Introduction

Optimal Labor

Suppose the production function is

$$X = \alpha L$$

Utility is

$$U(X, L) = X^{\beta} + (\bar{L} - L)$$

where $\bar{L} - L$ is leisure, $\beta < 1$.

Optimal labor if it might not rain

- Chance of rain

$$p \in [0,1]$$
$$V = p \times U(X, L) + (1 - p) \times U(0, L)$$

$$L^{*} = p^{\frac{1}{1-\beta}} \left\{ \beta^{\frac{1}{1-\beta}} \alpha^{\frac{\beta}{1-\beta}} \right\}$$

Estimation

$$r = \begin{bmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

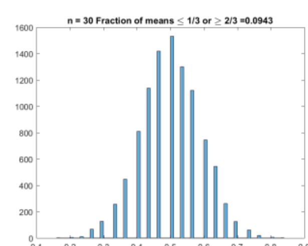
$$\hat{p} = \frac{1}{30} \sum_{i=1}^{30} r_i = \frac{2}{3}$$

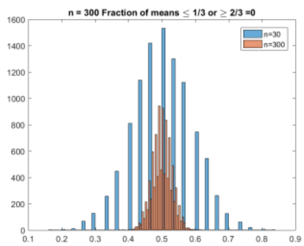
In a sample of size 30 with the probability of rain $p = .5$, what fraction of the time would you find $\hat{p} \geq 2/3$ or $\hat{p} \leq 1/3$?


```
function simulateRain
%{
simulate estimated mean of rain
Econ 241A
Dick Startz
June 2015
modified August 2016
%}

reset(RandStream.getGlobalStream);  % so you get the same random numbers
                                    % each time you run
n1 = 30;
n2 = 300;
p = 0.5;
nSims = 10000;
simulatedMeans1 = nan(nSims,1);
simulatedMeans2 = nan(nSims,1);
for iSim = 1:nSims
    simulatedMeans1(iSim) = mean(rand(n1,1)<p);
    simulatedMeans2(iSim) = mean(rand(n2,1)<p);
end
```



```
histogram(simulatedMeans1);
bigMisses = mean(simulatedMeans1 <= 1/3 | simulatedMeans1 >= 2/3);
title(['n = ',num2str(n1),' Fraction of means \leq 1/3 or \geq 2/3 =',...
    num2str(bigMisses)]);
print -dpng simulatedRain1;

figure;
histogram(simulatedMeans1);
hold on;
histogram(simulatedMeans2);
bigMisses = mean(simulatedMeans2 <= 1/3 | simulatedMeans2 >= 2/3);
title(['n = ',num2str(n2),' Fraction of means \leq 1/3 or \geq 2/3 =',...
    num2str(bigMisses)]);
legend(['n=',num2str(n1)],['n=',num2str(n2)]);
hold off;
print -dpng simulatedRain2;
end
```



Optimal labor if it might rain various amounts

$$V = \Pr(R=1) \times U(X, L) + \Pr(R = 1/2) \times U(.5 \cdot X, L) + \Pr(R=0) \times U(0, L)$$

$s.t.$

$$\Pr(R=1) + \Pr(R=1/2) + \Pr(R=0) = 1$$

- Probability of rain fraction r of normal equals $\Pr(R = r)$

$$V = \int_0^1 \Pr(R=r) \times U(r \cdot X, L) dr$$

$$s.t. \int_0^1 \Pr(R=r)\, dr = 1$$

Uniform chance of rain

$$\Pr(R=r) = 1$$

$$V = \int_0^1 1 \times U(r \cdot X, L) dr$$

$$V = \int_0^1 [(r\alpha L)^{\beta} + (\bar{L} - L)] dr$$

Sample space

Definition:

The set, S, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

$$Rain = \{yes, no\}$$
$$die = \{1,2,3,4,5,6\}$$
$$\hat{p} = [0,1] - sorta$$
$$\log(personal\ income) = \mathbb{R} - sorta$$

Event

Definition

An *event* is any collection of possible outcomes of an experiment, that is, any subset of S (including S itself).

$$Rain = yes$$
$$\hat{p} = \frac{20}{30}$$

Set operations

$$S = \{1,2,3,4,5,6\}, A = \{2,3\}, B = \{1,2,3\}, C = \{4\}, D = \{3,2\}$$

Containment (A is a subset of B) $A \subset B$ is met iff $x \in B$ whenever $x \in A$

$$A \subset B, C \subset S, S \subset S$$

Equality $A = B$ is met iff and $A \subset B\ and\ B \subset A$

$$A = D$$

Set operations continued

$$S = \{1,2,3,4,5,6\}, A = \{2,3\}, B = \{1,2,3\}, C = \{4\}, D = \{3,2\}$$

Union $A \cup B = \{x: x \in A\ or\ x \in B\}$

$$A \cup B = B$$

Intersection $A \cap B = \{x: x \in A\ and\ x \in B\}$

$$A \cap B = A$$

Complementation $A^C = \{x: x \notin A\}$

$$A^C = \{1,4,5,6\}$$
$$S^C = \emptyset$$

More laws

- Commutativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- De Morgan's Laws (Complementarity Laws)

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

More

$$\bigcup_{i=1}^{\infty} A_i = \{x \in S : x \in A_i \text{ for some } i\}$$
$$\bigcap_{i=1}^{\infty} A_i = \{x \in S : x \in A_i \ \forall\ i\}$$

A, B are *disjoint* if $A \cap B = \emptyset$

A_i are *pairwise disjoint* (mutually exclusive) if A_i, A_j are disjoint $\forall i \neq j$.

Informal definition of probability

- Probability of an event is the frequency of its occurrence when an event occurs
- Probability is a subjective belief in the chance of the event occurring.
- Looking for an axiomatic definition that maps every event $A \subset S$ into $[0,1]$, $p(A) \in [0,1]$.

sigma algebra

A collection of subsets of S is called a *sigma algebra* or $\sigma - algebra$ (or *Borel field*), $\mathcal{B}$, if is satisfies three properties

a) $\emptyset \in \mathcal{B}$

b) If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$. ($\mathcal{B}$ closed under complementation.)

c) If $A_1, A_2, \ldots \in \mathcal{B}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$. ($\mathcal{B}$ closed under uncountable unions.)

Axiomatic definition of probability

Given a sample space S and an associated sigma algebra $\mathcal{B}$, a probability function is a function P with domain $\mathcal{B}$ that satisfies

1. $P(A) \geq 0 \; \forall A \in \mathcal{B}$
2. $P(S) = 1$
3. If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Properties

Theorem 1.2.8 (CB) If P is a probability function and A is any set in $\mathcal{B}$, then

a. $P(\emptyset) = 0$

b. $P(A) \leq 1$

c. $P(A^c) = 1 - P(A)$

More properties

Theorem 1.2.9 (CB) If P is a probability function and A and B are any sets in $\mathcal{B}$, then

a. $P(B \cap A^c) = P(B) - P(A \cap B)$

b. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

c. If $A \subset B$, then $P(A) \leq P(B)$

Bonferroni's Inequality

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Combinatorics

- Factorial

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

Note:

$$n \times (n-1) \times (n-2) = \frac{n!}{(n-3)!}$$

$$= \frac{n \times (n-1) \times (n-2) \times \cancel{(n-3)} \times \cdots \times \cancel{1}}{\cancel{(n-3)} \times \cdots \times \cancel{1}}$$

Ordering and replacement

	Without replacement	With replacement
Ordered		
Unordered		

6 out of 44, ordered without replacement

$$44 \times 43 \times 42 \times 41 \times 40 \times 39$$

$$\frac{44!}{(44-6)!} = 5.082 \times 10^9$$

6 out of 44, ordered with replacement

$$44 \times 44 \times 44 \times 44 \times 44 \times 44$$

$$44^6 = 7.256 \times 10^9$$

6 out of 44, unordered without replacement

$$\frac{44 \times 43 \times 42 \times 41 \times 40 \times 39}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{44!}{6!\,38!}$$

$$= 7.059 \times 10^6$$

- $n\ choose\ r$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

6 out of 44, unordered with replacement

$$\binom{n+r-1}{r}$$

In this case 13.984×10^6

Combinatorics

	Without replacement	With replacement
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$

