Econ 241A Probability, Statistics and Econometrics
Fall 2017

Problem Set 2

1. Consider these three events:

   (a) $A = \{X = 2\}$
   (b) $B = \{X = 4\}$
   (c) $C = \{X = 5\}$

   Consider also these two discrete probability distributions:

   (a) Binomial with parameters $n = 6$, $p = 0.4$
   (b) Poisson with parameter $\lambda = 2$

   For each distribution calculate the probability of each event.

   For the $binomial(0.4, 6)$: $P(A) = .3110$; $P(B) = .1382$, $P(C) = .0369$
   For the $Possion(2)$: $P(A) = .2707$; $P(B) = .0902$, $P(C) = .0361$

2. Consider these three events:

   (a) $A = \{-2 \leq X \leq 2\}$
   (b) $B = \{0 \leq X \leq 3\}$
   (c) $C = \{1 < X < 6\}$

   For each of the following distributions calculate the probability of each event:

   (a) Exponential with parameter $\lambda = 3$, i.e., $f(x) = \lambda e^{-\lambda x}$, $0 < x < \infty$
   (b) Standard normal
   (c) Standard logistic ($\mu = 0, \sigma = 1$)

   For the $exp(3)$: $P(A) = .9975$; $P(B) = .9999$, $P(C) = .0498$
   For the $n(0, 1)$: $P(A) = .9545$; $P(B) = .4987$, $P(C) = .1587$
   For the Standard logistic ($\mu = 0, \sigma = 1$): $P(A) = .7616$; $P(B) = .4526$, $P(C) = .2664$

3. Suppose X has the exponential distribution with parameter $\lambda = 2$. Let $Z = \exp(X)$. Find $\mathbb{E}(Z)$, $\mathbb{E}(Z^2)$ and $\text{Var}(Z)$.

   $E[Z] = 2$. $E[Z^2]$ and $Var(Z) = 2$ do not exist.

4. A random variable X has a Weibull (extreme value) distribution if

   $$F_X(x|\alpha) = e^{-e^{-(x+\alpha)}}$$

   What is the cdf of $X - \nu$, where $\nu \in \mathbb{R}$ is a constant?
   Define $g(X) = X + \nu$. The support of the Weibull distribution is $[0, \infty)$, so the support of this transformation is $[\nu, \infty)$. Because we want the cdf, recall Theorem 2.1.3:

   $$F_Y(y) = F_X(g^{-1}(y))$$

Here $g(X) = X + \nu \Rightarrow g^{-1}(Y) = Y - \nu$. Thus

$$F_Y(y) = F_X(Y - \nu) = e^{-e^{-((y-\nu)+\alpha)}} = e^{-e^{-(y+(\alpha-\nu))}}$$

Observe that Y has a Weibull distribution as well, though with parameter $\alpha - \nu$.

In addition, solve the following problems from Casella and Berger: 2.1 (b), 2.17, 2.23 and 3.4.

2.1 In each of the following find the pdf of Y. Show that the pdf integrates to 1.

(b) $Y = 4X + 3$, $f_X(x) = 7e^{-7x}$, $0 < x < \infty$.
Here $Y = g(X) = 4X + 3$, which is monotonic (thus we will be able to apply Theorem 2.1.8 directly). The support of Y is $[3, \infty)$ (because the support of X is $[0, \infty)$ and $g(0) = 3$ and $g(\infty) = \infty$. Note:

$$g^{-1}(y) = \frac{y-3}{4}, \quad \frac{d}{dy}g^{-1}(y) = \frac{1}{4}$$

Applying Theorem 2.1.8, the pdf is:

$$f_Y(y) = 7e^{-\frac{7}{4}(y-3)} \cdot \frac{1}{4} = \begin{cases} \frac{7}{4}e^{-\frac{7}{4}(y-3)} & \text{if } y \in [3, \infty) \\ & 0 \text{ else} \end{cases}$$

Because $\int_3^\infty \frac{7}{4}e^{-\frac{7}{4}(y-3)}dy = e^{-\frac{7}{4}(y-3)}|_{y=3}^\infty = 1$, we've confirmed that this is a valid distribution.

2.17 Find the median of the following distributions

(a) $f(x) = 3x^2$, $x =\in (0, 1)$
$F(x) = x^3$; $F(median) = 0.5$, then $median = 0.5^{\frac{1}{3}}$.

(b) $f(x) = \frac{1}{\pi(1+x^2)}$, $x =\in \mathbb{R}$

$$F_X(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\, dt$$
$$F_X(x) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}$$

We know that $\tan^{-1}(0) = 0$, therefore $F_X(0) = \frac{1}{2}$ or $median(X) = 0$.
This is the pdf of a standard Cauchy random variable. Also note that the pdf is symmetric around zero (so zero is THE median as long the pdf integrates to 1)[1].

2.23 Let X have the pdf $f(x) = \frac{1}{2}(1 + x)$, $x \in (-1, 1)$.

(a) Find the pdf of $Y = X^2$
Let's find first the cdf of X.

$$F_X(x) = \frac{1}{2}x + \frac{1}{4}x^2, \quad x \in (-1, 1)$$
$$F_Y(y) = P[X^2 < y] = P[-\sqrt{y} < X < \sqrt{y}] = F(\sqrt{y}) - F(-\sqrt{y})$$
$$F_Y(y) = P[X^2 < y] = \frac{1}{2}(\sqrt{y} - (-\sqrt{y})) + \frac{1}{4}(y - y) = \sqrt{y}, \quad y \in [0, 1)$$

[1] Can a continuous random variable with a continuous pdf have multiple medians?

Taking derivates of the cdf to obtain the pdf

$$f_Y(y) = \frac{1}{2\sqrt{y}} \quad \text{for } y \in [0,1)$$

(b) Find $E[Y]$ and $Var(Y)$.

$$E[Y] = \int y \frac{1}{2\sqrt{y}} dy = \left[\frac{1}{3} y^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$$
$$E[Y^2] = \int y^2 \frac{1}{2\sqrt{y}} dy = \left[\frac{1}{5} y^{\frac{5}{2}}\right]_0^1 = \frac{1}{5}$$
$$Var(Y) = E[Y^2] - E[Y]^2 = \frac{4}{45}$$

3.4 A man with n keys wants to open his door and keys at random. Exactly one key will open the door. Find the mean number of trials if:

(a) unsuccessful keys are not eliminated from further trials

That is, this is sampling with replacement. Let X be the number of trials before opening the door. As in the baby problem, X will be geometric with probability of success $p = \frac{1}{n}$. Using the results from problem 2.20, we know $\mathbb{E}(X) = \frac{1}{p} = n$.

(b) unsuccessful keys are eliminated

That is, this is sampling without replacement. Let X be the number of trials before opening the door. Now X has a discrete uniform distribution: $f_X(x) = P(X = x) = \frac{1}{n}$.[2] This is just the solution from 2.24b above, $\mathbb{E}(X) = \frac{n+1}{2}$.

[2] To be truly formal, we would say that $P(X = 1) = \frac{1}{n}, P(X = 2) = (1 - P(X = 1)) \cdot P(X = 2 | X \neq 1) = (1 - \frac{1}{n})\frac{1}{n-1} = \frac{1}{n}$, etc.