Probability Review 3 - Univariate Random Variables and Distributions

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Random Variables

Random variable - function from sample space to $\Re$ or a subset of $\Re$

Discrete random variable - at most a countable number of values

Continuous random variable - uncountable set of possible values

Discrete Random Variables

Probability mass function: $p(x) = P(A = x)$

Cumulative distribution function: $F(a) = \sum_{x_i \le a} p(x_i)$

Bernoulli and Binomial Distributions

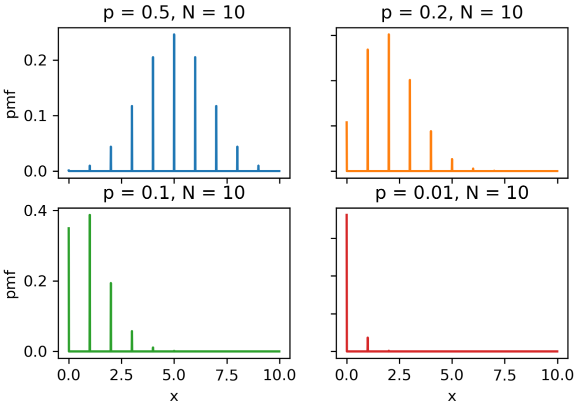
- Bernoulli:

$$X = \begin{cases} 1 & \text{with probability p} \\ 0 & \text{with probability 1-p} \end{cases}$$

- Binomial: Suppose independent Bernoulli trials with probability p, then the probability of x successes in N trials is

$$p(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

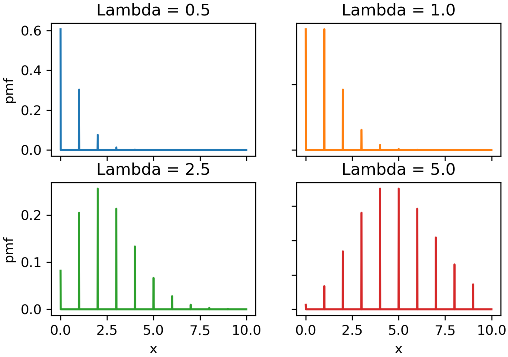
Binomial Distribution



Poisson Distribution

Poisson: For $\lambda > 0$ and some $x = 0, 1, \ldots,$

$$p(k) = e^{-\lambda} \frac{\lambda^x}{x!}$$



Continuous Random Variables

Probability density function : For $A \subset \Re$ with $f(x)$ such that

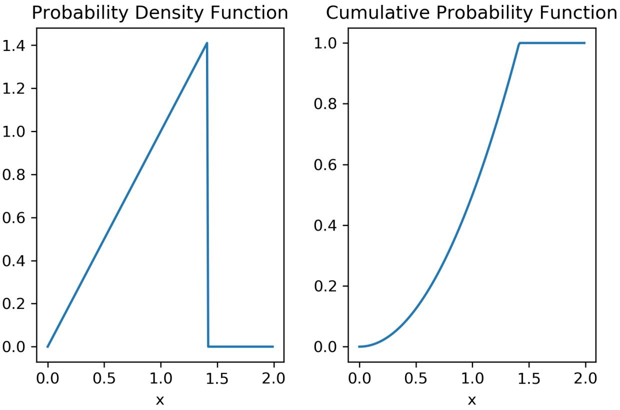
$$P(X \in A) = \int_A f(x)dx$$

Cumulative distribution function : $F(a) = \int_{-\infty}^{a} f(x)dx$

Example : Suppose $f(x) = x, x \in (0, \sqrt{2})$ and $f(x) = 0$ otherwise.

Graph $f(x)$ and $F(x)$

Example

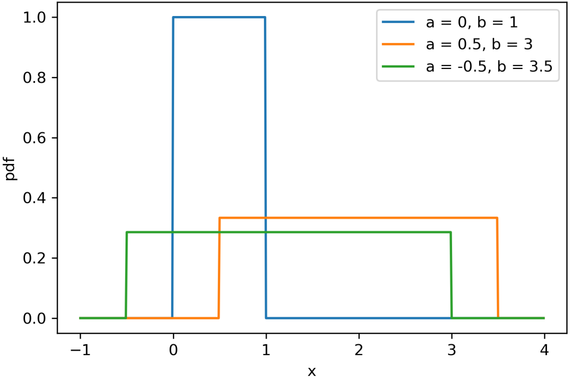

Probability Density Function
Cumulative Probability Function
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
x

Uniform Distribution

- Uniform:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Uniform Distribution



Beta Distribution

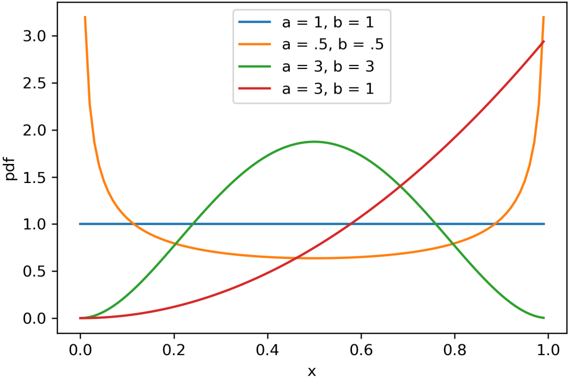
- Beta: For $a > 0, b > 0$ and $x \in [0, 1]$

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

Beta Distribution



Gamma Distribution

Univariate Probability
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Random Variables

Discrete Random Variables

Continuous Random Variables

- Gamma: For $\lambda > 0, \alpha > 0$ and $x \geq 0$

$$f(x) = \frac{\lambda^{\alpha} e^{-\lambda x} x^{\alpha - 1}}{\Gamma(\alpha)}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha - 1} dx$$

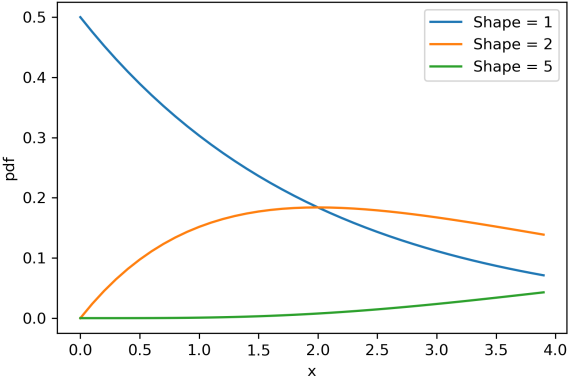
for $\alpha \in \mathbb{Z}^+$ (i.e., a positive integer)

$$\Gamma(\alpha) = (\alpha - 1)!$$

Also

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$$

Gamma Distribution



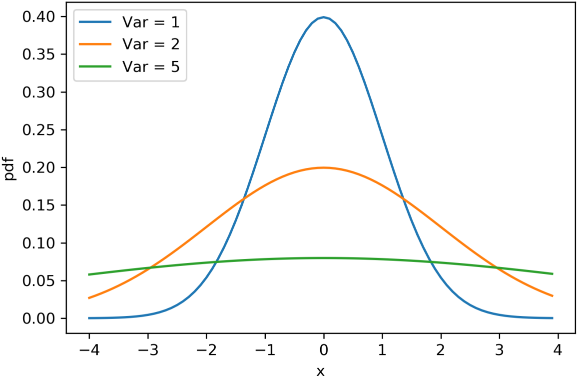
Gaussian Distribution

- Gaussian: For $\mu \in \Re, \sigma^2 > 0$ and $x \in \Re$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- Suppose $X \sim N(\mu, \sigma^2)$ what is the distribution of $Y = aX + b$?
- $Y \sim N(a\mu + b, a^2\sigma^2)$

Gaussian Distribution



t Distribution

- t Distribution: For $\mu \in \Re, \hat{\sigma}, \nu > 0$ and $x \in \Re$

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}\hat{\sigma}}\left(1 + \frac{1}{\nu}\frac{(x-\mu)^2}{\hat{\sigma}^2}\right)^{-\frac{\nu+1}{2}}$$

- μ is location, ν is the degrees of freedom, and $\hat{\sigma}$ is the scale
- As $\nu \to \infty$ the distribution converges to the Gaussian
- For $\nu = 1$ known as the Cauchy distribution

t Distribution