Econ 241A Probability, Statistics and Econometrics
Fall 2017

Problem Set 7a (Optional)

1. Let $X_1, X_2, \cdots, X_n$ be iid with pdf
$$f(x|\theta) = \theta x^{\theta-1}, \qquad 0 \le x \le 1, 0 < \theta < \infty$$
   (a) Find the MLE of θ. Show that the variance converges to zero as the sample size increases.
   (b) Find the methods of moments estimator of θ.

2. (Playing against a machine). Assume an individual receives w_t at time $t(= 1, 2, \dots)$ drawn from $U[\underline{\theta}, \overline{\theta}]$ where $\underline{\theta} > 0$. At each time the individual can either accept w_t or reject w_t. If she accepts then she receives w_t and the game ends. If she rejects then she does not receive w_t and she receives a new draw w_{t+1} from $U[\underline{\theta}, \overline{\theta}]$. What's the best strategy that the individual can follow (i.e. a plan of action for every $t > 0$)? What's the limit-in-probability payoff of such strategy?

3. In a study of household income distribution in SB the population of households is divided into two groups: households with a head older than 45 and households with head younger than or exactly 45. In fractions of these subpopulations in the full population of all households in SB are p and $1 - p$, respectively. In the two subpopulations random samples of size n_1 and n_2 are drawn. This is called a stratified random sample. The sample means are $\overline{X}_{1n_1}$ and $\overline{X}_{2n_2}$, respectively. The (population) mean and variance of income in the two subpopulations is μ_1, σ_1^2 and μ_2, σ_2^2, respectively.
   (a) Show that the population mean of income in the full population μ is equal to $p\mu_1 + (1-p)\mu_2$.
   (b) We estimate the mean income in the complete population by $\overline{X}_n = c\overline{X}_{1n_1} + d\overline{X}_{2n_2}$. Find (possibly unknown) values of c, d such that this estimator is an unbiased estimator of the population mean.
   (c) For a second independent random sample size of m from the population (without stratification) we observe whether the head of the household is older than (m_1 households) or younger than (m_2 households) 45. Use this estimation to estimate p.
   (d) Is the estimator of p ancillary for the population mean' How does this simplify the computation of the sampling distribution of the estimator of the population mean?
   (e) Derive the variance of the estimator of the population mean.
   (f) Minimize the variance with respect to the sample size in the two subsamples (treat the sample sizes as continuous variables) subject to the restriction that $n_1 + n_2 = n$ with n given. Can you use this result to optimize the sample design?

4. Find a minimal sufficient statistic for θ in the following cases.
   (a) $f(x;\theta) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}$, $-\infty < x < \infty$
   (b) $f(x;\theta) = \frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}$, $-\infty < x < \infty$