Samuelson conditions with benevolence

There are n consumers, one consumer and one public good. Each consumer i has a *private utility function* $v_i(x_i, y)$ that is continuously differentiable and quasi-concave.

Each consumer has a non-malevolent social utility function,

$$U_i\left(v_1(x_1, y), \ldots, v_n(x_n, y)\right)$$

that is non-decreasing in each of the v_i's.

Consumer i's *private marginal rate of substitution* between the public good and the private good is the ratio of the partial derivative of the function $v_i(x_i, y)$ with respect to y to its partial derivative with respect to x_i. While v_i does not depend on x_j for $j \neq i$. We define i's social *social marginal rate of substitution* between t the public good and private good as the ratio of the partial derivative of U_i with respect to y to its partial derivative with respect to x_i. Thus i's social marginal rate of substitution differs from her private marginal rate of substitution when i cares about the private utility of others who care about the amount of y.

We know that given our assumptions and if consumers are selfish, a necessary and sufficient condition for Pareto optimality is the Samuelson condition that the sum of the marginal rates of substitution equals the marginal cost of the public good. The counterpart of this result for the economy discussed here is that efficiency requires the sum of the *private* marginal rates of substitution (not the sum of the *social* marginal rates of substitution equals marginal cost.

We can show this, using Lagrangian multipliers in much the way Samuelson showed his condition for selfish consumers. A Pareto optimal allocation must maximize the utility of one consumer, say Consumer 1, subject to the constraints that each other consumer's utility achieves some fixed value and also subject to the resource constraint needed for feasibility. Here, lets show the proof for the case of 2 consumers. This generalizes easily to n consumers.

The Lagrangean for this problem is

$$U_1\left(v_1(x_1, y), v_2(x_n, y)\right) + \lambda\left(\bar{U}_2 - U_2(x_2, y)\right) + \mu\left(W - x_1 - x_2 - C(y)\right)$$

where W is the total initial wealth and $C(y)$ is the cost of producing y units of public good. Setting derivatives of the Lagrangian with respect to x_1 and x_2 equal to zero, we have

$$\frac{\partial v_1}{\partial x_1}\left(\frac{\partial U_1}{\partial v_1} + \lambda \frac{\partial U_2}{\partial v_1}\right) = \mu \tag{1}$$

and

$$\frac{\partial v_2}{\partial x_2}\left(\frac{\partial U_1}{\partial v_2} + \lambda \frac{\partial U_2}{\partial v_2}\right) = \mu \tag{2}$$

Setting the partial derivative of the Lagrangian with respect to y equal to zero gives us

$$\frac{\partial v_1}{\partial y}\left(\frac{\partial U_1}{\partial v_1}+\lambda\frac{\partial U_2}{\partial v_1}\right)+\frac{\partial v_2}{\partial y}\left(\frac{\partial U_1}{\partial v_2}+\lambda\frac{\partial U_2}{\partial v_2}\right)=\mu C'(y) \tag{3}$$

Substituting from Equations 1 and 2 into equation 3, we have

$$\frac{\mu\frac{\partial v_1}{\partial y}}{\frac{\partial v_1}{\partial x_1}}+\frac{\frac{\mu\partial v_2}{\partial y}}{\frac{\partial v_2}{\partial x_2}}=\mu C'(y) \tag{4}$$

Dividing both sides by $\mu > 0$, we have

$$\frac{\frac{\partial v_1}{\partial y}}{\frac{\partial v_1}{\partial x_1}}+\frac{\frac{\partial v_2}{\partial y}}{\frac{\partial v_2}{\partial x_2}}=C'(y) \tag{5}$$

which is simply the Samuelson condition in private utilities.