Econ 241A Probability, Statistics and Econometrics

Fall 2014

Midterm

- You have 1 hr 10 min to complete this midterm.
- The midterm has two parts. Part I requires to solve <u>all</u> problems. Part II allows you to choose between two problems. Please solve <u>just one</u> problem in Part II. If you answer both, only the lowest grade out of the two will be taken into account.
- The last page of the exam has a list of pmf's and pdf's that you may (or may not) need to use throughout the exam.
- Hint: You can have a perfect score without solving any integrals.

Part I

1. (4) Prove that $P(A \cap B|C) = P(B \cap C|A)\frac{P(A)}{P(C)}$

2. (4) Prove that $\text{Var}(X) = \mathbb{E}\left[\text{Var}(X|Y)\right] + \text{Var}\left(\mathbb{E}\left(X|Y\right)\right)$.

3. (4) Show that if X and Y are uncorrelated and $\mathbb{E}(X) = 0$, then $\text{Var}(XY) = \mathbb{E}(X^2Y^2)$

4. (4) Show that $f_X(x) = 3(1-x)^2$ for $x \in [0, 1]$ is a pdf.

Part II

Choose to answer one of the following two problems. If you answer both, only the lowest graded will be taken into account.

5. (14) In a bivariate populaton, let us define the *best proportional predictor* of Y given X as the ray through the origin, $\mathbb{E}^{**}(Y|X) = \gamma X$, with γ being the value for c that minimizes $\mathbb{E}(U^2)$, where now $U = Y - cX$.

(a) (3 points) Show that $\gamma = \mathbb{E}(XY)/\mathbb{E}(X^2)$

(b) (3 points) Find $\mathbb{E}(U)$ when $\mathbb{E}(X) = \mathbb{E}(Y) = 0$. Is the *best proportional predictor* "correct in average" (unbiased) when $\mathbb{E}(X) \neq 0$ or $\mathbb{E}(Y) \neq 0$? How does it compare with the linear predictor along this dimension?

(c) (3 points) What is the covariance between U and X?

(d) (2 points) Assume X and Y are independent and $\mathbb{E}(Y) = 1$. Is $\mathbb{E}\left(\mathbb{E}^{**}(Y|X)\right)$ larger or equal to 1?

(e) (3 points) Assume $X \sim$ exponential(λ). What is the pdf of the random variable $\mathbb{E}^{**}(Y|X)$?

6. (14) Gauss Oh is a student at UCSB – naturally – and enjoys studying statistics – also, quite naturally. In his not-so-ample free time, he moonlights as the mascot for his school, which naturally gets in the way of his statistics studying. Luckily for Gauss Oh, getting an "A" in his classes is not solely determined by his hard work (W); instead it's determined by work and luck (L). His loyal fan base wants to determine how often they can expect to see Gauss Oh moonlighting in his super-sweet mascot uniform, but they know that they only get to see him if he gets a certain number of A's. Unfortunately, no amount of "observing" him allows them to see his levels of work and luck, but they can discern his joint distribution:

		L		
		Unlucky (0)	Usual Luck (1)	Good Luck (2)
W	PhD in Econ (2)	.05	y	.15
	Usual Effort (1)	.18	.25	x
	Slacks (0)	.04	.03	.00

Furthermore, from previous experience, his fans know that $P(W = 1|L = 2) = 0.4$.

(a) (2) What is the value of x in the table above?

(b) (2) What is the probability that Gauss Oh works as hard as a PhD student in econ given that he gets his usual luck?

As it turns out, the number of A's that Gauss Oh earns during the school year is determined according to the transformation $X = 2W + L$.

(c) (3) What is the probability that Gauss Oh earns at least 5 A's?

Not only do Gauss Oh's fans care about his grades, but they also care about him graduating (G): they prefer him to stay at UCSB indefinitely. Oddly enough, the statistics department determines graduation strictly on the quantities of A's received during the current year: if Gauss Oh receives at least 5 A's this year, he will graduate ($G = 1$); otherwise, he will not ($G = 0$).

(d) (3) What is the probability mass function for G, $f_G(g)$?

(e) (3) What is the probability that Gauss Oh will not graduate until his 2nd year? What is the probability that he will not graduate until his 3rd year? What about not graduating until his 4th year? Assume each year is an independent try.

(f) (3) Let H represent the random variable associated with the year that Gauss Oh graduates, h. What is the probability mass function of H, $f_H(h)$?

Bernoulli

$P(X = x|p) = p^x(1-p)^{(1-x)}$; $x = 0, 1$; $0 \leq p \leq 1$ with $\mathbb{E}(X) = p$ and $\text{Var}(X) = p(1-p)$

Binomial

$P(X = x|n, p) = \left(\begin{array}{c} n \\ \\ x \end{array}\right) p^x(1-p)^{n-x}$; $x = 0, 1, 2, ..., n$; $0 \leq p \leq 1$, with $\mathbb{E}(X) = np$ and $\text{Var}(X) = np(1-p)$

Discrete uniform

$P(X = x|N) = \frac{1}{N}$; $x = 1, 2, ..., N$; $N = 1, 2, ...,$ with $\mathbb{E}(X) = \frac{N+1}{2}$ and $\text{Var}(X) = \frac{(N+1)(N-1)}{12}$

Poisson

$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, ...$; $0 \leq \lambda < \infty$ with $\mathbb{E}(X) = \lambda$ and $\text{Var}(X) = \lambda$

Exponential

$f(x|\beta) = \lambda e^{-\lambda x}$; $0 \leq x < \infty$ with $\mathbb{E}(X) = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$

Logistic

$f(x|\mu, \beta) = \frac{1}{\beta} \frac{\exp(-(x-\mu)/\beta)}{[1+\exp(-(x-\mu)/\beta)]^2}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\beta > 0$ with $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \frac{\pi^2\beta^2}{3}$

Normal

$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ with $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$