# Beta-Binomial Conjugate Priors

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# Agenda

# Definition of Conjugate Prior

Posterior has the same distribution with different parameters as the prior, so with

$$\underbrace{p(\boldsymbol{\theta}|\mathbf{X})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{X}|\boldsymbol{\theta})}^{}\,\overbrace{p(\boldsymbol{\theta})}^{}}{}$$

$$\underbrace{p(\boldsymbol{\theta}|\mathbf{X})}_{\text{posterior}} = \frac{\underbrace{p(\mathbf{X}|\boldsymbol{\theta})}_{\text{likelihood}}\ \underbrace{p(\boldsymbol{\theta})}_{\text{prior}}}{\underbrace{p(\mathbf{X})}_{\text{evidence}}}$$

Then

$$p(\boldsymbol{\theta}|\mathbf{X}) \sim p(\boldsymbol{\theta})$$

# Agenda



1. Conjugate Priors

2. Beta-Binomial

# Beta-Binomial

- Binomial likelihood with $N$ trials, $x = (x_1, \ldots, x_N)$

$$p(x|\theta) \propto \prod_{i=1}^{N} \theta^{x_i}(1-\theta)^{1-x_i}$$

# Beta-Binomial



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- Beta prior

$$h(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

# Beta-Binomial



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- Beta prior

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- Beta posterior

$$h(\theta|x) \propto \theta^{\sum_{i=1}^{N} x_i+\alpha-1}(1-\theta)^{N-\sum_{i=1}^{N} x_i+\beta-1}$$

$$c_1 = \frac{(\alpha+\beta+N-1)!}{(\alpha+\sum_{i=1}^{N} x_i-1)!(\beta+N-\sum_{i=1}^{N} x_i-1)!}$$

# Beta-Binomial Posterior Mean & Variance



- Posterior mean

$$E(\theta|x) = \frac{\sum_{i=1}^{N} x_i + \alpha}{N + \alpha + \beta}$$
$$= c_2 \frac{\alpha}{\alpha + \beta} + (1 - c_2)\bar{x}$$

for $c_2 = \frac{\alpha+\beta}{N+\alpha+\beta}$

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- Posterior variance

$$Var(\theta|x) = \frac{(\sum_{i=1}^{N} x_i + \alpha)(N - \sum_{i=1}^{N} x_i + \beta)}{(N + \alpha + \beta + 1)(N + \alpha + \beta)^2}$$

# Limit Results



- Mean as $N \to \infty$

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$$E[\theta|x] \to \frac{1}{N}\sum_{i=1}^{N} x_i$$

- Variance as $N \to \infty$

$$Var[\theta|x] \to 0$$

# Beta-Bernoulli Prediction



Since

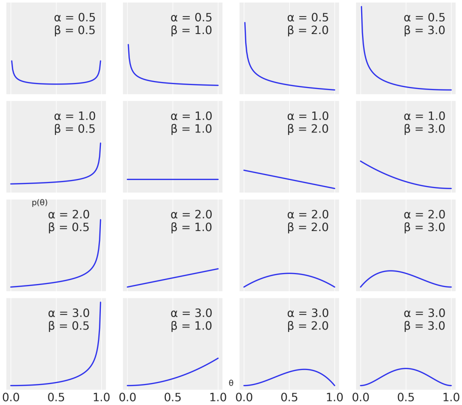
$$f(x_{new}|x) = \int f(x_{new}|\theta, x) f(\theta|x) d\theta$$

Show

$$f(x = 1|x) = \frac{\sum_{i=1}^{N} x_i + \alpha}{N + \alpha + \beta}$$
$$= E[\theta|x]$$

# Examples of Beta Distributions





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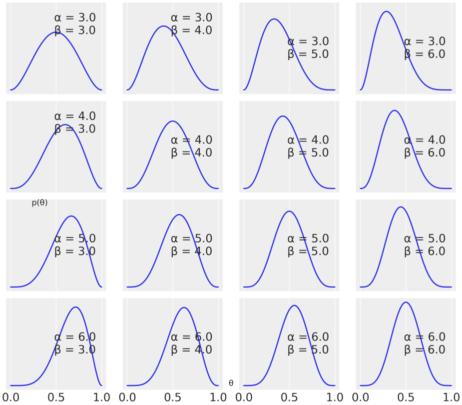

# Examples of Beta Posteriors



Priors: Be(1,1); Be(5,5); Be(1,6)