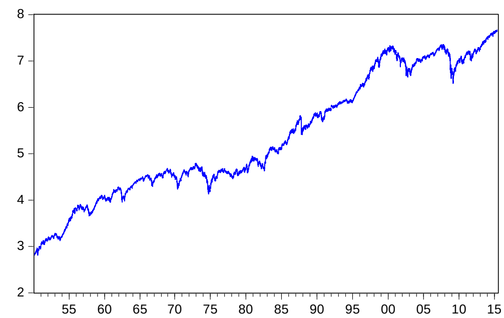
Hypothesis Testing

log S&P closing price adjusted for splits and dividends



Dependent Variable: LOGP
Method: Least Squares
Date: 11/18/15 Time: 08:23
Sample (adjusted): 1/04/1950 8/13/2015
Included observations: 16509 after adjustments

Variable	Coefficien...	Std. Error	t-Statistic	Prob.
C	0.002232	0.000834	2.674596	0.0075
LOGP(-1)	0.999391	0.000265	3777.368	0.0000
@TREND	1.61E-07	7.43E-08	2.165799	0.0303

R-squared	0.999947	Mean dependent var	5.366932
Adjusted R-squared	0.999947	S.D. dependent var	1.338434
S.E. of regression	0.009704	Akaike info criterion	-6.432431
Sum squared resid	1.554245	Schwarz criterion	-6.431029
Log likelihood	53099.50	Hannan-Quinn criter.	-6.431968
F-statistic	1.57E+08	Durbin-Watson stat	1.944332
Prob(F-statistic)	0.000000		

t-test on lagged stock prices

$$\hat{\beta} = 0.999391$$

$$\frac{0.999391 - 1.0}{0.000265} = -2.30$$

Statistical vs Economic Significance

- Statistical significance
  - ✓ Strong evidence that $\beta \neq 1$
- Economic significance
  - ✓ $\beta \approx 1$

Hypothesis

Definition:

A *hypothesis* is a statement about a population parameter.

Examples:

$$\beta = 1$$

$$\mu < 3.14159$$

Hypothesis test

$$H_0: \theta \in \Theta_0$$
$$H_1 \ (or\ H_A): \theta \in \Theta_0^C$$

Two-sided hypothesis:

$$H_0: \beta = \beta_0$$
$$H_A: \beta \neq \beta_0$$

One-sided hypothesis

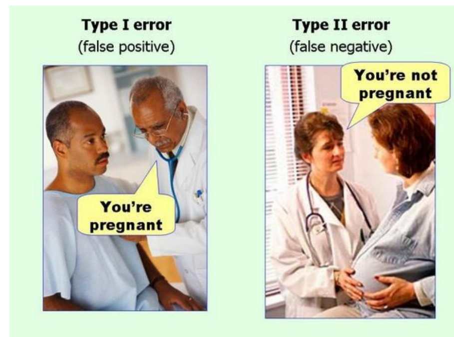
$$H_0: \beta \geq \beta_0$$
$$H_A: \beta < \beta_0$$

A *hypothesis test* is a decision rule that tells us for which sample values we should *reject* the null hypothesis.

Frequentist hypothesis tests tells us the strength of the evidence against the null—they say nothing about the alternative.

Error types

	Reject	Don’t reject
Null is true	Type I error	
Alternative is true		Type II error



Classic test on mean

If $x_i \sim iidN(\mu_0, \sigma^2)$, then if we take

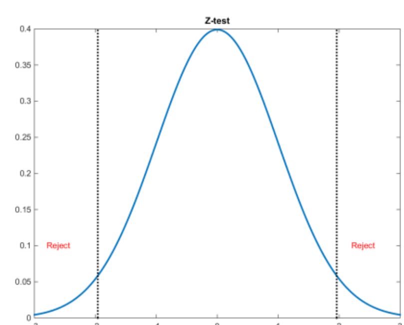
$$\hat{\mu} = \bar{x}$$

We know

$$z = \frac{\hat{\mu} - \mu_0}{\sqrt{\sigma^2/n}} \sim N(0,1)$$

Decision rule. Reject the null is $|z| > 1.96$.

Probability of Type 1 error is 0.05.



Size of a test

$$size = \alpha = P(\text{reject } H_0 | H_0 \text{ true})$$

In this example we have

$$0.05 = P\left(\left|\frac{\hat{\mu} - \mu_0}{\sqrt{\sigma^2/n}}\right| > 1.96 | \mu = \mu_0\right)$$

$$size = \alpha = \sup_{\theta_0 \in \Theta_0} P(\text{reject } H_0 | H_0 \text{ true})$$

Power of a test

$$power = \beta(\mu) = P(\text{reject } H_0 | \mu)$$

$$\beta(\mu) = P\left(\left|\frac{\hat{\mu} - \mu_0}{\sqrt{\sigma^2/n}}\right| > 1.96 | \mu\right)$$

Note that the size is the power evaluated at μ_0.

Write a program that computes the power function for our standard mean of iid normal variables, assuming the null hypothesis is $\mu_0 = 0$ and, that $\sigma^2 = 1$, that the sample is size n, and we do a 5 percent test. So we need to compute

$$\beta(\mu) = P\left(|z > 1.96| \middle| z \sim N\left(\mu, \frac{\sigma^2}{n}\right)\right)$$

Graph the power function for $n = 20$ and $n = 120$. Suppose you are "accept" the null with 20 observations. Is it convincing that the true $\mu \neq 1$? How about $\mu \neq .25$? What would your answer be for 120 observations?

Uniformly Most Powerful Test

Definition 8.3.11 Let $\mathcal{C}$ be a class of tests for testing $H_0: \theta \in \Theta_0$ versus $H_1: \theta \in \Theta_0^C$. A test in class $\mathcal{C}$, with power function $\beta(\theta)$, is a *uniformly most powerful* (UMP) *class $\mathcal{C}$ test* if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^C$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.

Neyman-Pearson Lemma

Theorem 8.3.12 (Neyman-Pearson Lemma)
Consider testing $H_0: \theta = \theta_0$ versus $H_1: \theta = \theta_1$ using a test where

Reject if $L(x|\theta_1) > kL(x|\theta_0)$

Don't reject if $L(x|\theta_1) < kL(x|\theta_0)$

For some $k \geq 0$, and

$$size = \alpha$$

Then

Every such test is a UMP size α test.

Hypothesis test

Two-sided hypothesis:

$$H_0: \beta = \beta_0$$
$$H_A: \beta \neq \beta_0$$

One-sided hypothesis

$$H_0: \beta \geq \beta_0$$
$$H_A: \beta < \beta_0$$

Critical value

Critical value c, *reject* if test statistic $> c$.

Examples:

$$|z| > c = 1.96$$

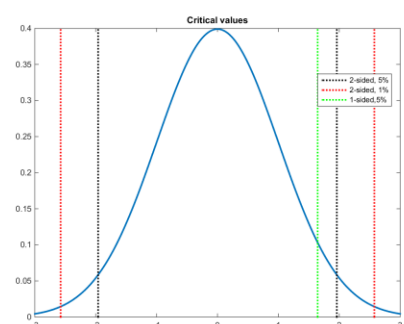
Two-sided test with $size = 0.05$

$$|z| > c = 2.58$$

Two-sided test with $size = 0.01$

$$z > c = 1.64$$

One-sided test with $size = 0.05$



Confidence interval

We *invert* a test statistic to get a *confidence interval*.

$$|z| < c$$
$$\Rightarrow$$
$$-c < z < c$$
$$\left|\frac{\hat{\mu} - \mu_0}{\sqrt{\sigma^2/n}}\right| < 1.96$$
$$\Rightarrow$$
$$\hat{\mu} - 1.96 \times \sqrt{\sigma^2/n} < \mu_0 < \hat{\mu} + 1.96 \times \sqrt{\sigma^2/n}$$

p-Value

Table A.2 t distribution: critical values of t

		Significance level					
Degrees of freedom	*Two-tailed test:*	10%	5%	2%	1%	0.2%	0.1%
	One-tailed test:	5%	2.5%	1%	0.5%	0.1%	0.05%
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.894	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587

$df = 10, t = 2.00,$
$p - value \approx 7.3\%$

$df = 10, t = 4.00,$
$p - value \approx 0.25\%$

p-value

At what significance level are we on the margin between rejecting and not rejecting?

Hypothesis Testing for FE
Date: 07/20/15 Time: 13:33
Sample: 1 89984
Included observations: 89984
Test of Hypothesis: Mean = 0.500000

Sample Mean = 0.515458
Sample Std. Dev. = 0.499764

Method	Value	Probability
t-statistic	9.278542	0.0000

95% confidence interval = $\left(.515 - 1.96 \times \frac{.49}{\sqrt{89984}}, .515 + 1.96 \times\right.$

Wald statistic, single restriction

$$\frac{amount\ hypothesis\ missed\ by}{\sqrt{\mathrm{var}(miss)}}$$
$$H_0: \theta = \theta^0$$
$$\hat{\theta} \sim N\left(\theta, \sigma^2/_n\right)$$

Under null

$$z = \frac{\hat{\theta} - \theta}{\sqrt{\sigma^2/_n}} \sim N(0,1)$$
$$t = \frac{\hat{\theta} - \theta}{\sqrt{s^2/_n}} \sim t_{n-1}$$

$$H_0: \theta_1 + \theta_2 = \theta^0$$
$$\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \sigma^2 \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \right)$$
$$z = \frac{\hat{\theta}_1 + \hat{\theta}_2 - \theta^0}{\sqrt{\operatorname{var}\left(\hat{\theta}_1 + \hat{\theta}_2 - \theta^0\right)}} \sim N(0,1)$$
$$\operatorname{var}\left(\hat{\theta}_1 + \hat{\theta}_2 - \theta^0\right) = \sigma^2 (V_{11} + V_{22} + 2V_{12})$$

$$t = \frac{\hat{\theta}_1 + \hat{\theta}_2 - \theta^0}{\sqrt{s^2 (V_{11} + V_{22} + 2V_{12})}} \sim t_{n-k}$$

Matrix version, single restriction

$$H_0: R\theta = r$$
$$R = [1 \quad 1]$$
$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$
$$r = \theta^0$$
$$[1 \quad 1] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \theta^0$$
$$\operatorname{var}\left(R\hat{\theta}\right) = R \operatorname{var}\left(\hat{\theta}\right) R'$$
$$= [1 \quad 1] \sigma^2 \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \sigma^2 (V_{11} + V_{22} + 2V_{12})$$

$$\frac{R\hat{\theta} - r}{\sqrt{R \operatorname{var}\left(\hat{\theta}\right) R'}} \sim N(0,1)$$
$$\frac{\left(R\hat{\theta} - r\right)^2}{R \operatorname{var}\left(\hat{\theta}\right) R'} \sim \chi_1^2$$

Wald test, multiple restrictions

$$H_0: \theta_1 = \theta_1^0$$
$$\theta_2 = \theta_2^0$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \theta_1^0 \\ \theta_2^0 \end{bmatrix}$$
$$R\theta = r$$

R is $q \times k$, θ is $k \times 1$ and r is $q \times 1$

$$\hat{\theta} \sim N(\theta, \Sigma)$$
$$\left(R\hat{\theta} - r\right)' [R \Sigma R']^{-1} \left(R\hat{\theta} - r\right) \sim \chi_q^2$$

Wald *F*-test

$$\left(R\hat{\theta} - r\right)' [R \Sigma R']^{-1} \left(R\hat{\theta} - r\right) \sim \chi_q^2$$
$$\frac{s^2}{\sigma^2} \sim \chi_{n-k}^2$$

$$\frac{\left(R\hat{\theta} - r\right)' [R \Sigma R']^{-1} \left(R\hat{\theta} - r\right) / q}{s^2 / \sigma^2}$$
$$\left(R\hat{\theta} - r\right)' \left[R \hat{\Sigma} R'\right]^{-1} \left(R\hat{\theta} - r\right) / q \sim F(q, n-k)$$

MLE Wald test

$$W = \frac{(\hat{\mu}_{mle} - \mu_H)^2}{-\operatorname{E}\left(\frac{\partial^2 \mathcal{L}}{\partial \mu^2}\right)^{-1}} \overset{A}{\sim} \chi_1^2$$

$$(\hat{\mu}_{mle} - \mu_H) I(\hat{\mu}_{mle}) (\hat{\mu}_{mle} - \mu_H) \overset{A}{\sim} \chi_1^2$$

MLE Wald test

$$H_0 : g(\theta) = 0$$
$$rank\left(\frac{\partial g}{\partial \theta}\right) = q$$

$$g(\hat{\theta})' \left(\frac{\partial g(\hat{\theta})}{\partial \theta} I^{-1}(\hat{\theta}) \frac{\partial g(\hat{\theta})'}{\partial \theta} \right)^{-1} g(\hat{\theta}) \overset{A}{\sim} \chi_q^2$$

Cobb-Douglass example

$$y = AL^{\alpha} K^{\beta} + \varepsilon$$

$$H_0 : g(A, \alpha, \beta, \sigma^2) = \alpha + \beta - 1 = 0$$

$$\frac{\partial g(\theta^{hat})}{\partial \theta} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Dependent Variable: Y
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 11/15/15 Time: 11:31
Sample: 1899 1922
Included observations: 24
Convergence achieved after 15 iterations
Coefficient covariance computed using outer product of gradients
Y=C(1)*L^C(2)*K^C(3)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.320637	0.576169	2.292098	0.0323
C(2)	0.661973	0.136539	4.848222	0.0001
C(3)	0.282077	0.061910	4.556265	0.0002

R-squared	0.940485	Mean dependent var	165.9167
Adjusted R-squared	0.934817	S.D. dependent var	43.75318
S.E. of regression	11.17063	Akaike info criterion	7.780921
Sum squared resid	2620.441	Schwarz criterion	7.928178
Log likelihood	-90.37105	Hannan-Quinn criter.	7.819988
Durbin-Watson stat	1.672870		

	C(1)	C(2)	C(3)
C(1)	0.33197116...	-0.07167473...	0.02010866...
C(2)	-0.07167473...	0.01864296...	-0.00721652...
C(3)	0.02010866...	-0.00721652...	0.00383280...

Wald = 0.389
$\chi_1^2(.95) = 3.84$

Likelihood ratio test (LRT)

$$\lambda(x) = \frac{\sup_{\Theta_0} L(\theta|x)}{\sup_{\Theta} L(\theta|x)}$$

Reject if $\lambda(x) \leq c, 0 \leq c \leq 1$

$$-2\left(\mathcal{L}(\theta_0) - \mathcal{L}(\hat{\theta}_{mle})\right) \overset{A}{\sim} \chi_q^2$$

Dependent Variable: Y
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 11/15/15 Time: 11:31
Sample: 1899 1922
Included observations: 24
Convergence achieved after 15 iterations
Coefficient covariance computed using outer product of gradients
Y=C(1)*L^C(2)*K^C(3)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.320637	0.576169	2.292098	0.0323
C(2)	0.661973	0.136539	4.848222	0.0001
C(3)	0.282077	0.061910	4.556265	0.0002

R-squared	0.940485	Mean dependent var	165.9167
Adjusted R-squared	0.934817	S.D. dependent var	43.75318
S.E. of regression	11.17063	Akaike info criterion	7.780921
Sum squared resid	2620.441	Schwarz criterion	7.928178
Log likelihood	-90.37105	Hannan-Quinn criter.	7.819988
Durbin-Watson stat	1.672870		

Dependent Variable: Y
Method: Least Squares (Gauss-Newton / Marquardt steps)
Date: 11/15/15 Time: 11:49
Sample: 1899 1922
Included observations: 24
Convergence achieved after 6 iterations
Coefficient covariance computed using outer product of gradients
Y=C(1)*L^(1-C(3))*K^C(3)

	Coefficient	Std. Error	t-Statistic	Prob.
C(1)	1.006646	0.027841	36.15666	0.0000
C(3)	0.258040	0.048815	5.286056	0.0000

R-squared	0.939363	Mean dependent var	165.9167
Adjusted R-squared	0.936607	S.D. dependent var	43.75318
S.E. of regression	11.01614	Akaike info criterion	7.716256
Sum squared resid	2669.819	Schwarz criterion	7.814427
Log likelihood	-90.59507	Hannan-Quinn criter.	7.742301
Durbin-Watson stat	1.524913		

$$LR = -2(-90.59507 - -90.37105)$$
$$= .44804$$

Lagrange multiplier (LM) or score test

$$LM = \frac{\partial \mathcal{L}(\theta)}{\partial \theta}\bigg|_{\theta_0}' \left[I\left(\theta_{H_0}\right)\right]^{-1} \frac{\partial \mathcal{L}(\theta)}{\partial \theta}\bigg|_{\theta_0} \overset{A}{\sim} \chi_q^2$$

Cobb-Douglas Example

$$\mathcal{L} = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum\left(y_i - AL_i^\alpha K_i^\beta\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \begin{bmatrix} \frac{1}{\sigma^2}\sum\left(y_i - AL_i^\alpha K_i^\beta\right)\left(L_i^\alpha K_i^\beta\right) \\ \frac{\alpha}{\sigma^2}\sum\left(y_i - AL_i^\alpha K_i^\beta\right)\left(AL_i^{\alpha-1} K_i^\beta\right) \\ \frac{\beta}{\sigma^2}\sum\left(y_i - AL_i^\alpha K_i^\beta\right)\left(AL_i^\alpha K_i^{\beta-1}\right) \\ -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^{2^2}}\sum\left(y_i - AL_i^\alpha K_i^\beta\right)^2 \end{bmatrix}$$

```
function LMTestCobbDouglas
%{
Compute LM test that Cobb-Douglas
coefficients show CRTS
using coefficients estimated by
EViews

y = A*l^alpha*k^beta + epsilon

Econ 241A
Dick Startz
November 2015
%}

load cobbdouglas_data.mat; %original
Cobb-Douglas Data
%coefficients estimated in EViews
A = 1.006646420473282;
beta = 0.2580401723789838;
alpha = 1 - beta;

n = length(y);
resid = y - A*(l.^alpha).*(k.^beta);
ssr = sum(resid.^2);
sigmaSqr = ssr/n;

% write out partial derivative
contributions, then put them together

d1 = resid.*(l.^alpha.*k.^beta)/sigmaSqr;
d2 = alpha*resid.*(A*l.^(alpha-
1).*k.^beta)/sigmaSqr;
d3 = beta*resid.*(A*l.^alpha.*k.^(beta-
1))/sigmaSqr;
d4 = (-1+resid.^2/sigmaSqr)/(2*sigmaSqr);

partial = [ d1 d2 d3 d4];
dLdTheta = sum(partial,1)';
info = partial'*partial;
LM = dLdTheta'*inv(info)*dLdTheta;

disp(['LM statistic for CRTS =
',num2str(LM)]);
end
```

LM statistic for CRTS = 0.26562

Suppose $x_i \sim iid(\mu, \sigma^2), i = 1, \ldots, n$ with σ^2 known. Show that the Wald, likelihood ratio, and Lagrange multiplier tests of

$$H_0: \mu = \mu_0$$
$$H_a: \mu \neq \mu_0$$

Are all identical, where

$$W = \frac{(\hat{\mu}_{mle} - \mu_0)^2}{I(\hat{\mu}_{mle})^{-1}}$$

$$LR = -2\left(\mathcal{L}^*(\mu_0) - \mathcal{L}(\hat{\mu}_{mle})\right)$$

$$LM = \frac{\partial \mathcal{L}(\mu)}{\partial \mu}\bigg|_{\mu_0}' [I(\mu_0)]^{-1} \frac{\partial \mathcal{L}(\mu)}{\partial \mu}\bigg|_{\mu_0}$$

Income is distributed (very) roughly log-normal, once people with zero income are dropped from the sample. The file cpsMarch2016Income.mat contains wage and salary income from the March 2016 current population survey in the variable wsal_val and gender in the variable fe (fe=1 for women and fe=0 for men.)

You may "remember" (or may have looked it up), that the pdf of a log normal can be written

$$f(y|\mu, \sigma^2) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\}$$

And that the mean is $\exp\left[\mu + \frac{\sigma^2}{2}\right]$, while the variance is $[\exp(\sigma^2) -$