Economics 241A

Chapter 4 Study Guide

1 Joint and Marginal Distributions

Definition 1.1. *An **n-dimensional random vector** is a function from a sample space S into $\mathbb{R}^n$, n-dimensional Euclidean space. [p.139]*

Definition 1.2. *Let (X, Y) be a discrete bivariate random vector. Then the function $f(x, y)$ from $\mathbb{R}^2$ into $\mathbb{R}$ defined by $f(x, y) = P(X = x, Y = y)$ is called the **joint probability mass function** or **joint pmf** of (X, Y). [p. 140]*

- Equivalently, we can use the notation $f_{X,Y}(x, y)$ if we want to stress the fact that f is the joint pmf of the vector (X,Y).

Theorem 1.1. *Let (X, Y) be a discrete bivariate random vector with joint pmf $f_{X,Y}(x, y)$. Then the marginal pmfs of X and Y, $f_X(x) = P(X = x)$ and $f_Y(y) = P(Y = y)$, are given by [p. 143]:*

$$f_X(x) = \sum_{y \in \mathbb{R}} f_{X,Y}(x, y) \quad \text{and} \quad f_Y(y) = \sum_{x \in \mathbb{R}} f_{X,Y}(x, y)$$

Definition 1.3. *A function $f(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}$ is called a **joint probability density function** if, for every $A \subset \mathbb{R}^2$ [p. 144]:*

$$P((X, Y) \in A) = \int_A \int f(x, y) dx dy$$

- Let $g(x, y)$ be a real-valued function. The **expected value** of $g(X, Y)$ is defined to be

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

- The **marginal probability density function** of X and Y for continuous random variables can be given by:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, x \in (-\infty, \infty)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx, y \in (-\infty, \infty)$$

- Any function $f(x, y)$ satisfying $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$ and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

  is the joint pdf of some continuous bivatiate random vector (X,Y).

- The joint cdf of (X,Y) is the function $F(x, y)$ defined by (for continuous bivariate random variables):

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t) dt ds$$

  for all $(x, y) \in \mathbb{R}^2$.

2 Conditional Distributions and Independence

Definition 2.1. *Let (X, Y) be a discrete bivariate random vector with joint pmf $f(x, y)$ and marginal pmfs $f_X(x)$ and $f_Y(y)$. For any x such that $P(X = x) = f_X(x) > 0$, the* ***conditional pmfs of Y given that X=x*** *is the function of y denoted by $f(y|x)$ and defined by [p. 150]:*

$$f(y|x) = P(Y = y|X = x) = \frac{f(x, y)}{f_X(x)}$$

- The definition of $f(x|y)$ follows similarly to definition 2.1.
- If $g(Y)$ is a function of Y, then the **conditional expected value of** $g(Y)$ **given that** $X = x$ is denoted by $\mathbb{E}[g(Y)|x]$ and is given by:

$$\mathbb{E}[g(Y)|x] = \sum_y g(y)f(y|x) \text{ and } \mathbb{E}[g(Y)|x] = \int_{-\infty}^{\infty} g(y)f(y|x)dy$$

- The variance of the probability distribution described by $f(y|x)$ is called the **conditional variance of Y given X = x**, is given by:

$$Var(Y|x) = \mathbb{E}[Y^2|x] - \big(\mathbb{E}[Y|x]\big)^2$$

Definition 2.2. *Let (X,Y) be a bivariate random vector with joint pdf or pmf $f(x, y)$ and marginal pdfs or pmfs $f_X(x)$ and $f_Y(y)$. Then X and Y are called* ***independent random variables*** *if, for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$ [p. 152]:*

$$f(x, y) = f_X(x)f_Y(y)$$

Lemma 2.1. *Let (X,Y) be a bivariate random vector with joint pdf or pmf f(x,y). Then X and Y are independent random variables if and only if there exist functions g(x) and h(y) such that, for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$ [p. 153]:*

$$f(x, y) = g(x)g(y)$$

Theorem 2.2. *Let X and Y be independent random variables.*

1. *For any $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$, $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$; that is, the events $X \in A$ and $Y \in B$ are independent events.*
2. *Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then [p. 154]:*

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Theorem 2.3. *Let X and Y be independent random variables with MFGs $M_X(t)$ and $M_Y(t)$. Then the MFG of the random variables $Z = X + Y$ is given by [p. 155]:*

$$M_Z(t) = M_X(t)M_Y(t)$$

Theorem 2.4. *Let $X \sim n(\mu, \sigma^2)$ and $Y \sim n(\mu, \sigma^2)$ be independent normal random variables. Then the random variable $Z = X + Y$ has a $n(\mu + \gamma, \sigma^2 + \tau^2)$ distribution.*

3 Bivariate Transformations

Theorem 3.1. *If $X \sim Poisson(\theta)$ and $Y \sim Poisson(\theta)$ and X and Y are independent, then $X + Y \sim Poisson(\theta + \lambda)$.*

- For more details on transformations and the Jacobian, refer to CB p. 156-162, and classnotes.

Theorem 3.2. *Let X and Y be independent random variables. Let g(x) be a function only of x and h(y) be a function only of y. Then the random variables $U = g(X)$ and $V = h(Y)$ are independent.*

4 Hierarchical Models and Mixture Distributions

Theorem 4.1. *If X and Y are any two random variables, then [p.164]*

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}(X|Y)]$$

provided that the expectations exist.

Theorem 4.2. *For any two random variables X and Y [p. 167]:*

$$Var(X) = \mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y])$$