Economics 241B Exercise 1 Proposed Solutions
Conditional Expectation Functions and Specification of Conditional Expectation Functions

1.

a. If $\mathbb{E}(y|x) = \beta_1 x + \beta_0$, find $\mathbb{E}(yx)$ as a function of the moments of x.

First, observe that $y = \mathbb{E}(y|x) + e = \beta_1 x + \beta_0 + e$. *Multiply both sides by* x

$$\mathbb{E}(yx) = \beta_1 \mathbb{E}\left(x^2\right) + \beta_0 \mathbb{E}(x) + \mathbb{E}(ex).$$

Observe that $\mathbb{E}(ex) = \mathbb{E}(x\mathbb{E}(e|x)) = 0$ *by construction.*

b. Suppose the random variables y and x take only the values 0 and 1 and have the following joint probability distribution

	$x=0$	$x=1$
$y=0$	a	c
$y=1$	b	d

To satisfy the properties of a joint distribution, what must be true of (a,b,c,d)? Find $\mathbb{E}(y|x)$, $\mathbb{E}(y^2|x)$, and $Var(y|x)$ for $x=0$ and $x=1$.

The values (a,b,c,d) *must all be non-negative and sum to 1. The conditional moments are formed as*

$$\mathbb{E}(y|x=0) = 0 \cdot \mathbb{P}(y=0|x=0) + 1 \cdot \mathbb{P}(y=1|x=0) = \mathbb{P}(y=1|x=0).$$

The conditional probability is

$$\mathbb{P}(y=1|x=0) = \frac{\mathbb{P}(y=1, x=0)}{\mathbb{P}(x=0)} = \frac{b}{a+b}.$$

Importantly, note that $\mathbb{P}(y=1|x=0) \neq b$. *Similarly*

$$\mathbb{P}(y=1|x=1) = \frac{d}{c+d}.$$

Here $\mathbb{E}(y^2|x) = \mathbb{E}(y|x)$ *and*

$$\begin{aligned} Var(y|x=0) &= \mathbb{E}(y|x=0)(1-\mathbb{E}(y|x=0)) = \frac{b}{a+b} \cdot \frac{a}{a+b} \\ Var(y|x=1) &= \frac{d}{c+d} \cdot \frac{c}{c+d}. \end{aligned}$$

2. Assume $\mathbb{E}\left|g(x)\,y\right| < \infty$.

Prove

$$\mathbb{E}\left(g(x)\,y|x\right) = g(x)\,\mathbb{E}\left(y|x\right).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left(g(x)\,y|x\right) &= \int g(x)\,y f_{y|x}(y|x)\,dy \\
&= g(x)\int y f_{y|x}(y|x)\,dy \\
&= g(x)\,\mathbb{E}\left(y|x\right).
\end{aligned}$$

3. If $y = x\beta + u$, $x \in \mathbb{R}$, then for each of the following statements either establish that they are true or provide a counterexample:

i) $\mathbb{E}\left(u|x\right) = 0$ implies $\mathbb{E}\left(x^2 u\right) = 0$,

True: $\mathbb{E}\left(x^2 u\right) = \mathbb{E}\left(x^2\mathbb{E}\left(u|x\right)\right) = 0$,

ii) $\mathbb{E}\left(xu\right) = 0$ implies $\mathbb{E}\left(x^2 u\right) = 0$,

False: Consider the joint distribution for $(x,u) = \begin{cases} (-1,1) \text{ with probability } .5 \\ (1,1) \text{ with probability } .5 \end{cases}$,
so that $\mathbb{E}\left(xu\right) = -1(.5) + 1(.5) = 0$ *but* $\mathbb{E}\left(x^2 u\right) = 1(.5) + 1(.5) = 1$,

iii) $\mathbb{E}\left(u|x\right) = 0$ implies $\mathbb{E}\left(y|x\right) = x\beta$,

True: $\mathbb{E}\left(y|x\right) = x\beta + \mathbb{E}\left(u|x\right)$,

iv) $\mathbb{E}\left(xu\right) = 0$ implies $\mathbb{E}\left(y|x\right) = x\beta$,

False: Consider $y = x\beta + x^2\gamma + e$ *with* $\mathbb{E}x^3 = 0$, *so that* $u = x^2\gamma + e$. *Here* $\mathbb{E}\left(y|x\right) = x\beta + x^2\gamma$ *and* $\mathbb{E}\left(xu\right) = \mathbb{E}x^3\gamma + \mathbb{E}\left(xe\right) = 0$. *(Recall* $\mathbb{E}\left(e|x\right) = 0$ *by construction.)*

4. Recall that the conditional variance is $\sigma^2(x) = Var\left(y|x\right) = \mathbb{E}\left(\left(y - \mathbb{E}\left(y|x\right)\right)^2|x\right)$. Show that the conditional variance can be written as

$$\sigma^2(x) = \mathbb{E}\left(y^2|x\right) - \left(\mathbb{E}\left(y|x\right)\right)^2.$$

We have

$$\begin{aligned}
\mathbb{E}\left(\left(y - \mathbb{E}\left(y|x\right)\right)^2|x\right) &= \mathbb{E}\left(\left(y^2 - 2y\mathbb{E}\left(y|x\right) + \mathbb{E}\left(y|x\right)^2\right)|x\right) \\
&= \mathbb{E}\left(y^2|x\right) - 2\mathbb{E}\left(y\mathbb{E}\left(y|x\right)|x\right) + \mathbb{E}\left(\mathbb{E}\left(y|x\right)^2|x\right) \\
&= \mathbb{E}\left(y^2|x\right) - \left(\mathbb{E}\left(y|x\right)\right)^2
\end{aligned}$$