Econ241b: PS 1

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1. (a) If $\mathbb{E}[y|x] = \beta_1 x + \beta_0$, find $\mathbb{E}[yx]$ as a function of the moments of x.

$$\begin{aligned}
y &= \mathbb{E}[y|x] + u = \beta_1 x + \beta_0 + u && \text{(standard CEF)}\\
\mathbb{E}[yx] &= \mathbb{E}[(\beta_1 x + \beta_0 + u)x] && \text{(substitute)}\\
&= \mathbb{E}[\beta_1 x^2 + \beta_0 x + ux] && \text{(distribute)}\\
&= \mathbb{E}[\beta_1 x^2] + \mathbb{E}[\beta_0 x] + \mathbb{E}[ux] && \text{(linearity of } \mathbb{E}[\cdot])\\
&= \beta_1 \mathbb{E}[x^2] + \beta_0 \mathbb{E}[x] + \mathbb{E}[ux] && \text{(pull out constants)}\\
\mathbb{E}[ux] &= \mathbb{E}[\mathbb{E}[ux|x]] = \mathbb{E}[x\mathbb{E}[u|x]] = 0 = \mathbb{E}[x \cdot 0] = 0 && (u, x \text{ uncorr.; } \mathbb{E}[u|x] = 0 \text{ by constr.})\\
\Rightarrow \mathbb{E}[yx] &= \beta_1 \mathbb{E}[x^2] + \beta_0 \mathbb{E}[x] && \blacksquare
\end{aligned}$$

First two moments of x are $\mathbb{E}[x]$ and $\mathbb{E}[x^2]$.

(b) Suppose the random variables y and x take only the values 0 and 1 and have the following joint probability distribution:

	$x = 0$	$x = 1$
$y = 0$	a	c
$y = 1$	b	d

- To satisfy the properties of a joint distribution, what must be true of (a; b; c; d)?
  - Total probability must add to 1: $a + b + c + d = 1$.
  - Each probability must be positive: $a, b, c, d \geq 0$.
- Find $\mathbb{E}[y|x]$, $\mathbb{E}[y^2|x]$, and $Var(y|x)$ for $x = 0$ and $x = 1$.

  Note that $a = P(y = 0, x = 0)$, $b = P(y = 1, x = 0)$, $c = P(y = 0, x = 1)$, $d = P(y = 1, x = 1)$. From this, we know $P(x = 0) = a + b, P(x = 1) = c + d)$.

  Conditional probability:
$$P(y = j|x = k) = \frac{P(y = j, x = k)}{P(x = k)}$$

  Conditional expected value is sum of conditional probabilities for each value of y at the given value of x:
$$\mathbb{E}[y|x = k] = \sum_{j=0}^{1} \frac{P(y = j, x = k)}{P(x = k)}$$

  From this, calculate $\mathbb{E}[y|x]$ for $x = 0$ and $x = 1$:

$$\mathbb{E}[y|x = 0] = \frac{0 \cdot a}{a + b} + \frac{1 \cdot b}{a + b} = \frac{b}{a + b}; \qquad \mathbb{E}[y|x = 1] = \frac{d}{c + d}$$

  Since y only takes values of 0 and 1, and $y = y^2$ for each of these, then $\mathbb{E}[y^2|x] = \mathbb{E}[y|x]$ in each case.

$$Var(y|x = 0) = \mathbb{E}[y^2|x = 0] - \mathbb{E}[y|x = 0]^2 = \frac{b^2}{(a + b)^2}; \quad Var(y|x = 1) = \frac{d^2}{(c + d)^2}$$

2. Assume $\mathbb{E}|g(x)y| < \infty$:

   Prove

$$\mathbb{E}[g(x)y|x] = g(x)\mathbb{E}[y|x]$$

$$\begin{aligned}
\mathbb{E}[g(x)y|x] &= \int_{-\infty}^{\infty} g(x)yf_{y|x}(y|x)dy && \text{(def of conditional expectation)} \\
&= g(x)\int_{-\infty}^{\infty} yf_{y|x}(y|x)dy && \text{($g(x)$ not function of y)} \\
&= g(x)\mathbb{E}[y|x] && \text{(def of conditional expectation)} \ \blacksquare
\end{aligned}$$

3. If $y = x\beta + u$, $x^2 \in \mathbb{R}$, then for each of the following statements either establish that they are true or provide a counterexample:

   (a) $\mathbb{E}[u|x] = 0$ implies $\mathbb{E}[x^2u] = 0$

$$\begin{aligned}
\mathbb{E}[x^2u] &= \mathbb{E}[\mathbb{E}[x^2u|x]] && \text{(iterated expectations)} \\
&= \mathbb{E}[x^2\mathbb{E}[u|x]] && \text{(conditionining thm)} \\
&= \mathbb{E}[x^2 \cdot 0] && \text{(given $\mathbb{E}[u|x] = 0$)} \\
\mathbb{E}[x^2u] &= 0 && \blacksquare
\end{aligned}$$

   (b) $\mathbb{E}[xu] = 0$ implies $\mathbb{E}[x^2u] = 0$

   For CEF, distribution of u constructed such that $\mathbb{E}[u] = 0$, but that is not given to us here. Suppose, toward contradiction, that $u\ n(1,1)$ (and thus $\mathbb{E}[u] = 1$, and u is independent of x). Further, suppose $x\ uniform(-1,1)$, so $\mathbb{E}[x] = 0$ and $\mathbb{E}[x^2] = 1/3$.

$$\begin{aligned}
\mathbb{E}[xu] &= \mathbb{E}[\mathbb{E}[xu|x]] && \text{(law of iterated expectations)} \\
&= \mathbb{E}[x\mathbb{E}[u|x]] && \text{(conditioning thm)} \\
\mathbb{E}[u|x] &= \mathbb{E}[u] = 1 && \text{(from supposition)} \\
\mathbb{E}[xu] &= \mathbb{E}[x \cdot 1] && \text{(substitute)} \\
&= \mathbb{E}[x] = 0 && \text{(from supposition)} \ \square
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[x^2u] &= \mathbb{E}[\mathbb{E}[x^2u|x]] && \text{(l.i.e.)} \\
&= \mathbb{E}[x^2\mathbb{E}[u|x]] && \text{(conditioning thm)} \\
\mathbb{E}[x^2u] &= \mathbb{E}[x \cdot 1] && \text{(substitute)} \\
&= \mathbb{E}[x^2] = 1/3 \neq 0 && \text{(from supposition)} \ \blacksquare
\end{aligned}$$

   Having identified a counterexample, we can see that $\mathbb{E}[xu] = 0$ does not imply $\mathbb{E}[x^2u] = 0$.

   (c) $\mathbb{E}[u|x] = 0$ implies $\mathbb{E}[y|x] = x\beta$

$$
\begin{aligned}
y &= x\beta + u && \text{(given)} \\
\mathbb{E}[y|x] &= \mathbb{E}[x\beta + u|x] && \text{(substitute)} \\
&= \mathbb{E}[x\beta|x] + \mathbb{E}[u|x] && \text{(linearity of } \mathbb{E}[\cdot]) \\
&= \beta\mathbb{E}[x|x] + 0 && \text{(lin. of } \mathbb{E}[\cdot] \text{ and subst.)} \\
&= \beta x && \text{(conditioning thm)} \blacksquare
\end{aligned}
$$

(d) $\mathbb{E}[xu] = 0$ implies $\mathbb{E}[y|x] = x\beta$
Suppose toward contradiction that $u\ n(1,1)$ and $x\ uniform(-1,1)$. As above, we can confirm that $\mathbb{E}[xu] = 0$ and $\mathbb{E}[u|x] = \mathbb{E}[u] = 1$ under these conditions.

$$
\begin{aligned}
y &= x\beta + u && \text{(given)} \\
\mathbb{E}[y|x] &= \mathbb{E}[x\beta + u|x] && \text{(substitute)} \\
&= \mathbb{E}[x\beta|x] + \mathbb{E}[u|x] && \text{(linearity of } \mathbb{E}[\cdot]) \\
&= \beta\mathbb{E}[x|x] + 1 && \text{(lin. of } \mathbb{E}[\cdot] \text{ and subst.)} \\
&= \beta x + 1 \neq x\beta && \text{(conditioning thm)} \blacksquare
\end{aligned}
$$

Having identified a counterexample, we see that $\mathbb{E}[xu] = 0$ does not necessarily imply $\mathbb{E}[y|x] = x\beta$.

4. Recall that the conditional variance is

$$\sigma^2(x) = Var(y|x) = \mathbb{E}[(y - \mathbb{E}[y|x])^2|x]$$

Show that the conditional variance can be written as

$$\sigma^2(x) = \mathbb{E}[y^2|x] - \mathbb{E}[y|x]^2$$

$$
\begin{aligned}
\sigma^2(x) &= Var(y|x) = \mathbb{E}[(y - \mathbb{E}[y|x])^2|x] && \text{(given)} \\
&= \mathbb{E}[(y^2 - 2y\mathbb{E}[y|x] + \mathbb{E}[y|x]^2)|x] && \text{(expand)} \\
&= \mathbb{E}[y^2|x] - \mathbb{E}[2y\mathbb{E}[y|x]|x] + \mathbb{E}[\mathbb{E}[y|x]^2|x] && \text{(lin. of } \mathbb{E}[\cdot])
\end{aligned}
$$

$$
\begin{aligned}
y &= \mathbb{E}[y|x] + e && \text{(def of CEF)} \\
\Rightarrow \mathbb{E}[2y\mathbb{E}[y|x]|x] &= 2\mathbb{E}[(\mathbb{E}[y|x] + e)\mathbb{E}[y|x]|x] && \text{(substitution)} \\
&= 2\mathbb{E}[\mathbb{E}[y|x]^2 + e\mathbb{E}[y|x]|x] && \text{(substitution)} \\
&= 2\mathbb{E}[\mathbb{E}[y|x]^2|x] + 2\mathbb{E}[e\mathbb{E}[y|x]|x] && \text{(lin. of } \mathbb{E}[\cdot]) \\
\text{But } \mathbb{E}[e\mathbb{E}[y|x]|x] &= \mathbb{E}[e|x]\mathbb{E}[\mathbb{E}[y|x]|x] = 0 && (\mathbb{E}[e|x] = 0 \text{ by constr.)} \\
\Rightarrow \mathbb{E}[2y\mathbb{E}[y|x]|x] &= 2\mathbb{E}[\mathbb{E}[y|x]^2|x] + 0 && \text{(substitution)} \\
&= 2\mathbb{E}[y|x]^2 && \text{(l.i.e.)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \sigma^2(x) &= \mathbb{E}[y^2|x] - 2\mathbb{E}[y|x]^2 + \mathbb{E}[y|x]^2 && \text{(subst., l.i.e.)} \\
\Rightarrow \sigma^2(x) &= \mathbb{E}[y^2|x] - \mathbb{E}[y|x]^2 && \blacksquare
\end{aligned}
$$