Econ 241A Prof. Startz

Final Fall 2016

Please answer all questions. Show your work.

The exam is open book/open note; closed any devices that can communicate. (No laptops, cell phones, Morse code keys, signal fires, etc.)

1 Let $y_i = \beta x_i + \epsilon_i$ with $\frac{1}{n}\sum x_i\epsilon_i \xrightarrow{p} \mathbb{E}[x_i\epsilon_i] = 0$ and $\frac{1}{n}\sum x_i^2 \xrightarrow{p} \mathbb{E}[x_i^2] = M < \infty$

Define $\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$

Prove that $\hat{\beta} \xrightarrow{p} \beta$

Answer:

$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum x_i(\beta x_i + \epsilon_i)}{\sum x_i^2} = \beta + \frac{\sum x_i\epsilon_i}{\sum x_i^2} = \beta + \frac{\frac{1}{n}\sum x_i\epsilon_i}{\frac{1}{n}\sum x_i^2} \xrightarrow{p} \beta + \frac{\mathbb{E}[x_i\epsilon_i]}{\mathbb{E}[x_i^2]} = \beta$

2 Let $x_1, x_2, \ldots$ be a sequence of iid random variables where:

$$x_n = \begin{cases} \mu & \text{with prob } \frac{n-2}{n} \\ 3\mu & \text{with prob } \frac{1}{n} \\ -\mu & \text{with prob } \frac{1}{n} \end{cases}$$

a) Find $\mathbb{E}[x_n]$

b) Does x_n converge in probability to μ?

c) Does x_n converge almost surely to μ?

Answer:

a) $\mathbb{E}[x_n] = \frac{n-2}{n}\mu + \frac{3\mu}{n} - \frac{\mu}{n} = \mu$

b) $\lim\limits_{n\to\infty} P(|x_n - \mu| < \epsilon) = \lim\limits_{n\to\infty} \frac{n-2}{n} = 1$

c) $P(\lim\limits_{n\to\infty} |x_n - \mu| < \epsilon) = \frac{n-2}{n} \neq 1$

3 You find yourself in a weird bar in Las Vegas. The bartender (who you trust) tells you there are only three types of coins allowed: $P(Heads) = 0.25$, $P(Heads) = 0.5$, and $P(Heads) = 0.75$. Denote the probability of Heads as μ. You observe 2 Heads and 1 Tails, let $X = \{H, H, T\}$.

Your uninformed prior is that $P(\mu) = \begin{cases} \frac{1}{3} & \text{if } \mu = 0.25 \\ \frac{1}{3} & \text{if } \mu = 0.5 \\ \frac{1}{3} & \text{if } \mu = 0.75 \end{cases}$

a) Calculate $P(\mu = 0.5|X)$

Your trustworthy bartender tells you a better prior is that $P(\mu) = \begin{cases} \frac{1}{4} & \text{if } \mu = 0.25 \\ \frac{1}{2} & \text{if } \mu = 0.5 \\ \frac{1}{4} & \text{if } \mu = 0.75 \end{cases}$

b) Calculate $P(\mu = 0.5|X)$

Suppose you were convinced that $P(0.5|X) = 0.05$, and that $P(\mu) = \begin{cases} \frac{1}{2}(1 - \frac{1}{z}) & \text{if } \mu = 0.25 \\ \frac{1}{z} & \text{if } \mu = 0.5 \\ \frac{1}{2}(1 - \frac{1}{z}) & \text{if } \mu = 0.75 \end{cases}$

c) Solve for z.

Answer:

$$P(X|\mu) = \begin{cases} 0.25^2 * 0.75 & \text{if } \mu = 0.25 \\ 0.5^3 & \text{if } \mu = 0.5 \\ 0.25 * 0.75^2 & \text{if } \mu = 0.75 \end{cases}$$

a) $P(\mu = 0.5|X) = \frac{P(X|0.5)P(0.5)}{\sum P(X|\mu)P(\mu)} = 0.4$

b) $P(\mu = 0.5|X) = \frac{P(X|0.5)P(0.5)}{\sum P(X|\mu)P(\mu)} = 0.57$

c) $0.05 = \frac{P(X|0.5)P(0.5)}{\sum P(X|\mu)P(\mu)} \Rightarrow z = 10.66$

4 Consider the following model in which s_i is an observed signal of person i's true ability, $a_i = a_i^0 + a_i^1$, and ϵ_i is an observational error. a_i^0 is inherent ability and a_i^1 is acquired ability. For convenience, we define the symbols $\bar{a} = \mathbb{E}[a_i]$ and $\sigma_a^2 = Var(a_i)$. Ability and the error are distributed joint normal

$$\begin{pmatrix} a_i^0 \\ a_i^1 \\ \epsilon_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \bar{a}^0 \\ \bar{a}^1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & \sigma_{01} & 0 \\ \sigma_{01} & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_\epsilon^2 \end{pmatrix} \right)$$

The signal is generated according to

$$s_i = a_i + \epsilon_i$$

Agents are paid according to their expected ability; income equals

$$y_i = w * \mathbb{E}[a_i|s_i]$$

Agents purchase ability at cost $C(a_i^1)$, which is the increasing function

$$C(a_i^1) = c_1 a_i^1 + \frac{1}{2} c_2 (a_i^1)^2$$

Note that income depends on total ability while costs depend only on acquired ability. While ϵ is observable by the econometrician, it is not observable by either the agent nor by the employer at the time decisions are made.

a) Find the formula for $\mathbb{E}[a_i|s_i]$ in terms of the parameters $\{\bar{a}, \sigma_a^2, \sigma_\epsilon^2\}$

b) Set up the agent's maximization problem and solve for a_i^1 in terms of $\{\bar{a}, \sigma_0^2, \sigma_1^2, \sigma_{01}, \sigma_\epsilon^2\}$

c) Using your results, solve for the values of $\bar{a}$ and σ_a^2

d) What is the expected value of net income?

e) Show that the expected value of net income is decreasing in σ_ϵ^2. Explain what this means in two sentences or less.

Answer:

a) $\mathbb{E}[a_i|s_i] = \mathbb{E}[a_i] + \frac{Cov(a_i, s_i)}{Var(s_i)}(s_i - \mathbb{E}[s_i]) = \bar{a} + \frac{\sigma_a^2}{\sigma_a^2 + \sigma_\epsilon^2}(s_i - \bar{a}) = (1-\beta)\bar{a} + \beta s_i$ where $\beta = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_\epsilon^2}$

b) $V(a_i^1) = w * \mathbb{E}[a_i|s_i] - C(a_i^1) = w * ((1-\beta)\bar{a} + \beta(a_i^0 + a_i^1 + \epsilon_i) - c_1 a_i^1 + \frac{1}{2} c_2 (a_i^1)^2$

$\frac{\partial V}{\partial a_i^1} = w\beta - c_1 - c_2 a_i^1 = 0 \Rightarrow a_i^1 = \frac{w\beta - c_1}{c_2}$

c) Note that acquired ability is the same for everyone. This gives us $\sigma_1^2 = \sigma_{01} = 0$, so $\sigma_a^2 = \sigma_0^2$ and $\bar{a} = \bar{a}^0 + a_i^1 = \bar{a}^0 + \frac{w\beta - c_1}{c_2}$

d) $\mathbb{E}[V] = w * \mathbb{E}[a_i] - C(a_i^1) = w(\bar{a}^0 + \frac{w\beta - c_1}{c_2}) - c_1 \frac{w\beta - c_1}{c_2} + \frac{1}{2} c_2 (\frac{w\beta - c_1}{c_2})^2$

e) $\frac{\partial \mathbb{E}[V]}{\partial \beta} = \frac{w^2}{c_2}(1-\beta) > 0$ and since $\frac{\partial \beta}{\partial \sigma_\epsilon^2} < 0 \Rightarrow \frac{\partial \mathbb{E}[V]}{\partial \sigma_\epsilon^2} < 0$

The noisier the signal is, the less ability the agent will acquire and so they will be paid less.

5 Your utility at time t is given by $u_t = \alpha + \gamma x_t + \delta y_t$

where x_t is a measure of your net wealth and is distributed iid $Pareto(1, \beta) \Rightarrow f_x(x_t) = \frac{\beta}{x_t^{\beta+1}}$ for $x_t \in [1, \infty)$

(Assume $\beta > 1$)

and y_t is a measure of your leisure time and is distributed iid $N(\mu, \sigma^2)$

Assume x_t and y_t are independent.

a) Find $\mathbb{E}[u_t]$

b) Find the Maximum Likelihood Estimator of β (Hint: this doesn't depend on y_t)

c) Find the Cramer Rao Lower Bound for an unbiased $\hat{\beta}$

d) Write down a Likelihood Ratio Test for $H_0 : \beta = \beta_0$

e) Find a Method of Moments Estimator for σ^2 (Hint: this doesn't depend on x_t)

f) Is it unbiased?

g) Find $Var(\hat{\sigma}^2_{MoM})$ (Hint: use that $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(n)}$)

h) Write down a Wald Test for $H_0 : \sigma^2 = \sigma_0^2$

Define $\hat{\theta} = \alpha + \delta\hat{\mu}$ where $\hat{\mu} = \frac{1}{n}\sum y_t$

i) Find the asymptotic distribution of $\hat{\theta}$

Answer:

a) $\mathbb{E}[u_t] = \alpha + \gamma\,\mathbb{E}[x_t] + \delta\,\mathbb{E}[y_t] = \alpha + \gamma\frac{\beta}{\beta-1} + \delta\mu$

b)

$$L = \frac{\beta^n}{\prod x_t^{\beta+1}}$$

$$\mathcal{L} = nlog(\beta) - (\beta + 1)\sum log(x_t)$$

$$FOC[\beta] : \frac{n}{\beta} - \sum log(x_t) = 0$$

$$\Rightarrow \hat{\beta} = \frac{n}{\sum log(x_t)}$$

c) $CRLB = \frac{1}{-n\,\mathbb{E}[\frac{\partial^2}{\partial\beta^2} f(x|\beta)]} = \frac{1}{-n(\frac{-1}{\beta^2})} = \frac{\beta^2}{n}$

d) $LRT = -2(\mathcal{L}(\beta_0) - \mathcal{L}(\hat{\beta})) = -2(n(log(\beta_0) - log(\hat{\beta})) - (\hat{\beta} - \beta_0)\sum log(x_t)) \sim \chi^2_{(1)}$

e) $Var(x_t) = \sigma^2 \Rightarrow \frac{1}{n}\sum(x_t - \bar{x})^2 = \hat{\sigma}^2$

f) $\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n}\,\mathbb{E}[\sum(x_t^2) - 2n\bar{x}^2 + \bar{x}^2] = \frac{1}{n}(n(\sigma^2 + \mu^2) - n\,\mathbb{E}[\bar{x}^2]) = \frac{1}{n}(n(\sigma^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)) = \frac{n-1}{n}\sigma^2$

g) $Var(\hat{\sigma}^2) = Var(\frac{\chi^2_{(n)}\sigma^2}{n}) = \frac{2n\sigma^4}{n^2} = \frac{2\sigma^4}{n}$

h) $\frac{\hat{\sigma}^2-\sigma_0^2}{\sqrt{\frac{2\hat{\sigma}^4}{n}}} \sim t_{(n-1)}$

i) $\sqrt{n}(\hat{\mu}-\mu) \sim N(0,\sigma^2)$ and $g(\mu) = \alpha + \delta\mu$

so by the Delta Method $\sqrt{n}(\hat{\theta} - (\alpha + \delta\mu)) \sim N(0, \sigma^2\delta^2)$

$\Rightarrow \hat{\theta} \sim N(\alpha + \delta\mu, \frac{\sigma^2\delta^2}{n})$