Potential Tempertature Investigations, Parker and Burton System

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1 Preliminary Results and Error Function Properties

We shall need to evaluate integrals I with the following form:

$$I(t, T, c) = \int_0^t H(t' - T) \exp\left(-\frac{(\xi - ct')^2}{2\sigma^2}\right) dt', \quad \forall t > 0. \tag{1}$$

Using the defintion of the Heavyside function I may be written :

$$I(t, T, c) = H(t - T) \int_T^t \exp\left(-\frac{(\xi - ct')^2}{2\sigma^2}\right) dt', \quad \forall t > 0, \tag{2}$$

which, on using the substitution $u = \frac{ct' - \xi}{\sqrt{2}\sigma}$ transforms to :

$$I(t, T, c) = H(t - T) \frac{\sqrt{2}\sigma}{c} \int_{\frac{cT - \xi}{\sqrt{2}\sigma}}^{\frac{ct - \xi}{\sqrt{2}\sigma}} \exp\left(-u^2\right) du, \quad \forall t > 0. \tag{3}$$

Clearly we need only consider the case of $t > T$. For $t > T$ it follows that the upper limit in the above definite integral is always greater than the lower limit, for $\xi > 0$ and $\xi < 0$. Hence we can write:

$$I(t, T, c) = H(t - T) \frac{\sqrt{2}\sigma}{c} \left(\int_0^{\frac{ct - \xi}{\sqrt{2}\sigma}} \exp\left(-u^2\right) du - \int_0^{\frac{cT - \xi}{\sqrt{2}\sigma}} \exp\left(-u^2\right) du \right). \tag{4}$$

Employing the defintion of the error function we may obtain an expression for the integrals in the above:

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-u^2\right) du \iff \int_0^x \exp\left(-u^2\right) du = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x), \tag{5}$$

we easily obtain the following:

$$I(t, T, c) = H(t - T) \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(-\operatorname{erf}\left(\frac{cT - \xi}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct - \xi}{\sqrt{2}\sigma}\right) \right). \tag{6}$$

We note some results which will be useful in what follows.

First, the error function is an odd function of its argument:

$$\operatorname{erf}(-x) = -\operatorname{erf}(x). \tag{7}$$

Second, set $T = 0$ and use that fact to obtain:

$$I(t, 0, c) = \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(\operatorname{erf}\left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf}\left(\frac{ct - \xi}{\sqrt{2}\sigma} \right) \right). \quad (8)$$

Third, change the sign of c (i.e. $c \to (-c)$) to obtain:

$$\begin{aligned} I(t, 0, -c) &= \sqrt{\frac{\pi}{2}} \frac{\sigma}{(-c)} \left(\operatorname{erf}\left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf}\left(\frac{-ct - \xi}{\sqrt{2}\sigma} \right) \right) \quad (9) \\ &= \sqrt{\frac{\pi}{2}} \frac{\sigma}{c} \left(-\operatorname{erf}\left(\frac{\xi}{\sqrt{2}\sigma} \right) + \operatorname{erf}\left(\frac{ct + \xi}{\sqrt{2}\sigma} \right) \right) \end{aligned}$$

2 Potential Temperature for a System with Transient Heating

We note that a steady response is a special case of a transient response. We therefore exploit previous experience and consider the unsteady directly. We shall assume a time variation of heating as:

$$H(t) - H(t - T) \quad (10)$$

Throughout we shall consider the Parker and Burton system and assume $s = 0$ (heat source is stationary) and $n = f = 0$. Also, we shall assume that the horizontal variation of heating is given by:

$$F(x) \equiv \exp\left(-\frac{x^2}{2\sigma^2} \right). \quad (11)$$

2.1 Previous Work, Steady Heating w-Response

The steady response, w_{m_z}, to the m_z mode of a steady (constant) heating function in a Parker and Burton systemmay be written:

$$\frac{w_{m_z}}{Q} = \frac{H^2}{m_z^2 \pi^2} \left(\frac{F(x)}{c^2} - \frac{1}{2} \frac{F(x - ct)}{c^2} - \frac{1}{2} \frac{F(x + ct)}{c^2} \right) \sin\left(\frac{m_z \pi}{H} z \right), \quad (12)$$

where:

$$c = \frac{NH}{m_z \pi}, \quad m_z \in \mathbb{Z}. \quad (13)$$

The above may be simplified:

$$\frac{w_{m_z}}{Q} = \frac{1}{N^2}\left(F(x) - \frac{1}{2}F(x-ct) - \frac{1}{2}F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right), \qquad (14)$$

For a steady heating function:

$$S(x,z,t) = QH(t)F(x)\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right), \qquad (15)$$

the aggregate w-response is obtained by superposing modes:

$$\frac{w}{Q} = \frac{1}{N^2}\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x) - \frac{1}{2}F(x-ct) - \frac{1}{2}F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right), \qquad (16)$$

which can be written, note:

$$\frac{w}{Q} = \frac{S_0}{QN^2} - \frac{1}{2N^2}\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-ct) + F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right), \qquad (17)$$

where we have defined:

$$S_0(x,z) \equiv QF(x)\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right), \qquad (18)$$

2.2 Previous Work, Unsteady Heating w-Response

Let us now consider transient heating, still with $s = 0$, $n = f = 0$. Using a Fourier series expansion to express the vertical variation, a transient heating function of duration T is:

$$S(x,z,t) = Q(H(t) - H(t-T))F(x)\sum_{m_z=1}^{\infty} b_n\sin\left(\frac{m_z\pi}{H}z\right), \qquad (19)$$

Using the delay theorem of Laplace transforms, the corresponding w-response may be computed:

$$\begin{aligned}\frac{w}{Q} &= \frac{S_0}{QN^2} - \frac{1}{2N^2}\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-ct) + F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right) \qquad (20)\\ &- \frac{S_0}{QN^2}H(t-T)\\ &+ \frac{1}{2N^2}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-c(t-T)) + F(x+c(t-T))\right)\sin\left(\frac{m_z\pi}{H}z\right)\end{aligned}$$

2.3 Unsteady Heating b-Response

Using equation (4) of Parker and Burton, the b-response may be written:

$$\frac{\partial}{\partial t}\left(\frac{b}{Q}\right) = \frac{S}{Q} - N^2\frac{w}{Q}, \tag{21}$$

which, on using equations 19 and 20 gives:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{b}{Q}\right) &= F(x)\sum_{m_z=1}^{\infty} b_n \sin\left(\frac{m_z\pi}{H}z\right) \qquad (22)\\
&- H(t-T)F(x)\sum_{m_z=1}^{\infty} b_n \sin\left(\frac{m_z\pi}{H}z\right)\\
&- \frac{S_0}{Q}\\
&+ \frac{1}{2}\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-ct)+F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right)\\
&+ H(t-T)\frac{S_0}{Q}\\
&- \frac{1}{2}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-c(t-T))+F(x+c(t-T))\right)\sin\left(\frac{m_z\pi}{H}z\right),
\end{aligned}$$

where we have suppressed a factor $H(t)$ in the first term, note. Consider the right hand side of the above. Using the definition of S_0 above the first and third cancel, as do the second and fifth:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{b}{Q}\right) &= \frac{1}{2}H(t)\sum_{m_z=1}^{\infty} b_{m_z}\left(F(x-ct)+F(x+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right) \qquad (23)\\
&- \frac{1}{2}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\left(F(\xi-ct)+F(\xi'+ct)\right)\sin\left(\frac{m_z\pi}{H}z\right),
\end{aligned}$$

where we have defined:

$$\xi = x + cT, \quad \xi' = x - cT, \tag{24}$$

and, recall $F(x) \equiv \exp\left(-\frac{x^2}{2\sigma^2}\right)$. Integrating on t now:

$$
\begin{aligned}
\frac{b}{Q} &= \frac{1}{2}\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right)\int_0^t H(t')\exp\left(-\frac{(x-ct')^2}{2\sigma^2}\right)dt' \\
&+ \frac{1}{2}\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right)\int_0^t H(t')\exp\left(-\frac{(x+ct')^2}{2\sigma^2}\right)dt' \\
&- \frac{1}{2}\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right)\int_0^t H(t'-T)\exp\left(-\frac{(\xi-ct')^2}{2\sigma^2}\right)dt' \\
&- \frac{1}{2}\sum_{m_z=1}^{\infty} b_{m_z}\sin\left(\frac{m_z\pi}{H}z\right)\int_0^t H(t'-T)\exp\left(-\frac{(\xi'+ct')^2}{2\sigma^2}\right)dt',
\end{aligned}
\tag{25}
$$

where we have used the initial condition that $b = 0$ at $t = 0$. Then, using the results in section 1, we have:

$$
\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right)\right) \\
&+ \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{cT-\xi}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct-\xi}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{cT+\xi'}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+\xi'}{\sqrt{2}\sigma}\right)\right),
\end{aligned}
\tag{26}
$$

where, e.g. in the second line on the right hand side we have used property 9.

2.3.1 Matlab Code

Now, the Matlab code uses the above result as follows. Substitute for ξ and ξ':

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right)\right) \\
&+ \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{-x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{c(t-T)+x}{\sqrt{2}\sigma}\right)\right),
\end{aligned} \tag{27}$$

and use the symmetry properties of the error function:

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right)\right) \\
&+ \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)H(t-T)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right)\right) \\
&- \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)H(t-T)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{c(t-T)+x}{\sqrt{2}\sigma}\right)\right),
\end{aligned} \tag{28}$$

and collect terms, adding another factor of σ to conserve total heating input (see below):

$$\frac{b}{Q} = \frac{\sigma^2}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(F_1(c)-F_1(-c)\right) \tag{29}$$

where:

$$F_1(c) \equiv \operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) - H(t-T)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{c(t-T)-x}{\sqrt{2}\sigma}\right)\right) \tag{30}$$

Note the presence of the factor $\frac{1}{c}$ in the summation in equation 28 , where, recall:

$$c = \frac{NH}{m_z \pi}, \quad m_z \in mathbbZ. \tag{31}$$

There are no corresponding factors in c in the w-response.

2.3.2 Compact Expression for b-Response

To obtain a compact expression for b let us return to equation 26. Cancelling terms and substituting for ξ and ξ':

$$\begin{aligned}
\frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\frac{b}{Q} &= \sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \\
&+ \sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{cT-x-cT}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct-x-cT}{\sqrt{2}\sigma}\right)\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty} b_{m_z}\frac{1}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{cT+x-cT}{\sqrt{2}\sigma}\right) + \operatorname{erf}\left(\frac{ct+x-cT}{\sqrt{2}\sigma}\right)\right),
\end{aligned} \tag{32}$$

it is possible to simplify the above further I THINK (CHECKING). Recalling that the error function is an odd function:

$$\begin{aligned}
\frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\frac{b}{Q} &= \sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \\
&+ \sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct-x-cT}{\sqrt{2}\sigma}\right)\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\left(-\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+x-cT}{\sqrt{2}\sigma}\right)\right),
\end{aligned} \tag{33}$$

whereupon we can gain another cancellation:

$$\begin{aligned}
\frac{2}{\sigma}\sqrt{\frac{2}{\pi}}\frac{b}{Q} &= \sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right) \\
&+ \sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct-x-cT}{\sqrt{2}\sigma}\right) \\
&- H(t-T)\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)\operatorname{erf}\left(\frac{ct+x-cT}{\sqrt{2}\sigma}\right).
\end{aligned} \tag{34}$$

Finally, for the simplified potential temperature response, we have:

$$\begin{aligned}
\frac{b}{Q} &= \frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)G(c,x,t) \\
&- H(t-T)\frac{\sigma}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)G(c,x,(t-T)),
\end{aligned} \tag{35}$$

where:

$$G(c,x,t) \equiv \operatorname{erf}\left(\frac{ct-x}{\sqrt{2}\sigma}\right)+\operatorname{erf}\left(\frac{ct+x}{\sqrt{2}\sigma}\right). \tag{36}$$

Note that to conserve the heating rate when σ is varied:

$$F(x) \to \sigma \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{37}$$

whereupon the b-response acquires an additional factor σ:

$$\begin{aligned} \frac{b}{Q} &= \frac{\sigma^2}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)G(c,x,t) \\ &- H(t-T)\frac{\sigma^2}{2}\sqrt{\frac{\pi}{2}}\sum_{m_z=1}^{\infty}\frac{b_{m_z}}{c}\sin\left(\frac{m_z\pi}{H}z\right)G(c,x,(t-T)), \end{aligned} \tag{38}$$