Econometrics 710 - Problem Set # 2

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Exercise 2.9

Define the following two dummy variables x_3, x_4

$$x_3 = \begin{cases} 1 \text{ if } x_2 = B \\ 0 \text{ if } x_2 \neq B \end{cases}$$

$$x_4 = \begin{cases} 1 \text{ if } x_2 = C \\ 0 \text{ if } x_2 \neq C \end{cases}$$

The conditional mean of y can only take six possible values;

$$\mathbb{E}(y|x_1, x_2) = \mathbb{E}(y|x_1, x_3, x_4) = \begin{cases} \mu_{01} \text{ when } x_1 = 0, x_2 = A & (x_3 = 0, x_4 = 0) \\ \mu_{02} \text{ when } x_1 = 0, x_2 = B & (x_3 = 1, x_4 = 0) \\ \mu_{03} \text{ when } x_1 = 0, x_2 = C & (x_3 = 0, x_4 = 1) \\ \mu_{11} \text{ when } x_1 = 1, x_2 = A & (x_3 = 0, x_4 = 0) \\ \mu_{12} \text{ when } x_1 = 1, x_2 = B & (x_3 = 1, x_4 = 0) \\ \mu_{13} \text{ when } x_1 = 1, x_2 = C & (x_3 = 0, x_4 = 1) \end{cases}$$

Then, we can write $\mathbb{E}(y|x_1, x_2)$ as follows

$$\mathbb{E}(y|x_1, x_2) = \mathbb{E}(y|x_1, x_3, x_4) = \alpha + \beta x_1 + \gamma x_3 + \delta x_4 + \eta x_1 x_3 + \zeta x_1 x_4$$

where, $\alpha = \mu_{01}, \beta = \mu_{11} - \mu_{01}, \gamma = \mu_{02} - \mu_{01}, \delta = \mu_{03} - \mu_{01}, \eta = \mu_{12} - \mu_{02} - \mu_{11} + \mu_{01}, \zeta = \mu_{13} - \mu_{03} - \mu_{11} + \mu_{01}$

Exercise 2.10

True. The mean independence condition $\mathbb{E}(e|x) = 0$ implies that $\mathbb{E}(h(x)e) = 0$ for any function $h(x)$ as long as $\mathbb{E}(|h(x)e|) < \infty$. Then, by the law of iterated expectations $\mathbb{E}(x^2 e) = \mathbb{E}(\mathbb{E}(x^2 e|x)) = \mathbb{E}(x^2 \mathbb{E}(e|x)) = 0$.

Exercise 2.11

False. Suppose that $x \sim F(x)$, where F is a symmetric distribution around zero (the odd moments of x are all zero); $y = x^2$; and consider linear projection model $y = \beta x + e$. Then $\beta = (\mathbb{E}xx')^{-1}\mathbb{E}(xy) = \frac{\mathbb{E}(x^3)}{\mathbb{E}(x^2)} = 0$, and $e = y - \beta x = x^2$. Therefore $\mathbb{E}(xe) = \mathbb{E}(x^3) = 0$, but $\mathbb{E}(x^2 e) = \mathbb{E}(x^4) \neq 0$, since x^4 is always positive.

Exercise 2.12

False. Mean independence does not imply full independence. There are many counter examples. Consider following example; $y = xu, u \perp x, \mathbb{E}(u) = 1$. Then, $\mathbb{E}(y|x) = x\mathbb{E}(u|x) = x$. Consider the CEF error, $e = y - \mathbb{E}(y|x) = x(u-1)$. Although $\mathbb{E}(e|x) = 0$, e is not independent of x by construction.

Exercise 2.13

False. You can use previous counter example of Exercise 2.11. Let's consider similar (vector) example. Suppose random variable $x \sim N(0,1), y = x^2$, and $\mathbf{x} = \begin{pmatrix} 1 \\ x \end{pmatrix}$. Consider projection model $y = \mathbf{x}'\beta + e$, where $\beta = (\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}\mathbf{x}y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e = x^2 - 1$. $\mathbb{E}\mathbf{x}e = \mathbb{E}\begin{pmatrix} e \\ xe \end{pmatrix} = 0$. However, $\mathbb{E}(e|\mathbf{x}) = \mathbb{E}(x^2 - 1|\mathbf{x}) = x^2 - 1 \neq 0$

Exercise 2.14

False. Mean independence and homoskedasticity do not imply that the random variables $\mathbf{x}$ and e are independent.

Consider the following counter example; $y = xu, \mathbb{E}(u|x) = 1, \mathrm{var}(u|x) = \sigma^2/x^2$. Consider the CEF error, $e = y - \mathbb{E}(y|x) = xu - x\mathbb{E}(u|x) = x(u-1)$. Even though $\mathbb{E}(e|x) = 0$ and $\mathbb{E}(e^2|x) = \mathbb{E}(x^2(u-1)^2|x) = x^2\mathbb{E}((u-1)^2|x) = x^2 var(u|x) = x^2(\sigma^2/x^2) = \sigma^2$, e and x is not independent by construction.

Exercise 2.15

The best linear predictor in the intercept-only model (same as linear projection coefficient in here) is defined as follows;

$$\alpha = \underset{\alpha\in\mathbb{R}}{\arg\min}\, S(\alpha) = \underset{\alpha\in\mathbb{R}}{\arg\min}\, \mathbb{E}(y-\alpha)^2 = \underset{\alpha\in\mathbb{R}}{\arg\min}\, \mathbb{E}y^2 - 2\alpha\mathbb{E}y + \alpha^2$$

FOC yields $\alpha = \mathbb{E}y$, which is unique and well-defined if $\mathbb{E}y^2 < \infty$

(We can also derive this from the general definition of the best linear predictor $\mathbf{x}'\beta = \mathbf{x}'(\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}(\mathbf{x}y)$ with $\mathbf{x} = 1$)

Exercise 2.16

The best linear predictor and the conditional mean are different in this exercise, since $m(x)$ is a non-linear function of x.

Since $f(y|x) = \frac{f(x,y)}{\int_0^1 f(x,y)dy} = \frac{(3/2)(x^2+y^2)}{(3/2)x^2+1/2} 1\{0 \leq y \leq 1\}$, the conditional mean function $m(x)$ is equal to $\mathbb{E}(y|x) = \int_0^1 yf(y|x)dy = \frac{(3/4)x^2+3/8}{(3/2)x^2+1/2}$. Also, $\mathbb{E}y = \mathbb{E}x = \int_0^1 xf(x)dx = 5/8, \mathbb{E}x^2 = \int x^2 f(x)dx = 7/15, \mathbb{E}xy = \int_0^1\int_0^1 xyf(x,y)dxdy = 3/8$. Therefore, we can compute the coefficients of the best linear predictor α, β as follows; $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\mathbb{E}x^2 - (\mathbb{E}x)^2} \begin{pmatrix} \mathbb{E}x^2\mathbb{E}y - \mathbb{E}x\mathbb{E}xy \\ \mathbb{E}xy - \mathbb{E}x\mathbb{E}y \end{pmatrix} = \begin{pmatrix} 55/73 \\ -15/73 \end{pmatrix}$. Then. the best linear predictor $\mathcal{P}(y|x) = \alpha + \beta x = \frac{55}{73} - \frac{15}{73}x$ and $m(x)$ are different.

Exercise 2.17

($\Leftarrow$) If $m = \mu$, $s = \sigma^2$, then

$$\mathbb{E}g(x|m,s) = \mathbb{E}\begin{pmatrix} x - m \\ (x-m)^2 - s \end{pmatrix} = \begin{pmatrix} \mathbb{E}(x) - \mu \\ \mathbb{E}(x-\mu)^2 - \sigma^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

($\Rightarrow$) If $\mathbb{E}g(x|m,s) = 0$, then

$$\mathbb{E}\begin{pmatrix} x - m \\ (x-m)^2 - s \end{pmatrix} = \begin{pmatrix} \mathbb{E}(x) - m \\ \mathbb{E}(x-m)^2 - s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, $m = \mathbb{E}(x) = \mu$ and $s = \mathbb{E}(x-\mu)^2 = \sigma^2$.

Exercise 2.18

(a)

$$Q_{\mathbf{xx}} = \mathbb{E}\mathbf{xx}' = \begin{pmatrix} 1 & \mathbb{E}(x_2) & \mathbb{E}(x_3) \\ \mathbb{E}(x_2) & \mathbb{E}(x_2^2) & \mathbb{E}(x_2x_3) \\ \mathbb{E}(x_3) & \mathbb{E}(x_2x_3) & \mathbb{E}(x_3^2) \end{pmatrix}.$$

Since $x_3 = \alpha_1 + \alpha_2 x_2$, the last column of $Q_{\mathbf{xx}}$ is a linear combination of the first two columns.

$$\begin{pmatrix} \mathbb{E}x_3 \\ \mathbb{E}x_2x_3 \\ \mathbb{E}x_3^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2\mathbb{E}x_2 \\ \alpha_1\mathbb{E}x_2 + \alpha_2\mathbb{E}x_2^2 \\ \alpha_1\mathbb{E}x_3 + \alpha_2 x_2x_3 \end{pmatrix} = \alpha_1\begin{pmatrix} 1 \\ \mathbb{E}x_2 \\ \mathbb{E}x_3 \end{pmatrix} + \alpha_2\begin{pmatrix} 1 \\ \mathbb{E}x_2 \\ \mathbb{E}x_3 \end{pmatrix}$$

Thus the rank of $Q_{\mathbf{xx}}$ is equal to 2; not full rank. Therefore, $Q_{\mathbf{xx}}$ is not invertible.

(b) $\mathbf{x} = A\mathbf{z}$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 1 \\ x_2 \end{pmatrix}$. Note that $Q_{\mathbf{zz}} = \mathbb{E}(\mathbf{zz}')$ is invertible.

The best linear predictor of y given $\mathbf{x}$ is

$$\begin{aligned} \mathcal{P}(y|\mathbf{x}) &= \mathbf{x}'\beta \quad \text{where} \quad \beta = \underset{\beta\in\mathbb{R}^3}{\arg\min}\ \mathbb{E}(y - \mathbf{x}'\beta)^2 \\ &= \mathbf{z}'(A'\beta) \quad \text{where} \quad \beta = \underset{\beta\in\mathbb{R}^3}{\arg\min}\ \mathbb{E}(y - \mathbf{z}'(A'\beta))^2 \\ &= \mathbf{z}'(A'\beta) \quad \text{where} \quad A'\beta = \underset{A'\beta\in\mathbb{R}^2}{\arg\min}\ \mathbb{E}(y - \mathbf{z}'(A'\beta))^2 \\ &= \mathbf{z}'\delta \quad \text{where} \quad \delta = \underset{\delta\in\mathbb{R}^2}{\arg\min}\ \mathbb{E}(y - \mathbf{z}'\delta)^2 \\ &= \mathbf{z}'(\mathbb{E}(\mathbf{zz}'))^{-1}\mathbb{E}\mathbf{z}y \\ &= \delta_1 + \delta_2 x_2 = \mathbb{E}y + \frac{\mathrm{cov}(x_2, y)}{\mathrm{var}(x_2)}(x_2 - \mathbb{E}x_2) \end{aligned}$$

The first and second equality holds by the definition of best linear predictor, and the way we define $A\mathbf{z}$. Note that minimizer β is not unique, and hence is not identified. Third equality immediately follows from the previous step. In the fourth equality, we define $\delta = A'\beta = \begin{pmatrix} \beta_1 + \alpha_1\beta_3 \\ \beta_2 + \alpha_2\beta_3 \end{pmatrix}$. Fifth equation uses uniqueness of the linear projection coeffcient of y given $\mathbf{z}$, i.e. δ, provided that $Q_{\mathbf{zz}}$ is invertible. The last equation uses the (usual) calculations $\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} \mathbb{E}y - \delta_2\mathbb{E}x_2 \\ \frac{\text{cov}(x_2,y)}{\text{var}(x_2)} \end{pmatrix}$

Since $\mathbf{x}$ is not full rank, we cannot identify all the parameters of β, however we can identify δ and the best linear predictor is well defined.

Exercise 2.19

$$\begin{aligned} \beta &= \arg\min_{\beta\in\mathbb{R}^k} \mathbb{E}(m(\mathbf{x}) - \mathbf{x}'\beta)^2 \\ &= \arg\min_{\beta\in\mathbb{R}^k} \mathbb{E}m(\mathbf{x})^2 - 2\mathbb{E}(m(\mathbf{x})\mathbf{x}')\beta + \beta'(\mathbb{E}\mathbf{x}\mathbf{x}')\beta \end{aligned}$$

FOC :

$$-2\mathbb{E}(\mathbf{x}m(\mathbf{x})) + 2(\mathbb{E}\mathbf{x}\mathbf{x}')\beta = 0$$

Thus,

$$\begin{aligned} \beta &= (\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}(\mathbf{x}m(\mathbf{x})) \quad \text{(eq 2.46)} \\ &= (\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}(\mathbf{x}\mathbb{E}(y|\mathbf{x})) \\ &= (\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}(\mathbb{E}(\mathbf{x}y|\mathbf{x})) \\ &= (\mathbb{E}\mathbf{x}\mathbf{x}')^{-1}\mathbb{E}(\mathbf{x}y) \quad \text{(eq 2.47)} \end{aligned}$$

The last equation holds by the law of iterated expectations.

Exercise 2.20

For all measurable sets $\mathcal{X} \subset \mathbb{R}^k$

$$\begin{aligned} \mathbb{E}(1(\mathbf{x}\in\mathcal{X})m(\mathbf{x})) &= \int_{\mathbb{R}^k} 1(\mathbf{x}\in\mathcal{X})m(\mathbf{x})f_x(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbb{R}^k} 1(\mathbf{x}\in\mathcal{X})\left(\int_{\mathbb{R}} yf_{y|\mathbf{x}}(y|\mathbf{x})dy\right)f_x(\mathbf{x})d\mathbf{x} \quad \text{(eq 2.7)} \\ &= \int_{\mathbb{R}^k}\int_{\mathbb{R}} 1(\mathbf{x}\in\mathcal{X})yf(y,\mathbf{x})dyd\mathbf{x} \\ &= \mathbb{E}(1(\mathbf{x}\in\mathcal{X})y) \end{aligned}$$

where the third equality holds by Fubini's theorem and $f_{y|\mathbf{x}}(y|\mathbf{x})f_{\mathbf{x}}(\mathbf{x}) = f(y,\mathbf{x})$