Econ 241A Prof. Startz

Midterm Fall 2016 Answers

Please answer all questions. Show your work.

The exam is open book/open note; closed any devices that can communicate. (No laptops, cell phones, Morse code keys, signal fires, etc.)

1 Consider a sample of $x_1, ..., x_n$ where $x_i \sim N(\mu, \sigma^2)$

Your TA asks you to consider an estimator given by $\tilde{x} = -\frac{1}{n_e} \sum_{i \text{ is even}} x_i + 2\frac{1}{n_o} \sum_{j \text{ is odd}} x_j$

a) Is $\tilde{x}$ an unbiased estimator of μ?

b) Find $Var(\tilde{x})$. How does it compare to $Var(\bar{x})$?

Answer:

a)

$$\mathbb{E}[\tilde{x}] = \mathbb{E}[-\frac{1}{n_e} \sum_{i \text{ is even}} x_i + 2\frac{1}{n_o} \sum_{j \text{ is odd}} x_j] = -\frac{1}{n_e} \sum_{i \text{ is even}} \mathbb{E}[x_i] + 2\frac{1}{n_o} \sum_{j \text{ is odd}} \mathbb{E}[x_j] = -\mu + 2\mu = \mu$$

b)

$$Var(\tilde{x}) = Var(-\frac{1}{n_e} \sum_{i \text{ is even}} x_i + 2\frac{1}{n_o} \sum_{j \text{ is odd}} x_j) = \frac{1}{n_e^2} n_e \sigma^2 + \frac{4}{n_o^2} n_o \sigma^2 = (\frac{1}{n_e} + \frac{4}{n_o})\sigma^2$$

$$Var(\bar{x}) = \frac{\sigma^2}{n}$$

$$n > n_e \Rightarrow \frac{1}{n} < \frac{1}{n_e} \Rightarrow Var(\tilde{x}) > Var(\bar{x})$$

2 Consider two independent variables X and Y with marginal densities given by:

$f_X(x) = \frac{x}{2}$ if $x \in [0, 2]$ and $f_Y(y) = \frac{3}{2} - y$ if $y \in [0, 2]$

Note: there was a typo in the domain of y. It should have read: $y \in [0, 1]$. This made both parts of the question give weird answers.

a) Find $f_{X,Y}(1, 2)$ and $f_{X,Y}(2, 0)$

b) Find $P(X - Y > 0)$

Answer:

a)

$$f_{xy}(x, y) = f_x f_y = \frac{x}{2}(\frac{3}{2} - y)$$

$$f_{xy}(1, 2) = \frac{1}{2}(\frac{3}{2} - 2) = -\frac{1}{4}$$

$$f_{xy}(2, 0) = 1 * \frac{3}{2} = \frac{3}{2}$$

b)

$$P(X - Y > 0) = P(Y < X)$$
$$0 \le x \le 2$$
$$0 \le y \le x$$
$$\int_0^2 \int_0^x \frac{x}{2}(\frac{3}{2} - y)dydx = \cdots = 1$$

2 corrected A better version of the problem would look like this:
Consider two independent variables X and Y with marginal densities given by:
$f_X(x) = \frac{x}{2}$ if $x \in [0, 2]$ and $f_Y(y) = \frac{3}{2} - y$ if $y \in [0, 1]$

a) Find $f_{X,Y}(1, 1)$ and $f_{X,Y}(2, 0)$
b) Find $P(X - Y > 0)$
Answer:
a)

$$f_{xy}(x, y) = f_x f_y = \frac{x}{2}(\frac{3}{2} - y)$$
$$f_{xy}(1, 1) = \frac{1}{2}(\frac{3}{2} - 1) = \frac{1}{4}$$
$$f_{xy}(2, 0) = 1 * \frac{3}{2} = \frac{3}{2}$$

b)

$$P(X - Y > 0) = P(Y < X)$$
$$0 \le x \le 2$$
$$0 \le y \le x$$

Be careful on the domain of y.

$$\int_0^1 \int_0^x \frac{x}{2}(\frac{3}{2} - y)dydx + \int_1^2 \int_0^1 \frac{x}{2}(\frac{3}{2} - y)dydx$$
$$= \frac{3}{16} + \frac{3}{4}$$
$$= \frac{15}{16}$$

3 The ranking General in the Prussian army just watched the movie Concussion and is concerned about the prevalence of CTE in his soldiers from being kicked in the head by bunnies. He asks you to model the distribution of the number of kicks that a soldier receives (per month) as poisson with parameter λ.

a) In terms of λ, on average, what percent of soldiers are kicked more than twice each month?

The General supplies you with the following data on kicking incidents:

# Kicks	# of Soldiers
0	50
1	10
2	5
3	5
4	3
10	1

b) Estimate and interpret $\hat{\lambda}$.

c) What is your estimate of Part a?

d) Suppose 50% of the soldiers are cavalry and are therefore not at risk of being kicked by bunnies. What other distribution would be more appropriate to model this data? Provide estimates for the parameters of this model (they don't have to be exact, just approximate).

Answer:

a)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - e^{-\lambda} \sum_{i=0}^{2} \frac{\lambda^i}{i!} = 1 - e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2})$$

b)

$$\lambda = \mathbb{E}[X] \Rightarrow \hat{\lambda} = \frac{1}{n} \sum X_i = \frac{1}{74}(0 * 50 + 1 * 10 + 2 * 5 + 3 * 5 + 4 * 3 + 10 * 1) = 0.77$$

On average, a soldier gets gets kicked in the head 0.77 times every month.

c)

$$P(X > 2) = 1 - e^{-\hat{\lambda}}(1 + \hat{\lambda} + \frac{\hat{\lambda}^2}{2}) = 0.043$$

d) A zero-inflated poisson would capture that some of the soldiers were not at risk of being kicked.

$$\hat{\pi} = 0.5,\ \hat{\lambda} = \frac{1}{37}(0 * 13 + 1 * 10 + 2 * 5 + 3 * 5 + 4 * 3 + 10 * 1) = 1.54$$

4 Consider three variables that are distributed jointly normal:

$$\begin{pmatrix} \theta \\ \nu \\ \epsilon \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \right)$$

Define $X = \theta + \nu$, $Y = \theta + \epsilon$, and $Z = \theta | X$.

a) Find $\mathbb{E}[Z]$.

b) Find $\mathbb{E}[Z|Y]$. (You can assume $Cov(Z, Y) = a$)

Answer:

a)

$$\mathbb{E}[Z] = \mathbb{E}[\theta|X] = \mathbb{E}[\theta] + \frac{Cov(\theta, X)}{Var(X)}(X - \mathbb{E}[X]) = \mu + \frac{a}{a+b}(X - \mu) = \frac{b}{a+b}\mu + \frac{a}{a+b}X$$

b)

$$\mathbb{E}[Z|Y] = \mathbb{E}[Z] + \frac{Cov(Z,Y)}{Var(Y)}(Y - \mathbb{E}[Y]) = \frac{b}{a+b}\mu + \frac{a}{a+b}X + \frac{a}{a+c}(Y-\mu) = (\frac{b}{a+b} + \frac{c}{a+c})\mu + \frac{a}{a+b}X + \frac{a}{a+c}Y$$

5 $X \sim U(-2,4)$

$g(x) = abs(x)$

$Y = g(X)$

Find $F_Y(y)$ and plot it.

Answer:

$$F_Y(y) = P(Y < y) = P(abs(X) < y) = P(-y < X < y) = P(X < y) - P(X < -y) = F_X(y) - F_X(-y)$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ F_X(y) - F_X(-y) & \text{if } 0 \leq y \leq 2 \\ F_X(y) - 0 & \text{if } 2 < y \leq 4 \\ 1 & \text{if } y > 4 \end{cases} = \begin{cases} 0 & \text{if } y < 0 \\ \frac{y}{3} & \text{if } 0 \leq y \leq 2 \\ \frac{y+2}{6} & \text{if } 2 < y \leq 4 \\ 1 & \text{if } y > 4 \end{cases}$$