Probability Review 4 - Expectation and Moments of Random Variables

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Agenda

1. Moments of a Random Variable
   - Mean
   - Variance

Expectation of a Discrete Random Variable

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- t Distribution: $E[X] = \mu, \nu > 1$

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- Find $E[X^3]$ for $X \sim U(0, 1)$

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- t Distribution: $Var[X] = \hat{\sigma}^2 \frac{\nu}{\nu-2}, \nu > 2$

Useful Formulas

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- Second Moment: $E[X^2] = (E[X])^2 + Var[X]$