Math Camp: PS 1 Linear Algebra

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1. **Consider the following matrices:**

$$A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

(a) **Calculate AB.**

$$AB = \begin{bmatrix} 2*2+8*3 & 2*0+8*8 \\ 3*2+0*3 & 3*0+0*8 \\ 5*2+1*3 & 5*0+1*8 \end{bmatrix}$$
$$AB = \begin{bmatrix} 28 & 64 \\ 6 & 0 \\ 13 & 8 \end{bmatrix}$$

(b) **Calculate CB.**

$$CB = \begin{bmatrix} 7*2+2*3 & 7*0+2*8 \\ 6*2+3*3 & 6*0+3*8 \end{bmatrix}$$
$$CB = \begin{bmatrix} 20 & 16 \\ 18 & 24 \end{bmatrix}$$

(c) **Is it true that $CB = BC$? Justify your response.**
No, $BC \neq CB$; matrix multiplication is not commutative. For a generalized example, consider the generic matrices:

$$B_g = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad C_g = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Comparing the first elements of B_gC_g and C_gB_g, we see that $BC_{11} = b_{11} * c_{11} + b_{12} * c_{21}$ while $CB_{11} = b_{11} * c_{11} + b_{21} * c_{12}$.

For the specific matrices B and C, this results in $BC_{11} = 2 * 7 + 0 * 6 = 14$ while $CB_{11} = 7 * 2 + 2 * 3 = 20$.

2. **Find the determinant of each of the following matrices:**

(a) $D = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

$|D| = 2 * -1 - (1 * 3) = -5$

(b) $E = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

Performing a Laplace expansion along the 2nd column, to take advantage of multiple zeros:

$$|E| = -\left(1 * \begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix}\right) + 0 * \begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} + -\left(0 * \begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}\right) = -6 + 0 - 0 = -6$$

3. **Find the inverse of the following matrix:**

$$F = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$$

First, find the determinant.

$$|F| = 5 * 1 - 0 * 2 = 5$$

Next, find the adjoint matrix as the transpose of the cofactor matrix: $\text{Adj}(F) = \boldsymbol{C}_F^T$.

$$\text{Adj}(F) = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Then the inverse F^{-1} is calculated as:

$$F^{-1} = \frac{1}{|F|}\text{Adj}(F)$$

$$F^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

Alternate method - row operations:

$$\begin{aligned}
[F|I_2] &= \left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 5 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right] && Row1 = Row1 - 2 * Row2 \\
&= \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & 1 \end{array}\right] && Row1 = Row1/5 \\
F^{-1} &= \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

4. **Find eigenvalues and eigenvectors associated with the matrix:**

$$G = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

Find $\vec{x}$ and λ such that $G\vec{x} = \lambda\vec{x}$ or $(G - I_n\lambda)\vec{x} = \vec{0}$.

The characteristic equation is $|G - I_n\lambda| = 0$ which results in the quadratic form:

$$\begin{aligned}
\lambda^2 - (g_{11} + g_{22})\lambda + g_{11}g_{22} - g_{12}g_{21} &= 0 \\
\lambda^2 - 7\lambda + 12 - 4 &= 0 \\
\lambda^2 - 7\lambda + 8 &= 0
\end{aligned}$$

Using the quadratic formula to factor this quadratic form, we get two eigenvalues:

$$\lambda_1 = \frac{7 - \sqrt{49 - 4 * 1 * 8}}{2} = \frac{7 - \sqrt{17}}{2}$$

$$\lambda_2 = \frac{7 + \sqrt{49 - 4 * 1 * 8}}{2} = \frac{7 + \sqrt{17}}{2}$$

Substituting $\lambda = \lambda_1$ into $(G - I_n\lambda)\vec{x} = \vec{0}$:

$$\begin{bmatrix} 4 - \frac{7-\sqrt{17}}{2} & 2 \\ 2 & 3 - \frac{7-\sqrt{17}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Simplify and multiply by 2:

$$\begin{bmatrix} 1 + \sqrt{17} & 4 \\ 4 & -1 + \sqrt{17} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Multiply row 1 by $(1 - \sqrt{17})$; note that $(1 + \sqrt{17})(1 - \sqrt{17}) = 1 - 17 = -16$

$$\begin{bmatrix} -16 & 4(1 - \sqrt{17}) \\ 4 & -(1 - \sqrt{17}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row 1 is simply -4 * row 2; no unique solution for singular matrix. $x_1 = \frac{1-\sqrt{17}}{4}x_2$ for $\lambda_1 = \frac{7-\sqrt{17}}{2}$, so:

$$vecx = \begin{bmatrix} \frac{1-\sqrt{17}}{4}x_2 \\ x_2 \end{bmatrix}$$

By a similar process, we find $x_1 = \frac{1+\sqrt{17}}{4}x_2$ for $\lambda_2 = \frac{7+\sqrt{17}}{2}$, and:

$$vecx = \begin{bmatrix} \frac{1+\sqrt{17}}{4}x_2 \\ x_2 \end{bmatrix}$$