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Econ 260A

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Econ 260A: Homework Challenge #1

The stock of a renewable resource at the beginning of period t is x_t and the harvest is h_t, leaving residual stock $y_t \equiv x_t - h_t$. The price is $p = 1$, assume initially that the cost of fishing is zero (so the period-t payoff is just h_t), and the discount factor is δ. The equation of motion is:

$$x_{t+1} = f(y_t) \tag{1}$$

Answer the following questions:

1. Write the period-t dynamic programming equation.

   x_t = stock at start of period t
   h_t = harvest during period t
   $y_t \equiv x_t - h_t$ = residual stock after harvest
   $p = 1, c = 0$ so $\pi_t = h_t$ = profit in period t
   $x_{t+1} = f(y_t)$ = equation of motion.

   $$\begin{aligned} V_t(x_t) &= \max_{y_t}[\pi(x_t, y_t) + \delta V_{t+1}(G(x_t, y_t))] && \text{(basic DPE)} \\ &= \max_{h_t}[h_t + \delta V_{t+1}(f(x_t - h_t))] && \text{(sub in given parameters)} \end{aligned}$$

2. Use backward induction to characterize the optimal policy function and value function at any arbitrary period $t < T$ in a T-period problem.

   Begin by assuming that at time T, there is no future value to the fishery, so $V_{T+1}(x_{T+1}) = 0$. This could be different if we assumed a positive salvage value.

   $$\begin{aligned} V_T(x_T) &= \max_{h_T}[h_T + \delta V_{T+1}(f(x_T - h_T))] && \text{(dpe in period } T\text{)} \\ &= \max_{h_T}[h_T + \delta \times 0] = \max_{h_T}[h_T] && \text{(sub in zero for } V_{T+1}(\cdot)\text{)} \\ \implies h_T^* &= x_T && (0 < h_T \leq x_T) \\ \implies V_T(x_T) &= x_T && \text{(value of stock in period } T\text{)} \end{aligned}$$

   So the optimal policy in period T is to harvest it all, taking all remaining stock x_T.

In the previous period $T-1$:

$$\begin{aligned}
V_{T-1}(x_{T-1}) &= \max_{h_{T-1}}[h_{T-1} + \delta V_T(f(x_{T-1} - h_{T-1}))] && \text{(dpe in period } T-1) \\
&= \max_{h_{T-1}}[h_{T-1} + \delta V_T(x_T)] && (x_T = f(x_{T-1} - h_{T-1})) \\
&= \max_{h_{T-1}}[h_{T-1} + \delta x_T] && (V_T(x_T) = x_T) \\
\frac{dV_{T-1}}{dH_{T-1}} &= 1 + \delta \frac{dx_T}{dh_{T-1}} = 0 && \text{(first-order condition)} \\
&= 1 + \delta f'(x_{T-1} - h_{T-1})(-1) = 0 \\
\implies f'(x_{T-1} - h_{T-1}) &= 1/\delta \\
\text{Define } y^* &\text{ such that } f' y^*_{T-1} = 1/\delta && \text{(optimal residual stock)} \\
\implies h^*_{T-1}(x_{T-1}) &= x_{T-1} - y^*_{T-1} \text{ so } y^*_{T-1} \text{ indep. of } x_{T-1} \\
\implies V_{T-1}(x_{T-1}) &= h^*_{T-1} + \delta x_T = h^*_{T-1} + \delta f(x_{T-1} - h^*_{T-1}) && (x_T = f(x_{T-1} - h^*_{T-1})) \\
&= x_{T-1} - y^*_{T-1} + \delta f(y^*_{T-1}) && (y^*_{T-1} = x_{T-1} - h^*_{T-1}) \\
\text{Since } y^*_{T-1} &\text{ is independent of } x_{T-1}, \text{ define} \\
C &\equiv -y^*_{T-1} + \delta f(y^*_{T-1}) = \text{constant} \\
\implies V_{T-1}(x_{T-1}) &= x_{T-1} + C
\end{aligned}$$

Stepping back one more time step to period $T-2$, and noting the similarities in equation form to those in period $T-1$:

$$\begin{aligned}
V_{T-2}(x_{T-2}) &= \max_{h_{T-2}}[h_{T-2} + \delta(x_{T-1} + C)] && (V_{T-1}(x_{T-1}) = x_{T-1} + C) \\
&= \max_{h_{T-2}}[h_{T-2} + \delta(f(x_{T-2} - h_{T-2}) + C)] && (x_{T-1} = f(x_{T-2} - h_{T-2})) \\
\frac{dV_{T-2}}{dh_{T-2}} &= 1 + \delta \frac{dx_T - 1}{dh_{T-2}} = 0 && \text{(first-order condition)} \\
&= 1 + \delta f'(x_{T-2} - h_{T-2})(-1) = 0 \\
\implies f'(x_{T-2} - h_{T-2}) &= 1/\delta \\
\text{Define } y^* &\text{ such that } f' y^*_{T-2} = 1/\delta && \text{(optimal residual stock)} \\
\implies h^*_{T-2}(x_{T-2}) &= x_{T-2} - y^*_{T-2} \text{ so } y^*_{T-2} \text{ indep. of } x_{T-2} \\
\text{by symmetry to } T-1, \text{ define } K &\equiv -y^*_{T-2} + \delta f(y^*_{T-2}) = \text{constant} \\
\implies V_{T-2}(x_{T-2}) &= x_{T-2} + K
\end{aligned}$$

Conjecture that the form of the value function at any time t takes the form

$$V_t(x_t) = x_t + \theta_t$$

where θ_t is a constant in each period t independent of x_t. Can we verify that this satisfies the DPE?

$$\begin{aligned}
V_t(x_t) &= \max_{h_t}[h_t + \delta V_{t+1}(x_{t+1})] && \text{(dpe)} \\
&= \max_{h_t}[h_t + \delta(x_{t+1} + \theta_{t+1})] && \text{(by hypothesis of } V_{t+1}) \\
&= \max_{h_t}[h_t + \delta(f(x_t - h_t) + \theta_{t+1})] && (x_{t+1} = f(x_t - h_t)) \\
\frac{dV_t}{dh_t} &= 1 + \delta(f'(x_t - h_t)(-1) + 0) = 0 && \text{(first order condition)} \\
\implies 1/\delta &= f'(x_t - h^*_t) = f'(y^*_t) \\
\implies V_t(x_t) &= x_t - y^*_t + \delta(f(y^*t) + \theta_{t+1}) \\
\text{Define } \theta_{t+1} &\equiv -y^*_t + \delta(f(y^*t) + \theta_{t+1} \\
\implies V_t(x_t) &= x_t + \theta_t && \blacksquare
\end{aligned}$$

Does this result (having a single MEY value at residual harvest y^*) depend on the growth function being strictly concave?

3. How does the period-t policy function depend on p?

   If the harvest is valued at price p then profit $\pi(h_t) = ph_t$. Using this value and quickly repeating the backward induction from period T as before (some steps elided for clarity):

$$\begin{aligned}
V_T(x_T) &= \max_{h_T}[ph_T + \delta V_{T+1}(f(x_T - h_T))] \\
&= \max_{h_T}[ph_T + \delta \times 0] = \max_{h_T}[ph_T] \\
\implies h_T^* &= x_T \\
\implies V_T(x_T) &= px_T
\end{aligned}$$

   Unsurprisingly, the policy in period T of harvesting all remaining stock is unchanged if $p \neq 1$. Continue backward induction! In period $T-1$:

$$\begin{aligned}
V_{T-1}(x_{T-1}) &= \max_{h_{T-1}}[ph_{T-1} + \delta V_T(x_T)] \\
&= \max_{h_{T-1}}[ph_{T-1} + \delta p x_T] \\
&= p \max_{h_{T-1}}[h_{T-1} + \delta x_T] && \text{(distributive property)} \\
&= p \max_{h_{T-1}}[h_{T-1} + \delta f(x_{T-1} - h_{T-1})] \\
&\ldots
\end{aligned}$$

   From here, the derivation is identical to that in question 2, multiplied by a scalar p term. The term to be maximized is independent of p, so the first-order conditions result in the same y^* as before, satisfying $f'(y^*) = 1/\delta$. Similarly, we find the value function $V_{T-1}(x_{T-1}) = px_{T-1} + C$.

   Does this carry through to period $T-2$?

$$\begin{aligned}
V_{T-2}(x_{T-2}) &= \max_{h_{T-2}}[ph_{T-2} + \delta V_{T-1}(x_{T-1})] \\
&= \max_{h_{T-2}}[ph_{T-2} + \delta(px_{T-1} + C)] \\
&= p \max_{h_{T-2}}[h_{T-2} + \delta(x_{T-1} + C/p)] && \text{(distributive property)} \\
&= p \max_{h_{T-2}}[h_{T-2} + \delta(f(x_{T-2} - h_{T-2}) + C/p)] \\
&\ldots
\end{aligned}$$

   Again, the term to be maximized is independent of p (note C/p is just a constant so drops out of the first order conditions). By similarity to question 2, we can see that the policy function is identical for any constant p, though the value function scales linearly with p.

   Note that if p were not constant but rather a function $p(h_t, x_t)$, e.g. a large harvest floods the market dropping the price, then the first order condition likely becomes considerably messier... Similarly if we were to include a cost term $c(h_t, x_t)$, the policy function would necessarily change.

4. Now suppose the current period payoff is $\alpha h_t - \beta h_t^2$ (for $\alpha > 0$ and $\beta > 0$). *Guess* that the period-t dynamic programming equation is a linear function of the state. Then try to verify that the Bellman equation is satisfied (or is not satisfied). In other words, try to prove or disprove that the DPE is linear in the state.

   Guess that the value function is linear function of state:

$$V_t(x_t) = \gamma x_t + \Theta_t$$

where γ is a linear scaling constant and θ is a time-dependent (but x_t-independent) constant.

Does this satisfy the DPE?

$$\begin{aligned}
V_t(x_t) &= \max_{h_t}[\alpha h_t - \beta h_t^2 + \delta V_{t+1}(x_{t+1})] && \text{(DPE with } \pi(h_t) = \alpha h_t - \beta h_t^2) \\
&= \max_{h_t}[\alpha h_t - \beta h_t^2 + \delta(\gamma x_{t+1} + \theta_{t+1})] && (V_{t+1}(x_{t+1}) = \gamma x_{t+1} + \theta_{t+1} \text{ by hyp.}) \\
&= \max_{h_t}[\alpha h_t - \beta h_t^2 + \delta(\gamma f(x_t - h_t) + \theta_{t+1})] && \text{(eqn of motion)} \\
\implies \frac{dV_t}{dh_t} &= \alpha - 2\beta h_t + \delta(\gamma f'(x_t - h_t)(-1) + 0)] = 0 && \text{(f.o.c.)} \\
\implies f'(x_t - h_t^*) &= \frac{\alpha - 2\beta h_t^*}{\delta\gamma} \\
\implies h_t^* &= \frac{\alpha - \delta\gamma f'(x_t - h_t)}{2\beta}
\end{aligned}$$

At this point, we can see that the slope of the growth function at the optimal residual stock $f'(x_t - h_t^*) = f'(y_t^*)$ is not constant, but instead dependent on the size of the optimal harvest itself. This shows that y^* is not independent of x: $y_t^* = y^*(x_t)$.

Define $h_t^* \equiv x_t - y^*(x_t)$, plug h_t^* into the DPE function to see if it can simplify to our conjectured value function:

$$\begin{aligned}
V_t(x_t) &= \alpha h_t^* - \beta h_t^* 2 + \delta V_{t+1}(x_{t+1})] \\
&= \alpha(x_t - y^*(x_t)) - \beta(x_t^2 - 2x_t y^*(x_t) + y^*(x_t)^2) + \delta V_{t+1}(f(y^*(x_t))) \\
&= x_t(\alpha + 2\beta y^*(x_t)) - x_t^2(\beta) + \Theta_t(y^*(x_t))
\end{aligned}$$

It is pretty clear that this, with x_t^2 terms and $y^*(x_t)$ terms, will not solve to $\gamma x_t + \theta_t$, unless $\beta = 0$.

For $\beta = 0$, our profit function becomes $\pi_t = \alpha h_t$, which is identical in form to problem 3, and results in a DPE that is linear in the state.

5. Retaining the non-linear payoff function, now assume the stock evolves in a stochastic fashion: $x_{t+1} = z_t f(y_t)$, where $z_t = \{1-\theta, 1+\theta\}$ (i.e. it takes one of those two values, each with probability 0.5), and $f(y_t) = y_t + ry_t(1 - y_t/K)$. Write a computer program that solves this problem numerically. You can use the following parameters: $\delta = .9$, $\alpha = 20$, $\beta = .6$, $r = .3$, $K = 100$, $\theta = .3$. How does the converged optimal policy function depend on θ, r, δ, and α?

   The policy function is based on the current stock x_t and the expectation of value of future stock $\mathbb{E}[V_{t+1}(x_{t+1})]$. The future shock is unobservable, so we must rely on our expectation of the value the stock after the stochastic shock has occurred.

$$V_t(x_t) = \max_{h_t}[\alpha h_t - \beta h_t^2 + \delta \mathbb{E}[V_{t+1}(x_{t+1})]]$$

   where

$$\mathbb{E}[V_{t+1}(x_{t+1})] = 0.5 V_{t+1}((1+\theta)x_{t+1}) + 0.5 V_{t+1}((1-\theta)x_{t+1})$$

   Fig. 1 shows the convergence of the value function and the policy function for this situation. The red dashed line shows the converged functions for the value and policy for the situation in which no stochastic shocks occur.

   See code in separate pdf.

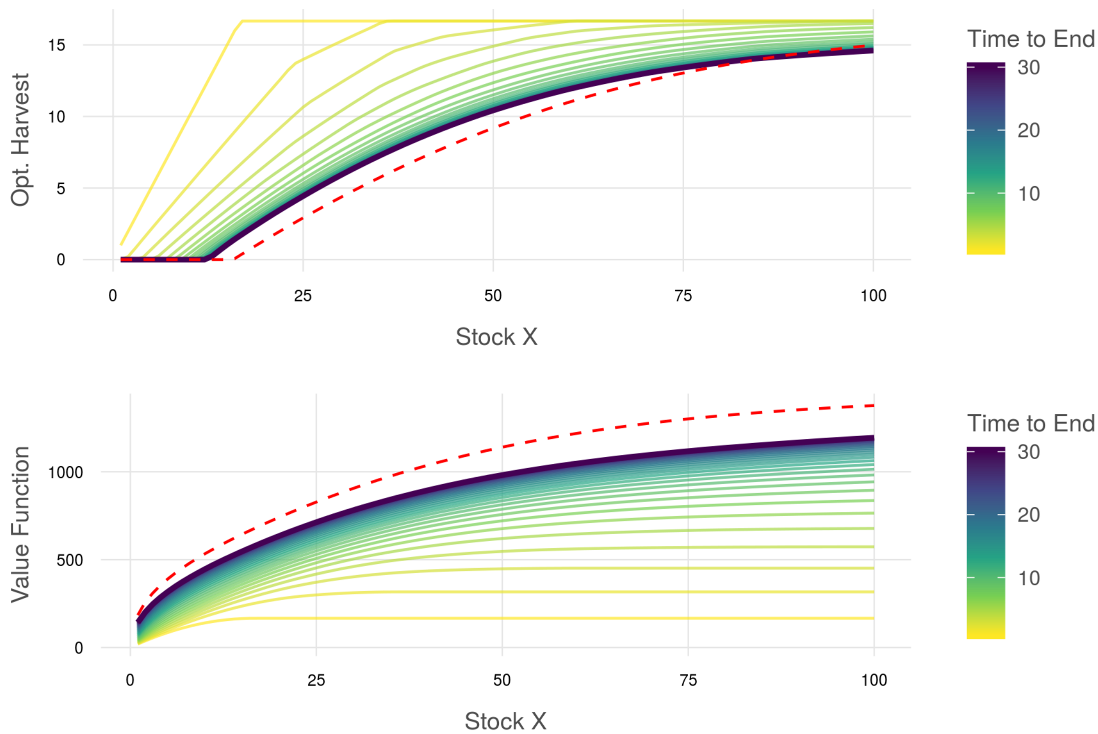

Figure 1: Convergence of value and harvest policy functions.

Dependence on θ: Since we are basing policy on the expected value of $V_t(z_t f(y^*))$, and $V_t(x_t)$ is concave, then by Jensen's inequality:

$$\mathbb{E}[V_t(x_t)] \leq V_t(\mathbb{E}[x_t])$$

increasing θ generally decreases the value of future stock growth, decreasing y^* and generally increasing the value of current harvest over the $\theta = 0$ situation.

Dependence on r: An increase in r will increase the optimal residual stock level (the minimum level at which harvest will occur), though once that residual stock is reached, harvests can be higher due to the higher growth rate of the stock. Since our optimal policy is based on setting $f'(x_t - h_t) = 1/\delta$, changing $f'(x_t - h_t)$ changes the optimal residual stock. For a given $\delta > 0$, y^* is increasing in r: a very low r results in a low y^* (value of leaving a fish in the water to reproduce is low) while high r results in high y^* (each fish left in the water will produce many more for future harvest).

Dependence on δ: An increase in δ increases the present value of future harvests, reducing current harvests (in favor of future harvests). For a given r, higher δ results in higher y^*: the reproductive value of a fish left in the water is increased.

Dependence on α: An increase in α with no increase in β represents an increase in sale price of harvest with no increase in cost. At higher stocks, the price of harvest greatly outweighs the cost to harvest, increasing the incentive to harvest at higher levels (Fig. 2). This also increases the value of future harvests, so it increases the optimal residual stock as well. These play against each other at stocks slightly above the optimal residual harvest, as seen in the plot where the policy curves intersect.

See code in separate pdf.

6. Suppose $x_0 = 15$. Using the infinite-horizon optimal policy function, simulate the optimized system forward for 20 years under baseline parameters above. Run 10 separate simulations and plot the

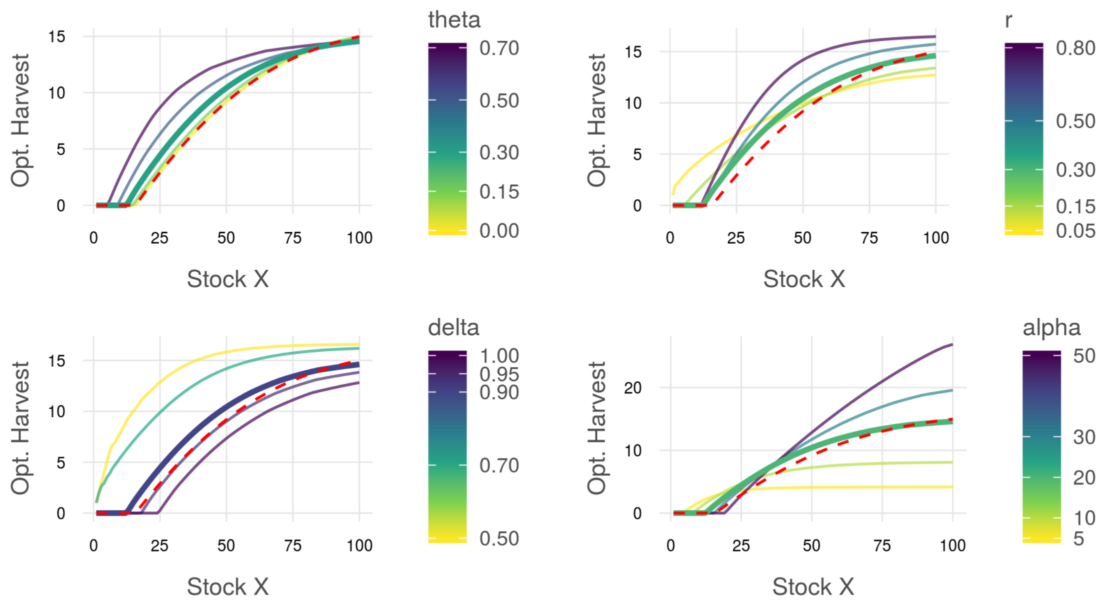

Figure 2: Effect of parameter change on policy functions.

trajectory of x_t and h_t over time for each simulation. (You should produce 2 plots, one for the 10 trajectories of x_t and one for the 10 trajectories of h_t).

Here (Fig. 3) I have incorporated 500 simulations and a time horizon of 50 years, and plotted the mean stock and harvest (across all simulations) on top. Note that the stock never drops to zero in any simulation, though in many simulations the harvest drops to zero for a time when the stock falls below the optimal residual stock. See code in separate pdf.

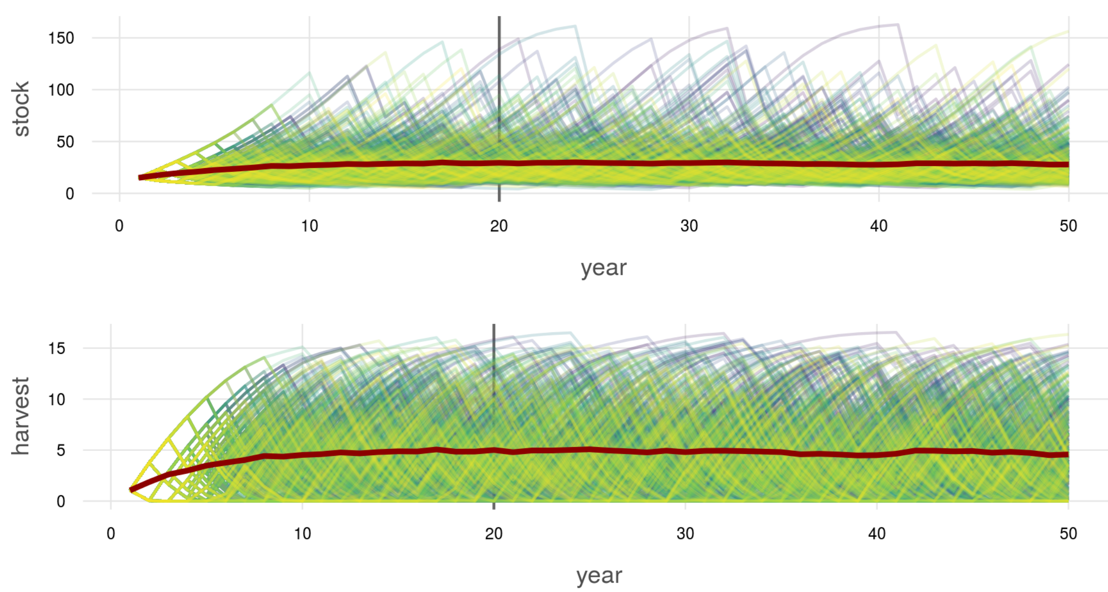

Figure 3: Simulations (500) over 50 time periods