Transformations and expectations

Functions of random variables

- A *random variable* maps a sample space into $\mathbb{R}^1$.
- The support set is the set of $x's$ with positive probability, $\{x: f_X(x) > 0\}$.

$$y = g(x)$$
$$g(x): \mathcal{X} \to \mathcal{Y}$$

Inverse mapping

- We can also talk about an inverse mapping, which maps subsets of $\mathcal{Y}$ into subsets of $\mathcal{X}$.

$$g^{-1}(A) = \{x \in \mathcal{X}: g(x) \in A\}$$

- For example, suppose

$$g(x) = x^2$$
$$g^{-1}(2) = \{-1.414, 1.414\}$$

Monotone functions

- $u > v \Longrightarrow g(u) > g(v)$ (increasing)
- $u > v \Longrightarrow g(u) < g(v)$ (decreasing)

Monotone functions

Theorem 2.1.3 Let X have cdf $F_X(x)$, let $Y = g(X)$, and let X and Y be defined as $X = \{x: f_X(x) > 0\}$ and $Y = \{y: y = g(x)\ for\ some\ x \in X\}$.

a. If g is an increasing function on X,

$$F_Y(y) = F_X(g^{-1}(y))\ for\ y \in \mathcal{Y}.$$

b. If g is a decreasing function on X and X is a *continuous* random variable,

$$F_Y(y) = 1 - F_X(g^{-1}(y))\ for\ y \in \mathcal{Y}.$$

Transformation pdf

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\big(g^{-1}(y)\big)$$
$$= \frac{d}{dx}F_X\big(g^{-1}(y)\big) \times \frac{dx}{dy}$$

using the chain rule.

$$= \frac{d}{dx}F_X\big(g^{-1}(y)\big) \times \frac{d}{dy}g^{-1}(y)$$

Transformation pdf

- **Theorem 2.1.5** Let X have continuous pdf $f_X(x)$ on $\mathcal{X}$ and let $Y = g(X)$, where $g(\cdot)$ is a monotone function and that $g^{-1}(\cdot)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X\big(g^{-1}(y)\big)\left|\frac{d}{dy}g^{-1}(y)\right|, y \in \mathcal{Y} \\ 0, \qquad \text{otherwise} \end{cases}$$

Derive log normal

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{d}{dy}g^{-1}(y)\right|$$
$$f_X(x) = \frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}x^2}$$
$$y = g(x) = e^x$$
$$g^{-1}(y) = \log y$$
$$f_Y(y) = \frac{1}{\sqrt{(2\pi)}}e^{-\frac{1}{2}(\log y)^2}\left|\frac{1}{y}\right|, y > 0$$

Expected value of a transformation

$$\mathrm{E}\big(g(x)\big) = \int_{-\infty}^{\infty} g(x) \cdot f(x)dx$$

Expected values don't always exist

- Consider a Bernoulli trial with probability p.

$$\mathrm{E}(x) = 1 \times p + 0 \times (1-p) = p$$

But if $g(x) = 1/x$

$$\mathrm{E}\big(g(x)\big) = \frac{1}{1} \times p + \frac{1}{0} \times (1-p) = \infty$$

Ratio of normals

- The ratio of two independent standard normals is distributed Cauchy.
- The Cauchy has no finite expected value.

Expectations of affine functions

Theorem 2.2.5 Let x be a random variable and let a, b and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

a) $\mathrm{E}(ag_1(x) + bg_2(x) + c\,) = a\,\mathrm{E}\big(g_1(x)\big) + b\,\mathrm{E}\big(g_2(\mathrm{x})\big) + c$

b) If $g_1(X) \geq 0$ for all x, then $\mathrm{E}(g_1(X)) \geq 0$

c) If $g_1(x) \geq g_2(x)$ for all x, then $\mathrm{E}\big(g_1(x)\big) \geq \mathrm{E}\big(g_2(x)\big)$

d) If $a \leq g_1(x) \leq b$ for all x, then $a \leq \mathrm{E}\,g_1(x) \leq b$.

Expectation of sum

$$\mathrm{E}(ax + by + c) = a \cdot \mathrm{E}(x) + b \cdot \mathrm{E}(y) + c$$

$$\mathrm{E}(x + y) = \mathrm{E}(x) + \mathrm{E}(y)$$

Affine transformations

$$\mathrm{var}(a + bx) = b^2\,\mathrm{var}(x)$$

$$std(a + bx) = |b| \cdot std(x)$$

Moments

For each integer n, the *nth moment* of x is

$$\mu'_n = \mathrm{E}[x^n]$$

The *nth central moment* is

$$\mu_n = \mathrm{E}[(x - \mu)^n]$$

where

$$\mu = \mu'_1 = \mathrm{E}(x)$$

Note that the variance is the central 2nd moment.

Moments of normal

- If $x \sim N(\mu, \sigma^2)$, then the first moment is μ, the second central moment is σ^2, all higher order odd moments are zero, and the n^{th} central moment is $\sigma^n (n-1)(n-3)\cdots 3 \cdot 1$

$$\mu_2 = \sigma^2$$
$$\mu_4 = 3\sigma^4$$
$$\mu_6 = 15\sigma^6$$

Moment generating functions

Let X be a random variable with cdf $F_X(x)$. The *moment generating function (mgf)* of X is

$$M_X(t) = \mathrm{E}(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

The *n^th^* moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at 0.

$$\mathrm{E}(x^n) \equiv M_X^{(n)}(0)$$

where

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}$$

Differentiating an integral

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx$$
$$= f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) \; - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{d}{d\theta} f(x,\theta)dx$$
$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{d}{d\theta} f(x,\theta)dx$$

- For indefinite integrals some regularity conditions are required.
- Assume they're satisfied.

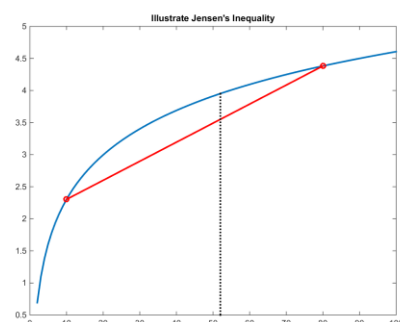
Jensen's Inequality

For any random variable X, if $g(x)$ is a convex function, then

$$\mathrm{E}\, g(x) \geq g(\mathrm{E}\, X)$$

For any random variable X, if $g(x)$ is a concave function, then

$$\mathrm{E}\, g(x) \leq g(\mathrm{E}\, X)$$



Assignment

- Consider a Bernoulli random variable with possible outcomes $x_1 = 10, x_2 = 80, p(X =$

Jensen's Inequality and Risk Aversion

- Utility is concave (diminishing marginal utility)
  - If we maximize expected utility, note

$$\mathrm{E}\big(U(x)\big) < U(\mathrm{E}(x))$$

Markov's inequality

- If $P(Y >= 0) = 1$ and $P(Y = 0) = 0$,

$$P(Y \geq r) \leq \frac{\mathrm{E}(Y)}{r}$$

Chebychev's inequality

- If X is a random variable and $g(x)$ is a non-negative function, then for $r > 0$

$$P(g(x) \geq r) \leq \frac{\mathrm{E}\big(g(x)\big)}{r}$$

Chebychev's inequality

$$P(g(x) \geq r) \leq \frac{\mathrm{E}(g(x))}{r}$$

Define $g(x) = (x-c)^2$, where c and $d > 0$ are constants

$$P\left((x-c)^2 \geq d^2\right) \leq \frac{\mathrm{E}[(x-c)^2]}{d^2}$$

$$P(|x-c| \geq d) \leq \frac{\mathrm{E}[(x-c)^2]}{d^2}$$

Lemma:

If $\mathrm{E}(x) = \mu$ and $\mathrm{var}(x) = \sigma^2$ and $d > 0$

$$P(|x-\mu| \geq d) \leq \sigma^2/d^2$$

- One version of Chebychev's inequality states that if a distribution has $\mathrm{E}(x) = \mu$ and $\mathrm{var}(x) = \sigma^2$ and $d \geq 0$, then
$$\mathrm{P}(|\mathrm{x}-\mu| \geq \mathrm{d}) \leq \sigma^2/\mathrm{d}^2$$
- Consider $x \sim N(0,1)$.

a) Prove that the inequality holds for $d = 1$, giving a proof that takes no more than five seconds.

b) Draw a graph in Matlab that plots both the bound from Chebychev's inequality and the actual value from the standard normal for interesting values of d.

Arrow-Pratt measure of absolute risk aversion

$$A(W) = -\frac{u''(W)}{u'(W)}$$

Constant absolute risk aversion (CARA):

$$u(W) = -e^{-\alpha W}$$
$$u'(W) = \alpha e^{-\alpha W}$$
$$u''(W) = -\alpha^2 e^{-\alpha W}$$
$$A(W) = \alpha$$

Initial wealth W_0, invest a fraction ω into a risky asset paying $\tilde{r}$ and the remaining $1-\omega$ into a safe asset paying r_f. Final wealth $\widetilde{W}$:

$$\widetilde{W} = \left((1-\omega)r_f + \omega\tilde{r}\right)W_0$$

$$\tilde{r} \sim N(\bar{r}, \sigma^2)$$

$$\widetilde{W} \sim N\left(\left((1-\omega)r_f + \omega\bar{r}\right)W_0, \sigma^2\omega^2W_0^2\right)$$

$$\widetilde{W} \sim N\left(\left((1-\omega)r_f + \omega\bar{r}\right)W_0, \sigma^2\omega^2W_0^2\right)$$

$$\mathrm{E}\left(U\left(\widetilde{W}\right)\right) = -\mathrm{E}\left(e^{-\alpha\widetilde{W}}\right)$$

$$= -e^{-\alpha\left((1-\omega)r_f + \omega\bar{r}\right)W_0 + \alpha^2\frac{1}{2}\sigma^2\omega^2W_0^2}$$

$$\frac{\partial\,\mathrm{E}\left(U\left(\widetilde{W}\right)\right)}{\partial\omega}$$

$$= -\left[-\alpha\left(-r_f + \bar{r}\right)W_0 + \alpha^2\sigma^2\omega W_0^2\right]e^{\{\cdot\}}$$

$$\omega^* = \frac{\bar{r} - r_f}{\alpha\sigma^2W_0}$$

