**Required Problems**

1. **Let $f : D \to R$ and $g : D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ and $R \subset \mathbb{R}$, be concave functions. Let $h : R \to \mathbb{R}$ be an increasing function. Show that each of the following propositions is true:**

   (a) $f(\mathbf{x}) + g(\mathbf{x})$ **is a concave function.**

   This proposition is equivalent to the theorem, "the sum of two concave functions is a concave function." Consider the following for $\mathbf{x}, \mathbf{y} \in D$ and $t \in [0, 1]$:

   $$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \quad \text{(by def. of concave)}$$

   $$g(t\mathbf{x} + (1-t)\mathbf{y}) \geq tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \quad \text{(by def. of concave)}$$

   $$f(t\mathbf{x} + (1-t)\mathbf{y}) + g(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) + tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \quad \text{(adding the two)}$$

   $$f(t\mathbf{x} + (1-t)\mathbf{y}) + g(t\mathbf{x} + (1-t)\mathbf{y}) \geq t\big[f(\mathbf{x}) + g(\mathbf{x})\big] + (1-t)\big[f(\mathbf{y}) + g(\mathbf{y})\big] \quad \text{(rearranging)}$$

   Thus, the sum of two concave functions is also concave. Note that for problems that require us to "show" a proposition is true, we need to present the argument, but we don't need to be quite as rigorous as if the question instructed us to "prove" a proposition.

   (b) $f(\mathbf{x})$ **is a quasiconcave function.**

   This proposition is equivalent to "concavity implies quasiconcavity." Consider the following for $\mathbf{x}, \mathbf{y} \in D$ and $t \in [0, 1]$:

   $$\begin{aligned} &\text{WLOG, let } f(\mathbf{x}) \geq f(\mathbf{y}). && \text{(by assumption)} \\ &f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) && \text{(by def. of concave)} \\ &\geq tf(\mathbf{y}) + (1-t)\mathbf{y} && \text{(by } f(\mathbf{x}) \geq f(\mathbf{y})\text{)} \\ &= f(\mathbf{y}) && \text{(simplifying)} \\ &= \min\{f(\mathbf{x}), f(\mathbf{y})\} && \text{(by } f(\mathbf{x}) \geq f(\mathbf{y})\text{)} \end{aligned}$$

   Thus, if $f$ satisfies the definition of concavity, it must satisfy the definition of quasiconcavity as well. Note the use of "without loss of generality"—since $\mathbf{x}$ and $\mathbf{y}$ are arbitrary, we simply say that $\mathbf{x}$ maps to the larger of the elements in the range. Be cautious when using "WLOG," as it can be easy to lose generality if you're not careful.

   (c) $(h \circ f)(\mathbf{x})$ **is a quasiconcave function.**

   This proposition is equivalent to "a monotonic transformation of a concave function is quasiconcave." The composite function is quasiconcave if and only if

   $$(h \circ f)(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{(h \circ f)(\mathbf{x}), (h \circ f)(\mathbf{y})\}$$

   for $\mathbf{x}, \mathbf{y} \in D$ and $t \in [0, 1]$. From part (b), recall that we have the inequality:

   $$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\} \quad \text{(by part (b))}$$

   We can then apply the function $h$ to both sides:

   $$h\big(f(t\mathbf{x} + (1-t)\mathbf{y})\big) \geq h\big(\min\{f(\mathbf{x}), f(\mathbf{y})\}\big) \quad (h \text{ is increasing})$$

   Note we can rearrange the order in applying the min function and $h$, as increasing functions preserve orderings:

   $$\begin{aligned} &h\big(f(t\mathbf{x} + (1-t)\mathbf{y})\big) \geq \min\big\{h\big(f(\mathbf{x})\big), h\big(f(\mathbf{y})\big)\big\} && (h \text{ is increasing}) \\ &(h \circ f)(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{(h \circ f)(\mathbf{x}), (h \circ f)(\mathbf{y})\} && \text{(by def. of } h \circ f\text{)} \end{aligned}$$

   Thus, the definition of quasiconcavity is satisifed.

2. **Find the extreme values of each of the following functions, then use the second-order conditions to determine whether they are maxima or minima.**

(a) $f(x, y) = x^2 + xy + 2y^2 + 3$

The first-order conditions are:

$$\frac{\partial f(x,y)}{\partial x} = 2x + y = 0 \qquad \text{(w.r.t. } x\text{)}$$

$$\frac{\partial f(x,y)}{\partial y} = x + 4y = 0 \qquad \text{(w.r.t. } y\text{)}$$

Solving both conditions for $y$, we can set the two conditions equal to arrive at the equation:

$$2x = \frac{1}{4}x \qquad \text{(combining the FOCs)}$$

$$\boxed{x^* = 0} \qquad \text{(solving for } x\text{)}$$

$$\boxed{y^* = 0} \qquad \text{(solving for } y\text{)}$$

The second order conditions require the Hessian matrix:

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \qquad \text{(the Hessian)}$$

$$|H_1| = 2 \qquad \text{(the first LPM)}$$

$$|H_2| = 7 \qquad \text{(the second LPM)}$$

Both leading principle minors are strictly positive, implying that the Hessian is positive definite and the critical point $(0, 0)$ is a global minimum.

(b) $g(x, y) = -x^2 - y^2 + 6x + 2y$

The first-order conditions are:

$$\frac{\partial f(x,y)}{\partial x} = -2x + 6 = 0 \qquad \text{(w.r.t. } x\text{)}$$

$$\frac{\partial f(x,y)}{\partial x} = -2y + 2 = 0 \qquad \text{(w.r.t. } y\text{)}$$

Solving these for $x$ and $y$, respectively:

$$\boxed{x^* = 3} \qquad \text{(solving for } x\text{)}$$

$$\boxed{y^* = 1}$$

The second order conditions require the Hessian matrix:

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \text{(the Hessian)}$$

$$|H_1| = -2 \qquad \text{(the first LPM)}$$

$$|H_2| = 4 \qquad \text{(the second LPM)}$$

The leading principle minors alternate in sign, beginning with a negative; thus, the Hessian is negative definite and the critical point $(3, 1)$ is a global maximum.

3. **Solve the following constrained utility-maximization problem.**

$$\max_{x_1, x_2} \; x_1^{\alpha} x_2^{\beta} \quad \text{s.t.} \quad M = p_1 x_1 + p_2 x_2$$

**where $\alpha > 0$, $\beta > 0$, $M > 0$, $p_1 > 0$, and $p_2 > 0$.**

The Lagrangian associated with this problem is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{\beta} + \lambda\big[M - p_1 x_1 - p_2 x_2\big]$$

The first-order conditions are:

$$\mathcal{L}_x = \alpha\left(\frac{x_2^{\beta}}{x_1^{1-\alpha}}\right) - p_1\lambda = 0 \qquad \text{(w.r.t. } x\text{)}$$

$$\mathcal{L}_y = \beta\left(\frac{x_1^{\alpha}}{x_1^{1-\beta}}\right) - p_2\lambda = 0 \qquad \text{(w.r.t. } y\text{)}$$

$$\mathcal{L}_\lambda = M - p_1 x_2 - p_2 x_2 = 0 \qquad \text{(w.r.t. } \lambda\text{)}$$

Solving the first two FOCs for $\lambda$ and setting them equal:

$$\frac{\alpha}{p_1}\left(\frac{x_2^{\beta}}{x_1^{1-\alpha}}\right) = \frac{\beta}{p_2}\left(\frac{x_1^{\alpha}}{x_1^{1-\beta}}\right) \qquad \text{(combining the 1st to FOCs)}$$

$$x_2 = \left(\frac{\beta}{p_2}\right)\left(\frac{p_1}{\alpha}\right)x_1 \qquad \text{(solving for } x_2\text{)}$$

Plugging this into the third FOC (the constraint):

$$M = p_1 x_1 + p_2\left[\left(\frac{\beta}{p_2}\right)\left(\frac{p_1}{\alpha}\right)x_1\right] \qquad \text{(plugging in for } x_2\text{)}$$

$$\boxed{x_1^* = \frac{\alpha M}{(\alpha+\beta)p_1}} \qquad \text{(solving for } x_1\text{)}$$

$$x_2 = \left(\frac{\beta}{p_2}\right)\left(\frac{p_1}{\alpha}\right)\left[\frac{\alpha M}{(\alpha+\beta)p_1}\right] \qquad \text{(plugging in for } x_1\text{)}$$

$$\boxed{x_2^* = \frac{\beta M}{(\alpha+\beta)p_2}} \qquad \text{(simplifying)}$$

$$\boxed{\lambda^* = \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{M}{\alpha+\beta}\right)^{\alpha+\beta-1}} \qquad \text{(solving for } \lambda\text{)}$$

It is possible, although quite tedious, to verify the second order conditions:

$$|\bar{H}| = \begin{bmatrix} 0 & p_1 & p_2 \\ p_1 & -\alpha(1-\alpha)x_1^{\alpha-2}x_2^{\beta} & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} \\ p_2 & \alpha\beta x_1^{\alpha-1}x_2^{\beta-1} & -\beta(1-\beta)x_1^{\alpha}x_2^{\beta-2} \end{bmatrix} \qquad \text{(the bordered Hessian)}$$

The bordered Hessian, evaluated at the value $(x_1^*, x_2^*)$ needs to have leading principle minors that are zero, negative, then positive. This is a difficult task, given the general nature of the utility function. Instead, it suffices to break the problem into smaller parts and employ several theorems:

- $\alpha\ln(x_1)+\beta\ln(x_2)$ is a concave function:

$$H=\begin{bmatrix}-\frac{\alpha}{x_1^2} & 0\\ 0 & -\frac{\beta}{x_2^2}\end{bmatrix} \qquad |H_1|=-\frac{\alpha}{x_1^2}<0 \qquad |H_2|=\frac{\alpha\beta}{x_1^2x_2^2}>0$$

  The LPMS are negative then positive in sign, implying the function is concave.
- $e^x$ is an increasing function:

$$\frac{d}{dx}\Big[e^x\Big]=e^x>0$$

- We can re-write our objection function in terms of an increasing function of a concave function:

$$x_1^\alpha x_2^\beta=\exp\{\alpha\ln(x_1)+\beta\ln(x_2)\}$$

  By the result in problem 1, part (a), $x_1^\alpha x_2^\beta$ is quasiconcave.
- The budget set is convex (no need to prove it unless explicitely asked).

Thus, we have a quasiconcave objective function, constrained to a convex set; thus, we know the optimum we found is a global maximum.

4. **Solve the following constrained utility-maximization problem.**

$$\max_{x_1,x_2}\ \alpha\ln(x_1)+x_2 \quad \textbf{s.t.} \quad M\geq p_1x_1+p_2x_2 \ \textbf{ and } \ x_1\geq 0, x_2\geq 0$$

**where $\alpha>0$, $M>0$, $p_1>0$, and $p_2>0$.**

The Lagrangian associated with this problem is:

$$\mathcal{L}(x_1,x_2,\lambda)=\alpha\ln(x_1)+x_2+\lambda\big[M-p_1x_1-p_2x_2\big]$$

The Kuhn-Tucker first-order conditions are:

$$\begin{aligned}
&\mathcal{L}_1=\frac{\alpha}{x_1}-p_1\lambda\leq 0 && x_1\geq 0 && x_1\mathcal{L}_1=0\\
&\mathcal{L}_2=1-p_2\lambda\leq 0 && x_2\geq 0 && x_2\mathcal{L}_2=0\\
&\mathcal{L}_\lambda=M-p_1x_2-p_2x_2\geq 0 && \lambda\geq 0 && \lambda\mathcal{L}_\lambda=0
\end{aligned}$$

We don't have to go through all of the possible combinations of binding/non-binding constraints if we use a little economic intution:

- The utility function is strictly increasing, so $M=p_1x_1+p_2x_2$
- $\ln(x_1)\to-\infty$ as $x\to 0$, so $x_1>0$

Thus, there are only two potential solutions. Assuming positive consumption of both goods (only the budget constraint binds):

$$\begin{aligned}
0&=\frac{\alpha}{x_1}-p_1\lambda && \text{(from the first FOC)}\\
0&=1-p_2\lambda && \text{(from the second FOC)}\\
M&=p_1x_1+p_2x_2 && \text{(from the third FOC)}
\end{aligned}$$

Manipulating the first two FOCs:

$$\begin{aligned}
&\frac{1}{p_2}=\frac{\alpha}{p_1x_1} && \text{(combining the 1st two FOCs)}\\
&\boxed{x_1^*=\alpha\left(\frac{p_2}{p_1}\right)} && \text{(solving for } x_1\text{)}
\end{aligned}$$

Note that we only needed two FOCs to pin down $x_1$; using the constraint to pin down $x_2$:

$$M = p_1 \left[\alpha \left(\frac{p_2}{p_1}\right)\right] + p_2 x_2 \qquad \text{(plugging in for } x_1\text{)}$$

$$\boxed{x_2^* = \frac{M}{p_2} - \alpha} \qquad \text{(solving for } x_2\text{)}$$

$$\boxed{\lambda^* = \frac{1}{p_2}} \qquad \text{(solving for } \lambda\text{)}$$

Even if we weren't using Kuhn-Tucker, this should be a tip-off that we need to consider corner solutions; the quantity $x_2^*$ is not unequivocally positive. Thus, we need to consider $x_2 = 0$:

$$0 = \frac{\alpha}{x_1} - p_1\lambda \qquad \text{(from the first FOC)}$$

$$M = p_1 x_1 \qquad \text{(from the third FOC)}$$

The 1st FOC doesn't give us any information about $x_1$. The constraint, however, lets us pin down $x_1$:

$$\boxed{x_1^* = \frac{M}{p_1}} \qquad \text{(solving for } x_1\text{)}$$

$$\boxed{x_2^* = 0} \qquad \text{(by assumption)}$$

$$\boxed{\lambda^* = \frac{\alpha}{M}} \qquad \text{(solving for } \lambda\text{)}$$

This is the solution when $M < \alpha p_2$. Thus, the complete solution is:

$$x_1^* = \begin{cases} \dfrac{M}{p_1} & \text{if } M < \alpha p_2 \\ \alpha\left(\dfrac{p_2}{p_1}\right) & \text{if } M \geq \alpha p_2 \end{cases} \qquad x_2^* = \begin{cases} 0 & \text{if } M < \alpha p_2 \\ \dfrac{M}{p_2} - \alpha & \text{if } M \geq \alpha p_2 \end{cases} \qquad \lambda^* = \begin{cases} \dfrac{1}{p_2} & \text{if } M < \alpha p_2 \\ \dfrac{\alpha}{M} & \text{if } M \geq \alpha p_2 \end{cases}$$

## Practice Problems

5. **Which of the following functions on $\mathbb{R}^n$ are concave or convex? Use the 2nd derivative test or the definiteness of the Hessian (for univariate and multivariate functions, respectively) to determine concavity/convexity.**

   (a) $f(x) = 3e^x + 5x^4 - \ln(x)$

   $$f'(x) = 3e^x + 20x^3 - \frac{1}{x} \qquad \text{(the first derivative)}$$

   $$f''(x) = 3e^x + 60x^2 + \frac{1}{x^2} \qquad \text{(the second derivative)}$$

   Note that the domain of this function is $\mathbb{R}^1_{++}$, e.g., $x > 0$. For strictly positive values of $x$, $f''(x) > 0$, implying the function is strictly concave.

   (b) $g(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$

   $$\nabla g(x, y) = \begin{bmatrix} -6x + 2y + 3 & 2x - 2y - 4 \end{bmatrix} \qquad \text{(the gradient)}$$

   $$H = \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix} \qquad \text{(the Hessian)}$$

   $$|H_1| = -6 < 0 \qquad \text{(the first LPM)}$$

   $$|H_2| = 8 > 0 \qquad \text{(the second LPM)}$$

   The LPMs alternate in sign, beginning with a negative; thus, the function is strictly concave.

(c) $h(x, y, z) = 3e^x + 5y^4 - \ln(z)$

$$\nabla h(x, y, z) = \begin{bmatrix} 3e^x & 20y^3 & -\frac{1}{x} \end{bmatrix} \qquad \text{(the gradient)}$$

$$H = \begin{bmatrix} 3e^x & 0 & 0 \\ 0 & 60y^2 & 0 \\ 0 & 0 & \frac{1}{z^2} \end{bmatrix} \qquad \text{(the Hessian)}$$

$$|H_1| = 3e^x > 0 \qquad \text{(the first LPM)}$$

$$|H_2| = 180e^x y^2 \geq 0 \qquad \text{(the second LPM)}$$

$$|H_3| = \frac{180e^x y^2}{z^2} \geq 0$$

Note that $z > 0$, but $x$ and $y$ can take on any numbers in $\mathbb{R}$. All three LPMs are non-negative; further, all non-leading principle minors are zero, so the function is convex (the Hessian is positive semi-definite).

6. **Determine whether or not the following functions are quasiconcave, quasiconvex, or neither on $\mathbb{R}^2_+$.**

(a) $f(x) = e^x$

$f(x) = e^x$ is strictly increasing, so the function is quasiconcave and quasiconvex. We can show this using the definition as well. WLOG, let $x \geq y$. If $t \in [0, 1]$, then:

$$tx + (1-t)y \geq \min\{x, y\} \qquad \text{(by } x \geq y\text{)}$$

$$e^{tx+(1-t)y} \geq e^{\min\{x,y\}} \qquad (e^x \text{ is increasing})$$

$$e^{tx+(1-t)y} \geq \min\left\{e^x, e^y\right\} \qquad \text{(increasing functions preserve orderings)}$$

We can make a similar argument that $e^x$ is convex:

$$tx + (1-t)y \leq \max\{x, y\} \qquad \text{(by } x \geq y\text{)}$$

$$e^{tx+(1-t)y} \leq e^{\max\{x,y\}} \qquad (e^x \text{ is increasing})$$

$$e^{tx+(1-t)y} \leq \max\left\{e^x, e^y\right\} \qquad \text{(increasing functions preserve orderings)}$$

Thus, the function is both quasiconcave and quasiconvex.

(b) $g(x) = x^3 - x$

$g(x) = x^3 - x$ is not quasiconcave, but it is quasiconvex. To see that the function is not quasiconcave, consider the points $x = 0$ and $y = 1$. Then if $t = 0.5$, $tx + (1-t)y = 0.5$:

$$g(0) = 0^3 - 0 = 0 \qquad \text{(evaluating at 0)}$$

$$g(1) = 1^3 - 1 = 0 \qquad \text{(evaluating at 1)}$$

$$g(0.5) = 0.5^3 - 0.5 = -0.375 \qquad \text{(evaluating at the convex combo.)}$$

Thus, we have a case where $g(tx + (1-t)y) \ngeq \min\{g(x), g(y)\}$, so the function is not quasiconcave. To show that the function is quasiconvex over $\mathbb{R}$, consider the second derivative:

$$g''(x) = 6x \geq 0 \qquad \text{(the second derivative)}$$

This is non-negative, so the function is convex; convexity implies quasiconvexity, so the function is quasiconvex.

(c) $h(x, y) = ye^{-x}$

Setting up the bordered Hessian and evaluating leading principle minors:

$$H = \begin{bmatrix} 0 & -ye^{-x} & e^{-x} \\ -ye^{-x} & ye^{-x} & -e^{-x} \\ e^{-x} & -e^{-x} & 0 \end{bmatrix} \qquad \text{(the bordered Hessian)}$$

$$|H_1| = 0 \qquad \text{(the first LPM)}$$

$$|H_2| = -y^2e^{-2x} \leq 0 \qquad \text{(the second LPM)}$$

$$|H_3| = ye^{-3x} \geq 0 \qquad \text{(the third LPM)}$$

Note that the third LPM is only non-negative on $\mathbb{R}^2_+$. Because the signs go 0, negative, positive, the function is quasiconcave on $\mathbb{R}^2$.

(d) $j(x, y) = (2x - 3y)^3$

Setting up the bordered Hessian and evaluating leading principle minors:

$$H = \begin{bmatrix} 0 & 6(2x-3y)^2 & -9(2x-3y)^2 \\ 6(2x-3y)^2 & 24(2x-3y) & -36(2x-3y) \\ -9(2x-3y)^2 & -36(2x-3y) & 54(2x-3y) \end{bmatrix} \qquad \text{(the bordered Hessian)}$$

$$|H_1| = 0 \qquad \text{(the first LPM)}$$

$$|H_2| = -36(2x-3y)^4 \leq 0 \qquad \text{(the second LPM)}$$

$$|H_3| = 0 \qquad \text{(the third LPM)}$$

Thus, this function matches the signing conventions for both quasiconcave and quasiconvex functions, so $j(x, y)$ is both quasiconcave and quasiconvex. Even without using the bordered Hessian, we could argue that this is true:

- Linear functions are both (weakly) concave and (weakly) convex
- The sum of concave functions is concave; the sum of convex functions is convex
- Increasing transformations of concave functions are concave; increasing transformations of convex functions are convex
- $z^3$ is an increasing function on $\mathbb{R}_+$

Using these four facts, we can see that $j(x, y)$ must be both quasiconcave and quasiconvex, as it is an increasing transformation of the sum of linear functions.

7. **Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?**

(a) $f(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^3$

This function is homogeneous of degree 6:

$$\begin{aligned} f(tx, ty) &= 3(tx)^5(ty) + 2(tx)^2(ty)^4 - 3(tx)^3(ty)^3 && \text{(scaling inputs by } t\text{)} \\ &= t^6\left[3x^5y + 2x^2y^4 - 3x^3y^3\right] && \text{(factoring out a } t^6\text{)} \\ &= t^6 f(x, y) && \text{(by def. of } f\text{)} \end{aligned}$$

(b) $g(x, y) = 3x^5y + 2x^2y^4 - 3x^3y^4$

This function is not homogeneous:

$$\begin{aligned} g(tx, ty) &= 3(tx)^5(ty) + 2(tx)^2(ty)^4 - 3(tx)^3(ty)^4 && \text{(scaling inputs by } t\text{)} \\ &= t^6x^5y + t^6 2x^2y^4 - t^7x^3y^4 && \text{(pulling } t^k \text{ out of each term)} \end{aligned}$$

In each term, $t$ is not raised to a uniform power (i.e., the first two terms have a $t^6$, but the third term has a $t^7$). Thus, the function is not homogeneous.

(c) $h(x, y) = x^{1/2}y^{-1/2} + 3xy^{-1} + 7$

This function is homogeneous of degree zero:

$$\begin{aligned} h(tx, ty) &= (tx)^{1/2}(ty)^{-1/2} + 3(tx)(ty)^{-1} + 7 && \text{(scaling inputs by } t\text{)} \\ &= t^0\left[x^{1/2}y^{-1/2} + 3xy^{-1} + 7\right] && \text{(factoring out a } t^0 = 1\text{)} \\ &= h(x, y) && \text{(by def. of } h\text{)} \end{aligned}$$

(d) $j(x, y) = x^{3/4}y^{1/4} + 6x + 4$

This function is not homogeneous. Like the function in part (b), each term in the polynomial is not of the same degree (the first two terms are of degree 1, but the third is of degree 0).

8. **Which of the following functions are homothetic? Give a reason for each answer.**

(a) $f(x, y) = e^{x^2y}e^{xy^2}$

This function is homothetic. Consider the function:

$$\begin{aligned} \tilde{f}(x, y) &= x^2y + xy^2 && \text{(defining a new function)} \\ \tilde{f}(tx, ty) &= (tx)^2(ty) + (tx)(ty)^2 && \text{(scaling the inputs by } t\text{)} \\ &= t^3[x^2y + xy^2] && \text{(factoring out a } t^3\text{)} \\ \tilde{f}(tx, ty) &= t^3\tilde{f}(x, y) && \text{(by the def. of } \tilde{f}\text{)} \end{aligned}$$

Thus, $\tilde{f}$ is HOD 3. Note then that $f$ is a monotonic transformation of $\tilde{f}$:

$$\begin{aligned} f(x, y) &= e^{\tilde{f}(x,y)} && \text{(taking an exponential transform)} \\ &= e^{x^2y + xy^2} && \text{(by def. of } \tilde{f}\text{)} \\ f(x, y) &= e^{x^2y}e^{xy^2} && \text{(rearranging)} \end{aligned}$$

Thus, $f$ is a monotonic transformation of a homogeneous function, implying that $f$ is homothetic.

(b) $g(x, y) = x^3y^6 + 3x^2y^4 + 6xy^2 + 9$

This function is not homothetic. It is a polynomial, but it's terms are not of a uniform degree: the first term is of degree 9, the second of degree 6, the third of degree 3, and the fourth of degree 0. Further, there is no transformation of this function that could make it homogeneous (e.g., we cannot raise the function to the 1/3 power or subtract 9 to make it homogeneous).

(c) $h(x, y) = 2\ln(x) + 3\ln(y)$

This function is homothetic. Consider the function:

$$\tilde{h}(x, y) = x^2 y^3$$

This function is clearly HOD 5. Further $h$ is simply the logarithmic transformation of $\tilde{h}$. Thus, $h$ is a monotonic transformation of a homogeneous function, implying that $h$ is homothetic.

9. **Consider the function**

$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

(a) **Show that $f(x_1, x_2)$ is homogeneous of degree 1**

$$\begin{aligned}
f(tx_1, tx_2) &= \sqrt{(tx_1)^2 + (tx_2)^2} && \text{(scaling inputs by } t) \\
&= \sqrt{t^2(x_1^2 + x_2^2)} && \text{(rearranging)} \\
&= t\sqrt{x_1^2 + x_2^2} && \text{(simplifying)} \\
f(tx_1, tx_2) &= t^1 f(x_1, x_2) && \text{(by def. of } f)
\end{aligned}$$

(b) **Verify that Euler's Theorem holds for this function.**

The first-order derivatives of this function are:

$$\begin{aligned}
\frac{\partial f(\cdot)}{\partial x_1} &= \frac{2x_1}{2(x_1^2 + x_2^2)^{1/2}} && \text{(differentiating w.r.t. } x_1) \\
\frac{\partial f(\cdot)}{\partial x_2} &= \frac{2x_2}{2(x_1^2 + x_2^2)^{1/2}} && \text{(differentiating w.r.t. } x_2)
\end{aligned}$$

Euler's Theorem states that

$$\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} \cdot x_i = k f(\mathbf{x})$$

if $f$ is homogeneous of degree $k$. Summing the relevant terms in this case:

$$\begin{aligned}
\frac{\partial f(\cdot)}{\partial x_1^2} \cdot x_1 + \frac{\partial f(\cdot)}{\partial x_2} \cdot x_2 &= \frac{x_1^2}{(x_1^2 + x_2^2)^{1/2}} + \frac{x_2^2}{(x_1^2 + x_2^2)^{1/2}} && \text{(summing)} \\
&= \frac{(x_1^2 + x_2^2)}{(x_1^2 + x_2^2)^{1/2}} && \text{(reducing)} \\
&= \sqrt{x_1^2 + x_2^2} && \text{(simplifying)} \\
\frac{\partial f(\cdot)}{\partial x_1^2} \cdot x_1 + \frac{\partial f(\cdot)}{\partial x_2} \cdot x_2 &= t^1 f(x_1, x_2) && \text{(by def. of } f)
\end{aligned}$$

10. **Let $f(\mathbf{x})$ be a convex function. Prove that $f(\mathbf{x})$ reaches a local minimum at $\mathbf{x}^*$ if and only if $f(\mathbf{x}^*)$ reaches a global minimum at $\mathbf{x}^*$.**

- Def. of convexity: $f$ is convex $\iff$ $f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{x})$
- Def. of local minimum: $\mathbf{x}^*$ is a local minimum $\iff$ $\exists\, \varepsilon > 0 \ni \forall \mathbf{x} \in B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f), f(\mathbf{x}^*) \leq f(\mathbf{x})$.
- Def. of global minimum: $\mathbf{x}^*$ is a global minimum $\iff$ $\forall \mathbf{x} \in \mathcal{D}(f), f(\mathbf{x}^*) \leq f(\mathbf{x})$

To show ($\Rightarrow$): $\mathbf{x}^*$ is a local minimum
Proof:

Let $\varepsilon > 0$ and let $\mathbf{x}^*$ be a global minimum (by hypothesis)

Let $\mathbf{x} \in B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f)$ (by hypothesis)

$\implies \mathbf{x} \in \mathcal{D}(f)$ (by def. of $\cap$)

$\implies f(\mathbf{x}) \geq f(\mathbf{x}^*)$ (by def. of global min)

Because $\mathbf{x}$ is an arbitrary point in $B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f)$, this implies that for every point in $B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f)$, $f(\mathbf{x}^*) \leq f(\mathbf{x})$. Thus, there exists an $\varepsilon$ such that $\mathbf{x}^*$ meets the definition of a local minimum.

Proof by contradiction to show: $\mathbf{x} \in B_\varepsilon(\mathbf{x}^*)$ implies $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ and $f(\mathbf{x}) < f(\mathbf{x}^*)$

Proof:

Let $f$ be convex and let $\mathbf{x}^*$ be a local minimum (by hypothesis)

$\implies \exists\, \varepsilon > 0 \ni x \in B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f) \implies f(\mathbf{x}) \geq f(\mathbf{x}^*)$ (by def. of local min)

Suppose $\mathbf{x}^*$ is not a global minimum (towards a contradiction)

$\implies \exists\, \tilde{\mathbf{x}} \in \mathcal{D}(f) \ni f(\tilde{\mathbf{x}}) < f(\mathbf{x}^*)$ ($\mathbf{x}^*$ is not a global min)

Let $t \in (0, 1)$ such that $t\tilde{\mathbf{x}} + (1-t)\mathbf{x}^* \in B_\varepsilon(\mathbf{x}^*) \cap \mathcal{D}(f)$ (defining a convex combo.)

$\implies f(t\tilde{\mathbf{x}} + (1-t)\mathbf{x}^*) \geq f(\mathbf{x}^*)$
and $f(t\tilde{\mathbf{x}} + (1-t)\mathbf{x}^*) \leq tf(\tilde{\mathbf{x}}) + (1-t)f(\mathbf{x}^*)$ (by def. of local min and convexity)

$\implies f(t\tilde{\mathbf{x}} + (1-t)\mathbf{x}^*) < tf(\mathbf{x}^*) + (1-t)f(\mathbf{x}^*)$ (by $f(\tilde{\mathbf{x}}) < f(\mathbf{x}^*)$)

$\implies f(t\tilde{\mathbf{x}} + (1-t)\mathbf{x}^*) < f(\mathbf{x}^*)$ (simplifying)

Thus, a contradiction

$\implies \mathbf{x}^*$ is a global minimum ■

11. **Find the local extreme values and classify the points as maxima, minima, or neither.**

(a) $f(x_1, x_2) = 2x_1 - x_1^2 - x_2^2$

The first-order conditions are:

$$0 = 2 - 2x_1 \qquad \text{(w.r.t. } x_1\text{)}$$

$$0 = -2x_2 \qquad \text{(w.r.t. } x_2\text{)}$$

Solving this system for $x_1$ and $x_2$:

$$\boxed{x_1^* = 1} \qquad \text{(solving for } x_1\text{)}$$

$$\boxed{x_2^* = 0} \qquad \text{(solving for } x_2\text{)}$$

The second-order conditions depend on the principle minors of the Hessian:

$$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \qquad |H_1| = -2 \qquad |H_2| = 4$$

Thus, the leading principle minors conform to our signing convention for strictly concave functions; thus, $(1, 0)$ is a maximum.

(b) $g(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_2$

The first-order conditions are:

$$0 = 2x_1 \qquad \text{(w.r.t. } x_1\text{)}$$

$$0 = 4x_2 - 4 \qquad \text{(w.r.t. } x_2\text{)}$$

Solving this system for $x_1$ and $x_2$:

$$\boxed{x_1^* = 0} \qquad \text{(solving for } x_1\text{)}$$

$$\boxed{x_2^* = 1} \qquad \text{(solving for } x_2\text{)}$$

The second-order conditions depend on the principle minors of the Hessian:

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad |H_1| = 2 \qquad |H_2| = 8$$

Thus, the leading principle minors conform to our signing convention for strictly convex functions; thus, $(0, 1)$ is a minimum.

(c) $h(x_1, x_2) = x_1^3 - x_2^2 + 2x_2$

The first-order conditions are:

$$0 = 3x_1^2 \qquad \text{(w.r.t. } x_1\text{)}$$

$$0 = -2x_2 + 2 \qquad \text{(w.r.t. } x_2\text{)}$$

Solving this system for $x_1$ and $x_2$:

$$\boxed{x_1^* = 0} \qquad \text{(solving for } x_1\text{)}$$

$$\boxed{x_2^* = 1} \qquad \text{(solving for } x_2\text{)}$$

The second-order conditions depend on the principle minors of the Hessian:

$$H = \begin{bmatrix} 6x_1 & 0 \\ 0 & -2 \end{bmatrix} \qquad |H_1| = 6x_1 \qquad |H_2| = -12x_1$$

In this case, the principle minors do not conform to any of our signing conventions; indeed, the signs on the leading principle minors depend on the values of $x_1$. Thus, $(0, 1)$ is neither a maximum nor a minimum.

12. **Solve the following constrained optimization problems.**

(a) $\min_{\mathbf{x}} \, (x_1^2 + x_2^2) \quad \text{s.t.} \quad x_1 x_2 = 1$

The Lagrangian associated with this problem is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^2 + x_2^2 + \lambda\big[1 - x_1 x_2\big]$$

The first-order conditions are:

$$\mathcal{L}_1 = 2x_1 - \lambda x_2 = 0 \qquad \text{(w.r.t. } x_1\text{)}$$

$$\mathcal{L}_2 = 2x_2 - \lambda x_1 = 0 \qquad \text{(w.r.t. } x_2\text{)}$$

$$\mathcal{L}_\lambda = 1 - x_1 x_2 = 0 \qquad \text{(w.r.t. } \lambda\text{)}$$

Solving the first two FOCS for $\lambda$ and setting them equal:

$$\frac{x_1}{x_2} = \frac{x_2}{x_1} \qquad \text{(from the 1st two FOCs)}$$

$$x_1^2 = x_2^2 \qquad \text{(rearranging)}$$

Rearranging the constraint:

$$x_2 = \frac{1}{x_1} \qquad \text{(from the 2nd FOC)}$$

Plugging this into our first relationship:

$$x_1^2 = \left(\frac{1}{x_2}\right)^2 \qquad \text{(plugging in for } x_2\text{)}$$

$$x_1 = \pm 1 \qquad \text{(solving for } x_1\text{)}$$

Thus, $x_1$ can be 1 or $-1$; from the constraint, the two potential optima are

$$\boxed{(1,1),\ \lambda^* = 2} \qquad \boxed{(-1,-1),\ \lambda^* = 2}$$

To determine whether or not these are minima, we need to check second-order conditions:

$$\bar{H} = \begin{bmatrix} 0 & x_2 & x_1 \\ x_2 & 2 & -\lambda \\ x_1 & -\lambda & 2 \end{bmatrix} \qquad \text{(the bordered Hessian)}$$

$$|\bar{H}_1| = 0 \qquad \text{(the first LPM)}$$

$$|\bar{H}_2| = -x_2^2 \qquad \text{(the second LPM)}$$

$$|\bar{H}_3| = -2x_1^2 - 2\lambda x_1 x_2 - 2x_2^2 \qquad \text{(the third LPM)}$$

Evaluated at $(1,1)$ and $(-1,-1)$, $|\bar{H}_2| = -1 < 0$ and $|\bar{H}_3| = -8 < 0$. Thus, the LPMs follow the signing convention for minima, implying both $\boxed{(1,1)}$ and $\boxed{(-1,-1)}$ are solutions.

(b) $\min\limits_{\mathbf{x}} \,(x_1 x_2) \quad \text{s.t.} \quad x_1^2 + x_2^2 = 1$

The Lagrangian associated with this problem is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda[1 - x_1^2 - x_2^2]$$

The first-order conditions are:

$$\mathcal{L}_1 = x_2 - 2\lambda x_1 = 0 \qquad \text{(w.r.t. } x_1\text{)}$$

$$\mathcal{L}_2 = x_1 - 2\lambda x_2 = 0 \qquad \text{(w.r.t. } x_2\text{)}$$

$$\mathcal{L}_\lambda = 1 - x_1^2 - x_2^2 = 0 \qquad \text{(w.r.t. } \lambda\text{)}$$

Solving the first two FOCS for $\lambda$ and setting them equal:

$$\frac{x_1}{2x_2} = \frac{x_2}{2x_1} \qquad \text{(from the 1st two FOCs)}$$

$$x_2^2 = x_1^2 \qquad \text{(rearranging)}$$

Plugging this into the constraint:

$$\begin{aligned} 1 &= x_1^2 + [x_1^2] && \text{(plugging in for } x_2^2\text{)} \\ x_1 &= \pm\frac{1}{\sqrt{2}} && \text{(solving for } x_1\text{)} \\ x_2 &= \pm\frac{1}{\sqrt{2}} && \text{(solving for } x_2\text{)} \end{aligned}$$

Thus, we have four potential optima to consider:

$$\boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \ \lambda = \frac{1}{2}} \qquad \boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \ \lambda = -\frac{1}{2}}$$

$$\boxed{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ \lambda = -\frac{1}{2}} \qquad \boxed{\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ \lambda = \frac{1}{2}}$$

To determine whether or not these are minima, we need to check the second-order conditions:

$$\begin{aligned} \bar{H} &= \begin{bmatrix} 0 & 2x_1 & 2x_2 \\ 2x_1 & -2\lambda & 1 \\ 2x_2 & 1 & -2\lambda \end{bmatrix} && \text{(the bordered Hessian)} \\ |\bar{H}_1| &= 0 && \text{(the first LPM)} \\ |\bar{H}_2| &= -4x_1^2 && \text{(the second LPM)} \\ |\bar{H}_3| &= 8\lambda x_1^2 + 8x_1x_2 + 8\lambda x_2^2 && \text{(the third LPM)} \end{aligned}$$

For the extrema $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, the LPMS are:

$$|\bar{H}_1| = 0 \qquad |\bar{H}_2| = -2 \qquad |\bar{H}_3| = 8$$

This follows the signing convention for maxima; thus, the extrema with the same sign of $x_1$ and $x_2$ are *not* solutions. For the extrema $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, however, the LPMS are:

$$|\bar{H}_1| = 0 \qquad |\bar{H}_2| = -2 \qquad |\bar{H}_3| = -8$$

This follows the signing convention for minima; thus, our solutions to this problem are:

$$\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \ \lambda = -\frac{1}{2}} \qquad \boxed{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \ \lambda = -\frac{1}{2}}$$

(c) $\max_{\mathbf{x}} \, (x_1 + x_2) \quad \text{s.t.} \quad x_1^4 + x_2^4 = 1$

The Lagrangian associated with this problem is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 + x_2 + \lambda[1 - x_1^4 - x_2^4]$$

The first-order conditions are:

$$\begin{aligned} \mathcal{L}_1 &= 1 - 4x_1^3\lambda = 0 && \text{(w.r.t. } x_1\text{)} \\ \mathcal{L}_2 &= 1 - 4x_2^3\lambda = 0 && \text{(w.r.t. } x_2\text{)} \\ \mathcal{L}_\lambda &= 1 - x_1^4 - x_2^4 = 0 && \text{(w.r.t. } \lambda\text{)} \end{aligned}$$

Solving the first two FOCS for $\lambda$ and setting them equal:

$$\frac{1}{4x_1^3} = \frac{1}{4x_2^3} \qquad \text{(from the 1st two FOCs)}$$

$$x_2 = x_1 \qquad \text{(simplifying)}$$

Plugging this into the constraint:

$$1 = x_1^4 = (x_1)^4 \qquad \text{(plugging in for } x_2\text{)}$$

$$x_1 = \pm\left(\frac{1}{2}\right)^{\frac{1}{4}}$$

From the first two FOCs, we know that $x_1$ and $x_2$ must be of the same sign (being raised to the third power). Thus, the two potential optima are:

$$\boxed{\left[\left(\frac{1}{2}\right)^{\frac{1}{4}} \quad \left(\frac{1}{2}\right)^{\frac{1}{4}}\right],\ \lambda = \left(\frac{1}{2}\right)^{\frac{5}{4}}} \qquad \boxed{\left[-\left(\frac{1}{2}\right)^{\frac{1}{4}} \quad -\left(\frac{1}{2}\right)^{\frac{1}{4}}\right],\ \lambda = -\left(\frac{1}{2}\right)^{\frac{5}{4}}}$$

To determine whether or not these are minima, we need to check the second-order conditions:

$$\bar{H} = \begin{bmatrix} 0 & 4x_1^3 & 4x_2^3 \\ 4x_1^3 & -12x_1^2\lambda & 0 \\ 4x_2^3 & 0 & -12x_2^2\lambda \end{bmatrix} \qquad \text{(the bordered Hessian)}$$

$$|\bar{H}_1| = 0 \qquad \text{(the first LPM)}$$

$$|\bar{H}_2| = -16x_1^6 \qquad \text{(the second LPM)}$$

$$|\bar{H}_3| = 192\, x_1^2\, x_2^2\, \lambda\, [x_1^4 + x_2^4] \qquad \text{(the third LPM)}$$

Note that the sign of $|H_3|$ is entirely determined by the sign of $\lambda$; thus, for the solution with $x_1 < 0$, $x_2 < 0$ and $\lambda < 0$, the LPMS have signs:

$$|\bar{H}_1| = 0 \qquad |\bar{H}_2| < 0 \qquad |\bar{H}_3| < 0$$

Which follows the signing convention for a minimum and *not* a solution. For the solutions with $x_1 > 0$, $x_2 > 0$ and $\lambda > 0$, the LPMS have signs:

$$|\bar{H}_1| = 0 \qquad |\bar{H}_2| < 0 \qquad |\bar{H}_3| > 0$$

Which follows the signing convextion for a maximum. Thus, the only solution to this probem is:

$$\boxed{\left[\left(\frac{1}{2}\right)^{\frac{1}{4}} \quad \left(\frac{1}{2}\right)^{\frac{1}{4}}\right],\ \lambda = \left(\frac{1}{2}\right)^{\frac{5}{4}}}$$

13. **State the Kuhn-Tucker theorem for the following minimization problem:**

$$\min_{x_1,x_2} f(x_1,x_2) \quad \text{s.t.} \quad g(x_1,x_2) \le 0 \text{ and } x_1 \ge 0, x_2 \ge 0$$

- Let $\mathcal{L}(x_1,x_2,\lambda) = f(x_1,x_2) + \lambda_0\big[0 - g(x_1,x_2)\big] + \lambda_1\big[x_1 - 0\big] + \lambda_2\big[x_2 - 0\big]$
- Let $f$ and $g$ be continuously differentiable real valued functions over $\mathbb{R}^2_+$
- Let $\mathbf{x}^*$ be an interior constrained minimum

- For each binding constraint $x = 0$, $y = 0$, and/or $g(x_1^*, x_2^*) = 0$, let $\begin{bmatrix}1 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 1\end{bmatrix}$ and/or $\nabla g(x_1^*, x_2^*)$ be linearly independent.

If these conditions are satisfied, then there exists a unique vector $\boldsymbol{\lambda}^* \in \mathbb{R}^3$ such that:

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} - \lambda_0^* \cdot \frac{\partial g(\mathbf{x}^*)}{\partial x_i} + \lambda_i^* = 0 \quad \text{for } i = 1, 2$$

where $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, and $\lambda_2 \geq 0$; $g(\mathbf{x}^*) \leq 0$; $x_1 \geq 0$ and $x_2 \geq 0$, and $\lambda_0 \mathcal{L}_{\lambda_0} = 0$, $\lambda_1 \mathcal{L}_{\lambda_1} = 0$, and $\lambda_2 \mathcal{L}_{\lambda_2} = 0$ (i.e., complementary slackness holds).

14. **Consider the following maximization problem:**

$$\max_{x,y}\ xy \quad \textbf{s.t.} \quad x + y \leq 100 \quad \textbf{and} \quad x, y \geq 0$$

**State the Kuhn-Tucker first order conditions and solve the maximization problem.**

The Lagrangian associated with this problem is:

$$\mathcal{L}(x, y, \lambda) = xy + \lambda[100 - x - y]$$

The Kuhn-Tucker first-order conditions are:

$$\begin{array}{lll} \mathcal{L}_x = y - \lambda \leq 0 & x \geq 0 & x\mathcal{L}_x = 0 \\ \mathcal{L}_y = x - \lambda \leq 0 & y \geq 0 & y\mathcal{L}_y = 0 \\ \mathcal{L}_\lambda = 100 - x - y \geq 0 & \lambda \geq 0 & \lambda \mathcal{L}_\lambda = 0 \end{array}$$

- Given the structure of the utility function, we know that $x > 0$ and $y > 0$.
- If $x > 0$ and $y > 0$, the first two FOCs imply that $\lambda > 0$, so the budget constraint will bind.

Thus, all of our FOCs will bind with equality. From the first two FOCs:

$$\begin{array}{ll} y = x & \text{(from the 1st two FOCs)} \\ 100 = x + (x) & \text{(plugging in for } y\text{)} \\ \boxed{x = 50} & \text{(solving for } x\text{)} \\ \boxed{y = 50} & \text{(solving for } y\text{)} \\ \boxed{\lambda = 50} & \text{(solving for } \lambda\text{)} \end{array}$$

15. **Suppose a consumer livers on an island where he produces two goods, $x$ and $y$, according to the production possibility frontier $x^2 + y^2 \leq 200$, and he consumes all goods himself. His utility function is**

$$u(x, y) = xy^3$$

**The consumer also faces an environment constraint on his total output of booth goods, given by $x + y \leq 20$.**

   (a) **Write out the Kuhn-Tucker first-order conditions.**

   The Lagrangian associated with this problem is:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2) = xy^3 + \lambda_1[200 - x^2 - y^2] + \lambda_2[20 - x - y]$$

$$\mathcal{L}_x = y^3 - 2\lambda_1 x - \lambda_2 \leq 0 \qquad x \geq 0 \qquad x\mathcal{L}_x = 0$$

$$\mathcal{L}_y = 3xy^2 - 2\lambda_1 y - \lambda_2 \leq 0 \qquad y \geq 0 \qquad y\mathcal{L}_y = 0$$

$$\mathcal{L}_{\lambda_1} = 200 - x^2 - y^2 \geq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1\mathcal{L}_{\lambda_1} = 0$$

$$\mathcal{L}_{\lambda_2} = 20 - x - y \geq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2\mathcal{L}_{\lambda_1} = 0$$

(b) **Find the consumer's optimal $x$ and $y$. Identify which constraints are binding.**

Again, given the structure of the utility function, we know that $x > 0$ and $y > 0$. Further, at least one of the constraints is binding. Consider case 1, where $20 = x + y$ and $\lambda_1 = 0$:

$$0 = y^3 - \lambda_2 \qquad \text{(from the FOC w.r.t } x\text{)}$$

$$0 = 3xy^2 - \lambda_2 \qquad \text{(from the FOC w.r.t. } y\text{)}$$

$$20 = x + y \qquad \text{(from the FOC w.r.t. } \lambda_2\text{)}$$

Combining the first two FOCs:

$$y^3 = 3xy^2 \qquad \text{(from the 1st two FOCs)}$$

$$y = 3x \qquad \text{(solving for } y\text{)}$$

Plugging this into the constraint:

$$20 = x + (3x) \qquad \text{(plugging in for } y\text{)}$$

$$\boxed{x = 5} \qquad \text{(solving for } x\text{)}$$

$$\boxed{y = 15} \qquad \text{(solving for } y\text{)}$$

$$\boxed{\lambda_2 = 3375} \qquad \text{(solving for } \lambda_2\text{)}$$

Checking this against the other constraint, however:

$$200 \ngeq 5^2 + 115^2$$

Thus, this cannot be a solution. Consider case 2, where $200 = x^2 + y^2$ and $\lambda_2 = 0$:

$$0 = y^3 - 2x\lambda_1 \qquad \text{(from the FOC w.r.t } x\text{)}$$

$$0 = 3xy^2 - 2y\lambda_1 \qquad \text{(from the FOC w.r.t. } y\text{)}$$

$$200 = x^2 + y^2 \qquad \text{(from the FOC w.r.t. } \lambda_1\text{)}$$

Combining the first two FOCs:

$$\frac{y^3}{2x} = \frac{3xy^2}{2y} \qquad \text{(from the 1st two FOCs)}$$

$$y^2 = 3x^2 \qquad \text{(rearranging)}$$

Plugging this into the constraint:

$$200 = x^2 + (3x^2) \qquad \text{(plugging in for } y^2\text{)}$$

$$\boxed{x = \sqrt{50}} \qquad \text{(solving for } x\text{)}$$

$$\boxed{y = \sqrt{150}} \qquad \text{(solving for } y\text{)}$$

$$\boxed{\lambda_1 = 75\sqrt{3}} \qquad \text{(solving for } \lambda_1\text{)}$$

Checking this against the other constraint:

$$20 > \sqrt{50} + \sqrt{150}$$

Thus, our constrained optimum is where only the production possibility frontier binds:

$$x = 5 \qquad y = 15 \qquad \lambda_1 = 75\sqrt{3} \qquad \lambda_2 = 0$$