



Beta-Binomial

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Conjugate
Priors

Beta-Binomial

Beta-Binomial Conjugate Priors

Donald E. Brown

School of Data Science
University of Virginia
Charlottesville, VA 22904



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Definition of Conjugate Prior

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Posterior has the same distribution with different parameters as the prior, so with

$$\underbrace{p(\boldsymbol{\theta}|\mathbf{X})}_{\text{posterior}} = \frac{\underbrace{p(\mathbf{X}|\boldsymbol{\theta})}_{\text{likelihood}} \underbrace{p(\boldsymbol{\theta})}_{\text{prior}}}{\underbrace{p(\mathbf{X})}_{\text{evidence}}}$$

Then

$$p(\boldsymbol{\theta}|\mathbf{X}) \sim p(\boldsymbol{\theta})$$



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- Binomial likelihood with N trials, $x = (x_1, \dots, x_N)$

$$p(x|\theta) \propto \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}$$



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- Binomial likelihood with N trials, $x = (x_1, \dots, x_N)$

$$p(x|\theta) \propto \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}$$

- Beta prior

$$h(\theta) \propto \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$



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- Binomial likelihood with N trials, $x = (x_1, \dots, x_N)$

$$p(x|\theta) \propto \prod_{i=1}^N \theta^{x_i} (1 - \theta)^{1-x_i}$$

- Beta prior

$$h(\theta) \propto \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

- Beta posterior

$$h(\theta|x) \propto \theta^{\sum_{i=1}^N x_i + \alpha - 1} (1 - \theta)^{N - \sum_{i=1}^N x_i + \beta - 1}$$
$$c_1 = \frac{(\alpha + \beta + N - 1)!}{(\alpha + \sum_{i=1}^N x_i - 1)! (\beta + N - \sum_{i=1}^N x_i - 1)!}$$



Beta-Binomial Posterior Mean & Variance

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- Posterior mean

$$\begin{aligned} E(\theta|x) &= \frac{\sum_{i=1}^N x_i + \alpha}{N + \alpha + \beta} \\ &= c_2 \frac{\alpha}{\alpha + \beta} + (1 - c_2) \bar{x} \end{aligned}$$

$$\text{for } c_2 = \frac{\alpha + \beta}{N + \alpha + \beta}$$



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- Posterior mean

$$\begin{aligned} E(\theta|x) &= \frac{\sum_{i=1}^N x_i + \alpha}{N + \alpha + \beta} \\ &= c_2 \frac{\alpha}{\alpha + \beta} + (1 - c_2) \bar{x} \end{aligned}$$

for $c_2 = \frac{\alpha + \beta}{N + \alpha + \beta}$

- Posterior variance

$$Var(\theta|x) = \frac{(\sum_{i=1}^N x_i + \alpha)(N - \sum_{i=1}^N x_i + \beta)}{(N + \alpha + \beta + 1)(N + \alpha + \beta)^2}$$



Limit Results

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- Mean as $N \rightarrow \infty$

$$E[\theta|x] \rightarrow \frac{1}{N} \sum_{i=1}^N x_i$$



Limit Results

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- Mean as $N \rightarrow \infty$

$$E[\theta|x] \rightarrow \frac{1}{N} \sum_{i=1}^N x_i$$

- Variance as $N \rightarrow \infty$

$$\text{Var}[\theta|x] \rightarrow 0$$



Beta-Bernoulli Prediction

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Since

$$f(x_{new}|x) = \int f(x_{new}|\theta, x)f(\theta|x)d\theta$$

Show

$$\begin{aligned} f(x = 1|x) &= \frac{\sum_{i=1}^N x_i + \alpha}{N + \alpha + \beta} \\ &= E[\theta|x] \end{aligned}$$



Examples of Beta Distributions

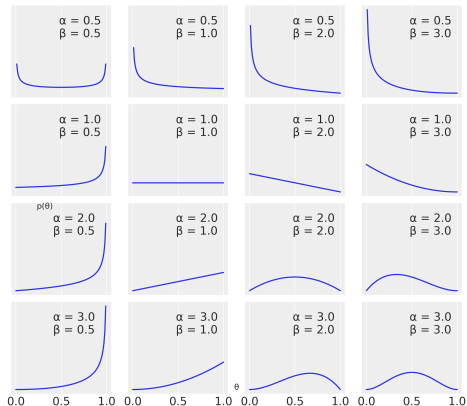
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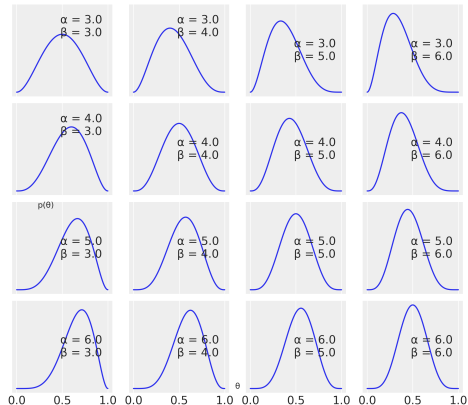
Examples of Beta Distributions

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Examples of Beta Posteriors

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Priors: $\text{Be}(1,1)$; $\text{Be}(5,5)$; $\text{Be}(1,6)$

