Midterm

Short Questions

1. Prove that $P(A \cap B|C) = P(A|B \cap C)P(B|C)$.

$P(A \cap B|C) = \frac{P((A\cap B)\cap C)}{P(C)}$	Baye's Rule
$P((A \cap B) \cap C) = P(A \cap B \cap C) = P(A \cap (B \cap C))$	Commutative
$P(A \cap (B \cap C)) = P(A|B \cap C)P(B \cap C)$	Baye's Rule
$P(B \cap C) = P(B|C)P(C)$	Baye's Rule
Since $P((A \cap B) \cap C) = P(A|B \cap C)P(B \cap C) \wedge P(B \cap C) = P(B|C)P(C)$, $P((A \cap B) \cap C) = P(A|B \cap C)P(B|C)P(C)$	Substitution
$\therefore$, $P(A \cap B|C) = \frac{P(A|B\cap C)P(B|C)P(C)}{P(C)} = P(A|B \cap C)P(B|C)$	Algebra & QED

2. Prove that if sets A and B in sample space S are mutually exclusive (disjoint), $P(A) > 0$, and $P(B) > 0$, then A and B cannot be independent.

A and B are mutually exclusive	Given
$\therefore$, $P(A \cap B) = 0$	Prop: M.E.
Assume $A \perp B$	Assumption
$\therefore$, $P(A \cap B) = P(A)P(B)$	Prop: $\perp$
$P(A) > 0$ and $P(B) > 0$	Given
$\therefore$, since $P(A \cap B) = P(A)P(B)$, $P(A \cap B) > 0$	Algebra
But $P(A \cap B) = 0$	Proved Fact
$\Longrightarrow$ a contradiction exists	Observation
$\Longrightarrow$ A and B are not independent under the givens	QED

3. X and Y are jointly distributed with pdf $f(x,y) = \frac{3}{4}(x^2 + y^2) + \frac{1}{2}$ with $0 < x < 1$ and $0 < y < 1$. Are X and Y independent? Justify your answer.

X and Y have joint pdf $f(x,y) = \begin{cases} \frac{3}{4}(x^2+y^2)+\frac{1}{2}, & 0 < x < 1 \wedge 0 < y < 1 \\ 0, & otherwise \end{cases}$. There exist two perfectly acceptable methods to solve this question:

i) $\forall x, y \in (0,1), f(x,y) = \frac{3}{4}(x^2 + y^2) + \frac{1}{2} = \frac{3}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{2} \neq f(x)f(y)$
$\Longleftrightarrow f(x,y) \neq f(x)f(y) \Longleftrightarrow X$ and Y are not independent.

ii) $f(x) = \int_0^1 \left[\frac{3}{4}(x^2 + y^2) + \frac{1}{2}\right] dy = \frac{3}{4}(x^2 + 1), \forall x \in (0,1)$
By symmetry, $f(y) = \frac{3}{4}(y^2 + 1), \forall y \in (0,1)$
$\therefore, \forall x, y \in (0,1), f(x)f(y) = \frac{9}{16}(x^2 + 1)(y^2 + 1) \neq \frac{3}{4}(x^2 + y^2) + \frac{1}{2} = f(x,y)$
$\therefore$, X and Y are not independent.

4. Prove that if Y and $\mathbb{E}(Y|X)$ are uncorrelated (have covariance equal to zero), then Y is mean independent of X. (Hint: what is the variance of a constant?)

Y and $\mathbb{E}(Y \mid X)$ are uncorrelated	Given
$\therefore$, Cov$[Y, \mathbb{E}(Y \mid X)] = 0$	Def: Covariance
$\therefore$, $\mathbb{E}[Y\mathbb{E}(Y \mid X)] - \mathbb{E}(Y)\mathbb{E}[\mathbb{E}(Y \mid X)] = 0$	Def: Covariance
$\mathbb{E}[Y\mathbb{E}(Y \mid X)] = \mathbb{E}\big[\mathbb{E}[Y\mathbb{E}(Y \mid X) \mid X]\big] \wedge \mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y \mid X)]$	LIE
$\therefore$, $0 = \mathbb{E}\big[\mathbb{E}[Y\mathbb{E}(Y \mid X) \mid X]\big] - \mathbb{E}[\mathbb{E}(Y \mid X)]\mathbb{E}[\mathbb{E}(Y \mid X)]$	Substitution
$= \mathbb{E}[\mathbb{E}(Y \mid X)\mathbb{E}(Y \mid X)] - \mathbb{E}[\mathbb{E}(Y \mid X)]\mathbb{E}[\mathbb{E}(Y \mid X)]$	Linearity of $\mathbb{E}$
$= \mathbb{E}[\mathbb{E}(Y \mid X)^2] - \mathbb{E}[\mathbb{E}(Y \mid X)]^2$	Algebra
$=$ Var$[\mathbb{E}(Y \mid X)]$	Def: Variance
Var$[\mathbb{E}(Y \mid X)] = 0 \Longrightarrow \mathbb{E}(Y \mid X) = c$, where $c \in \mathbb{R}$	Prop: Constant
$\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y \mid X)]$	Def: LIE
$\therefore$, since $\mathbb{E}(Y \mid X) = c$, $\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y \mid X)] = \mathbb{E}(c) = c = \mathbb{E}(Y \mid X)$	Algebra & Sub
$\Longrightarrow \mathbb{E}(Y \mid X) = \mathbb{E}(Y)$	Observation
$\therefore$, Y is mean independent of X	Def: M.I. & QED

Long Questions

5. [Hierarchical Model] Suppose that the random variable X has exponential distribution with parameter 1 (i.e. $\lambda = 1$) and the random variable $Y|X$ has uniform distribution with parameters 0 and X (i.e. $a = 0$ and $b = X$).

   a) What is the function for the conditional mean of Y given X, $\mathbb{E}(Y|X)$?

$$\mathbb{E}(Y|X) = \int_0^X y f_{Y|X}(y|X)\, dy = \int_0^X y\frac{1}{X}dy = \frac{1}{X}\int_0^X ydy = \frac{1}{X}\left[\frac{1}{2}y^2\right]_0^X = \frac{X}{2}$$

OR realize that $\mathbb{E}(Y|X) \sim U[0, X] \Longrightarrow \mathbb{E}(Y|X) = a + \frac{b-a}{2} = 0 + \frac{X-0}{2} = \frac{X}{2}$

   b) What is the mean of Y, $\mathbb{E}(Y)$?

$$\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y|x)] = \int_0^\infty \frac{x}{2}\lambda e^{-\lambda x}dx = \int_0^\infty \frac{x}{2}e^{-x}\, dx = -\int_0^\infty -\frac{x}{2}e^{-x}\, dx$$

Using u-substitution, where $u = \frac{x}{2} \Longrightarrow du = 1/2$ and $dv = -e^{-x}dx \Longrightarrow v = e^{-x}$,

$$\Longrightarrow \mathbb{E}(Y) = -\int_0^\infty -\frac{x}{2}e^{-x}\, dx = -\left[\frac{x}{2}e^{-x}\right]_0^\infty + \int_0^\infty \frac{1}{2}e^{-x}\, dx = 0 - \int_0^\infty -\frac{1}{2}e^{-x}\, dx$$

$$= \left[-\frac{1}{2}e^{-x}\right]_0^\infty = \frac{1}{2}$$

OR realize that $X \sim \exp(1) \Longrightarrow \mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(Y|X)] = \mathbb{E}\left(\frac{X}{2}\right) = \frac{1}{2}\mathbb{E}(X) = \frac{1}{2} * \frac{1}{\lambda} = \frac{1}{2}$

   c) What is the covariance between X and Y, Cov(X, Y)?

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}[\mathbb{E}(XY|X)] - \mathbb{E}(X)\mathbb{E}(Y)$$

From above, we know that $\mathbb{E}(Y) = \frac{1}{2}$ and $\mathbb{E}(X) = 2\mathbb{E}(Y) = 1$

$\mathbb{E}[\mathbb{E}(XY|X)] = \mathbb{E}[X\mathbb{E}(Y|X)] = \mathbb{E}\left(\frac{X^2}{2}\right) = \frac{1}{2}\mathbb{E}(X^2) = \frac{1}{2}\int_0^\infty x^2 e^{-x}dx = 1$. I'm not typing the integration for you because it takes two separate iterations of u-substitution using perfectly analogous steps as done in part (b). If you'd like to see the steps worked out, then let me know….

$\therefore$, Cov$(X,Y) = 1 - 1 * \frac{1}{2} = \frac{1}{2}$

OR, to avoid integration, realize that $\frac{1}{2}\mathbb{E}(X^2) = \frac{1}{2}Var(X) + \frac{1}{2}\mathbb{E}(X)^2 = \frac{1}{2}\left[\frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2\right] = 1$

d) What is the linear predictor of Y given X, $\mathbb{E}^*(Y|X)$?

$\mathbb{E}^*(Y|X) = \alpha + \beta X$, where $\beta = \frac{Cov(X,Y)}{Var(X)} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$ and $\alpha = E(Y) - \beta E(X) = \frac{1}{2} - \frac{1}{2} = 0$

$\therefore$, $\mathbb{E}^*(Y|X) = \frac{1}{2}X$

e) Derive the joint pdf of the bivariate random vector (X, Y).

$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = \frac{1}{x}e^{-x}$ for the usable range

$$\therefore, f_{X,Y}(x,y) = \begin{cases} \frac{1}{x}e^{-x}, & x > 0 \wedge 0 \leq y \leq x \\ 0, & otherwise \end{cases}$$

f) What is the pdf of $W = X^2$?

For the usable range of x, i.e. $x \in [0, \infty]$, the transformation of $W = g(X) = X^2$ satisfies all our necessary requisites to apply our theorem. Notice that this is a function of our usable range: if x could also descend below 0, then we'd be in trouble; however, since x is always nonnegative, $g(X)$ is monotone over the entire usable range.

$X = g^{-1}(W) = \pm\sqrt{W}$

Since x is always nonnegative, $X = \sqrt{W}$

$$\therefore, f_W(w) = \begin{cases} f_X[g^{-1}(w)]\left|\frac{d}{dw}\sqrt{w}\right|, & \forall w \in [0,\infty) \\ 0, & otherwise \end{cases}$$
$$= \begin{cases} f_X(\sqrt{w})\left|\frac{1}{2\sqrt{w}}\right|, & \forall w \in [0,\infty) \\ 0, & otherwise \end{cases}$$
$$= \begin{cases} \frac{1}{2\sqrt{w}}e^{-\sqrt{w}}, & \forall w \in [0,\infty) \\ 0, & otherwise \end{cases}$$

6. Gauss Oh is a student at UCSB – naturally – and enjoys studying statistics – also, quite naturally. In his not-so-ample free time, he moonlights as the mascot for his school, which naturally gets in the way of his statistics studying. Luckily for Gauss Oh, getting an "A" in his classes is not solely determined by his hard work (W); instead it's determined by work and luck (L). His loyal fan base wants to determine how often they can expect to see Gauss Oh moonlighting in his super-sweet mascot uniform, but they know that they only get to see him if he gets a certain number of A's. Unfortunately, no amount of "observing" him allows them to see his levels of work and luck, but they can discern his joint distribution:

		L		
		Unlucky (0)	Usual Luck (1)	Good Luck (2)
	PhD in Econ (2)	.05	y	.15
W	Usual Effort (1)	.18	.25	x
	Slacks (0)	.04	.03	.00

Furthermore, from previous experience, his fans know that $P(W = 1|L = 2) = .4$.

a) What is the value of x in the table above?

$$.4 = P(W = 1|L = 2) = \frac{P(W=1 \cap L=2)}{P(L=2)} = \frac{x}{.15+x+.00} = \frac{x}{.15+x}$$
$$\therefore, .4 = \frac{x}{.15+x} \Longrightarrow .06 + .4x = x \Longrightarrow .6x = .06$$
$$\Longrightarrow x = .1$$

b) What is the probability that Gauss Oh works as hard as a PhD student in econ given that he gets his usual luck?

From the table above, $y = 1 - .05 - .18 - .04 - .25 - .03 - .15 - .1 - .00$

$$\Longrightarrow y = .2$$
$$\therefore, P(W = 2|L = 1) = \frac{P(W=2 \cap L=1)}{P(L=1)} = \frac{.2}{.2+.25+.03} = \frac{.2}{.48}$$
$$\Longrightarrow P(W = 2|L = 1) = \frac{5}{12} = .41\bar{6}$$

As it turns out, the number of A's that Gauss Oh earns during the school year is determined according to the transformation $X = 2W + L$.

c) What is the probability that Gauss Oh earns at least 5 A's?

$$P(X \geq 5) = P(2W + L \geq 5) = .2 + .15$$
$$\Longrightarrow P(X \geq 5) = .35$$

Not only do Gauss Oh's fans care about his grades, but they also care about him graduating (G): they prefer him to stay at UCSB indefinitely. Oddly enough, the statistics department determines graduation strictly on the quantities of A's received during the current year: if Gauss Ho receives at least 5 A's this year, he will graduate ($G = 1$); otherwise, he will not ($G = 0$).

d) What is the probability mass function for G, $f_G(g)$?

$$P(G = 0) = P(X < 5) = 1 - P(X \geq 5) = 1 - .35 = .65$$
$$P(G = 1) = P(X \geq 5) = .35$$
$$\therefore, f_G(g) = \begin{cases} 0, & g > 1 \\ .35, & g = 1 \\ .65, & g = 0 \\ 0, & g < 0 \end{cases}$$

e) What is the probability that Gauss Oh will not graduate until his 2nd year? What is the probability that he will not graduate until his 3rd year? What about not graduating until his 4th year?

$$P(\text{graduate his } 2^{nd} \text{ year}) = P(\text{failing}) * P(\text{graduating}) = .65 * .35 = .2275$$
$$P(\text{graduating his } 3^{rd} \text{ year}) = P(\text{failing}) * P(\text{failing}) * P(\text{graduating})$$
$$= .65^2 * .35 = .1479$$
$$P(\text{graduating his } 4^{th} \text{ year}) = .65^3 * .35 = .0961$$

f) Let H represent the random variable associated with the year that Gauss Oh graduates h. What is the probability mass function of H, $f_H(h)$?

H is a geometric random variable and is characterized by

$$f_H(h) = \begin{cases} f_G(0)^{h-1} f_G(1), & \forall h > 0 \\ 0, & otherwise \end{cases}$$