Probability Review 6: Expectation by Conditioning

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Bayes Theorem

- We can now express Bayes theorem in terms of probability distributions
- For variables $\mathbf{X}$ and model parameters $\boldsymbol{\theta}$, Bayes theorem tells us how to obtain posterior beliefs about $\boldsymbol{\theta}$ after observing $\mathbf{X}$:

$$\underbrace{p(\boldsymbol{\theta}|\mathbf{X})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{X}|\boldsymbol{\theta})}^{}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\underbrace{p(\mathbf{X}|\boldsymbol{\theta})}_{\text{likelihood}}\;\underbrace{p(\boldsymbol{\theta})}_{\text{prior}}}{\underbrace{p(\mathbf{X})}_{\text{evidence}}}$$

Conditioning

- Notice we can find the denominator in Bayes theorem using the sum rule; sometimes called conditioning.
  - Discrete case

$$P(X = x) = \sum_y P(X = x|Y = y)P(Y = y)$$

  - Continuous case

$$f(x) = \int_{-\infty}^{\infty} f(x|y)f(y)dx$$

Expectation by Conditioning

- We can also find expectations by conditioning
  - $E[X] = E[E[X|Y]]$
  - Y discrete:

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

  - Y continuous:

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f(y)dy$$

Example

- Suppose N is the number of accidents per week on I95 and $X_i, i \in [1, N]$ is the number of injuries in accident i.
- What is the expected number of injuries per week, $E[I]$?
- Since $E[I] = \sum_{i=1}^{N} X_i$ solve by conditioning on N, So,

$$
\begin{aligned}
E[I] &= E[E[I|N]] \\
&= E\left[E\left[\sum_{i=1}^{N} X_i | N\right]\right]
\end{aligned}
$$

$$
E\left[\sum_{i=1}^{N} X_i | N\right] = NE[X]
$$

$$
E[I] = E[N]E[X]
$$