Econ 241A Prof. Startz

Final Exam Fall 2015

Please answer all questions. Show your work. The test is intended to take 2 hours, so you can have 2.5 hours.

The exam is open book/open note; closed any devices that can communicate.

1. a. Suppose that x_1, x_2 (i.e. $n = 2$) are distributed joint $N(\mu, \sigma^2)$ with correlation ρ. What is the distribution of the sample mean?

   Answer:

   Since the sample mean is $\bar{x} = \frac{x_1+x_2}{2}$, it will be normally distributed as linear combinations of joint normals as are normal. We have

$$\mathrm{E}\left(\frac{x_1+x_2}{2}\right) = \frac{1}{2}[\mathrm{E}(x_1) + \mathrm{E}(x_2)] = \frac{1}{2}[2\mu] = \mu$$

$$\mathrm{var}\left(\frac{x_1+x_2}{2}\right) = \frac{1}{4}[\mathrm{var}(x_1) + \mathrm{var}(x_2) + 2\,\mathrm{cov}(x_1, x_2)] = \frac{1}{4}\left[\sigma^2 + \sigma^2 + 2\rho\sqrt{\sigma^2}\sqrt{\sigma^2}\right]$$
$$= \frac{2(1+\rho)\sigma^2}{4} = \frac{(1+\rho)\sigma^2}{2}$$

   b. Now suppose we have an *n*-vector y which is jointly normally distributed with mean μ, variance σ^2 and pairwise correlations ρ. What is the distribution of the sample mean?

   Answer:

   We can write the sample mean as $\bar{y} = \frac{1}{n}Ly$ where $L = 1_{1\times n}$. We know that $\bar{y}$ will be normally distributed since it is a linear combination of joint normals. The mean will be

$$\frac{1}{n}L\mu = \frac{1}{n}\sum \mu = \mu$$

   If we let the variance-covariance matrix of y be Σ then the variance of $\bar{y}$ will be $\frac{1}{n^2}L\Sigma L'$. Write Σ as

$$\Sigma = \sigma^2\rho\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} + (\sigma^2 - \rho)I_n$$

   Then $L\Sigma L' = L\sigma^2\rho\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}L' + (\sigma^2-\rho)LL' = \sigma^2\rho n^2 + (\sigma^2 - \rho)n$

$$\mathrm{var}(\bar{y}) = \frac{\sigma^2\big(1 + \rho(n-1)\big)}{n}$$

2. Suppose that $x \sim \chi^2_{1000}$, find the 97.5th percentile of x. (Hint: think about a large sample approximation to the χ^2 distribution.)

   Answer:

We can think of x as being the sum of 1,000 independent standard normals. By the Central Limit Theorem that sum is approximately normal. We know from the rules about χ^2 that $\mathrm{E}(x) = 1000, \mathrm{var}(x) = 2 \times 1000$. So

$$P\left(\frac{(x - 1000)}{\sqrt{2000}} > 1.96\right) = .025$$

Which gives the percentile as $x = 1000 + 1.96 \times \sqrt{2000} \approx 1{,}088$

3. Suppose $x_i \sim iid\ exponential(\lambda), i = 1, \ldots, n.$
   a. Find the maximum likelihood estimator of λ and the asymptotic variance of that estimate.

Answer:

We have $f(x_i|\lambda) = \lambda e^{-\lambda x}$. So the likelihood function is

$$L(x|\lambda) = \lambda^n e^{-\lambda \sum x_i}$$

and the log-likelihood is

$$\mathcal{L}(x|\lambda) = n\log(\lambda) - \lambda \sum x_i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i$$

$$\hat{\lambda}_{mle} = \frac{n}{\sum x_i}$$

$$\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = -\frac{n}{\lambda^2}$$

So the asymptotic variance is $-\left(\mathrm{E}\left(-\frac{n}{\lambda^2}\right)\right)^{-1} = \frac{\lambda^2}{n}$

   b. Give the Wald, Likelihood ratio, and LM statistics for the hypothesis $\lambda = \lambda_0$.

Answer:

The Wald is

$$W = \frac{\left(\hat{\lambda}_{mle} - \lambda_0\right)^2}{\hat{\lambda}_{mle}^2/_n}$$

The likelihood ratio statistics is

$$LR = -2[\left(n\log(\lambda_0) - \lambda_0 \sum x_i\right) - \left(n\log\left(\frac{n}{\sum x_i}\right) - \frac{n}{\sum x_i}\sum x_i\right)$$

The LM statistic is

$$LM = \left(\frac{n}{\lambda_0} - \sum x_i\right)^2 \frac{n}{\lambda_0^2}$$

4. Suppose $x_i \sim iid\ Poisson(\lambda), i = 1, \ldots, n$. Show the Lagrange Multiplier statistic for $\lambda = \lambda_0$.

Answer:

The Poisson pdf is $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,\ldots,$ so the likelihood function is

$$L(x|\lambda) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}$$

The log likelihood is

$$\mathcal{L}(x|\lambda) = -n\lambda + (\log\lambda)\sum x_i - \sum \log(x_i!)$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = -n + \frac{1}{\lambda}\sum x_i$$

$$\hat{\lambda}_{mle} = \frac{\sum x_i}{n}$$

Note: $\frac{\partial\mathcal{L}}{\partial\lambda} = -n + \frac{n\hat{\lambda}_{mle}}{\lambda}$

$$\frac{\partial^2\mathcal{L}}{\partial\lambda^2} = -\frac{1}{\lambda^2}\sum x_i$$

Since $\mathrm{E}(x_i) = \lambda$

$$\mathrm{E}\left(\frac{\partial^2\mathcal{L}}{\partial\lambda^2}\right) = -\frac{1}{\lambda^2}n\lambda = -\frac{n}{\lambda}$$

$$I^{-1} = \frac{n}{\lambda}$$

The Lagrange Multiplier statistic is

$$\left(-n + \frac{n\hat{\lambda}_{mle}}{\lambda_0}\right)^2 \frac{n}{\lambda}$$

5. You have a coin that might be fair, $p(H) = 1/2$, or might have two heads, $p(H) = 1$. Your prior probability that the coin is fair is $\pi_F = 3/4$. You observe one toss. What is the posterior probability of the coin being fair if you observed a head? If you observe a tail?

Answer:
The posterior if you see a head is

$$p(Fair|H) = \frac{p(H|fair)\pi_F}{p(H|fair)\pi_F + p(H|cheat)(1-\pi_F)} = \frac{.5 \times .75}{.5 \times .75 + 1 \times .25} = \frac{1.5}{2.5} = .6$$

If you see a tail, then obviously the coin is fair.

6. Suppose that utility over consumption c is given by

$$u(c) = 1 - e^{-\alpha c}$$

And that $c \sim N(\mu, \sigma^2)$ (Don't worry about issues arising out of negative consumption. Find expected utility.

Answer:

Note that $\log(e^{-ac}) = -ac \sim N(\mu, \sigma^2)$, so e^{-ac} is distributed $lnN(a\mu, a^2\sigma^2)$

Thus $\mathrm{E}\big(u(c)\big) = 1 - e^{\left[a\mu + \frac{a^2\sigma^2}{2}\right]}$

7. Continuing from problem 6, suppose utility a function of wealth

$$u(\widetilde{W}) = 1 - e^{-\alpha\widetilde{W}}$$

The return to a risky asset is distributed $N(\bar{r}, \sigma^2)$. A safe asset pays at rate r_f. So if you invest W_r in the risky asset and $W - W_r$ in the safe asset the distribution of final wealth will be $\widetilde{W} \sim N\big(\bar{r}W_r + r_f(W - W_r), \sigma^2 W_r^2\big)$

How much should you invest in the risky asset to maximize expected utility?

Answer:

Following along from question 6 we have

$$\mathrm{E}\left(u(\widetilde{W})\right) = 1 - e^{\left[\bar{r}W_r + r_f(W - W_r) + \frac{\sigma^2 W_r^2}{2}\right]}$$

$$\frac{\partial\, \mathrm{E}\left(u(\widetilde{W})\right)}{\partial W_r} = -\left[\bar{r} - r_f - 2\frac{\sigma^2 W_r}{2}\right] e^{\left[\bar{r}W_r + r_f(W - W_r) + \frac{\sigma^2 W_r^2}{2}\right]} = 0$$

Or

$$W_r = \frac{\bar{r} - r_f}{\sigma^2}$$

8. Consider the "famous birthday problem." There are 365 days in a year, so the probability of a person having a particular birthday is $1/365$. Suppose there are 30 people in a room, what's the probability of at least one birthday in common (What's the chance that no one has a birthday in common?) Show how the calculation is set up; you needn't do the final arithmetic.

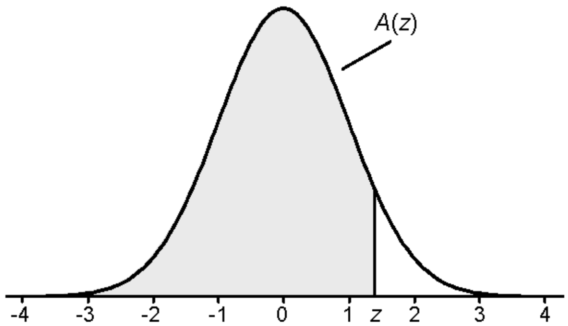
Answer:

The first person in the room has some birthday. The chance that the second person has a different birthday is $364/365$. The chance that a third person shares neither birthday is 363/365. Etc. So the chance that no one shares a birthday is

$$\frac{364}{365} \times \frac{363}{365} \times \dots \times \frac{365-29}{365}$$

Or the chance that at least someone shares a birthday is

$$1 - \frac{364!}{365^{29} \times (365-30)!} \approx .71$$



$A(z)$ is the integral of the standardized normal distribution from $-\infty$ to z (in other words, the area under the curve to the left of z). It gives the probability of a normal random variable not being more than z standard deviations above its mean. Values of z of particular importance:

z	$A(z)$	
1.645	0.9500	Lower limit of right 5% tail
1.960	0.9750	Lower limit of right 2.5% tail
2.326	0.9900	Lower limit of right 1% tail
2.576	0.9950	Lower limit of right 0.5% tail
3.090	0.9990	Lower limit of right 0.1% tail
3.291	0.9995	Lower limit of right 0.05% tail

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990