$N(z)$, $\rho_0(z)$ both vary

$$\frac{d}{dz}\left(\rho_0 \frac{d\phi_n}{dz}\right) + \frac{N(z)^2 \rho_0(z)}{c_n^2}\phi_n = 0 \qquad - (A)$$

$$\phi_n \equiv \int_0^z Z_n(u)\,du \iff \frac{d\phi_n}{dz} = Z_n$$

(A) transforms to :-

$$\frac{d}{dz}\left(\rho_0(z) Z_n(z)\right) + \frac{N(z)^2 \rho_0(z)}{c_n^2}\int_0^z Z_n(z')\,dz' = 0$$

d wrt z

$$\frac{d^2}{dz^2}\left(\rho_0(z)\, Z_n(z)\right) + \frac{N^2(z)\rho_0(z)}{c_n^2} Z_n(z) = 0$$

Not SL form

$N(z)$ varies, $\rho_0 \sim$ const

$$\frac{d}{dz}\frac{d\phi}{dz} + \frac{N(z)^2}{c_n^2}\phi_n = 0$$

$$\therefore \frac{1}{N^2}\frac{d}{dz}Z_n + \frac{1}{c_n^2}\int Z_n = 0 \qquad \text{d wrt } z$$

$$\therefore \frac{d}{dz}\left(\frac{1}{N^2}\frac{dZ_n}{dz}\right) + \frac{1}{c_n^2}Z_n = 0$$

Has SL form

/

$\underline{0 < z < D}$

$$\rho_0 \frac{d^2\phi_n}{dz^2} + \frac{d\rho_0}{dz}\frac{d\phi_n}{dz} + \frac{N_1^2 \rho_0}{c_n^2}\phi_n = 0$$

$$\frac{d^2\phi_n}{dz^2} - \frac{1}{D}\frac{d\phi_n}{dz} + \frac{N_1^2}{c_n^2}\phi_n = 0$$

$$m = \frac{\frac{1}{D} \pm \sqrt{\frac{1}{D^2} - 4\frac{N_1^2}{c_n^2}}}{2}$$

$$\phi_n(0) = 0$$

$$m \in \mathbb{C} \iff \frac{4N_1^2}{c_n^2} > \frac{1}{D^2}$$

$$m \in \mathbb{C} \iff \frac{4N_1^2 D^2}{c_n^2} > 1$$

$$\therefore \phi_n = A_n e^{+z/2D} \sin(k_n z)$$

$$m \in \mathbb{C} \iff 4N_1^2 D^2 x^2 > 1$$

$$x = \frac{1}{c_n^2}$$

$$k_n = \left(\frac{-1}{4D^2} + \frac{N_1^2}{c_n^2}\right)^{1/2}$$

$$m \in \mathbb{C} \iff x^2 > \frac{1}{4N_1^2 D^2}$$

$$x^2 = \frac{1}{c_n^2} > \frac{1}{4N_1^2 D^2}$$

$\underline{H > z > D}$

$$\phi(H) = 0$$

$$\phi_n = A_n' e^{z/2D} \sin(k_n'(z - H))$$

$$k_n = \left(\frac{-1}{4D^2} + \frac{N_2^2}{c_n^2}\right)$$

$$x^2 = \frac{1}{c_n^2} > \frac{1}{4N_2^2 D^2}$$

Matching

Based upon $\phi_n$ cts, $\frac{d\phi_n}{dz}$ cts. Note, it can be shown that these conditions are equivalent of $\xi_n$ cts, $\frac{1}{N}\frac{d\xi_n}{dz}$ cts in 2-layer constant $\rho$ base state case.

$\phi(D)$ cts ...

$$\Rightarrow A_n \sin(k_n D) - A_n' \sin(k_n'(D - H)) = 0 \quad -(1)$$

2) $\left.\frac{d\varphi}{dz}\right|_D$ cts

$$\Rightarrow A_n \cancel{e^{+z/D}} \cos\left(k_n D\right) k_n$$

$$+ \frac{1}{D} A_n \cancel{e^{-z/D}} \sin\left(k_n D\right)$$

$$= A_n' \cancel{e^{+z/D}} \cos\left(k_n'(D-H)\right) k_n' +$$

$$+ \frac{1}{D} A_n' \cancel{e^{+z/D}} \sin\left(k_n'(D-H)\right)$$

$$A_n\left(\cos(k_n D) k_n + \sin(k_n D)\frac{1}{D}\right)$$

$$- A_n'\left(\cos(k_n'(D-H)) k_n' + \sin(k_n'(D-H))\frac{1}{D}\right) = 0$$

$$\mathrm{Det}\left(\ \right) = 0$$

$$\Rightarrow \left(\cos(k_n D)\, k_n + \sin(k_n D)\frac{1}{D}\right)\sin\left(k_n'(D-H)\right)$$

$$- \left(\cos(k_n'(D-H))\, k_n' + \sin(k_n'(D-H))\frac{1}{D}\right)\sin\left(k_n D\right) = 0$$

$$\Rightarrow \left(k_n + \frac{1}{D}\tan(k_n D)\right)\sin\left(k_n'(D-H)\right)$$

$$- \tan(k_n D)\left(k_n' \cos\left(k_n'(D-H)\right) + \frac{1}{D}\sin\left(k_n'(D-H)\right)\right)$$

$$= 0$$

$$\Rightarrow \quad \left(k_n D + \cancel{\tan(k_n D)}\right) + $$
$$- \tan(k_n D)\left(k_n' D \cot\left(k_n'(D-H)\right) \cancel{+1}\right) = 0$$

$$\Rightarrow \quad k_n D - k_n' D \tan(k_n D) \cot\left(k_n'(D-H)\right) = 0$$

$$\therefore \quad \frac{k_n}{k_n'} - \tan(k_n D) \cot\left(k_n'(D-H)\right) = 0 \quad \text{—— } (*)$$

$$x^2 = \frac{1}{c_n^2} > \max\left(\frac{1}{4N_2^2D^2}, \frac{1}{4N_1^2D^2}\right)$$

Secular Equation is :-

$$\frac{k_n}{k_n'} - \tan(k_n D) \cot\left(k_n'(D-H)\right) = 0$$

$$k_n D = \left(N_1^2 x^2 - \frac{1}{4D^2}\right)^{1/2} D$$

$$= \left(N_1^2 H^2 x^2 - \frac{H^2}{4D^2}\right)^{1/2} \frac{D}{H}$$

$$= \left(N_1^2 \bar{x}^2 - \frac{1}{4\bar{R}^2}\right)^{1/2} \bar{R}$$

$$k_n D = \left(N_1^2 \bar{R}^2 \bar{x}^2 - \frac{1}{4}\right)$$

$$\& \quad k_n'(D-H) = \left(N_2^2 \bar{x}^2 - \frac{H^2}{4D^2}\right)^{1/2} (\bar{R}-1)$$

$$k_n'(D-H) = \left( (\bar{R}-1)^2 N_2^2 \bar{x}^2 - \frac{(\bar{R}-1)^2}{4\bar{R}^2} \right)^{1/2}$$

$$\therefore k_n'(D-H) = \left( (\bar{R}-1)^2 N_2^2 \bar{x}^2 - \frac{1}{4}\left(1-\frac{1}{\bar{R}}\right)^2 \right)^{1/2}.$$

- (*) will require numerical solution. For the seeds of an NR solver consider zeros of key terms:-

$$\cancel{\frac{k_n}{k_n'}}_{\text{neglect}} - \underbrace{\tan(k_n D)}_{\leftarrow \text{slow} \rightarrow} \underbrace{\cot(k_n'(D-H))}_{\leftarrow \text{fast} \rightarrow} = 0$$

seeds:- $\begin{cases} \tan(k_n D) \simeq 0 \Rightarrow k_n D = n\pi \\ \cot(k_n'(D-H)) \simeq 0 \Rightarrow k_n'(D-H) \simeq \frac{(2n-1)}{2}\pi . \end{cases}$

Hence, take for seeds:-

- $$(k_n D)^2 = \left(N_1^2 \bar{R}^2 \bar{x}^2 - 0.25\right) = n^2\pi^2$$

$$\Rightarrow \bar{x}_n = \pm \left( \frac{n^2\pi^2 + 0.25}{N_1^2 \bar{R}^2} \right)^{1/2}$$

- $$(k_n'(D-H)) = (\bar{R}-1)^2 N_2^2 \bar{x}^2 - \frac{1}{4}(1-1/\bar{R})^2 = \frac{(2n-1)^2\pi^2}{4}$$

$$\Rightarrow \bar{x}_n = \pm \left( \frac{(2n-1)^2\pi^2 + 0.25(1-1/\bar{R})^2}{4(\bar{R}-1)^2 N_2^2} \right)^{1/2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = u\,\frac{-1}{v^2}\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$$
$$= \left[\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right].$$

$$\frac{d}{dx}\left(\frac{Hk_n}{Hk_n'}\right) = \frac{(Hk_n')\,H\frac{dk_n}{dx} - (Hk_n)\,H\frac{dk_n'}{dx}}{k_n'^2\,H^2} = \frac{k_n'\frac{dk_n}{dx} - k_n\frac{dk_n'}{dx}}{k_n^2}$$

$$\therefore \frac{d}{dx}\left(\frac{Hk_n}{Hk_n'}\right) = \frac{k_n'}{k_n^2}\frac{dk_n}{dx} - \frac{k_n}{k_n^2}\frac{dk_n'}{dx} \qquad — (1)$$

$$\& \quad \frac{d}{dx}(Hk_n) = \cancel{\tfrac{1}{2}}(\ldots)^{-3/2} N_1^2 (-\cancel{2})x = \frac{-N_1^2 x}{(k_n)^3}.$$

$$\& \quad \frac{d}{dx}(Hk_n') = \tfrac{1}{2}(--)^{-3/2} N_2^2(-2)x = -\frac{N_2^2 x}{(k_n')^3}.$$