Are We There Yet?
Iterative Methods for Solving Linear Systems

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MTH 499-02

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Outline

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- Overview of methods of solution: direct, indirect

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- Theory on iterative methods for solving linear systems

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- Description and examples of Richardson, Jacobi, and Gauss-Seidel methods

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  - Example: finding solutions to linear systems (today's topic!)

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    - $\|x - x^{(k)}\| \to 0$ rapidly, and
    - $\left[x^{(k)}\right]$ is easy to compute
  - Note: often the initial vector $x^{(0)}$ is an estimate of the solution or arbitrary $(x = 0)$

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- Letting $k \to \infty$, solution is $Qx = (Q - A)x + b$
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- Thus, if $||I - Q^{-1}A|| < 1$, then $\lim_{k \to \infty} ||x^{(k)} - x|| = 0$ [[How does this follow from the last step?]]

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Corollary

The iteration formumla

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will produce a convergent sequence provided that $\rho(I - Q^{-1}A) < 1$.