Midterm 2012 Questions & Answers

Part I

1. **Prove that $\boldsymbol{P(A|B) = P(B|A)\frac{P(A)}{P(B)}}$.**

Answer:

$P(A \cap B) = P(B \cap A)$	Commutative
$P(B|A) = \frac{P(B \cap A)}{P(A)}$	Defn Cond Prob
$\therefore, P(B|A) = \frac{P(A \cap B)}{P(A)}$	Substitution
$\Longrightarrow P(B|A)P(A) = P(A \cap B)$	Algebra
$\Longrightarrow P(A \cap B) = P(B|A)P(A)$	Reflective
$P(A|B) = \frac{P(A \cap B)}{P(B)}$	Defn Cond Prob
$\therefore, P(A|B) = P(B|A)\frac{P(A)}{P(B)}$	Substitution
	$Q.E.D.$

2. **Prove that if sets $\boldsymbol{A}$ and $\boldsymbol{B}$ in sample space $\boldsymbol{S}$ are mutually exclusive (disjoint), $\boldsymbol{P(A) > 0}$, and $\boldsymbol{P(B) > 0}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ cannot be independent.**

Answer:

A and B are mutually exclusive	Given
$\Longrightarrow P(A \cap B) = 0$	Defn of ME
$P(A) > 0$ and $P(B) > 0$	Given
$\therefore, P(A)P(B) > 0$	Algebra
Assume A and B are independent	Assumption
$\Longrightarrow P(A \cap B) = P(A)P(B)$	Defn of $\perp$
Since $P(A)P(B) > 0$, $P(A \cap B) > 0$	Reflective Prop
But $P(A \cap B) = 0$	Contradiction
$\therefore$, assumption is false	Observation
$\Longrightarrow A$ and B are not independent	$Q.E.D.$

3. **$\boldsymbol{X}$ and $\boldsymbol{Y}$ are jointly distributed with *pdf* $\boldsymbol{f(x,y) = 9x^2y^2}$, where $\boldsymbol{0 < x < 1}$ and $\boldsymbol{0 < y < 1}$. Are $\boldsymbol{X}$ and $\boldsymbol{Y}$ independent? Justify your answer.**

Answer:

Let $g(x) = \begin{cases} 3x^2 & if\ 0 < x < 1 \\ 0 & otherwise \end{cases}$

Is $g(x)$ a *pdf*?

(a) $g(x) \geq 0\ \forall x \in (0,1)$

(b) $\int_{-\infty}^{\infty} g(x)dx = \int_0^1 3x^2 dx = 1$

$\therefore, g(x) = g_X(x)$ is a *pdf* — Theorem 1.6.5

Let $h(y) = \begin{cases} 3y^2 & if\ 0 < y < 1 \\ 0 & otherwise \end{cases}$

Is $h(y)$ a *pdf*?

(a) $h(y) \geq 0\ \forall y \in (0,1)$

(b) $\int_{\infty}^{\infty} h(y)dy = \int_0^1 3y^2 dy = 1$

$\therefore, h(y) = h_Y(y)$ is a *pdf* Theorem 1.6.5

$\therefore, g(x)h(y) = g_X(x)h_Y(y) = \begin{cases} 9x^2y^2 & if\ 0 < x < 1 \wedge 0 < y < 1 \\ 0 & otherwise \end{cases}$

$\therefore, f_{X,Y}(x,y) = g_X(x)h_Y(y)$

$\Longrightarrow X$ and Y are independent Definition 4.2.5

OR

Let $g(x) = \begin{cases} 9x^2 & if\ 0 < x < 1 \\ 0 & otherwise \end{cases}$

Let $h(y) = \begin{cases} y^2 & if\ 0 < y < 1 \\ 0 & otherwise \end{cases}$

$\therefore, g(x)h(y) = \begin{cases} 9x^2y^2 & if\ 0 < x < 1 \wedge 0 < y < 1 \\ 0 & otherwise \end{cases}$

$\therefore, f_{X,Y}(x,y) = g(x)h(y)$

$\Longrightarrow X$ and Y are independent Lemma 4.2.7

This method also works for letting $\begin{matrix} g(x) \\ h(y) \end{matrix} = \begin{cases} 3x^2 \\ 3y^2 \end{cases}$ or $\begin{matrix} g(x) \\ h(y) \end{matrix} = \begin{cases} x^2 \\ 9y^2 \end{cases}$.

4. **X and Y are independent and $\mathbb{E}(Z|X,Y) = X + Y$. Show that $cov(X,Z) = var(X)$.**

Answer:

$cov(X,Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z)$	Defn of Cov
$\Longrightarrow cov(X,Z) = \mathbb{E}[\mathbb{E}(XZ\|X,Y)] - \mathbb{E}(X)\mathbb{E}[\mathbb{E}(Z\|X,Y)]$	LIE
$\Longrightarrow cov(X,Z) = \mathbb{E}[X\mathbb{E}(Z\|X,Y)] - \mathbb{E}(X)\mathbb{E}[\mathbb{E}(Z\|X,Y)]$	Prop of $\mathbb{E}$
$\mathbb{E}(Z\|X,Y) = X + Y$	Given
$\therefore, cov(X,Z) = \mathbb{E}[X(X+Y)] - \mathbb{E}(X)\mathbb{E}(X+Y)$	Substitution
$\Longrightarrow cov(X,Z) = \mathbb{E}(X^2 + XY) - \mathbb{E}(X)\mathbb{E}(X+Y)$	Algebra
$\Longrightarrow cov(X,Z) = \mathbb{E}(X^2) + \mathbb{E}(XY) - \mathbb{E}(X)[\mathbb{E}(X) + \mathbb{E}(Y)]$	Prop of $\mathbb{E}$
$\Longrightarrow cov(X,Z) = \mathbb{E}(X^2) + \mathbb{E}(XY) - \mathbb{E}(X)^2 - \mathbb{E}(X)\mathbb{E}(Y)$	Algebra
$\Longrightarrow cov(X,Z) = [\mathbb{E}(X^2) - \mathbb{E}(X)^2] + [\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)]$	Commutative
$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$	Defn of Var
$cov(X,Y) = \mathbb{E}(X,Y) - \mathbb{E}(X)\mathbb{E}(Y)$	Defn of Cov
$\therefore, cov(X,Z) = var(X) + cov(X,Y)$	Substitution
X and Y are independent	Given
$\Longrightarrow cov(X,Y) = 0$	Theorem 4.5.5
$\therefore, cov(X,Z) = var(X)$	Substitution
	$Q.E.D.$

Part II

5. **The following table provides the joint pmf for the bivariate vector (X, Y), $f_{X,Y}(x, y)$. Empty cells correspond to (x, y) combinations outside the support of (X, Y). Answer questions (a), (b), (c), (d), (e) and (f) using this table. Hint: You can get full credit without any lengthy calculations as long as you explain your reasoning carefully.**

		X													
		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Y	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2	0	.0075	.0075	.0075	.0075	.0075	0	0	.0075	.0075	.0075	.0075	.0075	0
	3	0	.0075	.02	.01	.01	.0075	0	0	.0075	.01	.01	.02	.0075	0
	4	0	.0075	.02	.02	.01	.0075	0	0	.0075	.01	.02	.02	.0075	0
	5	0	.0075	.01	.02	.01	.0075	0	0	.0075	.01	.02	.01	.0075	0
	6	0	.0075	.0075	.0075	.0075	.0075	0	0	.0075	.0075	.0075	.0075	.0075	0
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	9	0	.0075	.0075	.0075	.0075	.0075	0	0	.0075	.0075	.0075	.0075	.0075	0
	10	0	.0075	.01	.02	.01	.0075	0	0	.0075	.01	.02	.01	.0075	0
	11	0	.0075	.01	.02	.02	.0075	0	0	.0075	.02	.02	.01	.0075	0
	12	0	.0075	.01	.01	.02	.0075	0	0	.0075	.02	.01	.01	.0075	0
	13	0	.0075	.0075	.0075	.0075	.0075	0	0	.0075	.0075	.0075	.0075	.0075	0
	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(a) Consider events $A = 1 \le X \le 7, B = 7 \le Y \le 14$. Are A and B independent? Provide careful argument for your answers.

Answer:

$P(A \cap B) = P(1 \le X \le 7 \cap 7 \le Y \le 14) = \sum_{x=1}^{7} \sum_{y=7}^{14} P(X = x \wedge Y = y) = .25$
$P(A) = P(1 \le X \le 7) = \sum_{x=1}^{7} \sum_{y=1}^{14} P(X = x \wedge Y = y) = .5$
$P(B) = P(7 \le Y \le 14) = \sum_{y=7}^{14} \sum_{x=1}^{14} P(X = x \wedge Y = y) = .5$
$\therefore, P(A \cap B) = P(A)P(B)$
$\Longrightarrow$ A and B are independent

(b) Are random variables X and Y independent? Provide a careful argument for your answer.

Answer:

$P(X \cap Y) = P(X)P(Y)$
$\Leftrightarrow P(X = x \wedge Y = y | x, y \in [1,14]) = P(X = x | x \in [1,14])P(Y = y | y \in [1,14])$
$\Leftrightarrow$ X and Y are independent
Let $x = 2$ and $y = 6$:
$P(X = 2 \wedge Y = 6) = .0075$
$P(X = 2) = .0075 * 10 = .075$
$P(Y = 6) = .0075 * 10 = .075$
$\therefore, P(X = 2)P(Y = 6) = .075 * .075 \approx .0056$
$\therefore, P(X = 2 \wedge Y = 6) \neq P(X = 2)P(Y = 6)$

$\Longrightarrow P(X \cap Y) \neq P(X)P(Y)$
$\Longrightarrow X$ and Y are NOT independent

(c) Are events A^c and B independent?

Answer: $A^c = X < 1 \cup X > 7$
$P(A^c \cap B) = P[(X < 1 \cup X > 7) \cap 7 \leq Y \leq 14]$
$= P[(X < 1 \cap 7 \leq Y \leq 14) \cup (X > 7 \cap 7 \leq Y \leq 14)]$
$= P(X < 1 \cap 7 \leq Y \leq 14) + P(X > 7 \cap 7 \leq Y \leq 14)$
$= 0 + \sum_{x=8}^{14} \sum_{y=7}^{14} P(X = x \wedge Y = y)$
$= .25$
$P(A^c) = P(X < 1 \cup 7 < X) = P(X < 1) + P(X > 7) = 0 + .5 = .5$
From (a), $P(B) = .5$
$\therefore, P(A^c \cap B) = P(A^c)P(B)$
$\Longrightarrow A^c$ and B are independent

OR

$A \perp B \Longrightarrow A^c \perp B$ Theorem 1.3.9

(d) Prove the following theorem: Let X and Y be independent random variables. For any $C \subset \mathbb{R}$ and $D \subset \mathbb{R}$, $P(X \in C, Y \in D) = P(X \in C)P(Y \in D)$.

Answer:	Let $X \in C$ and $Y \in D$ for some $C \subset \mathbb{R}$ and $D \subset \mathbb{R}$	Given
	$\therefore, P(X \in C \mid Y \in D) = \frac{P(X \in C \cap Y \in D)}{P(Y \in D)}$	Defn Cond Prob
	X and Y are independent	Given
	$\Longrightarrow P(X \in C \mid Y \in D) = P(X \in C)$	Prop of $\perp$
	$\therefore, P(X \in C) = \frac{P(X \in C \cap Y \in D)}{P(Y \in D)}$	Substitution
	$\Longrightarrow P(X \in C)P(Y \in D) = P(X \in C \cap Y \in D)$	Algebra
	$P(X \in C \cap Y \in D) = P(X \in C)P(Y \in D)$	Reflective Prop
		$Q.E.D.$

(e) Is Y mean independent of X?

Answer: $\mathbb{E}(Y|X) = \mathbb{E}(Y)$
$\Longleftrightarrow \mathbb{E}(Y|X = x \in [1,14]) = \mathbb{E}(Y)$
$\Longleftrightarrow Y$ is mean independent of X
Let $x = 5$
$\mathbb{E}(Y|x = 5) = \sum_{y=1}^{14} y P(Y = y|x = 5)$
$= \sum_{y=1}^{14} y \frac{P(Y=y \wedge X=5)}{P(X=5)}$
$= \sum_{y=1}^{14} y \frac{P(Y=y \wedge X=5)}{.11}$
≈ 8.23
Since Y is symmetrical around its midpoint, $\mathbb{E}(Y) = 7.5$
$\therefore, \mathbb{E}(Y|x = 5) \neq \mathbb{E}(Y)$
$\Longrightarrow \mathbb{E}(Y|X) \neq \mathbb{E}(Y)$
$\Longrightarrow Y$ is NOT mean independent of X

(f) Define the following random variable: $\boldsymbol{Z = 1_{X=Y}}$, where $\boldsymbol{1_A}$ is an indicator function that takes the value of one if statement $\boldsymbol{A}$ is true and the value $\boldsymbol{0}$ if $\boldsymbol{A}$ is false. Write the *pmf* of random variable $\boldsymbol{Z}$.

Answer:

$$P(Z = 1) = P(X = Y) = .12$$

$$\therefore, P(Z = 0) = 1 - P(X = Y) = 1 - .12 = .88$$

$$\therefore, f_Z(z) = \begin{cases} .12 & if\ z = 1 \\ .88 & if\ z = 0 \\ 0 & otherwise \end{cases}$$

6. **Suppose that random variable X is distributed logistic with parameters $\mu = 0$ and $\beta = 1$. Define the variable $Y = X^2$. Also, note that $\mathbb{E}(X^4) = \frac{26}{45}\pi^4$.**

(a) Is $\mathbb{E}(Y)$ larger, smaller or equal to $[\mathbb{E}(X)]^2$? Explain your answer.

Answer: $var(X) = \frac{\pi^2\beta^2}{3} = \frac{\pi^2}{3}$ and $\mathbb{E}(X) = \mu = 0$ Logistic Dist

$var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$\therefore, \frac{\pi^2}{3} = \mathbb{E}(X^2) - 0^2$

$\Longrightarrow \mathbb{E}(X^2) = \frac{\pi^2}{3}$

$\therefore, \mathbb{E}(Y) = \mathbb{E}(X^2) = \frac{\pi^2}{3} > 0 = \mathbb{E}(X)^2$

$\Longrightarrow \mathbb{E}(Y) > \mathbb{E}(X)^2$

(b) Is the variance of Y larger, smaller or equal to the variance of X? Explain your answer.

Answer: $var(X) = \frac{\pi^2}{3}$ and $\mathbb{E}(X) = 0$ Logistic Dist

$\Longrightarrow \mathbb{E}(X^2) = \frac{\pi^2}{3}$

$var(Y) = var(X^2) = \mathbb{E}(X^4) - \mathbb{E}(X^2)^2$

$\therefore, var(Y) = \frac{26\pi^4}{45} - \frac{\pi^4}{9} = \frac{21\pi^4}{45}$

$\therefore, var(Y) = \frac{21\pi^4}{45} > \frac{\pi^2}{3} = var(X)$

$\Longrightarrow var(Y) > var(X)$

(c) Characterize the support of Y.

Answer: $Y = X^2$ and $x \in (-\infty, \infty)$

$\therefore, x \in (-\infty, 0] \Longrightarrow y \in [0, \infty)$ and $x \in [0, \infty) \Longrightarrow y \in [0, \infty)$

$\therefore, y \in [0, \infty)$

(d) What is the *pdf* of Y? Hint: Note that because of symmetry about 0, $f_X(x) = f_X(-x)$.

Answer: Is $f_X(x)$ continuous? Yes.

Can you partition the range of X into continuous pieces? Yes, $A_1 = \{x | x \in (-\infty, 0)\}$, $A_2 = \{x | x \in (0, \infty)\}$, and $A_0 = \{x | x = 0\}$

Do A_1 and A_2 correspond to the same range of Y? Yes, $y \in (0, \infty)$ for both A_1 and A_2

Are A_1 and A_2 monotonic? Yes, both always increase

Is the transformation identical across partitions? Yes, $Y = X^2$

Is $\frac{d}{dy} g^{-1}(y)$ continuous on A_1 and A_2? Yes, $\frac{d}{dy}\left(-\sqrt{y}\right)$ for A_1 and $\frac{d}{dy}\sqrt{y}$ for A_2.

$$\therefore, f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X\left(g_i^{-1}(y)\right)\left|\frac{d}{dy} g_i^{-1}(y)\right|, & \forall y \in [0, \infty) \\ 0, & otherwise \end{cases}$$ Theorem 2.1.8

$$= \begin{cases} f_X\left(g_1^{-1}(y)\right)\left|\frac{d}{dy} g_1^{-1}(y)\right| + f_X\left(g_2^{-1}(y)\right)\left|\frac{d}{dy} g_2^{-1}(y)\right|, & \forall y \in [0, \infty) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} f_X(-\sqrt{y}) \left|\frac{d}{dy}(-\sqrt{y})\right| + f_X(\sqrt{y}) \left|\frac{d}{dy}\sqrt{y}\right|, & \forall y \in [0,\infty) \\ 0, & otherwise \end{cases}$$

$f_X(-\sqrt{y}) = f_X(\sqrt{y})$ Symmetry of f_X

$$\therefore, f_Y(y) = \begin{cases} f_X(\sqrt{y}) \left|\frac{d}{dy}(-\sqrt{y})\right| + f_X(\sqrt{y}) \left|\frac{d}{dy}\sqrt{y}\right|, & \forall y \in [0,\infty) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \frac{e^{-\sqrt{y}}}{\left(1+e^{-\sqrt{y}}\right)^2} \left|-\frac{1}{2\sqrt{y}}\right| + \frac{e^{-\sqrt{y}}}{\left(1+e^{-\sqrt{y}}\right)^2} \left|\frac{1}{2\sqrt{y}}\right|, & \forall y \in [0,\infty) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \frac{2e^{-\sqrt{y}}}{\left(1+e^{-\sqrt{y}}\right)^2} \frac{1}{2\sqrt{y}}, & \forall y \in [0,\infty) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{y}e^{\sqrt{y}}\left(1+e^{-\sqrt{y}}\right)^2}, & \forall y \in [0,\infty) \\ 0, & otherwise \end{cases}$$

(e) **Define the random variable $Z = F(x)$ where $F(.)$ Is the *cdf* of X, $F_X(x)$. What is the *pdf* of Z?**

Answer:

$Z = F_X(x) = \int \frac{e^{-x}}{(1+e^{-x})^2} dx$

Let $u = 1 + e^{-x}$

$\Longrightarrow du = -e^{-x} dx$

$\therefore, \int \frac{e^{-x}}{(1+e^{-x})^2} dx = \int -\frac{1}{u^2} du = \frac{1}{u} + C = \frac{1}{1+e^{-x}} + C$

pdf must integrate to 1 Prop of $\int$

$\therefore, \left.\frac{1}{1+e^{-x}}\right|_{-\infty}^{\infty} + C = 1$

$\Longrightarrow 1 - 0 + C = 1$

$\Longrightarrow C = 0$

$\therefore, Z = \frac{1}{1+e^{-X}}$, where $x \in (-\infty, \infty)$

$$\Longrightarrow \begin{cases} z \in (0,1) \\ X = g^{-1}(Z) = \ln\left(\frac{Z}{1-Z}\right) \end{cases}$$

Is $f_X(x)$ continuous on X? Yes

Is $g(x)$ monotone across the range of X? Yes, it is constantly decreasing

Does $g^{-1}(z)$ have a continuous derivative? Yes, it is $\frac{d}{dy} g^{-1}(z) = \frac{1}{z(1-z)}$

$$\therefore, f_Z(z) = \begin{cases} f_X\left(g^{-1}(z)\right) \left|\frac{d}{dz} g^{-1}(z)\right|, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$$ Theorem 2.1.5

$$= \begin{cases} \frac{e^{-\ln\left(\frac{z}{1-z}\right)}}{\left[1+e^{-\ln\left(\frac{z}{1-z}\right)}\right]^2} \left|\frac{1}{z(1-z)}\right|, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \frac{\frac{1-z}{z}}{\left(1+\frac{1-z}{z}\right)^2} \frac{1}{z(1-z)}, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} \frac{\frac{1-z}{z}}{\frac{1}{z^2}} \frac{1}{z(1-z)}, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$$

$$= \begin{cases} 1, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$$

$\therefore, Z \sim U(0,1)$

OR

$Z = \frac{1}{1+e^{-X}}$, where $x \in (-\infty, \infty)$

$\Longrightarrow z \in (0,1)$

Is $F_X(x)$ monotonically increasing in x? Yes, it's increasing.

$\therefore, F_Z(z) = F_X\left(F_X^{-1}(z)\right) = z, \forall z \in (0,1)$ Theorem 2.1.3

$\therefore, f_Z(z) = \frac{dF_X\left(F_X^{-1}(z)\right)}{dz} = \frac{dz}{dz} = 1, \forall z \in (0,1)$

$\therefore, f_Z(z) = \begin{cases} 1, & \forall z \in (0,1) \\ 0, & otherwise \end{cases}$

$\therefore, Z \sim U(0,1)$