You Perturb It, You Pay For It According to $\mathcal{K}(A)$: Norms and Error Analysis

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MTH 499-02

March 25, 2013

Outline

- Norms: properties, types, examples

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- Theorems, results on norms

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- Theorems, results on norms
- Relative/absolute error in perturbations, conditioning number

Definition

Let V be a vector space. A norm is a function $\|\cdot\| : V \to \mathbb{R}^+$ that has the following properties:

- $\|x\| > 0$ if $x \neq 0, \; x \in V$
- $\|\lambda x\| = |\lambda| \; \|x\|$ if $\lambda \in \mathbb{R}, \; x \in V$
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (triangle inequality)

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Other comments:

- Norm can be thought of as generalization of absolute value
- Interpretation: vector length in $\mathbb{R}^2$, $\mathbb{R}^3$

Types of Norms

Vector Norms:

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- l_k norm

$$l_k = \|x\|_k = \left(\sum_{i=1}^{n} |x_i|^k\right)^{1/k} \text{ where } k \in \{1, 2, \dots\} \text{ and } x = (x_1, \dots, x_n)^T$$

Vector Norms:

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- Some common instances of the l_k norm:

k	l_k
1	$\|x\|_1 = \sum_{i=1}^{n} \lvert x_i \rvert$
2	$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$
∞	$\|x\|_\infty = \max_{1 \le i \le n} \lvert x_i \rvert$

- Consider vectors in $\mathbb{R}^3$

$$x = (2, -1, 3)^T \qquad y = (7, 2, 5)^T \qquad z = (-3, 1, -9)^T$$

	$\|\| \cdot \|\|_1$	$\|\| \cdot \|\|_2$	$\|\| \cdot \|\|_\infty$
x			
y			
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	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
x	6		
y			
z			

$$||x||_1 = 2 + 1 + 3 = 6$$

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	$\|\| \cdot \|\|_1$	$\|\| \cdot \|\|_2$	$\|\| \cdot \|\|_\infty$
x	6	3.74	
y			
z			

$$||x||_2 = \sqrt{2^2 + 1^2 + 3^2} = 3.74$$

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$$||x||_\infty = 3$$

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	$\|\cdot\|_1$	$\|\cdot\|_2$	$\|\cdot\|_\infty$
x	6	3.74	3
y	14	8.83	7
z	13	9.54	9

Definition

A matrix norm subordinate to a vector norm $\|\cdot\|$ is defined as

$$\|A\| = \sup\left\{\|Au\| : u \in \mathbb{R}^n, \|u\| = 1\right\},$$

where A is an $n \times n$ matrix.

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    which is the maximum absolute row sum of A
  - Subordinate to vector norm $\|\cdot\|_2$ is

    $$\begin{aligned}\|A\|_2 &= \sup_{\|x\|_2=1} \{\|Ax\|_2\} \\ &= \sqrt{\rho(A^T A)},\end{aligned}$$

    where $\rho(A^T A)$ is spectral radius of $A^T A$ (i.e., largest eigenvalue in absolute value of $A^T A$)

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- For $A^T A$, eigenvalues are $\sigma_1^2 = 6.8541$ and $\sigma_2^2 = 0.1459$.
- So, $\|A\|_2 = \sqrt{6.8541} \approx 2.618$.

- Consider the matrix $B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

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- $\|B\|_\infty$ is simply the maximum absolute row sum, which is 9.

Theorem (Subordinate Matrix Norm)

If $\|\cdot\|$ is any norm in $\mathbb{R}^n$, then the equation

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defines a norm on the linear space of all $n \times n$ matrices.

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Follows from properties of vector norms. Questions? $\square$

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Other Results

- $\|Ax\| \leq \|A\|\|x\| \qquad (x \in \mathbb{R}^n)$
- $\|AB\| \leq \|A\|\|B\|$
- $\|I\| = 1$

Results

- Let $Ax = b$ be a linear system with A an $n \times n$ invertible matrix, and let B be a perturbation of A such that $B\tilde{x} = b$. Then the following holds:

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \|I - BA\|$$

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- Putting these together, we get
$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \mathcal{K}(A) \frac{\|b - \tilde{b}\|}{\|b\|}$$
where $\mathcal{K}(A) = \|A\| \, \|A^{-1}\|$ is called the condition number of A

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- $\mathcal{K}(A)$ large, A ill-conditioned, else A well-conditioned

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- $\mathcal{K}(A)$ large, A ill-conditioned, else A well-conditioned
- $\mathcal{K}(A) > 0$

Error and Condition Number Cont.

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We define the error e and residual r for perturbed systems as follows:

$$e = x - \tilde{x} \quad \text{and} \quad r = b - \tilde{b} = Ax - A\tilde{x} = Ae$$

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- For $Ax = b$, we have

$$\frac{1}{\mathcal{K}(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \mathcal{K}(A) \frac{\|r\|}{\|b\|}$$

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- Question: why useful?
- Answer: b, r, $\mathcal{K}(A)$ known, so can estimate the relative error in x

Consider $Ax = b$ perturbed to $A\tilde{x} = \tilde{b}$ where

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}, \quad b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix} \quad \tilde{b} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}$$

$$\|A\|_\infty = \|A^{-1}\|_\infty = 1999, \quad \mathcal{K}(A) \approx 3.996 \times 10^6$$

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Compute

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq \mathcal{K}(A) \frac{\|b - \tilde{b}\|_\infty}{\|b\|_\infty}$$