Math Camp: PS 2 Linear Algebra and Probability

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1. **Determine the definiteness of the following symmetric matrices:**

   (a) A $= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

   Inspect the leading principle minors: $|A_1| = 2$; $|A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$.

   Both LPMs > 0, thus A is positive definite.

   (b) B $= \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

   Inspect the leading principle minors:

   - $|B_1| = 1$
   - $|B_2| = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$
   - $|B_3| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{vmatrix} = 1 * 8 - 0 * 0 + 3 * -6 = -10.$

   No need for further calculations... we can already see:

   - B is NOT positive (semi-)definite: not all leading PMs $|A_i| > (\geq)0$.
   - B is NOT negative (semi-)definite: leading PMs do not meet $(-1)^i|A_i| > (\geq)0$.
   - so B is indefinite.

2. **Find the least squares solution to $X\mathbf{b} = \mathbf{y}$, i.e. by finding the estimate $\hat{\mathbf{b}}$ such that $X\hat{\mathbf{b}} = \hat{\mathbf{y}}$ using the following information:**

$$X = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

   $\hat{\mathbf{b}} = (X^T X)^{-1} X^T \mathbf{y}$; and $\hat{\mathbf{y}} = X\hat{\mathbf{b}}$.

$$\begin{aligned}
\hat{\mathbf{b}} &= \left(\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y} \\
&= \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & \frac{6}{11} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y} \\
&= \begin{bmatrix} 0 & 1 & 1 \\ \frac{1}{11} & \frac{4}{11} & \frac{7}{11} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 * 4 + 1 * 1 + 1 * 2 \\ \frac{1}{11} * 4 + \frac{4}{11} * 1 + \frac{7}{11} * 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\end{aligned}$$

Then $\hat{\mathbf{y}} = X\hat{\mathbf{b}}$:

$$\hat{\mathbf{y}} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1*3+2*2 \\ 2*3+-3*2 \\ -1*3+3*2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

3. **Dominoes: how many different pieces can be formed using the numbers 1, 2, ..., n?**
   Choosing $r = 2$ numbers out of n, with replacement (since doubles are allowed), we can find the number of unique combinations using:

$$\begin{aligned} N &= \frac{(n+r-1)!}{(n-1)!r!} \\ &= \frac{(n+1)!}{(n-1)!2!} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

   More intuitively for me, it can also be thought of as choosing $r = 2$ numbers out of n, without replacement, and then adding n tiles with doubled numbers. This results in:

$$\begin{aligned} N &= \frac{n!}{(n-r)!r!} + n \\ &= \frac{n!}{(n-2)!2!} + n \\ &= \frac{n(n-1)}{2} + n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

4. **Suppose 5% of men and 0.25% of women are colorblind. A person is chosen at random and that person is colorblind. What is the probability that the person is male (assuming males and females are equal in numbers)?**

$$\begin{aligned} \mathbb{P}(male|colorblind) &= \mathbb{P}(colorblind|male)\frac{\mathbb{P}(male)}{\mathbb{P}(colorblind)} \\ \mathbb{P}(colorblind|male) &= 0.05 \\ \mathbb{P}(male) &= 0.50 \\ \mathbb{P}(colorblind) &= 0.50*0.05 + 0.50*0.0025 = 0.02625 \\ \mathbb{P}(male|colorblind) &= 0.05*\frac{0.50}{0.02625} \\ &= 0.9524 = 95.24\% \end{aligned}$$

   Alternatively, we can consider the problem in terms of frequencies rather than probabilities: In a group of men, 1 in 20 will be colorblind, but in a group of women, only 1 in $\frac{1}{0.0025} = 400$ will be colorblind. So in a group of 800 people (400 men and 400 women), we'd find 20 colorblind men and 1 colorblind woman. Our sample is chosen at random and found to be in this group; so out of 21 colorblind people, 20 are men, so the probability is $\frac{20}{21} = 0.9524 = 95.24\%$.

5. **If the random variable X follows a geometric distribution, its PMF is given by**

$$f_X(x) = (1-p)^x p, \quad x \in \{0, 1, 2, 3, ...\}$$

   where the parameter $p \in (0, 1)$. Find the CDF of X.
   For a discrete function, $f_X(x) = \mathbb{P}(X = x)$ gives the probability of realizing a specific value x of the

random variable X. A CDF is, by its very name, cumulative, so CDF $F_X(x) = \mathbb{P}(X \leq x)$, or the probability of any value of x up to that point. Thus, $F_X(x)$ is just the sum of the PMF for x and the PMF for all lower values of x, across the entire support for x.

$$\text{CDF}(X) = F_X(x) = \sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} p(1-p)^x$$