Solutions Midterm - Econ 241A Probability, Statistics and Econometrics

October 31, 2017

1. Let $Y = exp(X)$, where $X \sim N(\mu, \sigma^2)$. Show that

$$\frac{\sigma_Y}{E[Y]} = \sqrt{e^{\sigma^2} - 1}$$

where σ_Y is the standard deviation of Y.

Answer.
Given that $Y \sim \ln N(\mu, \sigma^2)$ then

$$Var(Y) = \sigma_Y^2 = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

Given that $E[Y]^2 = e^{2\mu+\sigma^2}$

$$\frac{\sigma_Y}{E[Y]} = \sqrt{e^{\sigma^2} - 1}$$

2. Show that $E[\epsilon|X] = 0$ implies that $Cov(\epsilon, X) = 0$.

Answer.
$E[\epsilon X] = E[E[\epsilon X|X]] = E[XE[\epsilon|X]] = E[X \cdot 0] = 0$
$E[\epsilon] = E[E[\epsilon|X]] = E[0] = 0$
$Cov(\epsilon, X) = E[\epsilon X] - E[\epsilon]E[X] = 0 - 0 = 0$

3. Assume w is a random variable and $u(w)$ has a convergent Taylor expansion around $E[w] = \mu_w$, i.e.

$$u(w) = u(\mu_w) + u'(\mu_w)(w - \mu_w) + u''(\mu_w)(w - \mu_w)^2 + \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(\mu_w)(w - \mu_w)^n$$

(a) Give an exact expression for $E[u(w)]$ (state or not if any more assumptions are required).

Answer.

$$E[u(w)] = u(\mu_w) + u''(\mu_w)E[(w - \mu_w)^2] + \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(\mu_w)E[(w - \mu_w)^n]$$

It is required that expectations goes through the limit of the Taylor expansion.

(b) Assume that $w \sim N(0, \sigma_w^2)$. Give an even more detailed expression for $E[u(w)]$.

Answer.
$E[u(w)] = u(0) + u''(0)\sigma^2 + \frac{3}{4!}u^{(4)}(0)\sigma^4 + \frac{3\cdot5}{6!}u^{(6)}(0)\sigma^6 + \frac{3\cdot5\cdot7}{8!}u^{(8)}(0)\sigma^8 + \cdots$

(c) Assume that $u(w) = exp(w)$ and $w \sim N(\mu_w, \sigma_w^2)$. Give an *alternative* expression for $E[u(w)]$.

Answer.
$E[u(w)] = E[e^w] = e^{\mu+0.5\sigma^2}$

(d) Let $w_0 \sim N(\mu, \sigma_0^2)$ and $w_1 \sim N(\mu, \sigma_1^2)$ maintaining $u(w) = exp(w)$. Establish a condition over σ_0^2 and σ_1^2 such as $E[u(w_0)] \leq E[u(w_1)]$.

Answer.
Based on (c) it is sufficient and necessary that $\sigma_0^2 < \sigma_1^2$

4. Let X denote the math score on the ACT college entrance exam of a randomly selected student. Let Y denote the verbal score on the ACT college entrance exam of a randomly selected student. If X and Y are distributed jointly normal such as $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ and $corr(X, Y) = \rho$. State the following in terms of the given parameters $(\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2, \rho)$ and the standard normal cdf $\Phi(z)$.

(a) What is the probability that a randomly selected student's verbal ACT score is between 10 and 20 points?

Answer.
$P(10 \leq Y \leq 20) = P(\frac{10-\mu_Y}{\sigma_Y} \leq \frac{Y-\mu_Y}{\sigma_Y} \leq \frac{20-\mu_Y}{\sigma_Y}) = \Phi(\frac{20-\mu_Y}{\sigma_Y}) - \Phi(\frac{10-\mu_Y}{\sigma_Y})$

(b) What is the probability that a randomly selected student's verbal ACT score is between 10 and 20 points given that $X = 20$?

Answer.
$P(10 \leq Y \leq 20|X = 20) = \Phi\left(\frac{20-\mu_{Y|X}}{\sigma_{Y|X}}\right) - \Phi\left(\frac{10-\mu_{Y|X}}{\sigma_{Y|X}}\right)$
Where $\mu_{Y|X=20} = \mu_Y + \frac{cov(X,Y)}{\sigma_X^2}(20 - \mu_X)$ and $\sigma_{Y|X} = \sigma_Y\sqrt{1-\rho^2}$.

5. The Gini coefficient is commonly used to measure inequality of income. If income is represented by Y, a continuous random variable with cdf $F(y)$ with mean $E[Y] = \mu$, then the Gini coefficient is given by

$$G = \frac{1}{\mu}\int_0^{\infty} F(y)(1 - F(y))dy$$

(a) Assume that $Y \sim U[a, b]$ (clearly $a > 0$). Compute G.

Answer.
We know that

$$F(y) = \frac{y-a}{b-a}$$
$$\mu = \frac{b+a}{2}$$

So using the formula for the Gini coefficient

$$G = \frac{2}{a+b}\int_a^b \frac{y-a}{b-a}\left(1 - \frac{y-a}{b-a}\right)dy$$

If we make a change in variables such as $z = \frac{y-a}{b-a}$ and $dy = (b-a)dz$, then

$$\begin{aligned} G =& \frac{2(b-a)}{a+b}\int_0^1 z(1-z)dz \\ =& \frac{2(b-a)}{a+b}\left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^1 \\ =& \frac{2(b-a)}{a+b}\left[\frac{1}{2} - \frac{1}{3}\right]_0^1 \\ G =& \frac{b-a}{3(a+b)} \end{aligned}$$

(b) Assume that $Y \sim exp(\lambda)$ (i.e. $f_Y(y) = \lambda exp(-\lambda y)$, $y > 0$). Compute G.

Answer.
We know that

$$F(y) = 1 - e^{-\lambda y}$$
$$\mu = \frac{1}{\lambda}$$

So using the formula for the Gini coefficient

$$\begin{aligned} G =& \lambda\int_0^{\infty}(1 - e^{-\lambda y})(e^{-\lambda y})dy \\ G =& \lambda\int_0^{\infty}(e^{-\lambda y} - e^{-2\lambda y})dy \end{aligned}$$

Given that $f_Y(y) = \lambda e^{-\lambda y}$ is a pdf and integrates to 1, we have

$$\begin{aligned} G =& \lambda\left[\frac{1}{\lambda} - \frac{1}{2\lambda}\right] \\ G =& \frac{1}{2} \end{aligned}$$

6. A household has preferences represented by

$$u(w) = -\frac{1}{2}(w-a)^2$$

where w is random variable which represents wealth and we assume that a is high enough such as $0 \leq w < a$. The household maximizes expected utility $E[u(w)]$.

(a) Show that maximizing expected utility is equivalent to maximizing

$$aE[w] - \frac{1}{2}E[w]^2 - \frac{1}{2}Var(w)$$

Answer.

$$\begin{aligned}
E[u(w)] =& E\left[-\frac{1}{2}(w-a)^2\right] \\
E[u(w)] =& -\frac{1}{2}(E[w^2] - 2aE[w] + a^2) \\
E[u(w)] =& -\frac{1}{2}E[w]^2 - \frac{1}{2}Var(w) + aE[w] - \frac{1}{2}a^2
\end{aligned}$$

Adding $\frac{1}{2}a$ (an affine transformation) provides the answer.

(b) The household has an endowment $w_0 > 0$, and decides to invest nonnegative amounts ϕ in a risky asset and ϕ_f in a risk-free asset such as $w_0 = \phi_f + \phi$ (before knowing the risky asset's return). The household consumes wealth w after the return of the risky asset R is determined, i.e. $w = R_f\phi_f + R\phi$, where R_f is the return to the risk-free asset. What is the demand for the risky asset ϕ_f that maximizes expected utility?

$$\begin{aligned}
\max_{(\phi,\phi_f)\geq 0} \quad & aE[w] - \frac{1}{2}E[w]^2 - \frac{1}{2}Var(w) && (1) \\
\text{s. t.} \quad & w_0 = \phi_f + \phi && (2) \\
& w = R_f\phi_f + R\phi && (3)
\end{aligned}$$

Answer.
Solving for ϕ_f in Equation 2 and plugging into Equation 3, we get

$$w = (R - R_f)\phi + R_f w_0$$

Taking expected value and variance of w

$$\begin{aligned}
E[w] &= (\mu - R_f)\phi + R_f w_0 \\
Var[w] &= \phi^2\sigma^2
\end{aligned}$$

Where $\mu = E[R]$ and $\sigma = Var(R)$. Plugging back into the utility function (Equation 1)

$$\max_{\phi\geq 0} \quad a[(\mu - R_f)\phi + R_f w_0] - \frac{1}{2}[(\mu - R_f)\phi + R_f w_0]^2 - \frac{1}{2}\phi^2\sigma^2$$

The FOC wrt to ϕ is

$$\begin{aligned}
a(\mu - R_f) - [(\mu - R_f)\phi + R_f w_0](\mu - R_f) - \phi\sigma^2 = 0 \\
\phi = \frac{(\mu - R_f)(a - R_f w_0)}{(\mu - R_f)^2 + \sigma^2}
\end{aligned}$$

if $\mu > R_f$ and $\phi = 0$ otherwise.

(c) What is the effect of an increase in the endowment on the final demand for the risky asset (conditional on positive demand of the risky asset $\phi > 0$)? Discuss.

Answer.
The effect is negative since $\frac{\partial\phi}{\partial w_0} < 0$. The risky asset is an inferior good. Assuming these preferences is not appealing.