Horner's Method and Finding Roots of Polynomials

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MTH 499-02

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Our outline for today is...

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- Development of Horner's Algorithm

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- Pseudocode

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- Applications and Connections to Root Finding

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- Development of Horner's Algorithm
- Pseudocode
- Applications and Connections to Root Finding
- Related Ideas: Julia Sets and Fractals

Horner's Algorithm (Polynomials)

- Consider a polynomial p with degree n and some point x_0. We form the degree $n-1$ polynomial q as:

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- Expanding p and q:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1} + a_n x^n$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-2}x^{n-2} + b_{n-1}x^{n-1}.$$

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\text{constant}: &\quad p(x_0) = a_0 + b_0 x_0 \\
x: &\quad b_0 = a_1 + b_1 x_0 \\
x^2: &\quad b_1 = a_2 + b_2 x_0 \\
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x^{n-1}: &\quad b_{n-2} = a_{n-1} + b_{n-1} x_0 \\
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- Question: why should we care?

Horner's Algorithm: Pseudocode

function HORNER$(n, [a_0, \ldots, a_n], x_0)$
    $b_{n-1} \leftarrow a_n$
    for $k = 0$ **to** $n - 1$ **step** -1 **do**
        $b_{k-1} \leftarrow a_k + b_k x_0$
    return $[b_{-1}, \ldots, b_{n-1}]$

where

n: degree of p

a_i: coefficients of i^{th} degree term in p (degree n)

b_j: coefficients of j^{th} degree term in q (degree $n - 1$)

x_0: evaluation point

Applications

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- Thus, $p(x) = (x+1)(x^2 - 3x - 2) + 8$

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- Thus, the remainder terms from Horner's method are the coefficients we seek!

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- So, the Taylor expansion centered at $x = 2$ is

$$f(x) = (x-2)^3 + 4(x-2)^2 - (x-2) - 4.$$

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- Code: evaluate both points simultaneously (pg. 115)

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Extension: Julia Sets and Fractals

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- Demo!