Econ 241A Probability, Statistics and Econometrics

Fall 2010

Midterm

- You have 1 hr 10 min to complete this midterm.
- The midterm has two parts. Part I requires to solve all problems. Part II allows you to choose between two problems. Please solve just one problem in Part II. If you answer both, only the lowest grade out of the two will be taken into account.
- The last page of the exam has a list of pmf's and pdf's that you may (or may not) need to use throughout the exam.

Part I

Please answer all questions 1 to 4.

1. (4) Prove that if sets A and B in sample space S are mutually exclusive (disjoint) and $P(A) > 0, P(B) > 0$, then A and B cannot be independent.

A and B disjoint implies

$$P(A \cap B) = 0$$

For A and B to be independent, it needs to hold that

$$P(A|B) = P(A) > 0$$

However,

$$P(A|B) = P(A \cap B)/P(B) = 0$$

so A and B cannot be independent.

2. (4) X and Y are jointly distributed with pdf $f(x, y) = \frac{3}{4}(x^2 + y^2) + \frac{1}{2}$ with $0 < x < 1$ and $0 < y < 1$.

(a) Are X and Y independent? Justify your answer.

No. The pdf cannot be split in the product of the marginals (functions $h(x)$ and $g(y)$ cannot be factorized).

(b) How would you obtain the marginal pdf of X, $f(x)$?

$$f(x) = \int_0^1 f(x, y)dy = \int_0^1 \frac{3}{4}(x^2 + y^2) + \frac{1}{2}dy$$

3. (4) The joint pmf of X and Y is given in the following table:

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.1	0.1	0.2
$x = 1$	0.2	0.1	0.05
$x = 2$	0.1	0.1	0.05

(a) Calculate the conditional mean of Y given x, $\mathbb{E}(Y|x)$

$$\mathbb{E}(Y|x) = \begin{cases} \frac{5}{4} \text{ if } x = 0 \\ \frac{4}{7} \text{ if } x = 1 \\ \frac{4}{5} \text{ if } x = 2 \end{cases}$$

(b) Write the pmf of $\mathbb{E}(Y|X)$

$$f_{\mathbb{E}(Y|x)}(a) = \begin{cases} \frac{2}{5} \text{ if } a = \frac{5}{4} \\ \frac{7}{20} \text{ if } a = \frac{4}{7} \\ \frac{1}{4} \text{ if } a = \frac{4}{5} \end{cases}$$

4. (4) Prove that if Y and $\mathbb{E}(Y|X)$ are uncorrelated (have covariance equal to zero), then Y is mean independent of X. (Hint: what is the **variance** of a constant?)

$$\mathbb{E}\left[Y\mathbb{E}(Y|X)\right] - \mathbb{E}(Y)\mathbb{E}\left(\mathbb{E}(Y|X)\right) = 0$$

Using iterated expectations

$$\mathbb{E}\left[\mathbb{E}(Y|X)^2\right] - \left(\mathbb{E}\left[\mathbb{E}(Y|X)\right]\right)^2 = 0$$

From the definition of variance

$$Var(\mathbb{E}(Y|X)) = 0$$

Means that

$$\mathbb{E}(Y|X) = c$$

And since we know that $\mathbb{E}\left[\mathbb{E}(Y|X)\right] = \mathbb{E}(Y)$, then it must be the case that

$$\mathbb{E}(Y|X) = \mathbb{E}(Y)$$

Part II

Choose to answer one of the following two problems. If you answer both, only the lowest graded will be taken into account.

5. (14) [Hierarchical model] Suppose that the random variable X has exponential distribution with parameter 1 (e.g. $\lambda = 1$) and the random variable $Y|X$ has uniform distribution with parameters 0 and X (e.g. $a = 0$ and $b = X$).

(a) (3) What is the value of the conditional mean of Y given X, $\mathbb{E}(Y|X)$?

$$\mathbb{E}(Y|X) = \frac{X}{2}$$

(b) (2) What is the mean of Y, $\mathbb{E}(Y)$?

$$\mathbb{E}\left[\mathbb{E}(Y|X)\right] = \frac{1}{2\lambda} = \frac{1}{2}$$

(c) (3) What is the covariance between X and Y, $\text{Cov}(X, Y)$?

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$= \mathbb{E}(X\mathbb{E}(Y|X)) - (1)\left(\frac{1}{2}\right)$$

$$= \mathbb{E}\left(\frac{X^2}{2}\right) - \frac{1}{2}$$

$$= \frac{1}{2}\mathbb{E}(X^2) - \frac{1}{2}$$

$$= \frac{1}{2}\left[Var(X) + (\mathbb{E}(X))^2\right] - \frac{1}{2}$$

$$\frac{1}{2}(1+1) - \frac{1}{2}$$

$$= \frac{1}{2}$$

(d) (3) What is the linear predictor of Ygiven X, $\mathbb{E}^*(Y|X)$?

$$\mathbb{E}^*(Y|X) = \frac{1}{2}X$$

(e) (3) Derive the joint pdf of the bivariate random vector (X, Y)

$$f(x,y) = f(y|x)f(x) = \begin{cases} 0 \text{ for } x < 0 \text{ or } y < 0 \text{ or } y > x \\ \frac{1}{X}e^{-X} \text{ for } x > 0 \text{ and } 0 < y < x \end{cases}$$

6. (14) Suppose X and Y are <u>independent</u> and have identical standard <u>normal</u> distributions, *i.e.* normal distributions with parameters $\mu = 0$ and $\sigma^2 = 1$.

(a) (3) What is the joint pdf of X and Y

$$f(x,y) = \frac{1}{2\pi}e^{\frac{-x^2-y^2}{2}}$$

(b) (3) As a researcher, you are interested in U, where $U = Y - X$, and V, where $V = X$. What are the mean and variance of U?

$$\mathbb{E}(U) = \mathbb{E}(Y - X) = \mathbb{E}(Y) - \mathbb{E}(X) = 0$$

$$Var(U) = Var(Y) + Var(X) + 2Cov(XY) = 1 + 1 = 2$$

(c) (3) What is the joint pdf of U and V?

$$f(u,v) = \frac{1}{2\pi} e^{\frac{-(v+u)^2 - v^2}{2}} |-1|$$

$$-\infty < u < \infty$$

$$-\infty < v < \infty$$

(d) (2) Are U and V independent? Justify your answer.

$$f(u,v) = \frac{1}{2\pi} e^{\frac{-v^2 - 2vu - u^2 - v^2}{2}} |-1|$$

,

$$f(u,v) = \frac{1}{2\pi} e^{\frac{-2v^2 - 2vu - u^2}{2}}$$

$$-\infty < u < \infty$$

$$-\infty < v < \infty$$

Because of the term e^{-2uv}, we cannot separrate the pdf into two functions of u and v.

(e) (3) What is the conditional expectation of U given V, $\mathbb{E}(U|V)$? (Hint: Do not integrate, just stare at the conditional pdf)

$$f(u|v) = \frac{1}{2\pi} e^{\frac{-2v^2-2vu-u^2}{2}} / \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2-2vu-u^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{\frac{-(u+v)^2}{2}}$$

Because the above is a normal distribution with mean $-v$, then

$$\mathbb{E}(U|V) = -V$$

Bernoulli

$P(X = x|p) = p^x(1-p)^{(1-x)}$; $x = 0, 1$; $0 \leq p \leq 1$ with $\mathbb{E}(X) = p$ and $\text{Var}(X) = p(1-p)$

Binomial

$P(X = x|n, p) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x(1-p)^x$; $x = 0, 1, 2, ..., n$; $0 \leq p \leq 1$, with $\mathbb{E}(X) = np$ and $\text{Var}(X) = np(1-p)$

Discrete uniform

$P(X = x|N) = \frac{1}{N}$; $x = 1, 2, ..., N$; $N = 1, 2, ...,$ with $\mathbb{E}(X) = \frac{N+1}{2}$ and $\text{Var}(X) = \frac{(N+1)(N-1)}{12}$

Poisson

$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, ...$; $0 \leq \lambda < \infty$ with $\mathbb{E}(X) = \lambda$ and $\text{Var}(X) = \lambda$

Exponential

$f(x|\beta) = \lambda e^{-\lambda x}$; $0 \leq x < \infty$ with $\mathbb{E}(X) = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$

Logistic

$f(x|\mu, \beta) = \frac{1}{\beta} \frac{\exp(-(x-\mu)/\beta)}{[1+\exp(-(x-\mu)/\beta)]^2}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\beta > 0$ with $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \frac{\pi^2\beta^2}{3}$

Normal

$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$ with $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$