

Convergence, Parker and Burton System

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1 Limits of Spatial-Temporal Convergence

We aim to study the convergence of our forced forced, lidded system of Parker and Burton, with transient heating function:

$$\frac{Q(x, z, t)}{Q_0} = \sin\left(\frac{\pi z}{H_t}\right) (H(t) - H(t - T))(H(z) - H(z - H_t))F(x), \quad (1)$$

where:

$$F(x) = \sigma_0 \sigma \exp\left(-\frac{x^2}{2\sigma_0^2 \sigma^2}\right), \quad 0 < z < H_L. \quad (2)$$

In particular, we wish to find approximate limits (bounds) on the spatial domain and timescales, implicitly imposed by our use of a rigid lid upper boundary condition.

The partial Fourier spectrum of the above function:

$$\frac{Q(x, z, t)}{Q_0} = (H(t) - H(t - T)) \sum_{m=1}^M b_m \sin\left(\frac{m\pi}{H_L} z\right), \quad (3)$$

where:

$$b_m = \frac{2\bar{H}_t \pi \sin(\bar{H}_t \pi)}{1 - \bar{H}_t^2 m^2}, \quad \bar{H}_t = \frac{H_t}{H_L}, \quad M \propto H_L, \quad (4)$$

is represented in figure 1 below, for $H_L = 30H_t$, $\sigma = 1$. Here, to ensure convergence, the number of Fourier modes M used to express the above function scales with lid height H_L . The choice of lid location, relative to the top of heating, ensures convergence in the w -response to approximately one per-cent.

Let us denote the harmonic number of the Fourier series term in equation 3 with largest $|b_m|$ as m_{max} . For the data in figure 1 we find: find:

$$m_{max} = 23, \quad b_{m_{max}} = 0.035. \quad (5)$$

We observe that the w -response is comprised of bores of vertical velocity perturbation which propagate approximately radially, from the fixed source of heat to the right as they elongate radially, weaken and descend. This response is apparent in a plot of the logarithm of the w -response in figure 2 below. Let us consider only the principal bores which may be attributed to the subset of modes $m < 200$, that is to those modes which contribute most strongly to the Fourier spectrum.

Let us model the motion of the strongest bores on freely propagating gravity waves in our slab system, characterized by a vertical wavelength (wavenumber) λ_z (k_z). We obtain an estimate of this wavelength by considering the largest Fourier series term, namely $b_m \sin\left(\frac{m_{max}\pi z}{H_L}\right)$, which has a wavelength:

$$\lambda_z \approx \frac{H_L}{m_{max}\pi} \iff k_z = \frac{2\pi}{\lambda_z} = \frac{2m_{max}}{H_L}. \quad (6)$$

The horizontal wavelength and wavevector of the principal bores must also be estimated. We take the FWHM of the horizontal variation of heating, $F(x)$:

$$\lambda_x = \sigma\sigma_0 \iff k_x = \frac{2\pi}{\lambda_x} = \frac{2\pi}{\sigma\sigma_0}. \quad (7)$$

The above wavevector components are taken to be characteristic of the forced response in the Parker and Burton system.

Now, the group and phase velocity of our assumed heat-induced Boussinesq gravity waves is determined from the appropriate dispersion relation for free modes of the Parker and Burton system [GAFD equ 2.32]:

$$\underline{c}_g = \pm \frac{Nk_z}{(k_x^2 + k_z^2)^{3/2}}(k_z, 0, -k_x), \quad \underline{c}_p = \frac{N}{k_x^2 + k_z^2}(k_x, 0, k_z), \quad (8)$$

where it is understood that the above expression will be evaluated at those value of k_x and k_z which characterise the spectrum of the wavepacket in question. Hence we find the following group velocity by evaluating the above expression using estimates in equations 6 and 7:

$$\underline{c}_g = \pm \frac{NH_L m_{max}^2 R^3}{2(1 + R^2 m_{max}^2)^{3/2}} \left(1, 0, -\frac{1}{Rm_{max}}\right), \quad (9)$$

where we have defined the dimensionless ratio parameter

$$R = \frac{\sigma\sigma_0}{\pi H_L} = \frac{\sigma\sigma_0}{\pi H_t} \frac{H_t}{H_L}. \quad (10)$$

Considering the horizontal component of the advection of the w -perturbations, which is clearly positive, it is necessary to take the positive sign in the above. Note that as R increases (decreases) corresponding to less (more) localized heating, the vertical component of the group velocity will decrease (increase): the group velocity vector is tilted downwards as the heating becomes more localized.

To assess the influence of the lid on our calculations let us estimate the time, Δt taken for an "error" in the w -response caused by the lid i.e. located at $z = H_L$ to traverse the slab:

$$\Delta t = \frac{H_L}{c_{gz}} = \frac{2(1 + R^2 m_{max}^2)^{3/2}}{N m_{max} R^2} \quad (11)$$

Let us estimate Δt . For deep convection, which spans the troposphere (in some sense) the ratio of horizontal and vertical storm scales $\frac{\sigma\sigma_0}{H_t}$ will be small $\frac{\sigma\sigma_0}{H_t} = \frac{1}{10}$. Also, since for our converged data $H_L \approx 30H_t \iff \frac{H_t}{H_L} = \frac{1}{30}$ and we have already seen that $m_{max} \approx 20$. Therefore we can find an order of magnitude bound on the quantity Rm_{max} :

$$Rm_{max} = \frac{\sigma\sigma_0}{H_t} \times \frac{m_{max}}{\pi} \times \frac{H_t}{H_l} = \frac{1}{10} \times 10 \times \frac{1}{30} = \frac{1}{30}. \quad (12)$$

Therefore we neglect terms in $m_{max}^2 R^2$ to find:

$$\Delta t \approx \frac{2}{N m_{max} R^2} = \frac{2}{0.01 \times 20 \times 0.03^2} = 10000s. \quad (13)$$

The influence of the disturbance due to the lid will appear at the position:

$$\begin{aligned} \Delta x &= c_{cx} \Delta t = \frac{N H_L m_{max}^2 R^3}{2(1 + R^2 m_{max}^2)^{3/2}} \times \frac{2(1 + R^2 m_{max}^2)^{3/2}}{N m_{max} R^2}, \\ &= H_L m_{max} R \\ &= H_L m_{max} \frac{\sigma\sigma_0}{\pi H_L}. \end{aligned} \quad (14)$$

Expressing this distance in terms of the FWHM of the assumed horizontal variation of heating:

$$\frac{\Delta x}{\sigma\sigma_0} = \frac{m_{max}}{\pi} \approx \frac{20}{\pi} = 10. \quad (15)$$

All contributions to the w -response are of the form:

$$\frac{b_{m_z}}{N^2} \sin\left(\frac{m_z \pi}{H_l} z\right) \exp\left(-\frac{(x - \frac{N H_L}{m_z \pi})}{x}\right) \quad (16)$$

This structure will effectively propagate to the left at speed

$$\frac{dx}{dt} = \frac{NH_L}{\pi m_z}, \quad (17)$$

and will be centred on location:

$$x_0 = \frac{NH_L}{\pi m_z}, \quad (18)$$

which may or may not lie in the domain. Note, that (i) only high-order modes (with large m_z) stay inside the box $0 < x < x_{max}$ at long time and (ii) the larger H_L , the faster the modes leave the domain. Modes which have left the domain clearly won't contribute to our convergence calculations. Hence convergence might be said to be dominated by the larger-order modes. More quantitatively, for a Parker and Burton mode m_z to contribute to the w -response it must be centred on x_0 such that:

$$\frac{NH_L t}{\pi m_z} < x_{max} \iff m_z > \frac{NH_L t}{\pi x_{max}} \quad (19)$$

Put another way, if a PB mode has $m_z < \frac{NH_L t}{\pi x_{max}}$ it will have left our box before it can contribute to convergence calculations. We can count the number of PB modes, N , which remain inside the domain:

$$N = m_{max} - \frac{NH_L t}{\pi x_{max}} = kH_L - \frac{NH_L t}{\pi x_{max}} = H_L \left(k - \frac{Nt}{\pi x_{max}} \right), \quad (20)$$

since we are using the partial Fourier-series cut-off:

$$m_{max} = kH_L. \quad (21)$$

Hence, the number of PB modes used to assess convergence decreases for t increasing. Yet we see improved convergence as t increases ??!

Convergence within the domain therefore appears to be dominated (or limited) by contributions from larger m_z which travel slower and so do not escape.

We see in our short-time data the effect at the boundary of the domain of small m_z (which has the simplest vertical variation).

Since the number of modes used in computing the w -response increases in proportion to H_L ($m_{max} = cH_L$) there are always

Which of the above modes are inside the box at say $t = 1000$?

Figure 1: Convergence data. The Fourier spectrum of the heating function and the variation with z of the partial summation in equation 3.

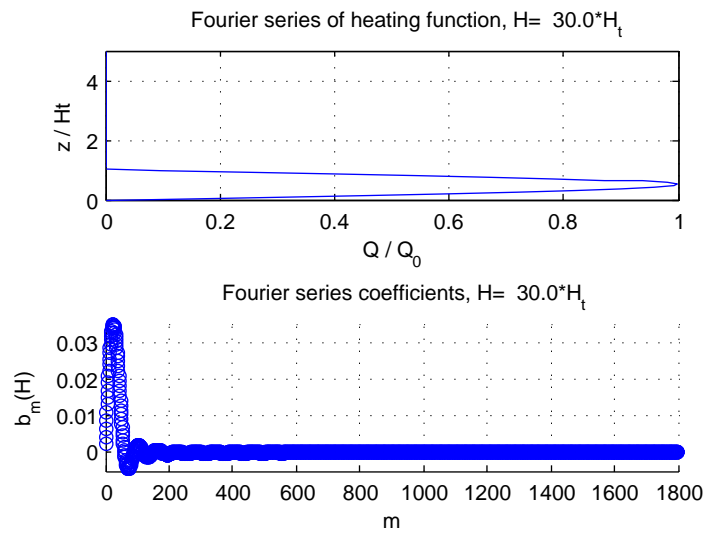


Figure 2: The w -response. Left panel is $\log_e(|w|)$, right panel w . The radial bores are visible in the plot on the left: the principal contribution is more clear on the light. The subset of b_m , $m < 200$ are mainly responsible for each the features in the panel on the right.

