Econometrics 710 - Problem Set 1

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Exercise 2.1

By applying the law of iterated expectations (Theorem 2.7.2) twice, we get

$$\mathbb{E}(\mathbb{E}(\mathbb{E}(y|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)|\mathbf{x}_1,\mathbf{x}_2)|\mathbf{x}_1) = \mathbb{E}(\mathbb{E}(y|\mathbf{x}_1,\mathbf{x}_2)|\mathbf{x}_1) = \mathbb{E}(y|\mathbf{x}_1)$$

Exercise 2.2

By the law of iterated expectations,

$$\mathbb{E}(yx) = \mathbb{E}(\mathbb{E}(yx|x)) = \mathbb{E}(x\mathbb{E}(y|x)) = \mathbb{E}(x(a+bx)) = a\mathbb{E}(x) + b\mathbb{E}(x^2).$$

Exercise 2.3

If $\mathbb{E}|h(\mathbf{x})e| < \infty$, by the law of iterated expectations it follows that $\mathbb{E}(h(\mathbf{x})e) = \mathbb{E}(\mathbb{E}(h(\mathbf{x})e)|\mathbf{x}) = \mathbb{E}(h(\mathbf{x})\mathbb{E}(e|\mathbf{x})) = 0$, since $\mathbb{E}(e|\mathbf{x}) = 0$.

Exercise 2.4

Note that $Pr(y=0|x=0) = \frac{Pr(y=0,x=0)}{Pr(x=0)} = \frac{0.1}{0.1+0.4} = 0.2$, $Pr(y=1|x=0) = 0.8, Pr(y=0|x=1) = 0.4, Pr(y=1|x=1) = 0.6$.

$$\begin{aligned}
\mathbb{E}(y|x=0) &= 0 \times Pr(y=0|x=0) + 1 \times Pr(y=1|x=0) = 0.8 \\
\mathbb{E}(y|x=1) &= 0 \times Pr(y=0|x=1) + 1 \times Pr(y=1|x=1) = 0.6 \\
\mathbb{E}(y^2|x=0) &= 0^2 \times Pr(y=0|x=0) + 1^2 \times Pr(y=1|x=0) = 0.8 \\
\mathbb{E}(y^2|x=1) &= 0^2 \times Pr(y=0|x=1) + 1^2 \times Pr(y=1|x=1) = 0.6 \\
\text{var}(y|x=0) &= \mathbb{E}(y^2|x=0) - (\mathbb{E}(y|x=0))^2 = 0.16 \\
\text{var}(y|x=1) &= \mathbb{E}(y^2|x=1) - (\mathbb{E}(y|x=1))^2 = 0.24
\end{aligned}$$

Exercise 2.5

(a) $\mathbb{E}(e^2 - h(\mathbf{x}))^2$
(b) Given a realization of the random variable $\mathbf{x}$, we guess the realization of the random variable e^2 with a function $h(\mathbf{x})$. In this case we assess the goodness of the prediction with the mean squared error of the prediction $\mathbb{E}(e^2 - h(\mathbf{x}))^2$.
(c) The mean squared error of a predictor $h(\mathbf{x})$ for e^2 is

$$\begin{aligned}
\mathbb{E}(e^2 - h(\mathbf{x}))^2 &= \mathbb{E}(e^2 - \sigma^2(\mathbf{x}) + \sigma^2(\mathbf{x}) - h(\mathbf{x}))^2 \\
&= \mathbb{E}(e^2 - \sigma^2(\mathbf{x}))^2 + 2\mathbb{E}(e^2 - \sigma^2(\mathbf{x}))(\sigma^2(\mathbf{x}) - h(\mathbf{x})) + \mathbb{E}(\sigma^2(\mathbf{x}) - h(\mathbf{x}))^2 \\
&= \mathbb{E}(e^2 - \sigma^2(\mathbf{x}))^2 + \mathbb{E}(\sigma^2(\mathbf{x}) - h(\mathbf{x}))^2 \\
&\geq \mathbb{E}(e^2 - \sigma^2(\mathbf{x}))^2
\end{aligned}$$

where the third equality holds since $\mathbb{E}(e^2 - \sigma^2(\mathbf{x}))(\sigma^2(\mathbf{x}) - h(\mathbf{x})) = 0$, which we will show later. The last inequality holds because of the non-negativity of $\mathbb{E}(\sigma^2(\mathbf{x}) - h(\mathbf{x}))^2$. The right-hand side after the inequality is minimized at $h(\mathbf{x}) = \sigma^2(\mathbf{x})$, thus $\sigma^2(\mathbf{x})$ is the best predictor.

It is enough to show that $\mathbb{E}(e^2 - \sigma^2(\mathbf{x}))(\sigma^2(\mathbf{x}) - h(\mathbf{x})) = 0$.

$$\begin{aligned}
\mathbb{E}(e^2 - \sigma^2(\mathbf{x}))(\sigma^2(\mathbf{x}) - h(\mathbf{x})) &= \mathbb{E}(\mathbb{E}((e^2 - \sigma^2(\mathbf{x}))(\sigma^2(\mathbf{x}) - h(\mathbf{x}))|\mathbf{x})) \\
&= \mathbb{E}((\sigma^2(\mathbf{x}) - h(\mathbf{x}))\mathbb{E}(e^2 - \sigma^2(\mathbf{x})|\mathbf{x})) \\
&= \mathbb{E}((\sigma^2(\mathbf{x}) - h(\mathbf{x}))(\mathbb{E}(e^2|\mathbf{x}) - \sigma^2(\mathbf{x}))) \\
&= \mathbb{E}((\sigma^2(\mathbf{x}) - h(\mathbf{x})) \cdot 0) = 0,
\end{aligned}$$

by the law of iterated expectations and by the definition of conditional variance $\sigma^2(\mathbf{x}) = \mathbb{E}(e^2|\mathbf{x})$

Exercise 2.6

$$\begin{aligned}
\mathrm{var}(y) &= \mathrm{var}(m(\mathbf{x}) + e) = \mathrm{var}(m(\mathbf{x})) + \mathrm{var}(e) + 2\mathrm{cov}(m(\mathbf{x}), e) \\
&= \mathrm{var}(m(\mathbf{x})) + \sigma^2 + 2\mathrm{cov}(m(\mathbf{x}), e) \\
&= \mathrm{var}(m(\mathbf{x})) + \sigma^2
\end{aligned}$$

where the last equality holds because $\mathrm{cov}(m(\mathbf{x}), e) = \mathbb{E}(m(\mathbf{x})e) - \mathbb{E}(m(\mathbf{x}))\mathbb{E}(e) = \mathbb{E}(m(\mathbf{x})\mathbb{E}(e|\mathbf{x})) - \mathbb{E}(m(\mathbf{x}))\mathbb{E}(e) = 0$ since $\mathbb{E}(e|\mathbf{x}) = 0$ and $\mathbb{E}(e) = 0$.

Exercise 2.7

$$\begin{aligned}
\sigma^2(\mathbf{x}) &= \mathrm{var}(y|\mathbf{x}) = \mathbb{E}(e^2|\mathbf{x}) \\
&= \mathbb{E}((y - m(\mathbf{x}))^2|\mathbf{x}) \\
&= \mathbb{E}(y^2 - 2m(\mathbf{x})y + m(\mathbf{x})^2|\mathbf{x}) \\
&= \mathbb{E}(y^2|\mathbf{x}) - 2m(\mathbf{x})\mathbb{E}(y|\mathbf{x}) + m(\mathbf{x})^2 \\
&= \mathbb{E}(y^2|\mathbf{x}) - 2m(\mathbf{x})^2 + 2m(\mathbf{x})\mathbb{E}(e|\mathbf{x}) + m(\mathbf{x})^2 \\
&= \mathbb{E}(y^2|\mathbf{x}) - (\mathbb{E}(y|\mathbf{x}))^2
\end{aligned}$$

Exercise 2.8

Since $y|\mathbf{x} \sim \text{Poisson}(\mathbf{x}'\boldsymbol{\beta})$, it follows that $\mathbb{E}(y|\mathbf{x}) = \mathrm{var}(y|\mathbf{x}) = \mathbf{x}'\boldsymbol{\beta}$. Therefore, we can justify a linear regression model $y = \mathbf{x}'\boldsymbol{\beta} + e$ with $\mathbb{E}(e|\mathbf{x}) = 0$, since the conditional expectation function is actually linear.