

Potential Temperature Investigations, Parker and Burton System

O. J. Halliday, S. D. Griffith, D. Parker,
University of Leeds

1 Preliminary Results and Error Function Properties

We shall need to evaluate integrals I with the following form:

$$I(t, T, c) = \int_0^t H(t' - T) \exp\left(-\frac{(\xi - ct')^2}{2\sigma^2}\right) dt', \quad \forall t > 0. \quad (1)$$

Using the definition of the Heavyside function I may be written :

$$I(t, T, c) = H(t - T) \int_T^t \exp\left(-\frac{(\xi - ct')^2}{2\sigma^2}\right) dt', \quad \forall t > 0, \quad (2)$$

which, on using the substitution $u = \frac{ct' - \xi}{\sqrt{2}\sigma}$ transforms to :

$$I(t, T, c) = H(t - T) \frac{\sqrt{2}\sigma}{c} \int_{\frac{cT - \xi}{\sqrt{2}\sigma}}^{\frac{ct - \xi}{\sqrt{2}\sigma}} \exp(-u^2) du, \quad \forall t > 0. \quad (3)$$

In order to appeal to the definition of the error function we must transform the right hand side in the above:

$$I(t, T, c) = H(t - T) \frac{\sqrt{2}\sigma}{c} \left(\int_{\frac{cT - \xi}{\sqrt{2}\sigma}}^0 \exp(-u^2) du + \int_0^{\frac{ct - \xi}{\sqrt{2}\sigma}} \exp(-u^2) du \right). \quad (4)$$

Reversing limits in the first term on the right hand side:

$$I(t, T, c) = H(t - T) \frac{\sqrt{2}\sigma}{c} \left(- \int_0^{\frac{cT - \xi}{\sqrt{2}\sigma}} \exp(-u^2) du + \int_0^{\frac{ct - \xi}{\sqrt{2}\sigma}} \exp(-u^2) du \right). \quad (5)$$

Employing the definition of the error function we may obtain an expression for the integrals in the above:

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) du \iff \int_0^x \exp(-u^2) du = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x), \quad (6)$$

we easily obtain the following:

$$I(t, T, c) = H(t - T) \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(-\operatorname{erf}\left(\frac{cT - \xi}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct - \xi}{\sqrt{2}\sigma}\right) \right). \quad (7)$$

We note some results which will be useful in what follows.

First, the error function is an odd function of its argument:

$$\operatorname{erf}(-x) = -\operatorname{erf}(x). \quad (8)$$

Second, set $T = 0$ and use that fact to obtain:

$$I(t, 0, c) = \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(\operatorname{erf} \left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct - \xi}{\sqrt{2}\sigma} \right) \right). \quad (9)$$

Third, change the sign of c (i.e. $c \rightarrow (-c)$) to obtain:

$$\begin{aligned} I(t, 0, -c) &= \sqrt{\frac{\pi}{2}} \frac{\sigma}{(-c)} \left(\operatorname{erf} \left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{-ct - \xi}{\sqrt{2}\sigma} \right) \right) \\ &= \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(-\operatorname{erf} \left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct + \xi}{\sqrt{2}\sigma} \right) \right) \end{aligned} \quad (10)$$

2 Potential Temperature for a System with Transient Heating

We note that a steady response is a special case of a transient response. We therefore exploit previous experience and consider the unsteady directly. We shall assume a time variation of heating as:

$$H(t) - H(t - T) \quad (11)$$

Throughout we shall consider the Parker and Burton system and assume $s = 0$ (heat source is stationary) and $n = f = 0$. Also, we shall assume that the horizontal variation of heating is given by:

$$F(x) \equiv \exp \left(-\frac{x^2}{2\sigma^2} \right). \quad (12)$$

2.1 Previous Work, Steady Heating w -Response

The steady response, w_{m_z} , to the m_z mode of a steady (constant) heating function in a Parker and Burton system may be written:

$$\frac{w_{m_z}}{Q} = \frac{H^2}{m_z^2 \pi^2} \left(\frac{F(x)}{c^2} - \frac{1}{2} \frac{F(x - ct)}{c^2} - \frac{1}{2} \frac{F(x + ct)}{c^2} \right) \sin \left(\frac{m_z \pi}{H} z \right), \quad (13)$$

where:

$$c = \frac{NH}{m_z \pi}, \quad m_z \in \mathbb{Z}. \quad (14)$$

The above may be simplified:

$$\frac{w_{m_z}}{Q} = \frac{1}{N^2} \left(F(x) - \frac{1}{2}F(x - ct) - \frac{1}{2}F(x + ct) \right) \sin \left(\frac{m_z \pi}{H} z \right), \quad (15)$$

For a steady heating function:

$$S(x, z, t) = QH(t)F(x) \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right), \quad (16)$$

the aggregate w -response is obtained by superposing modes:

$$\frac{w}{Q} = \frac{1}{N^2} \sum_{m_z=1}^{\infty} b_{m_z} \left(F(x) - \frac{1}{2}F(x - ct) - \frac{1}{2}F(x + ct) \right) \sin \left(\frac{m_z \pi}{H} z \right), \quad (17)$$

which can be written, note:

$$\frac{w}{Q} = \frac{S_0}{QN^2} - \frac{1}{2N^2} \sum_{m_z=1}^{\infty} b_{m_z} (F(x - ct) + F(x + ct)) \sin \left(\frac{m_z \pi}{H} z \right), \quad (18)$$

where we have defined:

$$S_0(x, z) \equiv QF(x) \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right), \quad (19)$$

2.2 Previous Work, Unsteady Heating w -Response

Let us now consider transient heating, still with $s = 0$, $n = f = 0$. Using a Fourier series expansion to express the vertical variation, a transient heating function of duration T is:

$$S(x, z, t) = Q(H(t) - H(t - T))F(x) \sum_{m_z=1}^{\infty} b_n \sin \left(\frac{m_z \pi}{H} z \right), \quad (20)$$

Using the delay theorem of Laplace transforms, the corresponding w -response may be computed:

$$\begin{aligned} \frac{w}{Q} &= \frac{S_0}{QN^2} - \frac{1}{2N^2} \sum_{m_z=1}^{\infty} b_{m_z} (F(x-ct) + F(x+ct)) \sin\left(\frac{m_z\pi}{H}z\right) \\ &\quad - \frac{S_0}{QN^2} H(t-T) \\ &\quad + \frac{1}{2N^2} H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} (F(x-c(t-T)) + F(x+c(t-T))) \sin\left(\frac{m_z\pi}{H}z\right) \end{aligned} \quad (21)$$

2.3 Unsteady Heating b -Response

Using equation (4) of Parker and Burton, the b -response may be written:

$$\frac{\partial}{\partial t} \left(\frac{b}{Q} \right) = \frac{S}{Q} - N^2 \frac{w}{Q}, \quad (22)$$

which, on using equations 20 and 21 gives:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{b}{Q} \right) &= F(x) \sum_{m_z=1}^{\infty} b_n \sin\left(\frac{m_z\pi}{H}z\right) \\ &\quad - H(t-T) F(x) \sum_{m_z=1}^{\infty} b_n \sin\left(\frac{m_z\pi}{H}z\right) \\ &\quad - \frac{S_0}{Q} \\ &\quad + \frac{1}{2} \sum_{m_z=1}^{\infty} b_{m_z} (F(x-ct) + F(x+ct)) \sin\left(\frac{m_z\pi}{H}z\right) \\ &\quad + H(t-T) \frac{S_0}{Q} \\ &\quad - \frac{1}{2} H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} (F(x-c(t-T)) + F(x+c(t-T))) \sin\left(\frac{m_z\pi}{H}z\right), \end{aligned} \quad (23)$$

where we have suppressed a factor $H(t)$ in the first term, note. Consider the right hand side of the above. Using the definition of S_0 above the first and

third cancel, as do the second and fifth:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{b}{Q} \right) &= \frac{1}{2} H(t) \sum_{m_z=1}^{\infty} b_{m_z} (F(x - ct) + F(x + ct)) \sin \left(\frac{m_z \pi}{H} z \right) \\ &\quad - \frac{1}{2} H(t - T) \sum_{m_z=1}^{\infty} b_{m_z} (F(\xi - ct) + F(\xi' + ct)) \sin \left(\frac{m_z \pi}{H} z \right), \end{aligned} \quad (24)$$

where we have defined:

$$\xi = x + cT, \quad \xi' = x - cT, \quad (25)$$

and, recall $F(x) \equiv \exp \left(-\frac{x^2}{2\sigma^2} \right)$. Integrating on t now:

$$\begin{aligned} \frac{b}{Q} &= \frac{1}{2} \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right) \int_0^t H(t') \exp \left(-\frac{(x - ct')^2}{2\sigma^2} \right) dt' \\ &\quad + \frac{1}{2} \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right) \int_0^t H(t') \exp \left(-\frac{(x + ct')^2}{2\sigma^2} \right) dt' \\ &\quad - \frac{1}{2} \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right) \int_0^t H(t' - T) \exp \left(-\frac{(\xi - ct')^2}{2\sigma^2} \right) dt' \\ &\quad - \frac{1}{2} \sum_{m_z=1}^{\infty} b_{m_z} \sin \left(\frac{m_z \pi}{H} z \right) \int_0^t H(t' - T) \exp \left(-\frac{(\xi' + ct')^2}{2\sigma^2} \right) dt', \end{aligned} \quad (26)$$

where we have used the initial condition that $b = 0$ at $t = 0$. Then, using the results in section 1, we have:

$$\begin{aligned} \frac{b}{Q} &= \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin \left(\frac{m_z \pi}{H} z \right) \left(\operatorname{erf} \left(\frac{x}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct - x}{\sqrt{2}\sigma} \right) \right) \\ &\quad + \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin \left(\frac{m_z \pi}{H} z \right) \left(-\operatorname{erf} \left(\frac{x}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct + x}{\sqrt{2}\sigma} \right) \right) \\ &\quad - \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} H(t - T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin \left(\frac{m_z \pi}{H} z \right) \left(-\operatorname{erf} \left(\frac{cT - \xi}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct - \xi}{\sqrt{2}\sigma} \right) \right) \\ &\quad - \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} H(t - T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin \left(\frac{m_z \pi}{H} z \right) \left(-\operatorname{erf} \left(\frac{cT + \xi'}{\sqrt{2}\sigma} \right) + \operatorname{erf} \left(\frac{ct + \xi'}{\sqrt{2}\sigma} \right) \right), \end{aligned} \quad (27)$$

where, e.g. in the second line on the right hand side we have used property 10.

2.3.1 Matlab Code

Now, the Matlab code uses the above result as follows. Substitute for ξ and ξ' :

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \right) \\
&+ \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \right) \\
&- \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{-x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right) \right) \\
&- \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)+x}{\sqrt{2}\sigma}\right) \right),
\end{aligned} \tag{28}$$

and use the symmetry properties of the error function:

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \right) \\
&+ \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \right) \\
&- \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) H(t-T) \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right) \right) \\
&- \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) H(t-T) \left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)+x}{\sqrt{2}\sigma}\right) \right),
\end{aligned} \tag{29}$$

and collect terms, adding another factor of σ to conserve total heating input (see below):

$$\frac{b}{Q} = \frac{\sigma^2}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) (F_1(c) - F_1(-c)) \tag{30}$$

where:

$$F_1(c) \equiv \operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) - H(t-T) \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right) \right) \quad (31)$$

2.3.2 Compact Expression for b -Response

To obtain a compact expression for b let us return to equation 27. Cancelling terms and substituting for ξ and ξ' :

$$\begin{aligned} \frac{2}{\sigma} \sqrt{\frac{2}{\pi}} \frac{b}{Q} &= \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \\ &+ \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \\ &- H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{cT-x-cT}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x-cT}{\sqrt{2}\sigma}\right) \right) \\ &- H(t-T) \sum_{m_z=1}^{\infty} b_{m_z} \frac{1}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{cT+x-cT}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct+x-cT}{\sqrt{2}\sigma}\right) \right), \end{aligned} \quad (32)$$

it is possible to simplify the above further I THINK (CHECKING). Recalling that the error function is an odd function:

$$\begin{aligned} \frac{2}{\sigma} \sqrt{\frac{2}{\pi}} \frac{b}{Q} &= \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \\ &+ \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \\ &- H(t-T) \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x-cT}{\sqrt{2}\sigma}\right) \right) \\ &- H(t-T) \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct+x-cT}{\sqrt{2}\sigma}\right) \right), \end{aligned} \quad (33)$$

whereupon we can gain another cancellation:

$$\begin{aligned}
\frac{2}{\sigma} \sqrt{\frac{2}{\pi}} \frac{b}{Q} &= \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct - x}{\sqrt{2}\sigma}\right) \\
&+ \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct + x}{\sqrt{2}\sigma}\right) \\
&- H(t - T) \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct - x - cT}{\sqrt{2}\sigma}\right) \\
&- H(t - T) \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) \operatorname{erf}\left(\frac{ct + x - cT}{\sqrt{2}\sigma}\right).
\end{aligned} \tag{34}$$

Finally, for the simplified potential temperature response, we have:

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) G(c, x, t) \\
&- H(t - T) \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) G(c, x, (t - T)),
\end{aligned} \tag{35}$$

where:

$$G(c, x, t) \equiv \operatorname{erf}\left(\frac{ct - x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct + x}{\sqrt{2}\sigma}\right). \tag{36}$$

Note that to conserve the heating rate when σ is varied:

$$F(x) \rightarrow \sigma \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{37}$$

whereupon the b -response acquires an additional factor σ :

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma^2}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) G(c, x, t) \\
&- H(t - T) \frac{\sigma^2}{2} \sqrt{\frac{\pi}{2}} \sum_{m_z=1}^{\infty} \frac{b_{m_z}}{c} \sin\left(\frac{m_z \pi}{H} z\right) G(c, x, (t - T)),
\end{aligned} \tag{38}$$