Chapter 60

QUANTAL RESPONSE EQUILIBRIA: A BRIEF SYNOPSIS

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1. Introduction

The quantal response equilibrium (QRE) is an extension of the standard model of Nash equilibrium which allows for errors in choice. The QRE can be viewed as a generalization of statistical models of discrete choice behavior of individuals to a game theoretic setting. Just like in discrete choice models of individual behavior, individuals are assumed to have an observed and unobserved component of their utility function. However, unlike in models of individual choice, the unobserved part of an individual's utility is unobserved not just to the econometrician, but also to the other players in the game. Thus, the game becomes a game of incomplete information. A QRE is simply a Bayesian equilibrium to this game.

In a QRE, individuals do not choose the strategy with the highest (observed) utility for sure, but rather choose it with a probability that is a function of the utility difference between that strategy and other strategies. Specifically, given a set of alternative choices, individuals choose probabilistically, choosing better alternatives more often than worse alternatives. The Nash equilibrium model corresponds to an extreme special case of this model, in which the probability of choosing an optimal alternative is equal to one and the probability of making a suboptimal choice is zero.

2. The Model

We begin by defining a game in its normal form in the standard way. Let $I = \{1, \dots, n\}$ be the set of *players*. For each $i \in I$ there is a strategy set A_i, which we assume to be finite, with J_i elements. Each player has a payoff function $u_i : A \to \mathcal{R}$, where $A = \prod_{i \in I} A_i$. Let S_i be the set of probability distributions over A_i and an element $s_i \in S_i$ is a *mixed strategy*. Given a strategy profile $s \in S = \prod_{i \in I} S_i$ player i's expected payoff is $v_i(s) = \sum_{a \in A} p(a) u_i(a)$, where $p(a) = \prod_{i \in I} s_i(a_i)$. A (mixed) strategy profile $s \in S$ is a *Nash equilibrium* if, for all $i \in I$ and for all $t_i \in S_i$, $v_i(s) \geqslant v_i(t_i, s_{-i})$.

For each i and each $j \in \{1, \dots, J_i\}$, and for any $s \in S$, denote by $v_{ij}(s)$ the expected utility to i of adopting the pure strategy a_{ij} when the other players use s_{-i}. For a quantal response equilibrium, it is assumed that for each pure strategy a_{ij}, player i receives an additional privately observed disturbance to their payoff, ε_{ij}. Thus i's payoff from

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adopting strategy a_{ij} under the strategy profile s is not $v_{ij}(s)$ but instead:

$$\hat{v}_{ij} = v_{ij}(s) + \varepsilon_{ij}.$$

Player i's profile of payoff disturbances, $\varepsilon_i = (\varepsilon_{i1}, \dots, \varepsilon_{iJ_i})$, is distributed according to a joint distribution with density function $f_i(\varepsilon_i)$. Assume that the marginal distribution of f_i exists for each ε_{ij} and $E(\varepsilon_i) = 0$. McKelvey and Palfrey (1995) call $f = (f_1, \dots, f_n)$ *admissible* if f_i satisfies the above properties for all i. The assumed choice behavior is that each player chooses strategy a_{ij} such that $\hat{v}_{ij} \geqslant \hat{v}_{ik}$ $\forall k = 1, \dots, J_i$. Given this choice behavior, v and f together induce a distribution over the actual choices by each player. To be more specific, for any v, define $B_{ij(v)}$ as the set of realizations of ε_i such that strategy a_{ij} has the highest disturbed expected payoff. So

$$P_{ij}(v) = \int_{B_{ij}(v)} f(\varepsilon)\, \mathrm{d}\varepsilon$$

is the induced probability that player i will select strategy j given v. Since $P(v) \in S$ and $v = v(s)$ is defined for any $s \in S$, $P \circ v(s) = P(v(s))$ defines a mapping from S into itself. Any fixed point s^* such that $s^* = P(v(s^*))$ is called a *quantal response equilibrium* of the game (I, A, u).

3. Properties of the QRE

The above definitions for normal form games can be extended to extensive form games. In the case of extensive form games, each for each information set of a player, there is assumed to be an independent payoff disturbance for each action. In the case where the payoff disturbance is known only to the "agent" in charge of each information set, the QRE can be defined in a similar fashion to the definitions for the normal form game, where instead one uses the agent normal form (see McKelvey and Palfrey, 1998). McKelvey and Palfrey (1995, 1998) prove the following results for the QRE and it's modification for extensive form games, the agent quantal response equilibrium (AQRE):

THEOREM 1. *For any admissible f, a QRE (AQRE) exists.*

In the remainder of the paper, we focus mainly on a specialized version of the model, where each ε_{ij} is independently and identically distributed according to the type I extreme value (or log Weibull) distribution with cumulative density $F(\varepsilon_{ij}) = e^{-e^{-\lambda \varepsilon_{ij}}}$.

This distribution of the disturbances leads to choice probabilities following a multinomial logit distribution (see, e.g., McFadden, 1975). We call the resulting equilibrium a logit QRE (AQRE), or just a logit equilibrium.

THEOREM 2. *For every finite normal (extensive) form game, every limit point of a sequence of logit QREs (AQREs) with λ going to infinity corresponds to the strategy of a Nash (sequential) equilibrium of the game.*

THEOREM 3. *For almost all finite normal (extensive) form games*:

1. *The logit QRE (AQRE) correspondence, $\mathcal{Q}$ is a one-dimensional manifold.*
2. *For almost all λ there are an odd number of logit QRE (AQRE).*
3. *There is a unique branch, $\mathcal{B}$ (the principal branch) of the logit QRE (AQRE) selection connected to the centroid of the game at $\lambda = 0$.*
4. *The principal branch of the logit QRE (AQRE) correspondence selects a unique Nash equilibrium (sequential equilibrium component) of the game as $\lambda \to \infty$.*

4. Fit to Experimental Data

We illustrate with three examples some features of experimental data that the QRE can help explain. These examples are presented in more detail in McKelvey and Palfrey (1995, 1998).

4.1. Learning to Play Nash Over Time

In some experiments, individuals take time to learn how to play the Nash equilibrium. Lieberman (1960) conducted experiments on the following two person zero sum game (the payoffs are to the row player):

	B_1	B_2	B_3
A_1	15	0	−2
A_2	0	−15	−1
A_3	1	2	0

The fit of the QRE to the data from this experiment illustrates a pattern of learning that is seen in some experiments. In Figure 1, the QRE correspondence is shown along with the aggregate data, broken down by time period. In this figure, the curved line represents the logistic QRE correspondence. For low values of λ (which is the precision of the error term) the QRE starts at the centroid, and then as λ increases, the QRE approaches a Nash equilibrium. The learning is be captured by the QRE through the estimation of decreasing variance (increasing precision) of the payoff disturbances as subjects become more experienced. Also, in this case, the QRE captures the fact that subjects learn fairly quickly to eliminate the dominated strategy, but more slowly to eliminate the iterated dominated strategy.

4.2. Systematic Bias Away from the Nash Equilibrium

In games with mixed strategy equilibria, the subjects are indifferent between their strategies, and hence there is no reason, in equilibrium, for them to adopt the pure strategies with the correct probabilities. In such experiments, one frequently sees systematic deviations away from the Nash predictions.

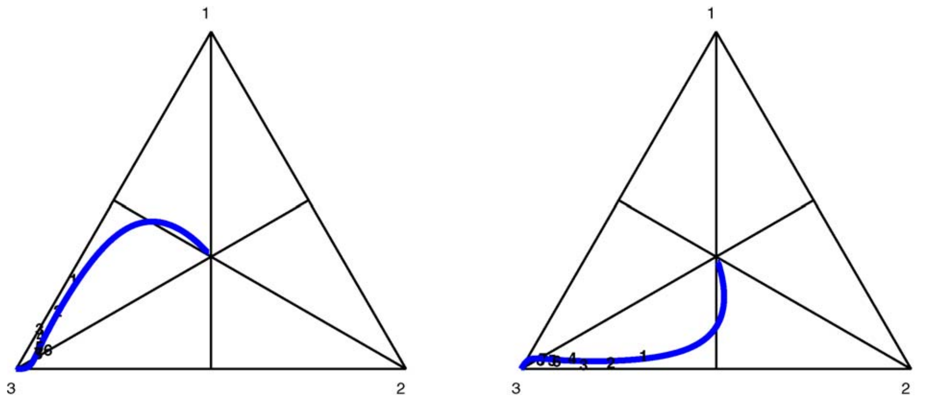

Figure 1. QRE as a function of λ for Lieberman experiment.

O'Neill (1987) conducted experiments on the following two person zero sum normal form game:

	B_1	B_2	B_3	B_4
A_1	5	−5	−5	−5
A_2	−5	−5	5	5
A_3	−5	5	−5	5
A_4	−5	5	5	−5

The data from this experiment illustrate the systematic bias away from equilibrium that is predicted by the QRE. In these experiments, the row players underplayed the first strategy, while the column players overplayed their first strategy. The QRE correspondence for this game is plotted in Figure 2. Here, the value of λ, which is the precision of the error, is on the horizontal axis, and the corresponding probabilities of selected strategies is on the vertical axis. The data is broken down by time periods, and the aggregate data for each time period is plotted at the value of λ that maximizes the likelihood function of the data. Each time period thus corresponds to four points on the graph, one for each of the selected probabilities.

As is evident from the figure, the QRE predicts that when there is error, player 1 will systematically (i.e., for all values of λ), underplay his first strategy, while player 2 will systematically overplay her first strategy except for very low values of λ. This clearly is borne out in data of Figure 2.

4.3. Nash Equilibrium Selection

A third feature of the QRE, which follows from Theorem 3, is that it can be used to select a unique Nash, or sequential equilibrium in a game. This is illustrated by an

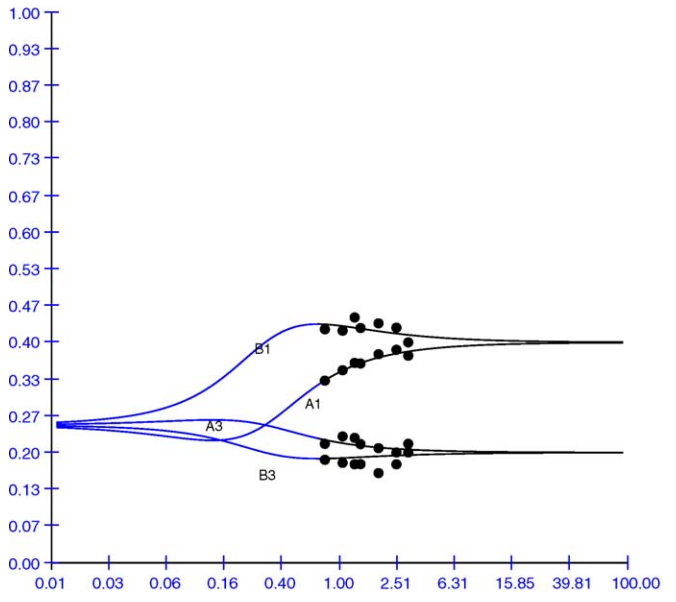

Figure 2. QRE as a function of λ for O'Neill experiment.

Table 1

BH #3 (sequential vs intuitive)							
Sequential (S, D, C)				Intuitive (I, C, D)			
$m = I$	C	D^n	E	$m = S$	C	D	E
A	45, 30	15, 0	30, 15	A	30, 90	0, 15	45, 15
B	30, 30	0, 45	30, 15	B	45, 0	15, 30	30, 15
BH #4 (sequential vs intuitive)							
Sequential (S, D, C)				Intuitive (I, C, D)			
$m = I$	C	D^n	E	$m = S$	C	D	E
A	30, 30	0, 0	50, 35	A	45, 90	15, 15	100, 30
B	30, 30	30, 45	30, 0	B	45, 0	0, 30	0, 15

experiment of Brandts and Holt (1993) who ran a pair of signaling experiments each of which contained a sequential and intuitive equilibrium.

The intuitive equilibrium is a more refined equilibrium concept than sequential equilibrium. Signaling games studied by Banks, Camerer, and Porter (1994) suggested that subjects go to the more refined equilibrium. The objective of Brandts and Holt was to

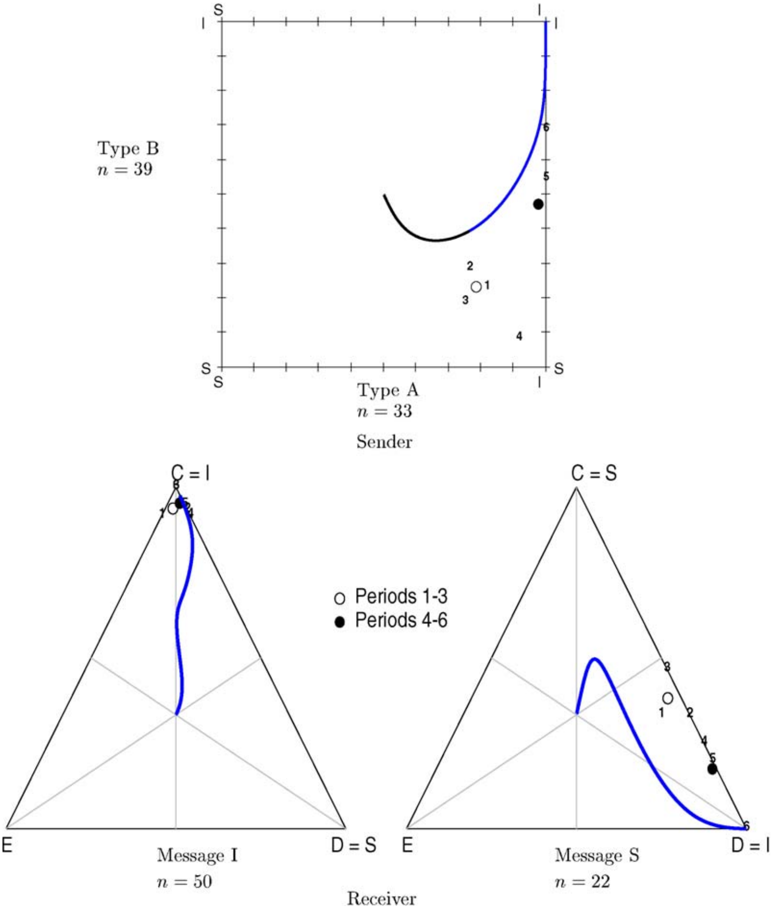

Figure 3. Brandts and Holt (1993) game 3. Sequential vs intuitive.

see if they could take a game with two equilibria, one sequential, and one intuitive, and by changing the payoffs (without affecting the properties of the equilibria), get subjects to choose a different equilibrium in the different games. In particular, could a game be designed where the subjects would choose a less refined (sequential) equilibrium over a more refined (intuitive) one.

The games that Brandts and Holt studied are given in Table 1. Each has a sequential and an intuitive equilibrium. The experiment was successful in achieving its objective – in the first game, subjects tended to select the intuitive equilibrium. and in the second, subjects tended to select the sequential equilibrium. However, standard refinement theory can not be used in general to predict which equilibrium would be selected. The AQRE offers an explanation of this failure. Figures 3 and 4 illustrate that the principal

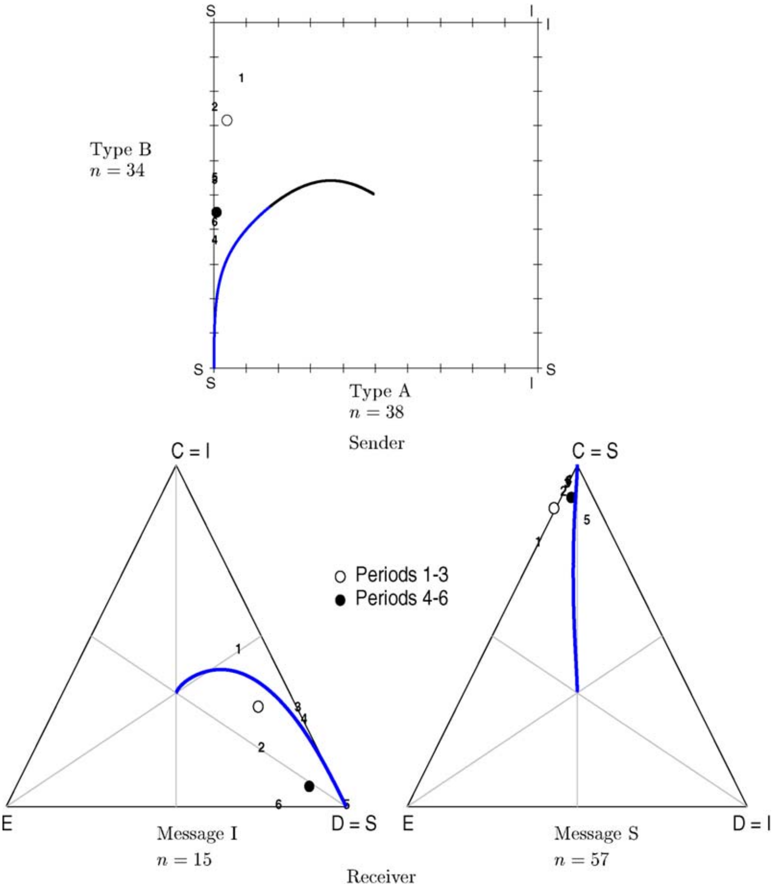

Figure 4. Brandts and Holt (1993) game 4. Sequential vs intuitive.

branch of the AQRE selects the non-intuitive sequential equilibrium in the first game, and the intuitive equilibrium in the second, and that it compares favorably with the experimental data.

Acknowledgement

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