Math Camp: PS 2 Topology

Casey O'Hara

2017-09-11

1. **Let $A = \{1, 2, 3, 4\}$. Describe a codomain B and a function $f : A \to B$ such that f is:**

   (a) onto B but not one-to-one.
   $f(x) = |x - 2|, x \in A$ and $B = \{0, 1, 2\}$; $f(x)$ could be any function that ends up returning the same result for multiple elements of A (e.g. $f(-1) = f(1) = 1$), and B contains all the values of $f(x)$ for $x \in A$ and nothing else.

   (b) one-to-one but not onto B.
   $f(x) = 2x$ and $B = \mathbb{R}$; for any value $x \in A$, there is a unique value of $f(x)$; however, B might contain other values as well (e.g. rational numbers between each integer values for x outside the domain of A).

   (c) both one-to-one and onto B.
   $f(x) = 2x, x \in A$ and $B = \{2, 4, 6, 8\}$

   (d) neither one-to-one nor onto B. $f(x) = |x - 2|, x \in A$ and $B = \mathbb{R}$

2. **Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that**

$$x_n = \frac{n+1}{n}$$

To what does this sequence converge? Prove that this sequence converges to this limit.

- The sequence converges to 1.
- To show: $|x_n - 1| < \epsilon$, so $\{x_n\}_{n=1}^{\infty} \to 1$.
- **Proof:**

Let $\epsilon > 0$.	(by hypothesis)
Let $N = 1/\epsilon$.	(by hypothesis)
Let $n > N$.	(by hypothesis)
$\Rightarrow \lvert x_n - 1\rvert = \lvert\frac{n+1}{n} - 1\rvert$	(definition of x_n)
$\Rightarrow \forall n \in [1, \infty), \frac{n+1}{n} > 1$	(succession: $n + 1 > n$)
$\Rightarrow \lvert x_n - 1\rvert = \frac{n+1}{n} - 1$	(simplify absolute value: all values positive)
$\Rightarrow \lvert x_n - 1\rvert < \frac{N+1}{N} - 1$	($N < n, \frac{N+1}{N} > \frac{n+1}{n}$)
$\Rightarrow \lvert x_n - 1\rvert < \frac{1/\epsilon + 1}{1/\epsilon} - 1$	($N = 1/\epsilon$)
$\Rightarrow \lvert x_n - 1\rvert < \frac{(1+\epsilon)/\epsilon}{1/\epsilon} - 1$	(simplify)
$\Rightarrow \lvert x_n - 1\rvert < \frac{(1+\epsilon)}{\epsilon} * \epsilon - 1$	(simplify)
$\Rightarrow \lvert x_n - 1\rvert < \epsilon$	(simplify)

■

3. **Let S and T be convex sets. Prove that the intersection of S and T is also a convex set.**
To show: For $A = S \cap T, \quad \forall(\vec{x}, \vec{y} \in A \wedge \alpha \in [0,1]), \quad \alpha\vec{x} + (1-\alpha)\vec{y} \in A$.
Proof:

$$
\begin{aligned}
&\text{Let } A = S \cap T. && \text{(by hypothesis)}\\
&\forall \vec{x}, \vec{y} \in A, \quad \vec{x}, \vec{y} \in S \wedge \vec{x}, \vec{y} \in T && \text{(definition of intersection)}\\
&\forall(\vec{x}, \vec{y} \in A \wedge \alpha \in [0,1]), \quad \alpha\vec{x} + (1-\alpha)\vec{y} \in S && \text{(definition of convex)}\\
&\forall(\vec{x}, \vec{y} \in A \wedge \alpha \in [0,1]), \quad \alpha\vec{x} + (1-\alpha)\vec{y} \in T && \text{(definition of convex)}\\
&\forall(\vec{x}, \vec{y} \in A \wedge \alpha \in [0,1]), \quad \alpha\vec{x} + (1-\alpha)\vec{y} \in A && \text{(definition of intersection)}
\end{aligned}
$$

■

4. **The set $S^{n-1} = \{\mathbf{x} | \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, ..., n\}$ is called the $(n-1)$-dimensional unit simplex.**

   - I had a hard time with this one, I think because I am having a hard time conceptualizing the S^{n-1} definition.
   - Prove that S^{n-1} is a convex set.
     To show: $\forall x_i, x_j \in S^{n-1} \wedge \alpha \in [0,1]), \quad \alpha x_i + (1-\alpha)x_j \in S^{n-1}$.
     Proof:
   - Prove that S^{n-1} is a compact set.
     To show: $\forall x_i \in S^{n-1},\ 0 \leq x_i \leq 1$ (closed and bounded).
     Proof:

$$
\begin{aligned}
&\forall x_i \in S^{n-1},\ \ x_i \geq 0 && \text{(definition of } S^{n-1})\\
&\sum_{i=1}^{n} x_i = 1 && \text{(definition of } S^{n-1})\\
&\forall x_i \in S^{n-1},\ \ x_i \leq 1 && (\sum_{i=1}^{n} x_i = 1,\ \ x_i \geq 0)\\
&\forall x_i \in S^{n-1},\ \ 0 \leq x_i \leq 1 && \text{(definition of compact)}
\end{aligned}
$$

■

Trying by the notes method:
To show S^{n-1} is bounded.

$$
\begin{aligned}
&\text{Let } M = 2. && \text{(by hypothesis)}\\
&B_M = \{\mathbf{x} | \sum_{i=1}^{n} x_i < 2, x_i \geq 0, i = 1, ..., n\} && \text{(define an M-ball)}\\
&S^{n-1} \subset B_M && \text{(by inspection)}\\
&S^{n-1} \text{ is bounded.} && \text{(definition of bounded)}
\end{aligned}
$$

■