Required Problems

1. Let $f : D \to R$ and $g : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ and $R \subset \mathbb{R}$, be concave functions. Let $h : R \to \mathbb{R}$ be an increasing function. Show that each of the following propositions is true:

   (a) $f(\mathbf{x}) + g(\mathbf{x})$ is a concave function.

   (b) $f(\mathbf{x})$ is a quasiconcave function.

   (c) $(h \circ f)(\mathbf{x})$ is a quasiconcave function.

2. Find the extreme values of each of the following functions, then use the second-order conditions to determine whether they are maxima or minima.

   (a) $f(x, y) = x^2 + xy + 2y^2 + 3$

   (b) $g(x, y) = -x^2 - y^2 + 6x + 2y$

3. Solve the following constrained utility-maximization problem.

$$\max_{x_1, x_2} x_1^{\alpha} x_2^{\beta} \quad \text{s.t.} \quad M = p_1 x_1 + p_2 x_2$$

4. Solve the following constrained utility-maximization problem.

$$\max_{x_1, x_2} \alpha \ln(x_1) + x_2 \quad \text{s.t.} \quad M \geq p_1 x_1 + p_2 x_2$$

Practice Problems

5. Which of the following functions on $\mathbb{R}^n$ are concave or convex? Use the 2nd derivative test or the definiteness of the Hessian (for univariate and multivariate functions, respectively) to determine concavity/convexity.

   (a) $f(x) = 3e^x + 5x^4 - \ln(x)$

   (b) $g(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$

   (c) $h(x, y, z) = 3e^x + 5y^4 - \ln(z)$

6. Determine whether or not the following functions are quasiconcave, quasiconvex, or neither on $\mathbb{R}^2_+$.

   (a) $f(x) = e^x$

   (b) $g(x) = x^3 - x$

   (c) $h(x, y) = ye^{-x}$

   (d) $j(x, y) = (2x - 3y)^3$

7. Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?

   (a) $f(x, y) = 3x^5 y + 2x^2 y^4 - 3x^3 y^3$

   (b) $g(x, y) = 3x^5 y + 2x^2 y^4 - 3x^3 y^4$

   (c) $h(x, y) = x^{1/2} y^{-1/2} + 3xy^{-1} + 7$

   (d) $j(x, y) = x^{3/4} y^{1/4} + 6x + 4$

8. Which of the following functions are homothetic? Give a reason for each answer.
   (a) $f(x, y) = e^{x^2 y} e^{xy^2}$
   (b) $g(x, y) = x^3y^6 + 3x^2y^4 + 6xy^2 + 9$
   (c) $h(x, y) = 2\ln(x) + 3\ln(y)$

9. Consider the function
$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$
   (a) Show that $f(x_1, x_2)$ is homogeneous of degree 1
   (b) Verify that Euler's Theorem holds for this function.

10. Let $f(\mathbf{x})$ be a convex function. Prove that $f(\mathbf{x})$ reaches a local minimum at $\mathbf{x}^*$ if and only if $f(\mathbf{x}^*)$ reaches a global minimum at $\mathbf{x}^*$.

11. Find the local extreme values and classify the points as maxima, minima, or neither
   (a) $f(x_1, x_2) = 2x_1 - x_1^2 - x_2^2$
   (b) $g(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_2$
   (c) $h(x_1, x_2) = x_1^3 - x_2^2 + 2x_2$

12. Solve the following constrained optimization problems.
   (a) $\min_{\mathbf{x}} (x_1^2 + x_2^2) \quad \text{s.t.} \quad x_1x_2 = 1$
   (b) $\min_{\mathbf{x}} (x_1x_2) \quad \text{s.t.} \quad x_1^2 + x_2^2 = 1$
   (c) $\max_{\mathbf{x}} (x_1 + x_2) \quad \text{s.t.} \quad x_1^4 + x_2^4 = 1$

13. State the Kuhn-Tucker theorem for the following minimization problem:
$$\min_{x_1, x_2} f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) \leq 0 \text{ and } x_1 \geq 0, x_2 \geq 0$$

14. Consider the following maximization problem:
$$\max_{x,y} xy \quad \text{s.t.} \quad x + y \leq 100 \quad \text{and} \quad x, y \geq 0$$
State the Kuhn-Tucker first order conditions and solve the maximiation problem.

15. Suppose a consumer livers on an island where he produces two goods, x and y, according to the production possibility frontier $x^2 + y^2 \leq 200$, and he consumes all goods himself. His utility function is
$$u(x, y) = xy^3$$
The consumer also faces an environment constraint on his total output of booth goods, given by $x + y \leq 20$.
   (a) Write out the Kuhn-Tucker first-order conditions.
   (b) Find the consumer's optimal x and y. Identify which constraints are binding.