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Econ 260A

2018-10-30

Econ 260A: Homework Challenge #2

An exhaustible resource is costlessly extracted in continuous time by competitive firms (assume all the other assumptions of the Hotelling model apply). The market inverse demand curve is $p(y) = a - by$ where $p(t)$ and $y(t)$ denote the price and quantity in time t and a and b are finite parameters. The initial stock of the resource is given by $x(0) = x_0$, the interest rate is r, and the time horizon is $0 \leq t \leq T$.

a) Use optimal control theory *to characterize the dynamic market equilibrium. Clearly state all of the first-order and transversality conditions that must hold.*

From the Nonrenewable Resources lecture notes, we know that we can obtain the competitive market equilibrium in the same way as we would determine the optimal program based on solving the planner problem.

Determine flow of benefits $U(y(t))$:

$$U(y(t)) = \int_0^{y(t)} p(s)ds = \int_0^{y(t)} (a + bs)ds = as + \frac{1}{2}bs^2 \Big|_0^{y(t)}$$
$$= ay(t) + \frac{1}{2}by(t)^2$$

The basic planner problem becomes:

$$\max_{y(t)} \int_0^T [ay(t) - \frac{1}{2}by(t)^2]e^{-rt}dt \quad \text{subj. to} \quad \dot{x}(t) = -y(t), \quad x(0) = x_0$$

Current-value Hamiltonian:

$$\tilde{H} = U(y(t)) - \mu(t)y(t) = ay(t) - \frac{1}{2}by(t)^2 - \mu(t)y(t)$$

The maximum principle for this problem:

$$\tilde{H}_y = 0 = a + by(t) - \mu(t) = p(y(t)) - \mu(t) \qquad \text{(maximum condition)}$$
$$-\tilde{H}_x = \dot{\mu}(t) - r\mu(t) = 0 \qquad \text{(adjoint equation)}$$
$$\tilde{H}_\mu = \dot{x}(t) = y(t) \qquad \text{(equation of motion)}$$
$$\tilde{H}(T) = 0 \qquad \text{(free time condition)}$$
$$\mu(T)x(T) = 0 \Rightarrow x(T) = 0 \qquad \text{(free state condition)}$$

From the maximum condition and adjoint equation:

$$\tilde{H}_y = 0 = p(y(t)) - \mu(t) = p(t) - \mu(t) \qquad \text{(apply Leibniz' rule to max cond'n)}$$
$$\Rightarrow p(t) = \mu(t)$$
$$\dot{\mu}(t) - r\mu(t) = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{(adjoint eq'n)}$$
$$\Rightarrow \frac{\dot{\mu}(t)}{\mu(t)} = r \Rightarrow \frac{\dot{p}(t)}{p(t)} = r \qquad\qquad \text{(substitute into adjoint eq'n)}$$
$$\Rightarrow p(t) = p(0)e^{rt} \qquad\qquad\qquad\qquad\qquad \text{(Hotelling's rule)}$$

So to characterize the dynamic market equilibrium, the price grows at the discount rate:

$$p(t) = p(0)e^{rt}$$

With the free time condition $\tilde{H}(T) = 0$ and free state condition $x(T) = 0$ we know, as stated in the notes:

$$\int_0^T y(t)dt = x_0$$

Additionally, we know from expanding free time condition $\tilde{H}(T) = 0$ that

$$y(T) = 0$$

b) Use your results from part (a) *to derive an implicit function for T in terms of the parameters of the problem (a, b, x_0, r). Show how T changes with marginal changes in each of the parameters and discuss the economic intuition for each comparative static result.*

Since $y(T) = 0$ and $p(t) = a - by(t)$:

$$p(T) = p(0)e^{rT} = a$$

We can also derive a function $y(p, t)$:

$$p(t) = a - by(t) \implies y(t) = \frac{a - p(t)}{b}$$

Hotelling (1931) follows these steps to determine an implicit function which we can use to determine T^*.

$$x_0 = \int_0^T y(t)dt = \int_0^T \frac{a - p(t)}{b} dt$$
$$= \frac{1}{b}\left(aT - \frac{p_0}{r}(e^{rT} - 1)\right)$$
$$\Rightarrow bx_0 = aT - \frac{a}{r}(1 - e^{-rT})$$
$$\Rightarrow \frac{a}{r}e^{-rT} = bx_0 - aT + \frac{a}{r}$$

From this, we can define two functions

$$z_1(T) = \frac{a}{r}e^{-rT} \quad \text{and} \quad z_2(T) = -aT + bx_0 + \frac{a}{r}$$

Plotting these on a (T, z) plot, the decreasing exponential curve $z_1(T)$ should intersect the negative slope line exactly once in $\mathbb{R}_2^+$.
Logic: at $T = 0$, both have same slope $-a$ but the intercept of the line is higher ($z_1(0) = \frac{a}{r}$, $z_2(0) = \frac{a}{r} + bx_0$). Since z_1 approaches zero asymptotically from above and z_2 crosses zero, there will be exactly one intersection. At this intersection we find the optimal T^* for exhaustion of the resource.

From this, we can examine the comparative statics for changes in a, b, r, x_0. Hotelling (1931) claims that a higher expected price when production rate is very small will result in a longer period of production:

In general, the higher the price anticipated when the rate of production becomes extremely small, compared with the price for a more rapid production, the more protracted will be the period of operation.

- As a increases (holding other parameters constant), T^* increases: $\frac{\Delta T^*}{\Delta a} > 0$.
  - By the equations, intercepts for both z_1, z_2 increase linearly, shifting them both upward by the same amount. Additionally, the slope of each at $T = 0$ is steeper (in a negative direction). The slope change would tend to reduce T^* while the intercept change would tend to increase it, making graphical analysis a little trickier.
  - However, we also know that $p(T) = a = p_0 e^{rT}$, which, holding constant p_0, means an increase in T.
  - But the optimal program is determined by choosing an initial price such that the resource is exhausted at the time when demand goes to zero. A higher price pushes up the demand curve, resulting in higher prices at all production rates, increasing T^* per Hotelling's prediction.
- As b increases, T^* should increase: $\frac{\Delta T^*}{\Delta b} > 0$.
  - Graphically, the z_1 curve is unchanged, but z_2 line shifts upward. This pushes the intersection point T^* farther to the right. Therefore increase in b results in a longer time to exhaustion T^*.

  - This term is basically a cost that increases with high extraction, which should reduce extraction in early years and push final extraction farther into the future.
- As x_0 increases, T^* increases: $\frac{\Delta T^*}{\Delta x_0} > 0$.
  - Graphically, z_2 shifts upward. This is similar to an increase in b since the terms are linked in the implicit function.

  - Intuitively, additional stock at the start will still be subject to extraction rates driven by demand, meaning the rate of extraction relative to the total initial stock will decrease, extending the time to final extraction,
- As r increases, T^* will decrease. . . ? $\frac{\Delta T^*}{\Delta r} < 0$.
  - Graphically, both intercepts shift downward by the same amount $\frac{a}{\Delta r}$, though z_2 drops *proportionally* less due to the bx_0 term. The slopes at $T = 0$ remain identical, at $-a$. This becomes difficult to predict graphically, though it seems as if it should move the intersection to the left, *decreasing T^**.
  - Conceptually, if r increases then the prices grow at a higher rate of interest, resulting in higher prices as extraction shrinks; by one reading of Hotelling this should extend the time until final extraction. This may be an inaccurate reading of Hotelling's prediction, since I believe he is discussing rate of extraction as a direct function of price (essentially our demand curve $y = \frac{a-p}{b}$); but here we are looking at extraction decreasing over time as price increases as a function of time, one step removed from the demand curve.
  - I'm inclined to stick with T^* decreasing in r.