

Multivariate Probability 1/14

Probability Review 5 - Multivariate **Probability Distributions**

Donald E. Brown

School of Data Science University of Virginia Charlottesville, VA 22904



Agenda

Multivariate Probability 2/14

Multivariate Probability Distributions



- Multivariate Probability Distributions
 - Random Vectors and Distributions
 - Multivariate Distributions



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Random Vectors

 A Random vector is an ordered set of random variables,

e.g.,
$$\mathbf{X} = (X_1, \dots, X_k)^T$$
 (note notation)



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Multivariat Probability Distribution

Random Vectors Multivariate Distributions

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- e.g., $\mathbf{X} = (X_1, \dots, X_k)^T$ (note notation) $f(\mathbf{x})$ is the joint density function (mass function for the discrete case) (short notation)



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Distribution
Random Vectors

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 $E[\mathbf{X}]$ is the mean vector (short notation) and

$$E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$$



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$$E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$$

• $Var(\mathbf{X}) = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T]$ is the variance-covariance matrix



Multinomial and Dirichlet Distributions

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Multivariate Probability Distributions Random Vectors Multivariate • **Multinomial**: Suppose N trials where each outcome belongs to one of k categories, i = 1, ..., k, with probabilities, $p_1, ..., p_k$. The random vector $X = (X_1, ..., X_k)$ gives the number of occurrences for each category is the multinomial density function:

$$f(\mathbf{x}|N,\mathbf{p}) = \frac{N!}{x_1!,\cdots,x_k!}p_1^{x_1}\cdots p_k^{x_k}$$



Multinomial and Dirichlet Distributions

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• **Multinomial**: Suppose N trials where each outcome belongs to one of k categories, $i = 1, \ldots, k$, with probabilities, p_1, \ldots, p_k . The random vector $X = (X_1, \ldots, X_k)$ gives the number of occurrences for each category is the multinomial density function:

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• **Dirichlet**: The random vector $\mathbf{p} = (p_1, \dots, p_k), p_i \in [0, 1], \sum_{i=1}^k p_i = 1$, with parametric vector $\mathbf{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i > 0, i = 1, \dots, k$ has density function

$$f(\mathbf{p}|\alpha) = \frac{\Gamma(\alpha_1 + \dots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}$$

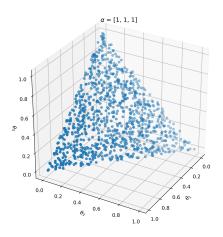


Dirichlet $\alpha_i = 1, \forall i$

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Multivariate Probability Distribution



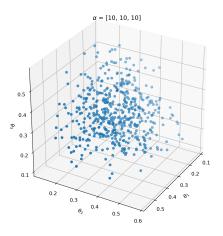


Dirichlet $\alpha_i = 10, \forall i$

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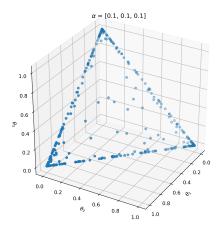


Dirichlet $\alpha_i = 0.1, \forall i$

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Dirichlet $\alpha = [10, 0.2, 0.2]$

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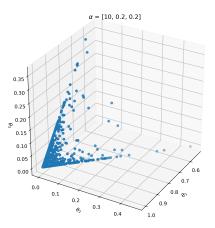
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Multivariate Probability Distribution Random Vectors

Random Vector

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Multivariate Gaussian

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• Multivariate Gaussian: The random vector $X=(X_1,\ldots,X_k), X_i\in\Re, i=1,\ldots,k,$ with parameters $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_k)$ and covariance matrix $\boldsymbol{\Sigma}$ has the density

$$f(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{k}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} exp[-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})]$$



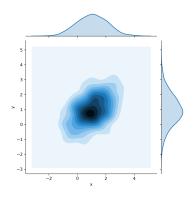
Multivariate Gaussian

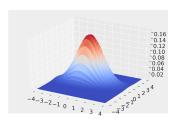
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Multivariate t Distribution

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Multivariate Probability

Distribution

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• Multivariate t Distribution: The random vector $X = (X_1, \dots, X_k), X_i \in \Re, i = 1, \dots, k$, with parameters: Location, $\mu = (\mu_1, \dots, \mu_k)$; Shape or scale matrix Σ ; and Degrees of Freedom, ν , has the density

$$\frac{\Gamma(\nu+k)}{\Gamma(\nu/2)(\nu\pi)^{k/2}|\mathbf{\Sigma}|^{1/2}}\left[1+\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(\nu+k)/2)}$$

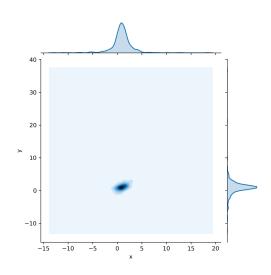


Mulltivariate t Distribution DF=2

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Probability Distribution



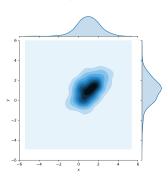


Mulltivariate t Distribution DF=10

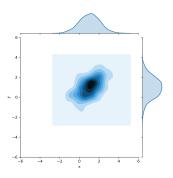
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Multivariate Probability Distributions Random Vectors t Distribution, DF = 10



Gaussian





Common Multivariate Distributions

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• Wishart: Let $X = (X_1, \ldots, X_n)$ be a random sample of k dimensional random vectors with $X_i \sim N_k(\mathbf{0}, \mathbf{\Sigma})$. Then $V = \sum_{i=1}^n X_i X_i^T$ is a random $k \times k$ matrix and V has the density

$$f(\mathbf{V}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = c|\boldsymbol{\Sigma}|^{n/2}|V|^{(n-k-1)/2}exp[-\frac{1}{2}tr(\boldsymbol{\Sigma}^{-1}V)]$$

$$c = \left[2^{nk/2}\pi^{k(k-1)/4}\prod_{j=1}^{k}\left(\frac{n+1-j}{2}\right)\right]^{-1}$$