Econ 241A Probability, Statistics and Econometrics

Fall 2014

Final Exam

- You have 2:30 hrs to complete this exam
- The exam has two parts. Part I requires to solve all problems. Part II allows you to choose between two problems. Please solve just one problem in Part II. If you answer both, only the lowest grade out of the two will be taken into account.
- The last page of the exam has a list of pmf's and pdf's that you may (or may not) need to use throughout the exam.

Part I

1. (5) Let X and Y be iid Poisson with parameter λ. What is the distribution of $X+Y$. Hint: There are two ways of solving this problem. One of them uses the following mathematical series result $\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} = 2^n$. Also, remember X and Y are discrete random variables.

$$f_{X,Y}(x,y) = \frac{e^{-2\lambda}\lambda^{x+y}}{x!y!}$$

Let $U = X+Y$ and $V = Y$. Then $h_1(u,v) = u - v$ and $h_2(u,v) = v$. Then

$$J = det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1$$

$$f_{U,V}(u,v) = \frac{e^{-2\lambda}\lambda^{u-v+v}}{(u-v)!v!}$$

for $u = 0, 1, 2, ...$ and $v \in [0, u]$ and $\lambda >= 0$. Then,

$$f_U(u) = \sum_{v=0}^{u} \frac{e^{-2\lambda}\lambda^{u}}{(u-v)!v!} = \frac{\lambda^u}{u!e^{2\lambda}} \sum_{v=0}^{u} \frac{u!}{(u-v)!v!} = \frac{e^{-2\lambda}(2\lambda)^u}{u!}$$

So $X+Y \sim Poisson(2\lambda)$.

2. (5) Let (X_n, Y_n) denote a sequence of random variables. Assume $X_n \rightarrow_d n(0, \sigma_X^2)$ and $Y_n \rightarrow_p a$. Find the limiting distribution of $\log(Y_n + X_n)$.

Give the students full credit if they wrote the first step:

$$Y_n + X_n \rightarrow_d n(a, \sigma_X^2)$$

Question should have said: Assume $\sqrt{n}(X_n - \theta) \rightarrow_d n(0, \sigma_X^2)$ and $Y_n \rightarrow_p a$. Find the limiting distribution of $\sqrt{n}\left(\log(X_n) - \log(\theta)\right) + Y_n$. Then we can answer it:

We can get the following limiting distribution

$$\sqrt{n}\left(\log(X_n) - \log(\theta)\right) \rightarrow_d n\left(0, \sigma_X^2 \frac{1}{\theta^2}\right)$$

and we know that $Y_n \rightarrow_p a$, so

$$\sqrt{n}\left(\log(X_n) - \log(\theta)\right) + Y_n \rightarrow_d n\left(a, \sigma_X^2 \frac{1}{\theta^2}\right)$$

3. (5) Assume $X_1, ..., X_n$ is a random sample with $X_i \sim \mathrm{n}(0,1)$. Consider the expression $kS_{k+1}^2 = (k-1)S_k^2 + \left(\frac{k}{k+1}\right)\left(X_{k+1} - \bar{X}_k\right)^2$, where S_k^2 (S_{k+1}^2) denotes the sample variance of the k ($k+1$) first observations, $\bar{X}_k$ denotes the sample mean of the first k observations and X_{k+1} denotes the $k+1$th observation in the sample. Show that if $(k-1)S_k^2 \sim \chi_{k-1}^2$, then $kS_{k+1}^2 \sim \chi_k^2$.

We know that $X_{k+1} \sim n(0,1)$ and $\bar{X}_k \sim n(0, 1/k)$. Thus, $X_{k+1} - \bar{X}_k \sim n\left(0, 1 + \frac{1}{k}\right)$. Then, $\frac{X_{k+1} - \bar{X}_k}{\sqrt{\frac{k+1}{k}}} \sim n(0,1)$ and $\left(\frac{X_{k+1} - \bar{X}_k}{\sqrt{\frac{k+1}{k}}}\right)^2 \sim \chi_1^2$. Since $(k-1)S_k^2 \sim \chi_{k-1}^2$ and $\left(\frac{k}{k+1}\right)\left(X_{k+1} - \bar{X}_k\right)^2 \sim \chi_1^2$, then $kS_{k+1}^2 = \chi_{k-1}^2 + \chi_1^2 \sim \chi_k^2$.

4. (5) Let the sample space S of an experiment be the closed interval $[0,1]$ with the uniform probability distribution. Define the sequence $X_1, X_2, ...$ as follows:

$$X_1(s) = s + I_{[0,1]}(s), \qquad X_2(s) = s + I_{[0,\frac{1}{2}]}(s), \qquad X_3(s) = s + I_{[\frac{1}{2},1]}(s),$$
$$X_4(s) = s + I_{[0,\frac{1}{3}]}(s), \qquad X_5(s) = s + I_{[\frac{1}{3},\frac{2}{3}]}(s), \qquad X_6(s) = s + I_{[\frac{2}{3},1]}(s),$$

etc. Let $X(s) = s$. Does X_n converge in probability to X? Discuss.

Note: $X_1(s) = s+1$, $X_2(s) = \begin{cases} s & p = 0.5 \\ s+1 & p = 0.5 \end{cases}$, $X_4(s) = \begin{cases} s & p = 0.67 \\ s+1 & p = 0.33 \end{cases}$. So that as $n \to \infty$, $X_n(s) = s$ with increasing probability so that $\lim_{n\to\infty} P\left(|X_n(s) - s| < \epsilon\right) = 1$ and X_n converges in probability to X.

5. (5) Let $X_1, X_2, ..., X_n$ be iid Weibull$(1, \beta)$. Derive the MLE of β.

$$f(\mathbf{x}|1,\beta) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-x_i/\beta}$$

$$L(\beta|\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-x_i/\beta}$$

$$= \left(\frac{1}{\beta}\right)^n e^{\frac{-1}{\beta}\sum_{i=1}^{n} x_i}$$

$$l(\beta|\mathbf{x}) = n \log\left(\frac{1}{\beta}\right) - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial}{\partial\beta} l(\beta|\mathbf{x}) = n\beta\left(\frac{-1}{\beta^2}\right) + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0$$

$$-n\frac{1}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i = 0$$

$$\beta^{MLE} = \bar{x}$$

6. (Extra Credit) Solve Question 1 in a different way.

You can also solve this using moment functions:

$$mgf(X+Y) = mgf\ W = e^{\lambda(e^t-1)}e^{\lambda(e^t-1)} = e^{2\lambda(e^t-1)}$$

Hence, $X+Y$ is distributed Poisson with parameter 2λ.

Part II

7. (15) **C**onsider a random sample, $X_1, X_2, ..., X_n$, where X_i are iid Binomial

$$f_{X_i}(x) = \binom{k}{x} p^x(1-p)^{k-x}$$

with k known and p unknown.

a) What is the method of moments estimator of p?

$$p_{MM} = \frac{1}{k}\frac{1}{n}\sum_{i=1}^{n} X_i$$

b) Find the MLE of p.

$$L(p|x,k) = \prod_{i=1}^{n} \binom{k}{x_i} p^{x_i}(1-p)^{k-x_i}$$

where $\binom{k}{x_i} = \frac{k!}{x_i!(k-x_i)!}$.

$$logL(p|x.k) = \sum_{i=1}^{n} \left[log \binom{k}{x_i} + x_i log(p) + (k-x_i) log(1-p) \right]$$

$$F.O.C : \frac{\partial logL}{\partial p} = \sum_{i=1}^{n} \left[\frac{x_i}{p} + \frac{k-x_i}{1-p} \right] = 0$$

$$p \sum (k - x_i) = (1-p) \sum x_i$$

$$p_{MLE} = \frac{\sum x_i}{kn}$$

c) What is the asymtotic distribution of $\sqrt{n}(p^{MLE} - p)$?

Since $p_{MLE} = \frac{1}{k}\bar{X}_n$ and $E[X_i] = kp$ and $Var(X_i) = kp(1-p)$. Let $g(\theta) = \frac{1}{k}\theta$ so that $g'(\theta) = \frac{1}{k}$. Then by CLT,

$$\sqrt{n}\left(\bar{X}_n - kp\right) \to_d n\left(0, kp(1-p)\right)$$

and using the delta method,

$$\sqrt{n}\left(\frac{1}{k}\bar{X}_n - p\right) \to_d n\left(0, kp(1-p)\left(\frac{1}{k}\right)^2\right)$$

$$\sqrt{n}\left(p_{MLE} - p\right) \to_d n\left(0, p(1-p)\right)$$

d) Assume that p is a random variable that is distributed uniformly between 0 and 1. What is the conditional distribution of X given p.

$$X|p \sim Binomial(k, p)$$

$$f_{X|p}(x|p) = \left(\begin{array}{c} k \\ x \end{array} \right) p^x (1-p)^{k-x}$$

e) What is the joint distribution of X and p?

$f_p(p) = 1$ and $f_{x,p}(x,p) = f(x|p) * f_p(p) = f_{x|p}(x|p) = Binomial(k,p)$.

f) What is the linear predictor of X given p? Hint: What is the conditional expectation of X given p?

$E^*\left[x|p\right] = \alpha + \beta p$ where $\alpha = E[x] - \beta E[p]$ and $\beta = \frac{Cov(x,p)}{Var(p)}$.

$$\begin{aligned} Cov(x,p) &= E[xp] - E[x]E[p] \\ &= E\left[E\left[xp|p\right]\right] - E\left[E\left[x|p\right]\right]E\left[p\right] \\ &= E\left[pE\left[x|p\right]\right] - E\left[kp\right]\frac{1}{2} \\ &= E\left[p^2 k\right] - \frac{k}{4} \\ &= k\left(Var\left(p\right) + \frac{1}{4}\right) - \frac{k}{4} \\ &= \frac{k}{12} \end{aligned}$$

So $\beta = (k/12) * 12 = k$ and $\alpha = E[kp] - k/2 = 0$. Thus, $E^*[x|p] = kp$.

8. (15) A remote sensing machine has been located at a highway exit. The remote sensing machine is capable of measuring vehicles emissions (in particular CO) of vehicles that go by, but does so somewhat imprecisely. The resulting emission records correspond to a random variable that will have different distributions for vehicles that have a working

catalytic converter (class A) than for vehicles that do not (class B). A researcher has been assigned with the task of using the data produced by this machine to estimate the proportion of vehicles that have a working catalytic converter. From previous analysis, the researcher knows that the variance of emission recordings from either class of vehicles is similar. However, the mean of emission recordings for class A vehicles is lower than for class B vehicles.

The researcher analyses a sample of size n of emission recordings, $Y_1, ..., Y_n$. Assume that the vehicles that go through the highway entrance are a random sample of the population of vehicles (i.e. $Y_1, ..., Y_n$ are independent). Assume also that the distribution of emission recordings is normal with mean μ_A and variance σ^2 for class A vehicles and μ_B and variance σ^2 for class B vehicles. The class of the vehicle is denoted with the random variable X_i, where X_i takes the value of 1 if the vehicle is class A (has a catalytic converter) and the value of 0 if the vehicle is class B (does not have a catalytic converter). The (unknown) probability that a vehicle belongs to class A is p. Thus X_i is distributed Bernoulli with parameter p. Summarizing, $Y_i = X_i Y_{Ai} + (1 - X_i) Y_{Bi}$, where $Y_{Ai} \perp Y_{Bi}$, $Y_{Ai} \perp X_i$, $Y_{Bi} \perp X_i$. $Y_{Ai} \sim normal(\mu_A, \sigma^2)$, $Y_{Bi} \sim normal(\mu_B, \sigma^2)$, and $X_i \sim Bernoulli(p)$.

(a) Write the conditional mean of Y_i given X_i, $\mathbb{E}(Y_i|X_i)$. Hint: this should be a function of X_i.

$$\mathbb{E}(Y_i|X_i) = X_i\mu_A + (1 - X_i)\mu_B$$

(b) Write the unconditional mean of Y_i, $\mathbb{E}(Y_i)$ as a function of parameters p, μ_A and μ_B.

$$\mathbb{E}(Y_i) = p\mu_A + (1 - p)\mu_B$$

(c) Show that$\mathrm{Var}(Y_i) = \sigma^2 + (\mu_A - \mu_B)^2(p - p^2)$.

$$\mathrm{Var}(Y_i) = \mathbb{E}\left((X_iY_{Ai} + (1-X_i)Y_{Bi})^2\right) - \left(p\mu_A + (1-p)\mu_B\right)^2$$

$$\mathrm{Var}(Y_i) = \mathbb{E}\left(X_i^2Y_{Ai}^2 + (1-X_i)^2Y_{Bi}^2 - 2\left(X_i(1-X_i)Y_{Ai}Y_{Bi}\right)\right) - \left(p^2\mu_A^2 + (1-p)^2\mu_B^2 + 2p(1-p)\mu_B\mu_A\right)$$

$$= \mathbb{E}\left(X_i^2Y_{Ai}^2\right) + \mathbb{E}\left((1-X_i)^2Y_{Bi}^2\right) - 2\mathbb{E}\left(X_i(1-X_i)Y_{Ai}Y_{Bi}\right) - \left(p^2\mu_A^2 + (1-p)^2\mu_B^2 + 2p(1-p)\mu_B\mu_A\right)$$

$$= p\mathbb{E}\left(Y_{Ai}^2\right) + (1-p)\mathbb{E}\left(Y_{Bi}^2\right) - 2\mu_A\mu_B\left(\mathbb{E}\left(X_i\right) - \mathbb{E}(X_i^2)\right) - \left(p^2\mu_A^2 + (1-p)^2\mu_B^2 + 2p(1-p)\mu_B\mu_A\right)$$

$$= p\left(\sigma^2 + \mu_A^2\right) + (1-p)\left(\sigma^2 + \mu_B^2\right) - \left(p^2\mu_A^2 + (1-p)^2\mu_B^2 + 2p(1-p)\mu_B\mu_A\right)$$

$$= \sigma^2 + p\mu_A^2 + (1-p)\mu_B^2 - p^2\mu_A^2 - (1-p)^2\mu_B^2 - 2p(1-p)\mu_B\mu_A$$

$$= \sigma^2 + \left(p - p^2\right)\mu_A^2 + (p-p^2)\mu_B^2 - 2(p-p^2)\mu_B\mu_A$$

$$= \sigma^2 + \left(p - p^2\right)\left(\mu_A - \mu_B\right)^2$$

(d) Write the method of moments estimators for $\mathbb{E}(Y_i)$ and for $\mathrm{Var}(Y_i)$. If μ_A, μ_B and σ^2 are unknown, can we use these two estimators to provide a method of moments estimator for p? Explain.

No. We need at least three matching-moment equations since we would have to get estimates of μ_A and μ_B first.

(e) Assume we obtain an independent, unbiased and consistent estimates of μ_A and σ^2, $\hat{\mu}_A$ and $\hat{\sigma}^2$, from a random sample of vehicles with catalytic converters. Write the method of moments estimators for μ_B and p given this new information.

Denote the estimators for μ_A and σ^2 as $\hat{\mu}_A$ and $\hat{\sigma}^2$ and the sample size they are based on as m. The MM estimator for μ_B and p solves the following system of equations:

$$\bar{y} = p(\hat{\mu}_A) + (1-p)\mu_B$$

$$S_Y^2 = \hat{\sigma}^2 + (\hat{\mu}_A - \mu_B)^2(p - p^2)$$

(f) Is the estimator for p unbiased?

No. $p^{MM}$is a non-linear function of unbiased estimates, and it is therefore biased.

Bernoulli

$P(X=x|p) = p^x(1-p)^{(1-x)}$; $x = 0, 1$; $0 \le p \le 1$

mgf $M_X(t) = (1-p) + pe^t$

Binomial

$P(X=x|n,p) = \left(\begin{array}{c} n \\ \\ x \end{array}\right) p^x(1-p)^{(n-x)}$; $x = 0, 1, 2, ..., n$; $0 \le p \le 1$

mgf $M_X(t) = [pe^t + (1-p)]^n$

Discrete uniform

$P(X=x|N) = \frac{1}{N}$; $x = 1, 2, ..., N$; $N = 1, 2, ...$

mgf $M_X(t) = \frac{1}{N}\sum_{i=1}^{N} e^{it}$

Poisson

$P(X=x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, ...$; $0 \le \lambda < \infty$

mgf $M_X(t) = e^{\lambda(e^t-1)}$

Uniform

$f(x|a,b) = \frac{1}{b-a}$; $x \in [a,b]$

mgf $M_X(t) = \frac{e^{bt}-e^{\alpha t}}{(b-a)t}$

Exponential

$f(x|\beta) = \lambda e^{-\lambda x}$; $0 \le x < \infty$, $\lambda > 0$

mgf $M_X(t) = \frac{1}{1-\beta t}$, $t < \frac{1}{\beta}$

Weibull

$f(x|\gamma,\beta) = \frac{\gamma}{\beta}x^{\gamma-1}e^{-x^\gamma/\beta}$; $0 < x < \infty$, $\gamma > 0$, $\beta > 0$

mgf (Only exists for $\gamma \ge 1$. Its form is not very useful.)

Normal

$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$

mgf $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$