Econ241a: PS 1

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Problems from assignment sheet

1. **The probability that it rains in city A is 0.5, the probability that it rains in city B is 0.3, and the probability that it rains in both is 0.15. Find the probability of each of these events:**

   (a) It does not rain in either city.
   Does not rain in $A = P(A^c) = 1 - P(A)$ (and similar for B)

   $$\begin{aligned} P(A^c \cap B^c) &= (1 - P(A)) * (1 - P(B)) \\ &= 0.5 * 0.7 = 0.35 \end{aligned}$$

   (b) It rains in both cities.

   $$\begin{aligned} P(A \cap B) &= P(A) * P(B) \\ &= 0.5 * 0.3 = 0.15 \end{aligned}$$

   (also, given...)

   (c) It rains in at least one city.

   $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.5 + 0.3 - 0.15 = 0.55 \end{aligned}$$

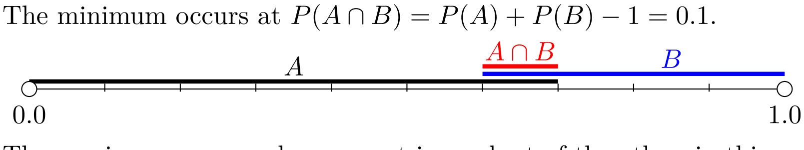
2. **Consider two events A and B such that $P(A) = 1/5$ and $P(B) = 1/3$. Find $P(B \cap A^c)$ for each of these cases:**

   (a) A and B are disjoint
   If A and B are disjoint, then $P(B \cap A^c) = P(B) = 1/3$.

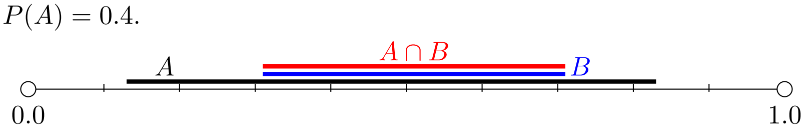
   (b) $B \subset A$
   For this specific circumstance, since $P(A) < P(B)$, B cannot be a subset of A. But in general, if $B \subset A$ then $\forall x \in B, x \in A$; so $\forall x \in B, x \notin A^c$, so $P(B \cap A^c) = 0$.

   (c) $P(B \cap A^c) = 1/7$
   $P(B \cap A^c) = P(B) - P(A \cap B)$ (from thm 1.2.9a) $= 1/3 - 1/7 = \frac{4}{21}$

3. **Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the minimum and maximum values of $P(A \cap B)$ and the conditions under which each is attained.**
   The minimum occurs at $P(A \cap B) = P(A) + P(B) - 1 = 0.1$.



   The maximum occurs when one set is a subset of the other; in this case when $A \subset B$, and $P(A \cap B) = P(A) = 0.4$.



Problems from Casella and Berger

1.2 **Verify the following identities.**

1. $A - B = A - (A \cap B) = A \cap B^c$

$$
\begin{aligned}
A - B &= \{x : x \in A \wedge x \notin B\} && \text{(def of set subtraction)} \\
&= A \cap B^c && \text{left hand side = right hand side}
\end{aligned}
$$

$$
\begin{aligned}
A - (A \cap B) &= \{x : x \in A \wedge \neg(x \in A \wedge x \in B)\} && \text{(def of set subtraction)} \\
&= A \cap (A \cap B)^c && \text{(write as set operations)} \\
&= A \cap (A^c \cup B^c) && \text{(DeMorgan's laws)} \\
&= (A \cap A^c) \cup (A \cap B^c) && \text{(distributivity)} \\
&= 0 \cup (A \cap B^c) = (A \cap B^c) && \text{middle = right hand side}
\end{aligned}
$$

2. $B = (B \cap A) \cup (B \cap A^c)$

$$
\begin{aligned}
(B \cap A) \cup (B \cap A^c) &= B \cap (A \cup A^c) && \text{(un-distribute)} \\
A \cup A^c &= \text{ universal set } \mathbb{U} && \text{(set + complement = all values)} \\
B \cap \mathbb{U} &= B
\end{aligned}
$$

3. $B - A = (B \cap A^c)$

$$
\begin{aligned}
B - A &= \{x : x \in B \wedge x \notin A\} && \text{(def of set subtraction)} \\
&= \{x : x \in B\} \cap \{x : x \notin A\} && \text{(split set definition)} \\
&= B \cap A^c && \text{(rewrite as set operations)}
\end{aligned}
$$

4. $A \cup B = A \cup (B \cap A^c)$

$$
\begin{aligned}
A \cup B &= (A \cup B) \cap (A \cup A^c) && \text{(distributivity)} \\
(A \cup A^c) &= \mathbb{U} && \text{(set + complement = all)} \\
A \cup B &= (A \cup B) \cap \mathbb{U} && \text{(substitute)} \\
&= (A \cup B)
\end{aligned}
$$

1.6 **Two pennies, one with $P(heads_1) = u$ and one with $P(heads_2) = w$, are to be tossed together independently. Define:**

- $p_0 = P(0 \text{ heads occur})$

$$
\begin{aligned}
p_0 &= P(\neg heads_1 \wedge \neg heads_2) \\
&= P(h_1^c) * P(h_2^c) \\
&= (1-u)(1-w) = 1 - u - w + uw
\end{aligned}
$$

- $p_1 = P(1 \text{ heads occurs})$

$$
\begin{aligned}
p_1 &= P(heads_1 \vee heads_2) \\
&= P(h_1) + P(h_2) - P(h_1 \cap h_2) \\
&= u + w - uw
\end{aligned}
$$

- $p_2 = P(\text{2 heads occur})$

$$\begin{aligned} p_2 &= P(heads_1 \wedge heads_2) \\ &= P(h_1) * P(h_2) \\ &= uw \end{aligned}$$

Can u and w be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

Proof by contradiction, to show: $\exists (u, w) \in [0, 1] \ni p_0 = p_1 = p_2$. **Proof:** Suppose (toward contradiction) $p_0 = p_1 = p_2$.

$$\begin{aligned} &\implies 1 - u - w + uw = u + w - uw = uw && \text{(from defs above)} \\ &\implies 1 - u - w = u + w = 0 && \text{(subtract uw)} \\ &\implies (u + w = 1) \wedge (u + w = 0) && \text{Aha! a contradiction!} \end{aligned} \quad \blacksquare$$

Thus, no... there are no values for u and $w \in [0, 1]$ that allow for $p_0 = p_1 = p_2$.

1.35 **Prove that if $P(\cdot)$ is a legit probability function and B is a set with $P(B) > 0$, then $P(\cdot|B)$ also satisfies Kolmogorov's Axioms.**

To show: $P(\cdot|B)$ satisfies Kolmogorov's Axioms. **Proof:** Let $P(\cdot)$ be a probability function that satisfies Kolmogorov's Axioms.

$$\begin{aligned} &\implies P(\cdot) \geq 0, \forall \cdot && \text{(by axiom 1)} \\ &\implies P(\cdot|B) = P(\cdot \cap B)/P(B) && \text{(Bayes thm)} \\ &\implies P(\cdot \cap B)/P(B) \geq 0 && \text{(all components +)} \\ &\implies P(\cdot|B) \geq 0 && \text{(satisfies axiom 1)} \end{aligned}$$

$$\begin{aligned} &\implies P(S) = 1 && \text{(by axiom 2)} \\ &\implies P(S|B) = P(S \cap B)/P(B) && \text{(Bayes thm)} \\ &\implies P(S|B) = P(B)/P(B) = 1 && \text{(satisfies axiom 2)} \end{aligned}$$

$$\begin{aligned} &\implies \text{if } A_1, A_2, \ldots \text{ are pairwise disjoint, } P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) && \text{(by axiom 3)} \\ &\implies P(\bigcup_{i=1}^{\infty} A_i|B) = \frac{P(\bigcup_{i=1}^{\infty} A_i \cap B)}{P(B)} && \text{(Bayes thm)} \\ &\implies P(\bigcup_{i=1}^{\infty} A_i|B) = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} && \text{(by axiom 3)} \\ &\implies P(\bigcup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} && \text{(distribute } 1/P(B) \text{ into sum)} \\ &\implies P(\bigcup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B) && \text{(satisfies axiom 3)} \end{aligned}$$

$\blacksquare$

1.39 **A pair of events A and B connot be simultaneously *mutually exclusive* and *independent.* Prove that if $P(A) > 0$ and $P(B) > 0$, then:**

- If A and B are mutually exclusive, they cannot be independent.
- If A and B are independent, they cannot be mutually exclusive.

Proof by contradiction, show that $(P(A|B) = \emptyset) \wedge (P(A|B) = P(A))$. Suppose toward contradiction that A and B are both mutually exclusive and independent.

$$
\begin{aligned}
&\iff A \cap B = 0 && \text{(def of mutually exclusive)} \\
&\iff P(A|B) = P(A) && \text{(def of independent)} \\
&\iff P(A|B) = \frac{P(A \cap B)}{P(B)} && \text{(Bayes thm)} \\
&\iff P(A|B) = \frac{0}{P(B)} && \text{(from mutually exclusive)} \\
&\iff P(A|B) = 0 = P(A) && \text{contradiction!}
\end{aligned}
$$

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If A and B are both mutually exclusive and independent, this results in a contradiction, proving both (a) and (b).

1.47 **Prove that the following functions are cdfs.** To be a cdf, a function must meet the criteria:

a. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.
b. $F_X(x)$ is non-decreasing function of x.
c. $F_X(x)$ is right-continuous.

- $F_X(x) = e^{-e^{-x}}, x \in (-\infty, \infty)$. To show $F_X(x)$ meets criteria a, b, c above, **Proof:**

$$
\begin{aligned}
&\implies \lim_{x \to -\infty} e^{-e^{-x}} = e^{-e^{\infty}} = e^{-\infty} = 0 \\
&\implies \lim_{x \to \infty} e^{-e^{-x}} = e^{-e^{-\infty}} = e^{0} = 1 \\
&\implies F_X(x) \text{ meets limits (criteria a)} \\
&\implies \frac{d(e^{-e^{-x}})}{dx} = e^{-e^{-x}-x} \\
&\implies \frac{d(e^{-e^{-x}})}{dx} > 0, \forall x \\
&\implies F_X(x) \text{ is increasing function of } x \\
&\implies F_X(x) \text{ is exponential, thus continuous over all } x
\end{aligned}
$$

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- $F_X(x) = 1 - e^{-x}, x \in (0, \infty)$. To show $F_X(x)$ meets criteria a, b, c above, **Proof:**

$\Longrightarrow$ approaching 0 from above, $\lim_{x\to 0} 1 - e^{-x} = 1 - e^0 = 0$

$\Longrightarrow$ below 0, $F_X(x) = 0$, so $\lim_{x\to -\infty} 0 = 0$

$\Longrightarrow \lim_{x\to\infty} 1 - e^{-x} = 1 - 0 = 1$

$\Longrightarrow F_X(x)$ meets limits (criteria a)

$\Longrightarrow \dfrac{d(1-e^{-x})}{dx} = e^{-x}$

$\Longrightarrow \dfrac{d(1-e^{-x})}{dx} > 0, \forall x$

$\Longrightarrow F_X(x)$ is increasing function of x

$\Longrightarrow F_X(x)$ is exponential for $x > 0$ and 0 else, thus continuous over all x

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1.49 ___A cdf F_X is *stochastically greater* than a cdf F_Y if $F_X(t) \le F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t. Prove that if $X\ F_X$ and $Y\ F_Y$, then

- $P(X > t) \ge P(Y > t)$ for every t, and
- $P(X > t) > P(Y > t)$ for some t.

Proof: Let $F_X(t) = P(X \le t) \le F_Y(t)$ for all t (by hypothesis).

$$
\begin{aligned}
&\Longrightarrow F_X(t) = P(X \le t) \le F_Y(t) = P(Y \le t) && \text{(def of CDF)}\\
&\Longrightarrow -F_X(t) = -P(X \le t) \ge -F_Y(t) = -P(Y \le t) && \text{(multiply by -1, flip inequality)}\\
&\Longrightarrow 1 - F_X(t) = 1 - P(X \le t) \ge 1 - F_Y(t) = 1 - P(Y \le t) && \text{(add 1 each side)}\\
&\Longrightarrow 1 - F_X(t) = P(X > t) \ge 1 - F_Y(t) = P(Y > t) && (1 - P(x) = P(x^c))\\
&\Longrightarrow P(X > t) \ge P(Y > t) \text{ for all } t\\
&\Longrightarrow P(X > t) > P(Y > t) \text{ for some } t && \text{(same process for $>$ as for $\ge$)}
\end{aligned}
$$

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1.51 **An appliance store receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 at random, without replacement, and tests to see if they are defective. Let $X =$ the number of defectives found. Calculate the pmf and cdf of X and plot the cdf.**

- To choose X defective means both X defective out of 5, and $4 - X$ good out of 25. This can be written as $\binom{5}{X}$ and $\binom{25}{4-X}$ out of $\binom{30}{4}$ possible combinations.

$$
f_X(x) = \begin{cases}
\frac{\binom{5}{0}\binom{25}{4}}{\binom{30}{4}} = \frac{\frac{5!}{0!5!} * \frac{25!}{4!21!}}{\frac{30!}{4!26!}} = 12650/27405 = .46159, & X = 0\\
\frac{\binom{5}{1}\binom{25}{3}}{\binom{30}{4}} = \frac{\frac{5!}{1!4!} * \frac{25!}{3!22!}}{\frac{30!}{4!26!}} = 11500/27405 = .41963, & X = 1\\
\frac{\binom{5}{2}\binom{25}{2}}{\binom{30}{4}} = \frac{\frac{5!}{2!3!} * \frac{25!}{2!23!}}{\frac{30!}{4!26!}} = 3000/27405 = .10947, & X = 2\\
\frac{\binom{5}{3}\binom{25}{1}}{\binom{30}{4}} = \frac{\frac{5!}{3!2!} * \frac{25!}{1!24!}}{\frac{30!}{4!26!}} = 250/27405 = .00912, & X = 3\\
\frac{\binom{5}{4}\binom{25}{0}}{\binom{30}{4}} = \frac{\frac{5!}{4!1!} * \frac{25!}{0!25!}}{\frac{30!}{4!26!}} = 5/27405 = .000182, & X = 4\\
0, & \text{else}
\end{cases}
$$

$$F_X(x) = \begin{cases} 0, & X < 0 \\ .46159, & X < 1 \\ .88122, & X < 2 \\ .99069, & X < 3 \\ .99981, & X < 4 \\ 1, & X \geq 4 \end{cases}$$

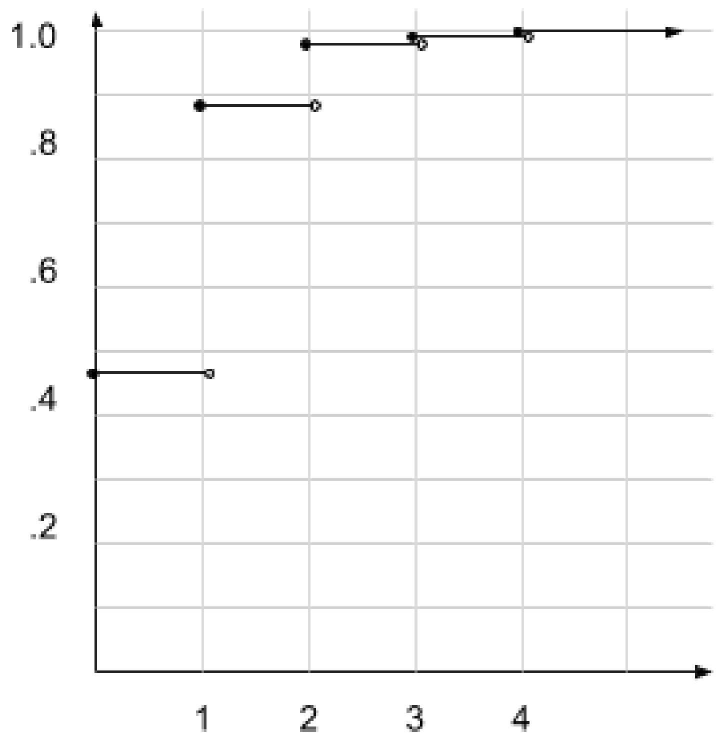

Figure 1: cdf for 1.51

1.54 (b) **Determine the value of c that makes $f(x)$ a pdf:**

- $f(x) = ce^{-|x|}, -\infty < x < \infty$

Integrate across the entire support, then determine c such that the integral sums to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_{-\infty}^{0} ce^x dx + \int_{0}^{\infty} ce^{-x} dx \\ &= ce^x|_{-\infty}^{0} + -ce^{-x}|_{0}^{\infty} \\ &= c(1-0) - c(0-1) = 2c \end{aligned}$$

So $c = 1/2$.