Solutions to the Exercises in Public Finance

1 Chapter 1

1.1

(a) see graph

(b) (10,15) is Pareto optimal since it represents Bruce's bliss point. Any movement away from this point would make him worse off.

The P.O. set is the line segment $T = 5+C$, where $10 \leq C \leq 20$. All tangency points of the two circles around A and B must be on the straight line connecting A and B, however, only the points between A and B are P.O.

(c) The set of situations Pareto superior to (9,14) is the set of all C, T such that

$$-[(C-10)^2 + (T-15)^2] < 2$$

which is a circle around Bruce's bliss point of radius $\sqrt{2}$. see graph

(d) For the indifference curves to be tangent, the condition $MRS^A = MRS^B$ must hold.

$$MRS^A = \frac{C-20}{T-25} = MRS^B = \frac{C-10}{T-15}$$

Consider the point (5,10), where $MRS^A = MRS^B$. At this point,

$$\tilde{U}^A(5,10) = -250$$

$$\tilde{U}^B(5,10) = -100$$

But at the point (10,15), we have

$$\begin{aligned} U^A(10,15) &= -200 > \tilde{U}^A \\ U^B(10,15) &= 0 > \tilde{U}^B \end{aligned}$$

Thus (5,10) is not P.O. To increase both $\tilde{U}^A$ and $\tilde{U}^B$, both T and C have to be increased. A necessary condition for P.O. is, however, that preferred directions of change be opposite. At (5,10) the indifference curves are tangent on the "wrong" side.

see graph

(e) see graph

Suppose a is P.O. Consider the point b, which is not P.O., since there is c, which is Pareto superior to b. Both a and c are P.O. and c is Pareto superior to b, but a is not Pareto superior to b, since moving from b to a would make B worse off.

1.2 see graph

1.3

(a) No, they cannot improve on the sure thing P.O. allocations by gambling, since we are in the negative quadrant and making the utility possibility set a convex set by allowing for gambling does not increase their utilities.

(b)

$$
\begin{aligned}
U_A &= \frac{1}{(C-20)^2+(T-25)^2} \\
U_B &= \frac{1}{(C-10)^2+(T-15)^2}
\end{aligned}
$$

1.4

If the indifference curves are "sloped", then less C can be compensated by more T or vice versa. They prefer much more playing in colder temperature to playing just a little more with a little higher temperature. The more "sloped" the indifference curves are, the more extreme is this trade off.

1.5

The boundary of the utility possibility set slopes uphill to the west of point B because these points are representations of the "bad" tangency points. Moving away from these points, i.e. increasing U^A, also means increasing U^B.

1.6 see graph

1.7 see graph

The Pareto set is the triangle connecting the bliss points A, B and C. Set up the Lagrangean:

$$U^A(T,C)+\lambda(U^B(T,C)-\bar{U}^B)+\mu(U^C(T,C)-\bar{U}^C)$$

$$
\begin{aligned}
\frac{\partial L}{\partial C} &= -2(C-20) - 2\lambda(C-10) - 2\mu(C-20) \\
\frac{\partial L}{\partial T} &= -2(T-25) - 2\lambda(T-15) - 2\mu(T-15) \\
\frac{\partial L}{\partial \lambda} &= U^B(T,C) \geq \bar{U}^B, > if \lambda = 0 \\
\frac{\partial L}{\partial \mu} &= U^C(T,C) \geq \bar{U}^C, > if \mu = 0
\end{aligned}
$$

Checking for the constraint qualification, which allows us to use the Kuhn-Tucker method, we find that the determinant of the matrix of constraints is zero. Thus the constraint qualification is not met.

2 Chapter 2

2.1

(a)

$$\begin{aligned}\sum_{i=1}^{1000} \frac{\partial U_i/\partial Y}{\partial U_i/\partial X_i} &= \frac{10}{1} \\ 1000\frac{100}{Y^2} &= 10\end{aligned}$$

(b)

$$\begin{aligned}Y^2 &= 10,000 \\ Y &= 100\end{aligned}$$

2.2

$$\begin{aligned}\sum_{i=1}^{1000} MRS_i &= \sum_{i=1}^{1000} \frac{i}{20,000\sqrt{Y}} = 1 \\ 500,500 &= 20,000\sqrt{Y} \\ Y &= 25,025^2\end{aligned}$$

2.3

(a)

$$\begin{aligned}U_L &= X_i^{\alpha} Y^{1-\alpha} \\ U_O &= X_i^{\beta} Y^{1-\beta}\end{aligned}$$

Samuelson condition:

$$\begin{aligned}\sum_{i=1}^{500} \frac{(1-\alpha)X_i^{\alpha}Y^{-\alpha}}{\alpha X_i^{\alpha-1}Y^{1-\alpha}} + \sum_{i=1}^{500} \frac{(1-\beta)X_i^{\beta}Y^{-\beta}}{\beta X_i^{\beta-1}Y^{1-\beta}} &= 1 \\ \sum_{i=1}^{500} \frac{(1-\alpha)X_i}{\alpha Y} + \sum_{i=1}^{500} \frac{(1-\beta)X_i}{\beta Y} &= 1\end{aligned}$$

(b) $\alpha = \beta$

$$\sum_{i=1}^{1000} \frac{(1-\alpha)X_i}{\alpha Y} = 1$$

budget constraint:

$$Y + \sum_{i=1}^{1000} X_i = W$$

$$\begin{aligned} \frac{(1-\alpha)}{\alpha}(W-Y) &= Y \\ (1-\alpha)W &= Y \end{aligned}$$

Thus the optimal level of Y depends on aggregate income only.

2.4

$$\begin{aligned} U_C &= X_C + 2\sqrt{Y} \\ U_D &= X_D + \sqrt{Y} \end{aligned}$$

(a) Samuelson condition:

$$\sum_i MRS_i = Y^{-\frac{1}{2}} + \frac{1}{2}Y^{-\frac{1}{2}} = 1$$

$$Y = \frac{9}{4}$$

(b) $X_C = 0$ and $X_D = W - Y$. Then $U_C = 2\sqrt{Y}$ and $U_D = W - Y + \sqrt{Y}$. Then for a Pareto optimum the following condition must hold:

$$Y^{-\frac{1}{2}} - 1 + \frac{1}{2}Y^{-\frac{1}{2}} = 1$$

or

$$Y^{-\frac{1}{2}} + \frac{1}{2}Y^{-\frac{1}{2}} = 2$$

which is not the same as the Samuelson condition.

(c) see graph

(d) At $Y = \frac{9}{4}$ we have

$$U_C = X_C + 3 = W - X_D - \frac{3}{4}$$

$$U_D = X_D + \frac{3}{2}$$

$$U_D = (W + \frac{3}{2}) - U_C$$

which is the equation that describes the straight line part of the utility possibility frontier.

(e) $X_C = 0$ and $X_D = W - Y$. From $U_C = 2\sqrt{Y}$ we get $Y = \frac{U_C^2}{4}$ and plugging this expression into

$$U_D = W - Y + \sqrt{Y}$$

we can describe the top curved part of the utility possibility frontier by

$$U_D = W - \frac{U_C^2}{4} + \frac{U_C}{2}.$$

For the bottom curved part we set $X_D = 0$ and $X_C = W - Y$:

$$U_C = W - Y + 2\sqrt{Y}$$

and from $U_D = \sqrt{Y}$ we get

$$U_C = W - U_D^2 + 2U_D.$$

3 Chapter 3

3.1

$$
\begin{aligned}
U_C &= X_C Y_2 \\
U_D &= X_D Y
\end{aligned}
$$

Samuelson condition:

$$
2\frac{X_C}{Y} + \frac{X_D}{Y} = 1
$$

Budget equation:

$$
X_C + X_D + Y = W
$$

Distribution:

$$
X_C = 2X_D
$$

Putting these three equations together, we get

$$
\begin{aligned}
Y &= \frac{5}{8}W \\
X_C &= \frac{1}{8}W \\
X_D &= \frac{1}{4}W
\end{aligned}
$$

3.2

$\alpha > 0, \beta > 0$

$$
U_i = (X_i)_i^{\alpha} Y^{\beta}
$$

monotonic transformation:

$$
\begin{aligned}
V_i &= \ln U_i = \alpha \ln X_i + \beta \ln Y \\
W_i &= \ln X_i + \frac{\beta}{\alpha} \ln Y = A(Y) X_i = B_i(Y),
\end{aligned}
$$

where $A(Y) = 1$ and $B_i = \frac{\beta}{\alpha}, \forall i$.

3.3

The Lagrangean is

$$
\max Y \sum X_i + \lambda (W - \sum X_i - Y)
$$

solving for Y, we get

$$
Y = \sum X_i = \frac{W}{2}
$$

The utility possibility set is

see graph

where the utility possibility frontier is the triangle between

$$(\frac{W^2}{4}, 0, 0), (0, \frac{W^2}{4}, 0), (0, 0, \frac{W^2}{4})$$

3.4

(a) Samuelson condition:

$$\frac{\sum_i (X_i + k_i)}{Y} = 1$$

Using the budget equation: $\sum X_i + Y = W$, we get

$$Y = \frac{W + \sum k_i}{2}.$$

(b) Take a corner solution $X_1 = 0, X_2 = W - Y$. Solving for the optimal amount of Y, we get $Y = \frac{W + k_2}{2}$, which is not Pareto efficient, since it is different from the outcome of the Samuelson condition.

(c) see graph The highlighted part of the curve is the utility possibility frontier. From the interior P.O. amount of Y, we can use the two utility functions and the feasibility constraint to solve U_2 as a function of U_1:

$$U_2 = \frac{(k_1 + k_2)^2}{4} + \frac{W(k_1 + k_2)}{2} - U_1$$

which describes the straight part of the utility possiblity frontier. The curved parts come from the corner solutions:
$\underline{X_1 = 0}$ (top curve)

$$\begin{aligned}
U_1 &= k_1 Y \\
Y &= \frac{U_1}{k_1} \\
U_2 &= \frac{U_1}{k_1}(W - \frac{U_1}{k_1} + k_2) \\
U_2 &= \frac{W + k_1}{k_2} U_1 - \frac{{U_1}^2}{{k_1}^2}
\end{aligned}$$

$\underline{X_2 = 0}$ (bottom curve)

$$\begin{aligned} U_2 &= k_2 Y \\ Y &= \frac{U_2}{k_2} \\ U_1 &= \frac{U_2}{k_2}(W - \frac{U_2}{k_2} + k_1) \\ U_1 &= \frac{W + k_2}{k_1} U_2 - \frac{{U_2}^2}{{k_2}^2} \end{aligned}$$

(d) If both $k_i < 0$, the optimal amount of public good decreases (looking at FOCs). If one of the k_i is negative, one person gets all income and doesn't take into account the reduction in utility for the other. Therefore, the amount of Y provided is higher than the Pareto efficient level.

3.5

$$MRS_{YX_i} = \frac{\alpha X_i + \beta_i Y + \gamma_i)}{Y} + \beta_i$$

Samuelson condition:

$$\frac{\sum_i (\alpha X_i + \beta_i Y + \gamma_i)}{Y} + \sum \beta_i = 1$$

using the global feasibility condition

$$\sum X_i + Y = W$$

we get

$$\begin{aligned} Y &= n\alpha(W - Y) + 2Y \sum \beta_i + \sum \gamma_i \\ Y &= \frac{n\alpha W + \sum \gamma_i}{1 - 2\sum \beta_i + n\alpha} \end{aligned}$$

3.6
(a) Proof: Suppose that $(\bar{X}, \bar{Y})$ is feasible and maximizes $A(Y)X + \sum B_i(Y)$. Suppose that the allocation (X', Y') is Pareto superior to $(\bar{X}, \bar{Y})$. Then

$$A(Y')X_i' + B_i(Y') \geq A(\bar{Y})\bar{X}_i + B_i(\bar{Y})$$

for all i, with strict inequality for some i. Then

$$A(Y')X' + \sum B_i(Y') > A(\bar{Y})\bar{X} + \sum B_i(\bar{Y})$$

But since $(\bar{X}, \bar{Y})$ is feasible and maximizes $A(Y)X + \sum B_i(Y)$, (X', Y') must not be feasible. Thus, $(\bar{X}, \bar{Y})$ must be Pareto optimal.

(b) Suppose that $(\bar{X}, \bar{Y})$ is feasible and maximizes $\sum_{i=1}^{n} a_i U_i(X_i, Y)$. Suppose that the allocation (X', Y') is Pareto superior to $(\bar{X}, \bar{Y})$. Then

$$U(X_i', Y') \geq U_i(\bar{X}_i, \bar{Y})$$

for all i and with strict inequality for at least one i. Then

$$\sum_{i=1}^{n} a_i U_i(X_i', Y') > \sum_{i=1}^{n} a_i U_i(\bar{X}_i, \bar{Y})$$

for $a > 0$. But if this is true, (X', Y') must not be feasible, since it contradicts $(\bar{X}, \bar{Y})$ maximizing $\sum_{i=1}^{n} a_i U_i(X_i, Y)$.

3.7

Set up the Lagrangean:

$$\begin{aligned}
L &= 2X_1 + X_2 + 2\sqrt{Y} + \lambda(3 - X_1 - X_2 - Y) \\
\frac{\partial L}{\partial X_1} &= 2 - \lambda \leq 0 \\
\frac{\partial L}{\partial X_2} &= 1 - \lambda \leq 0 \\
\frac{\partial L}{\partial Y} &= Y^{\frac{1}{2}} - \lambda \leq 0 \\
\frac{\partial L}{\partial \lambda} &= 3 - X_1 - X_2 - Y \geq 0
\end{aligned}$$

There is no interior solution. If we set $X_1 = 0$, we get $\lambda > 2$ and $\lambda = 1$, which is a contradiction. Set $X_2 = 0$. Then the conditions for λ are $\lambda = 2$ and $\lambda > 1$ and $Y = \frac{1}{4}$ and $X_1 = \frac{11}{4}$.

3.8

(a) Utility function of the Bergstrom-Cornes form:

$$U_i(X_i, Y) = A(Y)X_i + B_i(Y)$$

With the given utility functions for U_1 and U_2, $A(Y) = Y$, $B_1(Y) = Y$ and $B_2(Y) = -\frac{1}{2}Y^2$.

(b) For person two, the public good increases utility only if it is consumed with the private good, i.e. if $X_2 > 0$. Too much of the public good decreases utility (the optimal amount is where $Y = X_2$).

(c) interior solution:

$$\begin{aligned}
\max L &= (1+X_1)Y + X_2Y - \frac{1}{2}Y^2 + \lambda(W - X_1 - X_2 - Y) \\
\frac{\partial L}{\partial X_1} &= Y - \lambda = 0 \\
\frac{\partial L}{\partial X_2} &= Y - \lambda = 0 \\
\frac{\partial L}{\partial Y} &= 1 + X_1 + X_2 - Y - \lambda = 0 \Rightarrow Y = \frac{1 + X_1 + X_2}{2} \\
\frac{\partial L}{\partial \lambda} &= W - X_1 - X_2 - \frac{1 + X_1 + X_2}{2} = 0
\end{aligned}$$

Then for all $W > \frac{1}{2}$

$$X = X_1 + X_2 = \frac{1}{3}(2W - 1)$$

$$Y = \frac{1}{3}(W + 1)$$

corner solution: if $W \leq \frac{1}{2}$ then $X = 0$ and $Y = W$.

(d) The interior P.O. set consists of any allocation such that $X_1, X_2 \geq 0$ and $\{X_1 + X_2 = \frac{2W-1}{3}, Y = \frac{W+1}{3}\}$ In the case where $W = 4$, the interior P.O. set consists of any allocation such that $X_1, X_2 \geq 0$ and $\{X_1 + X_2 = \frac{7}{3}, Y = \frac{5}{3}\}$. To find the corner solutions, we maximize U_i subject to $X_i + Y = W$:

$$\begin{aligned}
\max L_1 &= (1 + X_1)Y + \lambda(W - Y - X_1) \\
\max L_2 &= X_2Y - \frac{1}{2}Y^2 + \lambda(W - Y - X_2)
\end{aligned}$$

We get $Y = \frac{W+1}{2}$ and $X_1 = \frac{W-1}{2}$ when $X_2 = 0$, and $Y = \frac{1}{3}W$ and $X_2 = \frac{2}{3}W$ when $X_1 = 0$. In the case where $W = 4$ the corner solutions are all allocations $(W - Y, 0, Y)$ such that $\frac{5}{3} \leq Y \leq \frac{5}{2}$ and $(0, W - Y, Y)$ such that $\frac{4}{3} \leq Y \leq \frac{5}{3}$.
The utility possibility set consists of three parts:
For the straight part, we have an interior P.O. amount of $Y = \frac{W+1}{3}$ and we get

$$\begin{aligned}
X_1 &= \frac{U_1}{Y} - 1 \\
X_2 &= W - Y - X_1 \\
U_2 &= (W - Y - X_1)Y - \frac{1}{2}Y^2 \\
&= \left(W - \frac{W+1}{3} - X_1\right)\frac{W+1}{3} - \frac{1}{2}\left(\frac{W+1}{3}\right)^2 \\
&= \frac{1}{9}(2W-1)(W+1) + \frac{1}{3}(W+1) - \frac{1}{18}(W+1)^2 - U_1
\end{aligned}$$

For the top curve where $X_1 = 0$ we get

$$\begin{aligned}
U_1 &= Y \\
X_2 &= W - Y \\
U_2 &= X_2 U_1 - \frac{1}{2}{U_1}^2 \\
&= W U_1 - \frac{3}{2}{U_1}^2
\end{aligned}$$

For the bottom curve where $X_2 = 0$ we get

$$\begin{aligned}
U_1 &= (1 + W - Y)Y \\
U_2 &= -\frac{1}{2}Y^2
\end{aligned}$$

Solving for Y from the first equation gives two possible values for Y:

$$Y_1 = \frac{(1+W) + \sqrt{(1+W)^2 - 4U_1}}{2}, \; Y_2 = \frac{(1+W) - \sqrt{(1+W)^2 - 4U_1}}{2}$$

where only Y_2 is feasible $(Y_1 > W)$. Plugging this into the equation for U_2, we get

$$U_2 = -\frac{1}{2}\left(\frac{1 + W - \sqrt{(1+W)^2 - 4U_1}}{2}\right)^2$$

Then the utility possibility frontier is the set of all U_2 such that

$$U_2 = \begin{cases} 4U_1 - \frac{3}{2}{U_1}^2 & \text{for } \frac{4}{3} \le U_1 \le \frac{5}{3} \\ \frac{75}{18} - U_1 & \text{for } \frac{5}{3} \le U_1 \le \frac{50}{9} \\ -\frac{1}{2}\left(\frac{5 - \sqrt{25 - 4U_1}}{2}\right)^2 & \text{for } \frac{50}{9} \le U_1 \le \frac{25}{4} \end{cases}$$

(f) If $W = \frac{1}{2}$, there is no interior P.O. The P.O. set is

$$(0, \frac{1}{2} - Y, Y) \text{ for } \tfrac{1}{6} \le Y \le \tfrac{1}{2}$$

The utility possibility frontier is given by

$$U_2 = -\frac{1}{2}\left(\frac{\frac{3}{2} - \sqrt{(\frac{3}{2})^2 - 4U_1}}{2}\right)^2 \text{ for } \tfrac{1}{6} \le U_1 \le \tfrac{1}{2}$$

see graph

(g) If $W = \frac{3}{2}$, the interior P.O. set consists of any allocation such that $X_1, X_2 \geq 0$ and $\{X_1 + X_2 = \frac{2}{3}, Y = \frac{5}{6}\}$. The corner solutions are all allocations $\left(\frac{3}{2} - Y, 0, Y\right)$ for $\frac{5}{6} \leq Y \leq \frac{5}{4}$ and $\left(0, \frac{3}{2} - Y, Y\right)$ for $\frac{1}{2} \leq Y \leq \frac{5}{6}$.
The utility possibility frontier is the set of all U_2 such that

$$U_2 = \begin{cases} \frac{3}{2}U_1 - \frac{3}{2}{U_1}^2 & \text{for } \frac{1}{2} \leq U_1 \leq \frac{5}{6} \\ \frac{75}{72} - U_1 & \text{for } \frac{5}{6} \leq U_1 \leq \frac{50}{36} \\ -\frac{1}{2}\left(\frac{\frac{5}{2} - \sqrt{\left(\frac{5}{2}\right)^2 - 4U_1}}{2}\right)^2 & \text{for } \frac{50}{36} \leq U_1 \leq \frac{25}{16} \end{cases}$$

see graph

(h) If both consumers consume some private goods, we have a unique P.O. amount for the public good which also maximizes the sum of the utilities.

3.9
Solving $U = A(Y) + B_i(Y)$ for X and taking the derivative with respect to Y we get:

$$MRS = \frac{dx}{dy} = \frac{-B_i'(Y)A(Y) - A'(Y)(\bar{U} - B_i(Y))}{A(Y)^2}$$

To get the conditions for the functions A and B_i that imply diminishing MRS (i.e. convex indifference curves), we have to show for which A and B_i we get quasiconcave utility functions. $Y = A(Y)X + B_i(Y)$ is quasiconcave iff the Hessian matrix is negative semidefinite. The necessary and sufficient conditions are:

$$\begin{aligned} 2U_Y U_X U_X Y - {U_Y}^2 U_X X - {U_X}^2 U_Y Y &\geq 0 \\ 2(A'(Y)X + {B_i}'(Y))A(Y)A'(Y) - A(Y)^2(A''(Y)X + {B_i}''(Y)) &\geq 0 \end{aligned}$$

3.10
(a) Suppose there is a monotonic transformation that makes it possible to write both utility functions in the Bergstrom-Cornes form. Then the sum of the MRS would not change after a redistribution of private goods. The sum of the MRS of the given utility functions is

$$\sum MRS = \frac{X_1 + 1}{Y + 1} + \frac{W - X_1 + 1}{Y - 2}$$

After a redistribution of private goods, where person 1 gets $X_1 + \epsilon$ and person 2 gets $X_2 - \epsilon = 1 - X_1 - \epsilon$ the new sum of the MRS will be

$$\sum MRS' = \sum MRS + \frac{\epsilon}{Y+1} - \frac{\epsilon}{Y-2}$$

which is not equal to the initial MRS. Thus, there is no monotonic transformation to write both utility functions in the Bergstrom-Cornes form.

(b) see graph

(c) Suppose there is an allocation (X, Y) with $X_i \geq 0$ and $\sum X = 1$ that is not P.O. Then there is another allocation $(\bar{X}, \bar{Y})$ such that

$$\begin{aligned} U_1(X,Y) &\leq U_1(\bar{X}, \bar{Y}) \\ U_2(X,Y) &\leq U_2(\bar{X}, \bar{Y}) \end{aligned}$$

with one of the inequalities strict. Since the public good can take only the value 0 or 1, consider the case where $\bar{Y} = Y$ (i.e. $Y = 0$ and $\bar{Y} = 0$ or $Y = 1$ and $\bar{Y} = 1$). Then for person 1

$$(1+Y)X_1 + Y + 1 \leq (1+\bar{Y})\bar{X}_1 + \bar{Y} + 1 = (1+Y)\bar{X}_1 + Y + 1$$

We get the condition $X_1 \leq \bar{X}_1$. For person 2 we have

$$(2-Y)X_2 - Y + 2 \leq (2-\bar{Y})\bar{X}_2 - \bar{Y} + 2 = (2-Y)\bar{X}_2 - Y + 2$$

and we get the condition $X_2 \leq \bar{X}_2$. Also, one of the inequalities has to be strict, i.e. $X_1 + X_2 < \bar{X}_1 + \bar{X}_2$. But from the feasibility condition, we have $X_1 + X_2 = \bar{X}_1 + \bar{X}_2 = 1$, which leads to a contradiction. In the case where $Y = 1$ and $\bar{Y} = 0$ (and similarly for $Y = 0$ and $\bar{Y} = 1$), if (X, Y) is not P.O. then

$$\begin{aligned} 2X_1 + 2 &\leq \bar{X}_1 + 1 \\ X_2 + 1 &\leq 2\bar{X}_2 + 2 \end{aligned}$$

with one inequality strict. From the first inequality we get the condition $X_1 \leq \frac{\bar{X}_1 - 1}{2}$. Since $X_1 \geq 0$, it must be that $\bar{X}_1 = 1$ and therefore $X_1 = 0$ and $\bar{X}_2 = 0$. But then none of the two inequalities can be stricy. Therefore any feasible (X, Y) must be P.O.

(d) see graph

The graph shows that by introducing a lottery both consumers can be made better off, if we consider only allocations where both consumers have some private goods. Thus, denoting by $U_i^{max}(U_i^{min})$ the highest (lowest) possible utility person i can get, if we introduce a lottery such that for $Y = 0$

$$\begin{aligned} pU_1{}^{max} + (1-p)U_1{}^{min} &> U_1 \\ pU_2{}^{min} + (1-p)U_1{}^{max} &> U_2 \end{aligned}$$

We get the condition $p > U_1 - 1$. In the case where $Y = 1$, we have

$$\begin{aligned} pU_1{}^{min} + (1-p)U_1{}^{max} &> U_1 \\ pU_2{}^{max} + (1-p)U_1{}^{min} &> U_2 \end{aligned}$$

We get the condition $p > U_2 - 1$. Then both consumers are better off.

(e) see graph

To show that the P.O. amount of Y depends on the distribution of income, consider the allocation where $X_1 = 2$ and $X_2 = 0$. The respective utilities in this case are

$$\begin{aligned} U_1(2,0) &= 3 \text{ and } U_2(0,0) = 2 \\ U_1(2,1) &= 6 \text{ and } U_2(0,1) = 1 \end{aligned}$$

Thus, $Y = 0$ is a P.O. amount of the public good given the distribution of private goods with $X_1 = 2$ and $X_2 = 0$. However, if we change the distribution of private goods, we find that $Y = 0$ is not P.O. in the following case:

$$\begin{aligned} U_1(X_1,1) = (X_1+1)2 &> U_1(2,0) = 3 \\ U_2(X_2,1) = (X_2+1) &> U_2(0,0) = 2 \end{aligned}$$

Then for any distribution of the private goods such that $\frac{1}{2} < X_1 < 1$ the P.O. amount of the public good is $Y = 1$.

4 Chapter 4

4.1

(a) Samuelson condition:

$$\begin{aligned}\sum_i MRS_{YX_i} &= MRT \\ MRS_{YX_i} &= \frac{\alpha X_i}{(1-\alpha)Y} \\ \sum_i MRS_{YX_i} &= \frac{\alpha \sum X_i}{(1-\alpha)Y}\end{aligned}$$

from the Cobb-Douglas utility function, we have $X_i = (1-\alpha)W_i$

$$\sum MRS_{YX_i} = \frac{\alpha \sum W_i}{Y}$$

then using the Samuelson condition,

$$\frac{\alpha \sum W_i}{Y} = p$$

and thus

$$Y = \frac{\alpha \sum W_i}{p}$$

(b) The Lindahl prices are

$$p_i^Y = \frac{\alpha W_i}{Y} = \frac{W_i}{\sum W_i} p$$

4.2

From the Cobb-Douglas utility function, demand for X_i and Y_i are

$$\begin{aligned}X_i &= (1-\alpha)W_i \\ Y_i &= \frac{\alpha W_i}{p_i^Y}\end{aligned}$$

The global feasibility condition is:

$$\sum X_i + pY = \sum W_i$$

and since Y is a public good, $Y_1 = \cdots = Y_N = Y$. Solving for Y, we get

$$Y = \frac{\sum W_i - \sum X_i}{p} = \frac{\alpha \sum W_i}{p}$$

Solving for Lindahl prices,

$$\begin{aligned} p_i^Y &= \frac{\alpha W_i}{Y} \\ &= \frac{\alpha W_i}{\sum Wi - \sum X_i} p \\ &= \frac{W_i}{\sum W_i} p \end{aligned}$$

4.3

(a) Denoting the total number of α's by a, the number of β's by b and the number of γ's by c, we have the following individual budget constraint:

$$\begin{aligned} W_\alpha &= X_{i\alpha} + p_{i\alpha}^Y Y \quad \text{for } i = 1, \cdots, a \\ W_\beta &= X_{i\beta} + p_{i\beta}^Y Y \quad \text{for } i = 1, \cdots, b \\ W_\gamma &= X_{i\gamma} + p_{i\gamma}^Y Y \quad \text{for } i = 1, \cdots, c \end{aligned}$$

From Cobb-Douglas utility functions, the individual demand for X is

$$\begin{aligned} X_{i\alpha} &= (1-\alpha) W_\alpha \\ X_{i\beta} &= (1-\beta) W_\beta \\ X_{i\gamma} &= (1-\gamma) W_\gamma \end{aligned}$$

from the Samuelson condition, we get

$$\sum_{i=1}^{a} \frac{\alpha X_{i\alpha}}{(1-\alpha)Y} + \sum_{i=1}^{b} \frac{\beta X_{i\beta}}{(1-\beta)Y} + \sum_{i=1}^{c} \frac{\gamma X_{i\gamma}}{(1-\gamma)Y} = p,$$

where $c = N - a - b$. Since all people of one type must be treated equally, we have

$$\begin{aligned} a\frac{\alpha X_{i\alpha}}{(1-\alpha)Y} + b\frac{\beta X_{i\beta}}{(1-\beta)Y} + c\frac{\gamma X_{i\gamma}}{(1-\gamma)Y} &= p \\ \frac{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma}{p} &= Y \end{aligned}$$

and solving for the Lindahl prices, we get

$$\begin{aligned} p_{i\alpha}^Y &= \frac{p\left(W_\alpha - X_{i\alpha}\right)}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\ &= \frac{p\alpha W_\alpha}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\ p_{i\beta}^Y &= \frac{p\beta W_\beta}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\ p_{i\gamma}^Y &= \frac{p\gamma W_\gamma}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \end{aligned}$$

(b) Since $X_i = \frac{1}{N}X$, we have

$$\begin{aligned}
U_{i\alpha}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\alpha)} Y^{\alpha} \\
U_{i\beta}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\beta)} Y^{\beta} \\
U_{i\gamma}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\gamma)} Y^{\gamma}
\end{aligned}$$

The individual budget constraints are

$$\frac{1}{N}X = W_\alpha - p_{i\alpha}^Y Y = W_\beta - p_{i\beta}^Y Y = W_\gamma - p_{i\gamma}^Y Y$$

The individual demand for X is

$$\frac{1}{N}X = (1-\alpha)W_\alpha = (1-\beta)W_\beta = (1-\gamma)W_\gamma$$

Then the total demand for X:

$$X = a(1-\alpha)W_\alpha + b(1-\beta)W_\beta + c(1-\gamma)W_\gamma$$

From the Samuelson condition

$$\begin{aligned}
a\frac{\alpha X}{(1-\alpha)NY} + b\frac{\beta X}{(1-\beta)NY} + c\frac{\gamma X}{(1-\gamma)NY} &= p \\
\frac{X}{NY}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right) &= p
\end{aligned}$$

The optimal amount of the public good is then

$$\begin{aligned}
Y &= \frac{X}{Np}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right) \\
&= \frac{a(1-\alpha)W_\alpha + b(1-\beta)W_\beta + c(1-\gamma)W_\gamma}{Np}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right)
\end{aligned}$$

4.4

$$U_i(X_i, Y) = Y_\alpha (X_i + k_i)$$

individual budget:

$$W_i = X_i + p_i^Y Y$$

global feasibility:

$$\sum W_i = \sum X_i + Y$$

Samuelson condition:

$$\sum MRS = \frac{\alpha}{Y}(X_i + k_i) = 1$$

substituting for X_i from global feasibility:

$$\frac{\alpha}{Y}\left(\sum W_i - Y + \sum k_i\right) = 1$$

$$Y = \frac{\alpha}{1+\alpha}\left(\sum W_i + \sum k_i\right)$$

To get Lindahl prices,

$$\begin{aligned} MRS_i &= \frac{\alpha}{Y}\left(X_i + k_i\right) + p_i^Y \\ p_i^Y &= \frac{W_i + k_i}{\sum W_i + \sum k_i} \end{aligned}$$

5 Chapter 5

5.1

(a) see graph

(b) At an interior solution: $MRS_E = MRS_F$ From their respective utility functions, we solve for S:

$$S = \sqrt{\frac{B_E}{2}}$$

Thus, the set of interior P.O. allocations is

$$\{B_E > 0, B_F > 0, S = S = \sqrt{\frac{B_E}{2}}\}.$$

The corner solutions are

$$\{B_E = 16, B_F = 0, 2\sqrt{2} < S < 4\} \quad \text{and} \quad \{B_E = 0, B_F = 16, S = 0\}$$

(c) From (b) we have the condition $B_E = 2S^2$, plugging this into $U_E = B_E S$, we get

$$U_E = 2S^3$$

From $U_F = B_F - S^2$, since $B_F = 16 - B_E$, we get

$$U_F = 16 - 3S^2,$$

then

$$U_F = 6 - 3\left(\frac{U_E}{2}\right)^{\frac{2}{3}}$$

see graph

(d) initial property rights forbid smoking

$$p_F = -p_E$$

The budget constraints are

$$\begin{aligned} B_E + p_E S &\leq W_E \\ (16 - B_E) - p_E S &\leq 16 - W_E \end{aligned}$$

or

$$W_E \leq B_E + p_E S$$

Setting up the Lagrangean for both Fiona and Ed

$$\begin{aligned} L_E &= B_E S + \lambda(W_E - B_E - p_E S) \\ L_F &= B_F - S^2 + \lambda(W_F - B_F + p_F S) \end{aligned}$$

and solving simultaneously for Lindahl equilibrium prices, we get

$$p_E = \sqrt{W_E}$$

$$p_F = -\sqrt{W_E}$$

and the efficient amount of smoking is

$$S = \frac{\sqrt{W_E}}{2}$$

(e) If Ed can smoke as much as he wishes, there will be only one price which Fiona pays Ed to smoke less. Setting up the Lagrangean for Ed and Fiona:

$$\begin{aligned} L_E &= B_E S + \lambda_1(W_E + pS - B_E) + \lambda_2(4 - S) \\ L_F &= 16 - B_E - S^2 + \lambda_3(B_E - W_E - pS) \\ \frac{\partial L_E}{\partial B_E} &= S - \lambda_1 = 0 \Rightarrow S = \lambda_1 && (1) \\ \frac{\partial L_F}{\partial B_E} &= -1 + \lambda_3 = 0 \Rightarrow \lambda_3 = 1 && (2) \\ \frac{\partial L_E}{\partial S} &= B_E + \lambda_1 p - \lambda_2 = 0 \Rightarrow \lambda_2 = B_E + p && (3) \\ \frac{\partial L_F}{\partial S} &= -2S - p\lambda = 0 \Rightarrow p = -2S && (4) \\ \frac{\partial L_E}{\partial \lambda_1} &= W_E + pS - B_E = 0 \Rightarrow B_E = W_E - 2S^2 && (5) \\ \frac{\partial L_E}{\partial \lambda_2} &= 4 - S \geq 0 && (6) \\ \frac{\partial L_E}{\partial \lambda_3} &= B_E - W_E - pS = 0 && (7) \\ & && (8) \end{aligned}$$

From condition (5) we have two possibilities:

If $S = 4$ then $\lambda_2 > 0$, then condition (4) gives $p = -8$ and condition (5) gives $B_E = W_E - 32$, which is not possible.
If $S < 4$ then $\lambda_2 = 0$ and we can solve for B_E using (3) and (5):

$$B_E = 2S = -2S^2 + W_E$$

$$2S^2 + 2S - W_E = 0 \Rightarrow S = \sqrt{\frac{1}{4} + \frac{1}{2} W_E}$$

and $p = -2S = -2\sqrt{\frac{1}{4} + \frac{1}{2} W_E}$.

5.2

(a) To solve for the P.O. set, we can maximize the sum of Jim and Tammy's utilities subject to their budget constraints. We get $x = 10$ and $y = \frac{25}{6}$ as optimal level of activities.

(b) $I^J = I^T = 500,000$ and no bargaining. Now we maximize each person's individual utility subject to his/her budget constraint, taking the other person's activity as given:

$$\begin{aligned} \max c_J &+ 500 \ln x - 20\bar{y} + \lambda(500,000 - c_J - 40x) \\ \max c_T &+ 500 \ln y - 10\bar{x} + \lambda(500,000 - c_T - 100y) \end{aligned}$$

Solving for the optimal amounts of x and y, we get $x = 12.5$ and $y = 5$.

(d) No x and no y allowed without the other's consent. Then the budget constraints are:

$$\begin{aligned} c_J + (40 + p_x)x + p_y y &\leq 500,000 \\ c_T + p_x x + (100 + p_y)y &\leq 500,000 \end{aligned}$$

where p_x is the price Jim has to pay Tammy for xing and p_y is the price Tammy has to pay Jim for ying. Setting up the individual maximization problems for each of them, and solving simultaneously for Lindahl prices and optimal quantities, we get $p_y = 20$, $p_x = 10$, $x = 10$ and $y = \frac{25}{6}$.

(e) Any amount of x and y allowed The the budget constraints are

$$\begin{aligned} c_J + (40 + p_x)x + p_y(y^{max} - y) &\leq 500,000 \\ c_T + +(100 + p_y)y + p_x(x^{max} - x) &\leq 500,000 \end{aligned}$$

where $y^{max}(x^{max})$ is the maximum amount Tammy(Jim) can y(x).

We get the same FOCs as in (d), since the utility functions are quasilinear in c_J and c_T. i.e., we get the same results for the Lindahl prices and the optimal amounts of xing and ying, no matter how the initial property rights are assigned.

5.3

(a)

$$U(c = 1, l = 1) = 0$$

(b)

$$U(c = \frac{3}{4}, l = \frac{3}{4}) = \frac{3}{16}$$

(c) In this case $c = l$, thus we have to solve

$$\max U = c - c^2$$

and get $c = \frac{1}{2}$.

(d) Suppose we look at some of the cottagers (without loss of generality)who live in a circle as in the following graph:

see graph

Take two cottagers L and M. Their utility is

$$\begin{aligned} U_L &= c - L - l_L L^2 \\ U_M &= c_M - {l_L}^2 \end{aligned}$$

Since cottager LL consumes 1 unit, L's utility is $U_L = c_L - 1$. Thus, in order for him to be better off, c_L must be greater than 1. To make M better off, we have the condition

$$\begin{aligned} U_M &= c_M - {c_L}^2 > 0 \\ c_M &> {c_L}^2 \end{aligned}$$

Since $c_L > 1$, so is c_M. On the other hand, $c_L + c_M \leq 2$, and we get a contradiction. Thus, they cannot both be better off by redistributing consumption.

(e) Take three cottagers L, M and R. From their utility functions we get the following conditions for their consumption:

$$\begin{aligned} U_L &= c_L - 1 \Rightarrow c_L > 1 \\ U_M &= c_M - {c_L}^2 \Rightarrow c_M > 1 \\ U_R &= c_R - {c_M}^2 \Rightarrow c_R > 1 \end{aligned}$$

But $c_L + c_M + c_R > 3$ is not feasible for this group of three people. Thus they cannot be made better off.

(f) It must be that all of the 100 cottagers cooperate in order to be better off, since, as shown in (d) and (e) as long as the circle is not closed, no subgroup can improve upon.

5.4

(a) If $\alpha = \frac{2}{3}$ then

$$U_R = S_R^{\frac{2}{3}} S_J^{\frac{1}{3}}, U_J = S_R^{\frac{1}{3}} S_J^{\frac{2}{3}}$$

If Romeo maximized his own utility, the problem would be

$$\max S_R^{\frac{2}{3}} S_J^{\frac{1}{3}}$$

subject to $S_R + S_J = 24$. From the Cobb-Douglas utility function, we know the demand functions:

$$S_R = \frac{2}{3} \cdot 24 = 16, S_J = \frac{1}{3} \cdot 24 = 8$$

Similarly, if Juliet maximized her own utility, we get $S_J = 16, S_R = 8$.

(b) The Pareto optimal allocations are all combinations of S_R and S_J between their individually optimal consumption points R^* and J^*, i.e. all points such that $8 \leq S_R \leq 16$ and $S_R + S_J = 24$.

(c) see graph

(d) By analogy to (a), if $\alpha = \frac{1}{3}$, we get $S_R = 8$ and $S_J = 16$ for Romeo maximizing his utility, and $S_R = 16$ and $S_J = 8$ when Juliet is maximizing her utility. In other words, Romeo wants Juliet to consume more than himself since he regards her consumption higher than his own, and vice versa.

5.5

(a) Denote the price that Romeo pays per unit of Juliet's consumption by p_{RJ}.

$$p_{RJ} + p_{JJ} = 1$$

Romeo's maximization problem:

$$\max S_R{}^a S_J{}^{1-a} + \lambda(18 - p_{RJ} S_J - p_{RR} S_R)$$

From the Cobb-Douglas utility function, we have the demands

$$S_R = \frac{a}{p_{RR}} \cdot 18, S_J = \frac{1-a}{p_{RJ}} \cdot 18$$

and from Juliet's maximization problem we get

$$S_R = \frac{1-a}{p_{JR}} \cdot 6, S_J = \frac{a}{p_{JJ}} \cdot 6$$

Since $p_{JJ} + p_{RJ} = 1 \Rightarrow p_{JJ} = 1 - p_{RJ}$. Since there can be only one quantity of S_J and one for S_R, we can solve for the prices:

$$\frac{6a}{p_J J} = \frac{18(1-a)}{1 - p_{JJ}}$$

$$
\begin{aligned}
p_{JJ} &= \frac{a}{3-2a} \\
p_{RJ} &= \frac{3(1-a)}{3-2a} \\
S_J &= 6(3-2a)
\end{aligned}
$$

Similarly for S_R:

$$\frac{18a}{p_{RR}} = \frac{6(1-a)}{1-p_{RR}}$$

$$
\begin{aligned}
p_{RR} &= \frac{3a}{2a+1} \\
p_{JR} &= \frac{a+1}{2a+1} \\
S_R &= 6(2a+1)
\end{aligned}
$$

6 Chapter 6

6.1

$$U_G = 48x - x^2 + z_G U_H = 60y - y^2 - xy + z_H$$

(a) To find the Pareto optimal allocations

$$\max U_G + U_H$$

$$\begin{aligned}\frac{\partial L}{\partial x} &= 48 - 2x - y = 0 \\ \frac{\partial L}{\partial y} &= 60 - 2y - x = 0\end{aligned}$$

We get $x^E = 12$ and $y^E = 24$

(b) For the non-cooperative Nash equilibrium, we maximize individual utility, taking the other person's input as given. FOC:

$$\begin{aligned}48 - 2x &= 0 \Rightarrow x^N = 24 \\ 60 - 2y &= x \Rightarrow y^N = 18\end{aligned}$$

(c) With a legal liability

$$\begin{aligned}U_G &= 48x - x^2 + W_G - xy \\ U_H &= 60y - y^2 + W_H\end{aligned}$$

FOC:

$$48 - 2x = y$$

$$60 - 2y = 0$$

Then $x^L = 9$ and $y^L = 30$.

6.2

(a)

$$\begin{aligned}U_G &= 48x - x^2 + z_G - tx \\ U_H &= 60y - y^2 - xy + z_H\end{aligned}$$

From the FOC $48 - 2x = t$ Since $x^E = 12$, $t = 24$.

(b)

$$\begin{aligned}U_G &= 48x - x^2 + z_G - tx \\ U_H &= 60y - y^2 - xy + z_H + tx\end{aligned}$$

We get the same FOC, and $t = 24$.

(c)

$$\begin{aligned} U_G &= 48x - x^2 + z_G - \frac{t}{2}x \\ U_H &= 60y - y^2 - xy + z_H + \frac{t}{2}x \end{aligned}$$

FOC:

$$48 - 2x^E - \frac{t}{2} = 0$$
$$t = 48$$

6.3

$$x^G(p) = \arg\max(U_G + px)$$

FOC:

$$48 - 2x^G + p = 0$$
$$x^G = 24 + \frac{p}{2}$$

$$x^H(p) = \arg\max(U_H - px)$$

FOC:

$$60 - 2y - x = 0$$
$$y = -p$$
$$x^H = 60 + 2p$$

set $x^G = x^H$:

$$24 + \frac{p}{2} = 60 + 2p$$

Then $p = -24$, $x^E = 12$ and $y^E = 24$.

6.4
The threat point is $x^N = 24$, $y^N = 18$

$$\max(U_G - \bar{U_G})(U_H - \bar{U_H}) - \lambda H(U_G, U_H)$$

where H is the utility possibility frontier. From the FOC we get

$$\frac{\partial H}{\partial U_G} / \frac{\partial H}{\partial U_H} = \frac{U_H - \bar{U}_H}{U_G - \bar{U}_G}$$

and

$$\frac{\partial H}{\partial U_G} dU_G + \frac{\partial H}{\partial U_H} dU_H = 0$$

Then we have an expression for the MRT:

$$\frac{dU_H}{dU_G} = \frac{U_H - \bar{U}_H}{U_G - \bar{U}_G}$$

Since by definition the bargaining outcome is efficient, only monetary transfers matter:

$$\frac{dU_H}{dU_G} = -1$$

since $\frac{\partial U_G}{\partial z_G} = 1$. Therefore, $U_H - \bar{U}_H = U_G - \bar{U}_G$

6.5

(a) $x^E = 12$, $y^E = 24$, $x^N = 24$, $y^N = 18$

$$\begin{aligned}
U_G - \bar{U}_G &= 48 \cdot 12 - 12^2 + z_G - (48 \cdot 24 - 24^2 + W_G) \\
U_H - \bar{U}_H &= 60 \cdot 24 - 24^2 - 12 \cdot 24 + z_H - (60 \cdot 18 - 18^2 - 24 \cdot 18 + W_H)
\end{aligned}$$

Set $U_G - \bar{U}_G = U_H - \bar{U}_H$

$$z_G = 198 + W_G$$

$$z_H = -198 + W_H$$

(b) $x^L = 9$, $y^L = 30$, $x^E = 12$, $y^E = 24$

$$\begin{aligned}
U_G - \bar{U}_G &= 48 \cdot 12 - 12^2 + z_G - (48 \cdot 9 - 9^2 + W_G) \\
U_H - \bar{U}_H &= 60 \cdot 24 - 24^2 - 12 \cdot 24 + z_H - (60 \cdot 30 - 30^2 - 9 \cdot 30 + W_H)
\end{aligned}$$

Solving for z_G and z_H, we get

$$\begin{aligned}
z_G &= -\frac{135}{2} + W_G \\
z_H &= W_H + \frac{135}{2}
\end{aligned}$$

6.6

(a) $t = 24$

$$\begin{aligned}
U_G &= 48x - x^2 + z_G - 24x \\
\bar{U}_G &= 48 \cdot 12 - 12^2 + W_G - 24 \cdot 12 \\
U_H &= 60y - y^2 - xy + z_H \\
\bar{U}_H &= 60 \cdot 24 - 24^2 - 12 \cdot 24 + W_H
\end{aligned}$$

Set $U_G - \bar{U}_G = U_H - \bar{U}_H$. Solving for z_H and z_G gives $z_H = W_H$ and $z_G = W_G$.

(b)

$$\begin{aligned} U_G &= 48x - x^2 + z_G - 48 \cdot 6 \\ U_H &= 60y - y^2 - xy + z_H - 48 \cdot 6 \end{aligned}$$

$z_H = w_H$ and $z_G = w_G$.

6.7

If Hazel proposes a contract that is not accepted by George, Hazel's payoff will be $60y^N - (y^N)^2 + W_H - x^N Y^N$.
If George accepts, his payoff will be

$$U_G = U_G(x^*) + z^*$$

and Hazel's payoff will be

$$60y^* - (y^*)^2 + W_H - z^* - x^* y^*$$

Then Hazel's maximization problem is

$$\max 60y^* - (y^*)^2 + W_H - z^* - x^* y^*) + (48x^* - (x^*)^2 + W_G + z^*)$$

Then $x^* = 12$ and $z^* = U_G(x^N) - U_G(x^*) = 144$.

6.8

(a) Strategy 1: Choose x* and z* so that George is sure to accept her offer. Assume $A = 60$

$$\max 60y^* - (y^*)^2 + W_H - z^* - x^* y^* + 60x^* - (x^*)^2 + W_G + z^*$$

Taking partial derivatives with respect to x* and y*, we get x*=20.
On the other hand, to find the non-cooperative Nash equilibrium, we have to maximize $U_G(x^N) = 60x^N - (x^N)^2 + W_G$, and we get $x^N = 30$. Then z* will be

$$z^* = U_G(x^N) - U_G(x^*) = 100$$

Then Hazel's best response to offering George a lump sum payment of z*=100 if his amount of Xing is only x*=20 will be

$$\max 60y^* - (y^*)^2 + W_H - 100 - 20y^*$$

And the optimal amount of Ying is then y*=20. Thus, if George accepts her offer, Hazel's utility will be

$$U_H = 300 + W_H$$

Strategy 2:Choose x* and z* so that George accepts if and only if $A = 36$.

$$\max 60y^* - (y^*)^2 + W_H - z^* - x^*y^* + 36x^* - (x^*)^2 + W_G + z^*$$

From the FOC we get x*=4. For the non-cooperative Nash equilibrium, we get $x^N = 18$. Then z* is given by

$$U_G(x^N) - U_G(x*) = 196$$

Then Hazel's best response to offering George a lump sum payment of z*=196 if his amount of Xing is only x*=4 will be

$$\max 60y^* - (y^*)^2 + W_H - 196 - 4y^*$$

and the optimal amount of Ying is then y*=28. Thus, if George accepts her offer, Hazel's utility will be

$$U_H = 588 + W_H$$

If George doesn't accept, both play their Nash strategies. Hazel's best reply to George playing Nash is

$$\max 60Y^N - (y^N)^2 + W_H - 30y^N$$

and we get $y^N = 15$, in this case Hazel's utility will be

$$U_H = 225 + W_H$$

Comparing the two strategies

$$.5(588 + w_H) + .5(225 + W_H) = 406.5 + W_H \geq 300 + W_H$$

Therefore Hazel should play strategy 2.

(b) With prob=.5, George is of type $A = 60$. Given that Hazel chooses strategy 1 and offers him z*=100, his dominant strategy is to accept and thus play the efficient amount of x. However, if he is of type $A = 36$, he won't play the efficient amount of x.

Given that Hazel chooses strategy 2, George will only accept and play the efficient amount of x, if he is of type $A = 36$, which occurs with prob=.5.

7 Chapter 7

7.1

(a) Samuelson condition for efficient D_i:

$$\begin{aligned}
\sum_i MRS_i^{D_i M_j} &= p_F = 0 \\
\frac{U_i^{D_i}}{U_i^{M_i}} + \sum_{j\neq i} \frac{U_j^{D_i}}{U_j^{M_j}} &= A_i - D_i - \frac{D}{H} - \frac{D_i}{H} - \sum_{j\neq i} \frac{D_j}{H} = 0 \\
A_i - D_I &= 2\frac{D}{H} \Rightarrow D_i = A_i - 2\frac{D}{H}
\end{aligned}$$

The condition for the efficient amount of H is:

$$\sum_j MRS_j^{HM_j} = \sum_j \frac{D_j D}{H^2} = \frac{D^2}{H^2} = p_H$$

From $H = \frac{D}{\sqrt{p_H}}$ we can go back to the efiicient amount of driving:

$$\begin{aligned}
D_i &= A_i - \frac{2D}{D/\sqrt{p_H}} = A_i - 2\sqrt{p_H} \\
D &= \sum A_i - 2n\sqrt{p_H} \\
H &= \frac{\sum A_i}{\sqrt{p_H}} - 2n
\end{aligned}$$

If $p_H = 1$ then $H = D$.

(b) Take D_{-i} as given. From the condition $MRS_i^{D_i M_i} = 0$ we get

$$A_i - D_i - \frac{D}{H} - \frac{D_i}{H} = 0 \Rightarrow D_i = \frac{HA_i - D}{H+1}$$

Then the total amount of driving

$$D = \sum D_i = \frac{H\sum A_i}{H+1} - \frac{nD}{H+1} = \frac{H\sum A_i}{H+1+n}$$

and the individual amount

$$D_i = \frac{HA_i}{H+1} - \frac{H\sum A_i}{(H+1)(H+1+n)}.$$

(c)

$$T = -p_H \frac{C_D}{C_H}$$

$$
\begin{aligned}
&= p_H \frac{1/H}{D/H^2} = p_H \frac{H}{D} \\
&= p_H \frac{\sum A_i/\sqrt{p_H} - 2n}{\sum A_i - 2n\sqrt{p_H}} = \sqrt{p_H}
\end{aligned}
$$

For $p_H = 1$, T is 1. To find out, how much driving each i will do, we have to set the MRS between driving and big macs equal to the respective price ratio:

$$A_i - D_i - \frac{D}{H} - \frac{D_i}{H} = \sqrt{p_H}$$

If $p_H = 1$, $D = H$ and

$$
\begin{aligned}
A_i - D_i - 2 - \frac{D_i}{D} &= 0 \\
A_i - D_i &= 2 + \frac{1}{n} \\
D_i &= A_i - 2 - \frac{1}{n}
\end{aligned}
$$

If the population n is large, $D_i = A_i - 2$.

7.2

(a)

$$20 + \frac{N_1}{100} = 45 \Rightarrow N_1 = 2500$$

Then the total number of minutes spent will be 270,000.

(b)

$$\min(20 + \frac{N_1}{100})N_1 + 45(6000 - N_1)$$

$$20 + \frac{N_1}{100} + \frac{N_1}{100} - 45 = 0$$

Solving for the optimal N_1, we get $N_1^* = 1250$, and it would take 32.5 min to commute.

(c)

$$\left(45 - \left(20 + \frac{N_1^*}{100}\right)\right) W = T$$

Then $T = 12.5W$.

7.3

The "demand curve" is

$$\begin{aligned} p &= \left(45 - \left(20 + \frac{N}{100}\right)\right) \\ N_1 &= 25 - \frac{100p}{W} = 2500 - \frac{100p}{W} \end{aligned}$$

From maximizing total revenue, we get

$$\max TR = pN_1 = 2500p - \frac{100p^2}{W}$$

$p = 12.5W$ and $N_1 = 1250$, which is the same number of commuters as in (b).

7.4

The government maximizes the value of the time saved minus the cost for road building

$$\max \left(45 - \left(20 + \frac{N_1}{100 + H}\right)\right) WN_1 - p$$

or, denoting be $H' = 100 + H$ the new size of the road:

$$\min \left(20 + \frac{N_1}{H'}\right) N_1 W + p(H' - 100)$$

The first order condition gives $\frac{N_1^2}{H'^2}W = p$ or $\frac{N_1}{H'} = \frac{\sqrt{p}}{\sqrt{W}}$. In order to save time, all 6,000 people must be able to use the road. Then

$$H = H' - 100 = \frac{N_1\sqrt{W}}{\sqrt{p}} = \frac{6000\sqrt{W}}{\sqrt{p}} - 100$$

The time it will take them to make the trip given the road is optimally widened is

$$20 + \frac{6000}{100 + 6000\sqrt{W}/\sqrt{p} - 100} = 20 + \frac{\sqrt{p}}{\sqrt{W}}$$

The condition for widening the road is that this will save time:

$$\begin{aligned} N_1\left(20 + \frac{\sqrt{p}}{\sqrt{W}}\right) W + p\left(N_1\frac{\sqrt{W}}{\sqrt{p}} - 100\right) &< 45N_1W \\ N_1(20W + \sqrt{pW}) + \sqrt{pW}N_1 - 100p &< 45N_1W \\ 2N_1\sqrt{pW} - 100p &< 25N_1W \\ 2\sqrt{pW} - \frac{p}{60} &< 25W \end{aligned}$$

7.5

The government wants to maximize the "profit" from providing more highway, which is the value of time saved minus the cost of adding more highway:

$$
\begin{aligned}
\max L &= 6000 \cdot 45 - \left[\left(20 + \frac{N_1}{100 = H}\right) N_1 + (600 - N_1) \cdot 45\right] W - pH \\
\frac{\partial L}{\partial H} &= \frac{N_1^2}{(100+H)^2} W - p = 0 \\
\frac{\partial L}{\partial N_1} &= \frac{-2N_1}{100+H} + 45W = 0
\end{aligned}
$$

Then we get

$$N_1 = \frac{45}{2}(100 + H)$$

$$\left(\frac{45}{2}\right)^2 \frac{W}{p} - 100 = H$$

The optimal toll is

$$W\left[25 - \frac{45}{2}\right] = 2.5W$$

and the amount of highway expenditure is

$$pH = \left(\frac{45}{2}\right)^2 W - 100p$$

8 Chapter on Voting

8.1

The utility function is:

$$U_i(x_i, y_1, y_2) = x_i + a_1 y_1 + a_2 y_2 - c_1 y_1^2 - c_2 y_2^2$$

and the budget constraint is:

$$W_i = x_i + t_{i1} y_1 + t_{i2} y_2$$

Then the unconstrained problem is

$$\max U_i(y_1, y_2) = W_i - t_{i1} y_1 - t_{i2} y_2 - a_1 y_1 - a_2 y_2 - c_1 y_1^2 - c_2 y_2^2$$

$$\begin{aligned}
\frac{\partial U}{\partial y_1} &= -t_{i1} + a_1 - 2c_1 y_1 = 0 \\
\frac{\partial U}{\partial y_2} &= -t_{i2} + a_2 - 2c_2 y_2 = 0 \\
y_1 &= \frac{a_1 - t_{i1}}{2c_1} \\
y_2 &= \frac{a_2 - t_{i2}}{2c_2}
\end{aligned}$$

Indifference curves:

$$\bar{U} - W_i = -y_1(t_{i1} - a_1) - y_2(t_{i2} - a_2) - c_1 y_1^2 - c_2 y 2^2$$

$$\underbrace{\bar{U} - W_i - \frac{t_{i1} - a_1}{4c_1} - \frac{t_{i2} - a_2}{4c_2}}_{D} = \underbrace{-c_1 \left(y_1 + \frac{t_{i1} - a_1}{2c_1} \right)^2}_{A(y_1 - \bar{y}_1)^2} \underbrace{- c_2 \left(y_2 + \frac{t_{i2} - a_2}{2c_2} \right)^2}_{C(y_2 - \bar{y}_2)^2}$$

If A=C (if $c_1 = c_2$), the indifference curves are circles.

8.2

$$\begin{aligned}
U_i(y_1, y_2) &= W_i - \frac{1}{10} y_1 - \frac{1}{10} y_2 + \frac{51}{10} y_1 + \frac{81}{10} y_2 + y_1 y_2 - y_1^2 - 2y_2^2 \\
&= W_i + 5y_1 + 8y_2 + y_1 y_2 - y_1^2 - 2y_2^2
\end{aligned}$$

To get to the desired form

$$A(y_1 - \bar{y}_1)^2 + B(y_1 - \bar{y}_1)(y_2 - \bar{y}_2) + C(y_2 - \bar{y}_2)^2 = D$$

of an ellipse, we have to complete the square:

$$2\bar{y}_1 - \bar{y}_2 = 5$$

$$4\bar{y}_1 - \bar{y}_2 = 8$$

And we get the following values for $\bar{y}_1$ and $\bar{y}_2$:

$$\bar{y}_1 = 4, \qquad \bar{y}_2 = 3$$

Then the indifference curves can be written as

$$\bar{U} - W_i = -(y_1 - 4)^2 - 2(y_2 - 3)^2 + (y_1 - 4)(y_2 - 3) + 22$$

To get the consumer's preferred combination of y_1 and y_2, take the FOC of $U_i(y_1, y_2)$ and find $y_1 = 7$ and $y_2 = 9$.

see graph

8.3.1

From the utility function we get the preferred amounts of y_{i1} and y_{i2}:

$$\frac{\partial U_i}{\partial y_{i1}} = 0 \quad \rightarrow y_{i1} = \frac{a_{i1}}{2}$$

$$\frac{\partial U_i}{\partial y_{i2}} = 0 \quad \rightarrow y_{i2} = \frac{a_{i2}}{2}$$

which are each independent of the amount of the other public good supplied.

If a_{i1} is uniformly distributed on [0,10] and a_{i2} on [0,20] then everybody's preferred amount will lie in the interval [0,5] for a_{i1} and [0.10] for a_{i2}. Take the median $y_1 = \frac{\hat{a}_{i1}}{2}$ and $y_2 = \frac{\hat{a}_{i2}}{2}$. Since the parameters are uniformly distributed, we know that median = mean. Then the outcome $y_1 = 2.5$, $y_2 = 5$ is efficient.

8.3.2

Take the median $\hat{y}_1$ and $\hat{y}_2$ of y_1 and y_2. Suppose majority voting for a change in y_1, given $\hat{y}_2$. Since voting is one issue at a time, we can look at one dimension at a time only. If the issue is to vote for an increase in y_1, everybody on one side of $\hat{a}_{i1}$ will vote against, including the median voter. So there cannot be a majority for an increase in y_1. Similarly for a decrease in y_1. Thus $\hat{y}_1$ cannot be defeated. Similarly for $\hat{y}_2$, taking y_1 as given.

8.4

Let M be the median of the y_{i1} and y_{i2}. Then by increasing y_2 and moving to V, both person 1 and 3 are made better off.

see graph

8.5 The graph of 8.4 also holds for this problem. V is an "equilibrium" in issue by issue voting.

NOTE: I COULD FIND ONLY ONE "EQUILIBRIUM".