Graduate Econometrics II
March 2017 Final

1. Suppose
$$\sqrt{n}\left(\widehat{\mu}-\mu\right) \rightsquigarrow \mathcal{N}\left(0, v^2\right). \tag{1}$$
Set $\beta=\mu^2$ and $\widehat{\beta}=\widehat{\mu}^2$.

a. Determine the asymptotic distribution for $\sqrt{n}\left(\widehat{\beta}-\beta\right)$.

Answer. By the Delta Method, $\sqrt{n}\left(g\left(\widehat{\mu}\right)-g\left(\mu\right)\right) \rightsquigarrow \mathcal{N}\left(0, g'\left(\mu\right)^2 v^2\right)$. Here $g'\left(u\right)=2u$, so
$$\sqrt{n}\left(g\left(\widehat{\mu}\right)-g\left(\mu\right)\right) \rightsquigarrow \mathcal{N}\left(0, 4\mu^2 v^2\right).$$

b. Now suppose $\mu=0$. Describe what happens to the asymptotic distribution from the previous part.

Answer. Mechanically, if $\mu=0$, then the variance of the asymptotic distribution from the previous part is zero. In essence, if $\widehat{\mu}$ is small, then $\widehat{\mu}^2$ is very small and, even when multiplied by $\sqrt{n}$, is collapsing to 0.

c. Under the assumption that $\mu=0$ we can improve on the approximation in the previous answer. Find the asymptotic distribution for $n\widehat{\beta}=n\widehat{\mu}^2$.

Answer. If $\mu=0$, then $\sqrt{n}\left(\widehat{\mu}-\mu\right)=\sqrt{n}\widehat{\mu}$. From (1) $\frac{\sqrt{n}\widehat{\mu}}{v} \rightsquigarrow \mathcal{N}\left(0,1\right)$,so by the Continuous Mapping Theorem
$$\frac{n\widehat{\mu}^2}{v^2} \rightsquigarrow \chi_1^2$$
and
$$n\widehat{\mu}^2 \rightsquigarrow v^2 \cdot \chi_1^2.$$

d. Use a Taylor expansion for $\widehat{\beta}=g\left(\widehat{\mu}\right)$ to comment on the differences between the answers in parts a and c.

Answer. The Taylor expansion is
$$\widehat{\mu}^2=\mu^2+2\mu\left(\widehat{\mu}-\mu\right)+\frac{1}{2}2\left(\widehat{\mu}-\mu\right)^2+R.$$

Note, $\sqrt{n}\left(\widehat{\mu}-\mu\right)=O_p\left(1\right)$ and $\sqrt{n}\left(\widehat{\mu}-\mu\right)^2=o_p\left(1\right)$, thus

$$\sqrt{n}\left(\widehat{\mu}^2-\mu^2\right)=2\mu\cdot\sqrt{n}\left(\widehat{\mu}-\mu\right)+o_P\left(1\right).$$

This is the first-order approximation that yields the delta rule

$$\sqrt{n}\left(\widehat{\beta}-\beta\right)\rightsquigarrow\mathcal{N}\left(0,4\mu^2v^2\right).$$

When $\mu=0$, we must use a second-order expansion. If $\mu=0$, then

$$\widehat{\mu}^2=\mu^2+\left(\widehat{\mu}-\mu\right)^2+R,$$

where $n\left(\widehat{\mu}-\mu\right)^2=O_p\left(1\right)$. Thus the second-order approximation is

$$n\left(\widehat{\mu}^2-\mu^2\right)=n\left(\widehat{\mu}-\mu\right)^2+o_P\left(1\right).$$

Because $\frac{\sqrt{n}(\widehat{\mu}-\mu)}{v}\rightsquigarrow\mathcal{N}\left(0,1\right)$, it follows that $\frac{n(\widehat{\mu}-\mu)^2}{v^2}\rightsquigarrow\chi^2_{(1)}$. Because $\mu=0$

$$\frac{n\left(\widehat{\mu}-\mu\right)^2}{v^2}=\frac{n\widehat{\mu}^2}{v^2}=\frac{n\widehat{\beta}}{v^2},$$

so that

$$n\widehat{\beta}\rightsquigarrow v^2\cdot\chi^2_1.$$

2. Consider the following model:

$$wage=\beta_1 ed+\beta_2 ab+u \qquad (2)$$

where $wage$ is log wage, ed is education, and ab is innate ability, and everything is measured in deviations from means.

a What assumption is needed for Equation 2 to be the CEF of wages given education and ability?

$\mathbb{E}[u|ed,ab]=0$

b What is the interpretation of β_1? In light of your answer, why might we think Equation 2 is not a good representation of the CEF?

An additional year of income increases your wage by $\beta_1 * 100\%$, holding innate ability constant. We might not expect the returns to education to be the same for all education levels.

c Propose an alternative specification of the conditional expectation of wages as a function of education (holding ability constant).

Possible answers include:

$\mathbb{E}[wage|ed] = \gamma_1 ed + \gamma_2 ed^2$

$\mathbb{E}[wage|ed] = \gamma_0 + \gamma_1 \mathbb{1}(ed > 8) + \gamma_2 \mathbb{1}(ed > 12) + \gamma_3 \mathbb{1}(ed > 16)$

Suppose we don't have data on innate ability so we rewrite our model as:

$$wage = \gamma ed + v \tag{3}$$

where γ is the linear projection coefficient.

d Solve for γ in terms of β_1 and β_2

$$\begin{aligned}
\gamma &= \frac{\mathbb{E}[ed * wage]}{\mathbb{E}[ed^2]} && \text{Def of linear projection coefficient} \\
&= \frac{\mathbb{E}[ed(\beta_1 ed + \beta_2 ab + u)]}{\mathbb{E}[ed^2]} && \text{plug in for wage} \\
&= \beta_1 + \beta_2 \frac{Cov(ed, ab)}{Var(ed)} + \frac{Cov(ed, u)}{Var(ed)} \\
&= \beta_1 + \beta_2 \frac{\mathbb{E}[ed * ab]}{\mathbb{E}[ed^2]} && \text{since } \mathbb{E}[ed * u] = 0
\end{aligned}$$

e Is γ larger, smaller or the same as β_1? State any assumptions you make.

If we assume $\beta_2 > 0$ and $Cov(ab, ed) > 0$ then $\gamma > \beta_1$

f What is $Cov(ed, v)$?

$$
\begin{aligned}
Cov(ed, v) &= \mathbb{E}[ed * v] \\
&= \mathbb{E}[ed(wage - \gamma ed)] && \text{plug in for v} \\
&= \mathbb{E}[ed(\beta_1 ed + \beta_2 ab + u - \beta_1 ed - \beta_2 \frac{\mathbb{E}[ed * ab]}{\mathbb{E}[ed^2]} ed) && \text{plug in for wage and } \gamma \\
&= \mathbb{E}[ed\beta_2(ab - \frac{\mathbb{E}[ed * ab]}{\mathbb{E}[ed^2]} ed)] + \mathbb{E}[ed * u] \\
&= \beta_2(\mathbb{E}[ed * ab] - \frac{\mathbb{E}[ed * ab]}{\mathbb{E}[ed^2]} \mathbb{E}[ed^2]) && \text{since } \mathbb{E}[ed * u] = 0 \\
&= 0
\end{aligned}
$$

g Now suppose we have data for a particular partition of wages and education. In particular, there are three groups of wages (Low, Medium, and High) and two groups of education (High School, College). The joint distribution is given by:

		ed	
		High School	College
wage	H	0.05	0.20
	M	0.05	0.50
	L	0.10	0.10

Suppose you meet two people, Chris and Bryan. Chris is in the High School education group and Bryan is in the College education group. Who is more likely to have High wages? Why?

They are equally likely. To see this, we need to construct the conditional probabilities. There are many more high wage earners with a college education than with a high school education, but there are also many more people with a college education.

$P(wage = H|ed = College) = \frac{P(wage=H, ed=College)}{P(ed=College)} = \frac{0.2}{0.8} = \frac{1}{4}$

$P(wage = H|ed = HighSchool) = \frac{P(wage=H, ed=HighSchool)}{P(ed=HighSchool)} = \frac{0.05}{0.2} = \frac{1}{4}$