A Summary of Methods of Solution for Linear Systems

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MTH 499-02

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- Types of methods

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- Direct methods and examples

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- Indirect methods and examples

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Note: methods can be general or exploit certain matrix characteristics

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    - Scaling
    - Addition

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- Recall: GE is equivalent to computing LU factorization

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    - Unscaled: compare absolute value of entries directly
    - Scaled: compare ratios of pivot entries to maximum entries in magnitude in each row

$$A_1 = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

$$p = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$$

1: **function** GE_PP$(n, (a_{ij}), (p_i))$
2:     **for** $k = 1$ **to** $n - 1$ **do**
3:         **for** $i = k + 1$ **to** n **do**
4:             $z \leftarrow a_{p_i k} / a_{p_k k}$
5:             $a_{p_i k} \leftarrow 0$
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7:                 $a_{p_i j} \leftarrow a_{p_i j} - z a_{p_k j}$
8:     **return** (a_{ij})

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Theorem

For the linear system $Ax = b$ with A invertible, define the iteration formula

$$x^{(k)} = Gx^{(k-1)} + c.$$

The sequence $\left\{ x^{(k)} \right\}$ will converge to $(I - G)^{-1}c$ provided that $\rho(G) < 1$.

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Methods

- Richardson
- Jacobi
- Gauss-Seidel
- SOR
- Steepest Descent
- Conjugate Gradient

Method

- Richardson

System Stipulations

- A invertible

Iteration Formula

- $x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$

Convergence Criteria

- $\rho(I - A) < 1$

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- After 12 iterations, $x^{(12)} \approx \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$.

Method

- Jacobi

System Stipulations

- A invertible
- A diagonally dominant

Iteration Formula

- $x^{(k)} = (I - D^{-1}A)x^{(k-1)} + D^{-1}b$
- $D = a_{ii}$

Convergence Criteria

- $\rho(I - D^{-1}A) < 1$

Method

- Gauss-Seidel

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Iteration Formula

- $x^{(k)} = (I - L^{-1}A)x^{(k-1)} + L^{-1}b$
- $L = a_{ij}, i \geq j$

Convergence Criteria

- $\rho(I - L^{-1}A) < 1$

Method

- SOR

System Stipulations

- Complex system
- A: positive definite, Hermitian

Iteration Formula

- $x^{(k+1)} = (I - (\alpha D - C)^{-1}A)x^{(k)} + (\alpha D - C)^{(-1)}b$
- D: positive definite, Hermitian
- $C : C + C^* = D - A$
- $\alpha \in \mathbb{R} \ni \alpha > \frac{1}{2}$

Convergence Criteria

- $\rho(I - (\alpha D - C)^{-1}A) < 1$

Typical values

- $D = a_{ii}, C = -a_{ij} \ni i \geq j$