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Problem set 5

Exercises lecture 8

8.1 Every morning, 6,000 commuters must travel from East Potato to West Potato. There are two ways to make the trip.

One way is to drive straight across town, through the heart of Middle Potato. The other way is to take the Beltline Freeway that circles the Potatoes. The Beltline Freeway is entirely uncongested, but the drive is roundabout and it takes 45 minutes to get from East Potato to West Potato by this means. The road through Middle Potato is much shorter and if it were uncongested it would take only 20 minutes to make the trip. Given the current size of the road through Middle Potato, if the number of commuters using the road is N_1, the time it takes to drive from East Potato to West Potato by this road is $20 + N_1/100$.

a) With the road at its current size and with no tolls, how many commuters will use the road through Middle Potato? What will be the total number of minutes per morning spent by all commuters going from East Potato to West Potato?

   - Commuters will decide which road to take based on their utility, which without additional information, we assume is directly related to how long it takes to get to West Potato, or WePo. Equilibrium will occur when the marginal commuter's WePo commute time (i.e. disutility) on the Beltway, or BF, is the same as the time it takes to drive through Middle Potato, or MoPo.

$$
\begin{aligned}
U_{i,BF} &= U_{i,MoPo} && \text{(at equilibrium)} \\
-45 &= -(20 + \frac{x}{100}) \\
\Rightarrow x &= (45 - 20) * 100 = 2500
\end{aligned}
$$

   - Once 2500 drivers have chosen the MoPo route, the two commute times will be equal and commuters will be indifferent between the two routes; if any more were to consider this route, then it would take longer and therefore they would opt to take the BF route.
   - All drivers will now take 45 minutes to commute to WePo, so $45 * 6000 = 270,000$ total driver-minutes each morning.

b) Suppose that a social planner controlled access to the road through Middle Potato and chose the number of commuters who were allowed to use the Middle Potato road in such a way as to minimize the sum of the number of minutes spent by commuters traveling from East Potato to West Potato. How many commuters would the social planner permit to use the Middle Potato road each morning? How long would it then take the commuters who used this road to make their morning commute?

   - The social planner will minimize the total commuting time (i.e. maximize utility), dividing commuters into x taking the MoPo route and $(6000 - x)$ taking the BF route:

$$
\begin{aligned}
U_{tot} = \sum U_i = \sum_{i-1}^{x} -(20 + \frac{x}{100}) + \sum_{i=x+1}^{6000} -45 \\
= -20x - \frac{x^2}{100} - 45(6000 - x) \\
\frac{\partial U_{tot}}{\partial x} = 25 - \frac{x}{50} = 0 && \text{(first order cond'ns)} \\
\Rightarrow x = 1250
\end{aligned}
$$

- The commute for drivers taking the BF route will still take 45 minutes; the commute for drivers assigned to take the MoPo will take only $20 + x/100 = 32.5$ minutes.
- Total commute time for the entire EPo community is $45(6000 - x) + 32.5x = 254,375$ minutes.

c) Suppose that commuters value time saved from commuting at \$$w$ per minute. Suppose that the government charges a toll for driving on the road through Middle Potato and divides the revenue from the toll equally among all 6,000 commuters. If the government's objective is to minimize the total amount of time that people spend commuting, how high should it set the toll? How much revenue will it collect? With this policy, how much better off (evaluated in dollars) is each commuter than they were in the equilibrium with no tolls?

- The toll should be set such that the marginal commuter is indifferent between the BF route and MoPo route at the time-minimizing levels, to ensure equilibrium.

$$
\begin{aligned}
U_{BF} &= -45w + \frac{tx}{6000} && \text{(time disutility plus rebate)} \\
U_{MoPo} &= -(20 + \frac{x}{100})w - t + \frac{tx}{6000} && \text{(time disutility minus tax plus rebate)} \\
\Rightarrow -45w + \frac{tx}{6000} &= -20w - \frac{wx}{100} - t + \frac{tx}{6000} \\
\Rightarrow t &= (25 - \frac{x}{100})w \\
&= 12.5w && \text{(substitute } x = 1250\text{)}
\end{aligned}
$$

- At this level, the tax will generate $tx = 12.5x \times 1250 = \$15,625w$ of revenue.
- Relative to the no-toll equilibrium, the BF drivers are $+\frac{tx}{6000} = \$2.6w$ better off; because the system is at equilibrium, the MoPo drivers are better off by the same amount relative to the no-toll equilibrium. We can also consider that the utility loss due to the tax on the MoPo drivers exactly cancels the utility gain of the reduced commute time; therefore, the net utility gain is just the rebate term.

8.2 Suppose that a monopolist controls the road through Middle Potato and sets the toll that maximizes total revenue.

What price will the monopolist set and how many people will use the road at this price? How does this price compare with the optimal toll?

> **Hint**: *First find the "demand curve" relating number of users to price. Then find the revenue-maximizing quantity and price.*

- Demand for access to the MoPo route is based on the price of the toll; demand will be at equilibrium when utility for commuter i are equal: $U_{i,BF} = U_{i,MoPo}$.

$$\begin{aligned}
U_{BF} &= -45w && \text{(time disutility only)}\\
U_{MoPo} &= -(20+\frac{x}{100})w - t && \text{(time disutility minus toll)}\\
\Rightarrow t &= (25-\frac{x}{100})w && \text{(same as 8.1c)}\\
\Rightarrow x &= 2500 - 100\frac{t}{w}
\end{aligned}$$

- Total revenue is then $R = xt = 2500t - \frac{100}{w}t^2$.

$$\begin{aligned}
\frac{\partial R}{\partial t} &= 2500 - \frac{200}{w}t = 0\\
\Rightarrow t &= 12.5w
\end{aligned}$$

- Therefore, marginal revenue = 0 when $t = 12.5w$, and $x = 1250$ commuters on the MoPo route. Profit to the monopolist = $\$15,625w$.
  - Note this is also the point where price elasticity of demand is equal to 1, therefore maximizing revenue; for $dx = +1$, $dt/dx = -1/100w$:

$$e_p = -\frac{dx/x}{dt/t} = -\frac{1/1250}{-\frac{w}{100}/12.50w} = 1$$

- This is identical to the toll and number of MoPo commuters in the tax situation (8.1c). A monopolist charging a profit-maximizing toll will reduce the traffic on the EPo/WePo commute system to minimize the total driving time across all commuters just as effectively as an optimal tax by the government (rebated or otherwise).

8.3 Suppose that the road through Middle Potato can be widened at a cost of $\$p$ per inch per day.

If the road is widened by H inches and if N_1 people use the road, the time that it would take to drive to from East Potato to West Potato by this road would be $20 + N_1/(100+H)$. Suppose that the government's objective function is to maximize the total value of time saved from commuting minus the sum of money spent on road building. Let w is the value per minute of time saved from commuting. Assuming that the government can not charge a toll for the road, or what values of the parameters p and w would it pay to widen the road at all. (Assume that there is no congestion for commuters coming home from work, only going to work.) If it pays to widen the road, how many commuters will be using the road when it is optimally widened? How long will it take them to make the trip, given that the road is optimally widened?

In equilibrium, we know the costs of each driver must be the same. So $45w = (20 + \frac{N_1}{100+H})w$. Then

$$N_1 = 2500 + 25H.$$

Each person's value of time saved is $w(45 - 20 + \frac{N_1}{100+H})$. The maximization problem is:

$$\begin{gathered}
\max_H \quad N_1 w(45 - 20 - \frac{N_1}{100+H}) - pH \;\; s.t. \;\; N_1 = 2500 + 25H\\
\Leftrightarrow\\
\max_H \quad -pH
\end{gathered}$$

Since this expression is strictly decreasing in H, the value of H that minimizes the expression is 0 unless H is so large that

$$45 > 20 + \frac{6000}{100+H}$$

This implies $H > 140$. The new maximization problem becomes:

$$\max_{H} \quad 6000w(25 - \frac{6000}{100 + H}) - pH$$

FOC: $\frac{6000}{100+H} = \sqrt{\frac{c}{w}}$.

Then $H^* = 600\sqrt{\frac{c}{w}} - 100$.

The corresponding number of commuters is 6000 and the time it takes is $20 + 10\frac{w}{c}$.

8.4 Suppose that as in the previous problem, the road through Middle Potato can be widened at a cost of $\$p$ per inch per day.

Suppose also that the government can charge an efficient toll for using this road. What is the best combination of toll and highway expenditures? How does total toll revenue compare with the cost of building the highway? How does the optimal amount of highway expenditures compare with the optimal amount if the government cannot charge a toll? *Hint: Think about two possible corner solutions.*

Recall the toll is 12.5w and the government will charge enough to cover the cost of expansion. The maximization problem becomes:

$$\max_{H} \quad N_1 w(25 - \frac{N_1}{100 + H}) - pH \;\; s.t. \;\; N_1 = \frac{1}{w}(25w - 12.5w)(100 + H)$$

Then $H^* = \frac{6000}{\sqrt{12.5}} - 100$.

8.5 Generalize the bottleneck problem so that the cost per minute of being late can take a different value γ from the cost per minute of being early.

Show what happens in the limit as γ gets large.

- Let t_1 be the departure time of the first commuter, t_N of the last, β the cost of arriving early and γ the cost of arriving late. Using the equal cost condition between first and last commuters:

$$\begin{aligned}
&\beta(t^* - t_1) = \gamma(t_N - t^*) \\
&\beta(t^* - t_1) = \gamma(\frac{N}{s} + t_1 - t^*) && \text{(since } t_N - t_1 = \frac{N}{s}\text{)} \\
&(\beta + \gamma)t^* - (\beta + \gamma)t_1 = \frac{\gamma N}{s} \\
&t^* - t_1 = \frac{\gamma N}{(\beta + \gamma)s} \\
&\boxed{t_1^* = t^* - \frac{\gamma}{(\beta + \gamma)}\frac{N}{s}} \\
\Rightarrow &\boxed{t_N^* = t^* + \frac{\beta}{(\beta + \gamma)}\frac{N}{s}}
\end{aligned}$$

- For the early and late time, we can see that when γ increases:

$$\frac{\partial t_1^*}{\partial \gamma} < 0$$
$$\frac{\partial t_N^*}{\partial \gamma} < 0$$

Hence, as the cost of arriving late increases, but the earliest and latest departure time reduce.

- For the commuter that arrives just in time:

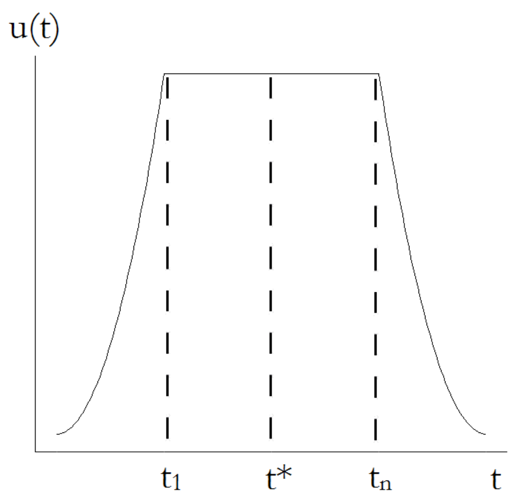
$$\boxed{\tilde{t}^* = t^* - \frac{\beta}{\alpha} \frac{\beta}{(\beta + \gamma)} \frac{N}{s}}$$
$$\frac{\partial \tilde{t}^*}{\partial \gamma} < 0$$

His departure time also reduces.

8.6 Generalize the bottleneck problem so that the everyone has a utility $u(t)$ for arriving at work at time t where $u(t)$ is a fairly arbitrary singlepeaked (or single-plateaued) function.

(Assume that $u'(t) > -s\alpha$ for all relevant t.) Which results from our earlier discussion still apply and which do not?

Let all commuters $i \in N$ have utility functions $u(t)$ for leaving to work at time t, with $N = \{1, 2, ..., n\}$ for n commuters in the city. Again let t^* be the time that work starts. The game is defined as $\langle N, t \in \mathbb{R}, u(t) \rangle$ where $t \in \mathbb{R}$ describes the possible action taken by a commuter. The utility function takes a single plateau, as represented below.



Since $u(t)$ is the same $\forall\ i \in N$, it must be that in Nash equilibrium, $u(t_1) = u(t_2) = ... = u(t_n)$. For simplicity assume that $t_1 < t_2 < ... < t_n$ so that each commuter leaves in order.

One solution is that the departure times $t_1, t_2, ..., t_n$ are uniformly distributed along the plateau. Then, as before, $t_1 = t^* - n/2s \ \wedge \ t_n = t^* + n/2s$. Visually, the portion of the utility curve that is at a plateau is of width n/s, or the length of the rush hour. We can verify this by observing that $t_n - t_1 = n/s$. This is the same result as when we gave explicit utility forms.

Without an explicit utility form we may still keep some of the same notation as before. For example, the length of the queue at time t may still be written as $D(t)$. We can again define the rate of growth for the queue as $\dot{D}(t) = r(t) - s$. But without an explicit utility form we no longer solve for $r(t)$. In the original model we were able to solve that $r(t) = \frac{s\alpha}{\alpha-\beta}$, which implied that the queue growed at a constant rate $r > s$ as long as the commuters arrived to work before t^*. In this model we might therefore relax the result that the queue grows at a constant rate. For example, on the single-plateaued utility function, the solutions $t \in \{t_1, t_2, ..., t_n\}$ need not be distributed uniformly, which means congestion could grow at different rates determined by $r(t)$.

We may also still define $\tilde{t}$ as the departure time that allows the worker to arrive to work at t^*. This again implies that $\tilde{t} + \frac{D(\tilde{t})}{s} = t^*$. $D(\tilde{t})$ is the maximum length that the queue takes. One important result from the original model was that $D(\tilde{t})$, in the original model, did not depend on the capacity of the bridge. Here, without a utility form, we do not solve for $D(\hat{t})$, so we lose that result.

8.7 What happens in the above generalization if $u'(t) < -s\alpha$?

In the above generalization, note that for the relevant t, $t \in \{t_1, t_2, ..., t_n\}$, we have that $u'(t) = 0$ on the plateau of the utility function. This gives $u'(t) = 0 > -\alpha s$. Recall that αs is comprised of α, the cost per minute of waiting in traffic, while s is the number of people let through the bottleneck per minute. We need that the marginal utility with respect to time is greater than the total cost per minute αs in the system. If the marginal utility is not greater than the cost of waiting, $u'(t) < -\alpha s$, no one should go to work.

8.8 Generalize the model to allow two types of commuters with different preferred times of arrival.

- Let type x be those who prefer an early arrival time, and type y be those who prefer a later arrival time. Let $t_y^* = t_x^* + \tau$ where $\tau > 0$. Let $N = N_x + N_y$. Let the traffic wait time cost α be identical for all people, and office arrival penalty β be identical for all people and identical for early and late arrivals.
- First note two degenerate cases; the first is when $\tau = 0$ which devolves to the basic case, and the second is when τ is very large, such that congestion from the two subpopulations do not overlap. These cases are both boring, so ignore them.
- Assume that despite the different types, the equilibrium situation will result in all commuters having an equal average (dis)utility of commuting.
- Note that the very first commuter, leaving at time t_1, must be a type x (a type y would suffer a greater penalty for arriving that much earlier), and the last commuter, leaving at time t_N, must be a type y (a type x would similarly suffer a greater penalty for arriving that much later). Neither of these will face a queue, so travel time is zero and cost is just arrival time penalty.
- Note that as long as there is a queue $D(t) > 0$ in the bottleneck at all times between t_1 and t_N, the total traffic jam will last N/s as in the basic problem. Commuter type is irrelevant in the flow rate through the bottleneck.

$$\begin{aligned}
t_N - t_1 &= \frac{N}{s} && \\
C_1 &= \beta(t_x^* - t_1) && \text{(cost; first commuter is } x\text{)} \\
C_N &= \beta(t_N - t_y^*) && \text{(cost; last commuter is } y\text{)} \\
C_1 = C_N \Rightarrow \beta(t_x^* - t_1) &= \beta(t_N - t_y^*) && \text{(equal costs)} \\
\Rightarrow \beta(t_x^* - t_1) &= \beta(t_1 + \frac{N}{s} - (t_x^* + \tau)) && \text{(sub for } t_y^* \text{ and } t_N\text{)} \\
\Rightarrow t_x^* - t_1 &= \frac{N/s - \tau}{2} && \\
\Rightarrow t_N - t_y^* &= \frac{N/s - \tau}{2} &&
\end{aligned}$$

- In the basic case, $t^* - t_1 = \frac{N}{2s}$. In this case, assuming some y commuters take pressure off the earliest commute, it seems OK that the time to leave becomes closer to t_x^*, but it seems like there should be some weighting by the number of x and y commuters. For example, imagine 1000 type x commuters; the first commuter leaves at $t_1 = t_x^* - 1000/2s$. Then add a single type y, with $\tau = 1$ hour later. This formulation says the first commuter should now leave at $t_1 = t_x^* - 1001/2s + \tau/2$, a half an hour later, just because of a single entrant who prefers a later time. This seems to indicate one of the assumptions is faulty.

Exercises Lecture 9

9.1 A city has 2 types of people, and 1000 people of each type. There is one private good and one public good.

Let X_i denote the amount of private consumption consumed by citizen i and let Y denote the amount of public good available in the city. All type 1s have the utility function $U(X_i, Y) = X_i Y$, type 2's have the utility function $U(X_i, Y) = X_i Y^2$. The price of private goods is \$1 per unit. Type 1s have an income of \$10,000 and Type 2s have an income of \$15,000. Public goods can be made from private goods with constant returns to scale. It takes 30 units of the private good to make one unit of the public good. The following questions relate to alternative arrangements for provision of public goods in this city.

a) Calculate the Lindahl equilibrium prices and quantities for this city.

- Type 1

$$\max_{X_i, Y} X_i Y \quad \text{s.t.} \quad 10,000 \geq p_1 Y + X_i$$

FOCs:

$$\begin{aligned}
&\frac{\partial L}{\partial X_i} = Y - \lambda = 0 \\
&\frac{\partial L}{\partial Y} = X_i - p_1 \lambda = 0 \\
\Rightarrow & p_1 Y = X_i \\
&\frac{\partial L}{\partial \lambda} = 10,000 = p_1 Y + X_i \\
\Rightarrow & 10,000 = p_1 Y + p_1 Y \\
&\frac{5000}{Y} = p_1
\end{aligned}$$

- Type 2

$$\max_{X_i, Y} X_i Y^2 \quad \text{s.t.} \quad 15,000 \geq p_2 Y + X_i$$

FOCs:

$$
\begin{aligned}
&\frac{\partial L}{\partial X_i} = Y^2 - \lambda = 0\\
&\frac{\partial L}{\partial Y} = 2X_iY - p_2\lambda = 0\\
\Rightarrow &\frac{p_2 Y}{2} = X_i\\
&\frac{\partial L}{\partial \lambda} = 15,000 = p_2Y + X_i\\
\Rightarrow &15,000 = p_2Y + \frac{p_2Y}{2}\\
&\frac{10,000}{Y} = p_2
\end{aligned}
$$

- Finding the Y^*, p_1^* and p_2^*

$$
\begin{aligned}
&\sum_{i\in \text{Type 1}} p_1 + \sum_{i\in \text{Type 2}} p_2 = 30\\
&1000 * \frac{10,000}{Y} + 1000 * \frac{10,000}{Y} = 30\\
&\boxed{Y^* = 500,000}\\
\Rightarrow\ &\boxed{p_1^* = 0.01} \quad \text{and} \quad \boxed{p_2^* = 0.02}
\end{aligned}
$$

b) Suppose that the public good is excludable and marketed competitively as in the Oakland (1974) model. In the Oakland competitive equilibrium with free entry for firms, how many units will be consumed by the type 1s? the type 2s? What will be the total number of units produced? What will the competitive prices be? How many units of the public good will the low price seller sell? How much will the high price seller sell.

- From (a) we know that the individual demands for Type 1 and Type 2 are respectively:

$$
\frac{5,000}{p} = Y \quad \text{and} \quad \frac{10,000}{p} = Y
$$

- The first firm charges such that it covers its cost of provision, i.e. $p_L = \frac{30}{2000} = 0.015$. At this price Type 1 folks will demand $\frac{1,000,000}{3}$ and Type 2 $\frac{2,000,000}{3}$, hence firm 1 just offers $\frac{1,000,000}{3}$ to cover its cost.
- Firm 2 now attempts to enter the market and cater to Type 2 folks by charging $p_H = \frac{30}{1000} = 0.03$. At this price, Type 2 demand the same amount provided under firm 1, hence firm 2 will sell 0 additional units.