Math Camp 2017 - Logic*

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1. **Propositions and Connectives**

   (a) Definition. A **proposition** is a statement that has exactly one truth value: either true (denoted T) or false (denoted F)

   (b) Examples.

      i. "$1 + 1 = 2$"

      ii. "The UCSB Math Camp Instructor for 2017 is named James"

      iii. "James won a Nobel Prize in economics in 2016"

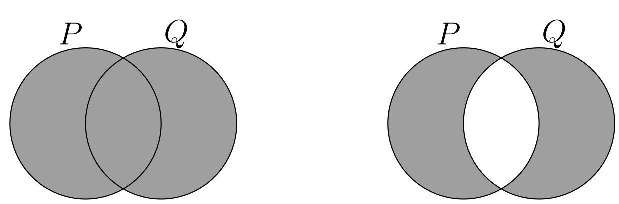
      iv. "$x^2 = 36$" (not a proposition)

   (c) Definition. The **negation** of the proposition P, denoted $\neg P$, is the proposition "not P." $\neg P$ is true when P is false.

   (d) Definition. Given propositions P and Q, the **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true when both P is true and Q is true.

   (e) Definition. Given propositions P and Q, the **disjunction** of P and Q, denoted $P \vee Q$ is the proposition "P or Q." $P \vee Q$ is true when P is true or Q is true.

   (f) Aside. This is known as the "inclusive or," e.g., "P or Q or both P and Q." There is also an "exclusive or," e.g., "P or Q but not both P and Q." For the purposes of the first-year sequences, you'll probably only have to deal with the "inclusive or." Presented as Venn Diagrams:



Inclusive Or: $P \vee Q$ Exclusive Or: $P \veebar Q$

   (g) Example. For the different combinations of truth values for the propositions P and Q, we can determine the truth values of our various more complicated compound propositions:

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P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	F

(h) Aside. This is known as a **truth table**. It is a helpful tool that lets us organize the truth value of propositions and connectives. You won't use them during the first year coursework, but it helps us get a sense for the form of logical statements and proofs.

(i) Definition. A **tautology** is a propositional form that is true for every assignment of truth values to its components.

(j) Example. Consider the propositional form $P \vee \neg P$:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(k) Definition. A **contradiction** is a propositional form that is false for every assignment of truth values to its components.

(l) Example. Consider the propositional form $P \wedge \neg P$:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(m) Aside. While these are trivial examples of tautologies and contradictions, they are important for proofs. "If and only if" statements are tautologies (which we frequently are trying to prove), while contradictions are a powerful tool for proving certain propositions.

(n) Definition. Two propositional forms are **equivalent** if they have the same truth tables.

(o) Example. P and $\neg(\neg P)$ are equivalent (draw a truth table for convincing!).

(p) Theorem (SES THM 1.1.1). The following propositional forms are equivalent:

i. Double Negation:
- P and $\neg(\neg P)$

ii. Commutative Laws
- $P \vee Q$ and $Q \vee P$
- $P \wedge Q$ and $Q \wedge P$

iii. Associative Laws
- $P \vee (Q \vee R)$ and $(P \vee Q) \vee R$
- $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$

iv. Distributive Laws
- $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$

v. DeMorgan's Laws
   - $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$
   - $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$

(q) Aside. These are important in formulating proofs, particularly in first quarter microeconomics. For example, imagine we have the statement "preferences are complete and transitive" is false. This is equivalent to saying "preferences are not complete or preferences are not transitive." Moving between equivalent propositional forms is important, because some forms are to write

2. **Conditionals and Biconditionals**

(a) Definition. For propositions P and Q, the **conditional sentence** $P \implies Q$ is the proposition "if P, then Q." The conditional sentence $P \implies Q$ is true exactly when P is false or Q is true. The truth table associated with $P \implies Q$ is:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Example. We can think of $P \implies Q$ in terms of a promise. For example, consider the promise, IF I draw [*name*]'s name, THEN I will pay [*name*] \$1. Then $P \equiv$ "I draw [*name*]'s name" and $Q \equiv$ "I pay [*name*] \$1." Suppose I draw [*name*]:

i. P is True

ii. If I pay, then Q is True
   - I didn't break the promise ($P \implies Q$ is True)

iii. If I don't pay, then Q is False
   - I broke the promise ($P \implies Q$ is False)

On the other hand, suppose I DON'T draw [*name*]:

i. P is False

ii. If I pay, then Q is True
   - I didn't break the promise ($P \implies Q$ is True)

iii. If I don't pay, then Q is False
   - I didn't break the promise ($P \implies Q$ is True)

(c) Aside. The conditional statement is the most important propositional form in mathematics and economics. Every "if then" and "implies" statement is a conditional sentence. "IF the minimum wage is binding, THEN there will be involuntary unemployment." "The existence of a utility function IMPLIES complete, transitive preferences."

Note that in common English $P \implies Q$ contains casual connotations, but this is not the case in mathematics. The proposition

$$\text{Michael Jordan was the first man on the Moon} \implies \Big(1+1=2\Big)$$

Is true in the mathematical sense, even though the proposition is nonsense (from line three on the truth table). Even though this may technically be true, however, during the first year we focus on the first two lines.

(d) Definition. For propositions P and Q, the **converse** of $P \implies Q$ is $Q \implies P$. The **contrapositive** of $P \implies Q$ is $\neg Q \implies \neg P$.

(e) Theorem. (SES THM 1.2.1) The proposition $P \implies Q$ is equivalent to its contrapositive $\neg Q \implies \neg P$. It is *not* equivalent to its converse $Q \implies P$.

Expanding the truth table for the conditional $P \implies Q$:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$Q \Rightarrow P$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

(f) Aside. This is an incredibly useful theorem, despite the fact that it may not be intuitive at first glance. As mentioned above, moving between equivalent propositional forms can be helpful when we're writing proofs; few (if any) are as frequently used as the contrapositive.

(g) Definition. For propositions P and Q, the **biconditional sentence** $P \iff Q$ is the proposition "P if and only if Q." $P \iff Q$ is true exactly when P and Q have the same truth values.

The truth table associated with $P \iff Q$ is:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

(h) Aside. Note that we frequently write "if and only if" as iff. Note that all biconditionals are tautologies; further, all (properly stated) definitions are biconditionals, as they lay out the exact conditions to meet the definition.

(i) Theorem (SES THM 1.2.2) For propositions P, Q, and R:

i. $P \implies Q$ is equivalent to $(\neg P) \vee Q$

ii. $P \iff Q$ is equivalent to $(P \implies Q) \wedge (Q \implies P)$

iii. $\neg(P \implies Q)$ is equivalent to $P \wedge (\neg Q)$

iv. $\neg(P \wedge Q)$ is equivalent to $(P \implies \neg Q)$ and to $(Q \implies \neg P)$

v. $P \implies (Q \implies R)$ is equivalent to $(P \wedge Q) \implies R)$

vi. $P \implies (Q \wedge R)$ is equivalent to $(P \implies Q) \wedge (P \implies R)$

vii. $(P \vee Q) \implies R$ is equivalent to $(P \implies R) \wedge (Q \implies R)$

viii. $P \implies (Q \vee R)$ is equivalent to $(P \wedge \neg R) \implies Q$ and to $(P \wedge \neg Q) \implies R$

3. **Quantifiers**

(a) Definition. A statement that contains a variable is called an **open sentence**. It becomes a proposition only when its variables are assigned specific values.

(b) Example.

i. $P(x) \equiv x \leq 10$

ii. $Q(Y) \equiv Y$ is wearing glasses

(c) Definition. Given an open sentence, the collection of permissible objects available for consideration is the **universe of discourse**. The collection of permissible objects that make an open sentence a true proposition is the **truth set**.

(d) Example.

i. For $P(x)$: if the universe of discourse $\mathbb{R}$, then the truth set is $(-\infty, 10]$.

ii. For $Q(Y)$: if the universe of discourse is the room, the set of people wearing glasses is the truth set.

(e) Aside. In logic and mathematics, specifying what universe you're considering can be quite important. In particular, you must be extremely careful in Econ 241A–you'll be docked points if you don't specify the relevant universes (known as supports; we'll talk more on this later) for distributions.

(f) Definition. For an open sentence $P(x)$, the sentence $\exists\ x \ni P(x)$ reads "there exists an x such that $P(x)$." The symbol $\exists$ is called the **existential qualifier**.

(g) Definition. For an open sentence $Q(x)$, the sentence $\forall\ x, P(x)$ reads "for all x, $P(x)$." The symbol $\forall$ is called the **universal qualifier**.

(h) Example. If $\mathbb{R}$ is the universe of discourse,

i. $\exists\ x \ni x + 3 = 0$

ii. $\forall x, x < x + 1$

(i) Aside. Essentially, $\exists$ and $\forall$ take open sentences and make them propositions. $\exists$ means that there is at least one value that makes an open sentence true; $\forall$ says the open sentence is true for every value. You will use these quantifiers a lot during the first year classes, so make sure you're comfortable with them.

(j) Theorem (SES THM 1.3.1). If $A(x)$ is an open sentence with variable x, then

i. $\neg(\forall\ x, A(x))$ is equivalent to $\exists\ x \ni \neg A(x)$

ii. $\neg(\exists\ x \ni A(x))$ is equivalent to $\forall\ x, \neg A(x)$

(k) Example.

- Our universe of discourse is "all social scientists"
- $Q(y) \equiv$ "bad at math"

Consider the statement:

$$\neg\Big[\forall y, \quad Q(y)\Big]$$

This statement reads, "not, for all social scientists y, y is bad at math," or, in plain English, "it is not true that all social scientists are bad at math."

What would it take for the statement "all social scientists are bad at math" to be false? At least one that is good at math! In other words: "there exists a social scientist who is good at math."

$$\exists\ y \ni Q(y)$$

(l) Example. Suppose you have an exam question: "Weakly dominated strategies cannot be part of Nash Equilibria. True or False. Provide a proof."

Knowing that $\forall$ and $\exists$ are the negations of each other is extremely important for proofs. In this example, the statement is false. Further, All that is required of the proof is to provide ONE example where a weakly dominated strategy is part of a NE.

(m) Aside. This gets at an important point: to disprove a "for all" proposition, we just need one counterexample to prove it's false. This type of question is very common for the first-quarter micro sequence!

(n) Example. Let $\mathbb{N}$ be the "natural numbers" (i.e., 1, 2, 3, ...). Find the negation of the proposition

$$\begin{aligned}
&\exists x \in \mathbb{N} \ni x < 2 \land x \neq 1 && \text{(a false propostion)}\\
&\neg\,(\exists\, x \in \mathbb{N} \ni x < 2 \land x \neq 1) && \text{(the negation)}\\
&\forall\, x \in \mathbb{N}, \neg(x < 2 \land x \neq 1) && \text{(by SES THM 1.3.1)}\\
&\forall\, x \in \mathbb{N}, x \geq 2 \lor x = 1 && \text{(by DeMorgan's Laws)}
\end{aligned}$$

This is a true proposition!

4. **Basics of Writing Proofs**.

(a) Definition. Initial sets of statements assumed to be true are called **axioms**.

(b) Outline of Proof Writing.

i. List definitions, axioms, previously proved results/theorems, or assumptions (be careful about assumptions).

ii. At any time, replace statements with equivalent statements

iii. At any time, state tautologies

(c) Form of Direct Proofs. Suppose we're trying to prove that $P \implies Q$. direct proofs frequently look something like:

- List relevant definitions, axioms, theorems, assumptions, etc.
- State "Direct proof to show Q" (or equivalent statement)
- Proof:

$$\begin{aligned}
&\text{Let } P \text{ be true} && \text{(by hypothesis)}\\
&\text{Then } \textit{replace} && \text{(by results/theorems)}\\
&\text{Consider } \textit{tautology} && \text{(by tautology rule)}\\
&\vdots &&\\
&\text{Then } Q &&
\end{aligned}$$

Thus, $P \implies Q$.

(d) Example. Let x be an integer. Prove that if x is odd, then $x + 1$ is even.

- $\mathbb{Z}$ is the set of integers
- Def. of even: $y \in \mathbb{Z}$ is even $\iff \exists\, k \in \mathbb{Z} \ni y = 2k$
- Def. of odd: $x \in \mathbb{Z}$ is odd $\iff \exists\, j \in \mathbb{Z} \ni x = 2j + 1$
- Closure property: the sum of two integers is an integer
- Successor property: If $x \in \mathbb{Z}$, x has a unique successor $x + 1$

Direct proof to show: $x+1$ is even.
Proof:

$$\begin{aligned}
&\text{Let } x \text{ be an odd integer} && \text{(by hypothesis)}\\
&\implies \exists\, k \in \mathbb{Z} \ni x = 2k+1 && \text{(by def. of odd)}\\
&\implies x+1 = (2k+1)+1 && \text{(by succession/closure)}\\
&\implies x+1 = 2k+2 && \text{(by associativity)}\\
&\implies x+1 = 2(k+1) && \text{(by distributivity)}\\
&(k+1) \text{is an integer} && \text{(by closure)}\\
&\implies x+1 \text{ is even} && \text{(by def. of even)}
\end{aligned}$$

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(e) Form of Proof by Controposition. Suppose we're trying to prove that $P \implies Q$. We can prove the contrapositive instead, as it is an equivalent statement:

- List relevant definitions, axioms, theorems, assumptions, etc.
- State the contrapositive
- State "Proof by contraposition to show $\neg P$"
- Proof:

$$\begin{aligned}
&\text{Let } \neg Q \text{ be true} && \text{(by hypothesis)}\\
&\text{Then } \textit{replace} && \text{(by results/theorems)}\\
&\text{Consider } \textit{tautology} && \text{(by tautology rule)}\\
&\vdots\\
&\text{Then } \neg P
\end{aligned}$$

Thus, $P \implies Q$.

(f) Example. Let m be an integer. Prove that if m^2 is even, then m is even.

- Contrapositive: if m is not even, then m^2 is not even
- Assumption: An integer is odd if and only if it is not even

Proof by contraposition to show: m^2 is not even
Proof:

$$\begin{aligned}
&\text{Let } m \text{ be a not even integer} && \text{(by hypothesis)}\\
&\implies m \text{ is odd} && \text{(by assumption)}\\
&\implies \exists\, k \in \mathbb{Z} \ni m = 2k+1 && \text{(by def. of odd)}\\
&\implies m^2 = (2k+1)^2 && \text{(squaring both sides)}\\
&\implies m^2 = 4k^2+4k+1 && \text{(expanding)}\\
&\implies m^2 = 2(2k^2+2k)+1 && \text{(rearranging)}\\
&2k^2+2k \text{ is an integer} && \text{(by closure)}\\
&\implies m^2 \text{ is odd} && \text{(by def. of odd)}\\
&\implies m^2 \text{ is not even} && \text{(by assumption)}
\end{aligned}$$

■

Thus, we have proved the contrapositive of "if m^2 is even, then m is even." Since the contrapositive is equivalent to the original statement, this is a sufficient proof.

(g) Aside. Note a few points: first, we assumed that an integer is even if and only if it isn't odd. We could have proved this, but you always need to take some things as given. Second, we assumed that the usual rules of algebra held (I didn't explictitly write down all the steps and associated assumptions).

When you're actually writing proofs during the first year, you will end up assuming quite a few things for convenience–just be careful you don't assume away the proof! Typically after the first week of so, you won't need to state things that seem obvious to us, like the rules of algebra (associativity, etc.) or that $0 \cdot a = 0$. Again, DO NOT make assumptions that make the proof trivially easy!

(h) Form of Proof by Contradiction. Suppose we're trying to prove a proposition P. A proof by contradiction essentially relies on the fact that if $\neg P$ must always be false, then P must be true:

- List relevant definitions, axioms, theorems, assumptions, etc.
- State "Proof by contradiction to show $\neg P \implies (Q \wedge \neg Q)$"
- Proof:

Suppose $\neg Q$ is true	(towards a contradiction)
Then *replace*	(by results/theorems)
Consider *tautology*	(by tautology rule)
$\vdots$	
Then Q	
$\vdots$	
Then $\neg Q$	
Thus, a contradiction	

Therefore, P must be true.

(i) Example. Prove that $\sqrt{2}$ is an irrational number.

- Def. of coprime: p and q are coprime if they have no common factors other than 1
- Def. of rational: If x is rational, then $\exists\ p, q \in \mathbb{Z} \ni \big(q \neq 0\big) \wedge \left(x = \frac{p}{q}\right) \wedge$ (p and q are coprime)

Proof by contradiction to show: if $\sqrt{2}$ is rational, then it has a fractional form p/q where p and q are coprime and not coprime.

Proof.

Suppose $\sqrt{2}$ is rational	(towards a contradiction)
$\implies \exists\ p, q \in \mathbb{Z} \ni (q \neq 0) \wedge (\sqrt{2} = p/q) \wedge (p \text{ and } q \text{ are coprime})$	(by def. of rational)
$\implies 2 = \dfrac{p^2}{q^2}$	(squaring both sides)
$\implies 2q^2 = p^2$	(multiplying by q^2)
$\implies p^2$ is an even integer	(by closure & def. of even)
$\implies p$ is even	(by previous proof)
$\implies \exists\ k \in \mathbb{Z} \ni p = 2k$	(by def. of even)
$\implies 2q^2 = (2k)^2$	(substituting $p = 2k$)
$\implies 2q^2 = 4k^2$	(squaring the RHS)

$\implies q^2 = 2k^2$ (dividing by 2)
$\implies q^2$ is even (by def. of even)
$\implies q$ is even (by previous proof)
$\implies \exists\, j \in \mathbb{Z} \ni q = 2j$ (by def. of even)
$\implies p$ and q have a common factor of 2
$\implies p$ and q are not coprime (by def. of coprime)
Thus, a contradiction ■

Therefore, $\sqrt{2}$ must be irrational.

5. **Applications**

   (a) Biconditional Proofs. Suppose we're trying to prove $P \iff Q$. There are two ways to approach the problem: in two parts, or biconditionally.

      i. Two-Part Proof (from SES THM 1.2.2)
         - Show $P \implies Q$
         - Show $Q \implies P$
         - Thus, $P \iff Q$

      ii. Biconditional Proof
         - Every line must involve a biconditional:

$$\begin{array}{ll} \text{Let } P \text{ be true} & \text{(by hypothesis)} \\ \iff R_1 & \text{(by results/theorems/definitions)} \\ \vdots & \\ \iff R_n & \text{(by results/theorems/definitions)} \\ \iff Q & \end{array}$$

   (b) Proofs with Quantifiers Suppose we're trying to prove a "for all" or a "there exists" proposition $P(x)$. Depending on the quantifier, there are two basic approaches:

      i. For-All Proofs
         - Define your universe X
         - Pick and arbitrary element in that universe, $x \in X$
         - Show that x makes $P(x)$ true
         - Because x was arbitrary, it's true for all $x \in X$

      ii. There-Exists Proofs
         - Define your universe X
         - Pick a specific element in that universe, $x \in X$
         - Show that x makes $P(x)$ true