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Econ 260A

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Consider the following two-period investment opportunity.

The cost of the investment is $I = 400$ and the revenues generated in year one are $V_1 = 200$. In year two, the investment will generate revenues of $V_2 = 600$ with probability p and $V_2 = 100$ with probability $1 - p$. The investment is irreversible once made, and the value of V_2 is revealed at the start of year two. Assume, for now, that the discount factor δ is equal to one.

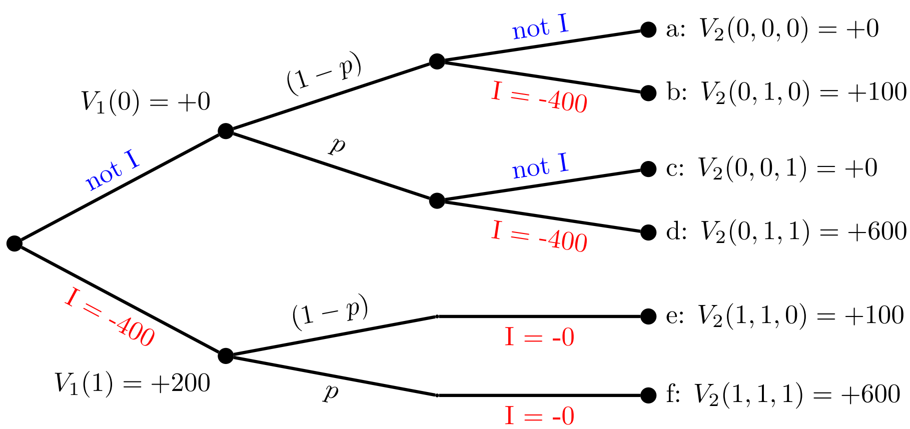

Figure 1: Structure of two-period investment opportunity

a. ***Derive an expression for the Dixit-Pindyck option value in terms of p.*** *Display this graphically and interpret.*

$$
\begin{aligned}
DPOV &= \max\{V^l(1), V^l(0)\} - \max\{V^n(1), V^n(0)\} \\
&\text{where:} \\
V^l(1) &= V^p(1) = V^n(1) = u_1(1) + \mathbb{E}[u_2(1,1,\tilde{\theta})] \\
&= -400 + 200 + 600p + 100(1-p) = 500p - 100 \\
V^l(0) &= u_1(0) + \mathbb{E}[\max_{x_2 \in 0,1}(u_2(0, x_2, \tilde{\theta}) - 400x_2)] \\
&= 0 + (600 - 400)p + (0-0)(1-p) = 200p \\
V^n(0) &= 0 \\
\implies DPOV &= \max\{500p - 100, 200p\} - \max\{500p - 100, 0\} \\
\implies DPOV &= \begin{cases} 0 & \text{for } p \geq 1/3 \\ 300 - 500p \ \ (> 0) & \text{for } 0.2 < p < 1/3 \\ 200p & \text{for } p \leq 0.2 \end{cases}
\end{aligned}
$$

Dixit-Pindyck option value $DPOV$ tells us the value of postponing our investment conditional on being able to learn the value of $\tilde{\theta}$ in the later period. A $DPOV$ of zero means the additional information later does not improve upon the value of foregone earlier investment.

In this case, for $p \geq 1/3$, $DPOV = 0$, otherwise $DPOV > 0$. Figure 2 shows the value of $DPOV$ for values of p. For $p \geq 1/3$, we gain nothing in postponing our investment, and should invest now, having a decent shot at high payoff branch f. For $p < 1/3$, the probability of a high outcome in the later period is small enough that it would be optimal to postpone (avoiding negative payoff branch e), observe the value of $\tilde{\theta}$ in period 2, and decide at that point whether our investment will be worthwhile (branch a if $\tilde{\theta} = 0$, branch d otherwise).

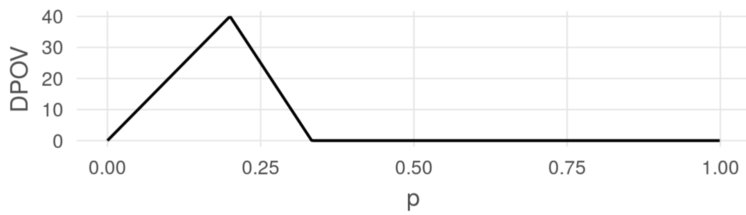

Figure 2: Variation of Dixit-Pindyck value option with p

b. ***Suspose there is a spread in the distribution of year two revenues.*** *Specifically,* $V_2 = 600 + 100u$ *with probability* p *and* $V_2 = 100 - 100u$ *with probability* $1 - p$ *where* $0 \leq u \leq 1$. *Derive an expression for the Dixit-Pindyck option value in terms of* p *and* u. *How does the option value change as* u *gets larger? Explain. How does the option value vary across* $p - u$ *space?*

$$
\begin{aligned}
DPOV &= \max\{V^l(1), V^l(0)\} - \max\{V^n(1), V^n(0)\} \\
&\quad \text{where:} \\
V^l(1) &= u_1(1) + \mathbb{E}[u_2(1, 1, \tilde{\theta})] \\
&= -400 + 200 + (600 + 100u)p + (100 - 100u)(1 - p) \\
&= 500p - 100 + 200up - 100u \\
V^l(0) &= u_1(0) + \mathbb{E}[\max_{x_2 \in 0,1}(u_2(0, x_2, \tilde{\theta}) - 400x_2)] \\
&= 0 + ((600 + 100u) - 400)p = 200p + 100up \\
V^n(0) &= 0 \\
\implies DPOV &= \max\{500p + 200up - 100u - 100, 200p + 100up\} \\
&\quad - \max\{500p + 200up - 100u - 100, 0\} \\
\implies DPOV &= \begin{cases} 0 & \text{for } p \geq \frac{u+1}{u+3} \\ 200p + 100up & \text{for } p \leq \frac{u+1}{2u+5} \\ 300 - 500p + 200u - 200up \ \ (> 0) & \text{else} \end{cases}
\end{aligned}
$$

When $u = 0$, this scenario is identical to that in part a. As u increases, the V_2 payoff both increases the value of the p payoff and decreases the value of the $(1 - p)$ payoff, so now the expected value of V_2 depends on both p and u. In order to make a period 1 investment optimal, we need to be confident that the probability of the high future payoff is high enough to ensure that $u_1(1) + \mathbb{E}[u_2(1, 1, \tilde{\theta})] > \mathbb{E}[\max_{x_2 \in 0,1}(u_2(0, x_2, \tilde{\theta}) - 400x_2)]$.

Figure 3 shows the values of $DPOV$ with respect to both p and u. The red curve indicates the values of p and v at which the $DPOV$ transitions from 0 (invest in period 1, right side of plot) to positive (wait until period 2, observe $\tilde{\theta}$, and then decide whether to invest, left side of plot).

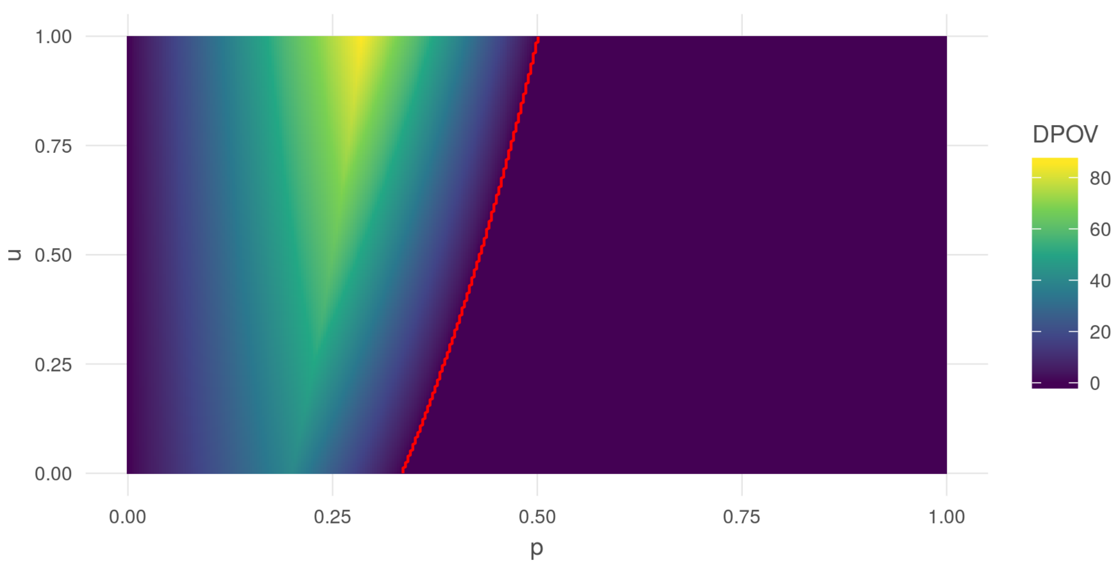

Figure 3: Variation of Dixit-Pindyck value option with p and u

c. ***Now suppose that*** $\delta \leq 1$ ***and*** $u = 0$***.*** *Derive an expression for the option value in terms of* p *and* δ*. How does the Dixit-Pindyck option value change as* δ *gets larger? Explain. How does the option value vary across* $p - \delta$ *space?*

$$
\begin{aligned}
DPOV &= \max\{V^l(1), V^l(0)\} - \max\{V^n(1), V^n(0)\} \\
&\quad \text{where:} \\
V^l(1) &= V^p(1) = V^n(1) = u_1(1) + \delta\mathbb{E}[u_2(1, 1, \tilde{\theta})] \\
&= -400 + 200 + \delta[600p + 100(1-p)] = 500p\delta + 100\delta - 200 \\
V^l(0) &= u_1(0) + \delta\mathbb{E}[\max_{x_2 \in 0,1}(u_2(0, x_2, \tilde{\theta}) - 400x_2)] \\
&= 0 + \delta(600 - 400)p = 200\delta p \\
V^n(0) &= 0 \\
\implies DPOV &= \max\{500p\delta + 100\delta - 200, 200\delta p\} - \max\{500p\delta + 100\delta - 200, 0\} \\
\implies DPOV &= \begin{cases} 0 & \text{for } p \geq \frac{2-\delta}{3\delta} \\ 200\delta & \text{for } p \leq \frac{2-\delta}{5\delta} \\ 200 - 500\delta p + 100\delta \quad (>0) & \text{else} \end{cases}
\end{aligned}
$$

As the discount rate increases (assuming $\delta = \frac{1}{1+r}$), or effectively as the time scale between periods 1 and 2 increases, δ decreases, reducing the present value of future investment.

As δ decreases, the p threshold at which a period 1 investment is optimal (i.e. $DPOV = 0$) shifts according to $p_{period1} = \frac{2-\delta}{3\delta}$, i.e. the probability of the high return must increase to ensure a better result than simply postponing. When $\delta = 0.5$ (i.e. $r = 1$), p must be 1 to make investment in period 1 optimal, and for any $\delta < 0.5$ (i.e. $r > 1$ which is a pretty usurious interest rate), then the investment should always be postponed.

As δ increases from 0.5 toward 1, the discounting of future values becomes less severe, and this scenario approaches the scenario in part a.

Figure 4 shows the relationship among $DPOV$, p, and δ. The upper right corner, above the red line indicating $DPOV = 0$, indicates the parameter space in which investment in period 1 is optimal.

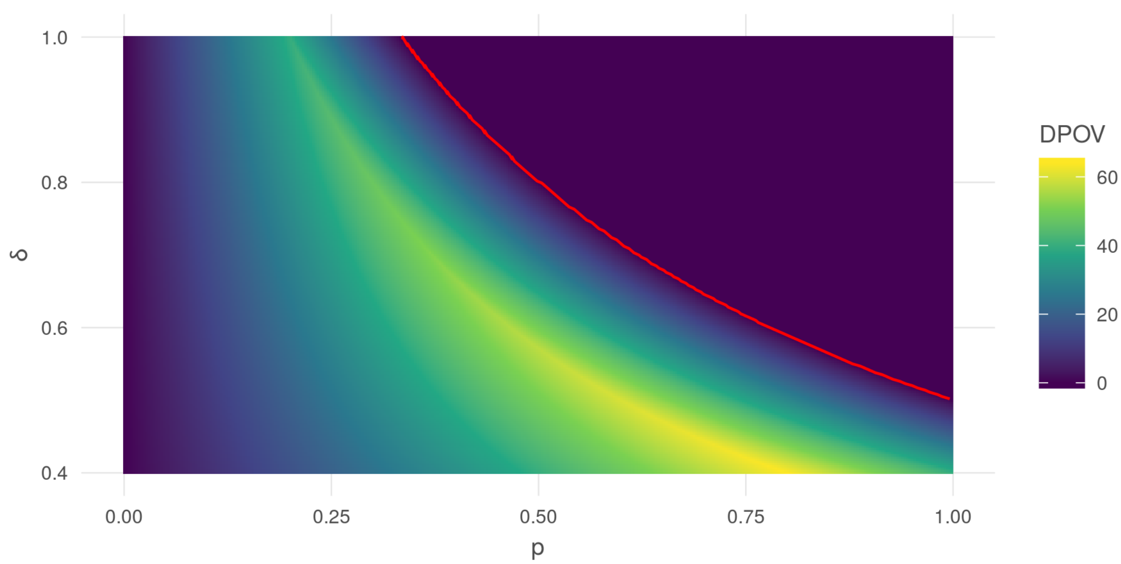

Figure 4: Variation of Dixit-Pindyck value option with p and δ