# You Perturb It, You Pay For It According to $\mathcal{K}(A)$ : Norms and Error Analysis

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MTH 499-02

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### Outline

 $\bullet$  Norms: properties, types, examples

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- $\bullet$  Theorems, results on norms

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- Theorems, results on norms
- Relative/absolute error in perturbations, conditioning number

### Vector Norms

#### Vector Norms

#### Definition

Let V be a vector space. A norm is a function  $\|\cdot\|:V\to\mathbb{R}^+$  that has the following properties:

- ||x|| > 0 if  $x \neq 0, x \in V$
- $\|\lambda x\| = |\lambda| \|x\|$  if  $\lambda \in \mathbb{R}, x \in V$
- $||x + y|| \le ||x|| + ||y||$  for all  $x, y \in V$  (triangle inequality)

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#### Other comments:

- Norm can be thought of as generalization of absolute value
- Interpretation: vector length in  $\mathbb{R}^2$ ,  $\mathbb{R}^3$

Vector Norms:

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•  $l_k$  norm

$$l_k = ||x||_k = \left(\sum_{i=1}^n |x_i|^k\right)^{1/k}$$
 where  $k \in \{1, 2, \dots\}$  and  $x = (x_1, \dots, x_n)^T$ 

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• Some common instances of the  $l_k$  norm:

$$\begin{array}{c|c} k & l_k \\ \hline 1 & \|x\|_1 = \sum\limits_{i=1}^n |x_i| \\ 2 & \|x\|_2 = \left(\sum\limits_{i=1}^n x_i^2\right)^{1/2} \\ \infty & \|x\|_\infty = \max\limits_{1 \le i \le n} |x_i| \end{array}$$

$$x = (2, -1, 3)^{T} y = (7, 2, 5)^{T} z = (-3, 1, -9)^{T}$$

$$\frac{|| || \cdot ||_{1} || \cdot ||_{2} || \cdot ||_{\infty}}{x}$$

$$y$$

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$$\frac{\| \| \cdot \|_{1} \| \cdot \|_{2} \| \cdot \|_{\infty}}{x 6}$$

$$y$$

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$$||x||_1 = 2 + 1 + 3 = 6$$

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$$\frac{\parallel ||\cdot||_{1} ||\cdot||_{2} ||\cdot||_{\infty}}{x || 6 3.74}$$

$$y$$

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$$||x||_2 = \sqrt{2^2 + 1^2 + 3^2} = 3.74$$

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$$\frac{\parallel ||\cdot||_{1} ||\cdot||_{2} ||\cdot||_{\infty}}{x || 6 3.74 3}$$

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$$||x||_{\infty} = 3$$

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	$     \cdot   _1$	$  \cdot  _2$	$  \cdot  _{\infty}$
x	6	3.74	3
y	14	8.83	7
z	13	9.54	9

#### Matrix Norms

#### Definition

A matrix norm subordinate to a vector norm  $\|\cdot\|$  is defined as

$$||A|| = \sup \{||Au|| : u \in \mathbb{R}^n, ||u|| = 1\},$$

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• Subordinate to vector norm  $\|\cdot\|_2$  is

$$||A||_2 = \sup_{\|x\|_2 = 1} \{||Ax||_2\}$$
$$= \sqrt{\rho(A^T A)},$$

where  $\rho(A^TA)$  is spectral radius of  $A^TA$ , largest eigenvalue of  $A^TA$ .



• Consider matrix 
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

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- So,  $||A||_2 = \sqrt{6.8541} \approx 2.618$ .

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$$B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

  
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•  $||B||_2$  is simply the maximum absolute row sum, which is 9.

#### Theorem (Subordinate Matrix Norm)

If  $\|\cdot\|$  is any norm in  $\mathbb{R}^n$ , then the equation

$$||A|| = \sup_{\|u\|=1} \{||Au|| : u \in \mathbb{R}^n\}$$

defines a norm on the linear space of all  $n \times n$  matrices.

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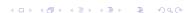
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#### Other Results

- $\bullet ||Ax|| \le ||A|| ||x|| \qquad (x \in \mathbb{R}^n)$
- $||AB|| \le ||A|| ||B||$
- ||I|| = 1

#### Results

• Let Ax = b be a linear system and let B be a perturbation of A such that  $B\tilde{x} = b$ . Then the following holds:

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \|I-BA\|$$

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- $\mathcal{K}(A)$  large, A ill-conditioned, else A well-conditioned
- $\mathcal{K}(A) > 0$



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$$e = x - \tilde{x}$$
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- Question: why useful?
- Answer:  $b, r, \mathcal{K}(A)$  known, so can estimate the relative error in x

## Error and Conditino Number Example

 ${\bf Consider}$