pi i |y|^2 w}\, \mathrm{d}w$$

And we find that it is almost the integral over the segment $(1, i)$!

If ϕ_0 is 1-periodic (a property we have already proven for modular forms), then

$$\phi_0\left(-1 - \frac{1}{w-1}\right) = \phi_0\left(1 + -1 - \frac{1}{w-1}\right) = \phi_0\left(-\frac{1}{w-1}\right)$$

In other words, the part composed of the integrals over the segments $(\pm 1, i)$ is essentially a eigenfunction (under the assumption that $\phi_0(z+1) = \phi_0(z)$).

$$
\begin{aligned}
a(r) = &\int_{-1}^{i} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
&-2\int_{0}^{i} \phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
&+\int_{1}^{i} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\,\mathrm{d}z \\
&+\int_{i}^{i\infty} \left[\phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{\pi i r^2 z}\right. \\
&\qquad\left. -2\phi_0\left(-\frac{1}{z}\right) z^2 e^{\pi i r^2 z} + \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{\pi i r^2 z}\right] \mathrm{d}z
\end{aligned}
$$

To ensure that $a(r)$ is entirely a eigenfunction, it must hold that the second part of the function, that is, the segments $(0, i)$ and $(i, i\infty)$, are also eigenfunctions.

We can apply the Fourier transform to it and compare, obtaining that if:

$$\phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 - 2\phi_0\left(-\frac{1}{z}\right)z^2 + \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 = 2\phi_0(z)$$

then the second part of $a(r)$ will also be a eigenfunction, meaning that $a(r)$ itself will be exactly the eigenfunction we are seeking!

A Function that Satisfies the Conditions

We now introduce j, also known as the *elliptic j-invariant*.

It is defined as follows:

$$j = \frac{1728E_4^3}{E_4^3 - E_6^2}$$

We can verify and find that indeed

$$j\left(-\frac{1}{z}\right) = \frac{1728E_4^3\left(-\frac{1}{z}\right)}{E_4^3\left(-\frac{1}{z}\right) - E_6^2\left(-\frac{1}{z}\right)} = \frac{1728z^{12}E_4^3(z)}{z^{12}E_4^3(z) - z^{12}E_6^2(z)} = j(z)$$

That is, j is indeed invariant in this sense.

Immediately, we can also obtain that $j(z) = j(z+1)$.

Similar to the function j, we define two helper functions:

$$\varphi_{-2} := \frac{-1728 E_4 E_6}{E_4^3 - E_6^2}, \qquad \varphi_{-4} := \frac{1728 E_4^2}{E_4^3 - E_6^2}$$

These are functions that have the properties

$$\varphi_{-2}(-1/z) = z^{-2}\varphi_{-2}(z)$$
$$\varphi_{-4}(-1/z) = z^{-4}\varphi_{-4}(z)$$

(and of course both are 1-periodic)

Using these functions, we define

$$\phi_0 := \varphi_{-4} E_2^2 + 2\varphi_{-2} E_2 + j - 1728$$

and ϕ_0 fulfills all the properties we need! The functional equation follows from:

$$\phi_0\left(-\frac{1}{z}\right) = \phi_0(z) - \frac{12i}{\pi} \cdot \frac{1}{z}(\varphi_{-4} E_2 + \varphi_{-2}) - \frac{36}{\pi^2} \frac{1}{z^2} \varphi_{-4}(z)$$

and most of the other properties can be proven using the Fourier expansion of ϕ_0, which can be computed using an automatic tool:

$$\phi_0(z) = 518400q + 31104000q^2 + 870912000q^3 + 15697152000q^4 + O(q^5)$$

in combination with asymptotic bounds on the coefficients.

The Function's Values on the Lattice and at the Origin

To find the values of $a(r)$ for $r = 0$ and $r = \sqrt{2}$, we use the following estimate of ϕ_0:

$$\phi_0\left(\frac{i}{t}\right)t^2 = \frac{36}{\pi^2}e^{2\pi t} - \frac{8640}{\pi}t + \frac{18144}{\pi^2} + O(t^2 e^{-2\pi t})$$

For $r > \sqrt{2}$ we can compute the appropriate integral over the first three terms and obtain:

$$\tilde{a}(r) = 4i\sin^2(\pi r^2/2)\left(\frac{36}{\pi^3(r^2-2)} - \ldots + \int_0^\infty \left(t^2\phi_0\left(\frac{i}{t}\right) - \frac{36}{\pi^2 e^{2\pi t}} + \ldots\right)e^{-\pi r^2 t}\,\mathrm{d}t\right)$$

$\tilde{a}(r)$ is analytic in some neighborhood of $[0, \infty)$ as is $a(r)$ and are equal. By taking a limit:

$$a\left(\sqrt{2}\right) = \lim_{r\to\sqrt{2}} 4i\sin^2(\pi r^2/2)\left(\frac{36}{\pi^3(r^2-2)} - \ldots\right) = 0, a(0) = \frac{-i8640}{\pi}$$

The Other Eigenfunction

We still need a function that satisfies $b(x) = -\hat{b}(x)$. We use the functions from before and denote

$$\psi_I := 128\frac{\theta_{00}^4 + \theta_{01}^4}{\theta_{10}^8} + 128\frac{\theta_{01}^4 - \theta_{10}^4}{\theta_{00}^8}$$

Using it, we define the following function

$$b(r) := -4\sin^2(\pi r^2/2)\int_0^{i\infty} \psi_I(z)e^{\pi i r^2 z}\,\mathrm{d}z$$

and $b(x)$ is the required eigenfunction, with $b(0) = 0, b\left(\sqrt{2}\right) = 0$.

Checking the Appropriate Bounds

Define

$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)$$

and this is the magic function we need!
We can substitute and obtain that $|x| = \sqrt{2} \Rightarrow g(x) = 0$.

Since $a(0) = \frac{-i8640}{\pi}$ and $b(0) = 0$, we also get that $g(0) = \hat{g}(0) = 1 > 0$.

We only need to prove the other two conditions, namely:

$$\hat{g}(y) \geq 0 \text{ for all } y \in \mathbb{R}^n$$

$$g(x) \leq 0 \text{ for all } |x| \geq r$$

Sketch of the proof that $g(x) \leq 0$ for all $|x| > \sqrt{2}$. We present the function as follows:

$$g(r) = \frac{\pi}{2160} \sin^2(\pi r^2/2) \int_0^\infty A(t) e^{-\pi r^2 t} \, \mathrm{d}t, \;\; A(t) = -t^2 \phi_0(i/t) - \frac{36}{\pi^2} \psi_I(it)$$

We prove that $A(t) < 0$ in the segment, so g does not take positive values in $r > \sqrt{2}$.

In $(0, 1]$ we denote $A_0^{(n)}$: $A(t) = A_0^{(n)}(t) + O\left(t^2 e^{-\pi n/t}\right)$, with error $\left|A(t) - A_0^{(m)}(t)\right|$.

Using bounds on the coefficients of the Fourier expansion, we can bound the error.

$A_0^{(6)}(t) < 0$, and $\left|A(t) - A_0^{(6)}(t)\right| \leq \left|A_0^{(6)}(t)\right|$, therefore $A(t) < 0$ when $t \in (0, 1]$.

Similar estimates can be obtained in the segment $(1, \infty)$, and a similar process allows us to prove that $\hat{g}(y) \geq 0$ for all $y \in \mathbb{R}^n$.

So, $g(x)$ is the required magic function, and therefore the maximal density in $\mathbb{R}^8$ is

$$\operatorname{vol}\left(B^8_{\sqrt{2}/2}\right) = \frac{\pi^4}{384} = 0.2538...$$

Since this is the density we obtained for E_8, we have proven that it is indeed the packing with the highest possible density in this dimension. $\square$