Final Exam 2012

Part I

1. **Show that if $\sqrt{n}(Y_n - \mu) \longrightarrow_d N(0, \sigma^2)$, then $Y_n \longrightarrow_p \mu$.**

Answer:

$\sqrt{n}(Y_n - \mu) \longrightarrow_d N(0, \sigma^2)$

$\therefore, \exists X_n$ such that $Y_n = \frac{1}{n}\sum X_n$, where $E(X_n) = \mu$, and $var(X_n) = \sigma^2$ — CLT

$Y_n = \frac{1}{n}\sum X_n \longrightarrow_p E(X_n) = \mu$ — WLLN

$\therefore, Y_n \longrightarrow_p \mu$

2. **Let $X_1, \dots, X_n$ be a random sample from an exponential(λ) distribution, i.e. $f_X(x) = \lambda \exp(-\lambda x)$. What is the Cramér-Rao lower bound for the variance of an unbiased estimator of λ?**

Answer:

(i) Let $\hat{\lambda}$ be an unbiased estimator for λ

(ii) $f_X(x)$ is well-behaved and twice differentiable

(iii) $X_1, \dots X_n$ is a random sample

$\therefore, var(\hat{\lambda}) \geq \frac{1}{-n\mathbb{E}\left(\left[\frac{\partial^2}{\partial\lambda^2}\log f(X_i|\lambda)\right]\right)}$ — CRLB

$\log f(X_i|\lambda) = \log \lambda e^{-\lambda x} = \log\lambda - \lambda x$

$\therefore, \frac{\partial^2}{\partial\lambda^2}\log f(X_i|\lambda) = \frac{\partial^2}{\partial\lambda^2}(\log\lambda - \lambda x) = -\frac{1}{\lambda^2}$

$\therefore, -n\mathbb{E}\left(\left[\frac{\partial^2}{\partial\lambda^2}\log f(X_i|\lambda)\right]\right) = -n\mathbb{E}\left(-\frac{1}{\lambda^2}\right) = -n\left(-\frac{1}{\lambda^2}\right) = \frac{n}{\lambda^2}$

$\therefore, \frac{1}{-n\mathbb{E}\left(\left[\frac{\partial^2}{\partial\lambda^2}\log f(X_i|\lambda)\right]\right)} = \frac{1}{\frac{n}{\lambda^2}} = \frac{\lambda^2}{n}$

$\therefore, var(\hat{\lambda}) \geq \frac{\lambda^2}{n}$

3. **Define β as the slope parameter of the best linear predictor of Y given X. Prove that $\hat{\beta} = \frac{S_{XY}}{S_X^2} \longrightarrow_p \beta$. You may assume (without proving) that S_{XY} and S_X^2 are consistent estimators of σ_{XY} and σ_X^2, respectively.**

Answer:

$S_X^2 \longrightarrow_p \sigma_X^2$ — Given

$\therefore, \frac{1}{S_X^2} \longrightarrow_p \frac{1}{\sigma_X^2}$ — Th 5.5.4 (CMT)

$S_{XY} \longrightarrow_p \sigma_{XY}$ — Given

$\therefore, S_{XY} \times \frac{1}{S_X^2} \longrightarrow_p \sigma_{XY} \times \frac{1}{\sigma_X^2}$ — Slutsky

$\beta = \frac{\sigma_{XY}}{\sigma_X^2}$ and $\hat{\beta} = \frac{S_{XY}}{S_X^2}$ — Given

$\therefore, \hat{\beta} \longrightarrow_p \beta$ — Substitution

4. **Let $X_n \longrightarrow_p a$ and $\sqrt{n}(Y_n - \mu) \longrightarrow_d N(0, \sigma^2)$. What is the asymptotic distribution of $W_n = b\sqrt{n}X_n^2(Y_n^2 - \mu^2) + c$?**

Answer:

$Y_n^2 = g(Y_n)$

$\sqrt{n}[g(Y_n) - g(\mu)] \longrightarrow_d N(0, \sigma^2[g'(\mu)]^2)$, — Delta Method

where $g(\mu) = \mu^2$ and $[g'(\mu)]^2 = \left[\frac{d}{d\mu}g(\mu)\right]^2 = 4\mu^2$

$\therefore, \sqrt{n}(Y_n^2 - \mu^2) \longrightarrow_d N(0, 4\sigma^2\mu^2)$

For simplicity, define $Z_n = \sqrt{n}(Y_n^2 - \mu^2)$

$\therefore, Z_n \longrightarrow_d Z$, where $Z \sim N(0, 4\sigma^2\mu^2)$

$X_n \longrightarrow_p a \Longrightarrow bX_n^2 \longrightarrow_p ba^2$ — Th 5.5.4 (CMT)

Since $bX_n^2 \longrightarrow_p a^2b$ and $Z_n \longrightarrow_d Z$, $bX_n^2 Z_n \longrightarrow_d a^2bZ$ — Slutsky

$\therefore, W_n = bX_n^2 Z_n + c \longrightarrow_d a^2bZ + c$ — Th 5.5.4 (CMT)

$\mathbb{E}(a^2bZ + c) = a^2b\mathbb{E}(Z) + c = c$

$var(a^2bZ + c) = var(a^2bZ) = a^4b^2var(Z) = 4a^4b^2\sigma^2\mu^2$

$\therefore, W_n \longrightarrow_d N(c, 4a^4b^2\sigma^2\mu^2)$

5. **Assume $\mathbb{E}(Y|X) = \theta X^2$, $\mathbb{E}(X) = 0$, and $var(X) = \sigma^2$. What is $\mathbb{E}(Y)$?**

Answer:

$\mathbb{E}(Y|X) = \theta X^2$

$\Longrightarrow \mathbb{E}[\mathbb{E}(Y|X)] = \mathbb{E}(\theta X^2)$

$\Longrightarrow \mathbb{E}(Y) = \theta\mathbb{E}(X^2)$

$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \sigma^2$

Since $\mathbb{E}(X) = 0$, $\mathbb{E}(X^2) = \sigma^2$

$\therefore, \mathbb{E}(Y) = \theta\sigma^2$

6. **Let $U \sim U(0,1)$. Show that $-\log U$ is an exponential random variable.**

Answer:

$$F_U(u) = \begin{cases} 1, & u \geq b \\ \frac{u-a}{b-a}, & u \in (a,b) \\ 0, & u \leq a \end{cases} \Longrightarrow F_U(u) = \begin{cases} 1, & u \geq 1 \\ u, & u \in (0,1) \\ 0, & u \leq 0 \end{cases}$$

Define $Z = g(U) = -\log U$

$\therefore, u \in (0,1) \Longrightarrow z \in (-\log 0, -\log 1) = (0, \infty)$

(i) U has a properly defined cdf

(ii) $g(U)$ is decreasing in U

$\therefore, F_Z(z) = 1 - F_U(g^{-1}(z)) = 1 - F_U(e^{-z}) = \begin{cases} 1 - e^{-z}, & z > 0 \\ 0, & z \leq 0 \end{cases}$ — Th 2.1.3

$\therefore, Z \sim exp(1)$

Part II

7. **Let $X_1, \ldots, X_n$ be iid with pdf $f(x|\theta) = \theta x^{\theta-1}$, where $0 < x \leq 1$ and $0 < \theta < \infty$.**

(a) Show that $\prod_{i=1}^{n} X_i$ is a sufficient statistic for θ.

Answer:

$L(\theta|x) = \prod\left(\theta x^{\theta-1}\right) 1_{x_{(1)}\in(0,\infty)} 1_{x_{(n)}\in(-\infty,1]}$
$\Longrightarrow L(\theta|x) = \theta^n\left(\prod x\right)^{\theta-1} I_{x_{(1)}\in(0,\infty)} I_{x_{(n)}\in(-\infty,1]}$
Let $h(x) = I_{x_{(1)}\in(0,\infty)} I_{x_{(n)}\in(-\infty,1]}$
Let $g(T(x)|\theta) = \theta^n\left(\prod x\right)^{\theta-1}$, where $T(x) = \prod_{i=1}^{n} x_i$
$\therefore, L(\theta|x) = g(T(x)|\theta)h(x)$
$\therefore, \prod_{i=1}^{n} x_i$ is a sufficient statistic Factorization Th

(b) Find the maximum likelihood estimate (MLE) of θ.

Answer:

$L(\theta|x) = \prod \theta x^{\theta-1}$
$\therefore, \log L(\theta|x) = \log\left(\prod \theta x^{\theta-1}\right) = \sum\left[\log\left(\theta x^{\theta-1}\right)\right]$
$\Longrightarrow \log L(\theta|x) = n\log\theta + (\theta-1)\sum\log x$
Maximize $\log L\left(\hat{\theta}\middle|x\right)$ $w.r.t.\,\hat{\theta}$
$\therefore, \frac{d}{d\theta}\log L\left(\hat{\theta}\middle|x\right) = \frac{n}{\hat{\theta}} + \sum\log x = 0$
$\Longrightarrow \hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$

(c) Write the Likelihood Ratio Test statistic, $\lambda(x_1, \ldots, x_n)$, for $H_0\colon \theta = \theta_0, H_1 = \theta \neq \theta_0$

Answer:

$\lambda(x) = \frac{L(\theta_0|x)}{L\left(\hat{\theta}_{MLE}\middle|x\right)} = \frac{\theta_0^n\left(\prod x\right)^{\theta_0-1}}{\hat{\theta}_{MLE}{}^n\left(\prod x\right)^{\hat{\theta}_{MLE}-1}} = \left(\frac{\theta_0}{-\frac{n}{\sum\log x}}\right)^n \left(\prod x\right)^{\theta_0-1-\left(-\frac{n}{\sum\log x}-1\right)}$
$\Longrightarrow \lambda(x) = \left(-\frac{\theta_0}{n}\sum_{i=1}^{n}\log x_i\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta_0+\frac{n}{\sum_{i=1}^{n}\log x_i}}$

It turns out that, given the above pdf for X_i, $\mathbb{E}(\log X_i) = -\frac{1}{\theta}$ and $var(\log X_i) = \frac{1}{\theta^2}$. Using the above facts, answer questions (d) and (f).

(d) Find two alternative method of moment estimators for θ. Which one is more efficient?

Answer:

The first method of moments estimator, $\hat{\theta}_{MoM}^1$:
$\mathbb{E}(\log X_i) = -\frac{1}{\theta}$
$\therefore, \frac{1}{n}\sum\log X_i = -\frac{1}{\hat{\theta}_{MoM}^1}$ Analogy Prin
$\Longrightarrow \hat{\theta}_{MoM}^1 = -\frac{n}{\sum_{i=1}^{n}\log X_i}$
The second method of moments estimator, $\hat{\theta}_{MoM}^2$:
$var(\log X_i) = \frac{1}{\theta^2}$
$\Longrightarrow \mathbb{E}[(\log X_i)^2] - \mathbb{E}(\log X_i)^2 = \frac{1}{\theta^2}$
Since $\mathbb{E}(\log X_i) = -\frac{1}{\theta}$, $\mathbb{E}[(\log X_i)^2] = \frac{1}{\theta^2} + \frac{1}{\theta^2} = \frac{2}{\theta^2}$

$\frac{1}{n}\sum[(\log X_i)^2] = \frac{2}{\left(\hat{\theta}^2_{MoM}\right)^2}$ Analogy Prin

$\Longrightarrow \hat{\theta}^2_{MoM} = \sqrt{\frac{2n}{\sum_{i=1}^{n}[(\log X_i)^2]}}$

Efficiency between estimators:

$\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n}\log x_i}$ achieves minimum variance Th 10.1.12

$\therefore$, $\hat{\theta}^1_{MoM} = \hat{\theta}_{MLE}$ also achieves minimum variance

$\hat{\theta}^2_{MoM} \neq g\left(\hat{\theta}_{MLE}\right)$

$\Longrightarrow \hat{\theta}^2_{MoM}$ does not achieve minimum variance Th 10.1.12

$\therefore$, $\hat{\theta}^1_{MoM}$ is more efficient than $\hat{\theta}^2_{MoM}$

(e) Derive the probability limit of $\boldsymbol{W(X) = (\prod_{i=1}^{n} X_i)^{\frac{1}{n}}}$.

Answer: $\log W(X) = \frac{1}{n}\log(\prod_{i=1}^{n} X_i) = \frac{1}{n}\sum \log X_i$

Define $T(X) = \frac{1}{n}\sum \log X_i$

$\therefore$, $\log W(X) = T(X)$

$\Longrightarrow W(X) = e^{T(X)}$

$T(X) \longrightarrow_p \mathbb{E}[T(X)]$ WLLN

$\therefore$, $e^{T(X)} \longrightarrow_p e^{\mathbb{E}[T(X)]}$ Th 5.5.4 (CMT)

Since $\mathbb{E}[\mathrm{T(X)}] = -\frac{1}{\theta}$, $W(X) \longrightarrow_p e^{-\frac{1}{\theta}}$

(f) What is the asymptotic distribution of $\boldsymbol{\sqrt{n}\left[\log(W(X)) + \frac{1}{\theta}\right]}$?

Answer: $\log W(X) = \frac{1}{n}\sum \log X_i$

$\sqrt{n}\left[\frac{1}{n}\sum \log X_i - \mathbb{E}(\log X_i)\right] \longrightarrow_d N\left(0, var(\log X_i)\right)$ CLT

$\mathbb{E}(\log X_i) = -\frac{1}{\theta}$ and $var(\log X_i) = \frac{1}{\theta^2}$

$\therefore$, $\sqrt{n}\left[\log(W(X)) + \frac{1}{\theta}\right] \longrightarrow_d N\left(0, \frac{1}{\theta^2}\right)$

8. (Intertemporal Investment Decision Model) Consider the following model of a two period investment decision: $\boldsymbol{(X_1, X_2)}$ is a random bivariate vector that denotes the cost of the production technology in period 1 and period 2. In period 1 the investor can observe the realization of the technology price for that period, $\boldsymbol{X_1 = x_1}$, but does not know what the price of the technology will be in period 2.

Define the choice variable $\boldsymbol{C}$, which takes the value of one if the investor chooses to purchase the technology in period 1 and the value of 0 if he chooses to purchase the technology in period 2. If the investor purchases the technology in period 1, then the present discounted value of the profit flow is $\boldsymbol{\Pi_{C=1} = r_1 - X_1 + \beta r_2}$, where $\boldsymbol{r_1}$ is revenue in period 1, $\boldsymbol{r_2}$ is revenue in period 2, and $\boldsymbol{\beta}$ is the time-discount factor. If the investor purchases the technology in period 2, then the present discounted value of the profit flow is $\boldsymbol{\Pi_{C=0} = \beta(r_2 - X_2)}$.

For simplicity, assume the investor has to purchase the technology in one of the two periods, i.e. he cannot choose not to invest. Answer the following questions assuming the investor is risk-neutral.

(a) Characterize the decision to invest in period 1 if X_1 and X_2 are independent (this should be an inequality condition). Note that in period 1 the investor only has information about X_1.

Answer: The investor invests in period 1 iff $\mathbb{E}(\Pi_{C=1}) \geq \mathbb{E}(\Pi_{C=0})$

$\therefore, \mathbb{E}(r_1 - X_1 + \beta r_2) \geq \mathbb{E}[\beta(r_2 - X_2)]$

$\Rightarrow r_1 - X_1 + \beta r_2 \geq \beta r_2 - \beta\mathbb{E}(X_2)$

$\Rightarrow X_1 \leq r_1 + \beta\mathbb{E}(X_2)$

(b) Characterize the decision to invest in period 1 if X_1 and X_2 are not independent (this should be an inequality condition). Note that in period 1 the investor only has information about X_1.

Answer: The investor invests in period 1 iff $\mathbb{E}(\Pi_{C=1}) \geq \mathbb{E}(\Pi_{C=0})$

$\therefore, \mathbb{E}(r_1 - X_1 + \beta r_2 | X_1) \geq \mathbb{E}[\beta(r_2 - X_2)|X_1]$

$\Rightarrow r_1 - X_1 + \beta r_2 \geq \beta r_2 - \beta\mathbb{E}(X_2|X_1)$

$\Rightarrow X_1 \leq r_1 + \beta\mathbb{E}(X_2|X_1)$

For the rest of the questions, assume (X_1, X_2) are bivariate normally distributed with parameters μ_{X_1}, μ_{X_2}, σ_X^2, σ_Y^2, and $\rho \neq 0$.

(c) Characterize the investor's decision to buy technology in period 1 as a function of the parameters above (this should be an inequality condition).

Answer: Since $\rho \neq 0$, part (b) characterizes the answer

$\therefore, X_1 \leq r_1 + \beta\mathbb{E}(X_2|X_1)$

$\mathbb{E}(X_2|X_1) = \mu_{X_2} + \frac{\rho\sigma_{X_1}}{\sigma_{X_2}}(X_1 - \mu_{X_1})$

$\therefore, X_1 \leq r_1 + \beta\left[\mu_{X_2} + \frac{\rho\sigma_{X_1}}{\sigma_{X_2}}(X_1 - \mu_{X_1})\right]$

$\Rightarrow X_1\left[1 - \beta\rho\frac{\sigma_{X_1}}{\sigma_{X_2}}\right] \leq r_1 + \beta\left[\mu_{X_2} - \rho\frac{\sigma_{X_1}}{\sigma_{X_2}}\mu_{X_1}\right]$

$\Rightarrow X_1 \leq \frac{r_1 + \beta\left[\mu_{X_2} - \rho\frac{\sigma_{X_1}}{\sigma_{X_2}}\mu_{X_1}\right]}{1 - \beta\rho\frac{\sigma_{X_1}}{\sigma_{X_2}}}$

$\Rightarrow X_1 \leq \frac{r_1\sigma_{X_2} + \beta[\sigma_{X_2}\mu_{X_2} - \rho\sigma_{X_1}\mu_{X_1}]}{\sigma_{X_2} - \beta\rho\sigma_{X_1}}$

(d) What is the probability density function of a discrete variable, C, that takes the value of 1 if an investor chooses to invest in period 1? Write the pdf as a function of the parameters of the joint normal distribution.

Answer:

$$f_C(c) = P(C = c) = \begin{cases} P\left(X_1 > \frac{r_1\sigma_{X_2} + \beta[\sigma_{X_2}\mu_{X_2} - \rho\sigma_{X_1}\mu_{X_1}]}{\sigma_{X_2} - \beta\rho\sigma_{X_1}}\right), & c = 0 \\ P\left(X_1 \leq \frac{r_1\sigma_{X_2} + \beta[\sigma_{X_2}\mu_{X_2} - \rho\sigma_{X_1}\mu_{X_1}]}{\sigma_{X_2} - \beta\rho\sigma_{X_1}}\right), & c = 1 \\ 0, & otherwise \end{cases}$$

This is just a binomial!

$$\therefore, f_C(c) = \begin{cases} F_X\left(\frac{r_1\sigma_{X_2}+\beta(\sigma_{X_2}\mu_{X_2}-\rho\sigma_{X_1}\mu_{X_1})}{\sigma_{X_2}-\beta\rho\sigma_{X_1}}\right)^c \times \left[1 - F_X\left(\frac{r_1\sigma_{X_2}+\beta(\sigma_{X_2}\mu_{X_2}-\rho\sigma_{X_1}\mu_{X_1})}{\sigma_{X_2}-\beta\rho\sigma_{X_1}}\right)\right]^{1-c}, & c = 0,1 \\ 0, & otherwise \end{cases}$$

(e) Consider a test for inside information about the technology price in period 2. The null hypothesis for this test is the lack of inside information. In other words, under the null an investor cannot use the realization of X_2 to make her investment decision. Note, however, that an investor can use information on X_1. Write an expression for the null hypothesis. Hint: think of the multivariate vector (C, X_1, X_2). The answer involves conditional pdf's.

Answer:

$H_0: f_{C|X_1,X_2}(c|x_1,x_2) = f_{C|X_1}(c|x_1)$

$\Longrightarrow H_0: P(c|x_1,x_2) = P(c|x_1)$

Define $h \equiv \frac{r_1\sigma_{X_2}+\beta[\sigma_{X_2}\mu_{X_2}-\rho\sigma_{X_1}\mu_{X_1}]}{\sigma_{X_2}-\beta\rho\sigma_{X_1}}$

$H_0: P(c|x_1,x_2) = I_{x_1\in(-\infty,h]}$