Econ 241A Probability, Statistics and Econometrics
Fall 2017

Solution Problem Set 1[1]
Updated: Oct 2017

1. The probability that it rains in city A is 0.5, the probability that it rains in city B is 0.3, and the probability that it rains in both is 0.15. Find the probability of each of these events:

    (a) It does not rain in either city.

    $$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - 0.5 - 0.3 + 0.15 = 0.35 \end{aligned}$$

    (b) It rains in both cities.

    $$P(A \cap B) = 0.15$$

    (c) It rains in at least one city.

    $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.5 + 0.3 - 0.15 = 0.65 \end{aligned}$$

2. Consider two events A and B such that $P(A) = 1/5$ and $P(B) = 1/3$. Find $P(B \cap A^c)$ for each of these cases:

    (a) A and B are disjoint.
    Then $P(A \cap B) = 0$ and $P(A^c \cap B) = P(B) = 1/3$.

    (b) $B \subset A$
    B cannot be a subset of A, since $P(A) < P(B)$. Nevertheless, (by a reasoning similar to problem 3. below we can argue that) $\frac{2}{15} \leq P(B \cap A^c) \leq \frac{1}{3}$.

    (c) $P(B \cap A) = 1/7$
    $P(B \cap A^c) = P(B) - P(B \cap A) = 1/3 - 1/7 = 4/21$

3. Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the minimum and maximum values of $P(A \cap B)$ and the conditions under which each is attained.

    Since $P(B) > P(A)$ and if $A \subset B$, then $P(A \cap B) = P(A) = 0.4$ which is the maximum for $P(A \cap B)$.
    By Bonferroni's inequality,

    $$\begin{aligned} P(A \cap B) &\geq P(A) + P(B) - 1 \\ P(A \cap B) &\geq 0.4 + 0.7 - 1 = 0.1 \end{aligned}$$

    Summarizing $0.1 \leq P(A \cap B) \leq 0.4$.

[1] Comments and suggestions please write to jramirezcuellar@umail.ucsb.edu

In addition, solve the following problems from Casella and Berger: 1.2, 1.6, 1.35, 1.39, 1.47 (c) and (d), 1.49, 1.51 and 1.54 (b).

1.2 (a) $A \backslash B = A \backslash (A \cap B) = A \cap B^c$

$$\begin{aligned} x \in A \backslash B &\Leftrightarrow x \in A \wedge x \notin B && \text{(def. of relative complement)} \\ &\Leftrightarrow x \in A \wedge x \notin A \cap B && \text{(def. of intersection)} \\ &\Leftrightarrow x \in A \backslash (A \cap B) && \text{(def. of relative complement)} \end{aligned}$$

$$\begin{aligned} x \in A \backslash B &\Leftrightarrow x \in A \wedge x \notin B && \text{(def. of relative complement)} \\ &\Leftrightarrow x \in A \wedge x \in B^c && \text{(def. of complement)} \\ &\Leftrightarrow x \in A \cap B^c && \text{(def. of intersection)} \end{aligned}$$

(b) $B = (B \cap A) \cup (B \cap A^c)$

$$\begin{aligned} x \in B &\Leftrightarrow x \in B \wedge x \in S && \text{(def. of } S\text{, universal set)} \\ &\Leftrightarrow x \in B \wedge (x \in A^c \vee x \in A) && (A \cup A^c = S) \\ &\Leftrightarrow (x \in B \wedge x \in A^c) \vee (x \in B \wedge x \in A) && \text{(distributive of "}\wedge\text{" operator)} \\ &\Leftrightarrow (x \in B \cap A^c) \vee (x \in B \cap A) && \text{(def. of intersection)} \end{aligned}$$

Then, $B = (B \cap A) \cup (B \cap A^c)$

(c) $B \backslash A = B \cap A^c$
Same as the second part of (a).

(d) $A \cup B = A \cup (B \cap A^c)$

$$\begin{aligned} A \cup (B \cap A^c) &= (A \cup B) \cap (A \cup A^c) && \text{(distributivity of union)} \\ &= (A \cup B) \cap (S) && \text{(def. of } S) \\ &= A \cup B && (A \cup B \subset S) \end{aligned}$$

1.6 [see text for question]

First, we have $p_0 = (1-u)(1-w)$, $p_1 = u(1-w) + (1-u)w$, and $p_2 = uw$. So we have two equations in two unknowns:

$$\begin{aligned} p_0 = p_2 &\Rightarrow 1 - u - w + uw = uw \\ p_1 = p_2 &\Rightarrow u - uw + w - uw = uw \end{aligned}$$

This gives $u + w = 1$ and $u + w = 3uw$. Combining yields $u(1-u) = 1/3$. However, $\max_{0 \leq x \leq 1} x(1-x) = 1/4$, so there is no valid probability assignment of u that satisfies these conditions.

1.35 [see text for question]

Clearly $P(A|B) \geq 0$ for any $A \in S$ and $P(S|B) = 1$. If $A_1, A_2, \ldots$ are mutually disjoint, then

$$\begin{aligned}
P\left(\bigcup_{i=1}^{\infty} A_i|B\right) &= \frac{(\bigcup_{i=1}^{\infty} A_i) \cap B}{P(B)} && \text{(def. of conditional prob.)} \\
&= \frac{P(\bigcup_{i=1}^{\infty}(A_i \cap B))}{P(B)} && \text{(distributivity of union)} \\
&= \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} && \text{(by } P \text{ being a probability measure)} \\
&= \sum_{i=1}^{\infty} P(A_i|B) && \text{(def. of conditional prob.)}
\end{aligned}$$

1.39 [see text for question]

For events with probabilities $P(A) > 0$ and $P(B) > 0$, prove that the events cannot be both mutually exclusive and independent.

(a) Prove $A \cap B = \varnothing \Rightarrow A$ and B cannot be independent.

Suppose A and B are mutually exclusive. Then $P(A \cap B) = 0$. Now suppose that A and B are also independent. Then $P(A \cap B) = P(A)P(B) = 0$. But $P(A) > 0$ and $P(B) > 0$, so it cannot be that $P(A)P(B) = 0$. Thus A and B cannot be independent.

(b) ibid.

1.47 [see text for question]

(c) $f(x) = e^{-e^{-x}}$, $x \in (-\infty, \infty)$

i. $\lim_{x\to-\infty} e^{-e^{-x}} = 0$ and $\lim_{x\to\infty} e^{-e^{-x}} = e^0 = 1$

ii. $f'(x) = e^{-e^{-x}}(-e^{-x})(-1) > 0$

iii. $f(x)$ is the composite of continuos functions and, therefore, right-continous.

(d) $f(x) = 1 - e^{-x}$, $x \in (0, \infty)$

i. $\lim_{x\to-\infty} 1 - e^{-x} = 1 - 1 = 0$ and $\lim_{x\to\infty} 1 - e^{-x} = 1 - 0 = 1$

ii. $f'(x) = e^{-x} > 0$

iii. $f(x)$ is the composite of continuos functions and, therefore, right-continous.

1.49 [see text for question]

$$\begin{aligned}
P(X > t) &\geq P(Y > t) \text{ for every } t \\
1 - P(X \leq t) &\geq 1 - P(Y \leq t) \\
P(X \leq t) &\leq P(Y \leq t)
\end{aligned}$$

Q.E.D.

1.51 [see text for question]

First, note that there a total of $\binom{30}{4}$ different ways of drawing a group of four microwaves from a total select of thirty. The number of ways of drawing k defective ones is $\binom{5}{k}\binom{25}{4-k}$ for $k = 0, 1, 2, 3, 4$. Thus the pmf is

$$f_X(x) = \begin{cases} \binom{5}{k}\binom{25}{4-k}/\binom{30}{4} & \text{if } x = 0, 1, 2, 3, 4 \\ 0 & \text{else} \end{cases}$$

and the cdf is a step function with cumulative probabilities given by recursively added elements of the pmf.

1.54 (b) [see text for question]
We need $f(x) > 0$ and $\int f(x)dx = 1$.

$$\begin{aligned} \int f(x) &= \int ce^{-|x|}dx \\ &= \int_0^{\infty} ce^{x}dx + \int_{-\infty}^{0} ce^{-x}dx \\ &= 2c = 1 \end{aligned}$$

Then $c = 1/2$.