Midterm - Econ 241A Probability, Statistics and Econometrics

October 31, 2017

1. Let $Y = exp(X)$, where $X \sim N(\mu, \sigma^2)$. Show that

$$\frac{\sigma_Y}{E[Y]} = \sqrt{e^{\sigma^2} - 1}$$

where σ_Y is the standard deviation of Y.

2. Show that $E[\epsilon|X] = 0$ implies that $cov(X, \epsilon) = 0$.

3. Assume w is a random variable and $u(w)$ has a convergent Taylor expansion around $E[w] = \mu_w$, i.e.

$$u(w) = u(\mu_w) + u'(\mu_w)(w - \mu_w) + \frac{u''(\mu_w)}{2}(w - \mu_w)^2 + \sum_{n=3}^{\infty} \frac{1}{n!} u^{(n)}(\mu_w)(w - \mu_w)^n$$

   (a) Give an exact expression for $E[u(w)]$ (state or not if any more assumptions are required).
   (b) Assume that $w \sim N(0, \sigma_w^2)$. Give an even more detailed expression for $E[u(w)]$.
   (c) Assume that $u(w) = exp(w)$ and $w \sim N(\mu_w, \sigma_w^2)$. Give an *alternative* expression for $E[u(w)]$.
   (d) Let $w_0 \sim N(\mu, \sigma_0^2)$ and $w_1 \sim N(\mu, \sigma_1^2)$ maintaing $u(w) = exp(w)$. Establish a condition over σ_0^2 and σ_1^2 such as $E[u(w_0)] \leq E[u(w_1)]$.

4. Let X denote the math score on the ACT college entrance exam of a randomly selected student. Let Y denote the verbal score on the ACT college entrance exam of a randomly selected student. If X and Y are distributed jointly normal such as $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ and $corr(X, Y) = \rho$. State the following in terms of the given parameters $(\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2, \rho)$ and the standard normal cdf $\Phi(z)$.

   (a) What is the probability that a randomly selected student's verbal ACT score is between 10 and 20 points?
   (b) What is the probability that a randomly selected student's verbal ACT score is between 10 and 20 points given that $X = 20$?

5. The Gini coefficient is commonly used to measure inequality of income. If income is represented by Y, a continuous random variable with cdf $F(y)$ with mean $E[Y] = \mu$, then the Gini coefficient is given by

$$G = \frac{1}{\mu} \int_0^{\infty} F(y)(1 - F(y))dy$$

   (a) Assume that $Y \sim U[a, b]$ (clearly $a > 0$). Compute G.
   (b) Assume that $Y \sim exp(\lambda)$ (i.e. $f_Y(y) = \lambda exp(-\lambda y)$, $y > 0$). Compute G.

6. A household has preferences represented by

$$u(w) = -\frac{1}{2}(w - a)^2$$

where w is random variable which represents wealth and we assume that a is high enough such as $0 \leq w < a$. The household maximizes expected utility $E[u(w)]$.

   (a) Show that maximizing expected utility is equivalent to maximizing

$$aE[w] - \frac{1}{2}E[w]^2 - \frac{1}{2}Var(w)$$

   (b) The household has an endowment $w_0 > 0$, and decides to invest nonnegative amounts ϕ in a risky asset and ϕ_f in a risk-free asset such as $w_0 = \phi_f + \phi$ (before knowing the risky asset's return). The household consumes wealth w after the return of the risky asset R is determined, i.e. $w = R_f\phi_f + R\phi$, where R_f is the return to the risk-free asset. What is the demand for the risky asset ϕ that maximizes expected utility?

$$\max_{(\phi, \phi_f) \geq 0} \quad aE[w] - \frac{1}{2}E[w]^2 - \frac{1}{2}Var(w) \tag{1}$$
$$\text{s. t.} \quad w_0 = \phi_f + \phi \tag{2}$$
$$w = R_f\phi_f + R\phi \tag{3}$$

   (c) What is the effect of an increase in the endowment on the final demand for the risky asset (conditional on positive demand of the risky asset $\phi > 0$)? Discuss.