Math Camp: PS 3 Probability and Statistics

Casey O'Hara

2017-09-14

1. **Let Y be a continuous random variable with PDF**

$$f_Y(y) = \begin{cases} (3/2)y^2 + y & \text{if } 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) Find the mean of Y.
Mean $= \mathbb{E}[Y]$

$$\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f(y) dy \\
&= \int_0^1 y((3/2)y^2 + y)dy + 0\Big|_{-\infty}^{0} + 0\Big|_1^{\infty} \\
&= \left(\frac{1}{4} * \frac{3}{2}y^4 + \frac{1}{3}y^3\right)\Big|_0^1 \\
&= \frac{3}{8} + \frac{1}{3} = \frac{17}{24}
\end{aligned}$$

(b) Find the variance of Y. Variance $= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$

$$\begin{aligned}
\mathbb{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f(y) dy \\
&= \int_0^1 y^2((3/2)y^2 + y)dy + 0\Big|_{-\infty}^{0} + 0\Big|_1^{\infty} \\
&= \left(\frac{1}{5} * \frac{3}{2}y^5 + \frac{1}{4}y^4\right)\Big|_0^1 \\
&= \frac{3}{10} + \frac{1}{4} = \frac{11}{20} \\
var(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \frac{11}{20} - \left(\frac{17}{24}\right)^2 \\
&= 0.0483
\end{aligned}$$

2. **Let Y be a random variable with probability density function given by**

$$f_Y(y) = 2(1-y), \qquad y \in [0,1]$$

(a) Find the PDF of $U = 2Y - 1$.
U is a strictly increasing function, so using transformation method.

$$\begin{aligned}
U = 2Y - 1 &\Rightarrow Y = \frac{u+1}{2} \\
f_U(u) &= f_y\left(\frac{u+1}{2}\right)\left|\frac{d}{du}\left(\frac{u+1}{2}\right)\right| \\
&= 2\left(1 - \frac{u+1}{2}\right)\left|\frac{1}{2}\right| \\
f_U(u) &= \frac{1-u}{2}, \qquad u \in [-1,1]
\end{aligned}$$

(b) Find the PDF of $W = 1 - 2Y$.

W is a strictly decreasing function, so using transformation method.

$$
\begin{aligned}
W = 1 - 2Y \Rightarrow Y &= \frac{1-w}{2} \\
f_W(w) &= f_y\left(\frac{1-w}{2}\right)\left|\frac{d}{dw}\left(\frac{1-w}{2}\right)\right| \\
&= 2\left(1 - \frac{1-w}{2}\right)\left|-\frac{1}{2}\right| \\
f_W(w) &= \frac{1+w}{2}, \qquad w \in [-1, 1]
\end{aligned}
$$

(c) Find the PDF of $Z = Y^2$.

Z is neither a strictly increasing nor decreasing function for all values of Y, though across the support of $Y \in [0, 1]$, it is strictly increasing; so using transformation method.

$$
\begin{aligned}
Z = Y^2 \Rightarrow Y &= \sqrt{Z} \\
f_Z(z) &= f_y(\sqrt{z})\left|\frac{d}{dz}\sqrt{z}\right| \\
&= 2(1 - \sqrt{z})\left|\frac{1}{2}z^{-1/2}\right| \\
f_Z(z) &= \frac{\sqrt{z}}{z} - 1, \qquad z \in [0, 1]
\end{aligned}
$$

3. Consider the multivariate distribution characterized by the PDF

$$
f_{XY} = 6(1-y), \qquad 0 \le x \le y \le 1
$$

(a) Find the conditional expectation of $\mathbb{E}[X|Y = y]$.

$$
\begin{aligned}
\mathbb{E}[X|Y = y] &= \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{\text{joint PDF}}{\text{marginal PDF}} \\
f_Y(y) &= \int_0^y 6(1-y)dx \\
&= 6y(1-y)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X|Y = y] &= \frac{f_{XY}(x, y)}{f_Y(y)} \\
&= \frac{6(1-y)}{6y(1-y)} \\
&= \frac{1}{y}
\end{aligned}
$$

(b) Find the covariance of X and Y.

$$
\begin{aligned}
cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
\\
\mathbb{E}[XY] &= \int_{y=0}^{1}\int_{x=0}^{y} xy * 6(1-y)dxdy \\
&= \int_{y=0}^{1} \left(\frac{1}{2} * 6x^2(y-y^2)\right)\Big|_0^y dy \\
&= \int_{y=0}^{1} (3y^3 - 3y^4))dy \\
&= \frac{3}{4}y^4 - \frac{3}{5}y^5\Big|_0^1 \\
&= \frac{3}{20}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x * f_X(x)dx \\
f_X(x) &= \int_x^1 f_{XY}(x,y)dy = 6y - 3y^2\Big|_x^1 \\
\mathbb{E}[X] &= \int_0^1 x(3 - 6x + 3x^2)dx \\
&= \frac{3}{2}x^2 - \frac{1}{3} * 6x^3 + \frac{1}{4} * 3x^4\Big|_0^1 \\
&= \frac{3}{2} - \frac{1}{3} + \frac{3}{4} = \frac{3}{12} = \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 y * f_Y(y)dy \\
f_Y(y) &= \int_0^y f_{XY}(x,y)dx = 6x - 6xy\Big|_0^y \\
\mathbb{E}[Y] &= \int_0^1 y(6y - 6y^2)dy \\
&= \frac{6}{3}y^3 - \frac{6}{4}y^4\Big|_0^1 \\
&= 2 - \frac{3}{2} = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
cov(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\
&= \frac{3}{20} - \frac{1}{4} * \frac{1}{2} = \frac{1}{40}
\end{aligned}
$$

4. **Let $X_1, ..., X_n$ be a random variable from a distribution with PMF**

$$
f(x_i|\theta) = \begin{cases} \theta(1-\theta)^{x_i-1} & \text{if } x = 1,2,3... \\ 0 & \text{else} \end{cases}
$$

where $\theta \in (0,1)$.

(a) Find the method of moments estimator for θ.

$$
\begin{aligned}
m_1 &= M_1(\hat{\theta}) \\
M_1(\hat{\theta}) = \mathbb{E}[X] &= \sum_{i=1}^{\infty} x_i \theta (1-\theta)^{x_i - 1} \\
&= \theta[1 + 2(1-\theta) + 3(1-\theta)^2 + 4(1-\theta)^3 ...] \\
&\text{similar to geometric series but not quite...} \\
\mathbb{E}[X] - (1-\theta)\mathbb{E}[X] &= \theta[1 + 2(1-\theta) + 3(1-\theta)^2 + 4(1-\theta)^3 ...] \\
&\quad - \theta[(1-\theta) + 2(1-\theta)^2 + 3(1-\theta)^3 ...] \\
&= \theta[1 + (1-\theta) + (1-\theta)^2 + (1-\theta)^3 ...] \\
&\text{Now it's a geom series; } \sum ar^x \rightarrow \frac{1}{1-r} ... \\
\mathbb{E}[X] - (1-\theta)\mathbb{E}[X] &= \theta(\frac{1}{1-(1-\theta)}) = 1 \\
m_1 = \mathbb{E}[X] &= \frac{1}{\theta}
\end{aligned}
$$