Probability Review 5 - Multivariate Probability Distributions

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Agenda

1. Multivariate Probability Distributions
   - Random Vectors and Distributions
   - Multivariate Distributions

Random Vectors

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  e.g., $\mathbf{X} = (X_1, \ldots, X_k)^T$ (note notation)

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  $$E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$$
- $\text{Var}(\mathbf{X}) = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T]$ is the variance-covariance matrix

Multinomial and Dirichlet Distributions

- **Multinomial**: Suppose N trials where each outcome belongs to one of k categories, $i = 1, \ldots, k$, with probabilities, $p_1, \ldots, p_k$. The random vector $\boldsymbol{X} = (X_1, \ldots, X_k)$ gives the number of occurrences for each category is the multinomial density function:

$$f(\mathbf{x}|N, \mathbf{p}) = \frac{N!}{x_1!, \cdots, x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

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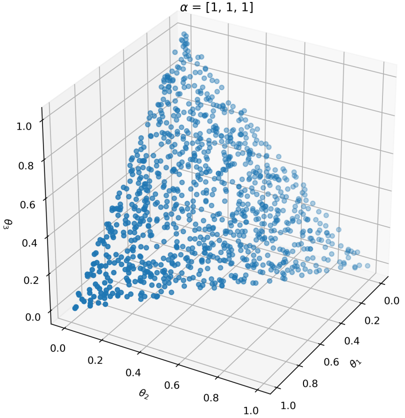
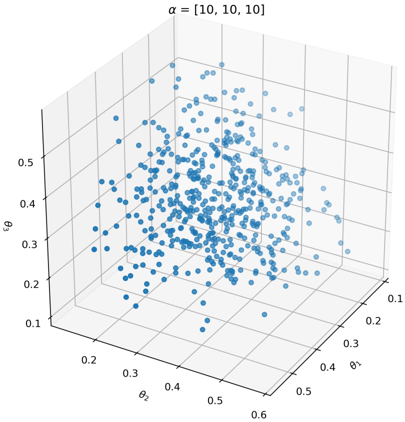
- **Dirichlet**: The random vector $\boldsymbol{p} = (p_1, \ldots, p_k), p_i \in [0, 1], \sum_{i=1}^{k} p_i = 1$, with parametric vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_i > 0, i = 1, \ldots, k$ has density function

$$f(\mathbf{p}|\boldsymbol{\alpha}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}$$

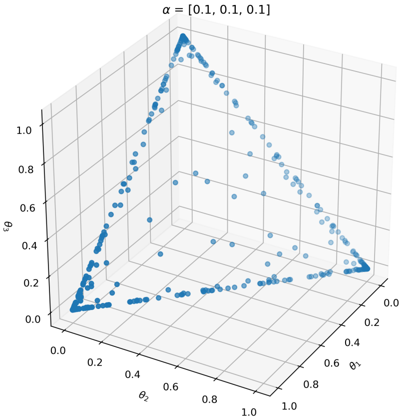
Dirichlet $\alpha_i = 1, \forall i$



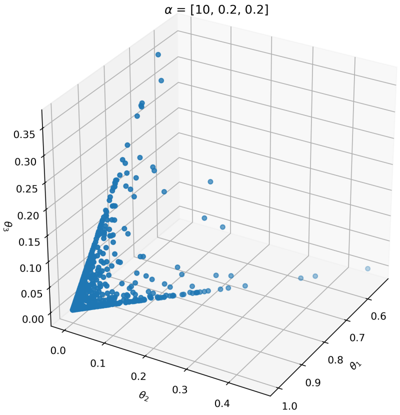
Dirichlet $\alpha_i = 10, \forall i$



Dirichlet $\alpha_i = 0.1, \forall i$



Dirichlet $\alpha = [10, 0.2, 0.2]$

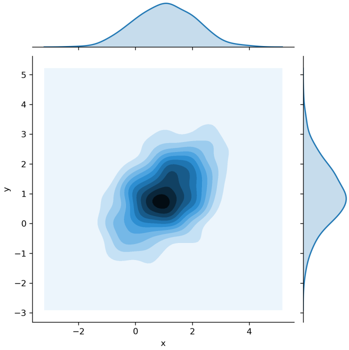


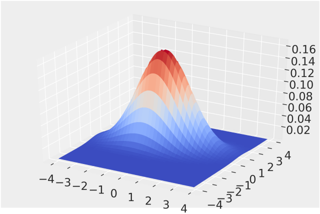
Multivariate Gaussian

- **Multivariate Gaussian**: The random vector $\boldsymbol{X} = (X_1, \ldots, X_k), X_i \in \Re, i = 1, \ldots, k$, with parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ and covariance matrix $\boldsymbol{\Sigma}$ has the density

$$f(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{k}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} exp[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})]$$

Multivariate Gaussian



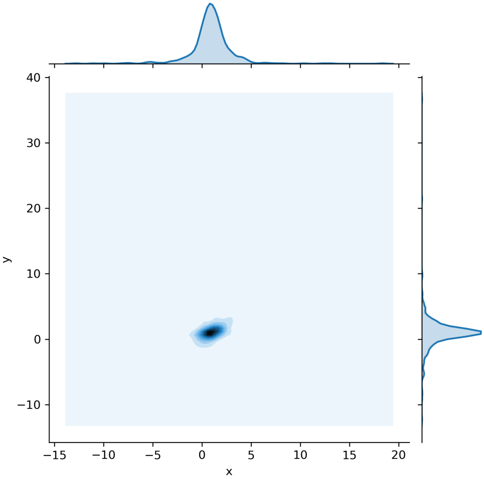


Multivariate t Distribution

- **Multivariate t Distribution**: The random vector $\boldsymbol{X} = (X_1, \ldots, X_k), X_i \in \Re, i = 1, \ldots, k$, with parameters: Location, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$; Shape or scale matrix $\boldsymbol{\Sigma}$; and Degrees of Freedom, ν, has the density

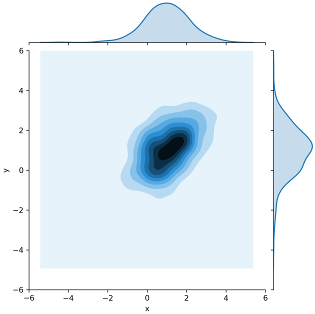
$$\frac{\Gamma(\nu + k)}{\Gamma(\nu/2)(\nu\pi)^{k/2}|\boldsymbol{\Sigma}|^{1/2}} \left[1 + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right]^{-(\nu+k)/2}$$

Mulltivariate t Distribution DF=2

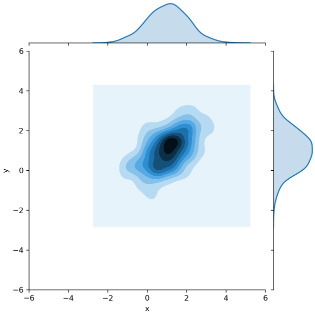


Mulltivariate t Distribution DF=10

t Distribution, DF = 10



Gaussian



Common Multivariate Distributions

- **Wishart**: Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a random sample of k dimensional random vectors with $X_i \sim N_k(\mathbf{0}, \boldsymbol{\Sigma})$. Then $\boldsymbol{V} = \sum_{i=1}^{n} X_i X_i^T$ is a random $k \times k$ matrix and V has the density

$$f(\mathbf{v}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = c|\boldsymbol{\Sigma}|^{n/2}|\boldsymbol{V}|^{(n-k-1)/2} exp[-\frac{1}{2} tr(\boldsymbol{\Sigma}^{-1}\boldsymbol{V})]$$

$$c = \left[2^{nk/2}\pi^{k(k-1)/4}\prod_{j=1}^{k}\left(\frac{n+1-j}{2}\right)\right]^{-1}$$