Required Problems

1. Let Y be a continuous random variable with PDF

$$f_Y(y) = \begin{cases} (3/2)y^2 + y & \text{if } 0 \le y \le 1\\ 0 & \text{else} \end{cases}$$

- (a) Find the mean of Y.
- (b) Find the variance of Y.

2. Let Y be a random variable with probability density function given by

$$f_Y(y) = 2(1-y), \qquad y \in [0,1]$$

- (a) Find the PDF of U = 2Y 1.
- (b) Find the PDF of W = 1 2Y.
- (c) Find the PDF of $Z = Y^2$.

3. Consider the multivariate distribution characterized by the PDF

$$f_{XY} = 6(1 - y), \qquad 0 \le x \le y \le 1$$

- (a) Find the conditional expectation $\mathbb{E}[X|Y=y]$.
- (b) Find the covariance of X and Y.

4. Let $X_1 \ldots X_n$ be a random sample from a distribution with PMF

$$f(x_i|\theta) = \begin{cases} \theta(1-\theta)^{x_i-1} & \text{if } x = 1, 2, 3 \dots \\ 0 & \text{else} \end{cases}$$

where $\theta \in (0, 1)$.

- (a) Find the method of moments estimator for θ .
- (b) Find the maximum likelihood estimator for θ .

Practice Problems

5. Let X be a discrete random variable with PMF $f_X(x)$, given in the following table.

Find the following:

- (a) $\mathbb{E}[X]$
- (b) $\mathbb{E}[1/X]$
- (c) $\mathbb{E}[X^2 1]$
- (d) Var[X]

6. A binomially-distributed random variable X has the PMF

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x \in \{0, 1, 2, \dots, n\}$$

where 0 .

- (a) Show that the expected value of X is np.
- (b) Show that the variance of X is np(1-p).

7. Let Y be a random variable with mean μ and variance σ^2 . If a and b are constants, show that:

- (a) $\mathbb{E}[aY + b] = a\mu + b$
- (b) $Var(aY + b) = a^2 \sigma^2$

8. Let X be a standard normal random variable with PDF

$$f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \qquad x \in (-\infty, \infty)$$

- (a) Show that the MGF of X is $M_X(t) = \exp\{(1/2)t^2\}$ (hint: complete the square in the exponent).
- (b) Use the MGF of X to calculate $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
- (c) Use the MGF to show that the sum of two independent standard normal random variables is distributed normally with $\mu = 0$ and $\sigma^2 = 2$.

9. For each of the following random variables, find the PDF of Y.

- (a) $f_X(x) = \frac{1}{2}(1+x)$, $x \in (-1,1)$ and $Y = X^2$
- (b) $f_X(x) = 60x^3(1-x)^2$, $x \in (0,1)$ and $Y = \log(X)$
- (c) $f_X(x) = (1+x)^2/9$, $x \in (-1,2)$ and $Y = (X-1)^2$

10. Consider the joint distribution of Y_1 and Y_2 given by the following table:

$$\begin{array}{c|cccc}
 & x_1 \\
 & 0 & 1 \\
 & 0 & 0.38 & 0.17 \\
 & x_2 & 1 & 0.14 & 0.02 \\
 & 2 & 0.24 & 0.05 \\
\end{array}$$

- (a) Find the marginal PMFs for X_1 and X_2 .
- (b) Find the conditional PMF for X_2 given $X_1 = 0$.
- 11. Consider the joint PDF:

$$f_{Y_1Y_2}(y_1, y_2) = 3y_1, \qquad 0 \le y_2 \le y_1 \le 1$$

- (a) Find the marginal PDFs for Y_1 and Y_2 .
- (b) Find the conditional PDF of Y_2 given $Y_1 = y_1$.
- (c) What is the probability that $Y_2 \ge 1/2$ given that $Y_1 = 3/4$?

- 12. For each of the following joint PDFs, determine whether or not the random variables x and y are independent.
 - (a) $f_{XY}(x,y) = 1$, where $x \in [0,1]$ and $y \in [0,1]$
 - (b) $f_{XY}(x,y) = e^{-(x+y)}$, where x > 0 and y > 0
 - (c) $f_{XY}(x,y) = 6(1-y)$, where $0 \le x \le y \le 1$
- 13. Show that $\mathbb{E}[(X \mathbb{E}[X])(Y \mathbb{E}[Y])] = \mathbb{E}[XY] \mathbb{E}[X]\mathbb{E}[Y]$.
- 14. Let X_1, \ldots, X_n be a random sample from a distribution with mean μ and $\sigma^2 < \infty$. Show that:
 - (a) $\mathbb{E}[\bar{X}] = \mu$
 - (b) $Var(\bar{X}) = \frac{\sigma^2}{n}$
 - (c) $\sum_{i=1}^{n} (X_i \bar{X})^2 = \left(\sum_{i=1}^{n} X_i^2\right) \bar{X}^2$
 - (d) $\mathbb{E}[S^2] = \sigma^2$
- 15. Let Y_1, \dots, Y_n be a random sample from a distribution with PDF

$$f(y_i|\theta) = \begin{cases} (\theta+1)y^{-(\theta+2)} & \text{if } y > 1\\ 0 & \text{else} \end{cases}$$

where $\theta \in (0, \infty)$.

- (a) Find the method of moments estimator for θ .
- (b) Find the maximum likelihood estimator for θ .
- 16. For each of the following PDFs, let X_1, \ldots, X_n be a random sample. Find a complete sufficient statistic for the unknown parameter(s).

(a)
$$f(x|\theta) = \frac{\theta}{(1+x)^{1+\theta}}$$
, where $0 < x < \infty$ and $\theta > 0$

(b)
$$f(x|\beta) = \frac{\ln[\beta]\beta^x}{\beta - 1}$$
, where $0 < x < 1$ and $\beta > 1$

(c)
$$f(x|\theta) = \exp\{-(x-\theta)\} \exp\{-e^{-(x-\theta)}\}$$
, where $-\infty < x < \infty$ and $-\infty < \theta < \infty$

(d)
$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$
 where $0 < x < 1$ and $\alpha,\beta > 0$