Required Problems

1. **Consider the following matrices:**

$$A = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) **Calculate AB.**

$$AB = \begin{bmatrix} 2 & 8 \\ 3 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} (4+24) & (0+64) \\ (6+0) & (0+0) \\ (10+3) & (0+8) \end{bmatrix}$$
$$AB = \begin{bmatrix} 28 & 64 \\ 6 & 0 \\ 13 & 8 \end{bmatrix}$$

   (b) **Calculate CB.**

$$CB = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix}$$
$$= \begin{bmatrix} (14+6) & (0+16) \\ (12+9) & (0+24) \end{bmatrix}$$
$$CB = \begin{bmatrix} 20 & 16 \\ 21 & 24 \end{bmatrix}$$

   (c) **Is it true that $CB = BC$? Justify your response**

$$BC = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} (14+0) & (4+0) \\ (21+48) & (6+24) \end{bmatrix}$$
$$BC = \begin{bmatrix} 14 & 4 \\ 69 & 30 \end{bmatrix}$$

   Matrix multiplication does not ha the commutativity property; clearly, the matrix we calculated for BC is not equal to the matrix we found for CB in part (b).

2. **Find the determinant of each of the following matrices.**

   (a) $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$

   Employing the formula for a 2×2 determinant:

$$|A| = (2)(-1) - (1)(3)$$
$$|A| = -5$$

(b) $B = \begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$

Performing the Laplace expansion using the second column:

$$\begin{aligned} |B| &= -(1)\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + (0)\begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} - (0)\begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix} \\ &= -1\big[12 - 6\big] \\ |B| &= -6 \end{aligned}$$

3. **Find the inverse of the following matrix:**

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$$

Using the formula for an inverse of a 2×2 matrix:

$$A^{-1} = \frac{1}{5-0}\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

4. **Find the eigenvalues and eigenvectors associated with the matrix:**

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

Recall the characteristic formula equation $|A - I_2\lambda| = 0$:

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} \qquad \text{(the characteristic polynomial)}$$

$$0 = (4-\lambda)(3-\lambda) - 4 \qquad \text{(taking the determinant)}$$

$$0 = 12 - 3\lambda - 4\lambda + \lambda^2 - 4 \qquad \text{(expanding)}$$

$$0 = \lambda^2 - 7\lambda + 8 \qquad \text{(simplifying)}$$

This does not factor; employing the quadratic equation obtains for us two roots:

$$\lambda_1 = \frac{7+\sqrt{17}}{2} \qquad \text{(root 1)}$$

$$\lambda_2 = \frac{7-\sqrt{17}}{2} \qquad \text{(root 2)}$$

Finding the first eigenvector $\mathbf{v}_1$, associated with λ_1, i.e., a vector that satisfies $(A - I_n\lambda_1)\mathbf{v} = \mathbf{0}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 - \frac{7+\sqrt{17}}{2} & 2 \\ 2 & 3 - \frac{7+\sqrt{17}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(plugging in } \lambda_1\text{)}$$

$$0 = 2x_1 + 3x_2 - \frac{7+\sqrt{17}}{2}x_2 \qquad \text{(from the second row)}$$

$$x_1 = \left(\frac{1}{4} + \frac{\sqrt{17}}{4}\right)x_2 \qquad \text{(solving for } x_1\text{)}$$

A non-normalized vector $\mathbf{u}_1$ that would satisfy this equation is:

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{bmatrix} \qquad \text{(from the equation above)}$$

$$\mathbf{v}_1 = \sqrt{\frac{8}{17+\sqrt{17}}} \begin{bmatrix} \frac{1}{4}(1+\sqrt{17}) \\ 1 \end{bmatrix} \qquad \text{(normalizing)}$$

Finding the second eigenvector $\mathbf{v}_2$, associated with λ_2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-\frac{7-\sqrt{17}}{2} & 2 \\ 2 & 3-\frac{7-\sqrt{17}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(plugging in } \lambda_2\text{)}$$

$$0 = 2x_1 + 3x_2 - \frac{7-\sqrt{17}}{2} x_2 \qquad \text{(from the second row)}$$

$$x_1 = \left(\frac{1}{4} - \frac{\sqrt{17}}{4}\right) x_2 \qquad \text{(solving for } x_1\text{)}$$

A non-normalized vector $\mathbf{u}_2$ that would satisfy this equation is:

$$\mathbf{u}_2 = \begin{bmatrix} \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{bmatrix} \qquad \text{(from the equation above)}$$

$$\mathbf{v}_2 = \sqrt{\frac{8}{17-\sqrt{17}}} \begin{bmatrix} \frac{1}{4}(1-\sqrt{17}) \\ 1 \end{bmatrix} \qquad \text{(normalizing)}$$

Practice Problems

5. **Consider the following column vectors:**

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 2 \end{bmatrix}$$

Are these vectors linearly independent? Justify your response.

Form a 4×4 by concatenating the vectors, then perform row operations to determine if the matrix is of full rank:

$$\mathbf{V} = \begin{bmatrix} 4 & -1 & 2 & 1 \\ -2 & 0 & 6 & -5 \\ 3 & -3 & 0 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix} \to \text{interchanging } R_1 \text{ and } R_4 \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ -2 & 0 & 6 & -5 \\ 3 & -3 & 0 & 3 \\ 4 & -1 & 2 & 1 \end{bmatrix}$$

Replacing R_2 with $R_2 + 2R_1$, R_3 with $R_3 - 3R_1$, and R_4 with $R_4 - 4R_1$:

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 10 & -1 \\ 0 & -6 & -6 & -3 \\ 0 & -5 & -6 & -7 \end{bmatrix} \to \begin{matrix} \text{replacing } R_3 \text{ with } R_3 + 2R_2 \\ \text{and } R_4 \text{ with } R_4 + 2.5R_2 \end{matrix} \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 10 & -1 \\ 0 & 0 & 24 & -6 \\ 0 & 0 & 19 & -9.5 \end{bmatrix}$$

Replacing R_4 with $R_4 - 19/24R_3$:

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 10 & -1 \\ 0 & 0 & 24 & -6 \\ 0 & 0 & 0 & -4.75 \end{bmatrix} \to \text{scaling each row} \to \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 5 & -1/2 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix is now in echelon form (with a leading ones in each row). There are four leading ones, which implies the matrix is of full rank, equivalent to the vectors being linearly independent.

6. **Find the determinanat of the following matrix:**

$$A = \begin{bmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{bmatrix}$$

Performing the Laplace expansion using the third column:

$$|A| = (0)\begin{vmatrix} 2 & 3 & 6 \\ 1 & 6 & -1 \\ 0 & -5 & 8 \end{vmatrix} - (4)\begin{vmatrix} 1 & 2 & 9 \\ 1 & 6 & -1 \\ 0 & -5 & 8 \end{vmatrix} + (0)\begin{vmatrix} 1 & 2 & 9 \\ 2 & 3 & 6 \\ 0 & -5 & 8 \end{vmatrix} - (0)\begin{vmatrix} 1 & 2 & 9 \\ 2 & 3 & 4 \\ 1 & 6 & -1 \end{vmatrix}$$

$$|A| = -(4)\begin{vmatrix} 1 & 2 & 9 \\ 1 & 6 & -1 \\ 0 & -5 & 8 \end{vmatrix}$$

Performing the Laplace expansion using the first column of the 3×3:

$$|A| = -(4)\left[(1)\begin{vmatrix} 6 & -1 \\ -5 & 8 \end{vmatrix} - (1)\begin{vmatrix} 2 & 9 \\ -5 & 8 \end{vmatrix} + (0)\begin{vmatrix} 2 & 9 \\ 6 & -1 \end{vmatrix}\right]$$

$$= (-4)\left[(1)\Big(48 - 5\Big) - (1)\Big(16 + 45\Big)\right]$$

$$|A| = 72$$

7. **Find the inverse of each of the following the matrices.**

(a) $B = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$ Using the formula for the inverse of a 2×2 matrix:

$$\mathbf{B}^{-1} = \frac{1}{-2-0}\begin{bmatrix} 2 & 0 \\ -9 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \frac{9}{2} & \frac{1}{2} \end{bmatrix}$$

(b) $C = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$

Note that the determinanat of the matrix is $|C| = 99$, so it is invertible. The cofactor matrix is

$$\mathcal{C}_C = \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} \qquad \text{(the cofactor matrix of } C\text{)}$$

$$\mathcal{C}_c = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix} \qquad \text{(simplifying)}$$

Recall that the adjoing matrix is the transpose of the cofactor matrix:

$$\text{adj}(C) = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix} \qquad \text{(the adjoint)}$$

Using our formula for the inverse:

$$C^{-1} = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

8. **Consider the linear system given by $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \varepsilon$, where $\mathbf{Y}$ and ε are $n \times 1$ vectors, $\mathbf{X}$ is a $n \times k$ matrix of full rank, and $\boldsymbol{\beta}$ is a $k \times 1$ vector.**

   (a) **Suppose that ε is orthogonal to each of the columns in $\mathbf{X}$ (i.e., $\mathbf{X}'\varepsilon = \mathbf{0}$). Using matrix algebra, solve for $\boldsymbol{\beta}$.**

$$\begin{aligned}
\mathbf{Y} &= \mathbf{X}\boldsymbol{\beta} + \varepsilon && \text{(given)} \\
\mathbf{X}'\mathbf{Y} &= \mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \mathbf{X}'\varepsilon && \text{(premultiplying by } \mathbf{X}'\text{)} \\
\mathbf{X}'\mathbf{Y} &= \mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \mathbf{0} && \text{(by } \boldsymbol{\varepsilon} \perp \mathbf{X}\text{)} \\
\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{X}'\mathbf{Y} && \text{(rearranging)} \\
(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} && \text{(premultiplying by } (\mathbf{X}'\mathbf{X})^{-1}\text{)} \\
\mathbf{I}_k\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} && \text{(by def. of the inverse)} \\
\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} && \text{(simplifying)}
\end{aligned}$$

   (b) **Further, suppose that $\boldsymbol{\beta}$ is 2×1 and the matrix $\mathbf{X}$ is $n \times 2$, such that:**

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

   In addition, assume that $\sum_{i=1}^{n} x_i = 0$. Find expressions for β_1 and β_2.

   Start by calculating each piece of our $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ matrix from (a):

$$\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} && \text{(multiplying matrices)} \\
&= \begin{bmatrix} n & 0 \\ 0 & \sum x_i^2 \end{bmatrix} && \text{(by } \textstyle\sum x_i = 0\text{)} \\
(\mathbf{X}'\mathbf{X})^{-1} &= \frac{1}{n\sum x_i^2}\begin{bmatrix} \sum x_i^2 & 0 \\ 0 & n \end{bmatrix} && \text{(inverting)}
\end{aligned}$$

The second piece of the formula:

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} \qquad \text{(multiplying matrices)}$$

Employing the formula from part (a)

$$\begin{aligned}
\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} && \text{(from above)} \\
&= \frac{1}{n\sum x_i^2} \begin{bmatrix} \sum x_i^2 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix} && \text{(plugging in matrices)} \\
&= \frac{1}{n\sum x_i^2} \begin{bmatrix} \left(\sum y_i\right)\left(\sum x_i^2\right) \\ n\left(\sum x_i y_i\right) \end{bmatrix} && \text{(multiplying)} \\
\boldsymbol{\beta} &= \begin{bmatrix} \frac{\sum y_i}{n} \\ \frac{\sum x_i y_i}{\sum x_i^2} \end{bmatrix} && \text{(simplifying)}
\end{aligned}$$

Thus, $\beta_1 = \bar{y}$ and $\beta_2 = \frac{\sum x_i y_i}{\sum x_i^2}$.

9. **Use Cramer's rule to solve the following equation systems:**

   (a) $8x_1 - x_2 = 16$
   $2x_2 + 5x_3 = 5$
   $2x_1 + 3x_3 = 7$

   Define a matrix of coefficients and a vector of constants:

$$A = \begin{bmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \qquad \mathbf{k}_a = \begin{bmatrix} 16 \\ 5 \\ 7 \end{bmatrix}$$

   Using Cramer's rule:

$$x_1^* = \frac{\begin{vmatrix} 16 & -1 & 0 \\ 5 & 2 & 5 \\ 7 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{76}{38} \qquad \text{(replacing column 1)}$$

$$x_2^* = \frac{\begin{vmatrix} 8 & 16 & 0 \\ 0 & 5 & 5 \\ 2 & 7 & 3 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{0}{38} \qquad \text{(replacing column 2)}$$

$$x_3^* = \frac{\begin{vmatrix} 8 & -1 & 16 \\ 0 & 2 & 5 \\ 2 & 0 & 7 \end{vmatrix}}{\begin{vmatrix} 8 & -1 & 0 \\ 0 & 2 & 5 \\ 2 & 0 & 3 \end{vmatrix}} = \frac{38}{38} \qquad \text{(replacing column 3)}$$

   Thus, $x_1^* = 2$, $x_2^* = 0$, and $x_3^* = 1$

(b) $-x_1 + 3x_2 + 2x_3 = 24$
$x_1 + x_3 = 6$
$5x_2 - x_3 = 8$

Define an augemented matrix of coefficients and the left-hand-side constants:

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & 24 \\ 1 & 0 & 1 & 6 \\ 0 & 5 & -1 & 8 \end{array}\right] \to \text{Interchanging } R_1 \text{ and } R_2 \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ -1 & 3 & 2 & 24 \\ 0 & 5 & -1 & 8 \end{array}\right]$$

Replacing R_2 with $R_2 + R_1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 3 & 3 & 30 \\ 0 & 5 & -1 & 8 \end{array}\right] \to \text{Scaling } R_2 \text{ by } 1/3\text{:} \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 10 \\ 0 & 5 & -1 & 8 \end{array}\right]$$

Replacing R_3 with $R_3 - 5R_2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & -6 & -42 \end{array}\right] \to \text{Scaling } R_3 \text{ by } -1/6\text{:} \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Replacing R_1 with $R_1 - R_3$ and R_2 with $R_2 - R_3$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Thus, $x_1^* = -1$, $x_2^* = 3$, and $x_3^* = 7$

(c) $4x + 3y - 2z = 1$
$x + 2y = 6$
$3x + z = 4$

Solving this equation via matrix inversion:

$$C = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad \text{(the coefficient matrix)}$$

$$|C| = (3)\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} + (1)\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} \qquad \text{(the determinant)}$$

$$|C| = 17 \qquad \text{(simplifying)}$$

$$\mathcal{C}_C = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & -2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix} \qquad \text{(the cofactor matrix)}$$

$$\mathcal{C}_C = \begin{bmatrix} 2 & -1 & -6 \\ -3 & 10 & 9 \\ 4 & -2 & 5 \end{bmatrix} \qquad \text{(simplifying)}$$

$$\text{adj}(C) = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 10 & -2 \\ -6 & 9 & 5 \end{bmatrix} \qquad \text{(the adjoint)}$$

Employing the inverse formula (dividing the adjoing by the determinant):

$$C^{-1} = \frac{1}{17}\begin{bmatrix} 2 & -3 & 4 \\ -1 & 10 & -2 \\ -6 & 9 & 5 \end{bmatrix} \qquad \text{(the inverse)}$$

Recall that there is a unique solution $\mathbf{x} = C^{-1}b$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 2 & -3 & 4 \\ -1 & 10 & -2 \\ -6 & 9 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \qquad \text{(plugging in matrices)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} (2-18+16) \\ (-1+60-8) \\ (-6+54+20) \end{bmatrix} \qquad \text{(multiplying)}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \qquad \text{(simplifying)}$$

Thus, $x^* = 0$, $y^* = 3$, and $z^* = 4$

(d) $-x + y + z = a$
$x - y + z = b$
$x + y - z = c$

Define a matrix of coefficients and a vector of constants:

$$\mathbf{D} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad \mathbf{k}_d = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Using Cramer's rule:

$$x^* = \frac{\begin{vmatrix} a & 1 & 1 \\ b & -1 & 1 \\ c & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{2(b+c)}{4} \qquad \text{(replacing column 1)}$$

$$y^* = \frac{\begin{vmatrix} -1 & a & 1 \\ 1 & b & 1 \\ 1 & c & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{2(a+c)}{4} \qquad \text{(replacing column 2)}$$

$$z^* = \frac{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{2(a+b)}{4} \qquad \text{(replacing column 3)}$$

Thus, $x^* = \frac{b+c}{2}$, $y^* = \frac{a+c}{2}$, and $z^* = \frac{a+b}{2}$

10. **Suppose there is a perfectly competitive firm with a production function $y = f(x_1, x_2)$, increasing in both arguments. The firm sells output y at the market price p. The firm purchases an input x_1 and price w_1; x_2, however, represents the entrepreneur's input and is limited to $\bar{x}_2$. Thus, the firms maximization problem, framed as a Lagrangian is given by**

$$\mathcal{L} = p \cdot f(x_1, x_2) - w_1 \cdot x_1 + \lambda(\bar{x}_2 - x_2)$$

This yields the first order conditions:

$$\begin{aligned}
\mathcal{L}_1 &= p \cdot f_1(x_1, x_2) - w_1 = 0 \\
\mathcal{L}_2 &= p \cdot f_2(x_1, x_2) - \lambda = 0 \\
\mathcal{L}_\lambda &= \bar{x}_2 - x_2 = 0
\end{aligned}$$

(a) **What is the sufficient second order condition?**

The bordered Hessian for this problem is:

$$\bar{\mathbf{H}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & pf_{11}(\cdot) & pf_{12} \\ 1 & pf_{12} & pf_{22} \end{bmatrix}$$

Since this is a constrained profit-maximization problem, we'd like this matrix to be negative definite. Unfortunately, $\bar{\mathbf{H}}$ is not negative definite (nor is it negative semi-definite). This is because the maximization doesn't really have two variables; from the FOC, we can see $x_2 = \bar{x}_2$, i.e., we're only choosing x_1. As a result, the only second-order condition we need for a maximum is that $pf_{11}(\cdot) < 0$.

Given that the profit-maximizing levels of x_1, x_2, and λ have been found as functions of p, w_1, and $\bar{x}_2$, e.g., $x_1^* = g(p, w_1, \bar{x}_2)$, the solutions may be plugged back into the first order conditions yielding:

$$\begin{aligned}
pf_1(x_1^*, x_2^*) - w_1 &\equiv 0 \\
pf_2(x_1^*, x_2^*) - \lambda^* &\equiv 0 \\
\bar{x}_2 - x_2^* &\equiv 0
\end{aligned}$$

(b) **Use Cramer's Rule to find a comparative statics prediction for $\frac{\partial x_1^*}{\partial w_1}$ (Hint: use the chain rule on the system of identities, differentiating w.r.t. w_1).**

Differentiating the system w.r.t. w_1 yields:

$$\begin{aligned}
pf_{11}(\cdot)\frac{\partial x_1^*}{\partial w_1} + pf_{12}(\cdot)\frac{\partial x_2^*}{\partial w_1} - 1 &= 0 \\
pf_{12}(\cdot)\frac{\partial x_1^*}{\partial w_1} + pf_{22}(\cdot)\frac{\partial x_2^*}{\partial w_1} - \frac{\partial \lambda^*}{\partial w_1} &= 0 \\
-\frac{\partial x_2^*}{\partial w_1} &= 0
\end{aligned}$$

Rewriting this in matrix notation:

$$\begin{bmatrix} pf_{11}(\cdot) & pf_{12}(\cdot) & 0 \\ pf_{12}(\cdot) & pf_{22}(\cdot) & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \\ \frac{\partial \lambda^*}{\partial w_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Note the similarity to the bordered Hessian from part (a)—the coefficient matrix here is just two row operations away from $\bar{\mathbf{H}}$. Using Cramer's Rule to find $\frac{\partial x^*}{\partial w_1}$:

$$\frac{\partial x_1^*}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12}(\cdot) & 0 \\ 0 & pf_{22}(\cdot) & -1 \\ 0 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} pf_{11}(\cdot) & pf_{12}(\cdot) & 0 \\ pf_{12}(\cdot) & pf_{22}(\cdot) & -1 \\ 0 & -1 & 0 \end{vmatrix}} = \frac{1}{pf_{11}(\cdot)}$$

If our second order condition from part (a) holds, then $\frac{1}{pf_{11}(\cdot)} < 0$, implying that $\frac{\partial x_1^*}{\partial w_1} < 0$. In other words, the demand for input x_1 slopes down.

11. **Find the eigenvalues and eigenvectors for each of the following matrices:**

(a) $A = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$

Finding the eigenvalues:

$$0 = \begin{vmatrix} -2-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} \qquad \text{(the characteristic polynomial)}$$

$$0 = (-2-\lambda)(-4-\lambda) - 4 \qquad \text{(taking the determinant)}$$

$$0 = 8 + 4\lambda + 2\lambda + \lambda^2 - 4 \qquad \text{(expanding)}$$

$$0 = \lambda^2 + 6\lambda + 4 \qquad \text{(simplifying)}$$

Once again employing the quadratic equation to find roots:

$$\lambda_1 = -3 - \sqrt{5} \qquad \text{(root 1)}$$

$$\lambda_2 = \sqrt{5} - 3 \qquad \text{(root 2)}$$

Finding the first eigenvector $\mathbf{v}_1$, associated with λ_1:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2-(-3-\sqrt{5}) & 2 \\ 2 & -4-(-3-\sqrt{5}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(plugging in } \lambda_1\text{)}$$

$$0 = 2x_1 - x_2 + \sqrt{5}x_2 \qquad \text{(from the second row)}$$

$$x_1 = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) x_2 \qquad \text{(solving for } x_1\text{)}$$

A non-normalized vector that would satisfy this equation is:

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{bmatrix} \qquad \text{(from the equation above)}$$

$$\mathbf{v}_1 = \sqrt{\frac{2}{5-\sqrt{5}}} \begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{bmatrix} \qquad \text{(normalizing)}$$

Finding the second eigenvector $\mathbf{v}_2$, associated with λ_2:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2-(\sqrt{5}-3) & 2 \\ 2 & -4-(\sqrt{5}-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(plugging in } \lambda_2\text{)}$$

$$0 = 2x_1 + -x_2 - \sqrt{5}x_2 \qquad \text{(from the second row)}$$

$$x_1 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) x_2 \qquad \text{(solving for } x_1\text{)}$$

A non-normalized vector that would satisfy this equation is:

$$\mathbf{u}_2 = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix} \qquad \text{(from the equation above)}$$

$$\mathbf{v}_2 = \sqrt{\frac{2}{5+\sqrt{5}}} \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix} \qquad \text{(normalizing)}$$

(b) $\mathbf{C} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Finding the eigenvalues:

$$0 = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \qquad \text{(the characteristic polynomial)}$$

$$0 = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} \qquad \text{(taking the determinant)}$$

$$0 = (3-\lambda)\big[-\lambda(3-\lambda)-4\big] - 2\big[2(3-\lambda)-8\big] + 4\big[4+4\lambda\big] \qquad \text{(multiplying)}$$

$$0 = -\lambda^3 + 6\lambda^2 + 15\lambda + 8 \qquad \text{(simplifying)}$$

$$0 = (1+\lambda)(1+\lambda)(8-\lambda) \qquad \text{(factoring)}$$

Thus, the roots of the polynomial are:

$$\lambda_1 = -1 \qquad \text{(root 1)}$$
$$\lambda_2 = -1 \qquad \text{(root 2)}$$
$$\lambda_3 = 8 \qquad \text{(root 3)}$$

Finding the first two eigvenvectors, associated with $\lambda_1 = \lambda_2 = -1$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-(-1) & 2 & 4 \\ 2 & -(-1) & 2 \\ 4 & 2 & 3-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{(plugging in } \lambda_1\text{)}$$

$$0 = 2x_1 + x_2 + 2x_3 \qquad \text{(from row 2)}$$

Setting $x_2 = 0$, we get:

$$x_1 = -x_3 \qquad \text{(solving for } x_1\text{)}$$

A non-normalized vector that would satisfy this equation is:

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T \qquad \text{(from the equation above)}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}^T \qquad \text{(normalizing)}$$

Setting $x_3 = 0$, we get:

$$x_2 = -2x_1 \qquad \text{(solving for } x_2\text{)}$$

A non-normalized vector that would satisfy this equation is:

$$\mathbf{u}_2 = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}^T \qquad \text{(from the equation above)}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \end{bmatrix}^T \qquad \text{(normalizing)}$$

Finding the third eigenvector $\mathbf{v}_3$, associated with λ_3:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-8 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{(plugging in } \lambda_3\text{)}$$

$$0 = -5x_1 + 2x_2 + 4x_3 \qquad \text{(from row 1)}$$

A non-normalized vector that would satisfy this equation is:

$$\mathbf{u}_3 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^T \qquad \text{(from the equation above)}$$

$$\mathbf{v}_3 = \begin{bmatrix} 2/3 & 1/3 & 2/3 \end{bmatrix}^T \qquad \text{(normalizing)}$$

12. **Given a quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a symmetric 2×2 matrix:**

$$\mathbf{A} = \begin{bmatrix} a & d \\ d & b \end{bmatrix}$$

Show that the following are true (Hint: use the quadratic equation):

(a) **Both eigenvalues must be real (i.e., they cannot involve $\sqrt{-1}$)**

$$0 = \begin{vmatrix} a-\lambda & d \\ d & b-\lambda \end{vmatrix} \qquad \text{(the characteristic equation)}$$

$$0 = (a-\lambda)(b-\lambda) - d^2 \qquad \text{(taking the determinant)}$$

$$0 = ab - (a+b)\lambda + \lambda^2 - d^2 \qquad \text{(expanding)}$$

$$0 = \lambda^2 - (a+b)^2 + (ab - d^2) \qquad \text{(rearranging)}$$

$$\lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab-d^2)}}{2} \qquad \text{(the quadratic equation)}$$

To ensure that the roots are real, we only need to focus on the terms under the radical sign:

$$\begin{aligned} (a+b)^2 - 4(ab-d^2) &= a^2 + 2ab + b^2 - 4ab + 4d^2 && \text{(expanding)} \\ &= a^2 - 2ab + b^2 + 4d^2 && \text{(simplifying)} \\ &= (a-b)^2 + 4d^2 && \text{(factoring)} \end{aligned}$$

Thus, the terms under the radial sign must be non-negative (we're summing values that are raised to the second power). As a result, the eigenvalues must be real.

(b) **A has repeated roots if and only if it is of the form $\mathbf{A} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$**

The determinant of $(\mathbf{A} - \mathbf{I}\lambda)$ will be $(c-\lambda)(c-\lambda)$, so it's clear that matrices of this form will have repeated roots. To show that if we have repeated roots then we must have a matrix of this form, we can look at the quadratic equation for the general symmetric 2×2 (from above):

$$\lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab-d^2)}}{2}$$

Repeated roots implies that the terms under the radical sign sum to zero:

$$\begin{aligned}
(a+b)^2 &= 4(ab-d^2) \\
a^2+2ab+b^2 &= 4ab-4d^2 && \text{(expanding)} \\
a^2-2ab+b^2 &= -4d^2 && \text{(rearranging)} \\
(a-b)^2 &= -4d^2 && \text{(factoring)}
\end{aligned}$$

The left hand side must be non-negative; the right hand side must be non-positive. The only for this equality to hold is if both are zero, i.e., $d = 0$ and $a = b$.

13. **Let A be a $m \times n$ matrix. Show that $(A^T)^T = A$.**

Let A be a $n \times m$ matrix, such that:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Taking the transpose, the element in the ith row and jth column in A becomes the element in the ith column and jth row in A^T:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

Again, taking the transpose, the element in the ith row and jth column in A^T becomes the element in the ith column and jth row in $(A^T)^T$:

$$(A^T)^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Which is our original matrix A.

14. **Let B and C be $m \times n$ matricies. Show that $k(A+B) = kA + kB$**

Let A and B be $n \times m$ matrices and let $k \in \mathbb{R}$. Consider the matrix $k(A+B)$. $k(a_{ij}+b_{ij})$ is its (i,j)th element. As a result:

$$k(A+B) = \begin{bmatrix} k(a_{11}+b_{11}) & k(a_{12}+b_{12}) & \dots & k(a_{1m}+b_{1m}) \\ k(a_{21}+b_{21}) & k(a_{22}+b_{22}) & \dots & k(a_{2m}+b_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ k(a_{n1}+b_{n1}) & k(a_{n2}+b_{n2}) & \dots & k(a_{nm}+b_{nm}) \end{bmatrix}$$

By the distributive property of addition:

$$k(A+B) = \begin{bmatrix} ka_{11}+kb_{11} & ka_{12}+kb_{12} & \dots & ka_{1m}+kb_{1m} \\ ka_{21}+kb_{21} & ka_{22}+kb_{22} & \dots & ka_{2m}+kb_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1}+kb_{n1} & ka_{n2}+kb_{n2} & \dots & ka_{nm}+kb_{nm} \end{bmatrix}$$

But ka_{ij} is the (i,j)th element of kA, while kb_{ij} is the (i,j)th element of kB. Thus, $ka_{ij} + kb_{ij}$ is the corresponding element in the matrix $kA + kB$. As a result:

$$k(A + B) = kA + kB$$

15. **Let D be a $n \times n$ invertible matrix. Show that D^{-1} is unique.**

$$\begin{aligned}
&\text{Let } D \text{ be an } n \times n \text{ invertible matrix} && \\
&\implies \exists\, D^{-1} \ni D^{-1}D = DD^{-1} = I_n && \text{(by def. of the inverse)} \\
&\text{Suppose } D^{-1} \text{ is not unique} && \text{(towards a contradiction)} \\
&\text{Let } B \neq D^{-1} \text{ be another inverse for } D && \\
&\implies BD = DB = I_n && \text{(by def. of the inverse)} \\
&DB = BD && \text{(by def. of the inverse)} \\
&D^{-1}DB = D^{-1}DB && \text{(pre-multiplying by } D\text{)} \\
&(D^{-1}D)B = D^{-1}(DB) && \text{(by the associative property)} \\
&I_nB = D^{-1}I_n && \text{(by def. of the inverse)} \\
&B = D^{-1} && \text{(by def. of the identity matrix)} \\
&\text{But this violates our assumption of non-uniqueness} && \\
&\text{Thus, a contradiction} && \\
&\implies D^{-1} \text{ is unique} &&
\end{aligned}$$

16. **Let E be a $n \times n$ matrix. Show that if E is idempotent, $I - E$ is idempotent as well.**

$$\begin{aligned}
&\text{Let } E \text{ be an } n \times n \text{ idempotent matrix} && \text{(by hypothesis)} \\
&\implies EE = E && \text{(by idempotence)} \\
&\text{Consider the matrix } (I_n - E)(I_n - E) && \\
&= I_nI_n - EI_n - EI_n + EE && \text{(expanding)} \\
&= I_n - E - E + EE && \text{(by def. of the identity matrix)} \\
&= I_n - E - E + E && \text{(by idempotence of } E\text{)} \\
&= I_n - E && \text{(simplifying)}
\end{aligned}$$

Thus, if E is idempotent then $I - E$ is idempotent as well.

17. **Show that for any 2×2 matrix A**

$$|A| = \frac{1}{2}\begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix}$$

To show the desired result, we will need to invoke three theorems. For an 2×2 matrix A, let λ_1 and λ_2 be its eigenvalues. Then for the left hand side of the equation, we have the theorem regarding the determinant's relationship to the eigenvalues:

$$|A| = \lambda_1\lambda_2$$

Further, we have two theorems regarding the trace of A and the eigenvalues:

$$\text{tr}(A) = \lambda_1 + \lambda_2$$

$$\text{tr}(AA) = \lambda_1^2 + \lambda_2^2$$

If we can show that the left hand side also equals $\lambda_1\lambda_2$, we will have shown the result. Manipulating the right hand side of the equation:

$$\frac{1}{2}\begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} (\lambda_1+\lambda_2) & 1 \\ (\lambda_1^2+\lambda_2^2) & (\lambda_1+\lambda_2) \end{vmatrix}$$
$$= \frac{1}{2}\left[(\lambda_1+\lambda_2)^2 - (\lambda_1^2+\lambda_2^2)\right]$$
$$= \frac{1}{2}\left[\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 - \lambda_1^2 - \lambda_2^2\right]$$
$$\frac{1}{2}\begin{vmatrix} \text{tr}(A) & 1 \\ \text{tr}(A^2) & \text{tr}(A) \end{vmatrix} = \lambda_1\lambda_2$$

Thus, the relationship holds for 2×2 matrices.

18. **State whether each of the following statements is true or false. If it is false, provide a counterexample.**

 (a) **No system of linear equations can have exactly k solutions for any $k \geq 2$.**

 True. Systems will have zero, one, or infinite solutions.

 (b) **Any system of n linear equations in n unknowns has at least one solution.**

 False. Consider $x_1 + x_2 = 1$ and $x_1 + x_2 = 2$. This is an inconsistent system with no solution.

 (c) **Any system of n linear equations in n unknowns has at most one solution.**

 False. $2x_1 + x_2 = 1$ and $4x_2 + 2x_2 = 2$. This has infinite solutions.

 (d) **If $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has a solution, then $A\mathbf{x} = \mathbf{b}$ has a solution.**

 False. We will always have the trivial solution to $Ax = 0$, so it implies nothing about a solution to $Ax = b$. If the question had read "a non-trival solution exists to $Ax = 0$", this would imply a unique solution exists for $Ax = b$.

 (e) **If an $n \times n$ matrix A is invertible, then $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.**

 True. This is actually an if and only if statement.

 (f) **If an $n \times n$ matrix A is full rank, then $A\mathbf{x} = \mathbf{b}$ has a solution.**

 True. This actually implies that there is a unique solution.

 (g) **If an $n \times n$ matrix A has rank less than n, then $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has no solution.**

 False. It's possible to have infinite solutions. See the counterexample in (c).