Econ 241A Prof. Startz

Final Fall 2016

Please answer all questions. Show your work.

The exam is open book/open note; closed any devices that can communicate. (No laptops, cell phones, Morse code keys, signal fires, etc.)

1 Let $y_i = \beta x_i + \epsilon_i$ with $\frac{1}{n}\sum x_i\epsilon_i \xrightarrow{p} \mathbb{E}[x_i\epsilon_i] = 0$ and $\frac{1}{n}\sum x_i^2 \xrightarrow{p} \mathbb{E}[x_i^2] = M < \infty$

Define $\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$

Prove that $\hat{\beta} \xrightarrow{p} \beta$

2 Let $x_1, x_2, \ldots$ be a sequence of iid random variables where:

$$x_n = \begin{cases} \mu & \text{with prob } \frac{n-2}{n} \\ 3\mu & \text{with prob } \frac{1}{n} \\ -\mu & \text{with prob } \frac{1}{n} \end{cases}$$

a) Find $\mathbb{E}[x_n]$

b) Does x_n converge in probability to μ?

c) Does x_n converge almost surely to μ?

3 You find yourself in a weird bar in Las Vegas. The bartender (who you trust) tells you there are only three types of coins allowed: $P(Heads) = 0.25$, $P(Heads) = 0.5$, and $P(Heads) = 0.75$. Denote the probability of Heads as μ. You observe 2 Heads and 1 Tails, let $X = \{H, H, T\}$.

Your uninformed prior is that $P(\mu) = \begin{cases} \frac{1}{3} & \text{if } \mu = 0.25 \\ \frac{1}{3} & \text{if } \mu = 0.5 \\ \frac{1}{3} & \text{if } \mu = 0.75 \end{cases}$

a) Calculate $P(\mu = 0.5|X)$

Your trustworthy bartender tells you a better prior is that $P(\mu) = \begin{cases} \frac{1}{4} & \text{if } \mu = 0.25 \\ \frac{1}{2} & \text{if } \mu = 0.5 \\ \frac{1}{4} & \text{if } \mu = 0.75 \end{cases}$

b) Calculate $P(\mu = 0.5|X)$

Suppose you were convinced that $P(0.5|X) = 0.05$, and that $P(\mu) = \begin{cases} \frac{1}{2}(1-\frac{1}{z}) & \text{if } \mu = 0.25 \\ \frac{1}{z} & \text{if } \mu = 0.5 \\ \frac{1}{2}(1-\frac{1}{z}) & \text{if } \mu = 0.75 \end{cases}$

c) Solve for z.

4 Consider the following model in which s_i is an observed signal of person i's true ability, $a_i = a_i^0 + a_i^1$, and ϵ_i is an observational error. a_i^0 is inherent ability and a_i^1 is acquired ability. For convenience, we define the symbols $\bar{a} = \mathbb{E}[a_i]$ and $\sigma_a^2 = Var(a_i)$. Ability and the error are distributed joint normal

$$\begin{pmatrix} a_i^0 \\ a_i^1 \\ \epsilon_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \bar{a}^0 \\ \bar{a}^1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & \sigma_{01} & 0 \\ \sigma_{01} & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_\epsilon^2 \end{pmatrix} \right)$$

The signal is generated according to

$$s_i = a_i + \epsilon_i$$

Agents are paid according to their expected ability; income equals

$$y_i = w * \mathbb{E}[a_i|s_i]$$

Agents purchase ability at cost $C(a_i^1)$, which is the increasing function

$$C(a_i^1) = c_1 a_i^1 + \frac{1}{2} c_2 (a_i^1)^2$$

Note that income depends on total ability while costs depend only on acquired ability. While ϵ is observable by the econometrician, it is not observable by either the agent nor by the employer at the time decisions are made.

a) Find the formula for $\mathbb{E}[a_i|s_i]$ in terms of the parameters $\{\bar{a}, \sigma_a^2, \sigma_\epsilon^2\}$

b) Set up the agent's maximization problem and solve for a_i^1 in terms of $\{\bar{a}, \sigma_0^2, \sigma_1^2, \sigma_{01}, \sigma_\epsilon^2\}$

c) Using your results, solve for the values of $\bar{a}$ and σ_a^2

d) What is the expected value of net income?

e) Show that the expected value of net income is decreasing in σ_ϵ^2. Explain what this means in two sentences or less.

5 Your utility at time t is given by $u_t = \alpha + \gamma x_t + \delta y_t$

where x_t is a measure of your net wealth and is distributed iid $Pareto(1, \beta) \Rightarrow f_x(x_t) = \frac{\beta}{x_t^{\beta+1}}$ for $x_t \in [1, \infty)$

(Assume $\beta > 1$)

and y_t is a measure of your leisure time and is distributed iid $N(\mu, \sigma^2)$

Assume x_t and y_t are independent.

a) Find $\mathbb{E}[u_t]$

b) Find the Maximum Likelihood Estimator of β (Hint: this doesn't depend on y_t)

c) Find the Cramer Rao Lower Bound for an unbiased $\hat{\beta}$

d) Write down a Likelihood Ratio Test for $H_0 : \beta = \beta_0$

e) Find a Method of Moments Estimator for σ^2 (Hint: this doesn't depend on x_t)

f) Is it unbiased?

g) Find $Var(\hat{\sigma}^2_{MoM})$ (Hint: use that $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(n)}$)

h) Write down a Wald Test for $H_0 : \sigma^2 = \sigma_0^2$

Define $\hat{\theta} = \alpha + \delta\hat{\mu}$ where $\hat{\mu} = \frac{1}{n}\sum y_t$

i) Find the asymptotic distribution of $\hat{\theta}$