Econ 241A Probability, Statistics and Econometrics
Fall 2017

Solution Problem Set 3
Updated: October 2017

1. Let $A = \{X \geq 2\}$ and $B = \{|X - \mu| \geq \sigma\}$.
   Consider these two distributions:

   (i) Rectangular on the interval [0,2]
   (ii) Exponential ($f(x) = \lambda e^{-\lambda x}$) with parameter $\lambda = 1$.

   For each distribution,

   (a) Use the Markov or Chebyshev Inequality to calculate an upper bound on $P(A)$ and $P(B)$.
   (b) Use the appropriate cdf to calculate the exact $P(A)$ and $P(B)$.
   (c) Comment on the usefulness of the inequalities.

Solution

(i) (a) Using the Chebyshev inequality we get $P(A) \leq \frac{1}{2}$. Using the Markov's Inequality $P(B) \leq 1$.
  (b) $P(A) = 0$ and $P(B) = 0.4226$

(ii) (a) Using the Chebyshev inequality we get $P(A) \leq \frac{1}{2}$. Using the Markov's Inequality $P(B) \leq 1$.
  (b) $P(A) = P(B) = 0.1353$.

  These inequalities are useful to get the big picture. $P(B) \leq 1$ is known from basic probability. Then, the inequalities may not be informative. Given powerful and cheap computing power, these inequalities are less relevant for purposes of knowing exact values. Nevertheless, they are useful in asymptotic theory.

In addition, solve the following problems from Casella and Berger: 2.33, 4.1, 4.2, 4.5, 4.6 and 4.10.

2.33 (a)

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^t \lambda x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$E[X] = \frac{d}{dt} M_X(t)|_{t=0} = e^{\lambda(e^t-1)} \lambda e^t|_{t=0} = \lambda$
$E[X^2] = \frac{d^2}{dt^2} M_X(t)|_{t=0} = \left[e^{\lambda(e^t-1)} \lambda^2 e^{2t} + e^{\lambda(e^t-1)} \lambda e^t \right]_{t=0} = \lambda|_{t=0} = \lambda^2 + \lambda$
$Var(X) = \lambda^2 + \lambda - \lambda = \lambda$

(b)

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} p(1-p)^x = p\sum_{x=0}^{\infty}(e^t(1-p))^x = \frac{p}{1-(1-p)e^t}$$

$E[X] = \frac{d}{dt}M_X(t)|_{t=0} = \frac{1-p}{p}$
$E[X^2] = \frac{d^2}{dt^2}M_X(t)|_{t=0} = \frac{p(1-p)+2(1-p)^2}{p^2}$
$Var(X) = \frac{1-p}{p^2}$

(c)

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}dx = e^{\mu t+\frac{\sigma^2 t^2}{2}}$$

$E[X] = \frac{d}{dt}M_X(t)|_{t=0} = \mu$
$E[X^2] = \frac{d^2}{dt^2}M_X(t)|_{t=0} = \mu^2+\sigma^2$
$Var(X) = \sigma^2$

4.1 (a) $X^2+Y^2<1$ $P(X^2+Y^2<1) = \frac{1}{4}\pi$

(b) $2X-Y>0$ $P(2X-Y>0) = \frac{1}{4}(2) = \frac{1}{2}$

(c) $|X+Y|<2$ $P(|X+Y|<2) = 1$

4.2 (a) $\mathbb{E}(ag_1(X,Y)+bg_2(X,Y)+c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[ag_1(X,Y)+bg_1(X,Y)+c\right]f_{X,Y}(x,y)dxdy$
$= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_1(X,Y)f_{X,Y}(x,y)dxdy$
$+b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_1(X,Y)f_{X,Y}(x,y)dxdy + c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X,Y}(x,y)dxdy =$
$a\mathbb{E}\left(g_1(X,Y)\right)+b\mathbb{E}\left(g_2(X,Y)\right)+c$

(b) $\mathbb{E}(g_1(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_1(x,y)f_{X,Y}(x,y)dxdy$. Since $g_1(x,y)\geq 0$ and $f_{X,Y}(x,y)\geq 0$, it should be the case that $g_1(x,y)f_{X,Y}(x,y)\geq 0$ for all values of x and y. Hence, the integral of $g_1(x,y)f_{X,Y}(x,y)$ should also be larger than or equal to 0.

(c) $\mathbb{E}(g_1(X,Y))-\mathbb{E}(g_2(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(g_1(x,y)-g_2(x,y)\right)f_{X,Y}(x,y)dxdy$ and $(g_1(x,y)-g_2(x,y))\geq 0$ for $-\infty<x<\infty, -\infty<y<\infty$. So proof follows as in (b)

(d) $a\leq g(X,Y)\leq b \implies a\leq\mathbb{E}\left(g(X,Y)\right)\leq b$. Make $g_2(X,Y)=b$ and apply (c) to get $\mathbb{E}(g_1(X,Y))\leq b$. Make $g_1(X,Y)=a$ and $g_2(X,Y)=g(X,Y)$ to get $\mathbb{E}(g_1(X,Y))\leq a$.

4.5 (a)

$$\begin{aligned} P(x>\sqrt{Y} &= \int_0^1\int_{\sqrt{y}}^1(x+y)dxdy \\ &= \int_0^1\left[\frac{1}{2}x^2+xy\right]_{\sqrt{y}}^1 dy \\ &= \int_0^1\left[\frac{1}{2}+\frac{1}{2}y-y^{\frac{3}{2}}\right]dy \\ &= \frac{7}{20} \end{aligned}$$

(b)

$$
\begin{aligned}
P(X^2 < Y < X) &= \int_0^1 \int_{x^2}^{x} 2x dy dx \\
&= \int_0^1 [2xy]_{x^2}^{x} dx \\
&= \int_0^1 (2x^2 - 2x^3) dx \\
&= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1 dx \\
&= \frac{1}{6}
\end{aligned}
$$

4.6 We need to calculate the cdf of the following random variable $W = B - A$ which is the difference between two uniform random variables. Clearly, $P(W \leq 0) = \frac{1}{2}$. We have

$$
\begin{aligned}
P(W > w) =& P(B - A > w) = 1 - P(B - A \leq w) = 1 - \int_1^{2-w} \int_{a+x}^{2} 1 db da \\
=& \frac{1}{2} - \frac{1}{2}w^2 + w
\end{aligned}
$$

for $1 > w > 0$.

4.10 (a) Y and X are independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. It is easy to see that $f_{X,Y}(x,y) = 0 \neq (1/3)(1/4) = f_X(x)f_Y(y)$. So the variables are not independent.

(b) Construct U and V using the following probability table

	$U = 1$	$U = 2$	$U = 3$	$f_V(v)$
$V = 2$	1/12	1/6	1/12	1/3
$V = 3$	1/12	1/6	1/12	1/3
$V = 4$	1/12	1/6	1/12	1/3
$f_U(u)$	1/4	1/2	1/4	

It is easy to check that the variables are indeed independent.