# Two-Layer, Deep Atmosphere Response to Gravest Mode of Tropospheric Heating, Including the Effects of Rotation in Three Dimensions and the Steady-State Response in Two Dimensions.

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# 1 Introduction

We have previously developed the method by which analytical solutions may be developed for the flow and thermodynamic response to steady and pulsed heat forcing, within a deep, two-dimensional atmosphere with vertical stratification i.e two layers and a stratified base state of density. We have recently extended this approach to a system the Coriolis acceleration.

Here we extend that method to a three-dimensional system with Coriolis acceleration. We outline first a methodology, based upon a Hankel transform and then present specimen results.

# 2 The Coriolis Acceleration

The Coriolis force must be inserted into our basic equation set. The left hand side of the momentum equations of the fluid, expressed in cylindrical polars, must be modified to contain the approximate acceleration $f\hat{e}_z \times \underline{v}$. Note, we are effectively using the hydrostatic approximation in the vertical equation. The appropriate Coriolis acceleration is:

$$\underline{a} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial r} & \frac{\partial}{\partial z} \\ u_r & ru_\theta & 0 \end{vmatrix} = -fu_\theta\hat{e}_r + fu_r\hat{e}_\theta.$$

# 3 Basic $w$ Equation

Following our previous analysis (Sept 2016) without rotation, we shall assume that rotation may be treated by approximating $u_\phi = 0$, within axial symmetry. Hence, we express the problem previously treated without rotation, within slab and 3D geometry using cylindrical polar co-ordinates. We assume the sensible heating occurs at $r = 0$. The basic set of flow equations, now expressed within cylindrical polars contain an axial flow and now read:

$$\frac{\partial u_r}{\partial t} - fu_\theta = -\frac{1}{\rho_0(z)}\frac{\partial p}{\partial r} \Longleftrightarrow \rho_0(z)\frac{\partial u_r}{\partial t} - \rho_0(z)fu_\theta = -\frac{\partial p}{\partial r}, \tag{1}$$

$$\frac{\partial u_\theta}{\partial t} + fu_r = 0, \tag{2}$$

$$b = \frac{1}{\rho_0(z)}\frac{\partial p}{\partial z}, \tag{3}$$

$$\frac{\partial b}{\partial t} + N(z)^2 w = s, \tag{4}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial w}{\partial z} = 0 \Longleftrightarrow \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial w}{\partial z} = 0. \tag{5}$$

Let us obtain an equation for $w$ by eliminating variables from our basic set as follows. First form $\frac{\partial}{\partial t}\left(\frac{\partial(3)}{\partial r} + \frac{\partial(1)}{\partial z}\right)$:

$$\frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial^2 u_r}{\partial t^2}\right) - f\frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial u_\theta}{\partial t}\right) = -\rho_0(z)\frac{\partial}{\partial r}\left(\frac{\partial b}{\partial t}\right), \tag{6}$$

substitute using equaitons 4 and 5:

$$\frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial^2 u_r}{\partial t^2}\right) + f^2\frac{\partial}{\partial z}\left(\rho_0(z) u_r\right) = -\rho_0(z)\frac{\partial s}{\partial r} + \rho_0(z)N(z)^2\frac{\partial w}{\partial r}, \tag{7}$$

multiply by $r$, differentiate partially on $r$, then multiply again by $\frac{1}{r}$:

$$\begin{aligned} \frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial^2}{\partial t^2}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right)\right) \quad &+ \quad f^2\frac{\partial}{\partial z}\left(\rho_0(z)\frac{1}{r}\frac{\partial r u_r}{\partial r}\right) \\ &= \quad -\rho_0(z)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \rho_0(z)N(z)^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \end{aligned} \tag{8}$$

use the continuity equation 5 now:

$$\begin{aligned} \frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial^2}{\partial t^2}\left(-\frac{\partial w}{\partial z}\right)\right) \quad &+ \quad f^2\frac{\partial}{\partial z}\left(-\rho_0(z)\frac{\partial w}{\partial z}\right) \\ &= \quad -\rho_0(z)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \rho_0(z)N(z)^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \end{aligned} \tag{9}$$

and finally simplify:

$$\begin{aligned} \frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial^2}{\partial t^2}\frac{\partial w}{\partial z}\right) \quad &+ \quad f^2\frac{\partial}{\partial z}\left(\rho_0(z)\frac{\partial w}{\partial z}\right) + \rho_0(z)N(z)^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \\ &= \quad \rho_0(z)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right), \end{aligned} \tag{10}$$

# 4 Solution by Modal Projection

The modal expansion applied in to the slab case is now applied to equation 10. We note that this procedure requires us to re-visit the physical interpretation of the constants $c_n$ which, in the slab case (without rotation) could be understood as free wave speeds. We return to this matter. Write:

$$w(r,z,t) = \sum_j w_j(r,t)\varphi_j(z), \quad s(r,z,t) = N(z)^2 \sum_j s_j(r,t)\varphi_j(z), \tag{11}$$

with the same equation and vertical boundary conditions for the orthonormal functions $\varphi_j(z)$ as in the previous slab case:

$$\frac{d}{dz}\left(\rho_0(z)\frac{d\varphi_n}{dz}\right) + \frac{\rho_0(z)N(z)^2}{c_n^2}\varphi_n = 0, \quad \varphi(0) = \varphi(H) = 0. \tag{12}$$

We also use the same matching conditions on the $\varphi_j(z)$ at the tropopause, $z = H_t$ i.e. continuity of the function and its first derivative. Accordingly the first term in equation (10) will transform as in the previous "slab" case when integration by parts and the boundary conditions are used:

$$\begin{aligned} &-\frac{1}{c_n^2}\int_0^H \rho_0(z)N(z)^2\varphi_n\frac{\partial^2 w}{\partial t^2}dz - \frac{f^2}{c_n^2}\int_0^H \rho_0(z)N(z)^2\varphi_n w dz + \\ &+\int_0^h \rho_0(z)N(z)^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)dz = \int_0^H \rho_0(z)N(z)^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)dz \end{aligned} \tag{13}$$

We now substitute the expansions in equations (11) and appeal to the orthogonality of the $\varphi_n(z)$:

$$-\frac{1}{c_n^2}\frac{\partial^2 w_n(r,t)}{\partial t^2} - \frac{f^2}{c_n^2}w_n(r,t) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_n(r,t)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s_n(r,t)}{\partial r}\right) \tag{14}$$

Note, the "slab" equivalent of (14) is $-\frac{1}{c_n^2}\frac{\partial^2 w_n(x,t)}{\partial t^2} - \frac{f^2}{c_n^2}w_n(x,t) + \frac{\partial^2 w_n(x,t)}{\partial x^2} = \frac{\partial^2 s_n(x,t)}{\partial x^2}$.

# 5 $w$ Response to Pulsed Heating

Seek $w$ by solving equation (14) using a Laplace-Hankel Transform approach. Other response fields will follow. We will consider the zero-order Hankel transforms of functions $f$ with axial symmetry i.e.:-

$$f = f(\sqrt{(}x^2 + y^2)) = f(r). \tag{15}$$

## 5.1 Zero-Order Bessel Function

$J_0$ is the zero-order Bessel function, which satisfies Bessel's equation with $n = 0$:

$$x\frac{d^2 J_0}{dx^2} + \frac{dJ_0}{dx} + xJ_0 = 0. \tag{16}$$

$J_0(k)$ may be conveniently written:

$$J_0(k) \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos(\theta)}d\theta. \tag{17}$$

Since $J_0(k)$ must be real we have from equation (17) $J_0(k) = \frac{1}{2\pi}\int_0^{2\pi}\cos(ikr\cos(\theta))d\theta$ hence:

$$J_0(k) = J_0(-k), \tag{18}$$

and that $J_0$ has an ortho-normaility property:

$$\int_0^\infty J_0(kr)J_0(kr')r'dr' = \frac{1}{k}\delta(r - r'), \tag{19}$$

where $\delta(r - r')$ is the Dirac delta function.

## 5.2 Zero-Order Hankel Transform

The zero-order order Hankel transform is defined:

$$\mathbf{Ha}_0\{f\} = F_0(\alpha) \equiv \int_0^\infty rf(r)J_0(\alpha r)dr. \tag{20}$$

The corresponding inverse Hankel transform is defined:

$$f(r) = \mathbf{Ha}_0^{-1}(F_0(k)) \equiv \int_0^\infty kF_0(k)J_0(kr)dk \tag{21}$$

Note, that the $n$-order order Hankel transform is:

$$\mathbf{Ha}_n\{f\} = F_n(\alpha) \equiv \int_0^\infty rf(r)J_n(\alpha r)dr, \tag{22}$$

( we need the zero-order case, note). Here $J_n(x)$ is the solution of the $n$-order Bessel equation.

## 5.3 Transformation of the $w$ Equation

In this section we take a Hankel transform and make use of parts and Bessel's equation to transform the resulting integrals, making the assumption that evaluated terms vanish. Hence it will be possible to follow our previous, "slab" approach, making $\hat{\bar{w}}_n$ subject, then performing a Laplace and Hankel inversions. The latter (to find expression for $w_n(r,t)$) must be performed numerically since no Hankel inversion theorem exists analogous to that used in the "slab" case (where the Fourier inversion could be written directly in terms of the assumed horizontal variation of the applied heating). We first apply the Laplace transform in the "vertical structure" equation:-

$$-\left(\frac{p^2+f^2}{c_n^2}\right)\bar{w}_n + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{w}_n}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{s}_n}{\partial r}\right). \tag{23}$$

Later we will consider the form for $\bar{s}_n$ for pulsed heating. Next, apply the Hankel transform (to the $r$ variable) by multiplying by $rJ_0(\alpha r)$, then integrating from 0 to $\infty$ over $r$.

$$\begin{aligned} &- \left(\frac{p^2+f^2}{c_n^2}\right)\int_0^\infty r\bar{w}_n(r,t)J_0(\alpha r)dr + \int_0^\infty J_0(\alpha r)\frac{\partial}{\partial r}\left(r\frac{\partial \bar{w}_n}{\partial r}\right)dr \\ &= \int_0^\infty J_0(\alpha r)\frac{\partial}{\partial r}\left(r\frac{\partial \bar{s}_n}{\partial r}\right)dr. \end{aligned} \tag{24}$$

Integrate by parts:-

$$\begin{aligned} &- \left(\frac{p^2+f^2}{c_n^2}\right)\mathbf{Ha}_0(\bar{w}_n) + \left[J_0(\alpha r)r\frac{\partial \bar{w}_n}{\partial r}\right]_0^\infty - \int_0^\infty r\frac{\partial \bar{w}_n}{\partial r}\frac{\partial J_0(\alpha r)}{\partial r} \\ &= \left[J_0(\alpha r)r\frac{\partial \bar{s}_n}{\partial r}\right]_0^\infty - \int_0^\infty r\frac{\partial \bar{s}_n}{\partial r}\frac{\partial J_0(\alpha r)}{\partial r}dr. \end{aligned} \tag{25}$$

Assuming that $J_0(\alpha r)$ decays to 0 more quickly than $r$ as $r$ tends to $\infty$ we eliminate the evaluated terms:

$$-\left(\frac{p^2+f^2}{c_n^2}\right)\mathbf{Ha}_0(\bar{w}_n) - \int_0^\infty r\frac{\partial \bar{w}_n}{\partial r}\frac{\partial J_0(\alpha r)}{\partial r}dr = -\int_0^\infty r\frac{\partial \bar{s}_n}{\partial r}\frac{\partial J_0(\alpha r)}{\partial r}dr. \tag{26}$$

Integrate by parts again, taking $u = r\frac{\partial J_0}{\partial r}$ and $\frac{\partial v}{\partial r} = \frac{\partial \bar{w}_n}{\partial r}$ or $\frac{\partial v}{\partial r} = \frac{\partial \bar{s}_n}{\partial r}$:

$$
\begin{aligned}
&- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) \\
&- \left\{ \left[ r\frac{\partial J_0(\alpha r)}{\partial r} . \bar{w}_n \right]_0^\infty - \int_0^\infty \bar{w}_n \frac{\partial}{\partial r} \left( r\frac{\partial J_0(\alpha r)}{\partial r} \right) dr \right\} \\
&= \left\{ \left[ r\frac{\partial J_0(\alpha r)}{\partial r} . \bar{s}_n \right]_0^\infty - \int_0^\infty \bar{s}_n \left( \frac{\partial}{\partial r} \left( r\frac{\partial J_0(\alpha r)}{\partial r} \right) \right) dr \right\}.
\end{aligned}
\tag{27}
$$

Assuming that the derivative of $J_0(\alpha r)$ decays to 0 more quickly than $r$ as $r$ tends to $\infty$:-

$$
\begin{aligned}
&- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) \\
&+ \int_0^\infty \bar{w}_n \frac{\partial}{\alpha \partial r} \left( \alpha r \frac{\partial J_0(\alpha r)}{\alpha \partial r} \right) \alpha dr \\
&= \int_0^\infty \bar{s}_n \left( \frac{\partial}{\alpha \partial r} \left( \alpha r \frac{\partial J_0(\alpha r)}{\alpha \partial r} \right) \right) \alpha dr,
\end{aligned}
\tag{28}
$$

and changing the variable, $x = \alpha r$ in the integrands:

$$
- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) + \int_0^\infty \bar{w}_n \frac{d}{dx} \left( x\frac{dJ_0}{dx} \right) dx = \int_0^\infty \bar{s}_n \frac{d}{dx} \left( x\frac{dJ_0}{dx} \right) dx.
\tag{29}
$$

We may now use Bessel's equation (16) to substitue $\frac{d}{dx}\left(x\frac{dJ_0}{dx}\right) = -xJ_0$ to obtain:

$$
- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) + \int_0^\infty -xJ_0 \bar{w}_n dx = \int_0^\infty -xJ_0 \bar{s}_n dx.
\tag{30}
$$

Resetting $x = \alpha r$ we find:-

$$
- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) - \alpha^2 \int_0^\infty \bar{w}_n r J_0(\alpha r) dr = -\alpha^2 \int_0^\infty \bar{s}_n r J_0(\alpha r) dr,
\tag{31}
$$

which, we note, is equivalent to:

$$
- \left(\frac{p^2 + f^2}{c_n^2}\right) \mathbf{Ha}_0(\bar{w}_n) - \alpha^2 \mathbf{Ha}_0(\bar{w}_n) = -\alpha^2 \mathbf{Ha}_0(\bar{s}_n)
\tag{32}
$$

Letting ^ denote the Hankel transform, we can solve for $\hat{\bar{w}}_n$ :-

$$\hat{\bar{w}}_n(\alpha, p) = \frac{c_n^2 \alpha^2}{(p^2 + f^2 + \alpha^2 c_n^2)} \hat{\bar{s}}_n(\alpha, p) = \frac{c_n^2 \alpha^2}{(p + ig_n)(p - ig_n)} \hat{\bar{s}}_n(\alpha, p), \qquad (33)$$

where we have defined:

$$g_n(\alpha) \equiv \sqrt{f^2 + \alpha^2 c_n^2}. \qquad (34)$$

Of course $\hat{\bar{s}}_n(\alpha, p)$ is the Hankel-Laplace transform of the horizontal variation of heating.

## 5.4 Pulsed Heating

We assume the following form for a pulsed, applied heating:

$$s(r, z, t) = s(r, t) S_z(z) = S_0 F(r) \left( \Theta(t) - \Theta(t - T) \right) \sum_j \sigma_j \varphi_j(z), \qquad (35)$$

implicit in which is the following identity for the expansion coefficients:

$$s_j(r, t) = S_0 F(r) \left( \Theta(t) - \Theta(t - T) \right) \sigma_j, \qquad \forall j \qquad (36)$$

with the $\sigma_j$ determined by the vertical variation of the heating:

$$\sigma_j = \int_0^{H_t} \rho_0(z) N(z)^2 \varphi_n(z) S_z(z) dz \qquad (37)$$

Accordingly we find:

$$\hat{\bar{s}}_j = \sigma_j S_0 \left( \frac{1 - e^{-pT}}{p} \right) \hat{F}(\alpha). \qquad (38)$$

From equation (33) therefore:

$$\frac{\hat{\bar{w}}_n(\alpha, p)}{S_0} = \sigma_n \frac{c_n^2 \alpha^2 \hat{F}(\alpha)(1 - e^{-pT})}{p(p + ig_n)(p - ig_n)}, \qquad (39)$$

in which, recall, $\hat{F}(\alpha)$ is the Hankel transform of $F(r)$. On using partial fractions, this may be written:

$$\begin{aligned}
\frac{\hat{\bar{w}}_n(\alpha,p)}{S_0} &= \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{g_n^2}\frac{1}{p} \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\frac{1}{(p+ig_n)} \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\frac{1}{(p-ig_n)} \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{g_n^2}\frac{e^{-pT}}{p} \\
&+ \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\frac{e^{-pT}}{(p+ig_n)} \\
&+ \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\frac{e^{-pT}}{(p-ig_n)}.
\end{aligned} \tag{40}$$

Let us now perform the Laplace inversion of the above, using the delay property of Laplace transforms etc.:

$$\begin{aligned}
\frac{\hat{w}_n(\alpha,t)}{S_0} &= \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{g_n^2}\Theta(t) \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\Theta(t)e^{ig_nt} \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\Theta(t)e^{-ig_nt} \\
&- \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{g_n^2}\Theta(t-T) \\
&+ \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\Theta(t-T)e^{ig_n(t-T)} \\
&+ \frac{\alpha^2\hat{F}(\alpha)c_n^2\sigma_n}{2g_n^2}\Theta(t-T)e^{-ig_n(t-T)}.
\end{aligned} \tag{41}$$

We proceed to the Hankel inversion, which will be comprised of expressions to be evaluated numerically. Introduce the following notation for the inverse Hankel transform of the terms in the above:

$$\mathbf{Ha}_0^{-1}\left(\frac{\alpha^2 e^{ki\alpha g_n t}\hat{F}(\alpha)}{g_n(\alpha)^2}\right) \equiv F'(r,t:k,g_n), \quad k \in [-1,0,1], \tag{42}$$

where we have defined:

$$F'(r,t:k,g_n) = \int_0^\infty \left(\frac{\alpha^3 e^{kig_n(\alpha)t}\hat{F}(\alpha)}{g_n(\alpha)^2}\right) J_0(\alpha r)d\alpha. \tag{43}$$

Clearly the integral above will be complex-valued in general, even if $\hat{F}(\alpha)$ is real. However, the structure of equation (41) ensures that the imaginary parts will always cancel.

Let us take for the horizontal variation of heating a Gaussian in $r$ which is normalised so as to conserve the total heat input into the model atmosphere. We will shortly establish the following function / Hankel transform correspondence for a horizontal variation of heating which conserves total heating:

$$F(r) \equiv \frac{1}{2\pi L^2}\exp\left(-\frac{r^2}{2L^2}\right) \iff \hat{F}(\alpha) = \frac{1}{2\pi}\exp\left(-\frac{L^2\alpha^2}{2}\right). \tag{44}$$

with which we obtain the following:

$$\begin{aligned}
\frac{w_n(r,t)}{S_0} &= c_n^2\sigma_n\Theta(t)F'(r,t:0,g_n) \\
&- \frac{c_n^2\sigma_n}{2}\Theta(t)F'(r,t:1,g_n) \\
&- \frac{c_n^2\sigma_n}{2}\Theta(t)F'(r,t:-1,g_n) \\
&- c_n^2\sigma_n\Theta(t-T)F'(r,t:0,g_n) \\
&+ \frac{c_n^2\sigma_n}{2}\Theta(t-T)F'(r,t-T:1,g_n) \\
&+ \frac{c_n^2\sigma_n}{2}\Theta(t-T)F'(r,t-T:-1,g_n).
\end{aligned} \tag{45}$$

where:

$$F'(r,t:k,g_n) = \frac{1}{2\pi}\int_0^\infty \left(\frac{\alpha^3}{g_n(\alpha)^2}\right) e^{-\frac{L^2\alpha^2}{2}+ikg_n(\alpha)t} J_0(\alpha r)d\alpha. \tag{46}$$

In the above, with $f=0$ we obtain $F'(r,t:k,c_n) = \frac{1}{2\pi c_n^2}\int_0^\infty \alpha e^{-\frac{L^2\alpha^2}{2}+ik\alpha c_n t} J_0(\alpha r) d\alpha$. Equation 45 may be written in a more efficient, explicitly real form:

$$\begin{aligned}
\frac{w_n(r,t)}{S_0} &= c_n^2\sigma_n\left(\Theta(t)-\Theta(t-T)\right)F''(r,t:g_n) \qquad (47)\\
&- c_n^2\sigma_n\Theta(t)F''(r,t:g_n)\\
&+ c_n^2\sigma_n\Theta(t-T)F''(r,t:g_n),
\end{aligned}$$

where:

$$F''(r,t:g_n) = \frac{1}{2\pi}\int_0^\infty \left(\frac{\alpha^3}{g_n(\alpha)^2}\right) e^{-\frac{L^2\alpha^2}{2}} \cos(g_n(\alpha)t) J_0(\alpha r) d\alpha. \qquad (48)$$

Hence, equations 11, 42, 45 and the chosen form of horizontal heating (see below) together determine the vertical flow response, $w(r,z,t)$.

# 6 Potential Temperature, $b$

The potential temperature response is found by integrating equation 4 , which we write in the convenient form:

$$b = \sum_n b_n N(z)^2\varphi_n(z), \quad b_n \equiv \int_0^t \left(s_n(r,t') - w_n(r,t')\right) dt'. \qquad (49)$$

Substituting in the above with equations 42, 45 etc. we have:

$$\begin{aligned}
b_n(r,t) &= S_0 F(r)\sigma_n \int_0^t \left(\Theta(t')-\Theta(t'-T)\right) dt' \qquad (50)\\
&- S_0\sigma_n c_n^2 \int_0^t F'(r,t:0,g_n)\left(\Theta(t')-\Theta(t'-T)\right) dt'\\
&+ \frac{1}{2}S_0\sigma_n c_n^2 \int_0^t \Theta(t')F'(r,t':1,g_n)dt'\\
&+ \frac{1}{2}S_0\sigma_n c_n^2 \int_0^t \Theta(t')F'(r,t':-1,g_n)dt'\\
&- \frac{1}{2}S_0\sigma_n c_n^2 \int_0^t \Theta(t'-T)F'(r,t'-T:1,g_n)dt'\\
&- \frac{1}{2}S_0\sigma_n c_n^2 \int_0^t \Theta(t'-T)F'(r,t'-T:-1,g_n)dt'.
\end{aligned}$$

Integrating, we obtain:

$$\begin{aligned}
b_n(r,t) &= S_0 F(r)\sigma_n \xi(t) - S_0 \sigma_n c_n^2 F'(r,t:0,g_n)\xi(t) \\
&+ \frac{1}{2} S_0 \sigma_n c_n^2 \Theta(t) G(r,t:1,g_n) \\
&+ \frac{1}{2} S_0 \sigma_n c_n^2 \Theta(t) G(r,t':-1,g_n) \\
&- \frac{1}{2} S_0 \sigma_n c_n^2 \Theta(t-T) G(r,t-T:1,g_n) \\
&- \frac{1}{2} S_0 \sigma_n c_n^2 \Theta(t-T) G(r,t-T:-1,g_n),
\end{aligned} \tag{51}$$

with:

$$\xi(t) = (\Theta(t) - \Theta(t-T))\, t + \Theta(t-T)T, \tag{52}$$

and:

$$G(r,t:k,g_n) = \frac{1}{2\pi} \int_0^\infty \frac{\alpha^3 e^{-\frac{L^2\alpha^2}{2}} \left(e^{ikg_n t} - 1\right)}{ikg_n^3} J_0(\alpha r) d\alpha. \tag{53}$$

The $b$ response may easily be set into a more computable, explicitly real form:

$$\begin{aligned}
b_n(r,t) &= S_0 F(r)\sigma_n \xi(t) - S_0 \sigma_n c_n^2 F''(r,t:g_n)\xi(t) \\
&+ S_0 \sigma_n c_n^2 \Theta(t) G'(r,t:1,g_n) \\
&- S_0 \sigma_n c_n^2 \Theta(t-T) G'(r,t-T:g_n),
\end{aligned} \tag{54}$$

with:

$$G'(r,t:g_n) = \frac{1}{2\pi} \int_0^\infty \frac{\alpha^3 e^{-\frac{L^2\alpha^2}{2}} \sin(g_n t)}{g_n^3} J_0(\alpha r) d\alpha. \tag{55}$$

This completes the calculation of the $b$ response.

# 7 Conserved Total Heating

We must now assume a particular form for the horizontal response for the 3D calculation. Let the horizontal variation of heating, $F(r)$, be given by a function similar to that used in our slab-geometry studies:

$$F(r) = k \exp\left(-\frac{r^2}{2L^2}\right), \tag{56}$$

where $k$ is a constant to be determined, which will conserve the total amount of heat input to the tropopause:

$$Q_T = kQ_0 \int_0^{H_t} \int_0^{\infty} \exp\left(-\frac{r^2}{2L^2}\right) 2\pi r dr dz. \tag{57}$$

Noting that the integrand in above may easily be made exact $\int_0^{\infty} \exp\left(-\frac{r^2}{2L^2}\right) 2\pi r dr = (-2L^2\pi) \int_0^{\infty} \left(-\frac{2r}{2L^2}\right) \exp\left(-\frac{r^2}{2L^2}\right) dr = (-2L^2\pi) \int_0^{\infty} \frac{d}{dr} \exp\left(-\frac{r^2}{2L^2}\right) dr = 2\pi L^2$, set $k = \frac{1}{2\pi L^2}$ to ensure conserved heating and since the Bessel transform of a Gaussian is a Gaussian:

$$F(r) = \exp\left(-\frac{r^2}{2L^2}\right) \iff \bar{F}(\alpha) = L^2 \exp\left(-\frac{L^2\alpha^2}{2}\right). \tag{58}$$

we can easily determine the Bessel transform of a horizontal heating function $F(r)$ which conserves total heating:

$$F(r) \equiv \frac{1}{2\pi L^2} \exp\left(-\frac{r^2}{2L^2}\right) \iff \bar{F}(\alpha) = \frac{1}{2\pi} \exp\left(-\frac{L^2\alpha^2}{2}\right). \tag{59}$$

# 8 Computation of Integrals

The integrals for $F''(r, t : g_n)$ (equation 48) and $G(r, t : k, g_n)$ (equation 55 ) are performed numerically, using Simpson's rule. The step, $\delta\alpha$ used must be chosen carefully to ensure an accurate quadrature.

The envelope factor in the integrand, $e^{-\frac{L^2\alpha^2}{2}}$, means it (the integrand) is small for $\alpha > 4/L$. Hence a range for the quadrature of $0 \leq \alpha \leq \frac{4}{L}$ is chosen. For most of this range of $\alpha$ these integrands oscillate due to factors $\sin(kg(\alpha)t)$ or $\cos(kg(\alpha)t)$ (recall, here $k = \pm 1$). With large $\alpha$ these factors become:

$$\sin(\pm(c_n t)\alpha), \quad \cos(\pm(c_n t)\alpha). \tag{60}$$

The above are both characterised by a wavelength in $\alpha$-space of:

$$\lambda = \frac{2\pi}{c_n t}, \tag{61}$$

so we see clearly that the oscillation of the inversion integrand is more rapid with large $c_n$ (which increase with $H_L$ note) and $t$. Figure 1 below illustrates this. For $H_L = 64H_t$, $f = 10^{-4}$, $t = 6 \times 60^2$s and constant $d\alpha = 0.01$ the inversion integrand for $\max(c_n)$ and $\min(c_n)$ is plotted for small and large horizontal distance, $x$. The data in the top panels corresponds to $\max(c_n)$ and is oscillating too fast for the chosen plotting step, $d\alpha$ (the structure of these two graphs is due to "beating" between $\lambda$ and $d\alpha$- to see accurately what is going-on one would need a much smaller step $d\alpha$). In contrast, in the bottom panels, for $\min(c_n)$, the integrand is well-resolved. **Clearly it is not possible adequately to resolve the integrand for all $c_n$ with the same step, $d\alpha$.** A suitable choice of inversion quadrature step which

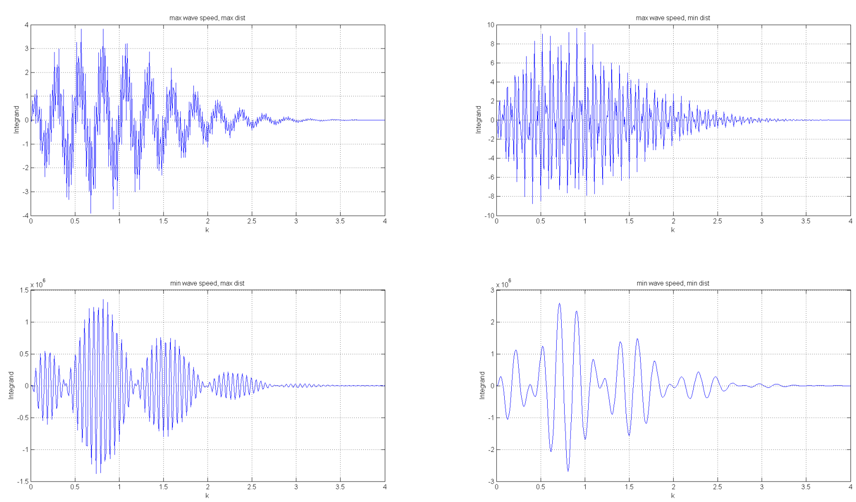


Figure 1: Data for $L = 1$. The variation with conjugate variable $k$ **should be** $\alpha$ of the integrand equation 48. Top panels correspond to $\max(c_n)$. Here the value of step $d\alpha$

places 20 quadrature points in each oscaillation of the integrand, for a given $c_n$, is:

$$d\alpha = \frac{2\pi}{20 c_n t}. \tag{62}$$

We remark that (i) range of integration required accurately to evaluate these quadratures is restricted by the localised, Gaussian nature of the envelope of the integrand (we choose $0 \leq \alpha \leq 4L$), (ii) the computer execution time taken to perform an inversion decreases as $c_n$ decreases and (iii) the results from equation 62 compare well with the much slower Matlab numerical integrator (which selects its own sampling rate and was set-up as follows for our example:

fun = @(kk) (defn of eq. 47); q = integral(fun,0,4/sigma);),

# 9 Results and Discussion

In this section we first we consider local, short-time adjustment to verify the method. Next we consider long-time remote response. In view of the discussions of section 8, the latter data is most challenging, due to convergence of the inversion integrals.

All data presented is for purposes of comparison only. In particular, the heating amplitude i.e calibration constant $Q_0 = 10^3$ has a value which is chosen arbitrarily. For all data presented in this sub-section, equations 11, 42, 45 and 58 together determine the vertical flow response, $w(r, z, t)$. Equations 11, 51, 55 and 58 together determine the potential temperature response, $b(r, z, t)$.

## 9.1 Effect of Rotation on Local Responses

Let us consider the response within the region of the heating, on relatively short times and seek to compare responses in order to observe the effects of rotation. Let $f = 10^{-4}, 0$, $H_L = 64km$, $H_t = 1.5$, $T = 2000$s, $L = 1$km and time $t = 5000$s. To obtain improved convergence onto the applied heating profile, a scaling of about 30 is required.

Figure 2 shows the $w$ response for a rotating (top panel) and non-rotating (middle panel) atmosphere and the difference between the two (bottom panel).

Figure 3 shows the $b$ response for a rotating and non-rotating atmosphere and the difference between the two. The persistence of the heating effect over the origin in this data might be due to the structure of $G'$ in equation 55. At long time (when $\xi(t) = T$, the pulse duration) the expression for $b_n(r, t)$ contains contributions in the assumed heating which partly cancel (the terms $S_0F(r)\sigma_n\xi(t) - S_0\sigma_n c_n^2 F''(r, t : g_n)$). When $f = 0$, $F''$ becomes an exact inverse Hankel transform. For large $f$, when $g_n(\alpha) \to f$ this cancellation will cease.

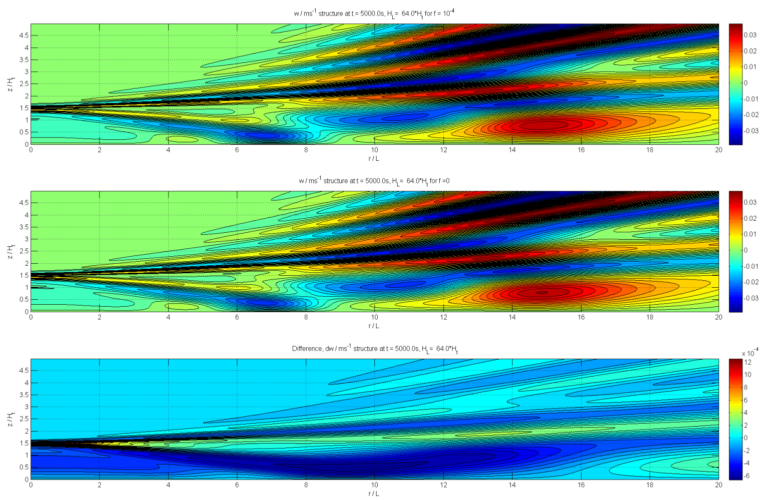

Figure 2: Local response to heating with and without Coriolis acceleration, in 3D. For this data, $f = 10^{-4}, 0$, $H_L = 64km$, $H_t = 1.5$, $T = 2000$s and time $t = 5000$s Figure shows the vertical velocity with (top panel) and without (middle panel) Coriolis acceleration. The bottom panel shows the difference .

### 9.2 Remote, Long-Time Response

Let us consider the remote response, at $t = 18$hrs for the system considered in figures 2 and 3. We consider the responses for $f = 10^{-4}, 0$, $H_L = 64km$, $H_t = 1.0$, $T = 2000$s and time $t = 18 \times 60 \times 60$s in figure 4 below. Note the increase in the horizontal scale.

## 10 Steady, Far-Field Response and the Thermal Wind

It is clear in figure 4 that there is steady-state response to the heating in the PT field. To relate the detail of the model used to any steady-state response, including the potential vorticity (PV) signature, we must return to treating system in 2D, $x - z$ plane slab geometry. We now use a Boussinesq

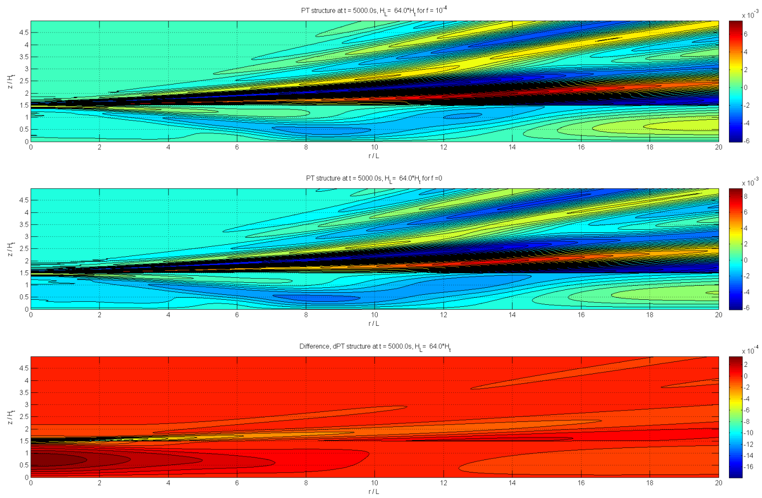


Figure 3: Vertical velocity without (top panel) and with (middle panel) Coriolis acceleration. The bottom panel shows the difference .

system similar to that of Parker and Burton, slightly modified as follows. To treat Coriolis effects consistently we must take full account of motion perpendicular to the slab, hence the continuity equation reflects motion in $y$ direction, $v$:

$$\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \\
b &= \frac{1}{\rho_0}\frac{\partial p}{\partial z}, \\
\frac{\partial b}{\partial t} + N(z)^2 w &= s, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned} \tag{63}$$

where:

$$b \equiv g\frac{\theta'}{\theta_0}, \quad \bar{b} = g\frac{\bar{\theta}}{\theta_0}, \quad N^2 = \frac{d\bar{b}}{dz}, \quad \rho_0 = constant. \tag{64}$$

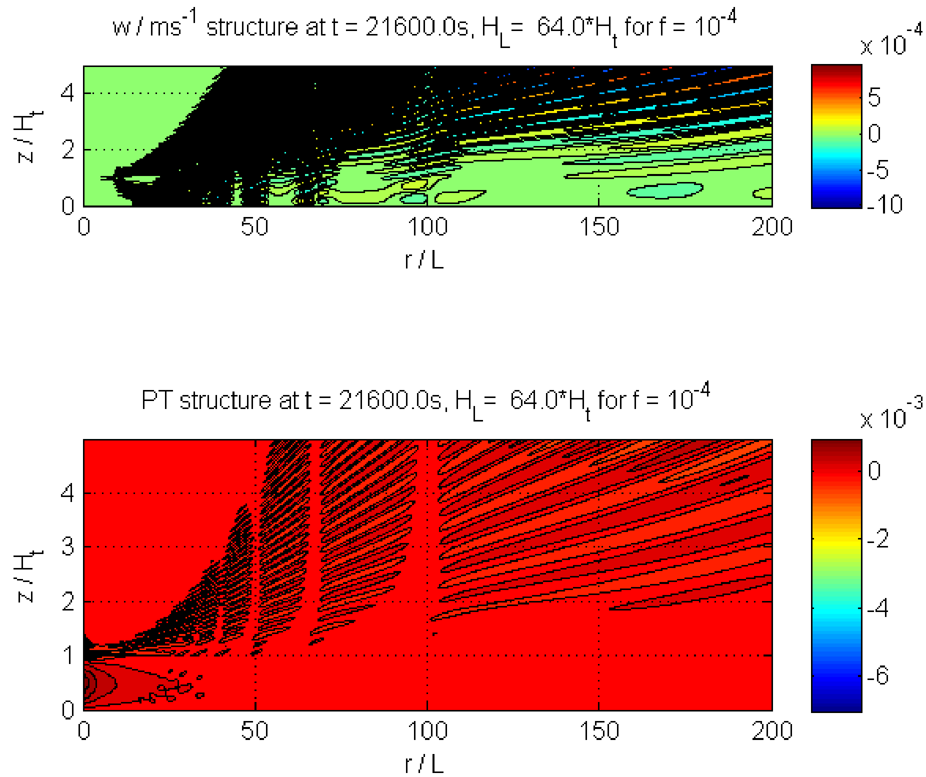


Figure 4: Vertical velocity (top panel) and PT for a 3D system with Coriolis acceleration. For this data $f = 10^{-4}, 0$, $H_L = 64km$, $H_t = 1.0$, $T = 2000$s and time $t = 18 \times 60 \times 60$s.

Here $\theta_0$ is a reference potential temperature which is constant and $\bar{b}(z)$ the base state of potential temperature. The base state of the flow is rest.

## 10.1 Potential Vorticity

Let us determine an approximate expression for the potential vorticity (PV) in the system of Parker and Burton.

PV is the absolute circulation of an air parcel that is enclosed between two isentropic (constant entropy) surfaces. The derivative following the motion of that parcel of air can only be changed by diabatic heating (such as latent heat released from condensation) or frictional processes. PV is defined as follows:

$$PV \equiv \frac{1}{\rho}\underline{\zeta} \cdot \underline{\nabla}\theta, \quad \underline{\zeta} = \underline{\nabla} \times \underline{v} + f\hat{e}_z \tag{65}$$

where $\underline{\zeta}$ is the absolute vorticity. Within the troposphere, the values of potential vorticity (PV) are usually low. However, the potential vorticity increases rapidly from the troposphere to the stratosphere.

Write:

$$PV = \frac{1}{\rho}\underline{\zeta} \cdot \underline{\nabla}\left(\theta \ + \bar{\theta}\right) = \frac{1}{\rho}\underline{\zeta} \cdot \underline{\nabla}\left(\frac{\theta_0}{g}\left[b + \bar{b}(z)\right]\right), \tag{66}$$

and seek a first-order expression for PV by setting $\rho = \rho_0 + \rho'$ (note, in the system of Parker and Burton the base state of density is constant) and retaining terms linear in perturbation quantities:

$$\begin{aligned} PV &= \frac{\theta_0}{g(\rho_0 + \rho)}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \cdot \left(\frac{\partial b}{\partial x}, \frac{\partial b}{\partial y}, \frac{d\bar{b}}{dz} + \frac{\partial b}{\partial z}\right) \\ &\approx \frac{\theta_0}{g\rho_0}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \cdot \left(\frac{d\bar{b}}{dz} + \frac{\partial b}{\partial z}\right). \end{aligned} \tag{67}$$

The above still contains second order terms which we shall remove now:

$$\begin{aligned} PV &= \frac{\theta_0}{g\rho_0}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \cdot \left(N^2 + \frac{\partial b}{\partial z}\right), \\ &= \frac{\theta_0 N^2}{\rho_0 g}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{f}{N^2}\frac{\partial b}{\partial z}\right). \end{aligned} \tag{68}$$

Now, seek and alternative expression for the right hand side of equation 68. By eliminating variables from the basic set in equation 63 it is possible to obtain the following:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\frac{\partial}{\partial z}\left(\frac{b}{N^2}\right)\right) = f\frac{\partial}{\partial z}\left(\frac{s}{N^2}\right). \tag{69}$$

Let us now assume a pulsed heating of duration $T$ (see below). Integrating over a time $t$ long compared with $T$, and using quiescent initial conditions ($u = v = b = 0$, $t = 0$) we find:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\frac{\partial}{\partial z}\left(\frac{b}{N^2}\right) = \int_0^T f\frac{\partial}{\partial z}\left(\frac{s}{N^2}\right) dt, \tag{70}$$

The right hand side of equation 68 and the left hand side of equation 70 are similar. Accordingly, a compact, analytic expression for PV may possibly be obtained.

To obtain an exact expression for PV however, $N$ must be taken to be constant and $s$ must have additional properties.

## 10.2 Steady Responses for Pulsed Heating and $N$ Constant

We seek steady-state $u$, $v$, $w$, $b$ and PV in this section. We neglect $y$ variation and treat buoyancy frequency as constant now:

$$N^2 = constant. \tag{71}$$

At steady state, our basic set 63, the PV 68 and equation 70 respectively simplify as follows:

$$u = w = s = 0, \quad fv = \frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad b = \frac{1}{\rho_0}\frac{\partial p}{\partial z}. \tag{72}$$

$$PV = \frac{\theta_0}{\rho_0 g}\left(N^2\frac{\partial v}{\partial x} + f\frac{\partial b}{\partial z}\right). \tag{73}$$

$$N^2\frac{\partial v}{\partial x} + f\frac{\partial b}{\partial z} = f\int_0^T \frac{\partial s}{\partial z}dt, \tag{74}$$

### 10.2.1 $p$, $v$ and $b$ Response

Using equations 72 and equation 74 we easily obtain an equation for $p$ from which we can eventually obtain $b$, $v$ (using equations 72) and, eventually, PV:

$$\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2} + \frac{f^2}{N^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial p}{\partial z}\right) = \frac{f^2}{N^2}\int_0^T \frac{\partial s}{\partial z}dt. \tag{75}$$

Recalling the base state of density is constant:

$$\frac{\partial^2 p}{\partial x^2} + \frac{f^2}{N^2}\frac{\partial^2 p}{\partial z^2} = \frac{\rho_0 f^2}{N^2}\int_0^T \frac{\partial s}{\partial z}dt. \tag{76}$$

We now define a separable, pulsed heating structure within the tropopause:

$$s(x,z,t) \equiv (\Theta(t) - \Theta(t-T))\,F(x)\sin\left(\frac{\pi z}{H_t}\right)(\Theta(z) - \Theta(z-H_t)), \tag{77}$$

with $H_t$ the tropopause height and $F(x)$ a horizontal variation chosen below. Note that the time dependance of our chosen $s$ may be factored. We return to this point.

Following Parker and Burton we seek solutions by modal expansion:

$$
\begin{aligned}
p(x,z) &= \sum_j p_j(x)\cos\left(\frac{j\pi z}{H}\right), \\
b(x,z) &= \sum_j b_j(x)\sin\left(\frac{j\pi z}{H}\right), \\
v(x,z) &= \sum_j v_j(x)\cos\left(\frac{j\pi z}{H}\right), \\
s(x,z,t) &= \left(\Theta(t)-\Theta(t-T)\right)F(x)\sum_j \sigma_j \sin\left(\frac{j\pi z}{H}\right),
\end{aligned}
\tag{78}
$$

here $H \gg H_t$ is the lid height and $\sigma_j$ are the Fourier coefficients the vertical variation assumed in equation 77, which are determined below.

As usual, we use integration by parts in equation 76 to avoid differentiating a series. Multiply equation 76 by $\cos\left(\frac{j\pi z}{H}\right)$ and apply the boundary conditions on the perturbation pressure:

$$
\frac{\partial p}{\partial z}_{z=0} = \frac{\partial p}{\partial z}_{z=H} = 0 \tag{79}
$$

**(because $b$ must vanish at these locations- a parcel cannot be buoyant at the surface and lid?)** and parts (twice in the left hand side, note), straightforwardly to obtain:

$$
\begin{aligned}
&\int_0^H \frac{\partial^2 p}{\partial x^2}\cos\left(\frac{j\pi z}{H}\right)dz - \frac{f^2}{N^2}\frac{j^2\pi^2}{H^2}\int_0^H p\cos\left(\frac{j\pi z}{H}\right)dz \\
=\;& \frac{\rho_0 f^2}{N^2}\frac{j\pi}{H}\left(\int_0^T \left(\Theta(t)-\Theta(t-T)\right)dt\right)\int_0^H s\sin\left(\frac{j\pi z}{H}\right)dz.
\end{aligned}
\tag{80}
$$

Substituting expansions 78 and using the orthonormality properties of the trigonometric functions:

$$
\frac{d^2 p_j}{dx^2} - \alpha_j^2 p_j = \frac{\rho_0 f T}{N}\alpha_j\sigma_j F(x), \quad \alpha_j \equiv \frac{jf\pi}{NH}, \tag{81}
$$

No confusion should arise between the $\alpha_j$ above, which is not related to the reciprocal space variable, $\alpha$, used in the analysis of the 3D heating in the first part of this document. To determine the particular integral of ODE 81 we take a Fourier transform:

$$\hat{p}_j(k) = -\alpha_j \sigma_j \frac{\rho_0 f T}{N} \left( \frac{1}{k^2 + \alpha_j^2} \right) \hat{F}(k). \tag{82}$$

To proceed we must assume a particular horizontal variation, **notwithstanding a Fourier theorem unknown to me**. Consider a horizontal variation of heating:

$$F(x) = e^{-\frac{|x|}{\sigma}} \iff \hat{F}(k) = \sqrt{\frac{2}{\pi}} \left( \frac{1/\sigma}{k^2 + 1/\sigma^2} \right). \tag{83}$$

Note that we now use parameter $\sigma$ to scale the horizontal heating. From the last equation:

$$\hat{p}_j(k) = \sqrt{\frac{2}{\pi}} \frac{C \alpha_j \sigma_j}{(ik + \alpha_j)(ik - \alpha_j)(ik + 1/\sigma)(-ik - 1/\sigma)}, \tag{84}$$

where:

$$C = \frac{\rho_0 f T}{N \sigma}. \tag{85}$$

Using the usual partial fraction decomposition we find:

$$\begin{aligned} p_j(x) &= \frac{\alpha_j \sigma_j C}{\alpha_j(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \mathbf{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \frac{1}{(ik + \alpha_j)} \right) \\ &- \frac{\alpha_j \sigma_j C}{\alpha_j(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \mathbf{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \frac{1}{(ik - \alpha_j)} \right) \\ &+ \frac{\alpha_j \sigma_j C}{1/\sigma(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \mathbf{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \frac{1}{(ik - 1/\sigma)} \right) \\ &- \frac{\alpha_j \sigma_j C}{1/\sigma(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \mathbf{F}^{-1} \left( \frac{1}{\sqrt{2\pi}} \frac{1}{(ik + 1/\sigma)} \right), \end{aligned} \tag{86}$$

Now, it is easy to show by direct calculation that:

$$\mathbf{F} \left( \Theta(\mp x) e^{\frac{\pm x}{\sigma}} \right) = \frac{1}{\sqrt{2\pi}} \left( \frac{\pm 1}{ik \pm 1/\sigma} \right), \tag{87}$$

hence we have from equation 86 the following:

$$\begin{aligned} p_j(x) &= \frac{\alpha_j \sigma_j C}{\alpha_j(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \left(\Theta(-x)e^{\alpha_j x} + \Theta(x)e^{-\alpha_j x}\right) && (88) \\ &- \frac{\alpha_j \sigma_j C}{1/\sigma(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} \left(\Theta(x)e^{-\frac{x}{\sigma}} + \Theta(-x)e^{\frac{x}{\sigma}}\right), \end{aligned}$$

and simplifying, the final complimentary function for equation 81 is therefore:

$$p_j(x) = \frac{\sigma_j C}{(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\alpha_j |x|} - \frac{\sigma \alpha_j \sigma_j C}{(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\frac{|x|}{\sigma}}. \quad (89)$$

The steady pressure, potential temperature and velocity may all now be written down, with $b$ and $v$ obtained using equations 72:

$$\begin{aligned} p(x,z) &= \sum_j \frac{\sigma_j C}{(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\alpha_j |x|} \cos\left(\frac{j\pi z}{H}\right) && (90) \\ &- \sum_i \frac{\sigma \alpha_j \sigma_j C}{(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\frac{|x|}{\sigma}} \cos\left(\frac{j\pi z}{H}\right), \\ b(x,z) &= \sum_i \frac{j\pi \sigma \alpha_j \sigma_j C}{\rho_0 H(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\frac{|x|}{\sigma}} \sin\left(\frac{j\pi z}{H}\right) && (91) \\ &- \sum_j \frac{j\pi \sigma_j C}{\rho_0 H(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\alpha_j |x|} \sin\left(\frac{j\pi z}{H}\right), \\ v(x,z) &= (2\Theta(x) - 1) \sum_i \frac{\alpha_j \sigma_j C}{\rho_0 f(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\frac{|x|}{\sigma}} \cos\left(\frac{j\pi z}{H}\right) && (92) \\ &- (2\Theta(x) - 1) \sum_j \frac{\alpha_j \sigma_j C}{\rho_0 f(\alpha_j + 1/\sigma)(\alpha_j - 1/\sigma)} e^{-\alpha_j |x|} \cos\left(\frac{j\pi z}{H}\right). \end{aligned}$$

$v$ is obtained straightforwardly however to obtain $b$ care must be taken to avoid differentiating a series as follows. Multiply the the last of equations 76 by $\sin\left(\frac{j\pi z}{H}\right)$, integrate the result on $z$ over $[0, H]$, use parts in the right hand side (the evaluated term is zero) then substitute the modal expansions for $b$ and $p$ and use the orthonormality properties of the trigonometric functions to find $b_j(x) = -\frac{j\pi}{\rho_0 H} p_j(x)$. Nota also that it is possible to add to the above solution for $p$ a solution of the associated homogeneous equation i.e. a complimentary function. Put another way, our solution for $p$ above corresponds to the particular integral of equation 81.

**Remark** In fact, in the Matlab codes, the buoyancy ($b$) and thermal wind ($v$) are simply obtained by numerical differentiation of the pressure field, after equations 72.

**Remark** Subject to approximations made above, $b$, $v$ and $p$ are obtained by series expansions and depend on $H$ and $H_t$. It is meaningful to compare trapped ($H = H_t$) and radiating solutions ($H \gg H_t$). (Recall, the number of terms in these series should be increased as $H$ is increased).

#### 10.2.2 PV Response

PV is obtained from equations 73 and 74 by substituting for heating, $s$, using our assumed heating variation defined in equation 77:

$$\begin{aligned} PV &= \frac{\theta_0 f}{\rho_0 g} \int_0^T \frac{\partial s}{\partial z} dt, \\ &= \frac{\theta_0 f}{\rho_0 g} T F(x) \frac{d}{dz} \left( \sin\left(\frac{\pi z}{H_t}\right) \theta(z - H_t) \right), \\ &= \frac{\theta_0 f}{\rho_0 g} T F(x) \left( \sin\left(\frac{\pi z}{H_t}\right) \delta(z - H_t) + \frac{\pi}{H_t} \cos\left(\frac{\pi z}{H_t}\right) \Theta(z - H_t) \right). \end{aligned} \tag{93}$$

**Remark** Apparently, it is possible to obtain an analytical expression for the steady PV associated with our chosen, pulsed heating, within the system of Parker and Burton. An assumed heating with (i) separable time-dependance and (ii) differentiable vertical variation should allow us to arrive at a similarly exact expression for PV.

**Remark** Whilst it is necessary to assume a form for $F(x)$ to find $p$ (and hence $v$ and $b$) it is possible to determine the steady PV for any $F(x)$.

**Remark** Subject to approximations made above PV is determined exactly. It appears to be independent of the lid height, $H$.

#### 10.2.3 Heating Coefficients

Recall, we suppose a vertical variation of the heating $\sin\left(\frac{\pi}{H_t} z\right) (\Theta(z) - \Theta(z - H_t))$. Then, using the orthonormality of the trigonometric functions, the Fourier series coefficients of this vertical heating variation are:

$$\sigma_j = \frac{2\pi H}{H_t} \frac{\sin\left(\frac{j\pi H_t}{H}\right)}{\pi^2 \left(\frac{H^2}{H_t^2} - 1\right)}. \tag{94}$$

### 10.2.4 Sample Results

Figure 5 represents sample data corresponding to $f = 10^{-4}$, $\rho_0 = 1$, $T = 6 \times 60 \times 60$s and $\sigma = 1$km. We see the steady-state responses to the heating with the z-variation defined in equation 94 and the horizontal variation defined.

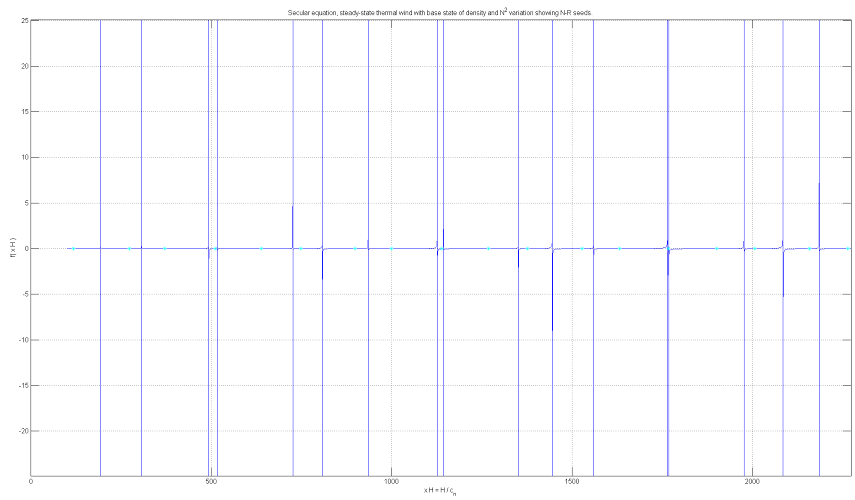


Figure 5: Example of the thermal wind for data $f = 10^{-4}$, $\rho_0 = 1$, $T = 6 \times 60 \times 60$s, $L = 1$km.

# 11 Steady, Far-Field Response and the Thermal Wind for Variable $\rho$, $N^2$

Let us obtain the steady-state response in 2D, $x - z$ plane slab geometry with variable base state density and buoyancy frequency. The basic equations are still those of equation 63:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \\
b &= \frac{1}{\rho_0}\frac{\partial p}{\partial z}, \\
\frac{\partial b}{\partial t} + N(z)^2 w &= s, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned}
\tag{95}
$$

We will again consider the steady response to a pulsed source of heating, with duration $T$.

## 11.1 Pressure Equation

In the present case, equation 75 becomes:

$$
\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2} + f^2\frac{\partial}{\partial z}\left(\frac{1}{\rho_0 N^2}\frac{\partial p}{\partial z}\right) = f^2\int_0^T \frac{\partial}{\partial z}\left(\frac{s}{N^2}\right)dt = f^2 T\frac{\partial}{\partial z}\left(\frac{s}{N^2}\right). \tag{96}
$$

Let $Z_n(z)$ be a Sturm-Liouville eigenfunction subject to boundary condition $\left[\frac{dZ_n}{dz}\right]_{0,H} = 0$ to be discussed below. Multiply equation 96 by $Z_n(z)$, integrate on $z$ over the interval $[0, H]$ to obtain:

$$
\frac{\partial^2}{\partial x^2}\int_0^H \frac{1}{\rho_0} p Z_n dz + f^2\int_0^H \frac{\partial}{\partial z}\left(\frac{1}{\rho_0 N^2}\frac{\partial p}{\partial z}\right) Z_n dz = f^2 T\int_0^H \frac{\partial}{\partial z}\left(\frac{s}{N^2}\right) Z_n dz. \tag{97}
$$

Use parts twice in the second term on the left hand side. The evaluated terms are eliminated by the pressure boundary condition $\left[\frac{dp}{dz}\right]_{0,H} = 0$ and the condition $\left[\frac{dZ_n}{dz}\right]_{0,H} = 0$. Use parts in the right hand side, where the condition $s(0) = s(h) = 0$ eliminated the evaluated term. We obtain:

$$
\frac{\partial^2}{\partial x^2}\int_0^H \frac{1}{\rho_0} p Z_n dz + f^2\int_0^H \frac{\partial}{\partial z}\left(\frac{1}{\rho_0 N^2}\frac{\partial Z_n}{\partial z}\right) p dz = f^2 T\int_0^H \frac{s}{N^2}\frac{dZ_n}{dz}dz. \tag{98}
$$

### 11.1.1 Modal Expansion

We shall use a modal expansion based upon eigenfunctions $Z_n$ such that:

$$\frac{d}{dz}\left(\frac{1}{\rho_0 N^2}\frac{dZ_n}{dz}\right) + \frac{1}{\rho_0 c_n^2} Z_n = 0, \quad \left[\frac{dZ_n}{dz}\right]_{0,H} = 0, \tag{99}$$

which corresponds to the choice $w = \frac{1}{\rho_0}$, $p = \frac{1}{\rho_0 N^2}$ in Arfken's notation. Let:

$$\phi_n(z) = \int^z \frac{1}{\rho_0(z')} Z_n(z')dz', \iff \frac{d\phi_n}{dz} = \frac{1}{\rho_0(z)} Z_n(z). \tag{100}$$

It is straightforward to prove the following results:

$$\begin{aligned}
\phi_n(0) = \phi_n(H) &= 0, \\
\frac{dZ_n}{dz} + \frac{\rho_o N^2}{c_n^2}\phi_n &= 0, \\
\int_0^H \frac{1}{\rho_0} Z_n Z_m dz &= H\delta_{nm}, \\
\int_0^H \rho_0 \frac{d\phi_n}{dz}\frac{d\phi_m}{dz} dz &= H\delta_{nm}, \\
\int_0^H \frac{1}{\rho_0 N^2}\frac{dZ_n}{dz}\frac{dZ_m}{dz} dz &= \frac{H}{c_n^2}\delta_{nm}.
\end{aligned} \tag{101}$$

Clearly, it will be possible to expand functions in terms of the $Z_n$, $\phi_n$ and the derivatives of $Z_n$, namely $\frac{dZ_n}{dz}$. Bearing in mind its structure we take the following expansions:

$$\begin{aligned}
p(x,z) &= \sum_j p_j(x) Z_j(z), \\
s(x,z) &= \frac{1}{\rho_0} F(x) \sum_j \sigma_j \frac{dZ_j}{dz}, \\
v(x,z) &= \frac{1}{\rho_0} \sum_j v_j(x) Z_j(z), \\
b(x,z) &= \frac{1}{\rho_0} \sum_j b_j(x) \frac{dZ_j}{dz},
\end{aligned} \tag{102}$$

which, when substituted into equation 98 yield the following ordinary differential equation:

$$\frac{d^2}{dx^2}p_n(x) - \frac{f^2}{c_n^2}p_n(x) = -\frac{f^2 T\sigma_n}{c_n^2}F(x), \tag{103}$$

where we have used the orthonormality properties among equations 101.

We return to the solution for $b$ and $v$ shortly, and continue to consider $p$ now. The above is clearly similar to equation 82. To determine the particular integral of ODE 81 we again take a Fourier transform:

$$\hat{p}_n(k) = \frac{f^2 T\sigma_n}{c_n^2}\left(\frac{1}{k^2 + \frac{f^2}{c_n^2}}\right)\hat{F}(k), \tag{104}$$

To proceed we must again assume a particular horizontal variation. For consistency, we consider a horizontal variation of heating:

$$F(x) = e^{-\frac{|x|}{\sigma}} \iff \hat{F}(k) = \sqrt{\frac{2}{\pi}}\left(\frac{1/\sigma}{k^2 + 1/\sigma^2}\right). \tag{105}$$

whence we obtain:

$$\hat{p}_n(k) = \sqrt{\frac{2}{\pi}}\frac{f^2 T\sigma_n}{\sigma c_n^2}\left(\frac{1}{k^2 + f^2/c_n^2}\right)\left(\frac{1}{k^2 + 1/\sigma^2}\right), \tag{106}$$

and upon using partial fractions in the right hand side, this gives:

$$\begin{aligned}\hat{p}_n(k) = \; & + \; \alpha_n\left(\sqrt{\frac{1}{2\pi}}\frac{1}{ik - f/c_n}\right) \\ & - \; \alpha_n\left(\sqrt{\frac{1}{2\pi}}\frac{1}{ik + f/c_n}\right) \\ & - \; \beta_n\left(\sqrt{\frac{1}{2\pi}}\frac{1}{ik - 1/\sigma}\right) \\ & + \; \beta_n\left(\sqrt{\frac{1}{2\pi}}\frac{1}{ik + 1/\sigma}\right),\end{aligned} \tag{107}$$

where we have defined:

$$\alpha_n \equiv \frac{fT\sigma_n c_n\sigma}{(\sigma^2 f^2 - c_n^2)}, \quad \beta_n \equiv \frac{f^2 T\sigma_n\sigma^2}{(\sigma^2 f^2 - c_n^2)}. \tag{108}$$

We can now inverse Fourier transform. Using the result in equation 87, namely:

$$\mathbf{F}\left(\Theta(\mp x)e^{\frac{\pm x}{\sigma}}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{\pm 1}{ik \pm 1/\sigma}\right), \tag{109}$$

equation 107 may yields:

$$\begin{aligned} p_n(x) = & - \alpha_n \left(\Theta(x)e^{-\frac{f}{c_n}x} + \Theta(-x)e^{\frac{f}{c_n}x}\right) \\ & + \beta_n \left(\Theta(x)e^{-\frac{1}{\sigma}x} + \Theta(-x)e^{\frac{1}{\sigma}x}\right), \end{aligned} \tag{110}$$

which clearly simplifies to give the following particular integral:

$$p_n(x) = -\alpha_n e^{-\frac{f}{c_n}|x|} + \beta_n e^{-\frac{1}{\sigma}|x|}. \tag{111}$$

There remains the problem of finding the heating coefficient $\sigma_n$, the $Z_n(z)$ and, of course the eigenvalues or wave speeds, $c_n$.

## 11.2 Model Stratification

Let us assume that the vertical variation of Buoyancy frequency is that previously used:

$$N(z) = (\Theta(z) - \Theta(z - H_t))N_1 + \Theta(z - H_t)N_2, \tag{112}$$

and the corresponding variation of the base state of density is also that used previously:

$$\rho_0(z) = (\Theta(z) - \Theta(z - H_t))\rho_s e^{-\frac{z}{H_1}} + \Theta(z - H_t)\rho_s e^{-\frac{H_t}{H_1}} e^{-\frac{(z-H_t)}{H_2}}. \tag{113}$$

Let us derive appropriate eigen-funcitons for the representation of the response fields.

### 11.2.1 Eigenfunctions

With our choice of density and buoyancy frequency variation, equation 100 gives for the $Z_n(z)$ the ODEs to solve:

$$\begin{aligned} \frac{d^2 Z_n^{(1)}}{dz^2} + \frac{1}{H_1}\frac{dZ_n^{(1)}}{dz} + \frac{N_1^2}{c_n^2}Z_n^{(1)} &= 0, z < H_t \\ \frac{d^2 Z_n^{(2)}}{dz^2} + \frac{1}{H_2}\frac{dZ_n^{(2)}}{dz} + \frac{N_2^2}{c_n^2}Z_n^{(2)} &= 0, H > z \geq H_t. \end{aligned} \tag{114}$$

To match the surface and lid boundary conditions we require oscillatory solutions, hence, to ensure complex roots to the auxiliary equations we assume:

$$\frac{N_i^2}{c_n} > \frac{1}{4H_i^2}, \quad i = 1, 2 \tag{115}$$

and accordingly:

$$\begin{aligned} Z_n^{(1)}(z) &= e^{-\frac{1}{2H_1}z}\left(A_n^{(1)}\cos\left(k_n^{(1)}z\right) + B_n^{(1)}\sin\left(k_n^{(1)}z\right)\right), z < H_t \quad (116)\\ Z_n^{(2)}(z) &= e^{-\frac{1}{2H_2}z}\left(A_n^{(2)}\cos\left(k_n^{(2)}z\right) + B_n^{(2)}\sin\left(k_n^{(2)}z\right)\right), z \geq H_t, \end{aligned}$$

where $A_n^{(1)}..B_n(2)$ are integration constants. Applying boundary conditions:

$$\left[\frac{dZ_n^{(1)}}{dz}\right]_{z=0} = 0, \quad \left[\frac{dZ_n^{(2)}}{dz}\right]_{z=H} = 0, \tag{117}$$

we straightforwardly obtain for the model troposphere, $z < H_t$:

$$Z_n^{(1)}(z) = A_n^{(1)}e^{-\frac{1}{2H_1}z}\sin\left(k_n^{(1)}z + \phi_n^{(1)}\right), \tag{118}$$

and for the model stratosphere $z \geq H_t$:

$$Z_n^{(2)}(z) = A_n^{(2)}e^{-\frac{1}{2H_2}z}\sin\left(k_n^{(2)}z + \phi_n^{(2)}\right), \tag{119}$$

where we have defined:

$$\begin{aligned} \phi_n^{(1)} &= \tan^{-1}(2H_1k_n^{(1)}), \quad (120)\\ \phi_n^{(2)} &= \tan^{-1}\left(\frac{2H_2k_n^{(2)} - \tan(k_n^{(2)}H)}{1 + 2H_2k_n^{(2)}\tan(k_n^{(2)}H)}\right),\\ &= \tan^{-1}\left(\tan\left(\tan^{-1}\left(2H_2k_n^{(2)}\right) - k_n^{(2)}H\right)\right),\\ &= \tan^{-1}\left(2H_2k_n^{(2)}\right) - k_n^{(2)}H,\\ k_n^{(i)} &= \sqrt{\frac{N_i^2}{c_n^2} - \frac{1}{4H_i^2}}, i = 1, 2. \end{aligned}$$

We also give here the corresponding functions $\frac{dZ_n}{dz}$, obtained by differentiation of the $Z_n$. Using the product rule and a number of trigonometric identities we obtain for the model troposphere, $z < H_t$:

$$\frac{dZ_n^{(1)}}{dz}(z) = -A_n^{(1)}\left(k_n^{(1)2} + \frac{1}{4H_1^2}\right)^{1/2} e^{-\frac{1}{2H_1}z}\sin\left(k_n^{(1)}z\right), \tag{121}$$

and for the model stratosphere $z \geq H_t$:

$$\frac{dZ_n^{(2)}}{dz}(z) = A_n^{(2)} \left( k_n^{(2)2} + \frac{1}{4H_2^2} \right)^{1/2} e^{-\frac{1}{2H_2}z} \sin\left(k_n^{(2)}(H-z)\right). \tag{122}$$

### 11.2.2 Matching Conditions and Secular Equation

Let us consider the matching conditions on $Z_n^{(1)}$ and $Z_n^{(2)}$ at $z = H_t$. Considering that pressure is expanded in the $Z_n$, continuity of pressure at the tropopause imposes the condition:

$$Z_n^{(1)}(H_t) = Z_n^{(2)}(H_t). \tag{123}$$

To derive a second condition we integrate equation 99 over a narrow range of $z$ spanning the tropopause to obtain:

$$\left[ \frac{1}{\rho_0 N^2} \frac{dZ_n}{dz} \right]_{H_t^-}^{H_t^+} + \frac{1}{c_n^2} \int_{H_t^i}^{H_t^+} \frac{1}{\rho_0} Z_n = 0. \tag{124}$$

As $H_t^- \to H_t^+$ the integral vanishes (it's integrand is continuous by assumption) and so, from the evaluated term in the above we have $\left[\frac{1}{\rho_0 N_1^2}\frac{dZ_n^{(1)}}{dz}\right]_{H_t^-} - \left[\frac{1}{\rho_0 N_2^2}\frac{dZ_n^{(2)}}{dz}\right]_{H_t^+} = 0$, which, since $\rho_0$ is continuous, yields a second condition:

$$\left[ \frac{1}{N_1^2} \frac{dZ_n^{(1)}}{dz} \right]_{H_t} = \left[ \frac{1}{N_2^2} \frac{dZ_n^{(2)}}{dz} \right]_{H_t}. \tag{125}$$

From equation 123 we therefore obtain:

$$A_n^{(1)} e^{-\frac{H_t}{2H_1}} \sin(k_n^{(1)} H_t + \phi_n^{(1)}) - A_n^{(2)} e^{-\frac{H_t}{2H_2}} \sin(k_n^{(2)} H_t + \phi_n^{(2))}) = 0. \tag{126}$$

In passing, we note that the above equation allows us to write normalisation constant $A_n^{(2)}$ in terms of $A_n^{(1)}$. From 125 (and the product rule) we also obtain a second, lengthier condition:

$$\begin{aligned} & A_n^{(1)} N_2^2 e^{-\frac{H_t}{2H_1}} \left( k_n^{(1)} \cos(k_n^{(1)} H_t + \phi_n^{(1)}) - \frac{1}{2H_1} \sin(k_n^{(1)} H_t + \phi_n^{(1)}) \right) \\ - \quad & A_n^{(2)} N_1^2 e^{-\frac{H_t}{2H_2}} \left( k_n^{(2)} \cos(k_n^{(2)} H_t + \phi_n^{(2)}) - \frac{1}{2H_2} \sin(k_n^{(2)} H_t + \phi_n^{(2)}) \right) = 0. \end{aligned} \tag{127}$$

For a non-trivial solution to equations 126 and 127 (for $A_n^{(1)}$ and $A_n^{(2)}$) to exist the determinant of the matrix of coefficients of the two unknowns $A_n^{(1)}$ and $A_n^{(2)}$ must vanish:

$$\begin{aligned} & N_1^2 \sin(k_n^{(1)} H_t + \phi_n^{(1)}) \left( k_n^{(2)} \cos(k_n^{(2)} H_t + \phi_n^{(2)}) - \frac{1}{2H_2} \sin(k_n^{(2)} H_t + \phi_n^{(2)}) \right) \quad (128) \\ - \; & N_2^2 \sin(k_n^{(2)} H_t + \phi_n^{(2)}) \left( k_n^{(1)} \cos(k_n^{(1)} H_t + \phi_n^{(1)}) - \frac{1}{2H_1} \sin(k_n^{(1)} H_t + \phi_n^{(1)}) \right) = 0, \end{aligned}$$

in which we have cancelled an exponential factor $e^{-H_t\left(\frac{1}{2H_1}+\frac{1}{2H_2}\right)}$. After some algebra secular equation 128 may written:

$$N_1^2 k_n^{(2)} \cot(k_n^{(2)} H_t + \phi_n^{(2)}) - N_2^2 k_n^{(1)} \cot(k_n^{(1)} H_t + \phi_n^{(1)}) - \frac{N_1^2 H_1 - N_2^2 H_2}{2H_1 H_2} = 0. \quad (129)$$

It is the above equation which must be solved numerically for the $c_n$s. Recall, $k_n^{(i)} = \sqrt{\frac{N_i^2}{c_n^2} - \frac{1}{4H_i^2}}$. See figure **??** below.

### 11.2.3 Normalization of the $Z_n(z)$ and the heating coefficients $\sigma_n$

Given that the $Z_n$ is defined piecewise, our assumed normalisation condition (equation 101 i.e. $\int_0^H \frac{1}{\rho_0} Z_n Z_m dz = H\delta_{nm}$) requires:

$$\begin{aligned} & A_n^{(1)2} \int_0^{H_t} \frac{1}{\rho_s e^{-\frac{z}{H_1}}} \left( e^{-\frac{z}{2H_1}} \sin(k_n^{(1)} z + \phi_n^{(1)}) \right)^2 dz \quad (130) \\ + \; & A_n^{(2)2} \int_{H_t}^{H} \frac{1}{\rho_s e^{-\frac{H_t}{H_1}} e^{-\frac{(z-H_t)}{H_2}}} \left( e^{-\frac{z}{2H_2}} \sin(k_n^{(2)} z + \phi_n^{(2)}) \right)^2 dz = H. \end{aligned}$$

We must be careful to handle the exponentials correctly in this normalisation integral:

$$\begin{aligned} & \frac{A_n^{(1)2}}{\rho_s} \int_0^{H_t} \sin^2(k_n^{(1)} z + \phi_n^{(1)}) dz \quad (131) \\ + \; & \frac{A_n^{(2)2}}{\rho_s e^{\frac{H_t}{H_2} - \frac{H_t}{H_1}}} \int_{H_t}^{H} \sin^2(k_n^{(2)} z + \phi_n^{(2)}) dz = H. \end{aligned}$$

Using condition 126 to eliminate $A_n^{(2)}$ and noting the exponentials in both integrands cancel we obtain, after some algebra, the following:

$$\frac{A_n^{(1)2}}{\rho_s}\left(\int_0^{H_t}\sin^2(k_n^{(1)}z+\phi_n^{(1)})dz+g^2\int_{H_t}^{H}\sin^2(k_n^{(2)}z+\phi_n^{(2)})dz\right)=H,$$

where:

$$g^2\equiv\frac{\sin^2(k_n^{(1)}H_t+\phi_n^{(1)})}{\sin^2(k_n^{(2)}H_t+\phi_n^{(2)})}. \qquad (132)$$

Performing the integrals using the trig. substitution $\sin^2(x)=\frac{1}{2}(1-2\cos(2x))$ we have:

$$\begin{aligned}
A_n^{(1)2} &= \sqrt{\frac{2\rho_s H}{I_1+g^2I_2}},\\
I_1 &\equiv H_t-\frac{\sin(2k_n^{(1)}H_t+2\phi_n^{(1)})}{2k_n^{(1)}}+\frac{\sin(2\phi_n^{(1)})}{2k_n^{(1)}},\\
I_2 &\equiv (H-H_t)-\frac{\sin(2k_n^{(2)}H+2\phi_n^{(2)})}{2k_n^{(2)}}+\frac{\sin(2k_n^{(2)}H_t+2\phi_n^{(2)})}{2k_n^{(2)}}.
\end{aligned}$$

We can now consider the heating coefficients $\sigma_n$. Using equation 102 we have:

$$s(x,z)\equiv F(x)\sin\left(\frac{\pi z}{H_t}\right)(\Theta(z)-\Theta(z-H_t))=F(x)\frac{1}{\rho_0}\sum_j\sigma_j\frac{dZ_j}{dz},$$

Cancel $F(x)$, multiply by function $\frac{dZ_n}{dz}$, multiply by $\frac{1}{N^2}$ and integrate $\int_0^H$ to obtain:

$$\int_0^{H_t}\frac{1}{N^2}\sin\left(\frac{\pi z}{H_t}\right)\frac{dZ_n}{dz}dz=\sum_j\sigma_j\int_0^H\frac{1}{\rho_0N^2}\frac{dZ_j}{dz}\frac{dZ_n}{dz}dz=\frac{H}{c_n^2}\sigma_n.$$

Substituting for $Z_n$ and noting $N=N_1$ over the range of integration used, we obtain the following:

$$\sigma_n=\frac{c_n^2A_n^{(1)}}{HN_1^2}\int_0^{H_t}\sin\left(\frac{\pi z}{H_t}\right)\frac{d}{dz}\left(e^{-\frac{z}{2H_1}}\sin(k_n^{(1)}z+\phi_n^{(1)})\right)dz.$$

Performing the differentiation in the integrand (using the product rule) and replacing all trig functions using complex exponentials, we have, after some straightforward algebra:

$$\sigma_n = \frac{c_n^2 A_n^{(1)}}{2HN_1^2}\left(k_n^{(1)}\left(\Im(\omega_1)+\Im(\omega_2)\right)+\frac{1}{2H_1}\left(\Re(\omega_1)+\Re(\omega_2)\right)\right), \tag{133}$$

where:

$$\omega_1 = \frac{\left(e^{ik_n^{(1)}H_t-\frac{H_t}{2H_1}}+1\right)e^{i\phi_n^{(1)}}}{\left(-i\frac{\pi}{H_t}+ik_n^{(1)}-\frac{1}{2H_1}\right)}, \quad \omega_2 = \frac{\left(e^{-ik_n^{(1)}H_t-\frac{H_t}{2H_1}}+1\right)e^{-i\phi_n^{(1)}}}{\left(-i\frac{\pi}{H_t}-ik_n^{(1)}-\frac{1}{2H_1}\right)}. \tag{134}$$

### 11.2.4 Solutions for b and v

Having determined the solution for the steady pressure response, $p(x,z)$, we return to equations 63 and 102. In the first (third) of equations 63 we substitute the following modal expansions respectively:

$$\begin{aligned} v(x,z) &= \frac{1}{\rho_0}\sum_j v_j(x)Z_j(z), \\ b(x,z) &= \frac{1}{\rho_0}\sum_j b_j(x)\frac{dZ_j}{dz}. \end{aligned} \tag{135}$$

In the case of $v$, multiply the first of equations 63 by $Z_n$, substitute for $p$, integrate on $z$ over $[0,H]$ and apply the orthonormality of the $Z_n$s to obtain:

$$\int_0^H vZ_n(z')dz' = \frac{1}{f}\sum_j \frac{d}{dx}p_j(x)\int_0^H \frac{1}{\rho_0}Z_jZ_n dz' = \frac{H}{f}\frac{d}{dx}p_n(x), \tag{136}$$

and using the second of equations 135 in the left and side expression an the orthonormality property we have:

$$Hv_n(x) = \frac{H}{f}\frac{d}{dx}p_n(x) \implies v_n(x) = \frac{1}{f}p_n'(x). \tag{137}$$

In the case of $b$, first multiply the third of equations 63 by $\frac{1}{N^2}\frac{dZ_n}{dz}$, integrate on $z$ over $[0,H]$ and use integration by parts to obtain:

$$\int_0^H \frac{1}{N^2}b\frac{dZ_n}{dz'}dz' = \left[p\frac{1}{\rho_0 N^2}\frac{dZ_n}{dz'}\right]_0^H - \int_0^H p\frac{d}{dz'}\left(\frac{1}{\rho_0 N^2}\frac{dZ_n}{dz'}\right)dz', \tag{138}$$

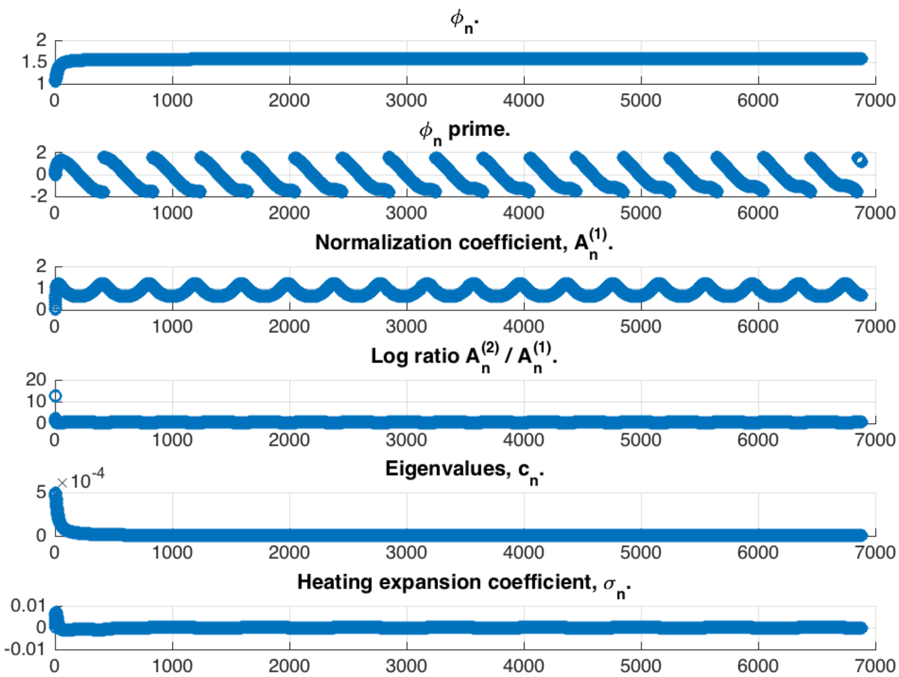


Figure 6: Plot showing the variation with index $n$ of quantities $\phi_n$, $\phi_n'$, normalization coefficient $A_n^{(1)}$, ratio $\frac{A_n^{(2)}}{A_n^{(1)}}$, eignevalue $c_n$ and heating expansion coefficient for a very hight lid $H_L = 200$ with $f = 10^{-4}$.

and using the boundary condition on $Z_n$ to eliminate the evaluated term and transforming the integrand on the right hand side using 99 we have:

$$\int_0^H \frac{1}{N^2} b \frac{dZ_n}{dz'} dz' = \frac{1}{c_n^2} \int_0^H \frac{1}{\rho_0} p Z_n dz'. \tag{139}$$

Finally, substituting the modal expansions for $p$ and $b$ and using orthonormality we have:

$$\frac{H}{c_n^2} b_n(x) = \frac{1}{c_n^2} H p_n(x) \implies b_n(x) = p_n(x). \tag{140}$$

### 11.2.5 Solutions for Potential Vorticity, PV

Define the potential vorticity:

$$PV = \frac{1}{\rho} \underline{\zeta} \cdot \underline{\nabla} \theta, \tag{141}$$

where all symbols have their usual meaning. Expanding about base states, we have:

$$\begin{aligned} PV + PV' &= \frac{1}{(\rho_0 + \rho')} \underline{\zeta} \cdot \underline{\nabla} (\theta_0 + \theta'), \\ &\approx \frac{1}{\rho_0} \underline{\zeta} \cdot \frac{d\theta_0}{dz} \hat{e}_z + \frac{1}{\rho_0} \underline{\zeta} \cdot \underline{\nabla} \theta' - \frac{1}{\rho_0^2} \rho' \underline{\zeta} \cdot \frac{d\theta_0}{dz} \hat{e}_z + O(2), , \end{aligned} \tag{142}$$

bearing in mind that the $\underline{\zeta}$ does not contain only $O(1)$ terms. Note that some of the above contain $O(0)$ terms. Substitute for $\underline{\zeta} = (v_x - u_y + f)\hat{e}_z$ to obtain:

$$\begin{aligned} PV + PV' \approx\ & \frac{1}{\rho_0} (v_x - u_y + f) \frac{d\theta_0}{dz} \\ +\ & \frac{1}{\rho_0} (v_x - u_y + f) \frac{\partial \theta'}{\partial z} \\ -\ & \frac{1}{\rho_0^2} \rho' (v_x - u_y + f) \frac{d\theta_0}{dz}, \end{aligned} \tag{143}$$

and retain only $O(0)$ and $O(1)$ terms, which we now separate:

$$\begin{aligned} PV &= \frac{f}{\rho_0} \frac{d\theta_0}{dz}, \\ PV' &= \frac{1}{\rho_0} (v_x - u_y) \frac{d\theta_0}{dz} + \frac{f}{\rho_0} \frac{\partial \theta'}{\partial z} - \frac{f \rho'}{\rho_0^2} \frac{d\theta_0}{dz}. \end{aligned} \tag{144}$$

Now:

$$N^2 \equiv g\frac{d}{dz}\log_e(\theta_0) = \frac{g}{\theta_0}\frac{d\theta_0}{dz} \iff \frac{d\theta_0}{dz} = \frac{N^2\theta_0}{g}, \tag{145}$$

and so, substituting:

$$PV' = \frac{1}{\rho_0}(v_x - u_y)\frac{N^2\theta_0}{g} + \frac{f}{\rho_0}\frac{\partial\theta'}{\partial z} - \frac{f\rho'}{\rho_0^2}\frac{N^2\theta_0}{g}. \tag{146}$$

Now, we will substitute using the following relations:

$$b \equiv \frac{g\theta'}{\theta_0}, \quad \frac{\rho'}{\rho_0} = -\frac{\theta'}{\theta_0}. \tag{147}$$

hence we obtain:

$$\begin{aligned} PV' &= \frac{N^2\theta_0}{g}\frac{1}{\rho_0}(v_x - u_x) + \frac{f}{g\rho_0}\frac{\partial}{\partial z}(b\theta_0) + \frac{fN^2}{g\rho_0}\theta', \\ &= \frac{N^2\theta_0}{g}\frac{1}{\rho_0}(v_x - u_x) \\ &+ \frac{f}{g\rho_0}\left(\theta_0\frac{\partial b}{\partial z} + b\frac{d\theta_0}{dz}\right) \\ &+ \frac{fN^2}{g\rho_0}\theta', \\ &= \frac{N^2\theta_0}{g\rho_0}(v_x - u_x) \\ &+ \frac{fN^2\theta_0}{g^2\rho_0}b + \frac{f\theta_0}{\rho_0 g}\frac{\partial b}{\partial z} \\ &+ \frac{fN^2\theta_0}{g^2\rho_0}b, \end{aligned} \tag{148}$$

Hence we have:

$$\frac{\rho_0 g}{\theta_0}(PV') = N^2(v_x - u_x) + f\frac{\partial b}{\partial z} + 2\frac{fN^2}{g}b. \tag{149}$$

This result is largely a direct consequence of definitions. Note, the terms on the right hand side are similar to those derived by SDG for the case of $PV'$ in a constant density system.

Let us now assume constant $N^2$ (but still retain $\rho_0$ variable) and consider the steady state of our system, long after the application of a heat pulse of duration $T$ ($s = u = w = 0$). Now we have:

$$\frac{\rho_0 g}{\theta_0 N^2}(PV') = v_x + \frac{f}{N^2}\frac{\partial b}{\partial z} + 2\frac{f}{g}b, \tag{150}$$

in which we may use equation 70 to eliminate terms on the right hand side:

$$\frac{\rho_0 g}{\theta_0 N^2}(PV') = \frac{fT}{N^2}\frac{\partial s}{\partial z} + 2\frac{f}{g}b, \tag{151}$$

We need not seek a modal expansion for (PV'). We can now substitute for our assumed $s$ (differentiated of course) and our $b$ solution. The above then gives the (scaled) potential vorticity adjustment at steady state.

#### 11.2.6 Specimen Data

The eigenvalues $c_n$ were determined using a numerical method (interval bisection) and the modal expansions plotted over a region of physical space using Matlab. The specimen results below correspond to the following paramterization.

$$\begin{aligned}
T &= 2000s \\
s &= 30 \\
\sigma &= 1km \\
f &= 0.0001; \\
N_2 &= 2(N_1 = 0.01) \\
H &= 2000km \\
H_v &= 50km \\
H_t &= 10km \\
H_1 &= 10^{-3}g/N_1^2 km \\
H_2 &= 10^{-3}g/N_2^2 km
\end{aligned} \tag{152}$$

in which the factor $10^{-4}$ in the scale heights is present to convert $g$ into units of $kms^{-2}$.

As a check, the applied heating profile was reconstructed from its modal expansion. This result has the correct functional form in the tropopause however we must conclude there is an error the specimen data presented here: see caption in figures below. **Note: data obtained from version in the file Untitled-message-11 in directory Steady Potential Vorticity, on Macbook desktop.**

### 11.2.7 Debugging : Constant $N$ Model

The discontinuity at the tropopause in the heating profile, which is expanded in the $\frac{dZ_n}{dz}$, suggests an issue with matching condition which may be removed by setting $N_1 = N_2$, whereupon the condition 125 is removed along with the need to partition the solution and the need to seek a numerical solution for the wave speeds $c_n$, as we shall now show. Note that the base state of density is now given by:

$$\rho_0(z) = \rho_s e^{-\frac{z}{H_1}}, \quad 0 < z < H. \tag{153}$$

The vertical variation in our modal expansion (the $Z_n$) now satisfies a single equation over the whole vertical domain:

$$\frac{d^2 Z_n}{dz^2} + \frac{1}{H_1}\frac{dZ_n}{dz} + \frac{N_1^2}{c_n^2} Z_n = 0, \quad z < H. \tag{154}$$

To match the surface and lid boundary conditions we still require oscillatory solutions, hence, to ensure complex roots to the auxiliary equation of the above ODE, we again assume $\frac{N_1^2}{c_n} > \frac{1}{4H_1^2}$ and accordingly:

$$\begin{aligned} Z_n(z) &= e^{-\frac{1}{2H_1}z}\left(A_n' \cos(k_n z) + B_n' \sin(k_n z)\right), \\ k_n &\equiv \sqrt{\frac{N_1^2}{c_n^2} - \frac{1}{4H_1^2}} \in R^+. \end{aligned} \tag{155}$$

Here $A_n', B_n'$ are integration constants. We choose to re-express $Z_n(z)$ as follows:

$$\begin{aligned} Z_n(z) &= A_n e^{-\frac{1}{2H_1}z}\cos(k_n z + \phi_n), \\ k_n &\equiv \sqrt{\frac{N_1^2}{c_n^2} - \frac{1}{4H_s^2}} \in R^+. \end{aligned} \tag{156}$$

Of course, $A_n \neq A_n'$.

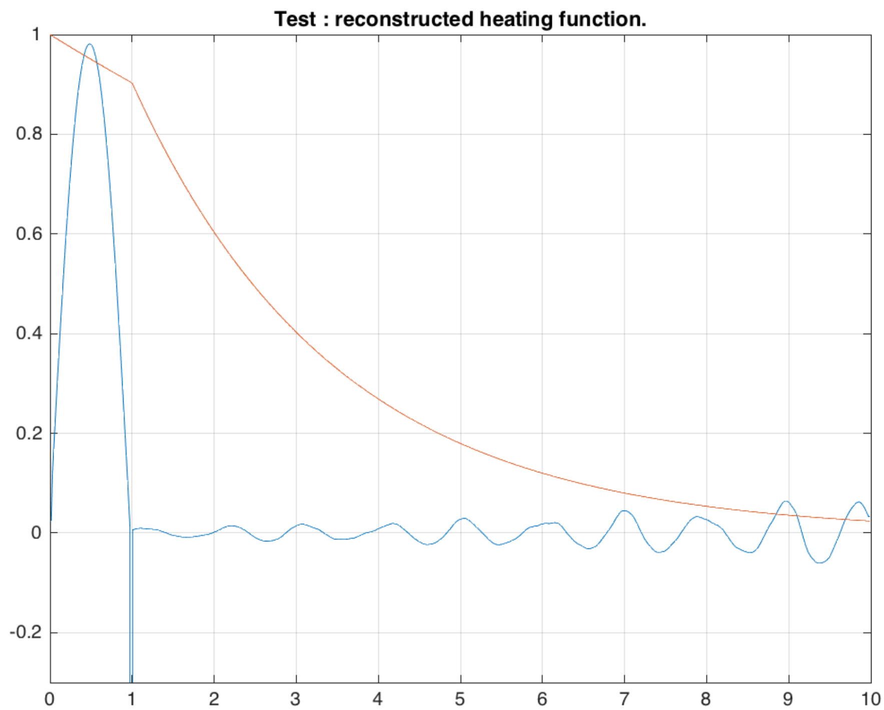


Figure 7: Applied heating profile reconstructed from the eigen-basis. We note (i) the discontinuity at the tropopause and (ii) the failure to converge to zero in the stratosphere.

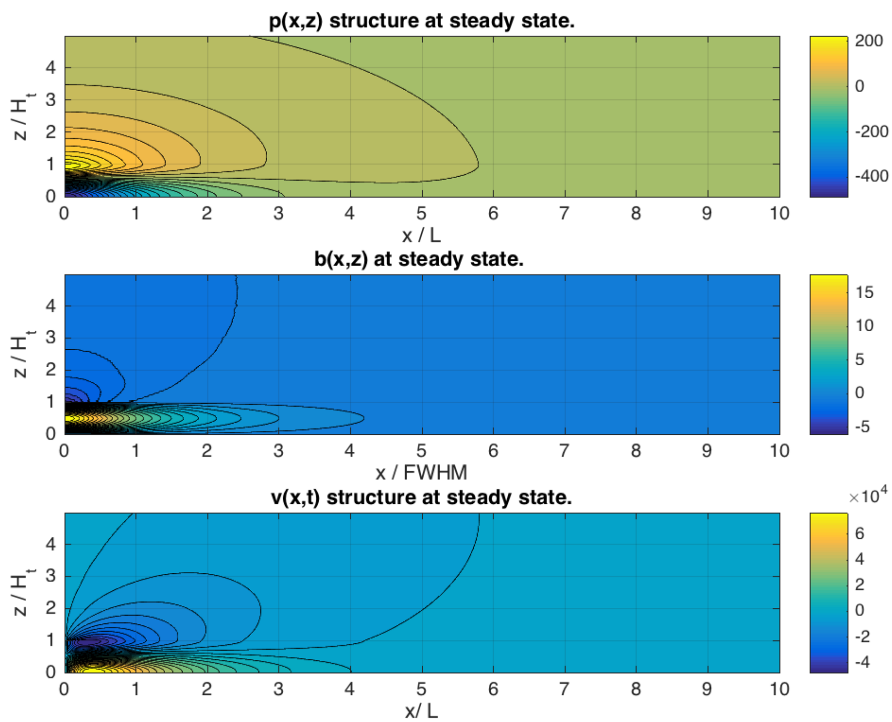


Figure 8: Specimen steady responses for the $p$, $b$ and $v$ fields obtained with the modal expansions developed here. This data should be treated with caution considering the artefacts identified in figure **??**

Hence, we can now obtain for our second set of functions the following:

$$\begin{aligned}
\frac{dZ_n}{dz} &= A_n e^{-\frac{1}{2H_1}z}\left(-k_n \sin\left(k_n z+\phi_n\right)-\frac{1}{2H_1}\sin(k_n z+\phi_n)\right), \quad (157)\\
&= -A_n e^{-\frac{1}{2H_1}z}\sqrt{k_n^2+\frac{1}{4H_1^2}}\sin\left(k_n z+\phi_n+\theta_n\right),\\
&= -A_n\frac{N_1}{c_n}e^{-\frac{1}{2H_1}z}\sin\left(k_n z+\phi_n+\theta_n\right),\\
\theta_n &\equiv \tan^{-1}\left(\frac{1}{2H_1 k_n}\right),
\end{aligned}$$

where we have used the definition of the $k_n$ in the third line. Applying boundary conditions:

$$\left[\frac{dZ_n}{dz}\right]_{z=0}=\left[\frac{dZ_n}{dz}\right]_{z=H}=0, \tag{158}$$

we straightforwardly obtain:

$$\phi_n=-\theta_n=-\tan^{-1}\left(\frac{1}{2H_1 k_n}\right), \quad k_n=\frac{n\pi}{H}. \tag{159}$$

We can now summarise the full solution:

$$\begin{aligned}
Z_n(z) &= A_n e^{-\frac{1}{2H_1}z}\cos\left(k_n^{(1)}z-\theta_n\right), \quad (160)\\
\frac{dZ_n}{dz} &= -A_n e^{-\frac{1}{2H_1}z}\sin\left(k_n^{(1)}z\right),\\
k_n &= \frac{n\pi}{H}.
\end{aligned}$$

It is now possible to write-down an analytical expression for the the wave-speeds, $c_n$ from the definition of $k_n$:

$$\frac{n^2\pi^2}{H^2}=\frac{N_1^2}{c_n^2}-\frac{1}{4H_1^2} \iff c_n=\left(\frac{n^2\pi^2}{H^2N_1^2}+\frac{1}{4H_1^2N_1^2}\right)^{-1/2}. \tag{161}$$

The normalisation coefficient $A_n$ may be obtained from the orthogonality condition $\int_0^H \frac{1}{\rho_0}Z_nZ_m dz = H\delta_{nm}$ as follows: exponentials cancel in the integrand to leave $\frac{A_n^2}{\rho_s}\int_0^H \cos^2\left(k_n^{(1)}z-\theta_n\right)dz = H$ and using the trigonometric identity $\cos(2x)=\frac{1}{2}(2\cos^2(x)-1)$, properties of trig functions and some straightforward algebra we easily obtain:

$$A_n=\sqrt{2\rho_s}, \quad \forall n. \tag{162}$$