Econ 241A Probability, Statistics and Econometrics
Fall 2017

Problem Set 7

1. (Consistency of the sample slope)

   Recall that we introduced earlier on what was the best linear predictor for Y given X, when Y and X were jointly distributed. We defined the best linear predictor as

$$\mathbb{E}^*(Y|X) = \alpha + \beta X,$$

   where

$$\beta = \frac{\sigma_{XY}}{\sigma_X^2}$$

$$\alpha = \mu_Y - \beta \mu_X$$

   Generally σ_{XY} and σ_X^2 are unknown (if the joint distribution of X and Y is unknown). However, we can estimate σ_{XY} and σ_X^2 using a random sample.

   Use Theorem 5.5.4 and Slutsky Theorem to prove that

$$\hat{\beta} = \frac{S_{XY}}{S_X^2} \rightarrow_p \beta$$

2. (Uniform MLE) Assume you are given the following random sample from a uniform(a, b) distribution

   {0.6849204, 3.216103, 2.789009, 3.023975, 3.42088, 0.5433397, 3.092291, 0.3053189, 2.776194, 4.357245}

   a) Recall that the uniform pdf is

$$f_X(x) = \begin{cases} \frac{1}{b-a} \text{ if } x \in [a, b] \\ 0 \text{ if } x \notin [a, b] \end{cases}$$

   with $b > a$.

   Therefore, the likelihood function for the uniform distribution is

$$L(a, b|x_1, .., x_n) = \prod_{i=1}^{n} \frac{1}{b-a} I_{[a,b]}(x_i)$$

   Recall that the $I_{[a,b]}(x_i)$ function takes the value of one if $x_i \in [a, b]$ and 0 otherwise.

   (a) Using the random sample values provided, evaluate the likelihood function above at the following parameter values:

      (i) $a = 2.7, b = 4.3$

      (ii) $a = 0.30$, $b = 3.02$.

(iii) $a = 0.68,\ b = 4.5$

(iv) $a = 0.2,\ b = 4.5$

(v) $a = -2, b = 6$

(b) If you had to choose a and b out of the five choices in (a), which values would maximize the likelihood function?

(c) Notice that we can write the likelihood function as

$$L(a, b|x_1, .., x_n) = \left[\frac{1}{b-a}\right]^n I_{[a,\infty]}(x_{(1)}) I_{[-\infty,b]}(x_{(n)}),$$

where $x_{(1)} = \min(x_1, ..., x_n)$ and $x_{(n)} = \max(x_1, ..., x_n)$. Explain why.

(d) What is the MLE for a and b in the general case with a random sample realization given by $x_1, ..., x_n$ that is distributed uniform$[a, b]$.

In addition, solve the following problem from Casella and Berger: 7.12.