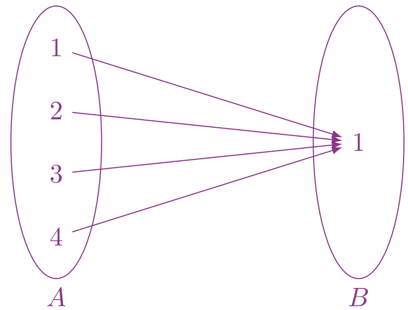
Required Problems

1. **Let $A = \{1, 2, 3, 4\}$. Discribe a codomain B and a function $f : A \to B$ such that f is**

   (a) **onto B but not one-to one.**

   - Let the codomain be $B = \{1\}$
   - Let the function be $f(x) = 1$

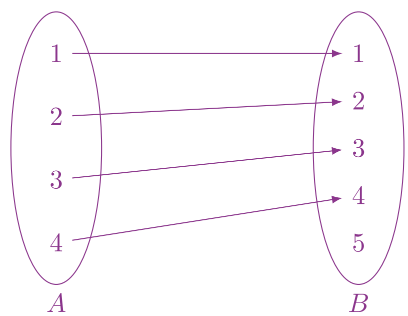
   The range $\mathcal{R}(f) = \{1\}$ is the same as the codomain, so the function is onto. More than one element maps to 1, however, so the function *is not* one-to-one:



   (b) **one-to-one but not onto B.**

   - Let the codomain be $B = \{1, 2, 3, 4, 5\}$
   - Let the function be $f(x) = x$

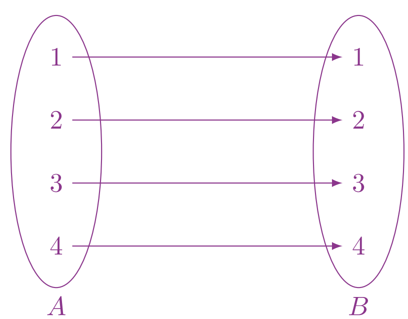
   Every element in B has only one corresponding element in A, satisfying the definition of one-to-one. The range of the function is $\mathcal{R}(f) = \{1, 2, 3, 4\}$; this is smaller than the codomain, so the function *is not* onto:



   (c) **both one-to-one and onto B.**

   - Let the codomain be $B = \{1, 2, 3, 4\}$
   - Let the function be $f(x) = x$

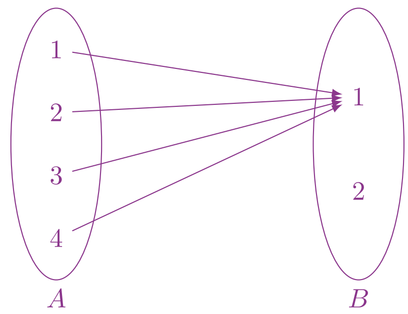
   Every element in B has only one corresponding element in A; the range, $\mathcal{R}(f) = \{1, 2, 3, 4\}$, is now the same as the codomain. Thus, the function is one-to-one and onto:



(d) **neither one-to-one nor onto B.**

- Let the codomain be $B = \{1, 2\}$
- Let the function be $f(x) = 1$

Now the function maps multiple elements from the domain to the same element in the range; further, the codomain contains more points than the range. Thus, the function is neither one-to-one nor onto:



2. **Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that**

$$x_n = \frac{n+1}{n}$$

To what does this sequence converge? Prove that this sequence converges to that limit.

This sequence converges to 1, i.e., $x_n \to 1$.

To show: $|\frac{n+1}{n} - 1| < \varepsilon$.

Proof:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 && \text{(by hypothesis)}\\
&\text{Let } N > \frac{1}{\varepsilon} \text{ and } n > N && \text{(by hypothesis)}\\
&\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| && \text{(simplifying)}\\
&= \frac{1}{n} && \text{(by } n > 0)\\
&< \frac{1}{N} && \text{(by } n > N)\\
&< \frac{1}{1/\varepsilon} && \text{(by } N > 1/\varepsilon)\\
&= \varepsilon && \text{(simplifying)}
\end{aligned}$$

■

3. **Let S and T be convex sets. Prove that the intersection of S and T is also a convex set.**

To show: $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S \cap T$

Proof:

$$\begin{aligned}
&\text{Let } S \text{ and } T \text{ be convex sets, } \mathbf{x}_1, \mathbf{x}_2 \in S \cap T \text{, and } t \in [0,1] && \text{(by hypothesis)}\\
&\implies \Big(\mathbf{x}_1 \in S \wedge \mathbf{x}_1 \in T\Big) \wedge \Big(\mathbf{x}_2 \in S \wedge \mathbf{x}_2 \in T\Big) && \text{(by def. of } \cap)\\
&\implies \Big(\mathbf{x}_1 \in S \wedge \mathbf{x}_2 \in S\Big) \wedge \Big(\mathbf{x}_1 \in T \wedge \mathbf{x}_2 \in T\Big) && \text{(by associativity)}\\
&\implies \Big(t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S\Big) \wedge \Big(t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in T\Big) && \text{(by def. of convex)}\\
&\implies t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S \cap T && \text{(by def. of } \cap)
\end{aligned}$$

■

4. **The set** $S^{n-1} = \{\mathbf{x} | \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n\}$ **is the** $(n-1)$**-dimensional unit simplex.**

(a) **Prove that** S^{n-1} **is a convex set.**

To show: $t\mathbf{x} + (1-t)\mathbf{y} \in S$

Proof:

$$
\begin{aligned}
&\text{Let } \mathbf{x}, \mathbf{y} \in S \text{ and } t \in [0,1] && \text{(by hypothesis)}\\
&\text{Consider } t\mathbf{x} + (1-t)\mathbf{y} && \text{(the convex combo.)}\\
&0 \leq tx_i + (1-t)y_i < 1 \quad \forall i = 1, \ldots, n && \text{(by } t \in [0,1])\\
&\sum_{i=1}^{n} \Big(tx_i + (1-t)y_i \Big) && \text{(summing the elements)}\\
&= \sum_{i=1}^{n} tx_i + \sum_{i=1}^{n} (1-t)y_i && \text{(by associativity)}\\
&= t\sum_{i=1}^{n} x_i + (1-t)\sum_{i=1}^{n} y_i && \text{(by distributivity)}\\
&= t \cdot 1 + (1-t) \cdot 1 && \text{(by } \mathbf{x}, \mathbf{y} \in S)\\
&= 1 && \text{(simplifying)}\\
&\implies t\mathbf{x} + (1-t)\mathbf{y} \in S && \text{(by def. of } S)
\end{aligned}
$$

■

(b) **Prove that** S^{n-1} **is a compact set.**

- Theorem (T1): $\mathbf{x}_k \to \mathbf{c} \iff x_{ik} \to c_i$ for all i (each element of the vector converges)
- Theorem (T2): $a_k \to a$ and $b_k \to b$ implies $a_k + b_k \to a + b$
- Theorem (T3): Weak inequalities are preserved in in the limit
- Lemma (L1): Constant sequences converge, i.e., $\{d, d, d \ldots\} \to d$

To show: $\mathbf{c} \in S$ Proof:

$$
\begin{aligned}
&\text{Let } \{\mathbf{x}_k\}_{k=0}^{\infty} \text{ be a sequence in } S \ni \big(\mathbf{x}_k \to \mathbf{c}\big) \wedge \big(\mathbf{x}_k \in S \;\forall\, k\big) && \text{(by hypothesis)}\\
&\implies \Big(x_{ik} \to c_i, c_i \geq 0 \;\forall\, i\Big) \wedge \left(\sum_{i=1}^{n} x_{ik} = 1 \;\forall\, k\right) \wedge \Big(x_{ik} \geq 0 \;\forall\, i, k\Big) && \text{(by T1, T3, and } \mathbf{x}_k \in S)\\
&\implies \left(\sum_{i=1}^{n} x_{ik} \to \sum_{i=1}^{n} c_i\right) \wedge \left(\sum_{i=1}^{n} x_{ik} \to 1\right) && \text{(by T2 and L1)}\\
&\implies \sum_{i=1}^{n} c_i = 1 && \text{(limits are unique)}\\
&\implies \mathbf{c} \in S && \text{(by def. of } S)
\end{aligned}
$$

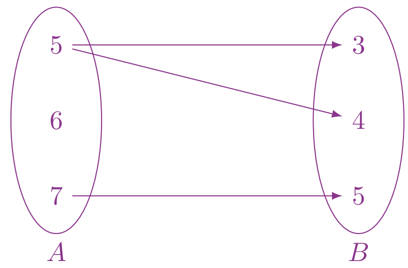
Because the convergent sequence and limit were arbitrary, it must be the case that the limit point of every convergent sequence is in S. Thus, S is closed.

To show that the sequence is bounded, we need only come up with one example for M such that $B_M(\mathbf{0})$ contains S. Note that $||\mathbf{x}|| \leq 1$ (by $0 \leq x_i \leq 1$). Thus, if $M = 2$ (or indeed, anything greater than 1), every point in S will be wholly contained in $B_M(\mathbf{0})$. Thus, S is closed and bounded, implying it is compact. ■

Practice Problems

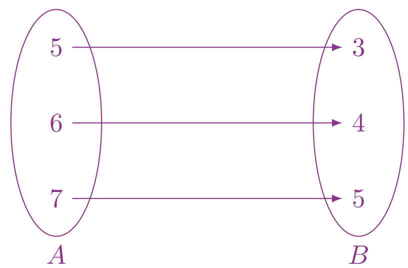
5. **Give a relation r from $A = \{5, 6, 7\}$ to $B = \{3, 4, 5\}$ such that**

   (a) r **is not a function**



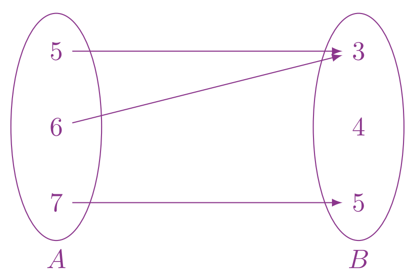
   This is not a function, as one of the points in the domain is not mapped to the range; further, 5 is mapped to two different elements in the range.

   (b) r **is a function from A to B with the range $\mathcal{R}(r) = B$**



   This relation is a function; every element in the codomain has a corresponding element in the domain, so $\mathcal{R}(r) = B$.

   (c) r **is a function from A to B with the range $\mathcal{R}(r) \neq B$**



   This relation is a function, but one element in the codomain does not have a corresponding element in the domain, so $\mathcal{R}(r) \neq B$.

6. **Identify the domain and range of each of the following mappings:**

   (a) $\left\{(x, y) \in \mathbb{R}^2 \middle| y = \frac{1}{x+1}\right\}$

   - Domain: $D = \mathbb{R} - \{1\}$
   - Range: $R = \mathbb{R} - \{0\}$

   (b) $\{(x, y) \in \mathbb{N} \times \mathbb{N} | y = x + 5\}$

   - Domain: $D = \mathbb{N}$
   - Range: $R = \mathbb{N} - \{1, 2, 3, 4, 5\}$

(c) $\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \middle| y = \frac{x^2-4}{x-2}\right\}$

- Domain: $D = \mathbb{Z} - \{2\}$
- Range: $R = \mathbb{Z} - \{4\}$

7. **Recall the definition of the inverse image associated with the function $f : X \to Y$, i.e.,**

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

If $B \subset Y$ and $C \subset Y$, prove that $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$.

To show: $x \in f^{-1}(B \cup C) \iff x \in f^{-1}(B) \cup f^{-1}(C)$

Proof:

$$\begin{aligned}
&\text{Let } x \in f^{-1}(B \cup C) && \text{(by hypothesis)}\\
&\iff f(x) \in B \cup C && \text{(by def of } f^{-1})\\
&\iff f(x) \in B \vee f(x) \in C && \text{(by def. of } \cup)\\
&\iff x \in f^{-1}(B) \vee x \in f^{-1}(C) && \text{(by def. of } f^{-1})\\
&\iff x \in f^{-1}(B) \cup f^{-1}(C) && \text{(by def. of } \cup)
\end{aligned}$$

■

8. **For each of the following sequences, list the first three terms:**

(a) $a_n = \frac{n+1}{2n+3}$

$$\left\{\frac{2}{5}, \quad \frac{3}{7}, \quad \frac{4}{11}, \dots\right\}$$

(b) $b_n = \frac{1}{n!}$

$$\left\{1, \quad \frac{1}{2}, \quad \frac{1}{6}, \dots\right\}$$

(c) $c_n = 1 - 2^{-n}$

$$\left\{\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \dots\right\}$$

9. **Prove that if $x_n \to L$ and $y_n \to M$, then $x_n + y_n \to L + M$.**

To show: $|(x_n + y_n) - (L + M)| < \varepsilon$

Proof:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 && \text{(by hypothesis)}\\
&\Longrightarrow \left(\exists N_x \ni n > N_x \Rightarrow |x_n - L| < \frac{\varepsilon}{2}\right) \wedge \left(\exists N_y \ni n > N_x \Rightarrow |y_n - M| < \frac{\varepsilon}{2}\right) && \text{(by def. of convergence)}\\
&\text{Let } n > \max\{N_x, N_y\} && \text{(defining } n)\\
&\text{Consider } \big|(x_n + y_n) - (L + M)\big| && \text{(summing the sequences \& limits)}\\
&= |(x_n - L) + (y_n - M)| && \text{(by associativity)}\\
&\leq |x_n - L| + |y_n - M| && \text{(by the triangle inequality)}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(by } n > \max\{N_x, N_y\})\\
&= \varepsilon && \text{(simplifying)}
\end{aligned}$$

■

10. **Prove that if $a_n \to a$ and $a_n \le b$ for all n, then $a \le b$.**

Proof by contradiction <u>to show</u>: $a_n \le b \;\forall\; n$ and $\exists a_n > b$

<u>Proof</u>:

Let $\big(a_n \to a\big) \wedge \big(a_n \le b \;\forall\; n\big)$	(by hypothesis)
Suppose $a > b$	(towards a contradiction)
Let $\varepsilon = a - b$	(defining ε)
$\implies \exists N \ni n > N \implies \lvert a_n - a \rvert < \varepsilon$	(by def. of convergence)
Consider $\lvert a_n - a \rvert < \varepsilon$	
$\implies -\varepsilon < a_n - a < \varepsilon$	(by the absolute value)
$\implies a - \varepsilon < a_n < a + \varepsilon$	(rearranging)
$\implies a - (a - b) < a_n$	(by def. of ε)
$\implies b < a_n$	(simplifying)
Buy $a_n \le b$ by assumption; thus, a contradiction	
$\implies a \le b$	■

11. **Consider the following intervals in $\mathbb{R}$. For each, determine if it is closed. If so, give a proof:**

(a) $(-\infty, b]$

This interval is closed. <u>To show</u>: $B_\varepsilon(x) \subset (b, \infty)$.

<u>Proof</u>:

Let $\big[x \in (b, \infty)\big] \wedge \big[\varepsilon = x - b\big]$	(by hypothesis)
Consider $B_\varepsilon(x)$	(defining an ε-ball)
Let $y \in B_\varepsilon(x)$	(picking a point in $B_\varepsilon(x)$)
$\implies x - \varepsilon < y < x + \varepsilon$	(by def. of $B_\varepsilon(x)$)
$\implies x - (x - b) < y < x + (x - b)$	(by def. of ε)
$\implies b < y < 2x + b$	(simplifying)
$\implies b < y < \infty$	($2x + b$ is finite)
$\implies y \in (b, \infty)$	(by def. of (b, ∞))
$\implies B_\varepsilon(x) \subset (b, \infty)$	(by def. of subset)

Because x was an arbitrary point and $B_\varepsilon(x) \subset (b, \infty)$, the interval is open. Thus, its complement $(-\infty, b]$ is closed. ■

(b) $(a, b]$

This interval is neither open nor closed. Consider a sequence $a_n = a + \frac{1}{kn}$, where k is a constant large enough such that $a_n \in (a, b]$ for all n. This sequence converges to a, but a is not in the set. Thus, the set is not closed.

(c) $[a, \infty)$

This interval is closed.

- Theorem (T1): weak inequalities are preserved in the limit (see problem 10)

<u>To show</u>: $x \in [a, \infty)$.

<u>Proof</u>:

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence $\ni \big(x_n \to x\big) \wedge \big(x_n \in [a, \infty) \;\forall\; n\big)$	(by hypothesis)
$\implies x_n \ge a \;\forall\; n$	(by $x_n \in [a, \infty)$)
$\implies x \ge a$	(by T1)

Because the convergent sequence and the limit point were arbitrary, it must be the case that the limit point of every convergent sequence is in $[a, \infty)$. ■

(d) $[a, b)$

As in part (b), this interval is neither open nor closed. Consider a sequence $b_n = b - \frac{1}{kn}$, where k is a constant large enough such that $b_n \in [a, b)$ for all n. This sequence converges to b, but b is not in the set. Thus, the set is not closed.

12. **Consider the following sets. If the set is bounded, provide an M and a x such that $B_M(\mathbf{x})$ contains the set.**

(a) $A = \{x | x \in \mathbb{R} \wedge x^2 \leq 10\}$

This set is bounded above and below by $\sqrt{10}$ and $-\sqrt{10}$, respectively. Thus, let $x = 0$ and $M = 4$. Then $B_M(0)$ contains the entire set.

(b) $B = \left\{x | x \in \mathbb{R} \wedge x + \frac{1}{x} < 5\right\}$

The function is bounded above by $\frac{5+\sqrt{21}}{2}$, but is not bounded below—x can take on any value in the real numbers less than zero.

(c) $C = \{(x, y) | (x, y) \in \mathbb{R}^2_+ \wedge xy < 1\}$

This set is not bounded. No matter how large x gets, there exists a y such that $xy < 1$ (and vice versa).

(d) $D = \{(x, y) | (x, y) \in \mathbb{R} \wedge |x| + |y| \leq 10\}$

Both x and y must fall between -10 and 10. Thus, let $x = 0$ and $M = 11$. Then $B_M(0)$ contains the entire set.

13. **Prove that the following functions are continuous using epsilon-delta proofs.**

(a) $f(x) = x + 3$

<u>To show</u>: $|(x + 3) - (x_0) + 3| < \varepsilon$
<u>Proof</u>:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(f) && \text{(by hypothesis)}\\
&\text{Let } \delta = \varepsilon && \text{(defining } \delta)\\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta &&\\
&\implies |x - x_0| < \varepsilon && \text{(by def. of } \delta)\\
&\implies |x - x_0 + 3 - 3| < \varepsilon && \text{(adding zero)}\\
&\implies |(x - 3) - (x_0 - 3)| < \varepsilon && \text{(by associativity)}
\end{aligned}$$

■

This is the basic format of a simple continuity proof: pick an arbitrary ε and an arbitrary point in the domain; pick a specific δ (typically a function of ε); show that if x is within δ of x_0, then $f(x)$ must be within ε of $f(x_0)$.

(b) $g(x) = x^2$

<u>To show</u>: $|x^2 - x_0^2| < \varepsilon$
<u>Proof</u>:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(g) && \text{(by hypothesis)}\\
&\text{Let } \delta \leq \min\left\{1, \frac{\varepsilon}{2 + 2|x_0|}\right\} && \text{(defining } \delta)\\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta &&
\end{aligned}$$

$$\begin{aligned}
&\implies |x - x_0| < \frac{\varepsilon}{2 + 2|x_0|} && \text{(by def. of } \delta\text{)}\\
&\implies |x - x_0| < \frac{\varepsilon}{1 + \delta + 2|x_0|} && \text{(by } \delta < 1\text{)}\\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x - x_0| + 2|x_0|} && \text{(by } |x - x_0| < \delta\text{)}\\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x - x_0 + 2x_0|} && \text{(by the triangle inequality)}\\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x + x_0|} && \text{(simplifying)}\\
&\implies |x - x_0||x + x_0| < \varepsilon && \text{(by } |x + x_0| \geq 0\text{)}\\
&\implies |(x - x_0)(x + x_0)| < \varepsilon && \text{(by } |ab| = |a||b|\text{)}\\
&\implies |x^2 - x_0^2| < \varepsilon && \text{(simplifying)}
\end{aligned}$$

$\blacksquare$

Note that throughout, the denominator is slightly more complicated than seems necessary (e.g., there's always a "$1 + \ldots$" on the bottom of a fraction). This is to handle the case where $x_0 = x = 0$.

(c) $h(x) = |x|$

<u>To show</u>: $||x| - |x_0|| < \varepsilon$
<u>Proof</u>:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(h) && \text{(by hypothesis)}\\
&\text{Let } \delta = \varepsilon && \text{(defining } \delta\text{)}\\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta && \\
&\implies |x - x_0| < \varepsilon && \text{(by def. of } \delta\text{)}\\
&\implies ||x| - |x_0|| < \varepsilon && \text{(by the reverse triangle ineq.)}
\end{aligned}$$

$\blacksquare$

This relies on the "reverse triangle inequality," which is easy to show:

$$\begin{aligned}
&|y| = |x + y - x| \leq |x| + |y - x| && \text{(by the triangle inequalit)}\\
&\implies |y| - |x| \leq |y - x| && \text{(rearranging)}\\
&|x| = |y + x - y| \leq |y| + |x - y| && \text{(by the triangle inequality)}\\
&\implies |x| - |y| \leq |x - y| && \text{(rearranging)}
\end{aligned}$$

Noting that $|x - y| = |y - x|$, and $|y| - |x| = -(|x| - |y|)$ we then have two conditions:

$$\begin{aligned}
&\Big(|x| - |y| \leq |x - y|\Big) \wedge \Big(-(|x| - |y|) \leq |x - y|\Big) && \text{(restating inequalities)}\\
&\implies \big||x| - |y|\big| \leq |x - y| && \text{(combining)}
\end{aligned}$$