Econ 241A Probability, Statistics and Econometrics

Fall 2013

Midterm

- You have 1 hr 10 min to complete this midterm.
- The midterm has two parts. Part I requires to solve all problems. Part II allows you to choose between two problems. Please solve just one problem in Part II. If you answer both, only the lowest grade out of the two will be taken into account.
- The last page of the exam has a list of pmf's and pdf's that you may (or may not) need to use throughout the exam.

Part I

1. (3) The joint pmf of X and Y is given by the following table.

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.2	0.05	0.1
$x = 1$	0.1	0.2	0.1
$x = 2$	0.1	0.05	0.1

Write the conditional pmf $f_{Y|X}(y|x = 0)$.

2. (3) Prove that $\text{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}(X))Y\right]$

3. (3) Complete the prove for Markov's inequality: $\Pr(g(X) \geq r) \leq \frac{\mathbb{E}(g(X))}{r}$, for non-negative $g(X)$. The first step of the prove is provided below.

- First note that $\{x : g(x) \geq r\} \in \mathcal{X}$, where $\mathcal{X}$ is the support of X. Hence,

$$\mathbb{E}(g(X)) = \int_{\mathcal{X}} g(x) f_X(x) dx \geq \int_{\{x:g(x)\geq r\}} g(x) f_X(x) dx$$

4. (3) Let X be $\text{n}(\mu, \sigma^2)$. Define $W = X \times I_{[a,b]}(X)$, where the indicator function, $I_{[a,b]}(x)$ is defined as

$$I_{[a,b]}(x) = \begin{cases} 0 & \text{if } x \notin [a, b] \\ \\ 1 & \text{if } x \in [a, b] \end{cases}$$

In other words, W is distributed truncated normal, $\text{tn}(\mu, \sigma, a, b)$, where the interval $[a, b]$ determines the support of W. Write the pdf of W. Hint: What are the conditions that guarantee that a function is a pdf?

5. (3) Prove that $\text{Cov}(X, \mathbb{E}(Y|X)) = \text{Cov}(X, Y)$.

6. (Extra Credit: 3 pts.) Define two random variables as $U = \mathbb{E}(Y|X)$ and $W = Y - \mathbb{E}(Y|X)$. Prove that U and W are uncorrelated.

Part II

Choose to answer one of the following two problems. If you answer both, only the lowest graded will be taken into account.

7. (15) **Taken from lecture's example.** Forest owners in the Amazon have an opportunity cost of preserving q hectares as forest given by

$$c(Q; W) = a + \frac{W}{2}q^2$$

Note that the marginal cost of preserving q hectares of forest is proportional to the random variable W. A Payments for Ecosystem Services (PES) program pays \$2 for each hectare of land preserved. Farmers decide how many hectares to submit by setting the marginal cost of preservation equal to the per-hectare compensation. Hence,

$$Q^* = \frac{2}{W}$$

Assume a researcher learns that the farmer's distribution of hectares submited to the PES program, Q^*, is exponential with $\lambda = \frac{1}{2}$.

(a) What is the cdf of W?

(b) What is the pdf of W?

For parts (c), (d), (e) and (f) assume that $Q = Q^* + U$, where $U \sim \text{uniform}[-1, 1]$. Continue to assume that Q^* is distributed exponential with $\lambda = \frac{1}{2}$. Assume U and Q^* are independent.

(c) What is the mean of Q?

(d) What is the variance of Q?

(e) What is the conditional pdf of Q given W?

(f) What is the joint distribution of Q and Q^*?

8. (15) **Willingness to pay for time (inspired by Garrido and Gutierrez, 2013)**. Individuals that show up at a clinic to get checked up for cataracts wait T minutes to get seen by a doctor. Individuals differ by their willingness to pay for time. Their willingness to pay for a minute of time is B.

For parts (a), (b), (c), and (d):

- Assume $B \sim \text{uniform}[1, 11]$.
- Assume that $T|B \sim \text{exponential}\left(\frac{B}{10}\right)$.

(a) What is the joint pdf of T and B?

(b) What is the expected wait for someone whose willingness to pay for a minute of time is b?

(c) Choose the statement that is most consistent with the above distributional assumptions and explain:

(i) Individuals with high willingness to pay for time come at hours of the day when wait times are longer in average.

(ii) Individuals with high willingness to pay for time come at hours of the day when wait times are shorter in average.

(iii) Average wait time is the same among individuals with different willingesses to pay for time

(d) What is the expected wait time for an individual randomly taken out of the population?

For parts (e) through (g) the vector (T, B) (wait time and willingness to pay) have an unknown joint pdf given by $f_{T,B}(t, b)$ and marginal pdf's given by $f_T(t)$ and $f_B(b)$, respectively. I.e., do not use the uniform and exponential functional forms given for parts a) through d); instead, use only generic distribution notation.

Individuals are offered m dollars to skip the line, where m is a constant. Their utility is assumed to be $U(m, T) = \left(\bar{M} - m\right) + B \times \left(\bar{T} - T\right)$, where $\bar{M}$ is their income, $\bar{T}$ is their time constraint, T is the wait they face in minutes, and B is their willingness to pay for one minute of time. Individuals opt for paying m dollars to skip the wait if

$$U(m, 0) = \bar{M} - m + B \times (\bar{T}) > \bar{M} + B\left(\bar{T} - T\right) = U(0, T)$$

(e) Write the correct integral expression for the conditional cdf of B given T, $F_{B|T}(b|t)$ as a function of the joint and marginal pdf's of (T, B). Assume the support of T is given by the interval $(0, \infty)$, and the support of B is given by the interval [1,11].

(f) Write an inequality condition for B that characterizes the decision to pay m dollars to skip the wait of T minutes. Using this condition and the conditional cdf of B given T, write an expression for the probability that an individual chooses to pay m for avoiding a wait of t minutes. Denote this probability as $P(m, t)$.

(g) (Extra Credit: 3 pts.) How would the expression for $P(m, t)$ change under independence of T and B?

Bernoulli

$P(X = x|p) = p^x(1-p)^{(1-x)}$; $x = 0, 1$; $0 \leq p \leq 1$ with $\mathbb{E}(X) = p$ and $\mathrm{Var}(X) = p(1-p)$

Binomial

$P(X = x|n, p) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x(1-p)^x$; $x = 0, 1, 2, ..., n$; $0 \leq p \leq 1$, with $\mathbb{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$

Discrete uniform

$P(X = x|N) = \frac{1}{N}$; $x = 1, 2, ..., N$; $N = 1, 2, ...$, with $\mathbb{E}(X) = \frac{N+1}{2}$ and $\mathrm{Var}(X) = \frac{(N+1)(N-1)}{12}$

Poisson

$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, ...$; $0 \leq \lambda < \infty$ with $\mathbb{E}(X) = \lambda$ and $\mathrm{Var}(X) = \lambda$

Uniform

$f(x|a, b) = \frac{1}{b-a}$; $x \in [a, b]$; with $\mathbb{E}(X) = \frac{b+a}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

Exponential

$f(x|\beta) = \lambda e^{-\lambda x}$; $0 \leq x < \infty$ with $\mathbb{E}(X) = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$

Logistic

$f(x|\mu, \beta) = \frac{1}{\beta} \frac{\exp(-(x-\mu)/\beta)}{[1+\exp(-(x-\mu)/\beta)]^2}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\beta > 0$ with $\mathbb{E}(X) = \mu$ and $\mathrm{Var}(X) = \frac{\pi^2\beta^2}{3}$

Normal

$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ with $\mathbb{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$