Better Explanation of Sum and Difference of Uniforms

$$0 \le u \le 2$$
$$-1 \le v \le 1$$

$$f_{u,v}(u,v) = \begin{cases} \frac{1}{2}, 0 \le u \le 2, -1 \le v \le 1 \\ 0, otherwise \end{cases}$$


```
function testJointDensity
%{
Numerically test the joint density of the sum and difference
of independent random variables.

Econ 241A
Dick Startz
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%}

rng('default');
nSample = 1000000;
x = rand(nSample,1);
y = rand(nSample,1);

u = x+y;
v = x-y;

histogram2(u,v,'normalization','pdf');
xlabel('u=x+y');
ylabel('v=x-y');
print -dpng testJointDensity
title('Joint density of sum and difference of independent uniform');
end
```
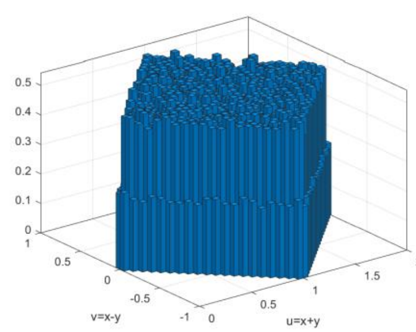




Joint density

$$0 \le u \le 2$$
$$-1 \le v \le 1$$
$$f_{x,y}\left(h_x(u,v), h_y(u,v)\right)$$
$$= \begin{cases} 1\,, 0 \le u \le 2, -1 \le v \le 1 \\ 0, otherwise \end{cases}$$
$$f_{u,v}(u,v) = \begin{cases} 1 \times \left|-\frac{1}{2}\right|, 0 \le u \le 2, -1 \le v \le 1 \\ 0, otherwise \end{cases}$$

Marginal distribution of the sum

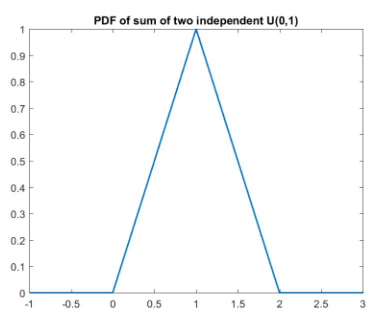
$$f_U = \int_{-\infty}^{\infty} f_{U,V}(u,v)dv$$

Suppose $0 \le u \le 1$, then $-u \le v \le u$.

$$f_U = \int_{-u}^{u} \frac{1}{2} dv = \frac{1}{2}[v]_{-u}^{u} = u$$

Suppose $1 \le u \le 2$, then $-(2-u) \le v \le (2-u)$

$$f_U = \int_{-(2-u)}^{(2-u)} \frac{1}{2} dv = \frac{1}{2}[v]_{-(2-u)}^{(2-u)} = 2-u$$



Marginal distribution of the difference

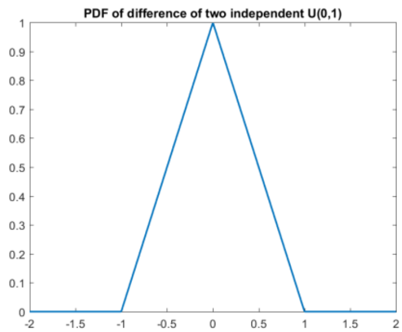
$$f_V = \int_{-\infty}^{\infty} f_{U,V}(u,v)du$$

If $-1 < v < 0$, then let's look at $-2v < u < 2$

$$f_V(v) = \left[\frac{1}{2}u\right]_{-2v}^{2} = 1 + v$$

If $0 < v < 1$, then let's look at $2v < u < 2$

$$f_V(v) = \left[\frac{1}{2}u\right]_{2v}^{2} = 1 - v$$



Independence of sum and difference of independent uniforms

$$f_{u,v}(u,v) = \begin{cases} \frac{1}{2}, 0 \leq u \leq 2, -1 \leq v \leq 1 \\ 0, otherwise \end{cases}$$

If $0 \leq u \leq 1$, $f_U = u$, If $1 \leq u \leq 2$, $f_U = 2 - u$

If $-1 < v < 0$, $f_V(v) = 1 + v$, if $0 < v < 1$, $f_V(v) = 1 - v$

Now consider

$$f_{u,v}(1,1) = \frac{1}{2} \neq f_u(1) \times f_v(1) = 1 \times 0 = 0$$

Therefore u, v are not independent.

