Best Linear Prediction

Econometrics II

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Overview

Reference: B. Hansen Econometrics Chapter 2.18-2.19, 2.24, 2.33

How to approximate $\mathbb{E}(y|x)$*?*

- if x is continuous, $\mathbb{E}(y|x)$ is generally unknown
  - linear approximation $x^{\mathrm{T}}\beta$
    - ★ β is the linear predictor (or projection) coefficient (β_{lpc})
    - ★ β is not $\nabla_x \mathbb{E}(y|x)$
  - β is identified if $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is invertible
- linear prediction error u is uncorrelated with x by construction

Approximate the CEF

- conditional mean $\mathbb{E}\left(y|x\right)$
  - "best" predictor (mean squared prediction error)
  - functional form generally unknown
    - ★ unless x discrete (and low dimension)
- approximate $\mathbb{E}\left(y|x\right)$ with $x^{\mathrm{T}}\beta$
  - linear approximation, thus a linear predictor

1. select β to form "best" linear predictor of y: $\mathcal{P}\left(y|x\right)$
2. select β to form "best" linear approximation to $\mathbb{E}\left(y|x\right)$

- 1 and 2 yield identical β
  - either criterion could be used to define β
  - we use 1 and refer to $x^{\mathrm{T}}\beta$ as the best linear predictor

Best Linear Predictor Coefficient

1. select β to minimize mean-square prediction error

$$S(\beta) = \mathbb{E}\left(y - x^{\mathrm{T}}\beta\right)^2$$

$\beta := \beta_{lpc}$ satisfies

$$\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta_{lpc} = \mathbb{E}(xy) \quad \textit{Solution}$$

2. select β to minimize mean-square approximation error

$$d(\beta) = \mathbb{E}_x\left(\mathbb{E}(y|x) - x^{\mathrm{T}}\beta\right)^2$$

solution satisfies

$$\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta_{lac} = \mathbb{E}(xy) \ \textit{Solution}$$

Identification

- Identification (General)
  - θ and θ' are separately identified iff $\theta \neq \theta' \Rightarrow \mathbb{P}_\theta \neq \mathbb{P}_{\theta'}$

Identification - Background

- Identification (Best Linear Predictor)
  - β and β' are separately identified iff $\left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$ from $\mathbb{P}_\beta$ does not equal $\left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$ from $\mathbb{P}_{\beta'}$
  - i.e. there is a unique solution to $\beta_{lpc} = \left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$
  - i.e. $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is invertible

Identification 2

Can we uniquely determine β_{lpc}?

$$\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta_{lpc} = \mathbb{E}\left(xy\right)$$

- if $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is invertible
  - there is a unique value of β_{lpc} that solves the equation
    - β_{lpc} is identified as there is a unique solution
- if $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is not invertible
  - there are multiple values of β_{lpc} that solve the equation
    - β_{lpc} is not identified as there is not a unique solution
    - mathematically $\beta_{lpc} = \left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-}\mathbb{E}\left(xy\right)$ *Generalized Inverse*
  - all solutions yield an equivalent best linear predictor $x^{\mathrm{T}}\beta_{lpc}$
    - best linear predictor is identified

Invertibility

Required assumption: $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ *is positive definite*

- for any non-zero $\alpha \in \mathbb{R}^k$:

$$\alpha^{\mathrm{T}}\mathbb{E}\left(xx^{\mathrm{T}}\right)\alpha = \mathbb{E}\left(\alpha^{\mathrm{T}}xx^{\mathrm{T}}\alpha\right) = \mathbb{E}\left(\alpha^{\mathrm{T}}x\right)^2 \geq 0$$

- so $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is positive semi-definite by construction
- positive semi-definite matrices are invertible IFF they are positive definite
- if we assume $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is positive definite, then
  - $\mathbb{E}\left(\alpha^{\mathrm{T}}x\right)^2 > 0$
  - there is no non-zero α for which $\alpha^T x = 0$
    - implies there are no redundant variables in x
    - i.e. all columns are linearly independent

Best Linear Predictor: Error

- best linear predictor (linear projection)

$$\mathcal{P}(y|x) = x^{\mathrm{T}}\beta_{lpc}$$

- decomposition

$$y = x^{\mathrm{T}}\beta_{lpc} + u \qquad u = e + \left(\mathbb{E}(y|x) - x^{\mathrm{T}}\beta_{lpc}\right)$$

- choice of β_{lpc} implies $\mathbb{E}(xu) = 0$
  - $\mathbb{E}(xu) = \mathbb{E}\left(x\left(y - x^{\mathrm{T}}\beta_{lpc}\right)\right) =$
    $\mathbb{E}(xy) - \mathbb{E}(xx^{\mathrm{T}})\left(\mathbb{E}(xx^{\mathrm{T}})\right)^{-1}\mathbb{E}(xy) = 0$
  - error from projection onto x is orthogonal to x

Best Linear Predictor: Error Variance

Variance of u equals the variance of the error from a linear projection

- Variance of u
  - $\mathbb{E}u^2 = \mathbb{E}\left(y - x^{\mathrm{T}}\beta\right)^2 = \mathbb{E}y^2 - \mathbb{E}\left(yx^{\mathrm{T}}\right)\beta$
    - because $\mathbb{E}\left(x^{\mathrm{T}}\beta\right)^2 = \mathbb{E}\left(yx^{\mathrm{T}}\right)\beta$
- Variance of projection error
  - projection error is defined as $\|u\| = \|y\| - \left\|x^{\mathrm{T}}\beta\right\|$
  - because $y^2 = \|y\|^2$

$$Var\,(u) = Var\,(\|u\|)$$

Best Linear Predictor: Covariate Error Correlation

- $\mathbb{E}(xu) = 0$ is a set of k equations, as
$$\mathbb{E}(x_j u) = 0$$
    - if x includes an intercept, $\mathbb{E}u = 0$
- because
$$Cov(x_j, u) = \mathbb{E}(x_j u) - \mathbb{E}x_j \cdot \mathbb{E}u$$
    - covariates are uncorrelated with u by construction
- for $r \geq 2$ if $\mathbb{E}|y|^r < \infty$ and $\mathbb{E}\|x\|^r < \infty$ then $\mathbb{E}|u|^r < \infty$
    - if y and x have finite second moments then the variance of u exists
    - note: $\mathbb{E}|y|^r < \infty \Rightarrow \mathbb{E}|y|^s < \infty$ for all $s \leq r$ (Liapunov's Inequality)

Linear Projection Model

linear projection model is

$$y = x^{\mathrm{T}}\beta + u \qquad \mathbb{E}\left(xu\right) = 0 \qquad \beta = \left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$$

- $x^{\mathrm{T}}\beta$ is the best linear predictor
  - not necessarily the conditional mean $\mathbb{E}\left(y|x\right)$
- β is the linear prediction coefficient
  - not the conditional mean coefficient if $\mathbb{E}\left(y|x\right) \neq x^{\mathrm{T}}\beta$
  - not a causal (structural) effect if:
    - ★ $\mathbb{E}\left(y|x\right) \neq x^{\mathrm{T}}\beta$
    - ★ $\mathbb{E}\left(y|x\right) = x^{\mathrm{T}}\beta$ but $\nabla_x e \neq 0$

How Does the Linear Projection Differ from the CEF?

Example 1

- CEF of log(wage) as a function of x (black and female indicators)
- discrete covariates, small number of values, compute CEF

$$\mathbb{E}\left(\log\left(wage\right)|x\right) = -.20\,black - .24\,female + .10\,inter + 3.06$$

  - $inter = black \cdot female$
    - ★ 20% male race gap (black males 20% below white males)
    - ★ 10% female race gap

- Linear Projection of log(wage) on x (black and female indicators)

$$\mathcal{P}\left(\log(wage)|x\right) = -.15\,black - .23\,female + 3.06$$

  - 15% race gap
    - ★ average race gap across males and females
    - ★ ignores the role of gender in race gap, even though gender is included

How Does the Linear Projection Differ from the CEF?

Example 2

CEF of white male log(wage) as a function of years of education (*ed*)

- discrete covariate with multiple values
  - could use categorical variables to compute CEF
    - large number of values leads to cumbersome estimation

approximate CEF with linear projections

Approximate CEF of Wage as a Function of Education

Approximation 1

- Linear Projection of log(wage) on $x = ed$

$$\mathcal{P}\left(\log(wage)|x\right) = 0.11\, ed + 1.50$$

  - 11% increase in mean wages for every year of education

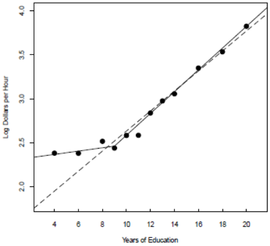


Figure 2.8: Projections of $\log(wage)$ onto Education

- works well for $ed \geq 9$, under predicts if education is lower

Approximate CEF of Wage as a Function of Education

Approximation 2: Linear Spline

- Linear Projection of log(wage) on $x = (ed, spline)$

$$\mathcal{P}\left(\log(wage)|x\right) = 0.02\,ed + 0.10\,spline + 2.30$$

  - $spline = (ed - 9) \cdot 1\,(ed)$
    - ★ 2% increase in mean wages for each year of education below 9
    - ★ 12% increase in mean wages for each year of education above 9

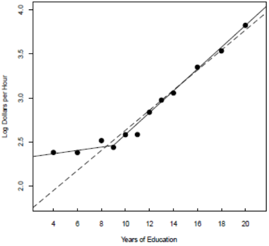


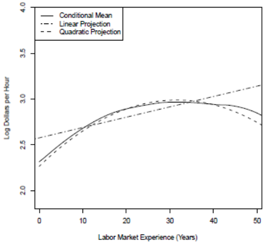
Figure 2.8: Projections of log($wage$) onto Education

How Does the Linear Projection Differ from the CEF?

Example 3

CEF of white male (with 12 years of education) log(wage) as a function of years of experience (ex)

- discrete covariate with large number of values
  - approximate CEF with linear projections
- Linear Projection of log(wage) on $x = ex$
  - $\mathcal{P}\left(\log(wage)|x\right) = 0.011\, ex + 2.50$



over predicts wage for young and old

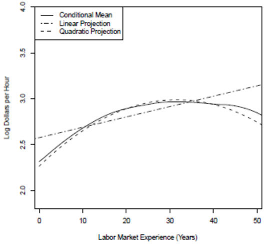
Approximate CEF of Wage as a Function of Experience

Approximation 2: Quadratic Projection

- Linear Projection of log(wage) on $x = (ex, ex^2)$

$$\mathcal{P}\left(\log(wage)|x\right) = 0.046\,ex - 0.001\,ex^2 + 2.30$$

  - $\nabla\mathcal{P} = .046 - .001 \cdot ex$
    - ★ captures strong downturn in mean wage for older workers



Properties of the Linear Projection Model

- Assumption 1
  - $\mathbb{E}y^2 < \infty \quad \mathbb{E}\left\|x\right\|^2 < \infty \quad Q_{xx} = \mathbb{E}\left(xx^{\mathrm{T}}\right)$ is positive definite
- Theorem: Under Assumption 1

1. $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ and $\mathbb{E}\left(xy\right)$ exist with finite elements
2. The linear projection coefficient exists, is unique, and equals

$$\beta = \left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$$

3. $\mathcal{P}\left(y|x\right) = x^{\mathrm{T}}\left(\mathbb{E}\left(xx^{\mathrm{T}}\right)\right)^{-1}\mathbb{E}\left(xy\right)$
4. For $u = y - x^{\mathrm{T}}\beta$, $\mathbb{E}\left(xu\right) = 0$ and $\mathbb{E}\left(u^2\right) < \infty$
5. If x contains a constant, $\mathbb{E}u = 0$
6. If $\mathbb{E}\left|y\right|^r < \infty$ and $\mathbb{E}\left\|x\right\|^r < \infty$ for $r \geq 2$, then $\mathbb{E}\left|u\right|^r < \infty$

Proof

Review

- How do we approximate $\mathbb{E}(y|x)$?
- $x^{\mathrm{T}}\beta$

How to do you interpret β?

- the linear projection coefficient, which is not generally equal to $\nabla_x \mathbb{E}(y|x)$

What is required for identification of β?

- $\mathbb{E}\left(xx^{\mathrm{T}}\right)$ is invertible

What is the correlation between x and u?

- 0 by construction!

Best Linear Predictor Coefficient Solution

- β_{lpc} is the value of β that minimizes
- $S(\beta) = \mathbb{E}y^2 - 2\beta^{\mathrm{T}}\mathbb{E}(xy) + \beta^{\mathrm{T}}\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta$ *Vector Calculus*
  - first derivative $-2\mathbb{E}(xy) + 2\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta$
- solution (linear projection coefficient)

$$\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta_{lpc} = \mathbb{E}(xy)$$

- required assumption
  - $\mathbb{E}y^2 < \infty \quad \mathbb{E}\|x\|^2 < \infty$ *Euclidean Length*

Return to Best Linear Predictor Coefficient

Best Linear Approximation Coefficient Solution

let $m(x) := \mathbb{E}(y|x)$

- β_{lac} is the value of β that minimizes

$$d(\beta) = \int_{\mathbb{R}^k} \left(m(x) - x^{\mathrm{T}}\beta\right)^2 f_x(x)\, dx$$

- $d(\beta) = \mathbb{E}m(x)^2 - 2\beta^{\mathrm{T}}\mathbb{E}(xm(x)) + \beta^{\mathrm{T}}\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta$
  - first derivative $-2\mathbb{E}(xm(x)) + 2\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta$
  - $\mathbb{E}(xm(x)) = \mathbb{E}(x\mathbb{E}(y|x)) = \mathbb{E}(\mathbb{E}(xy|x)) = \mathbb{E}(xy)$
- solution (linear approximation coefficient)

$$\mathbb{E}\left(xx^{\mathrm{T}}\right)\beta_{lac} = \mathbb{E}(xy)$$

Return to Best Linear Predictor Coefficient

Vector Calculus

- vector derivative: inner product
  - (2×1) vectors: B and C
  - $B^{\mathrm{T}} C = B_1 C_1 + B_2 C_2$
  - $\frac{\partial B^{\mathrm{T}} C}{\partial B} = \left[\begin{array}{c} \frac{\partial B^{\mathrm{T}} C}{\partial B_1} \\ \frac{\partial B^{\mathrm{T}} C}{\partial B_2} \end{array} \right] = \left[\begin{array}{c} C_1 \\ C_2 \end{array} \right] = C$
- vector derivative: quadratic form
  - (2×2) matrix: D
  - $B^{\mathrm{T}} D B = B_1^2 D_{11} + B_1 B_2 D_{12} + B_1 B_2 D_{21} + B_2^2 D_{22}$
  - $\frac{\partial B^{\mathrm{T}} D B}{\partial B} = \left[\begin{array}{c} (D_{11} + D_{11}) B_1 + (D_{12} + D_{21}) B_2 \\ (D_{21} + D_{12}) B_1 + (D_{22} + D_{22}) B_2 \end{array} \right] = \left(D + D^{\mathrm{T}}\right) B$

Return to Solution

Euclidean Length

- Pythagorean Theorem
  - $a^2 + b^2 = c^2$ so the length of the hypotenuse is $c = \left(a^2 + b^2\right)^{1/2}$
- c is a vector of dimension 2, so for x a vector of dimension n
  - the Euclidean length (norm) is $\|x\| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$
- therefore
  - $\mathbb{E}\,\|x\|^2 = \mathbb{E}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)$
  - $B^{\mathrm{T}}DB = B_1^2 D_{11} + B_1 B_2 D_{12} + B_1 B_2 D_{21} + B_2^2 D_{22}$

Return Again to Solution

Identification - Background

- identification is important in structural econometric modeling
  - F distribution of observed data (for example (y, x))
  - $\mathcal{F}$ a collection of distributions F
  - θ a parameter of interest (for example $\mathbb{E}y$)
    - ★ identification means that a parameter is uniquely determined by the distribution of the observed variables

Definition

A parameter $\theta \in \mathbb{R}$ is identified on $\mathcal{F}$ if for all $F \in \mathcal{F}$ there is a uniquely determined value of θ.

- equivalently, θ is identified if we can write out a mapping $\theta = g(F)$ on the set $\mathcal{F}$
  - restriction to $\mathcal{F}$ is important
    - ★ most parameters are identified only on a strict subset of the space of all distributions

Identification - Moments of Observed Data

- consider identification of the mean $\mu = \mathbb{E}y$
  - μ is uniquely determined if $\mathbb{E}y < \infty$
    - μ is identified for the set $\mathcal{F} = \left\{ F : \int_{-\infty}^{\infty} |y| \, dF(y) < \infty \right\}$
- identification of the conditional mean

  Theorem: If $\mathbb{E}y < \infty$, the conditional mean $m(x) = \mathbb{E}(y|x)$ is identified almost everywhere.

- generally, moments of observed data are identified as long as we exclude degenerate cases

Identification - More Complicated Models

- consider the context of censoring
  - y is a random variable with distribution F
  - we observe y^* defined by the censoring rule

$$y^* = \begin{cases} y & \text{if } y \leq \tau \\ \tau & \text{if } y > \tau \end{cases}$$

    - ★ applies to income surveys, where incomes above the top code are recorded as equal to the top code ("top coded" data)
- observed variable y^* has distribution

$$F^*(u) = \begin{cases} F(u) & \text{if } u < \tau \\ 1 & \text{if } u \geq \tau \end{cases}$$

- we are interested in the features of F not the censored distribution F^*
  - we cannot calculate $\mu = \mathbb{E}y$ from F^* except in the trivial case where there is no censoring $\mathbb{P}(y \geq \tau) = 0$
    - ★ μ is not generically identified from F^*

Assumptions to Restore Identification

- parametric identification
  - assume a parametric distribution $\left(y \sim \mathcal{N}\left(\mu, \sigma^2\right)\right)$
    - so $\mathcal{F}$ is the set of normal distributions
    - can show that $\left(\mu, \sigma^2\right)$ are identified for all $F \in \mathcal{F}$
  - not ideal - identification achieved only through use of an arbitrary and unverifiable parametric assumption
- nonparametric identification
  - quantiles q_α of F, for $\alpha \leq \mathbb{P}\left(y \leq \tau\right)$ are identified
    - if 20% of the distribution is censored, can identify all quantiles for $\alpha \in (0, 0.8)$
- study of identification focuses attention on what can be learned from the data distributions available

Return to General Identification

Generalized Inverse

- for any matrix A
  - A^- (Moore-Penrose generalized inverse) exists and is unique
- A^- satisfies
  - $AA^-A = A$
  - $A^-AA^- = A$
  - AA^- and A^-A are symmetric
- example, if A_{11}^{-1} exists and $A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$
- then $A^- = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

Return to Identification

Proof of Theorem 1

1. $\left\|\mathbb{E}\left(xx^{\mathrm{T}}\right)\right\| \leq \mathbb{E}\left\|xx^{\mathrm{T}}\right\|$ (Expectation Inequality)

$\mathbb{E}\left\|xx^{\mathrm{T}}\right\| = \mathbb{E}\|x\|^2 < \infty$ (Assumption 1)

- A^- satisfies
  - $AA^-A = A$
  - $A^-AA^- = A$
  - AA^- and A^-A are symmetric
- example, if A_{11}^{-1} exists and $A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}$
- then $A^- = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

Return to Properties of the LPM