Econ 241A Probability, Statistics and Econometrics
Fall 2017

Problem Set 4

1. Assume X and Y are jointly distributed with pdf $f(x,y) > 0$ for $(x,y) \in \mathcal{W}$, $\mathcal{W} \subset \mathbb{R}^2$. The marginals of X and Y are given by $f(x)$ with support $\mathcal{X}$ and $f(y)$ with support $\mathcal{Y}$. Define $g(X)$ as a function only of X. Prove that $\mathbb{E}(g(x)) = \int_{x:x\in\mathcal{X}} g(x)f(x)dx$.

$$\begin{aligned}
\mathbb{E}(g(X)) &= \int_{\mathcal{X}}\int_{\mathcal{Y}} g(x)f_{X,Y}(x,y)dydx \\
&= \int_{\mathcal{X}} g(x)\left[\int_{\mathcal{Y}} f_{X,Y}(x,y)dy\right]dx \\
&= \int_{\mathcal{X}} g(x)f_X(x)dx
\end{aligned}$$

2. For the joint pmf in the table below:

	$x=1$	$x=2$	$x=3$
$y=0$	0.10	0.10	0.10
$y=1$	0.10	0.40	0.20

(a) Find the conditional expectation function $\mathbb{E}(Y|X)$
First, find the conditional distributions (pmfs) of Y|X for different realizations of the random variable X:

$$f_{Y|X}(y|x=1) = \begin{cases} 0.5 & \text{if } y=0 \\ 0.5 & \text{if } y=1 \end{cases}$$

$$f_{Y|X}(y|x=2) = \begin{cases} 0.2 & \text{if } y=0 \\ 0.8 & \text{if } y=1 \end{cases}$$

$$f_{Y|X}(y|x=3) = \begin{cases} \frac{1}{3} & \text{if } y=0 \\ \frac{2}{3} & \text{if } y=1 \end{cases}$$

And of course, 0 anywhere else. So we can say $\mathbb{E}(Y|x=1) = 0.5$, $\mathbb{E}(Y|x=2) = 0.8$, and $\mathbb{E}(Y|x=3) = 2/3$. However, if $E = \mathbb{E}(Y|X)$ is itself a random variable, so we may be interested in $f_E(e)$. To figure out what this is, note that E only takes on three values: 0.5, 0.75 and 1/3, and it takes on these values according to the marginal frequency of X. From the table, we can deduce $f_X(1) = 0.2$, $f_X(2) = 0.5$, and $f_X(3) = 0.3$. Thus, we can conclude:

$$f_{\mathbb{E}(Y|X)}(e) = \begin{cases} 0.2 & \text{if } e = 0.5 \\ 0.5 & \text{if } e = 0.75 \\ 0.3 & \text{if } e = 1/3 \end{cases}$$

(b) Find the best linear predictor $\mathbb{E}^*(Y|X)$

The best linear predictor is given by $\mathbb{E}^*(Y|X) = \alpha+\beta X$, where $\beta = Cov(X,Y)/Var(X)$ and $\alpha = \mathbb{E}(Y) - \beta\mathbb{E}(X)$. Start of calculating β: for the numerator, $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$, and

$$\mathbb{E}(XY) = \sum_x \sum_y xyf_{X,Y}(x,y) = 0.10 + 2 \times 0.40 + 3 \times 0.2 = 1.50$$

And where $\mathbb{E}(X) = \sum_x \sum_y xf_{X,Y}(x,y) = 0.20 + 2 \times 0.50 + 3 \times 0.30 = 2.1$ and $\mathbb{E}(Y) = \sum_x \sum_y yf_{X,Y}(x,y) = 0.7$. Thus $Cov(X,Y) = 1.50 - 1.47 = 0.03$.

Similarly, $Var(X) = 0.49$. Then $\beta = 0.03/0.49 = 0.0612$ and $\alpha = 0.7 - 0.0612 \times 2.1 = -0.5714$.

(c) Prepare a table that gives $\mathbb{E}(Y|x)$ and $\mathbb{E}^*(Y|x)$ for $x = 1,2,3$.

Use the info from part (a) for $\mathbb{E}(Y|x)$ and from part (b) for $\mathbb{E}^*(Y|x)$, and we get:

	$x=1$	$x=2$	$x=3$
$\mathbb{E}(Y\|x)$	0.5	0.75	2/3
$\mathbb{E}^*(Y\|x)$	0.6327	0.6939	0.7551

3. Assume X and Y are joinly distributed with pdf $f(x,y) = x+xy$, $0 \le x \le 1$ and $0 \le y \le 1$. Define the bivariate random vector (U,V) as $U = X$ and $V = \sqrt{Y}$.

(a) Are X and Y independent?

Since $f(x,y) = x+xy = x(1+y)$, one might feel tempted to say they are independent. Nevertheless, $f(x,y)$ does not integrate to 1, then it is not the case that f is a pdf.

(b) Are U and V independent?

Similarly for this case.

(c) Find the marginal pdf of V.

$$f_V(v) = v + v^3$$

In addition, solve the following problems from Casella and Berger: 4.19 (a) (Hint: What is the distribution of the square of a standard normal rv (Ch 2)? Does this result surprise you given that X_1 and X_2 are iid?), 4.20, 4.22, 4.26, 4.30 (Hint for part b: does the pdf of $Y|x$ change for different values of x?) , 4.44, 4.47, 4.50 and 4.58 (a), (b) and (c).

4.19 If X_1 and X_2 are independent, standard normal random variables, what is the pdf of $(X_1 - X_2)^2/2$

First, note that because these are independent normal random variables, we have tricks that help us. In fact, we know:

$$\frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)$$

because the means are subtracted $(0-0)$, the variances add and are then reduced by the denominator. Finally, we know from the hint and Chapter 2 that a standard normal random variable squared is distributed chi-squared with one degree of freedom:

$$\frac{(X_1 - X_2)^2}{2} \sim \chi_1^2$$

4.20 If X_1 and X_2 are independent random variables distributed $N(0, \sigma^2)$ then (a) what is the joint pdf $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_1/\sqrt{Y_1}$, and (b) are they independent?

(a) This problem is somewhat more subtle than many we've seen. Namely, the function mapping into Y_2 can defined on three over three partitions, $A_0 = \{x_1 \in \mathbb{R}, x_2 = 0\}$, $A_1 = \{x_1 \in \mathbb{R}, x_2 > 0\}$, and $A_2 = \{x_1 \in \mathbb{R}, x_2 < 0\}$. As we've partitioned the space, the inverse transformation will have two monotonic sections. The support of (Y_1, Y_2) is $B = \{y_1 \in [0, \infty), y_2 \in [-1, 1]\}$.

The inverse map from B to A_1 is $x_1 = y_2\sqrt{y_1}$ and $x_2 = \sqrt{y_1 - y_1 y_2^2}$. The Jacobian has determinant $\frac{1}{2\sqrt{1-y_2^2}}$. The other inverse map (from B to A_2) has $x_1 = y_2\sqrt{y_1}$ and $x_2 = -\sqrt{y_1 - y_1 y_2^2}$, so $|J_2| = -|J_1|$. This is not dissimilar from Example 4.3.6. in the book.

Recall that we must sum over the r partitions, i.e. $f_{Y_1,Y_2} = \sum_r f_{X_1,X_2}(h_{1,r}, h_{2,r})|J_r|$. In this case, the two partitions yield the same expression, so we have:

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= 2\left(\frac{1}{2\pi\sigma} e^{-y_1/(2\sigma^2)} \frac{1}{2\sqrt{1-y_2^2}}\right) \\
&= \frac{1}{2\pi\sigma\sqrt{1-y_2^2}} e^{-y_1/(2\sigma^2)} \text{ for } Y_1, Y_2 \text{ in the support, 0 else.}
\end{aligned}
$$

(b) The above expression is easily factorable, thus the random variables are independent. The geometric interpretation is weird. What is it?

4.22 see book

Transformation are monotonic in this case, so no need to partition. We define the inverse maps as $h_1(u, v) = (u - b)/a$ and $h_2(u, v) = (v - d)/c$. The determinant of the Jacobian is:

$$
|J| = \begin{vmatrix} 1/a & 0 \\ 0 & 1/c \end{vmatrix} = \frac{1}{ac}
$$

and thus we can say:

$$
f_{U,V}(u, v) = \frac{1}{ac} f_{X,Y}\left(\frac{u-b}{a}, \frac{v-d}{c}\right)
$$

4.26 (a)

$$
\begin{aligned}
P(Z \le z, W = 0) =& P(\min(X, Y) \le z, Y \le X) = P(Y \le z, Y \le X) \\
=& \int_0^z \int_y^\infty \lambda e^{-\lambda x} \mu e^{-\mu y} dx dy \\
=& \frac{\mu}{\mu + \lambda} (1 - exp(-(\mu + \lambda) z)
\end{aligned}
$$

Similarly,

$$
P(Z \le z, W = 1) = \frac{\lambda}{\mu + \lambda} (1 - exp(-(\mu + \lambda) z)
$$

(b)

$$P(W=0) = P(Y \le X) = \int_0^0 \int_y^\infty \lambda e^{-\lambda x} \mu e^{-\mu y} dx dy = \frac{\mu}{\mu+\lambda}$$
$$P(W=1) = \frac{\lambda}{\mu+\lambda}$$
$$P(Z \le z) = P(Z \le z, W=1) + P(Z \le z, W=0) = 1 - (1 - exp(-(\mu+\lambda)z)$$

Therefore, Z and W are independent.

4.30 (a) Find $E[Y]$, $Var[Y]$, and $Cov[X, Y]$.
From the problem setup, we know $Y|X \sim N(X, X^2)$ and $X \sim U[0, 1]$. So we have:

$$\begin{aligned} E[Y] &= E[E[Y|X]] = E[X] = 1/2 \\ Var(Y) &= Var(E[Y|X]) + E[Var(Y|X)] = Var(X) + E[X^2] = 5/12 \\ Cav(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[E[XY|X]] - E[X]E[Y] \\ &= E[XE[Y|X]] - E[X]E[Y] \\ &= E[X]^2 - E[X]E[Y] = 1/12 \end{aligned}$$

(b) Prove that Y/X and X are independent.
Let $Z = Y/X$. Well, $Z \sim N(1, 1)$, which is independent of X. Formally, bivariate transformations should yield the appropriate result.

4.44

$$Var\left(\sum_{i=1}^n X_i\right) = Var(L'X)$$

where $L = (1, 1, \ldots, 1)'$ is a n-dimensional vector of ones

$$\begin{aligned} &= E[(L'X - E[L'X])^2] \\ &= E[(L'(X - E[X]))^2] \\ &= E[L'(X - E[X])(X - E[X])'L] \\ &= L'E[(X - E[X])(X - E[X])']L \end{aligned}$$

where $E[(X - E[X])(X - E[X])']$ is a $n \times n$ matrix with $Var(X_i)$ on the diagonal positions and $Cov(X_i, X_j)$ are the off-diagonal positions. Given that L is a vector of ones it is easy to see that

$$Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i) + 2\sum_{i=1}^n \sum_{j>i}^n Cov(X_i, X_j)$$

4.47 (a)

$$\begin{aligned} P(Z \le z) =& P(X \le z, XY > 0) + P(-X \le z, XY < 0) && \text{(by definition of } Z) \\ =& P(X \le z, Y < 0) + P(X \ge -z, Y < 0) && \text{(by } z < 0) \\ =& P(X \le z)P(Y < 0) + P(X \ge -z)P(Y < 0) && \text{(by independence)} \\ =& P(X \le z)0.5 + P(X \le z)0.5 && \text{(by symmetry and } med(Y) = 0) \\ =& P(X \le z) \end{aligned}$$

Similarly for $z > 0$. Then $Z \sim n(0,1)$.

(b) $Z > 0$ if and only if i. $X < 0$ and $Y > 0$, or ii. $X > 0$ and $Y > 0$. So Z and Y have the same sign which implies they cannot be bivariate normal.

4.50 We know that

$$\begin{aligned}
E[Y|X] &= \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(X - u_X) = \rho X \\
Var(Y|X) &= (1-\rho^2)\sigma_Y = (1-\rho^2) \\
Cov(X,Y) &= Cov(X, E[Y|X]) = cov(X, \rho X) = \rho Var(X) = \rho \\
Corr(X,Y) &= \rho
\end{aligned}$$

Similarly,

$$\begin{aligned}
Cov(X^2, Y^2) =& Cov(X^2, E[Y^2|X]) \\
=& Cov(X^2, Var(Y|X) + (E[Y|X])^2) \\
=& Cov(X^2, (1-\rho^2) + (\rho X)^2) \\
=& Cov(X^2, \rho^2 X^2) \\
=& \rho^2 Var(X^2) \\
=& \rho^2(E[X^4] - (E[X^2])^2) \\
=& \rho^2(3-1) \\
=& 2\rho^2
\end{aligned}$$

So

$$Corr(X^2, Y^2) = 2\rho^2/\sqrt{4} = \rho^2$$

4.58 (a) By definition of Cov and LIE, we have

$$\begin{aligned}
Cov(X, E(Y|X)) &= E[(X-\mu_X)(E[Y|X]-\mu_Y)] \\
&= E[XE[Y|X] + \mu_X\mu_Y - \mu_X E[Y|X] - \mu_Y E[X]] \\
&= E[E[XY|X]] + \mu_X\mu_Y - \mu_X E[E[Y|X]] - \mu_Y\mu_X && \text{(conditioning theorem)} \\
&= E[XY] + \mu_X\mu_Y - \mu_X\mu_Y - \mu_Y\mu_X && \text{(LIE)} \\
&= E[XY] - \mu_X\mu_Y \\
&= Cov(X,Y) && \text{(definition of } Cov\text{)}
\end{aligned}$$

(b)

$$Cov(X, Y - E(Y|X)) = Cov(X, E[Y|X] - E[Y|X])$$

where 4.58 (a), linearity of $E[\cdot|X]$ and conditioning theorem were used

$$= Cov(X, 0) = 0 \qquad \text{(Covariance of a RV and a constant is zero)}$$

(c)

$$\begin{aligned}
Var(Y - E[Y|X]) &= E[(Y - E[Y|X])^2] && \text{(def. of variance and } E[Y] = E[E[Y|X]]\text{)} \\
&= E[E[(Y - E[Y|X])^2]|X] && \text{(LIE)} \\
&= E[Var(Y|X)] && \text{(definition of conditional variance)}
\end{aligned}$$