Chapter 2 Transformations and Expectations

October 30, 2017

- **Theorem 2.1.3 - Page 51**
  Let X have cdf $F_X(x)$, let $Y = g(X)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be defined as the support sets as in (2.17), then
  - If g is an increasing function on $\mathcal{X}$ then $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$
  - If g is a decreasing function on $\mathcal{X}$ and X is a continuous random variable then $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
- **Theorem 2.1.5 - Page 51**
  Let X have pdf $f_X(x)$ and let $Y = g(X)$, where g is a monotone function. Let $\mathcal{X}$ and $\mathcal{Y}$ be defined by (2.1.7). Suppose that $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

- **Inverted gamma pdf - Example 2.1.6 - page 51**
- **Theorem 2.1.8 - page 53**
  Let X have pdf $f_X(x)$, let $Y = g(X)$, and define the sample space $\mathcal{X}$ as in (2.1.7). Suppose there exists a partition, $A_0, A_1, ..., A_k$ of $\mathcal{X}$ such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further, suppose there exists functions $g_1(x), ..., g_k(x)$, defined on $A_1, ..., A_k$, respectively, satisfying:
  - $g(x) = g_i(x)$, for $x \in A_i$
  - $g_i(x)$ is monotone on A_i
  - the set $\mathcal{Y} = \{y : y = g_i(x) \text{ for some } x \in A_i\}$
  - $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$, for each $i = 1, \ldots, k$

  Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & \text{if } y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

- **Theorem 2.1.10 (Probability integral transformation) - page 54**
  Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(x)$. Then Y is uniformly distributed on (0,1), that is, $P(Y \leq y) = y$, where $0 < y < 1$.
- **Definition of Expected Value of a Random Variable - page 55**
- **Cauchy mean - page 56**
- **Theorem 2.2.5 - page 57**
  Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist:

a $E[ag_1(X) + bg_2(X) + c] = aE[g_1(X)] + bE[g_2(X)] + c$
b If $g_1(x) \geq 0$ for all x, then $E[g_1(X)] \geq 0$
c If $g_1(x) \geq g_2(x)$ for all x, then $E[g_1(X)] \geq E[g_2(X)]$
d If $a \leq g_1(x) \leq b$ for all x, then $a \leq E[g_1(X)] \leq b$

- **$E[X] = b$ minimizes the distance $(X - b)^2$ (univariate) - page 58 Example 2.2.6**
- **Definition 2.3.1 (Moments) - page 59**
- **Theorem 2.3.4 - page 60**
  If X is a random variable with finite variance, then for any constants a and b,
  $$Var(aX + b) = a^2 Var(X)$$
- **Definition 2.3.6 (Moment Generating Function) page 62**
  Let X be a random variable with cdf F_X. The mgf of X, denoted by $M_X(t)$, is
  $$M_X(t) = E[e^{tX}]$$
- **Theorem 2.3.7 - page 62**
  If X has mgf $M_X(t)$, then
  $$E[X^n] = M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}$$
  That is, the nth moment is equal to the nth derivative of $M_X(t)$ evaluated at $t = 0$.
- **Theorem 2.3.11 - page 65**
  Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

  a If X and Y have bounded support, then $F_X(u) = F_Y(u)$ for all u if and only if $E[X^r] = E[Y^r]$ for all integers $r = 0, 1, 2, ...$
  b If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.
- **Theorem 2.3.12 (Convergence of mgfs) - page 66**
  Suppose $X_i, i = 1, 2, ...$ is a sequence of random variables, each with mgf $M_X(t)$. Furthermore, suppose that
  $$\lim_{i \to \infty} M_{X_i}(t) = M_X(t)$$
  for all t in a neighborhood of 0 and $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have
  $$\lim_{i \to \infty} F_{X_i}(x) = F_X(x)$$
  That is, *convergence for $|t| < h$, of mgfs to an mgf implies convergence of cdfs.*
- **Theorem 2.3.15 - page 67**
  For any constants a and b, the mgf of the random variable $aX + b$ is given by
  $$M_{aX+b}(t) = e^{bt} M_X(at)$$
- **Section 2.4 Differentiating Under an Integral Sign - page 68**
  1. Theorem 2.4.1 (Leibnitz's Rule) - page 69
  2. Theorem 2.4.2 - page 69: Limits under an integral.
  3. Theorem 2.4.3 - page 70: Order of operations when evaluating a differentiation of an integral.
  4. Theorem 2.4.8 - page 74: Differentiating under an infinite sum.
  5. Theorem 2.4.10 - page 75: Integrating an infinite sum.