241B Lecture
Stochastic Processes

Martingales

Let X_t be an element of Z_t. Then X_t is a martingale with respect to Z_t if

$$E\left[X_t | Z_{t-1}, Z_{t-2}, \ldots, Z_1\right] = X_{t-1} \text{ for all } t \geq 2.$$

- The collection $(Z_{t-1}, Z_{t-2}, \ldots)$ is called the information set at $t-1$
- If the conditioning information set is $(X_{t-1}, X_{t-2}, \ldots)$, then X_t is a martingale (it is implicit that X_t is a martingale with respect to X)
- If X_t is a martingale with respect to Z_t then X_t is a martingale (because Z_t contains X_t)
- The vector Z_t is a martingale if $E\left[Z_t | Z_{t-1}, Z_{t-2}, \ldots, Z_1\right] = Z_{t-1}$ for all $t \geq 2$
- If the process started in the infinite past, there is no need to include the qualifier "for all $t \geq 2$"
- For our large-sample results, it does not matter if the process started in the infinite past (simply that the process started before the first observation)

Random Walks

A leading example of a martingale process is a random walk. Let $\{U_t\}$ be vector independent white noise, so $EU_t = 0$ and the covariance matrix of U_t is finite. A random walk is a sequence of cumulative sums

$$Z_1 = U_1, Z_2 = U_1 + U_2, \ldots.$$

As the underlying white noise can be deduced from $\{Z_t\}$ via

$$U_1 = Z_1, U_2 = Z_2 - Z_1, \ldots$$

the two processes contain the same information (and the first difference of a random walk is independent white noise).

That a random walk is a martingale is shown as

$$\begin{aligned} E\left[Z_t|Z_{t-1},Z_{t-2},\ldots,Z_1\right] &= E\left[Z_t|U_{t-1},U_{t-2},\ldots,U_1\right] \\ &= E\left[U_1+\cdots U_t|U_{t-1},U_{t-2},\ldots,U_1\right] \\ &= U_1+\cdots+U_{t-1} \text{ because } E\left[U_t|U_{t-1},U_{t-2},\ldots,U_1\right]=0 \\ &= Z_{t-1}. \end{aligned}$$

Martingale Difference Sequences

A vector process $\{U_t\}$ is called a martingale difference sequence (m.d.s.) if

$$E\left[U_t|U_{t-1},U_{t-2},\ldots,U_1\right]=0 \text{ for } t\geq 2.$$

The process is so called because the cumulative sum formed from an m.d.s. is a martingale. Conversely, if $\{Z_t\}$ is a martingale, the first difference of Z_t forms a martingale difference sequence.

- A martingale difference sequence has no serial correlation.

  $$E\left(U_tU'_{t-j}\right)=E\left(E\left(U_t|U_{t-j}\right)U'_{t-j}\right).$$

  Note

  $$E\left(U_t|U_{t-j}\right)=E\left(E\left(U_t|U_{t-1},\ldots,U_{t-j},\ldots U_1\right)|U_{t-j}\right)=0.$$

ARCH Processes

An important class of martingale difference sequences are ARCH sequences, which are commonly used to model financial asset prices. Introduced by Engle, an ARCH(1) process is

$$U_t=\sqrt{\delta+\alpha U_{t-1}^2}\cdot V_t,$$

where $\{V_t\}$ is i.i.d. with mean 0 and variance 1. If U_1 is the initial value of the process, then U_t is a function of U_1 and $(V_2,\ldots,V_t)$. Therefore V_t is independent of $(U_1,\ldots,U_{t-1})$.

To show that an ARCH sequence is an m.d.s.,

$$\begin{aligned} E\left[U_t|U_{t-1},U_{t-2},\ldots,U_1\right] &= E\left[\sqrt{\delta+\alpha U_{t-1}^2}\cdot V_t|U_{t-1},U_{t-2},\ldots,U_1\right] \\ &= \sqrt{\delta+\alpha U_{t-1}^2}E\left[V_t|U_{t-1},U_{t-2},\ldots,U_1\right] \\ &= 0 \text{ because } V_t \text{ is independent of } \left(U_1,\ldots,U_{t-1}\right). \end{aligned}$$

- An ARCH process is conditionally heteroskedastic

$$E\left[U_t^2|U_{t-1}, U_{t-2}, \ldots, U_1\right] = \delta + \alpha U_{t-1}^2.$$

If $|\alpha| < 1$ the process is strictly stationary and ergodic (provided the process started in the infinite past or that U_1 is drawn from an appropriate distribution).

- If the process is stationary, the unconditional variance is

$$EU_t^2 = \delta + \alpha EU_{t-1}^2.$$

Because $EU_t^2 = EU_{t-1}^2$ under stationarity,

$$EU_t^2 = \frac{\delta}{1-\alpha}.$$

Formulations of Serial Uncorrelatedness

For zero mean, covariance stationary processes, we have three different strengths of lack of serial correlation. From strongest to weakest:

1. $\{Z_t\}$ is independent white noise
2. $\{Z_t\}$ is a stationary m.d.s. with finite variance
3. $\{Z_t\}$ is white noise.

Level 2 allows processes that are dependent, although serially uncorrelated. For example, the ARCH process in which the conditional variance of Z_t depends on Z_{t-1}. Level 3 allows processes that have conditional means that are not zero, although the unconditional mean remains zero. Consider the example process in which the cosine function is used

$$Z_t = \cos(tw) \ \ (t = 1, 2, \ldots).$$

We have

$$E(Z_2|Z_1) = E(\cos(2w)\,|\cos(w)) = 2\cos(w)^2.$$

The CLT for Ergodic Stationary Martingale Difference Sequences

The following CLT extends the Lindberg-Levy CLT to stationary and ergodic m.d.s.

Ergodic Stationary Martingale Difference CLT: *Let* $\{U_t\}$ *be a vector martingale difference sequence that is stationary and ergodic with* $E\left(U_t U_t'\right) = \Omega$ *and* $\bar{U} = \frac{1}{n}\sum_{t=1}^{n} U_t$. *Then*

$$\sqrt{n}\bar{U} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} U_t \underset{d}{\rightarrow} N\left(0, \Omega\right).$$

This CLT is applicable not just to i.i.d. sequences, but also to stationary martingale difference sequences, such as ARCH processes (although we have not yet allowed for serially correlated processes).

1. Because $\{U_t\}$ is an m.d.s. with mean zero, there is no need to subtract a mean from $\bar{U}$.

2. Because $\{U_t\}$ is stationary, the covariance matrix does not depend on t. It is implicit that the moments exist and are finite.