Probability Review 2 - Bayes Theorem

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Product Rule

- Conditional Probability

$$P(B|A) = \frac{P(A, B)}{P(A)}$$

- Product Rule

$$P(A, B) = P(B|A)P(A)$$

Bayes Theorem

$$
\begin{aligned}
P(B|A) &= \frac{P(A,B)}{P(A)} && \text{cond. prob.} \\
&= \frac{P(A|B)P(B)}{P(A)} && \text{prod. rule} \\
&= \frac{P(A|B)P(B)}{P(A,B)+P(A,B^c)} && \text{sum rule}
\end{aligned}
$$

$$\boxed{P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|B^c)(1-P(B))}} \qquad \text{prod. rule}$$

A can be data and B can be a parameter or model.

Monty Hall Problem

- Suppose you're a contestant on Let's Make A Deal, and Monty Hall gives you the choice of three doors: Behind one door is the grand prize: a new car; behind the other two doors are goats.
- You pick a door. Then Monty opens one of the remaining doors to reveal a goat.
- Should you keep the door you selected or change?

A "Supposedly" Bayesian Solution

Suppose you choose door 1 and learn that a goat is behind door 3. Should you change doors? Let A be the award door.

$$
\begin{aligned}
P(A = 1 | A \in \{1, 2\}) &= \frac{P(A = 1) P(A \in \{1, 2\} | A = 1)}{P(A \in \{1, 2\})} \\
&= \frac{(1/3) \times 1}{2/3} \\
&= 0.5
\end{aligned}
$$

So it does not matter. **But this is wrong!**

Correct Formulation

Let A = award, M = Monty's selection, and C = your first choice. Then $P(A = 1|M = 3, C = 1)$ is

$$
\begin{aligned}
&= \frac{P(A = 1|C = 1)P(M = 3|A = 1, C = 1)}{P(M = 3|C = 1)} \\
&= \frac{(1/3) \times (1/2)}{(1/3) \times (1/2) + (1/3) \times 1 + (1/3) \times 0} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P(M = 3|C = 1) = {} & P(A = 1)P(M = 3|A = 1, C = 1) \\
& + P(A = 2)P(M = 3|A = 2, C = 1) \\
& + P(A = 3)P(M = 3|A = 3, C = 1)
\end{aligned}
$$

So, always switch doors!