Heteroskedasticity-Consistent Standard Errors

Hayashi 2.5

Goal: Accurate standard error estimates with conditional heteroskedasticity

$$Var\left(\hat{\beta}\right) = \mathbb{E}\left[\left(\hat{\beta} - \beta\right)^2\right]$$

Scalar model

$$\left(\hat{\beta} - \beta\right)^2 = \frac{\left(\sum x_t u_t\right)^2}{\left(\sum x_t^2\right)^2}$$

General model

$$\left(\hat{\beta} - \beta\right)^2 = \left(\sum x_t x_t'\right)^{-1} \sum u_t^2 x_t x_t' \left(\sum x_t x_t'\right)^{-1}$$

Asymptotic Variance

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \sim\longrightarrow N\left(\mathbf{0}, \mathbf{\Sigma}_{xx}^{-1}\, S\, \mathbf{\Sigma}_{xx}^{-1}\right)$$

$\frac{1}{n}\sum x_t x_t' \xrightarrow{P} \mathbf{\Sigma}_{xx} \equiv \mathbb{E}\left[x_t x_t'\right]$

$\frac{1}{n}\sum u_t^2 x_t x_t' \xrightarrow{P} S$

- form of S depends on assumptions
  - martingale difference sequence assumption (no correlation)

$$S = \mathbb{E}\left[(x_t u_t)^2\right]$$

  - serial correlation yields

$$S = \sum_{j=0}^{J} \mathbb{E}\left[u_t u_{t-j} x_t x_{t-j}'\right]$$

Scalar to General Model

Scalar

$$Var\hat{\beta} = \frac{1}{\sum x_t^2} \sum x_t^2 u_t^2 \frac{1}{\sum x_t^2}$$

General

$$Var\hat{\beta} = \left(\sum x_t x_t'\right)^{-1} \sum u_t^2 x_t x_t' \left(\sum x_t x_t'\right)^{-1}$$

$$= \frac{1}{n} \left(\frac{1}{n} \sum x_t x_t'\right)^{-1} \frac{1}{n} \sum u_t^2 x_t x_t' \left(\frac{1}{n} \sum x_t x_t'\right)^{-1}$$

$$\equiv \frac{1}{n} S_{xx}^{-1} S \, S_{xx}^{-1}$$

Eicker-White Estimator

Need to estimate

$$S = \frac{1}{n} \sum u_t^2 x_t x_t'$$

Eicker-White estimator

$$\hat{S} = \frac{1}{n} \sum \hat{u}_t^2 x_t x_t'$$

$\hat{u}_t = y_t - x_t' \hat{\beta}$

$\hat{\beta}$ consistent for β (e.g. OLSE)

Heteroskedasticity-consistent standard errors

$$\widehat{se} = \sqrt{\frac{1}{n} S_{xx}^{-1} \hat{S} \, S_{xx}^{-1}}$$

Finite-Sample Accuracy

Test $H_0 : \beta_k = 0$

Statistic

$$\frac{\hat{\beta}_k}{\widehat{se}}$$

Size

$$\Pr\left(\text{reject } H_0 | H_0 \text{ true}\right)$$

in practice: over-rejection problem

$$\text{nominal size 5\%,empirical size} > 5\%$$

reason: estimated standard error is too small

$\hat{u}_t^2$ is a downward biased estimator of u_t^2

Modification 1: Degrees-of-Freedom

Replace $\hat{S}$ with

$$\tilde{S} = \frac{1}{n-k} \sum \hat{u}_t^2 x_t x_t'$$

$n - k < n \Rightarrow$ estimated standard errors are larger

$$\widetilde{se} = \sqrt{\frac{1}{n} S_{xx}^{-1} \hat{S} \, S_{xx}^{-1}}$$

Modification 2: Influence

Under homoskedasticity

$$E\hat{u}_t^2 = \sigma^2 (1 - p_t)$$

Therefore, replace $\hat{u}_t^2$ with

$$\frac{\hat{u}_t^2}{1 - p_t}$$

yielding

$$\tilde{S} = \frac{1}{n} \sum \frac{\hat{u}_t^2}{1 - p_t} x_t x_t'$$

Influence Calculation

Observation t influence: p_t

Scalar model

$$p_t = \frac{x_t^2}{\sum x_t^2}$$

General model

$$p_t = x_t' \left(X'X \right)^{-1} x_t$$

Asymptotic Theory

Goal: Establish

$$\hat{S} \xrightarrow{p} S$$

Approach (outline for scalar case)

$$\sum \hat{u}_t^2 x_t^2 - \sum u_t^2 x_t^2 \xrightarrow{p} 0$$

Step 1: Algebra

$$\begin{aligned}
\hat{u}_t^2 &= \left((y_t - \beta x_t) - \left(\hat{\beta} - \beta\right) x_t\right)^2 \\
&= u_t^2 - 2\left(\hat{\beta} - \beta\right) x_t u_t + \left(\hat{\beta} - \beta\right)^2 x_t^2
\end{aligned}$$

Asymptotic Theory Outline

Step 2: Moment form

$$\tfrac{1}{n}\sum \hat{u}_t^2 x_t^2 - \tfrac{1}{n}\sum u_t^2 x_t^2 =$$

$$-2\left(\hat{\beta}-\beta\right)\tfrac{1}{n}\sum u_t x_t^3 + \left(\hat{\beta}-\beta\right)^2 \tfrac{1}{n}\sum x_t^4$$

Key issue: convergence of

$$\frac{1}{n}\sum x_t^4 \text{ and } \frac{1}{n}\sum u_t x_t^3$$

Steps for convergence

- Establish moments exist and are finite
- Ergodic stationarity ensures sample moments converge

Asymptotic Theory Moment Existence 1

For

$$\frac{1}{n}\sum x_t^4$$

Assume

$$Ex_t^4 \text{ exists and is finite.}$$

General Assumption (2.6 in Hayashi)

$$E\left[\left(x_{tk}x_{tj}\right)^2\right] \text{ exists and is finite for all } k, j$$

Moment Existence 2

For $\frac{1}{n}\sum u_t x_t^3$

$$E\left|u_t x_t^3\right| = E\left|f \cdot h\right|$$

with $f = u_t x_t$ and $h = x_t^2$

Cauchy-Schwarz inequality

$$E\left|f \cdot h\right| \leq \sqrt{E\left(f^2\right)E\left(h^2\right)}$$

Hence

$$E\left|u_t x_t^3\right| \leq \sqrt{E\left(u_t^2 x_t^2\right)E\left(x_t^4\right)}$$

Already assumed

$$E\left(x_t^2 u_t^2\right) \text{ exists and is finite}$$

Therefore $E\left|u_t x_t^3\right|$ exists and is finite

Asymptotic Theory: Final Step

$$\frac{1}{n}\sum \hat{u}_t^2 x_t^2 - \frac{1}{n}\sum u_t^2 x_t^2 =$$

$$-2\left(\hat{\beta}-\beta\right)\frac{1}{n}\sum u_t x_t^3 + \left(\hat{\beta}-\beta\right)^2 \frac{1}{n}\sum x_t^4$$

Because $\frac{1}{n}\sum x_t^4 \xrightarrow{p} c_1$ and $\frac{1}{n}\sum u_t x_t^3 \xrightarrow{p} c_2$

$$-2\left(\hat{\beta}-\beta\right)\frac{1}{n}\sum u_t x_t^3 + \left(\hat{\beta}-\beta\right)^2 \frac{1}{n}\sum x_t^4 \xrightarrow{p} 0$$

Hence

$$\frac{1}{n}\sum \hat{u}_t^2 x_t^2 - \frac{1}{n}\sum u_t^2 x_t^2 \xrightarrow{p} 0$$

$$\hat{S} - S \xrightarrow{p} 0$$