Econ 241A Probability, Statistics and Econometrics
Fall 2017

Problem Set 6

1. (Convergence in distribution does not imply convergence in probability). Define the following sequence of variables over the sample space generated by under a fair default coin toss, $S = \{H, T\}$:
$$X_n = \begin{cases} 1 \text{ if the coin toss is heads } s = H \\ 0 \text{ if the coin toss is tails } s = T \end{cases}$$
for all n (this means that $X_1 = X_2 = ... = X_n$). Now define the variable X such that
$$X = \begin{cases} 0 \text{ if the coin toss is heads } s = H \\ 1 \text{ if the coin toss is tails } s = T \end{cases}$$
   (a) What is the cdf of X_1, X_2 and X_3? What is the cdf of X_n?
   (b) What is the cdf of X?
   (c) Does X_n converge in distribution to X?
   (d) What is the cdf of $|X_n - X|$?
   (e) Using your answer to (d) find $P\left(|X_n - X| \geq \frac{1}{2}\right)$
   (f) Does X_n converge in probability to X?

2. Assume $X_1, ..., X_{20}$ is a random sample where $X_i \sim \text{exponential}(2)$ $\left(f_X(x) = \lambda e^{-\lambda x}\right)$. The first 4 uncentered moments of the exponential(2) distribution are given by: $\mathbb{E}(X_i) = \frac{1}{2}$, $\mathbb{E}(X_i^2) = \frac{1}{2}$, $\mathbb{E}(X_i^3) = \frac{3}{4}$ and $\mathbb{E}(X_i^4) = \frac{3}{2}$. Let $T = \frac{1}{n}\sum_{i=1}^{n} X_i^2$
   (a) What is the mean of T?
   (b) What is the variance of T?
   (c) What is the asymptotic distribution of T?
   (d) What is the approximate probability that $T \leq 1$?

3. Let $\bar{X}$ denote the sample mean from a random sample of size n, from a population with exponential(λ) distribution. For convenience, let $\theta = \mathbb{E}(X) = \frac{1}{\lambda}$. So $\mathbb{E}(\bar{X}) = \theta$, $\text{Var}(\bar{X}) = \theta^2/n$, $\bar{X} \to_p \theta$ and $\frac{\sqrt{n}}{\theta^2}(\bar{X} - \theta) \to_d \text{n}(0, 1)$. Consider the sample statistic $U = \frac{1}{\bar{X}}$ (n.b. this is $1/\bar{X}$).
   (a) Use Slutsky theorem to show that $U \to_p \lambda$.
   (b) Use the Delta method to find the limiting distribution of $\sqrt{n}(U - \lambda)$.
   (c) Use your result to approximate $P(U \leq 5/2)$ with a random sample of size 16, from an exponential population with $\lambda = 2$.
   (d) Use the following result to find the exact value for $P(U \leq 5/2)$ with a random sample of size 16, from an exponential population with $\lambda = 2$. Let $X_1, X_2, ..., X_n$ be a random sample from an exponential(2) distribution. Define $\bar{X} = \frac{1}{n}\sum_i X_i$, then:
$$2n\lambda\bar{X} \sim \chi^2_{2n}$$

4. In a population, the random variable X = length of unemployment (in months) has the exponential distribution with parameter $\lambda = 2$. Consider a random sample of unemployment lengths where the sample size is $n = 21$. Let T be the proportion of the sampled persons who have been unemployed between 0.4158 and 1 months.

   Approximate the probability that T lies between 0.4 and 0.5. Hint: define the random variable

$$U_i = \begin{cases} 1 \text{ if } 0.4158 \leq X_i \leq 1 \\ 0 \text{ otherwise} \end{cases}$$

In addition, solve the following problems from Casella and Berger: 5.21 and 5.31.