Math Camp 2017 - Set Theory[*]

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1. **A Primer on Sets**

   (a) Definition. A **set** is a collection of objects, called **elements**, or members of the set. When the objects x is in the set A, we write $x \in A$. We may use set-builder notation to write a set as

   $$A = \{x|P(x)\}$$

   which reads "the set of x such that $P(x)$."

   (b) Example. Consider the open sentence $P(x) \equiv$ "x is an odd integer between 0 and 6." If A is the set described by $P(x)$, then

   $$A = \{x|P(x)\} = \{1, 3, 5\}$$

   (c) Example. Consider several of the most well-known sets:

   - The set of **natural numbers** is defined as $\{1, 2, 3, \dots\}$ and is denoted $\mathbb{N}$.
   - The set of **integers** is defined as $\{\dots, -2, -1, 0, 1, 2, \dots\}$ and is denoted $\mathbb{Z}$.
   - The set of **rational numbers** is defined as $\left\{q \middle| \exists\, p \in \mathbb{Z} \wedge \exists\, r \in \mathbb{Z} \ni q = \frac{p}{q}\right\}$ and is denoted $\mathbb{Q}$.
   - The set of **real numbers** is the set of all numbers along the number line, represented by finite or infinite decimals and is denoted $\mathbb{R}$.
   - The **empty set** (also known as the null set) can be defined as $= \{x|x \neq x\}$ and contains no elements.
   - Given real numbers a and b (or $a, b \in \mathbb{R}$) such that $a < b$, the **open inverval** from a to b is defined as

     $$(a, b) = \{x|x \in \mathbb{R} \wedge a < x < b\}$$

   - Similarly, the **closed interval** from a to b is defined as

     $$[a, b] = \{x|x \in \mathbb{R} \wedge a \leq x \leq b\}$$

   (d) Aside. Note that $\emptyset$ being a set is an *axiom* which we will take as given. It has no elements, it can be defined differently, but we'll just leave it as is for now.

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[*]These lecture notes are drawn principally from *A Transition to Advanced Mathematics*, 7th ed., by Douglas Smith, Maurice Eggen, and Richard St. Andre. The material posted on this website is for personal use only and is not intended for reproduction, distribution, or citation.

(e) Definition. Given two sets A and B, A is a **subset** of B, denoted $A \subset B$, if and only if every element of A is also an element of B. The sets A and B are **equal** if $A \subset B$ and $B \subset A$.

(f) Example. We can write out our definition for subsets using logical symbols. For sets A and B,

$$A \subset B \iff \forall\, x, (x \in A \implies x \in B)$$

To prove a set A is a subset of set B, we typically pick an element (usually arbitrary) and show that if it is in A, it must also be in B. Given $\mathbb{R}$ is our universe of discourse, consider the sets

$$A = \{x | a < x < b\} \quad \text{and} \quad B = \{y | a \leq y \leq b\}$$

We can prove $A \subset B$ directly:

To show: if $x \in A$, then $x \in B$ (note this could be stated $x \in A \implies X \in B$)
Proof:

$$\begin{array}{ll} \text{Let } x \in A & \text{(by hypothesis)} \\ \implies a < x < b & \text{(by def. of } A) \\ \implies a \leq x \leq b & \text{(by def. of } \leq) \\ \implies x \in B & \text{(by def. of } B) \end{array}$$

Thus, since x was chosen arbitrarily, $A \subset B$. Now, we can also show that $B \not\subset A$ by way of counter example.

To show: $\exists\, x \in B \ni x \notin A$.
Proof:

$$\begin{array}{ll} \text{Let } x = b & \text{(by hypothesis)} \\ \implies x \in B & \text{(by def. of } B) \\ \text{But } x \not< b & \text{(by def. of } <) \\ \implies x \in A & \text{(by def. of } A) \end{array}$$

Thus, there is an element in B that is not in A, so $B \not\subset A$.

(g) Aside. Like we did previously, the first proof assumes that the first proposition, $x \in A$ is true. We then use definitions, based on that initial assumption that the first part is true, to show that the second part, $x \in B$ must be true as well. In general, this is the form we use when dealing with proofs of $P \implies Q$.

Note also that we don't usually have to be this formal. For counter examples, it usually sufficies to simply assert that we found a point in B that isn't in A and leave it at that.

(h) Definition. Let A be a set. The **power set** of A is the set whose elements are the subsets of A and is denoted $\mathcal{P}(A)$:

$$\mathcal{P}(A) = \{B | B \subset A\}$$

(i) Example. Consider the set $A = \{a, b, c\}$. Then the power set of A is:

$$\mathcal{P}(A) = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\Big\}$$

(j) Aside. Note that per the definition, the elements of the power sets are themselves sets. Note that we would say that $A \in \mathcal{P}(A)$ (i.e., the set A is in the power set), but not $A \subset \mathcal{P}(A)$; rather, $\{A\} \subset \mathcal{P}(A)$. Further, the empty set $\emptyset$ is a subset of every set.

(k) Example. Let A and B be sets. We can prove that $A \subset B$ if and only if $\mathcal{P}(A) \subset \mathcal{P}(B)$. First, we can write this using logical notation:

$$A \subset B \iff \mathcal{P}(A) \subset \mathcal{P}(B)$$

Then, recall that we can break this down into two easier statements:

$$A \subset B \implies \mathcal{P}(A) \subset \mathcal{P}(B)$$
$$\mathcal{P}(A) \subset \mathcal{P}(B) \implies A \subset B$$

Following our layout from before:

- Definition of Subset: $A \subset B \iff x \in A \implies x \in B$
- Definition of Power Set: $\mathcal{P}(A) = \{C | C \subset A\}$
- Lemma (L1): If $A \subset B$ and $B \subset C$, then $A \subset C$
- Note that $A \subset B \implies \mathcal{P}(A) \subset \mathcal{P}(B)$ can be rewritten as:

$$A \subset B \implies \left[X \in \mathcal{P}(A) \implies X \in \mathcal{P}(\mathcal{B})\right]$$

  Which is our "hypothesis in the conclusion" logical form and is equivalent to:

$$\left[(A \subset B) \wedge X \in \mathcal{P}(A)\right] \implies X \in \mathcal{P}(B)$$

To show ($\Rightarrow$): $X \in \mathcal{P}(X)$
Proof:

$$\begin{array}{ll} \text{Let } A \subset B \wedge X \in \mathcal{P}(A) & \text{(by hypothesis)} \\ \implies X \subset A & \text{(by def. of the power set)} \\ \implies X \subset B & \text{(by L1)} \\ \implies X \in \mathcal{P}(X) & \text{(by def. of the power set)} \end{array}$$

Since X was an arbitrary element of $\mathcal{P}(A)$, this satifies the definition of of $\mathcal{P}(A) \subset \mathcal{P}(B)$. Further, since we employed an equivalent logical form, this is sufficient to prove $A \subset B \implies \mathcal{P}(A) \subset \mathcal{P}(B)$.
To show ($\Leftarrow$): $A \subset B$
Proof:

$$\begin{array}{ll} \text{Let}\mathcal{P}(A) \subset \mathcal{P}(B) & \text{(by hypothesis)} \\ A \in \mathcal{P}(A) & \text{(by def. of the power set)} \\ \implies A \in \mathcal{P}(B) & \text{(by def. of subset)} \\ \implies A \subset B & \text{(by def. of the power set)} \end{array}$$

Since we have proven that the implication goes in both directions, this is logically equivalent to having proven the biconditional; thus, $A \subset B \iff \mathcal{P}(A) \subset \mathcal{P}(B)$. ■

2. **Set Operations**

(a) Definition. Let A and B be sets. The **union** of A and B, denoted $A \cup B$, is the set

$$A \cup B \equiv \{x | x \in A \vee x \in B\}$$

The **intersection** of A and B, denoted $A \cap B$, is the set

$$A \cap B \equiv \{x | x \in A \wedge x \in B\}$$

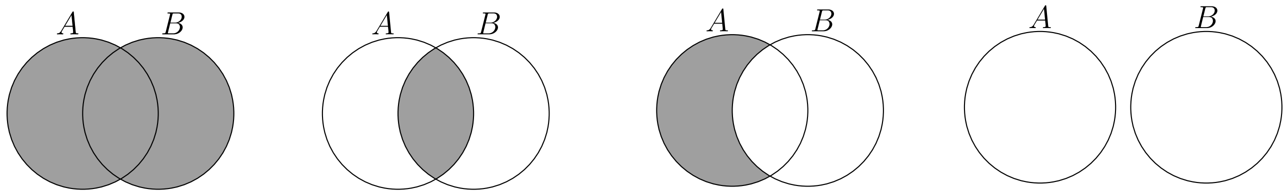
The **difference** of A and B, denoted $A - B$ or $A \backslash B$, is the set

$$A - B \equiv \{x | x \in A \wedge x \notin B\}$$

The sets A and B are **disjoint** if the intersection is empty

$$A \cap B = \emptyset$$

In terms of Venn diagrams:



Union: $A \cup B$ Intersection: $A \cap B$ Difference: $A - B$ Disjoint Sets

(b) Example. Consider the sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then

- $A \cup B = \{1, 2, 3, 4\}$
- $A \cap B = \{2, 3\}$
- $A - B = \{4\}$
- $B - A = \{1\}$

(c) Theorem (SES THM 2.2.1) For all sets A, B, C:

i. Commutative Laws
   - $A \cup B = B \cup A$
   - $A \cap B = B \cap A$

ii. Associate Laws
   - $A \cup (B \cup C) = (A \cup B) \cup C$
   - $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributive Laws
   - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) Example. Consider the first of the distributive laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We can prove this statement by showing that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$, then showing that the reverse is true as well. Instead, we can also do a biconditional proof.

- Distributive Laws (Connectives): $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$
- Definitions of set union ($\cup$) and intersection ($\cap$)

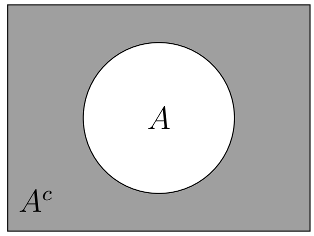
To show: $x \in (A \cap B) \cup (A \cap C)$
Proof:

$$\begin{aligned}
&\text{Let } x \in \big(A \cap (B \cup C)\big) && \text{(by hypothesis)}\\
\iff\ & \big(x \in A\big) \wedge \big(x \in B \vee x \in C\big) && \text{(by def. of } \cap \text{ and } \cup)\\
\iff\ & \big(x \in A \wedge x \in B\big) \vee \big(x \in A \wedge x \in C\big) && \text{(by distributivity)}\\
\iff\ & \big(x \in A \cap B\big) \vee \big(x \in A \cap C\big) && \text{(by def. of } \cap)\\
\iff\ & x \in (A \cap B) \cup (A \cap C) && \text{(by def. of } \cup)
\end{aligned}$$

■

Since x is an arbitrary element, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Note that the implications go in both directions; we could have started at the bottom and written the same proof in the opposite order.

(e) Definition. Let U be the universe of discourse and $A \subset U$. The **complement** of A is the set $A^c = U - A$. Graphically:



(f) Theorem (SES THM 2.2.2). Let U be the universe of discourse and let A and B be subsets of U. Then

i. $(A^c)^c = A$

ii. $A \cup A^c = U$

iii. $A \cap A^c = \emptyset$

iv. DeMorgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

(g) Aside. Note that these look very similar to our logical connective definitions, including have the same name (DeMoran's Laws)! This should come as no surprise, given how our definitinos of $\cap$ and $\cup$ include our logical connectives.

(h) Definition. A set of sets is known as a **family** of sets. If $\mathcal{A}$ is a family of sets, then the **union** over $\mathcal{A}$ is

$$\bigcup_{A \in \mathcal{A}} A = \{x | x \in A \text{ for some } A \in \mathcal{A}\}$$

Similarly, the **intersection** over $\mathcal{A}$ is

$$\bigcap_{A \in \mathcal{A}} = \{x | x \in A \text{ for every} A \in \mathcal{A}\}$$

(i) Definition. An **ordered pair** is an object formed from two entities a and b, and is denoted (a, b). Ordered pairs have the **coordinate property**, where (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

(j) Aside. We can extend this property to the idea of ordered n-tuples, which also have the coordinate property, just in higher dimensions. These come up a lot in economics: vectors, like the vector of goods an agent buys or a price vector, are ordered n-tuples. Also, note that vector-based programming languages like MatLab and R build everything on ordered n-tuples.

(k) Definition. Let A and B be sets. The **cross product** of A and B is

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

which is read "A cross B."

(l) Example. Consider the set $\mathbb{R} \times \mathbb{R} = \{(x, y) | x \in \mathbb{R} \wedge y \in \mathbb{R}\}$ This is exactly our two-dimensional Cartesian coordinate system, $\mathbb{R}^2$. Note that we do not need to limit ourselves to two dimensions; indeed, $\mathbb{R}^n$ is simply a a cross product of $\mathbb{R}$ n times.

3. **Cardinality**

(a) Aside. We won't spend too much time on cardinality, although it is a concept that comes up in utility theory, particularly the distinction between finite and infinite sets.

(b) Definition. Two sets A and B are **equivalent** $A \approx B$ iff there exists a one-to-one function from A to B. If two sets are equivalent, they have the same **cardinality**.

(c) Example. The sets defined by

$$A = \{\text{cheese}, \text{yogurt}, \text{butter}\} \quad \text{and} \quad B = \{1, 2, 3\}$$

Are equivalent sets. For finite sets such as these, the definition is not very helpful (we can just count elements in each); any 1-1 mapping between the sets is sufficient to show that they're equivalent.

(d) Definition. Let $\mathbb{N}_k = \{1, 2, 3, ..., k\}$. A set S is **finite** if and only if $S = \emptyset$ or $S \equiv \mathbb{N}_k$ for some $k \in \mathbb{N}$. A set is **infinite** if and only if it is not finite. A set S is countable if and only if it has the same cardinality as a subset of the natural numbers $\mathbb{R}$.

(e) Example. Consider the set of integers, $\mathbb{Z}$. It has the same cardinality as the natural numbers, $\mathbb{N}$. We can show this by coming up with a one-to-one mapping between $\mathbb{N}$ to $\mathbb{Z}$. Define a function $f : \mathbb{N} \to \mathbb{R}$ to be

$$f(x) = \begin{cases} \dfrac{x-1}{2} & \text{(if } x \text{ is odd)} \\ -\dfrac{x}{2} & \text{(if } x \text{ is even)} \end{cases}$$

Using this function, we can map to every element in $\mathbb{Z}$ from an element in $\mathbb{N}$! The natural numbers are a proper subset of the integers, yet we can map them one-to-one. Even more surprising, $N \times N$ has the same cardinality as $\mathbb{N}$, proven by Gregory Cantor.

(f) Theorem. The numbers in the open inverval $(0, 1)$ form an uncountable set.

(g) Aside. Basically, since you can keep dividing things infinitely in an interval, there is no way to map the natural numbers to the interval $(0, 1)$. This can be proven rigorously, but we'll take this as given for now.

(h) Aside. Previously, we discussed the existence of a utility function based on a few assumptions about the preference relation. In particular, a utility function exists when the preference relation is complete (you can rank everything) and transitive if the universe of discourse is a *finite set*. When it's infinite, then what? If it's countable, we're still okay, a utility function exists! When we're dealing with an *uncountably* infinite set (e.g., $\mathbb{R}^2$), things get more difficult. To guarantee the existence of a utility function, we need to add an additional assumption (known as continuity) about the preference relation.