RESEARCH ASSISTANT - CRYPTO

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Math Section

1. (Please show your workings). Over all real numbers, find the minimum value of a positive real number, y such that

$$y = \sqrt{(x+6)^2 + 25} + \sqrt{(x-\ 6)^2 + 121}$$

SOLUTION

Given that $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{d^2x}$

A **critical point** of a function of a single real variable f(x), is a value x_0 in the domain of f where it is not differentiable or its derivative is 0 ($y'(x_0) = 0$).

The **minimum value** of a function $y'(x)$ exists at a point x_o if $y'(x_0) = 0$ and if $y''(x_0) > 0$.

An **extraneous solution** is a solution, such as that to an equation, that emerges from the process of solving the problem but is not a valid solution to the problem.

Steps Involved

1. Find the first derivative of the function, $y'(x)$
2. Find the critical points, by setting $y'(x_0) = 0$
3. Remove extraneous solution(s) by substituting results into original equation
4. Determine if critical point is a minimum by checking if $y''(x_0) > 0$ (a positive number)

Step 1, solving for $y'(x)$

$$y = \sqrt{(x+6)^2 + 25} + \sqrt{(x-\ 6)^2 + 121}$$

$$y' = \left(2(x+6) \times \frac{1}{2\sqrt{(x+6)^2+25}}\right) + \left(2(x-6) \times \frac{1}{2\sqrt{(x-6)^2+121}}\right)$$

$$y' = \left(\frac{x+6}{\sqrt{(x+6)^2+25}}\right) + \left(\frac{x-6}{\sqrt{(x-6)^2+121}}\right)$$

Step 2, getting values of x for which $y'(x) = 0$.

$$y' = \left(\frac{x+6}{\sqrt{(x+6)^2+25}}\right) + \left(\frac{x-6}{\sqrt{(x-6)^2+121}}\right) = 0$$

$$\left(\frac{x+6}{\sqrt{(x+6)^2+25}}\right) + \left(\frac{x-6}{\sqrt{(x-6)^2+121}}\right) = 0$$

$$\frac{x+6}{\sqrt{(x+6)^2+25}} = -\frac{x-6}{\sqrt{(x-6)^2+121}}$$

Square both sides

$$\frac{(x+6)^2}{(x+6)^2+25} = \frac{(x-6)^2}{(x-6)^2+121}$$

Cross multiply

$$(x+6)^2 \times \left((x-6)^2 + 121\right) = (x-6)^2 \times \left((x+6)^2 + 25\right)$$

Open Brackets

$$(x+6)^2.(x-6)^2 + 121.(x+6)^2 = (x+6)^2.(x-6)^2 + 25.(x-6)^2$$

Subtract $(x+6)^2.(x-6)^2$ from both sides

$$121(x+6)^2 = 25(x-6)^2$$

$$121\,(x^2 + 12x + 36) = 25\,(x^2 - 12x + 36)$$

$$121x^2 + 1452x + 4356 = 25x^2 - 300x + 900$$

Simplify by collecting like-terms

$$96x^2 + 1752x + 3456 = 0$$

$$24\left(4x^2 + 73x + 144\right) = 0$$

$$4x^2 + 73x + 144 = 0$$

Solve for x using the quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-73 \pm \sqrt{(73)^2 - 4 \times 4 \times 144}}{2 \times 4}$$

$$x = \frac{-73 \pm \sqrt{3025}}{8}$$

$$x = \frac{-73 \pm 55}{8}$$

Therefore,

$$x_1 = \frac{-73+55}{8} = -\frac{18}{8} = -2\tfrac{1}{4}$$

$$x_2 = \frac{-73-55}{8} = -\frac{138}{8} = -16$$

Step 3, remove extraneous solution(s) by substituting x_1 and x_2 back into y' to verify that $y' = 0$ at x_1 and x_2

$$y'(x_1) = \left(\frac{-2\frac{1}{4}+6}{\sqrt{(-2\frac{1}{4}+6)^2+25}}\right) + \left(\frac{-2\frac{1}{4}-6}{\sqrt{(-2\frac{1}{4}-6)^2+121}}\right) = 0$$

$$y'(x_2) = \left(\frac{-16+6}{\sqrt{(-16+6)^2+25}}\right) + \left(\frac{-16-6}{\sqrt{(-16-6)^2+121}}\right) = -\frac{4\sqrt{5}}{5}$$

$$y'(x_2) \neq 0$$

Therefore, $x_2 = -16$ is an extraneous solution and is not a valid solution. Hence, the **critical point is** $-2\frac{1}{4}$

Step 4, check whether it is a **minimum or maximum point**, we evaluate the second derivative at this point

Applying the quotient rule to each term in y';

The **quotient rule**: $d\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).d(f(x)) - f(x).d(g(x))}{(g(x))^2}$

$$y'' = \frac{\sqrt{(x+6)^2+25} - (x+6).\frac{(x+6)}{\sqrt{(x+6)^2+25}}}{(x+6)^2+25} + \frac{\sqrt{(x-6)^2+121} - (x-6).\frac{(x-6)}{\sqrt{(x-6)^2+121}}}{(x-6)^2+121}$$

$$y'' = \frac{1}{\sqrt{(x+6)^2+25}}.\frac{(x+6)^2+25-(x+6)^2}{(x+6)^2+25} + \frac{1}{\sqrt{(x-6)^2+121}}.\frac{(x-6)^2+121-(x-6)^2}{(x-6)^2+121}$$

Simplify the equation above,

$$y'' = \frac{25}{((x+6)^2+25)^{\frac{3}{2}}} + \frac{121}{((x-6)^2+121)^{\frac{3}{2}}}$$

- Evaluate y'' at the critical point $x_1 = -2\frac{1}{4}$

$$y''(-2\tfrac{1}{4}) = \frac{25}{\left((-2\frac{1}{4}+6)^2+25\right)^{\frac{3}{2}}} + \frac{121}{\left((-2\frac{1}{4}-6)^2+121\right)^{\frac{3}{2}}}$$

$$y''(-2\tfrac{1}{4}) = 0.1489$$

Since $y'(-2\frac{1}{4}) = 0$ and $y''(-2\frac{1}{4}) > 0$ This tells us **algebraically** that the function has a **minimum** at $x = -2\frac{1}{4}$

The minimum value of y is therefore gotten by solving for the value of y at $x = -2\frac{1}{4}$,

$$y(-2\tfrac{1}{4}) = \sqrt{(-2\tfrac{1}{4}+6)^2+25} + \sqrt{(-2\tfrac{1}{4}-6)^2+121}$$

$$y(-2\tfrac{1}{4}) = \sqrt{14.0625+25} + \sqrt{68.0625 + 121}$$

$$y(-2\tfrac{1}{4}) = 20$$

Minimum point is $(-2\frac{1}{4}, 20)$

The Minimum value of the function $y = \sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121}$ **is 20.**

This result is confirmed graphically as shown below (Graph plotted using Google Spreadsheet), Minimum point is $\left(-2\frac{1}{4}, 20\right)$