Required Problems

1. **Determine the definiteness of the following symmetric matrices:**

   (a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

   We can determine definiteness by evaluating principle minors:

$$|A_1| = |2| \qquad \text{(the 1st LPM)}$$
$$|A_1| = 2 \qquad \text{(simplifying)}$$
$$|A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \qquad \text{(the 2nd LPM)}$$
$$|A_2| = 1 \qquad \text{(simplifying)}$$

   Both LPMs are strictly positive, so the matrix is positive definite.

   (b) $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

   Once again, we can determine definiteness by using our principle minor test:

$$|B_1| = |1| \qquad \text{(the 1st LPM)}$$
$$|B_1| = 1 \qquad \text{(simplifying)}$$
$$|B_2| = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \qquad \text{(the 2nd LPM)}$$
$$|B_2| = 2 \qquad \text{(simplifying)}$$
$$|B_3| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{vmatrix} \qquad \text{(the 3rd LPM)}$$
$$|B_3| = -10 \qquad \text{(simplifying)}$$

   Note that we do not need to go any further. The first two LPMs are positive; the third is negative. Thus, the matrix is indefinite.

2. **Find the least squares solution to $X\mathbf{b} = \mathbf{y}$, i.e., by finding the estimate $\hat{\mathbf{b}}$ such that $X\hat{\mathbf{b}} = \hat{\mathbf{y}}$ (where $\hat{\mathbf{y}}$ is the projection of $\mathbf{y}$ onto col(X)) using the following information:**

$$X = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

   Recall the formula for our OLS estimator:

$$\hat{\mathbf{b}} = (X^T X)^{-1} X^T \mathbf{y} \qquad \text{(the OLS solution)}$$

   Finding the first half of the formula:

$$X^T X = \begin{bmatrix} 9 & -2 \\ -2 & 22 \end{bmatrix} \qquad \text{(multiplying)}$$
$$(X^T X)^{-1} = \frac{1}{194} \begin{bmatrix} 22 & 2 \\ 2 & 9 \end{bmatrix} \qquad \text{(inverting)}$$

Finding the second half of the formula:

$$X^T\mathbf{y} = \begin{bmatrix} 2 \\ 11 \end{bmatrix} \qquad \text{(multiplying)}$$

"Assembling" all of the pieces:

$$\hat{\mathbf{b}} = \frac{1}{194}\begin{bmatrix} 22 & 2 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 11 \end{bmatrix} \qquad \text{(putting together the pieces)}$$

$$\hat{\mathbf{b}} = \frac{1}{194}\begin{bmatrix} 66 \\ 103 \end{bmatrix} \qquad \text{(simplifying)}$$

3. **In the game of dominoes, each piece is marked with two numbers. The pieces are symmetrical, so that the number pair is not ordered (e.g., (2, 6) = (6, 2)). Further, duplicated are allowed (e.g., (2, 2)). How many different pieces can be formed using the numbers $1, 2, \ldots, n$?**

   There are n numbers total, 2 are chosen with replacement, and order doesn't matter. Thus, to find the total number of different pieces, employ the unordered with replacement formula:

$$\begin{aligned} N &= \binom{n+2-1}{2} && \text{(unordered, with replacement)} \\ &= \frac{(n+1)!}{(n-1)!2!} && \text{(writing out the function)} \\ N &= \frac{(n+1)n}{2} && \text{(simplifying)} \end{aligned}$$

   Note that we can cancel $(n-1)!$ from both the numerator and the denominator when we're simplifying. Thus, there are $\frac{(n+1)n}{2}$ ways to form dominoes.

4. **Suppose that 5% of men and 0.25% of women are color blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male (assuming males and females are equal in number)?**

   Let m denote "male," f denote "female," and cb denote "color-blind." The question is asking for $P(m|cb)$. First, we need the find the probability of being male and color-blind:

$$\begin{aligned} \mathbb{P}(m \cap cb) &= P(m)P(cb|m) && \text{(be def. of cond. prob.)} \\ &= (0.5)(0.05) && \text{(plugging in values)} \\ &= 0.025 && \text{(simplifying)} \end{aligned}$$

   Next, we need to calculate that unconditional probability of being color-blind:

$$\begin{aligned} \mathbb{P}(cb) &= P(m)P(cb|m) + P(f)P(cb|f) && \text{(by law of total prob.)} \\ &= (0.5)(0.05) + (0.5)(0.0025) && \text{(plugging in values)} \\ &= 0.02625 && \text{(simplifying)} \end{aligned}$$

   Finally, we can put the pieces together using our definition, $P(m|cb) = P(m \cap cb)/P(cb)$:

$$\begin{aligned} \mathbb{P}(m|cb) &= \frac{0.025}{0.02625} && \text{(plugging in values)} \\ &= \frac{20}{21} && \text{(simplifying)} \end{aligned}$$

5. **If the random variable X follows a geometric distribution, its PMF is given by**

$$f_X(x) = (1-p)^x p, \qquad x \in \{0, 1, 2, 3 \dots\}$$

where the parameter $p \in (0,1)$. Find the CDF of X.

To find the CDF, we need to take the PMF and sum over all possible values up to and including x:

$$\begin{aligned} F_X(x) &= \sum_{k=0}^{x} (1-p)^k p && \text{(summing)} \\ &= p \sum_{k=0}^{x} (1-p)^k && \text{(pulling out a constant)} \end{aligned}$$

The sum is now clearly a geometric series (hence the name of the distribution). Consider the sum:

$$\begin{aligned} s &= \sum_{k=0}^{x} (1-p)^k && \text{(defining } s\text{)} \\ s &= (1-p)^0 + (1-p)^1 + (1-p)^2 + \cdots + (1-p)^x && \text{(expanding)} \\ (1-p)s &= (1-p)^1 + (1-p)^2 + \cdots + (1-p)^{x+1} && \text{(multiplying by } (1-p)\text{)} \\ [s - (1-p)s] &= (1-p)^0 - (1-p)^{x+1} && \text{(subtracting the two lines)} \\ s &= \frac{1-(1-p)^{x+1}}{p} && \text{(solving for } s\text{)} \end{aligned}$$

To find the CDF, we can plug in our expression for s:

$$\begin{aligned} F_X(x) &= p \sum_{k=0}^{x} (1-p)^k && \text{(the CDF)} \\ F_X(x) &= ps && \text{(by def. of } s\text{)} \\ F_X(x) &= 1 - (1-p)^{x+1} && \text{(simplifying)} \end{aligned}$$

This is how the CDF of a geometric RV is almost always stated, where $x \in \{0, 1, 2, \dots\}$. This would be a sufficient answer in any of the first-year classes. Notice, however, that if we wanted to define the CDF more rigorously, we would need to define this function over all of $\mathbb{R}$, rather than just the non-negative integers:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1-p)^{\lfloor x \rfloor + 1} & \text{if } x \geq 0 \end{cases}$$

Where $\lfloor x \rfloor$ is a function that rounds x to the closest integer less than or equal to x (e.g., if $x = 3.13$, then $\lfloor x \rfloor = 3$).

Practice Problems

6. **Determine the definiteness of the following symmetric matrices:**

   (a) $A = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

Again, we can determine definiteness by evaluating principle minors:

$$|A_1| = |-3| \qquad \text{(the 1st LPM)}$$
$$|A_1| = -3 \qquad \text{(simplifying)}$$
$$|A_2| = \begin{vmatrix} -3 & 4 \\ 4 & -5 \end{vmatrix} \qquad \text{(the 2nd PM)}$$
$$|A_2| = -1 \qquad \text{(simplifying)}$$

Both LPMs are negative, violating all of our signing conventions (negative definite, negative semi-definite, etc.). Thus, the matrix is indefinite.

(b) $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

$$|B_1| = |1| \qquad \text{(the 1st LPM)}$$
$$|B_1| = 1$$
$$|B_2| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \qquad \text{(the 2nd LPM)}$$
$$|B_2| = 0$$
$$|B_3| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{vmatrix} \qquad \text{(the 3rd LPM)}|B_3| = -25$$

The first LPM is positive; the second is zero; the third is negative. Thus, B is indefinite.

(c) $C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$|C_1| = |-1| \qquad \text{(the 1st LPM)}$$
$$|C_1| = -1$$
$$|C_2| = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} \qquad \text{(the 2nd LPM)}$$
$$|C_2| = 0$$
$$|C_3| = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \qquad \text{(the 3rd LPM)}$$
$$|C_3| = 0$$

The first LPM is negative; the next two are zero. Thus, the matrix is not negative definite, but it might be negative semidefinite. The other first order principle minors are:

$$|-1| = -1 \qquad |-2| = -2$$

The other second order principle minors are:

$$\begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2 \qquad \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} = 2$$

Thus, all of the first order principle minors are non-positive; all of the second order principle minors are non-negative; and the third order principle minor is non-positive. Thus, the matrix is negative semi-definite.

7. **Recall the conditions for $d(x,y)$ to quality as a metric. Verify that the function d is a metric where**

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

Note that it will be helpful to define the absolute value function here:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This implies that $|\cdot| : \mathbb{R} \to \mathbb{R}_+$. That is, the absolute value function maps the real numbers onto the non-negative real numbers. Verifying the conditions for $d(x,y)$:

- First Condition: $d(x,y) \geq 0$

$$\begin{aligned} |x-y| &\geq 0 && \text{(by def. of } |x|) \\ \frac{|x-y|}{1+|x-y|} &\geq 0 && \text{(the ratios of positives is postive)} \end{aligned}$$

- Second Condition: $d(x,y) = 0 \iff x = y$.

  If $x = y$, then:

$$\begin{aligned} d(x,y) &= \frac{|x-y|}{1+|x-y|} && \text{(by def. of } d(x,y)) \\ &= \frac{|x-x|}{1+|x-x|} && \text{(by } x = y) \\ d(x,y) &= 0 && \text{(simplifying)} \end{aligned}$$

  If $d(x,y) = 0$, then:

$$\begin{aligned} 0 &= \frac{|x-y|}{1+|x-y|} && \text{(by def. of } d(x,y)) \\ 0 &= |x-y| && \text{(multiplying by } 1+|x-y|) \\ 0 &= x-y && \text{(by def. of the absolute value)} \\ y &= x && \text{(solving for } y) \end{aligned}$$

- Third Condition: $d(x,y) = d(y,x)$

$$\begin{aligned} d(x,y) &= \frac{|x-y|}{1+|x-y|} && \text{(by def. of } d(x,y)) \\ &= \frac{|y-x|}{1+|y-x|} && \text{(by } |x-y| = |y-x|) \\ &= d(y,x) && \text{(by def. of } d(y,x)) \end{aligned}$$

- Fourth Condition: $d(x,z) \leq d(x,y) + d(y,z)$.

  A Lemma and a theorem will help:

  (a) Lemma (L1): If we have non-negative values a and b where $a \leq b$, then $\frac{a}{1+a} \leq \frac{b}{1+b}$

  $$\begin{aligned} &\text{Let } a \leq b \\ &\implies a + ab \leq b + ab && \text{(adding } ab \text{ to both sides)} \\ &\implies a(1+b) \leq b(1+a) && \text{(by the distributive law)} \\ &\implies \frac{a}{1+a} \leq \frac{b}{1+b} && \text{(dividing by } (1+b)(1+a)) \end{aligned}$$

  (b) Theorem (T1): $|a+b| \leq |a| + |b|$

  Showing the fourth condition holds:

  $$\begin{aligned} &|x-z| = |x-y+y-z| && \text{(adding zero)} \\ &= |(x-y)+(y-z)| && \text{(by associativity)} \\ &\leq |x-y| + |y-z| && \text{(by T1)} \\ &\implies \frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} && \text{(by L1)} \\ &= \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} && \text{(rearranging)} \\ &\leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} && \text{(by a smaller denominator)} \\ &\implies d(x,z) \leq d(x,y) + d(y,z) && \text{(by def. of } d(x,z)) \end{aligned}$$

8. **Consider the following vectors:**

$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \textbf{and} \qquad \mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Find the orthogonal projection of y onto u.

Recall that the idea is to decompose $\mathbf{y}$ into two orthogonal vectors, i.e., $\mathbf{y} = \alpha\mathbf{u} + \mathbf{z}$. As a result, we require $(\mathbf{y} - \alpha\mathbf{u})'\mathbf{u} = \mathbf{0}$, $\implies \alpha = \frac{\mathbf{y}'\mathbf{u}}{\mathbf{u}'\mathbf{u}}$. Thus:

$$\begin{aligned} \alpha &= \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}' \begin{bmatrix} -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}' \begin{bmatrix} -1 \\ 3 \end{bmatrix}} && \text{(using the formula)} \\ \alpha &= -\frac{2}{5} && \text{(simplifying)} \\ \hat{\mathbf{y}} &= \alpha\mathbf{u} && \text{(the projection)} \\ &= -\frac{2}{5}\begin{bmatrix} -1 \\ 3 \end{bmatrix} && \text{(plugging in values)} \\ \hat{\mathbf{y}} &= \begin{bmatrix} 2/5 \\ -6/5 \end{bmatrix} && \text{(simplifying)} \end{aligned}$$

9. **Let S be the span of a $n \times m$ matrix X, and let its orthogonal complement be $S^{\perp}$. Define $P_x = X(X^TX)^{-1}X^T$. For $X \in S$, $E \in S^{\perp}$, show each of the following holds:**

   (a) $(I_n - P_x)X = 0$

$$\begin{aligned}
(I_n - P_x)X &= \left(I_n - X(X^TX)^{-1}X^T\right)X && \text{(plugging in for } P_x\text{)} \\
&= I_nX - X(X^TX)^{-1}X^TX && \text{(expanding)} \\
&= I_nX - XI_n && \text{(by def. of the inverse)} \\
&= X - X && \text{(by def. of the identity matrix)} \\
(I_n - P_x)X &= \mathbf{0}
\end{aligned}$$

   (b) $(I_n - P_x)E = E$

$$\begin{aligned}
(I_n - P_x)E &= (I_n - X(X'X)^{-1}X')E && \text{(plugging in for } P_x\text{)} \\
&= I_nE - X(X'X)^{-1}X'E && \text{(expanding)} \\
&= I_nE - X(X'X)^{-1}\mathbf{0} && (X \perp E) \\
&= E - \mathbf{0} && \text{(by def. of the identity matrix)} \\
(I_n - P_x)E &= E
\end{aligned}$$

   (c) $P_xP_x = P_x$

$$\begin{aligned}
P_xP_x &= X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T && \text{(plugging in for } P_x\text{)} \\
&= X(X^TX)^{-1}I_mX^T && \text{(by def. of the inverse)} \\
&= X(X'X)^{-1}X^T && \text{(by def. of the identity matrix)} \\
P_xP_x &= P_x
\end{aligned}$$

   (d) $(I_n - P_x)P_x = 0$

$$\begin{aligned}
(I_n - P_x)P_x &= I_nP_x - P_xP_x && \text{(expanding)} \\
&= I_nP_x - P_x && \text{(by part (c))} \\
&= P_x - P_x && \text{(by def. of the identity matrix)} \\
(I_n - P_x)P_x &= \mathbf{0}
\end{aligned}$$

10. **People possess the blood types A, B, AB, or O. Further, each individual has a Rhesus factor (+) or does not (−). A medical technician is recording blood types and Rhesus factors. List the sample space for this "experiment."**

$$S = \left\{A^+,\, A^-,\, B^+,\, B^-,\, AB^+,\, AB^-,\, O^+,\, O^-\right\}$$

11. **An oil prospecting firm hits oil or gas on 10% of its drillings. If each drilling is an independent event (and we assume all potential wells produce with equal probability), what is the probability that the oil company will hit oil or gas?**

   (a) **on both drillings?**

   Because the events are independent:

$$\begin{aligned}\mathbb{P}(\text{hit}, \text{hit}) &= \mathbb{P}(\text{hit})\mathbb{P}(\text{hit}) && \text{(by independence)}\\ &= (0.1)(0.1) && \text{(plugging in values)}\\ &= 0.01 && \text{(simplifying)}\end{aligned}$$

   (b) **on the first drilling but not the second?**

$$\begin{aligned}\mathbb{P}(\text{hit}, \text{miss}) &= \mathbb{P}(\text{hit})\mathbb{P}(\text{miss}) && \text{(by independence)}\\ &= (0.1)(0.9) && \text{(plugging in values)}\\ &= 0.09 && \text{(simplifying)}\end{aligned}$$

   (c) **on at least one of the drillings?**

   This is the complement of missing on both drillings:

$$\begin{aligned}\mathbb{P}(\text{at least one}) &= 1 - \mathbb{P}(\text{miss}, \text{miss}) && \text{(by the prob. func. theorems)}\\ &= 1 - \mathbb{P}(\text{miss})\mathbb{P}(\text{miss}) && \text{(by independence)}\\ &= 1 - (0.9)(0.9) && \text{(plugging in values)}\\ &= 0.19 && \text{(simplifying)}\end{aligned}$$

12. **If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint? Explain.**

   A and B cannot be disjoint. Given that $P(B^c) = 1/4$, it must be that $P(B) = 3/4$. If A and B are disjoint, then by part (iii) of the definition of the probability function, $P(A \cup B) = P(A) + P(B)$. With the values given here, $P(A) + P(B) = 1/3 + 3/4 = 13/12$, which is greater than one.

13. **How many different sets of initials can be formed (using the English alphabet) if every person has one surname (last name) and**

   (a) **Exactly two given names (first names)?**

   If there are two first names and one last name, each set of initials has three letters. Further, order matters and draws are made with replacement. If we're using the English alphabet:

$$N = 26 \times 26 \times 26 = 17,576$$

   There are 26 ways to pick the first letter, 26 ways to pick the second, and 26 ways to pick the third.

   (b) **Either one or two given names?**

   Again, draws are made where order matters, with replacement. If there are two names (first and last):

$$N_2 = 26 \times 26$$

If there are three names (first, first, last):

$$N_3 = 26 \times 26 \times 26$$

Thus, in total

$$N = 26^2 + 26^3 = 18,252$$

(c) **Either one or two or three given names?**

Following the same logic as above:

$$N = 26^4 + 26^3 + 26^2 = 475,228$$

14. **If two events A and B are such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, find the following:**

(a) $P(A|B)$

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} && \text{(by def. of cond. prob.)} \\ &= \frac{0.1}{0.3} && \text{(plugging in values)} \\ &= \frac{1}{3} && \text{(simplifying)} \end{aligned}$$

(b) $P(B|A)$

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} && \text{(by def. of cond. prob.)} \\ &= \frac{0.1}{0.5} && \text{(plugging in values)} \\ &= \frac{1}{5} && \text{(simplifying)} \end{aligned}$$

(c) $P(A|A \cup B)$

$$\begin{aligned} P(A|A \cup B) &= \frac{P\big(A \cap (A \cup B)\big)}{P(A \cup B)} && \text{(by def. of cond. prob.)} \\ &= \frac{P(A)}{P(A) + P(B) - P(A \cap B)} && (A \subset (A \cup B)) \\ &= \frac{0.5}{0.5 + 0.3 - 0.1} && \text{(plugging in values)} \\ &= \frac{5}{7} && \text{(simplifying)} \end{aligned}$$

(d) $P(A|A \cap B)$

$$\begin{aligned} P(A|A \cap B) &= \frac{P\big(A \cap (A \cap B)\big)}{P(A \cap B)} && \text{(by def. of cond. prob.)} \\ &= \frac{P(A \cap B)}{P(A \cap B)} && ((A \cap B) \subset A) \\ &= 1 && \text{(simplifying)} \end{aligned}$$

(e) $P(A \cap B | A \cup B)$

$$\begin{aligned}
P(A \cap B | A \cup B) &= \frac{P\big((A \cap B) \cap (A \cup B)\big)}{P(A \cup B)} && \text{(by def. of cond. prob.)} \\
&= \frac{P(A \cap B)}{P(A) + P(B) - P(A \cap B)} && ((A \cap B) \subset (A \cup B)) \\
&= \frac{0.1}{0.5 + 0.3 - 0.1} && \text{(plugging in values)} \\
&= \frac{1}{7} && \text{(simplifying)}
\end{aligned}$$

15. **Let Y be a random variable with $p(y)$ given in the table below:**

y	**1**	**2**	**3**	**4**
$p(y)$	**0.4**	**0.3**	**0.2**	**0.1**

Give the cumulative distribution function $F_Y(y)$. Be sure to specify the value of $F_Y(y)$ for all $y \in (-\infty, \infty)$.

Recall that the cumulative probability function gives $P(Y \leq y)$. Then the CDF is

$$F_Y(y) = \begin{cases} 0 & \text{if } -\infty < y < 1 \\ 0.4 & \text{if } 1 \leq y < 2 \\ 0.7 & \text{if } 2 \leq y < 3 \\ 0.9 & \text{if } 3 \leq y < 4 \\ 1 & \text{if } 4 \leq y < \infty \end{cases}$$