Economics 241B Exercise 2
Best Linear Prediction and Regression Lectures

1. Assume $\mathbb{E}(y) < \infty$.

a. Prove
$$\mathbb{E}(\mathbb{E}(y|x)) = \mathbb{E}(y).$$

b. Prove
$$\mathbb{E}(\mathbb{E}(y|x_1, x_2)|x_1) = \mathbb{E}(y|x_1).$$

2. (2016 Final: Lecture 3 BLP) Consider a dependent variable y for which

$$\begin{aligned}\mathbb{E}(y|x) &= \beta_2 x^2 + \beta_1 x + \beta_0,\\ y &= \beta_2 x^2 + \beta_1 x + \beta_0 + e,\end{aligned}$$

where $e \sim \mathcal{N}(0, \sigma^2(x))$.

a. Determine the distribution of y given x.

b. For any $h(x)$ such that $\mathbb{E}|h(x)e| < \infty$, prove the following statements:

$$\begin{aligned} i)\ \mathbb{E}(e|x) &= 0,\\ ii)\ \mathbb{E}(h(x)e) &= 0.\end{aligned}$$

Clearly state why the condition $\mathbb{E}|h(x)e| < \infty$ is needed. Do these statements imply that the covariate x is uncorrelated with the (conditional expectation function) error e?

c. We have shown (in class) that $\beta_2 x^2 + \beta_1 x + \beta_0 := m(x)$ is the predictor of y that minimizes the mean-squared prediction error. Consider predicting e^2 and write the mean-squared error of a predictor $g(x)$. Show that $\sigma^2(x)$ minimizes this mean-squared error.

3. Let $g(\cdot) : \mathbb{R}^m \to \mathbb{R}$ be a convex function.

a. For any random vector x, if $\mathbb{E}\|x\| < \infty$ and $\mathbb{E}|g(x)| < \infty$, prove (Jensen's Inequality)
$$g(\mathbb{E}(x)) \leq \mathbb{E}(g(x)).$$

b. With $m = 1$, use Jensen's Inequality to bound $(\mathbb{E}(x))^2$.

c. For any random vectors (y, x), if $\mathbb{E}\|y\| < \infty$ and $\mathbb{E}|g(y)| < \infty$, prove (Conditional Jensen's Inequality)

$$g(\mathbb{E}(y|x)) \leq \mathbb{E}(g(y)|x).$$

d. With $m = 1$, use the Conditional Jensen's Inequality to bound $(\mathbb{E}(y|x))^2$.

4) (2017 Prelim) You are asked to determine how the conditional mean of a discrete variable y depends on a (continuous) conditioning variable x. With a discrete dependent variable, the assumption about the form of the conditional mean is replaced with an assumption about the entire conditional distribution for y. You need to consider two cases.

Case 1: y takes only 2 values, $y \in \{0, 1\}$. Assume

$$\mathbb{P}(y = 1|x) = x^{\mathrm{T}}\beta_0.$$

(The conditional distribution of y given x is Bernoulli.)

Case 2: y takes positive integer values, $y \in \{0, 1, 2, \ldots\}$. Assume

$$\mathbb{P}(y = k|x) = \frac{\exp\left(-x^{\mathrm{T}}\beta_0\right)\left(x^{\mathrm{T}}\beta_0\right)^k}{k!} \quad k = 0, 1, 2, \ldots$$

(The conditional distribution of y given x is Poisson.)

a) For Case 1, compute $\mathbb{E}(y|x)$. Does this justify a linear regression model of the form $y = x^{\mathrm{T}}\beta_0 + u$?

b) For Case 1, compute $Var(y|x)$. Does this justify an alternative estimator to OLS?

c) For Case 2, compute $\mathbb{E}(y|x)$. Does this justify a linear regression model of the form $y = x^{\mathrm{T}}\beta_0 + u$? Hint:

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \exp(\lambda).$$