Econ 241A Prof. Startz

Midterm Fall 2015

Please answer all questions. Show your work.

The exam is open book/open note; closed any devices that can communicate. (No laptops, cell phones, Morse code keys.)

1. Suppose $x \sim U(0,1)$. Find the covariance between x and x^2.
   Answer:
   First we need the expected values. $\mathrm{E}(x) = .5$, obviously. $\mathrm{E}(x^2) = \int_0^1 x^2 \times 1dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$. So the covariance is $\mathrm{E}(x \times x^2) - \mathrm{E}(x)\,\mathrm{E}(x^2)$. The first term is $\mathrm{E}(x \times x^2) = \int_0^1 x^3 \times 1dx = \frac{1}{4}x^4|_0^1 = \frac{1}{4}$. So the answer is $\frac{1}{4} - \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$.

2. There were two kinds of Sneetches in the world, the Star-Belly Sneetches had bellies with stars. The Plain-Belly Sneetches had none upon thars. 90 percent of Sneetches had a star. The Star-Belly Sneetches believed that when they saw a bad thing, two-thirds of the time it was due to a Plain-Belly Sneetch. In other words, people believed that 2/3rd of bad Sneetches were Plain-Bellies. (The Star-Belly Sneetches were wrong about this, but for the purpose of the problem pretend they were right.) Among all Sneetches, only 1 percent were really bad.

   If a Star-Belly comes upon a Plain-Belly, what is the probability that the Plain-Belly is bad?

   Answer:
   Bayes law tells us that
$$p(bad|PlainBelly) = p(PlainBelly|bad) \times \frac{P(bad)}{P(PlainBelly)} = \frac{2}{3} \times \frac{.01}{.1} = \frac{2}{30} = \frac{1}{15}$$

3. The probability of dying is distributed exponentially with expected number of years $1/\lambda$. Consider an annuity that pays out continuously a rate p per year and stops payment at death. If the continuously compounded interest rate is r, (so a dollar at time t is worth e^{-rt} dollars now, then the present value of payments through year τ is
$$\frac{p}{r}[1 - e^{-r\tau}]$$
   What is the expected net present value of the annuity?

Answer:

The pdf of the exponential is $f(t) = \lambda \exp(-\lambda t)$. So the expected present value of payments is

$$A = \int_0^\infty \left(\frac{p}{r}[1 - e^{-r\tau}]\right) \lambda \exp(-\lambda t)\, dt = \frac{p\lambda}{r}\left[\int_0^\infty e^{-\lambda t} dt - \int_0^\infty e^{-(r+\lambda)t} dt\right]$$

With appropriate constants, both integrals are integrals of an exponential pdf equaling 1.

$$A = \frac{p\lambda}{r}\left[\frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda t} dt - \frac{1}{r+\lambda}\int_0^\infty (r+\lambda) e^{-(r+\lambda)t} dt\right] = \frac{p\lambda}{r}\left[\frac{1}{\lambda} - \frac{1}{r+\lambda}\right] = \frac{p}{r+\lambda}$$

4. Suppose that $x, \varepsilon,$ and v are jointly normally distributed

$$\begin{bmatrix} x \\ \varepsilon \\ v \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_\varepsilon^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{bmatrix}\right)$$

Further, $y = \beta x + \varepsilon$ and $z = x + v$.

Find

$$\frac{\mathrm{cov}(y,z)}{\mathrm{var}(z)}$$

Answer:

$$\mathrm{var}(z) = \sigma_x^2 + \sigma_v^2 + 2 \times 0$$
$$\mathrm{cov}(y,z) = \mathrm{cov}(\beta x + \varepsilon, x + v) = \beta\sigma_x^2 + \beta\sigma_{x\varepsilon} + \sigma_{\varepsilon x} + \sigma_{\varepsilon v} = \beta\sigma_x^2 + 0 + 0 + 0$$
$$\frac{\mathrm{cov}(y,z)}{\mathrm{var}(z)} = \beta\frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2}$$

5. Consider a simulation that produces a yes/no answer where the probability of "yes" is p. The total number of independent Monte Carlo trials is n. If we observe k yeses, we estimate

$$\hat{p} = \frac{k}{n}$$

(a) Find mean, μ, and the variance, V, of $\hat{p}$ in terms of $p, k,$ and n.

(b) In a large number of trials, $\hat{p}$ is approximately normally distributed. Taking $\hat{p} \sim N(\mu, V)$, then it can be shown $P\left(|\hat{p} - \mu| > 1.96\sqrt{V}\right) = .05$. If we think $p = .1$, how many observations do we need do that the probability $\hat{p}$ is off by 0.01 is five percent?

Answer:

The expected value of $\hat{p} = p$ since we are looking at the sum of n trials each with expectation p and then we are dividing by n. The variance of a single draw is $p(1-p)$. We have n draws so the variance of the sum is $np(1-p)$. We divide by n which divides the variance by n^2. So $V = p(1-p)/n$.

We have

$$P\left(|\hat{p} - p| > 1.96\sqrt{p(1-p)/n}\right) = .05$$

So we want $1.96\sqrt{p(1-p)/n} = 0.01$. For $p \approx .9$ that's $1.96\sqrt{.9 \times .1/n} = 0.01$.

$$n = \left(\frac{1.96\sqrt{.9 \times .1}}{.01}\right)^2 = n = 1.96 \times 900 \approx 3{,}457$$