Chapter 1

Preference Revelation Mechanisms for Public Goods

1.1 The Groves-Clarke Mechanism

In the late 1960's, the idea stork delivered similar good ideas to two different economics graduate students, Ed Clarke at the University of Chicago and Ted Groves at UC Berkeley. Each of them independently proposed a taxation scheme that would induce rational selfish consumers to reveal their true preferences for public goods, while the government would supply a Pareto optimal quantity of public goods based on this information.[1] The clearest presentation of the Groves-Clarke idea that I have come across is in a paper by Groves and Loeb [?]. The Groves-Loeb paper is motivated as a problem in which several firms share a public good as a factor of production. Each firm knows its own production function but not that of others. A central authority will decide the amount of the public factor of production to purchase and the way to allocate its cost based on information supplied by the firms. This problem is formally the same as a public goods problem with quasi-linear utility.

The Groves-Clarke mechanism for providing public goods is well-defined only for the case of quasi-linear utility. We will consider the following model. There is one private good and one public good. Consumer i has the utility function

$$U_i(X_i, Y) = X_i + F_i(Y) \tag{1.1}$$

where X_i is his private good consumption and Y is the amount of public good. Each i has an initial endowment of W_i units of private good. Public good must be produced using private goods as an input. The total amount of private

[1] Clarke's solution to this problem was published in [?]. Groves' solution appeared in his unpublished 1969 ph.d. thesis.

goods needed to produce Y units of public good is a function $C(Y)$. Assume that F_i is a strictly concave function and C a convex function. If we consider only allocations in which everyone receives at least some private good, then for this economy there is a unique Pareto optimal quantity of public good. This quantity maximizes

$$\sum_i F_i(Y) - C(Y) \tag{1.2}$$

Consumers are asked to reveal their functions F_i to the government. Let M_i (possibly different from F_i) be the function that consumer i claims. Let $M = (M_1, \cdots, M_n)$ be the vector of functions claimed by the population. If the reported vector is M, the government chooses a quantity of public goods $Y(M)$ that would be Pareto optimal if everyone were telling the truth about their utilities. That is, the government chooses $Y(M)$ such that:

$$\sum_i M_i(Y(M)) - C(Y(M)) \geq \sum_i M_i(Y) - C(Y) \tag{1.3}$$

for all $Y \geq 0$.

Taxes $T_i(M)$ are then assigned to each consumer i according to the formula

$$T_i(M) = C(Y(M)) - \sum_{j \neq i} M_j(Y(M)) - R_i(M), \tag{1.4}$$

where $R_i(M)$ is some function that may depend on the functions, M_j, reported by consumers other than i but is constant with respect to M_i.

If the vector of functions reported to the government is $M = (M_1, \cdots, M_n)$, then Consumer i's private consumption is

$$X_i(M) = W_i - T_i(M) \tag{1.5}$$

and his utility is

$$X_i(M) + F_i\left(Y(M)\right) = W_i + \sum_{j \neq i} M_j\left(Y(M)\right) + F_i\left(Y(M)\right) - C\left(Y(M)\right) + R_i(M) \tag{1.6}$$

Since $W_i + R_i(M)$ is independent of M_i, we notice that the only way in which i's stated function M_i affects his utility is through the dependence of $Y(M)$ on M_i. We see, therefore from 1.6 that given any choice of strategies by the other players, the best choice of M_i for i is the one that leads the government to choose $Y(M)$ so as to maximize

$$\sum_{j \neq i} M_j(Y) + F_i(Y) - C(Y). \tag{1.7}$$

But recall from expression 1.3 that the government attempts to maximize

$$\sum_{j=1}^{n} M_j(Y) - C(Y). \tag{1.8}$$

Therefore if consumer i reports his true function, so that, $M_i = F_i$, then when the government is maximizing 1.8 it maximize 1.7. It follows that the consumer can not do better and could do worse than to report the truth. Honest revelation is therefore a dominant strategy.

Since everyone chooses his dominant strategy, true preferences are revealed and the government's choice of $Y(M)$ is the value of Y that maximizes

$$\sum_{j=1}^{n} F_j(Y) - C(Y) \tag{1.9}$$

This leads to the correct amount of public goods. Of course for the device to be feasible, it must be that total taxes collected are at least as large as the total cost of the public goods. If the outcome is to be Pareto optimal, the amount of taxes collected must be no greater than the total cost of public goods. Otherwise private goods are wasted. We are left, therefore, with the task of trying to rig the functions $R_i(M)$ in such a way to establish this balance. In general, it turns out to be impossible to find functions $R_i(M)$ that are independent of M_i for each i and such that

$$\sum_i T_i(M) = C(Y(M)) \tag{1.10}$$

However, Clarke and also Groves and Loeb found functions $R_i(M)$ that guarantee that tax revenues at least cover total costs. Their idea can be explained as follows. Suppose that for each i, the government sets a "target share" $\theta_i \geq 0$ where $\sum_i \theta_i = 1$. The government then tries to fix $R_i(M)$ so that $T_i(M) \geq \theta_i C(Y(M))$ for each i. Then, of course, $\sum_i T_i(M) \geq C(Y(M))$. From equation (3), it follows that

$$T_i(M) - \theta_i C(Y(M)) = [(1-\theta_i)C(Y(M)) - \sum_{j \neq i} M_j(Y(M))] - R_i(M). \tag{1.11}$$

Therefore the government could set $T_i(M) = \theta_i C(Y(M))$ if and only if it could set

$$R_i(M) = (1-\theta_i)C(Y(M)) - \sum_{j \neq i} M_j(Y(M)). \tag{1.12}$$

But in general such a choice of $R_i(M)$ would be inadmissible for our purpose because $R_i(M)$ depends on M_i, since $Y(M)$ depends on M_i.

Suppose that the government sets

$$R_i(M) = \min_Y [(1-\theta_i)C(Y) - \sum_{j \neq i} M_j(Y)]. \tag{1.13}$$

Then $R_i(M)$ depends on the M_j's for $j \neq i$ but is independent of M_i. From (10) it follows that with this choice of $R_i(M)$ we have:

$$T_i(M) - \theta_i C(Y(M)) \geq 0 \text{forall } i \tag{1.14}$$

Therefore

$$\sum_i T_i(M) \geq C(Y(M)). \tag{1.15}$$

This establishes the claim we made for the Clarke tax.

1.2 The Groves-Ledyard Mechanism

Groves and Ledyard propose a demand revealing mechanism which they call "An Optimal Government". The mechanism formulates rules of a game in which the amount of public goods and the distribution of taxes is determined by the government as a result of messages which the citizens choose to communicate. Although the government has no independent knowledge of preferences, and citizens are aware that sending deceptive signals might possibly be beneficial, it turns out that Nash equilibrium for this game is Pareto optimal. The Groves–Ledyard mechanism is defined for general equilibrium and applies to arbitrary smooth convex preferences.

In contrast, the Clarke tax (discovered independently by Clarke [1971] and Groves and Loeb [1975]) is well defined only for economies in which relative prices are exogenously determined and where utility of all consumers takes the quasi–linear form:

$$U_i(X_i, Y) = X_i + F_i(Y). \tag{1.16}$$

The Clarke tax has the advantage that for each consumer, equilibrium is a dominant strategy equilibrium rather than just a Nash equilibrium. Thus there are no complications related to stability or multiple equilibria. On the other hand, the Clarke tax has the disadvantages that although it leads to a Pareto efficient amount of public goods it generally will lead to some waste of private goods.

Suppose that there are n consumers, and one public good and one private good. Each consumer has an initial endowment of W_i units of private good. Public good is produced at a constant unit cost of q.

The government asks each consumer i to submit a number, (positive or negative) m_i. The government will supply an amount of public goods $Y = \sum_i m_i$. To describe the Groves-Ledyard mechanism efficiently it is useful to define the following bits of notation: Define

$$\bar{m}_{\tilde{}i} = \frac{1}{n-1} \sum_{j \neq i} m_j \tag{1.17}$$

to be the average of the numbers submitted by persons other than i. The variance of the messages sent by persons other than i is

$$R_i(m) = \frac{1}{n-2} \sum_{j \neq i} (m_j - \bar{m}_{\tilde{}i})^2 \tag{1.18}$$

Notice that $R_i(m)$ depends on the m_j's for $j \neq i$, but does not depend on m_i. As we will see, when we tack these variance terms onto each player's tax bill, we make budgets balance.

When the vector of messages sent by individuals is $m = (m_1, \ldots, m_n)$, the Groves-Ledyard mechanism will impose a tax on individual i equal to

$$T^i(m) = \alpha_i q \sum_{k=1}^{n} m_k + \frac{\gamma}{2}\left[\left(\frac{n-1}{n}\right)(m_i - \bar{m}_{\sim i})^2 - R_i(m)\right] \tag{1.19}$$

where the α_i's and γ are arbitrarily chosen positive parameters and $\sum_k \alpha_k = 1$. (Though Expression 1.19 looks nasty, remember that it is only a quadratic, and we are soon going to defang this beast by differentiating it.)

If the vector of messages is $m = (m_1, \cdots, m_n)$, consumer i's utility will be

$$W_i - T^i(m) + F_i(\sum_{k=1}^{n} m_k). \tag{1.20}$$

Therefore in a Nash equilibrium each consumer i would be choosing m_i to maximize 1.20. The first order condition for maximizing 1.20, is the relatively meek-looking expression:

$$F_i'(\sum_k m_k) = \gamma\left(\frac{n-1}{n}\right)(m_i - \bar{m}_{\sim i}) + \alpha_i q \tag{1.21}$$

It is helpful to notice that

$$\frac{n-1}{n}(m_i - \bar{m}_{\sim i}) = m_i - \bar{m}. \tag{1.22}$$

Thus Equation 1.21 can be written equivalently as

$$F_i'(\sum_k m_k) = \gamma(m_i - \bar{m}) + \alpha_i q \tag{1.23}$$

Summing the equations in 1.23 and recalling that $\sum_k \alpha_k = 1$, we see that

$$\sum_k F_k'(\sum_k m_k) = q. \tag{1.24}$$

This is the Samuelson condition for efficient provision of public goods.

The trickiest thing to show is that total revenue collected by the Groves-Ledyard tax equals the total costs of the public good. To find this out, we sum the taxes collected from each i to find that

$$\sum_{i=1}^{n} T_i(m) = \sum_{i=1}^{n} \alpha_i q \sum_{k=1}^{n} m_k + \frac{\gamma}{2}\left[\sum_{i=1}^{n}\left(\frac{n}{n-1}\right)(m_i - \bar{m}_{\sim i})^2 - \sum_{i=1}^{n} R_i(m)\right] \tag{1.25}$$

Some fiddling with sums of quadratics will give us the result that

$$\sum_{i=1}^{n} \frac{n}{n-1}(m_i - \bar{m}_{\sim i})^2 = \sum_{i=1}^{n} R_i(m) \tag{1.26}$$

Therefore Equation 1.25 simplifies to:

$$\sum_{i=1}^{n} T_i(m) = \sum_{i=1}^{n} \alpha_i q \sum_{k=1}^{n} m_k \tag{1.27}$$

Since $\sum_{k=1}^{n} m_k = Y$,and $\sum_{i=1}^{n} \alpha_i = 1$, this expression simplifies further to

$$\sum_{i=1}^{n} T_i(m) = qY \tag{1.28}$$

which means that revenue exactly covers the cost of the public good.

The Groves-Ledyard Mechanism with Quasi-linear Utility

It is interesting to examine the Groves–Ledyard mechanism in the special case of quasi-linear utility, where each consumer i has a utility function

$$U_i(X_i, Y) = X_i + F_i(Y). \tag{1.29}$$

In this discussion, we will assume that the public good is produced at constant unit cost q and that $\alpha_i = \frac{1}{n}$ for all i. Studying the quasilinear case will help us to develop some feel for the device by seeing how it performs in a manageable environment. It also is useful to compare the merits of this system with the Vickery-Clarke-Groves mechanism when both are operating on VCG's home turf. (remember that the VCG mechanism is defined *only* for quasilinear utility.)

We are able to show quite generally that when there is quasi-linear utility, the Groves-Ledyard mechanism has exactly one Nash equilibrium. Furthermore, this equilibrium is fairly easily computed and described. This is of interest because, in general, little is known about the uniqueness of Groves-Ledyard equilibrium and the question of the existence of equilibrium is also less than satisfactorily resolved.

Since $F_k''(\cdot) < 0$, Equation 1.24 has a unique solution for $\sum_k m_k$. Let Y denote this solution. In equilibrium, $Y = \sum m_i$, so $\bar{m} = \frac{Y}{n}$. Thus 1.23 can be rewritten as

$$F_i'(Y) = \gamma \left(m_i - \frac{1}{n} Y \right) + \frac{q}{n}. \tag{1.30}$$

Thus we solve for m_i as follows:

$$m_i = \frac{1}{\gamma}(F_i'(Y) - \frac{q}{n}) + \frac{Y}{n}. \tag{1.31}$$

Since Y is uniquely determined by the first-order condition in Equation 1.23, the message sent by each consumer is determined by Equation 1.31. Having solved for the messages sent by each player, we can find the total amount of taxes paid by each player, as well as the quantity of public good supplied.

A very special case

Archie, Betty, and Veronica are friends. They want to throw a party, but have different tastes about how many people to invite. Archie likes smaller parties than Betty, and Betty likes smaller parties then Veronica. Where y is the number of people invited and x_i is private goods consumed by person i, they have quasi-linear preferences, described by utility functions

$$
\begin{aligned}
U_A(x_A, y) &= x_A + 20y - \frac{1}{2}y^2 \\
U_B(x_B, y) &= x_B + 40y - \frac{1}{2}y^2 \\
U_V(x_V, y) &= x_V + 60y - \frac{1}{2}y^2
\end{aligned}
$$

Let us find the messages and payments for each of them in Groves-Ledyard equilibrium. In this example, the public good y is provided at zero cost. We have shown that in the unique Nash equilibrium for the Groves-Ledyard system, the amount of public good provided is Pareto efficient. In this quasi-linear case, we see that there is a unique Pareto efficient size for the party. At this size, the sum of the three friends' marginal rates of substitution for party size is zero. This happens where

$$Y = \frac{\sum_i A_i}{3} = 40 \tag{1.32}$$

In this example, Equation 1.31 simplifies to

$$m_i = \frac{1}{\gamma}(A_i - 40) + \frac{40}{3}. \tag{1.33}$$

Thus we the messages sent by Archie, Betty, and Veronica, respectively are $m_A = \frac{40}{3} - \frac{20}{\gamma}$, $m_B = \frac{40}{3}$, and $m_V = \frac{40}{3} + \frac{20}{\gamma}$. We also have

$$m_i - \bar{m} = \frac{1}{\gamma}(A_i - 40). \tag{1.34}$$

Making use of equations 1.19, 1.22, and 1.34, we find that the tax paid by player i is

$$
\begin{aligned}
T_i(m) &= \frac{\gamma}{2}\left[\left(\frac{n-1}{n}\right)(m_i - \bar{m}_{\text{-}i})^2 - R_i(m)\right] \\
&= \frac{\gamma}{2}\left[\left(\frac{n}{n-1}\right)\left(\frac{A_i - 40}{\gamma}\right)^2 - R_i(m)\right] \\
&= \frac{3}{4\gamma}(A_i - 40)^2 - \frac{\gamma}{2}R_i(m) \qquad (1.35)
\end{aligned}
$$

where $R_i(m)$ is the variance of the messages sent by players other than i.

For three different players, i, j, and k, we must have

$$\begin{aligned} R_i(m) &= \left(m_j - \frac{m_j + m_k}{2}\right)^2 + \left(m_k - \frac{m_j + m_k}{2}\right)^2 \\ &= \frac{1}{2}\left(m_j - m_k\right)^2 \\ &= \frac{1}{2\gamma^2}\left(A_j - A_k\right)^2 \end{aligned} \tag{1.36}$$

From Equations 1.35 and 1.36, it follows that

$$T_i(m) = \frac{1}{\gamma}\left[\frac{3}{4}(A_i - 40)^2 - \frac{1}{2}(A_j - A_k)^2\right] \tag{1.37}$$

Thus we have

$$T_A(m) = \frac{100}{\gamma} \tag{1.38}$$

$$T_B(m) = \frac{-200}{\gamma} \tag{1.39}$$

$$T_V(m) = \frac{100}{\gamma} \tag{1.40}$$

Thus Archie and Veronica, who prefer outcomes different from the mean, each pay for their influence, while Betty, who prefers the mean gets a positive payment (negative tax.)