Econometrics 710 Problem Set #10

April 29, 2015

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Exercise 13.1

We write the moment equations as

$$\mathbb{E}(\boldsymbol{x}_i(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})) = 0$$
$$\mathbb{E}(\boldsymbol{z}_i((y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2 - \boldsymbol{z}_i'\boldsymbol{\gamma})) = 0$$

The method of moments estimators $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$ for $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is obtained by replacing the population moments with the sample moments, an solving for the parameters; therefore, $(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$ satisfy following equations;

$$\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i(y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}}) = 0$$
$$\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i((y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}})^2 - \boldsymbol{z}_i'\hat{\boldsymbol{\gamma}}) = 0$$

Parameter estimates are $\hat{\boldsymbol{\beta}} = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\boldsymbol{x}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i y_i\right), \hat{\boldsymbol{\gamma}} = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\hat{e}_i^2\right)$ where $\hat{e}_i = y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}}$. These are equivalent to the LS estimator of y on $\boldsymbol{x}$, and $\hat{e}^2$ on $\boldsymbol{z}$, since the model is just-identified.

Exercise 13.2

Let's consider over identified case where dimension of $\boldsymbol{z}_i$ is greater or equal than dimension of $\boldsymbol{x}_i$. Moreover, consider the following unconditional moment equations implied by the conditional moment equations $\mathbb{E}(e_i|\boldsymbol{z}_i) = 0$

$$\mathbb{E}g_i(\boldsymbol{\beta}) = 0, \quad g_i(\boldsymbol{\beta}) = \boldsymbol{z}_i(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})$$

The GMM estimator $\hat{\boldsymbol{\beta}}_{GMM}$ minimizes $J_n(\boldsymbol{\beta}) = n \cdot \bar{g}_n(\boldsymbol{\beta})' W_n \bar{g}_n(\boldsymbol{\beta})$ with weight matrix W_n, where $\bar{g}_n(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n} g_i(\boldsymbol{\beta})$.

If we use $W_n = (Z'Z)^{-1}$, we get $\hat{\boldsymbol{\beta}} = \left((X'Z)(Z'Z)^{-1}(Z'X)\right)^{-1}\left((X'Z)(Z'Z)^{-1}(Z'y)\right) = (X'P_Z X)^{-1}(X'P_Z y)$ by proposition 13.2.1. (This is equivalent to the 2SLS estimator).

Note that $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (X'P_Z X)^{-1}(X'P_Z e)$.

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \left((\frac{1}{n}Z'X)'(\frac{1}{n}Z'Z)^{-1}(\frac{1}{n}Z'X)\right)^{-1}\left((\frac{1}{n}Z'X)'(\frac{1}{n}Z'Z)^{-1}(\frac{1}{\sqrt{n}}Z'e)\right)$$

Since $\frac{1}{n}Z'X \xrightarrow{p} \mathbb{E}\boldsymbol{z}_i\boldsymbol{x}_i' \equiv Q, \frac{1}{n}Z'Z \xrightarrow{p} \mathbb{E}\boldsymbol{z}_i\boldsymbol{z}_i' \equiv M$ by WLLN, and $\frac{1}{\sqrt{n}}Z'e \xrightarrow{p} N(0, \mathbb{E}\boldsymbol{z}_i\boldsymbol{z}_i'e_i^2)$ by CLT. By the assumption $\mathbb{E}(e_i^2|\boldsymbol{z}_i) = \sigma^2$, we get $\mathbb{E}\boldsymbol{z}_i\boldsymbol{z}_i'e_i^2 = \mathbb{E}(\mathbb{E}(\boldsymbol{z}_i\boldsymbol{z}_i'e_i^2|\boldsymbol{z}_i) = \sigma^2 M$. Therefore,

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} (Q'M^{-1}Q)^{-1}Q'M^{-1}N(0, \sigma^2 M) = N(0, \sigma^2(Q'M^{-1}Q)^{-1})$$

by the continuous mapping theorem and Slutzky's lemma.

Exercise 13.3

Assume that the 4th moment of $\boldsymbol{x}_i, y_i$, and $\boldsymbol{z}_i$ exist.

$$W_n^{-1} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'\hat{e}_i^2 = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i' e_i^2 + \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'(\hat{e}_i^2 - e_i^2)$$

Since $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i' e_i^2 \xrightarrow{p} \mathbb{E}(\boldsymbol{z}_i\boldsymbol{z}_i' e_i^2) = \Omega$, it is enough to show that $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'(\hat{e}_i^2 - e_i^2) \xrightarrow{p} 0$ to show the consistency of W_n^{-1}. Then, $W_n \xrightarrow{p} \Omega^{-1}$ follows immediately from the continuous mapping theorem.

We want to show that $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'(\hat{e}_i^2 - e_i^2) \xrightarrow{p} 0$. This can be shown by the similar arguments as in chapter 6.7. Specifically,

$$\begin{aligned}
||\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{z}_i\boldsymbol{z}_i'(\hat{e}_i^2 - e_i^2)|| &\leq \frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{z}_i\boldsymbol{z}_i'(\hat{e}_i^2 - e_i^2)|| \\
&= \frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{z}_i||^2|\hat{e}_i^2 - e_i^2| \\
&\leq \frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{z}_i||^2\left(2|e_i|\ \ ||\boldsymbol{x}_i||\ \ ||\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}|| + ||\boldsymbol{x}_i||^2||\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}||^2\right) \quad \text{(Using inequality(6.31))} \\
&= 2\left(\frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{z}_i||^2||\boldsymbol{x}_i|||e_i|\right)||\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}|| + \left(\frac{1}{n}\sum_{i=1}^{n} ||\boldsymbol{z}_i||^2||\boldsymbol{x}_i||^2\right)||\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}||^2 \\
&= o_p(1)
\end{aligned}$$

The last equality follows from the $||\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}|| \xrightarrow{p} 0$ and $\mathbb{E}\left(||\boldsymbol{z}_i||^2||\boldsymbol{x}_i|||e_i|\right) < \infty, \mathbb{E}\left(||\boldsymbol{z}_i||^2||\boldsymbol{x}_i||^2\right) < \infty$. (The last two immediately follows from Cauchy-Schwartz inequality under the assumption of fourth finite moments of $\boldsymbol{x}_i, \boldsymbol{z}_i, y_i$.)

Exercise 13.4

(a)

$$V_0 = (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}\Omega\Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1} = (Q'\Omega^{-1}Q)^{-1}$$

(b)

$$V_0 = B'\Omega B$$

where $B = \Omega^{-1}Q(Q'\Omega^{-1}Q)^{-1}$.

$$V = A'\Omega A$$

where $A = WQ(Q'WQ)^{-1}$

(c)

$$\begin{aligned} B'\Omega A &= (Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}\Omega WQ(Q'WQ)^{-1} \\ &= (Q'\Omega^{-1}Q)^{-1} = V_0 = B'\Omega B \end{aligned}$$

Thus $B'\Omega(A-B) = 0$

(d)

$$\begin{aligned} V &= A'\Omega A = (B+A-B)'\Omega(B+A-B) \\ &= B'\Omega B + B'\Omega(A-B) + (A-B)'\Omega B + (A-B)'\Omega(A-B) \\ &= V_0 + (A-B)'\Omega(A-B) \quad (\text{from}(c), B'\Omega(A-B) = 0) \end{aligned}$$

Since Ω is positive definite, $(A-B)'\Omega(A-B)$ is positive semi-definite. Thus $V \geq V_0$

Exercise 13.5

The moment condition is,

$$\mathbb{E}g_i(\boldsymbol{\beta}) = 0, \quad g_i(\boldsymbol{\beta}) = \boldsymbol{z}_i(y_i - m(\boldsymbol{x}_i, \boldsymbol{\beta}))$$

Let $\bar{g}_n(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^n g_i(\boldsymbol{\beta})$. To compute the efficient GMM estimator, we use the following procedure. (Note that a closed form solution for $\hat{\boldsymbol{\beta}}$ usually doesn't exist for a nonlinear function $m(\boldsymbol{x}, \boldsymbol{\beta})$.)

(i) Find a preliminary consistent estimator $\tilde{\beta}$. One typical choice will be the GMM estimator with identity weight matrix $W_n = I$, i.e. $\tilde{\beta} = \arg\min_{\boldsymbol{\beta}} n \cdot \bar{g}_n(\boldsymbol{\beta})'\bar{g}_n(\boldsymbol{\beta})$.

(ii) Construct the residuals using the estimator from the previous step. $\hat{e}_i = y_i - m(x_i, \tilde{\boldsymbol{\beta}}), \hat{g}_i = z_i\hat{e}_i$. Then, compute the second step GMM estimator $\hat{\boldsymbol{\beta}}_{GMM}$ that minimizes $J_n(\boldsymbol{\beta}) = n \cdot \bar{g}_n(\boldsymbol{\beta})'W_n\bar{g}_n(\boldsymbol{\beta})$ with weight matrix,

$$W_n = \left(\frac{1}{n}\sum_{i=1}^n \hat{g}_i\hat{g}_i' - (\frac{1}{n}\sum_{i=1}^n \hat{g}_i)(\frac{1}{n}\sum_{i=1}^n \hat{g}_i)'\right)^{-1}$$

(iii) Iterate (ii) for a refinement if possible, updating the weight matrix. (Two-step GMM is asymptotically efficient anyway, but iterated GMM performs better.)

Exercise 13.6

(a)

$$\begin{aligned}
J_n(\boldsymbol{\beta}) &= \frac{1}{n}(y-X\beta)'X\hat{\Omega}^{-1}X'(y-X\beta) \\
&= \frac{1}{n}(X\hat{\beta}+\hat{e}-X\beta)'X\hat{\Omega}^{-1}X'(X\hat{\beta}+\hat{e}-X\beta) \\
&= \frac{1}{n}(\hat{\beta}-\beta)'X'X\hat{\Omega}^{-1}X'X(\hat{\beta}-\beta) \quad (\because X'\hat{e}=0) \\
&= n(\hat{\beta}-\beta)'\left((\frac{1}{n}X'X)^{-1}(\frac{1}{n}X'DX)(\frac{1}{n}X'X)^{-1}\right)^{-1}(\hat{\beta}-\beta) \\
&= n(\hat{\beta}-\beta)'\hat{V}_{\beta}^{-1}(\hat{\beta}-\beta)
\end{aligned}$$

where $D = \mathrm{diag}(\hat{e}_1^2, \cdots, \hat{e}_n^2)$. Thus $\tilde{\boldsymbol{\beta}} = \underset{h(\boldsymbol{\beta})=0}{\arg\min}\ J_n(\boldsymbol{\beta})$ is the same as the minimum distance estimator with weight matrix $\hat{V}_{\beta}^{-1}$

(b)

Case 1) $h(\beta)$ is linear in $\boldsymbol{\beta}$ $(h(\boldsymbol{\beta}) = R'\boldsymbol{\beta} - c)$

From the equation (7.22)(or exercise 7.7),

$$\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} - \hat{V}_{\boldsymbol{\beta}}R(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}(R'\hat{\boldsymbol{\beta}} - c)$$

Therefore,

$$\begin{aligned}
D_n &= J_n(\tilde{\boldsymbol{\beta}}) - J_n(\hat{\boldsymbol{\beta}}) = n(\hat{\boldsymbol{\beta}}-\tilde{\boldsymbol{\beta}})'\hat{V}_{\boldsymbol{\beta}}^{-1}(\hat{\boldsymbol{\beta}}-\tilde{\boldsymbol{\beta}}) \quad (\because (a) \text{ and } J_n(\hat{\boldsymbol{\beta}})=0) \\
&= n\left(\hat{V}_{\boldsymbol{\beta}}R(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}(R'\hat{\boldsymbol{\beta}}-c)\right)'\hat{V}_{\boldsymbol{\beta}}^{-1}\left(\hat{V}_{\boldsymbol{\beta}}R(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}(R'\hat{\boldsymbol{\beta}}-c)\right) \\
&= n(R'\hat{\boldsymbol{\beta}}-c)'(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}R'\hat{V}_{\boldsymbol{\beta}}'\hat{V}_{\boldsymbol{\beta}}^{-1}\hat{V}_{\boldsymbol{\beta}}R(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}(R'\hat{\boldsymbol{\beta}}-c) \\
&= n(R'\hat{\boldsymbol{\beta}}-c)'(R'\hat{V}_{\boldsymbol{\beta}}R)^{-1}(R'\hat{\boldsymbol{\beta}}-c)
\end{aligned}$$

Thus D_n equals to the Wald statistic to test $H_0 : h(\boldsymbol{\beta}) = 0 (R'\boldsymbol{\beta} = c)$.

Case 2) $h(\boldsymbol{\beta})$ is nonlinear

We can show D_n and Wald statistics are generally different, but they are asymptotically equivalent, since they converge in distribution to χ_q^2, where $q = \mathrm{rank}(\frac{\partial}{\partial\boldsymbol{\beta}}h(\boldsymbol{\beta})')$. If you want to learn more about this, see the Handbook of Econometrics, chapter 36, "*Large Sample Estimation and Hypothesis Testing*", by Newey and McFadden. (And take Econ 715!)

Exercise 13.7

(a) Since Ω is positive definite, Ω^{-1} is also positive definite. By the spectral decomposition (see chapter A.8) $\Omega^{-1} = H\Lambda H'$ where $H'H = I$ and Λ is diagonal matrix with strictly positive diagonal elements. Thus we can write $\Omega^{-1} = CC'$ where $C = H\Lambda^{1/2}$, and $\Omega = C'^{-1}C^{-1}$

(b)

$$J_n = n\bar{g}_n(\hat{\boldsymbol{\beta}})'\hat{\Omega}^{-1}\bar{g}_n(\hat{\boldsymbol{\beta}}) = n\bar{g}_n(\hat{\boldsymbol{\beta}})'CC^{-1}\hat{\Omega}^{-1}C'^{-1}C'\bar{g}_n(\hat{\boldsymbol{\beta}}) = n\left(C'\bar{g}_n(\hat{\boldsymbol{\beta}})\right)'\left(C'\hat{\Omega}C\right)^{-1}C'\bar{g}_n(\hat{\boldsymbol{\beta}})$$

(c)

$$\begin{aligned}
C'\bar{g}_n(\boldsymbol{\beta}) &= C'\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{z}_i(y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}}) = C'\frac{1}{n}(Z'y - Z'X\hat{\boldsymbol{\beta}}) \\
&= C'\frac{1}{n}(Z'e - Z'X(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)) \quad (y = X\boldsymbol{\beta}_0 + e) \\
&= C'\frac{1}{n}(Z'e - Z'X(X'Z\hat{\Omega}^{-1}Z'X)^{-1}X'Z\hat{\Omega}^{-1}Z'e) \\
&\quad (\because \hat{\boldsymbol{\beta}} = (X'Z\hat{\Omega}^{-1}Z'X)^{-1}X'Z\hat{\Omega}^{-1}Z'y \quad \text{by proposition 13.2.1}) \\
&= (I_l - C'Z'X(X'Z\hat{\Omega}^{-1}Z'X)^{-1}X'Z\hat{\Omega}^{-1}C'^{-1})C'\frac{1}{n}Z'e \\
&= \left(I_l - C'\left(\frac{1}{n}Z'X\right)\left((\frac{1}{n}X'Z)\hat{\Omega}^{-1}(\frac{1}{n}Z'X)\right)^{-1}(\frac{1}{n}X'Z)\hat{\Omega}^{-1}C'^{-1}\right)C'\frac{1}{n}Z'e \\
&= D_nC'\bar{g}_n(\boldsymbol{\beta}_0)
\end{aligned}$$

(d) Note that $\frac{1}{n}Z'X \xrightarrow{p} \mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'), \hat{\Omega}^{-1} \xrightarrow{p} \Omega^{-1} = CC'$ by WLLN.

$$\begin{aligned}
D_n &= I_l - C'\left(\frac{1}{n}Z'X\right)\left((\frac{1}{n}X'Z)\hat{\Omega}^{-1}(\frac{1}{n}Z'X)\right)^{-1}(\frac{1}{n}X'Z)\hat{\Omega}^{-1}C'^{-1} \\
&\xrightarrow{p} I_l - C'\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i')\left((\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))'\Omega^{-1}(\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))\right)^{-1}(\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))'\Omega^{-1}C'^{-1} \\
&\quad = I_l - C'\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i')\left((\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))'CC'(\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))\right)^{-1}(\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i'))'C \\
&\quad = I_l - R(R'R)^{-1}R'
\end{aligned}$$

where $R = C'\mathbb{E}(\boldsymbol{z}_i\boldsymbol{x}_i')$

(e)

$$\begin{aligned}
n^{1/2}C'\bar{g}_n(\boldsymbol{\beta}_0) &= \sqrt{n}C'\frac{1}{n}Z'e = C'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{z}_ie_i \\
&\xrightarrow{d} C'N(0, \mathbb{E}\boldsymbol{z}_i\boldsymbol{z}_i'e_i^2) \quad \text{(by CLT)} \\
&= N(0, C'\Omega C) = N(0, I_l) \equiv u \quad (\text{Since } \Omega = C'^{-1}C^{-1})
\end{aligned}$$

(f)

$$
\begin{aligned}
J_n &= n\left(C'\bar{g}_n(\hat{\boldsymbol{\beta}})\right)'\left(C'\hat{\Omega}C\right)^{-1}C'\bar{g}_n(\hat{\boldsymbol{\beta}}) \quad \text{(from (b))} \\
&= \left(D_n n^{1/2}C'\bar{g}_n(\boldsymbol{\beta}_0)\right)'\left(C'\hat{\Omega}C\right)^{-1}D_n n^{1/2}C'\bar{g}_n(\boldsymbol{\beta}_0) \quad \text{(from (c))} \\
&\xrightarrow{d} (Du)'\,(C'\Omega C)^{-1}\,(Du) \quad \text{(from (d), (e), and CMT)} \\
&= u'Du
\end{aligned}
$$

where $D = I_l - R(R'R)^{-1}R'$. The last equality holds because $C'\Omega C = I, D^2 = D$.(Note that $D = I_l - R(R'R)^{-1}R'$ is a projection matrix, thus D is idempotent)

(g) $u'Du \sim \chi^2_{rank(D)}$. Since D is symmetric idempotent matrix, rank$(D) = tr(D) = tr(I_l - R(R'R)R') = tr(I_l) - tr((R'R)^{-1}R'R) = l - k$. Therefore, $u'Du \sim \chi^2_{l-k}$

(a)-(g) imply that $J_n \xrightarrow{d} \chi^2_{l-k}$.