Required Problems

1. **Using truth tables, prove both of DeMorgan's Laws for logical connectives.**

   (a) $\neg(P \wedge Q)$ **is logically equivalent to** $\neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$(P \wedge Q)$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

   The last two columns, for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ respectively, have the same truth values for all truth assignments of P and Q; thus, they are logically equivalent.

   (b) $\neg(P \vee Q)$ **is logically equivalent to** $\neg P \wedge \neg Q$

P	Q	$\neg P$	$\neg Q$	$(P \vee Q)$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

   Once again, the last two columns, for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ respectively, have the same truth values for all truth assignments of P and Q; thus, they are logically equivalent.

2. **Find the contrapositive and converse of each of the following statements:**

   (a) **"If squares have four sides, then triangles have four sides."**

   This has the form: (squares have four sides) $\Longrightarrow$ (trianges have four sides).

   - Contrapositive: "If triangles do not have four sides, then squares do not have four sides."
   - Converse: "If triangles have four sides, then squares have four sides."

   (b) **"A sequence a is bounded whenever a is convergent."**

   This has the form: (a is convergent) $\Longrightarrow$ (a is bounded).

   - Contrapositive: "If a is not bounded, then a is not convergent."
   - Converse: "If a is bounded, then a is convergent."

   (c) **"The differentiability of a function f is sufficient for f to be continuous.**

   This has the form: (f is differentiable) $\Longrightarrow$ (f is continuous)

   - Contrapositive: "If f is not continuous, then f is not differentiable."
   - Converse: "If f is continuous, then f is differentiable."

3. **Let x and y be integers. Prove that if x and y are even, then $x + y$ is even.**

   - $\mathbb{Z}$ is the set of integers
   - Def. of even: $x \in \mathbb{Z}$ is even iff $\exists\, k \in \mathbb{Z} \ni x = 2k$

   <u>To show</u>: There exists an integer h such that $x + y = 2h$

Proof:

$$
\begin{aligned}
&\text{Let } x, y \in \mathbb{Z} \text{ be even} && \text{(by hypothesis)}\\
&\implies \Big(\exists\, k \in \mathbb{Z} \ni x = 2k\Big) \wedge \Big(\exists\, j \in \mathbb{Z} \ni y = 2j\Big) && \text{(by def. of even)}\\
&\implies x + y = 2k + 2j && \text{(summing)}\\
&\implies x + y = 2(k + j) && \text{(by distributivity)}\\
&k + j \in \mathbb{Z} && \text{(by closure)}\\
&\text{Let } h = k + j && \text{(defining an integer } h\text{)}\\
&\implies \exists\, h \in \mathbb{Z} \ni x + y = 2h && \text{(substituting for } j + k\text{)}\\
&\implies x + y \text{ is even} && \text{(by def. of even)}
\end{aligned}
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4. **Let A and B be sets. Prove that $A \subset B$ if and only if $A - B = \emptyset$.**

- Statement using biconditional: $(A \subset B) \iff (A - B = \emptyset)$
- Def. of subset: $A \subset B \iff \big(x \in A \implies x \in B\big)$
- Def. of set difference: $A - B = \{x | x \in A \wedge x \notin B\}$
- Theorem (T1): $(P \implies Q)$ is logically equivalent to $(\neg P) \vee Q$
- Theorem (T2): For all sets A, $\emptyset \subset A$
- Contrapositive of $(\Rightarrow)$: $A - B \neq \emptyset \implies A \not\subset B$

Proof by contraposition, to show: $A \not\subset B$.
Proof $(\Rightarrow)$:

$$
\begin{aligned}
&\text{Let } A - B \neq \emptyset && \text{(by hypothesis)}\\
&\implies \exists\, x \in A - B && \text{(by def. of non-empty)}\\
&\implies x \in A \wedge x \notin B && \text{(by def. of set difference)}\\
&\implies \neg\big(x \notin A \vee x \in B\big) && \text{(by negation)}\\
&\implies \neg\big(x \in A \implies \in B\big) && \text{(by T1)}\\
&\implies \neg(A \subset B) && \text{(by def. of subset)}\\
&\implies A \not\subset B && \text{(by negation)}
\end{aligned}
$$

To show: $A \subset B$.
Proof $(\Leftarrow)$:

$$
\begin{aligned}
&\text{Let } A - B = \emptyset && \text{(by hypothesis)}\\
&\text{Case 1: } A = \emptyset && \text{(by hypothesis)}\\
&\implies A \subset B && \text{(by T2)}\\
&\text{Case 2: } A \neq \emptyset && \text{(by hypothesis)}\\
&\implies \exists\, x \in A && \text{(by def. of non-empty)}\\
&\implies x \notin A - B && \text{(by def. of empty)}\\
&\implies \neg(x \in A \wedge x \notin B) && \text{(by def. of set difference)}\\
&\implies x \notin A \vee x \in B && \text{(by negation)}\\
&\implies (x \in A \implies x \in B) && \text{(by T1)}\\
&\implies A \subset B && \text{(by def. of subset)}
\end{aligned}
$$

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Practice Problems

5. **If P, Q, and R are true while S and T are false, which of the following are true?**

   (a) $Q \wedge (R \wedge S)$

Q	R	S	$(R \wedge S)$	$Q \wedge (R \wedge S)$
T	T	F	F	F

   Thus, $Q \wedge (R \wedge S)$ is false.

   (b) $(P \vee Q) \wedge (R \vee S)$

P	Q	R	S	$(P \vee Q)$	$(R \vee S)$	$(P \vee Q) \wedge (R \vee S)$
T	T	T	F	T	T	T

   Thus, $(P \vee Q) \wedge (R \vee S)$ is true.

   (c) $(P \vee S) \wedge (P \vee T)$

P	S	T	$(P \vee S)$	$(P \vee T)$	$(P \vee Q) \wedge (P \vee T)$
T	F	F	T	T	T

   Thus, $(P \vee Q) \wedge (P \vee T)$ is true.

6. **Make truth tables for these propositional forms:**

   (a) $P \implies (Q \wedge P)$

P	Q	$Q \wedge P$	$P \implies (Q \wedge P)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

   (b) $(\neg P \implies Q) \vee (Q \implies P)$

P	Q	$\neg P$	$\neg P \implies Q$	$Q \implies P$	$(\neg P \implies Q) \vee (Q \implies P)$
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	T
F	F	T	F	T	T

   (c) $\neg Q \implies (Q \iff P)$

P	Q	$\neg Q$	$Q \iff P$	$\neg Q \implies (Q \iff P)$
T	T	F	T	T
T	F	T	F	F
F	T	F	F	T
F	F	T	T	T

7. **Rewrite each of the following sentences to be symbolic sentences using logical connectives.**

   (a) **If $x = 1$ or $x = -1$, then $|x| = 1$.**

$$\big(x = 1 \vee x = -1\big) \implies |x| = 1$$

(b) **B is invertible is a necessary and sufficient condition for $|B| \neq 0$.**

$$\exists\, B^{-1} \iff |B| \neq 0$$

(c) **$6 \geq n - 3$ only if $n > 4$ or $n > 10$.**

$$\neg\big(n > 4 \vee n > 10\big) \implies \neg\big(6 \geq n - 3\big)$$

(d) **S is compact iff S is closed and bounded.**

$$(S \text{ is compact}) \iff (S \text{ is closed and bounded})$$

8. **Rewrite each of the following sentences to be symbolic sentences using quantifiers. The universe of discourse for each is given in paratheses.**

(a) **Every nonzero real number is positive or negative. (Real Numbers)**

$$\forall\, x \in \mathbb{R}, (x \geq 0 \wedge x \leq 0)$$

(b) **Every integer is greater than some integer. (Integers)**

$$\forall\, x \in \mathbb{Z}\; \exists\, y \ni \mathbb{Z} \ni x > y$$

(c) **There is a smallest positive real number. (Real Numbers)**

$$\exists\, x \in \mathbb{R} \ni x \geq 0 \wedge \forall\, y \in \mathbb{R} \ni y \geq 0, x \leq y$$

9. **The qualifier $\exists!$ is defined as follows:**

$$\exists!\; x \ni A(x) \iff \big(\exists x \ni A(x)\big) \wedge \big(\forall y \wedge \forall z, A(y) \wedge A(z) \implies y = z\big)$$

Describe in plain english the qualifer $\exists!$.

The qualifier $\exists!$ states that there exists an element making $A(x)$ true and if $A(x)$ is true for two values, those values must be the same. in other words, the qualifier $\exists!$ reads "there exsits a uniqe," so it is the unique existential qualifier.

10. **Let x and y be integers. Prove the following propositions:**

(a) **If x and y are even, then xy is even.**

<u>To show</u>: there exists an integer h such that $xy = 2h$.
<u>Proof</u>:

Let x and y be even integers	(by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j\Big)$	(by def. of even)
$\implies xy = (2j)(2k)$	(multiplying)
$\implies xy = 2(2jk)$	(by associativity)
$2jk \in \mathbb{Z}$	(by closure)
Let $h = 2jk$	(defining an integer h)
$\implies xy = 2h$	(substituting for $2jk$)
$\implies xy$ is even	(by def. of even)

$\blacksquare$

(b) **If x and y are odd, then $x + y$ is even.**

To show: there exists an integer h such that $x + y = 2h$
Proof:

Let x and y be odd integers	(by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k + 1\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j + 1\Big)$	(by def. of odd)
$\implies x + y = (2k + 1) + (2j + 1)$	(summing)
$\implies x + y = 2k + 2j + 2$	(by associativity/commutativity)
$\implies x + y = 2(k + j + 1)$	(by distributivity)
$k + j + 1 \in \mathbb{Z}$	(by closure)
Let $h = k + j + 1$	(defining an integer h)
$\implies x + y = 2h$	(substituting for $k + j + 1$)
$\implies x + y$ is even	(by def. of even)

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(c) **If x is even and y is odd, then $x + y$ is odd.**

To show: there exists an integer h such that $x + y = 2h + 1$
Proof:

Let x be an even integer and let y be an odd integer	(by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j + 1\Big)$	(by def. of even and odd)
$\implies x + y = 2k + 2j + 1$	(summing)
$\implies x + y = 2(k + j) + 1$	(by distributivity)
$k + j \in \mathbb{Z}$	(by closure)
Let $h = k + j$	(defining an integer h)
$\implies x + y = 2h + 1$	(substituting for $k + j$)
$\implies x + y$ is odd	(by def. of odd)

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11. **Let a and b be real numbers. Prove that $|a + b| \leq |a| + |b|$.**

- Lemma (L1): if a is a real number, then $-|a| \leq a \leq |a|$
- Lemma (L2): $|b| \leq c \iff -c \leq b \leq c$

To show: $|a + b| \leq |a| + |b|$
Proof:

Let a and b be real numbers	(by hypothesis)
$\implies \Big(-|a| \leq a \leq |a|\Big) \wedge \Big(-|b| \leq b \leq |b|\Big)$	(by L1)
$\implies -|a| - |b| \leq a + b \leq |a| + |b|$	(summing)
$\implies |a + b| \leq |a| + |b|$	(by L2)

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12. **Let x be an integer. Write a proof by contraposition to show that if x is even, then $x + 1$ is odd.**

- Contrapositive: if $x + 1$ is not odd, then x is not even

- Theorem (T1): x is not even if and only if x is odd.

Proof by contraposition to show: x is not even
Proof:

Let x be an integer such that $x+1$ is not odd	(by hypothesis)
$\implies x+1$ is even	(by T1)
$\implies \exists\, k \in \mathbb{Z} \ni x+1 = 2k$	(by def. of even)
$\implies x = 2k-1$	(subtracting 1 from both sides)
$\implies x = 2k-2+1$	(rearranging the r.h.s)
$\implies x = 2(k-1)+1$	(by distributivity)
$k-1$ is an integer	(by closure)
let $j = k-1$	(defining an integer j)
$\implies x = 2j+1$	(subsituting for $k-1$)
$\implies x$ is odd	(by def. of odd)
$\implies x$ is not even	(by T1)

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13. **Suppose a and b are positive integers. Write a proof by contradiction to show that if ab is odd, the both a and b are odd.**

- Theorem (T1): x is not even if and only if x is odd.

Proof by contradiction to show: if a or b is not odd, then ab is odd and not odd.
Proof:

Let $ab \in \mathbb{Z}_+ \ni ab$ is odd and a or b is not odd	(towards a contradiction)
$\implies a$ is even or b is even	(by T1)
Case 1: a is even	
$\implies \exists\, k \in \mathbb{Z} \ni a = 2k$	(by def. of even)
$\implies ab = (2k)b$	(substituting for a)
$\implies ab = 2(kb)$	(by associativity)
kb is an integer	(by closure)
$\implies ab$ is even	(by def. of even)
$\implies ab$is not odd	
Thus, a contradiction	
Case 2: b is even	
(similar to Case 1)	
Case 3: both a and b are even	
$\implies ab$ is even	(by proof in problem 10(a))
$\implies ab$ is not odd	(by T1)
Thus, a contradiction	

Because every case produces the contradiction ab is odd and not odd, it must be the case that a and b are both odd. ■

14. **Prove that if $x \notin B$ and $A \subset B$, then $x \notin A$.**

To show: if $x \notin B$, then $x \notin A$:
Proof:

Let $A \subset B$	(by hypothesis)
$\implies (x \in A \implies x \in B)$	(by def. of subset)
$\implies [(\neg x \in B) \implies (\neg x \in A)]$	(the contrapositive)
$\implies [x \notin B \implies x \notin A]$	(by negation)

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15. **Let $A = \{1,3,5,7,9\}$, $B = \{0,2,4,6,8\}$, and $C = \{1,2,4,5,7,8\}$ and $D = \{1,2,3,5,6,7,8,9,10\}$. Find the following:**

(a) $A \cup B$

$$A \cup B = \{0,1,2,3,4,5,6,7,8,9\}$$

(b) $A - B$

$$A - B = \{1,3,5,7,9\}$$

(c) $(A \cap C) \cap D$

$$(A \cap C) \cap D = \{1,5,7\}$$

(d) $A \cup (C \cap D)$

$$A \cup (C \cap D) = \{1,2,3,5,7,8,9\}$$

16. **Let A, B, C, and D be sets prove that if $C \subset A$ and $D \subset B$ and A and B are disjoint, then C and D are disjoint.**

- Def. of disjoint: A and B are disjoint $\iff A \cap B = \emptyset$
- Contrapositive: C and D are not disjoint implies A and B are not disjoint

Proof by contraposition to show: A and B are not disjoint
Proof:

Let $C \subset A$ and $D \subset B$ such that C and D are not disjoint	(by hypothesis)
$\implies C \cap D \neq \emptyset$	(by def. of disjoint)
$\implies \exists\, x \in C \cap D$	(by def. of non-empty)
$\implies x \in C \wedge x \in D$	(by def. of $\cap$)
$\implies x \in A \wedge x \in B$	(by def. of subset)
$\implies x \in A \cap B$	(by def. of $\cap$)
$\implies A \cap B \neq \emptyset$	(by def. of non-empty)
$\implies A$ and B are not disjoint	(by def. of disjoint)

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