Econ 241A Probability, Statistics and Econometrics

Fall 2013

Final Exam

- You have 3 hrs to complete this exam
- The exam has two parts. Part I requires to solve all problems. Part II allows you to choose between two problems. Please solve just one problem in Part II. If you answer both, only the lowest grade out of the two will be taken into account.
- The last page of the exam has a list of pmf's and pdf's that you may (or may not) need to use throughout the exam.

Part I

1. (5) Let $X_1, ..., X_n$ be a random sample from a Pareto distribution with parameter $\alpha > 2$:

$$f_{X_i}(x) = \frac{\alpha}{x^{\alpha+1}} \text{ for } x \geq 1$$

with $\mathbb{E}(X_i) = \frac{\alpha}{\alpha-1}$ and $\text{Var}(X_i) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)}$. What is the Cramér-Rao lower bound for the variance of an unbiased estimator of α?

2. (5) Assume $U \sim \text{uniform}(2, 5]$ and X has a Pareto distribution (given in question 1) with parameter equal to random variable U (i.e. $\alpha = U$). What is the joint pdf of (X, U)?

3. (5) Assume $(X_1, Y_1), ..., (X_n, Y_n)$ is a bivariate random sample. Show that $M_{11} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$ converges in probability to σ_{XY}. Hint: Note that you can write M_{11} as a function of $M_{11}^* = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_X)(Y_i - \mu_Y)$ (no need to prove this statement):

$$M_{11} = M_{11}^* - (\bar{X} - \mu_X)(\bar{Y} - \mu_Y).$$

4. (5) Show that $\sqrt{n}\,(M_{11} - \sigma_{XY})$ converges in distribution to a normal, where M_{11} is defined as in question 3.

5. (5) Let $X_1, X_2, ..., X_n$ be a sequence of independent random variables. Each random variable is drawn from a an exponential distribution with parameter λ_i: $X_i \sim \exp(\lambda_i)$, where $\lambda_i > 0\ \forall i$. Define the statistic $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$. Derive the CDF of $X_{(1)}$. What distribution does this CDF represent?

6. (Extra Credit) Prove that $\text{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}(X))Y\right]$

Part II

7. (15) Consider a random sample, $X_1, X_2, ..., X_n$, where X_i is distributed normal with mean $\mu > 0$, and variance of one.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{(x_i - \mu)^2}{2}\right]$$

a) What is the pdf of $W = n\left(\bar{X} - \mu\right)^2$, where $\bar{X}$ is the sample average? Hint: What is the distribution of $\sqrt{n}(\bar{X} - \mu)$?

b) Write the formula for the pdf of $Y = \ln\left(\bar{X}\right)$, which should be characterized by parameters μ and n.

c) Assume you observe a random sample, $Y_1, Y_2, ..., Y_m$, i.e. a random sample from the distribution derived in part (b). What is the method of moments estimator for parameters μ and n? Hint: note that $\bar{X} = \exp(Y)$, and $\bar{X}$ has a simpler distribution than Y.

d) I the method of moments estimator for n you proposed in part (c) unbiased? If no, what is the direction of the bias.

e) Show that the method of moments estimator for μ derived in part (c) is consistent.

f) Is the method of moments estimator for n consistent? No need to provide a proof. Instead, provide a valid argument that uses results seen in class.

8. (15) **Taken from lecture's example.** A researcher is interested in learning about the distribution of opportunity costs of preserving hectares of forest in the Amazon. He proposes a model for preservation costs given by

$$c(Q; W) = a + \frac{W}{2}q^2, \tag{1}$$

where q are the number of hectares preserved and $a > 0$.

Note that the marginal cost of preserving q hectares of forest is proportional to the random variable W. A Payments for Ecosystem Services (PES) program pays p for each hectare of land preserved, where p is a known constant. Farmers decide how many hectares to submit by setting the marginal cost of preservation equal to the per-hectare compensation. Hence,

$$Q^* = \frac{p}{W} \tag{2}$$

The researcher observes the number of hectares submitted to the program for a random sample of farmers, $Q_1^*, ..., Q_n^*$.

(a) Derive the method of moments estimator for the mean marginal cost determinant W, μ_W, that is consistent with the cost model in (1) and (2).

(b) Derive an unbiased method of moments estimator for the variance of W, σ_W^2.

(c) Write the pdf of Q^* under the new assumption for the distribution of W.

(d) What is the MLE estimator for λ?

(e) What is the MLE estimator for the variance of W?

(f) Provided that the distributional assumption on W holds, which estimator for the variance of W is more efficient? Choose between the one you derived in part (b) and the one you derived in part (d). Explain.

Bernoulli

$P(X = x|p) = p^x (1-p)^{(1-x)}$; $x = 0, 1$; $0 \leq p \leq 1$

Binomial

$P(X = x|n, p) = \left(\begin{array}{c} n \\ \\ x \end{array} \right) p^x (1-p)^{(n-x)}$; $x = 0, 1, 2, ..., n$; $0 \leq p \leq 1$

Discrete uniform

$P(X = x|N) = \frac{1}{N}$; $x = 1, 2, ..., N$; $N = 1, 2, ...$

Poisson

$P(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, ...$; $0 \leq \lambda < \infty$

Uniform

$f(x|a, b) = \frac{1}{b-a}$; $x \in [a, b]$

Exponential

$f(x|\beta) = \lambda e^{-\lambda x}$; $0 \leq x < \infty$, $\lambda > 0$

Logistic

$f(x|\mu, \beta) = \frac{1}{\beta} \frac{\exp(-(x-\mu)/\beta)}{[1+\exp(-(x-\mu)/\beta)]^2}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\beta > 0$

Normal

$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$