A Summary of Methods of Solution for Linear Systems

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MTH 499-02

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Outline

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- Types of methods

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- Direct methods and examples

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- Indirect methods and examples

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Note: methods can be general or exploit certain matrix characteristics

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  - Scaling
  - Addition

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- Recall: GE is equivalent to computing LU factorization

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    - Partial: analyze one row
    - Complete: analyze entire submatrix
  - Unscaled/scaled: pertains to how the potential pivot locations are compared
    - Unscaled: compare absolute value of entries directly
    - Scaled: compare ratios of pivot entries to maximum entries in magnitude in each row

Direct Methods: Pivoting Examples

$$A_1 = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

$$p = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$$

function GE_PP$(n, (a_{ij}), (p_i))$
    for $k = 1$ **to** $n - 1$ **do**
        for $i = k + 1$ **to** n **do**
            $z \leftarrow a_{p_i k} / a_{p_k k}$
            $a_{p_i k} \leftarrow 0$
            for $j = k + 1$ **to** n **do**
                $a_{p_i j} \leftarrow a_{p_i j} - z a_{p_k j}$
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$$R_2 \leftarrow \frac{1}{2} R_2$$

Direct Methods: Pivoting Examples

$$A_2 = \begin{bmatrix} 0 & 2.5 & 5.5 \\ 1 & 0.5 & 0.5 \\ 0 & 2.5 & 2.5 \end{bmatrix}$$

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Direct Methods: Pivoting Examples

$$A_3 = \begin{bmatrix} 0 & 1 & 2.2 \\ 1 & 0.5 & -0.6 \\ 0 & 0 & -3 \end{bmatrix}$$

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Theorem

For the linear system $Ax = b$ with A invertible, define the iteration formula

$$x^{(k)} = Gx^{(k-1)} + c.$$

The sequence $\left\{x^{(k)}\right\}$ will converge to $(I - G)^{-1}c$ provided that $\rho(G) < 1$.

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Method	Iteration Formula: $x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$
Richardson	$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$
Jacobi	$x^{(k)} = (I - D^{-1}A)x^{(k-1)} + D^{-1}b$
Gauss-Seidel	$x^{(k)} = (I - L^{-1}A)x^{(k-1)} + L^{-1}b$
SOR	$x^{(k)} = (I - (\alpha D - C)^{-1}A)x^{(k-1)} + (\alpha D - C)^{-1}b$

where

- D: diagonal matrix where $d_{ii} = a_{ii}$
- L: lower triangular matrix where $l_{ij} = a_{ij}, i \geq j$
- C: $C + C^* = D - A$, C^* Hermitian, $\alpha \in \mathbb{R} \ni \alpha > \frac{1}{2}$

Method

- Richardson

System Stipulations

- A invertible

Iteration Formula

- $x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$

Convergence Criteria

- $\rho(I - A) < 1$

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$$\begin{aligned} x^{(1)} &= \begin{bmatrix} 0 & -1/6 & -1/6 \\ -1/3 & 1/3 & 0 \\ -1/6 & -1/3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 4/3 & 0 & 0 \end{bmatrix}^T \end{aligned}$$

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- After 12 iterations, $x^{(12)} \approx \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$.

Method

- Jacobi

System Stipulations

- A invertible
- A diagonally dominant

Iteration Formula

- $x^{(k)} = (I - D^{-1}A)x^{(k-1)} + D^{-1}b$
- $D = a_{ii}$

Convergence Criteria

- $\rho(I - D^{-1}A) < 1$

Method

- Gauss-Seidel

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Iteration Formula

- $x^{(k)} = (I - L^{-1}A)x^{(k-1)} + L^{-1}b$
- $L = a_{ij}, i \geq j$

Convergence Criteria

- $\rho(I - L^{-1}A) < 1$

Method

- SOR

System Stipulations

- Complex system
- A: positive definite, Hermitian

Iteration Formula

- $x^{(k+1)} = (I - (\alpha D - C)^{-1} A) x^{(k)} + (\alpha D - C)^{(-1)} b$
- D: positive definite, Hermitian
- $C : C + C^* = D - A$
- $\alpha \in \mathbb{R} \ni \alpha > \frac{1}{2}$

Convergence Criteria

- $\rho(I - (\alpha D - C)^{-1} A) < 1$

Typical values

- $D = a_{ii}, C = -a_{ij} \ni i \geq j$