4 Chapter 4

4.1

(a) Samuelson condition:

$$\begin{aligned}\sum_i MRS_{YX_i} &= MRT \\ MRS_{YX_i} &= \frac{\alpha X_i}{(1-\alpha)Y} \\ \sum_i MRS_{YX_i} &= \frac{\alpha \sum X_i}{(1-\alpha)Y}\end{aligned}$$

from the Cobb-Douglas utility function, we have $X_i = (1-\alpha)W_i$

$$\sum MRS_{YX_i} = \frac{\alpha \sum W_i}{Y}$$

then using the Samuelson condition,

$$\frac{\alpha \sum W_i}{Y} = p$$

and thus

$$Y = \frac{\alpha \sum W_i}{p}$$

(b) The Lindahl prices are

$$p_i^Y = \frac{\alpha W_i}{Y} = \frac{W_i}{\sum W_i} p$$

4.2

From the Cobb-Douglas utility function, demand for X_i and Y_i are

$$\begin{aligned}X_i &= (1-\alpha)W_i \\ Y_i &= \frac{\alpha W_i}{p_i^Y}\end{aligned}$$

The global feasibility condition is:

$$\sum X_i + pY = \sum W_i$$

and since Y is a public good, $Y_1 = \cdots = Y_N = Y$. Solving for Y, we get

$$Y = \frac{\sum W_i - \sum X_i}{p} = \frac{\alpha \sum W_i}{p}$$

Solving for Lindahl prices,

$$
\begin{aligned}
p_i^Y &= \frac{\alpha W_i}{Y} \\
&= \frac{\alpha W_i}{\sum Wi - \sum X_i} p \\
&= \frac{W_i}{\sum W_i} p
\end{aligned}
$$

4.3

(a) Denoting the total number of α's by a, the number of β's by b and the number of γ's by c, we have the following individual budget constraint:

$$
\begin{aligned}
W_\alpha &= X_{i\alpha} + p_{i\alpha}^Y Y \quad \text{for } i = 1, \cdots, a \\
W_\beta &= X_{i\beta} + p_{i\beta}^Y Y \quad \text{for } i = 1, \cdots, b \\
W_\gamma &= X_{i\gamma} + p_{i\gamma}^Y Y \quad \text{for } i = 1, \cdots, c
\end{aligned}
$$

From Cobb-Douglas utility functions, the individual demand for X is

$$
\begin{aligned}
X_{i\alpha} &= (1-\alpha) W_\alpha \\
X_{i\beta} &= (1-\beta) W_\beta \\
X_{i\gamma} &= (1-\gamma) W_\gamma
\end{aligned}
$$

from the Samuelson condition, we get

$$
\sum_{i=1}^{a} \frac{\alpha X_{i\alpha}}{(1-\alpha)Y} + \sum_{i=1}^{b} \frac{\beta X_{i\beta}}{(1-\beta)Y} + \sum_{i=1}^{c} \frac{\gamma X_{i\gamma}}{(1-\gamma)Y} = p,
$$

where $c = N - a - b$. Since all people of one type must be treated equally, we have

$$
\begin{aligned}
a\frac{\alpha X_{i\alpha}}{(1-\alpha)Y} + b\frac{\beta X_{i\beta}}{(1-\beta)Y} + c\frac{\gamma X_{i\gamma}}{(1-\gamma)Y} &= p \\
\frac{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma}{p} &= Y
\end{aligned}
$$

and solving for the Lindahl prices, we get

$$
\begin{aligned}
p_{i\alpha}^Y &= \frac{p\left(W_\alpha - X_{i\alpha}\right)}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\
&= \frac{p\alpha W_\alpha}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\
p_{i\beta}^Y &= \frac{p\beta W_\beta}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma} \\
p_{i\gamma}^Y &= \frac{p\gamma W_\gamma}{a\alpha W_\alpha + b\beta W_\beta + c\gamma W_\gamma}
\end{aligned}
$$

(b) Since $X_i = \frac{1}{N}X$, we have

$$
\begin{aligned}
U_{i\alpha}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\alpha)} Y^{\alpha} \\
U_{i\beta}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\beta)} Y^{\beta} \\
U_{i\gamma}(X_i, Y) &= \left(\frac{1}{N}X\right)^{(1-\gamma)} Y^{\gamma}
\end{aligned}
$$

The individual budget constraints are

$$\frac{1}{N}X = W_\alpha - p^Y_{i\alpha}Y = W_\beta - p^Y_{i\beta}Y = W_\gamma - p^Y_{i\gamma}Y$$

The individual demand for X is

$$\frac{1}{N}X = (1-\alpha)W_\alpha = (1-\beta)W_\beta = (1-\gamma)W_\gamma$$

Then the total demand for X:

$$X = a(1-\alpha)W_\alpha + b(1-\beta)W_\beta + c(1-\gamma)W_\gamma$$

From the Samuelson condition

$$
\begin{aligned}
a\frac{\alpha X}{(1-\alpha)NY} + b\frac{\beta X}{(1-\beta)NY} + c\frac{\gamma X}{(1-\gamma)NY} &= p \\
\frac{X}{NY}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right) &= p
\end{aligned}
$$

The optimal amount of the public good is then

$$
\begin{aligned}
Y &= \frac{X}{Np}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right) \\
&= \frac{a(1-\alpha)W_\alpha + b(1-\beta)W_\beta + c(1-\gamma)W_\gamma}{Np}\left(a\frac{\alpha}{1-\alpha} + b\frac{\beta}{1-\beta} + c\frac{\gamma}{1-\gamma}\right)
\end{aligned}
$$

4.4

$$U_i(X_i, Y) = Y_\alpha\,(X_i + k_i)$$

individual budget:

$$W_i = X_i + p_i^Y Y$$

global feasibility:

$$\sum W_i = \sum X_i + Y$$

Samuelson condition:

$$\sum MRS = \frac{\alpha}{Y}(X_i + k_i) = 1$$

substituting for X_i from global feasibility:

$$\frac{\alpha}{Y}\left(\sum W_i - Y + \sum k_i\right) = 1$$

$$Y = \frac{\alpha}{1+\alpha}\left(\sum W_i + \sum k_i\right)$$

To get Lindahl prices,

$$\begin{aligned} MRS_i &= \frac{\alpha}{Y}\left(X_i + k_i\right) + p_i^Y \\ p_i^Y &= \frac{W_i + k_i}{\sum W_i + \sum k_i} \end{aligned}$$