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- *Reduction of Compound lotteries:* We assume that preferences over a compound lottery are equivalent to those over the "reduced" lottery formed by the compound lottery, e.g., if $L_k(1)$ is the probability that lottery k assigns to prize 1, then the probability of the first prize in the reduced lottery is $p_1 = \sum_{k=1}^{K} \alpha_k L_k(1)$.

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We write $\alpha L_1 + (1 - \alpha)L_2$ to be the compound lottery that gives L_1 with probability α and L_2 with complementary probability. [NB Rubinstein and some other authors use the symbol $\oplus$, e.g., $\alpha L_1 \oplus (1 - \alpha)L_2$, for such "probabilistic mixtures".]

Preferences over Lotteries

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Assumption (Continuity)

For every $L, L', L'' \in \Delta(Z)$, such that $L' \succ L$ there exist $\overline{\alpha}, \overline{\beta} \in (0,1)$ such that

$$\alpha L' + (1-\alpha)\, L'' \succ L$$

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Assumption (Continuity*)

For every $L, L', L'' \in \Delta(Z)$, such that $L' \succ L \succ L''$ there exist $\gamma \in (0,1)$ such that

$$L \sim \gamma L' + (1-\gamma) L''$$

Independence Axiom

Assumption (Independence Axiom)

For every $L, L', L'' \in \Delta(Z)$ *and* $\alpha \in (0,1]$, $L \succsim L'$ *iff*

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Someone who prefers lottery L to L' prefers a fixed probabilistic mixture of L and L'' to the same probabilistic mixture of L' and L''

Expected Utility

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Adding the Independence Axiom puts more structure on preferences.

Definition

The utility function $U : \Delta(Z) \rightarrow \mathbb{R}$ is a *von Neumann Morgenstern (vNM) expected-utility function* if there exists a vector of utilities for the prizes $(u_1, ..., u_N)$ such that for every simple lottery $L = (p_1, \ldots, p_N) \in \Delta(Z)$,

$$U(L) = p_1 u_1 + \ldots + p_N u_N.$$

The vNM utility of a lottery equals the *expected utility* of the outcomes. vNM preferences are *linear* in the prize probabilities.

Expected-Utility Theorem

This gives us the basis for our representation theorem.

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Proposition (Expected-Utility Theorem)

Preferences satisfy rationality, continuity and the Independence Axiom iff they have a von Neumann Morgenstern (vNM) expected utility representation.

Risk Aversion

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Let's take the prize space to comprise money payments, and for simplicity $Z = \mathbb{R}$.

Proposition

A vNM DM is risk averse iff her Bernoulli utility function u is concave.

A Calibration Argument (Rabin 2000)

Assume

1. People are risk-averse EU maximisers
2. People reject small-scale gambles that real people appear to reject

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1. People are risk-averse EU maximisers
2. People reject small-scale gambles that real people appear to reject

Show that this implies implausible preferences over large-scale risk. Conclude that EU theory does not provide a satisfactory, integrated account of both small- and large-scale risk attitudes.

Suppose an expected-utility (EU) maximiser with strictly increasing concave Bernoulli utility function over money $u(.)$ rejects an additive 50-50 gain $\$1$ lose $\$\mathcal{L} < \$1$ bet from any starting wealth.

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Rejecting 50-50 gain \$1 lose $\$\mathcal{L}$ bet at w then implies that $u(w+1) - u(w) \leq 1$

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$$\begin{aligned}
\Rightarrow u(w+2) - u(w) &= u(w+2) - u(w+1) + u(w+1) - u(w) \\
&\leq \mathcal{L} + 1.
\end{aligned}$$

Iterating,

$$u(w+k) - u(w) \leq \sum_{j=0}^{k-1} \mathcal{L}^j$$

Hence for any x,

$$u(x) \leq u(w) + \sum_{j=0}^{\infty} \mathcal{L}^j = u(w) + \frac{1}{1-\mathcal{L}}.$$

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Suppose $\mathcal{L} = 0.95$. Then $u(x) \leq u(w) + \frac{1}{1-0.95} = u(w) + 20$.

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For $\mathcal{L} = 0.95$, our bettor then rejects a 50-50 lose $\$20$ win $\$\infty$ bet! Such behaviour is crazy and implausible.

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Rabin (2000) makes this and stronger statements.

Alternatively, read Rabin and Thaler in *Journal of Economic Perspectives* (2001)

Concave Utility of Wealth Produces Second-Order Risk Aversion

- Concave utility of wealth is a theory of "second-order" risk aversion that derives from curvature of the utility function—it cannot explain "first-order" risk aversion, namely that over small stakes for plausible degrees of risk aversion.
- The intuition is simply that the even the most concave function is approximately linear in a small neighbourhood (that's what it means to be differentiable) and hence approximately risk neutral in that neighbourhood.
- When we estimate people's degree of risk aversion using small stakes, they appear to be much more risk averse than when we do it using large stakes.
- No concave utility of wealth function can fit people's risk attitudes towards both small and large stakes.

Evidence of First-Order Risk Aversion

Sydnor (2008) looks at American homeowner house insurance

- Typical homeowner chooses to pay \$100 to reduce deductible from \$1000 to \$500
- Chance of event where such coverage pays out is less than 5%, i.e., actuarially fair price of extra insurance is less than \$25

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Someone who declined the insurance and saved the \$100 premium to self insure would be on average \$2000 richer after 30 years, with only 1.6% chance of ending poorer.

Loss Aversion (Kahneman and Tversky 1979)

Kahneman and Tversky propose the reference-dependent Bernoulli utility function

$$u(r, z) = w(r) + v(z - r)$$

Bernoulli utility defined over *reference point* r and money outcome z: it is *reference utility* (w) plus a *value function* (v) evaluating gain or loss from the reference point that satisfies four properties:

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$w(r)$ is strictly increasing in r

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Losses loom larger than same-sized gains—let's see how Properties 1-4 get this

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Also we can write

$$v(y) - v(x) < v(-x) - v(-y),$$

and so

$$\frac{v(y) - v(x)}{y - x} < \frac{v(-x) - v(-y)}{y - x}.$$

Letting $y \to x$ gives $v'(x) \leq v'(-y) = v'(-x)$.

The value function is always steeper in the losses domain.

Property 2 implies:

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This does not imply that v not differentiable (has a kink) at 0, for which we need Property 4.

The Kink at the Reference Point

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because $\alpha > 1$ and $x < y$.

$v'(0^+) = 1 < v'(0^-) = \alpha$, satisfying Property 4.

The S-shape

Finally, piecewise-linear v is concave in gains and convex in losses (since linear functions are both convex and concave).

Note though that it is not just concave in the gains domain but concave overall, as the function f is concave if for any x, y and any $\beta \in [0, 1]$,

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Someone with such a v is risk averse.

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For $t \in (0, 1)$, the lottery $(tx, p; ty, 1-p)$ scales $(x, p; y, 1-p)$ by the factor t: e.g., rather than get x with probability p, the bettor now gets tx with probability p

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For $t \in (0, 1)$, the lottery $(tx, p; ty, 1-p)$ scales $(x, p; y, 1-p)$ by the factor t: e.g., rather than get x with probability p, the bettor now gets tx with probability p

If $(x, p; y, 1-p)$ is better than fair, then a risk-averse DM with concave continuously differentiable Bernoulli utility function prefers $(tx, p; ty, 1-p)$ to [0] for positive t small enough. Intuitively, for t small, the range of outcomes in lottery shrinks to zero, as $ty - tx \to 0$. Over small stakes, the agent is approximately risk neutral and will therefore accept a better-than-fair lottery.

What Does the Kink Do?

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This suggests how loss aversion can explain our puzzle: people are "first-order risk averse" around their reference points without being too globally risk averse.

Risk Lovingness over Losses

Prop 3 says risk-loving over losses.

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From K&T 1979

4000, .8	3000	$-4000, .8$	-3000
20	80	92	8
4000, .2	3000, .25	$-4000, .2$	$-3000, .25$
65	35	42	52
3000, .9	6000, .45	$-3000, .9$	$-6000, .45$
86	14	8	92
3000, .002	6000, .001	$-3000, .002$	$-6000, .001$
27	73	70	30