Chapter 3: Common Families of Distributions

*This only convers distributions which are not included in the Math 4200 notes.

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3.2 Discrete Distributions

- **Discrete Uniform Distribution (Pg 86):** A RV, X, has a discrete uniform $(1, N)$ distribution if:

$$P(X = x|N) = \frac{1}{N}, \quad x = 1, 2, ..., N$$

- **Zero Inflated Poisson (Slide 11 - Common Dist.):**
  - Allows for frequent zero valued observations.

$$\begin{aligned} f(x|x = 0) &= \pi + (1 - \pi)e^{-\lambda} \\ f(x|x > 0) &= (1 - \pi)\frac{e^{-\lambda}\lambda^x}{x!} \\ E(X) &= (1 - \pi)\lambda \\ Var(X) &= (1 + \pi\lambda)(1 - \pi)\lambda \end{aligned}$$

3.3 Continuous Distributions

- **Gamma Distribution (Pg 99):**
  - α is shape parameter and β is scale parameter
- **Chi Squared (Pg 101):**
  - Square of Normal is χ^2
- **Exponential (Pg 101):**
  - Memoryless
  - Continuous analogue of geometric distribution in discrete case.
  - Example (Slide 25): Suppose we are to receive x dollars in t years discounted at r. So $PV = xe^{-rt}$. Payment date is uncertain with exponential distribution.

$$\begin{aligned} \implies EPV &= \int_0^\infty xe^{-rt} f(t)dt \\ &= \int_0^\infty xe^{-rt}\lambda e^{-\lambda t} dt \\ &= x\frac{\lambda}{\lambda + r}\int_0^\infty (\lambda + r)e^{-(\lambda+r)t} dt \qquad \text{The integrand is pdf of exponential} \\ \implies EPV &= x\frac{\lambda}{\lambda + r} \times 1 \end{aligned}$$

- **Weibull (Pg 102):** If $X \sim exp(\beta)$ then $Y = X^{1/\gamma}$ has Weibull(γ, β) distribution.

- Used for analysis of failure time data and modeling hazard functions.

- **Normal (Pg 102):** Symmetric around and has maximum at μ. (Theres nothing normal about it...)
  - If $X \sim binom(n, p)$, then X can be approximated by normal, $n(np, np(1-p))$ (pg 104).
  - If $X_1, .., X_n$ are iid Bernoulli(p), then $\bar{X} \sim n(p, p(1-p)/n)$

- **Standard Normal (Pg 102):** If $X \sim n(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$ is distributed $n(0, 1)$.

$$\begin{aligned}
\phi(x) &\equiv f_N(x|0,1) \\
\Phi(x) &\equiv F_N(x|0,1) \\
f(x|\mu,\sigma^2) &= \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}) \\
F(x|\mu,\sigma^2) &= \frac{1}{\sigma}\Phi(\frac{x-\mu}{\sigma})
\end{aligned}$$

- **Beta Distribution (Pg 106):**
- **Cauchy Distribution (Pg 107):**
- **Double Exponential Distribution (Pg 109):**
- **Log Normal Distribution (Pg 109):** If $log(X) \sim n(\mu, \sigma^2)$, then $X \sim \ln n(\mu, \sigma^2)$
  - Income are skewed right, modeling with log normal allows for use of normal-theory on log(income).
  - Example (Slide 31): Cobb-Douglas with error: $Y = AK^{\beta}L^{1-\beta}\mathcal{E}$. If $\mathcal{E} \sim \ln n(\cdot)$ then $\log Y = \log A + \beta \log K + (1-\beta)\log L + \log \mathcal{E}$

- **Logistic Distribution**: (See table of Common Distributions or Slide 36). Looks similar to normal but has a closed form CDF.

- **Standard Logistic Distribution (Slide 37)**: An application of Standard Logistic is if we think something happens if $x < x_0$ and otherwise it doesn't happen. The odds of it happening are:

$$\frac{F(x_0)}{1-F(x_0)} = e^{x_0}, \quad \log\frac{F(x_0)}{1-F(x_0)} = x_0, \quad x_0 \text{ gives log odds}$$

- **Extreme Value (Gumbel) Distribution (Slide 38):**

- **Exponential Families (Pg 111):** A family of pdfs is called an exponential family if it can be written as

$$f(x|\theta) = h(x)c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(\theta)\right)$$

  Includes: Normal, gamma, beta, binomial, poisson.

- **Location and Scale Families (Pg. 116):** Let $f(x)$ be any pdf and let $\mu, \sigma > 0$ be any given constants. Then

$$\begin{aligned}
g(x|\mu,\sigma) &= \frac{1}{\sigma}f\left(\frac{x-\mu}{\sigma}\right) \quad \text{is a pdf} \\
G(x|\mu,\sigma) &= F\left(\frac{x-\mu}{\sigma}\right) \quad \text{is a cdf}
\end{aligned}$$

  - The family of pdfs $f(x-\mu)$ is called the location family with standard pdf $f(x)$ with μ location parameter.

- The family of pdfs $1/\sigma f(x/\sigma)$ is called the scale family with standard pdf $f(x)$ with σ scale parameter.
- **Theorem 3.5.7 (Pg 121):** Let Z be a RV with pdf $f(z)$. Suppose EZ and $VarZ$ exist. If X is a RV with pdf $\frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$, then:

  $$EX = \sigma EZ + \mu \quad \text{and} \quad VarX = \sigma^2 VarZ$$

  In particular, if $EZ = 0$ and $VarZ = 1$, then $EX = \mu$ and $VarX = \sigma^2$.

- **Inequalities and Identities (Pg. 121):**
  - **Theorem 3.6.1 (Chebychev's Inequality, Pg 122):** Let X be a RV and let $g(x)$ be a nonnegative function. Then, for any $r > 0$:

    $$P(g(X) \geq r) \leq \frac{Eg(X)}{r}$$

  - If X is Poisson(λ), then $P(X = x+1) = \frac{\lambda}{\lambda+1}P(X = x)$
  - **Stein's Lemma (Pg. 124)**: Useful for calculating higher order moments. (pg 125)
  - **Theorem 3.6.7 (Pg. 125)**: Let χ_p^2 be chi squared with p degrees of freedom. For any function $h(x)$,

    $$Eh(\chi_p^2) = pE\left(\frac{h(\chi_{p+2}^2)}{\chi_{p+2}^2}\right)$$

    $\implies E\chi_p^2 = p$, $E(\chi_p^2)^2 = p(p+2)$, $Var\chi_p^2 = p(p+2) - p^2 = 2p$. Results from using the theorem's formula.
  - **Theorem 3.6.8 (Hwang) (pg 126) -** Useful for calculating higher order Poisson moments.