4.7 Filtering Periodic Signals

The Fourier series representation of a periodic signal makes it easy to determine how a linear, time-invariant filter reshapes such signals **in general**. The fundamental property of a linear system is that its input-output relation obeys superposition: $L(a_1 s_1(t) + a_2 s_2(t)) = a_1 L(s_1(t)) + a_2 L(s_2(t))$. Because the Fourier series represents a periodic signal as a linear combination of complex exponentials, we can exploit the superposition property. Furthermore, we found for linear circuits that their output to a complex exponential input is just the frequency response evaluated at the signal's frequency times the complex exponential. Said mathematically, if

$$x(t) = e^{j\frac{2\pi kt}{T}}$$

, then the output

$$y(t) = H\left(\frac{k}{T}\right)e^{j\frac{2\pi kt}{T}}$$

because

$$f = \frac{k}{T}$$

. Thus, if $x(t)$ is periodic thereby

having a Fourier series, a linear circuit's output to this signal will be the superposition of the output to each component.

$$y(t) = \sum_{k=-\infty}^{\infty} \left(c_k H\left(\frac{k}{T}\right) e^{j\frac{2\pi kt}{T}}\right)$$

Thus, the output has a Fourier series, which means that it too is periodic. Its Fourier coefficients equal $c_k H\left(\frac{k}{T}\right)$. **To obtain the spectrum of the output, we simply multiply the input spectrum by the frequency response**. The circuit modifies the magnitude and phase of each Fourier coefficient. Note especially that while the Fourier coefficients do not depend on the signal's period, the circuit's transfer function does depend on frequency, which means that the circuit's output will differ as the period varies.