You Perturb It, You Pay For It According to $\mathcal{K}(A)$: Norms and Error Analysis

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MTH 499-02

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Outline

 \bullet Norms: properties, types, examples

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- \bullet Theorems, results on norms

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- Theorems, results on norms
- Relative/absolute error in perturbations, conditioning number

Vector Norms

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Definition

Let V be a vector space. A norm is a function $\|\cdot\|:V\to\mathbb{R}^+$ that has the following properties:

- ||x|| > 0 if $x \neq 0, x \in V$
- $\|\lambda x\| = |\lambda| \|x\|$ if $\lambda \in \mathbb{R}, x \in V$
- $||x + y|| \le ||x|| + ||y||$ for all $x, y \in V$ (triangle inequality)

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Other comments:

- Norm can be thought of as generalization of absolute value
- Interpretation: vector length in \mathbb{R}^2 , \mathbb{R}^3

Vector Norms:

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• l_k norm

$$l_k = ||x||_k = \left(\sum_{i=1}^n |x_i|^k\right)^{1/k}$$
 where $k \in \{1, 2, \dots\}$ and $x = (x_1, \dots, x_n)^T$

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• Some common instances of the l_k norm:

$$\begin{array}{c|c} k & l_k \\ \hline 1 & \|x\|_1 = \sum\limits_{i=1}^n |x_i| \\ 2 & \|x\|_2 = \left(\sum\limits_{i=1}^n x_i^2\right)^{1/2} \\ \infty & \|x\|_\infty = \max\limits_{1 \le i \le n} |x_i| \end{array}$$

$$x = (2, -1, 3)^{T} y = (7, 2, 5)^{T} z = (-3, 1, -9)^{T}$$

$$\frac{|| || \cdot ||_{1} || \cdot ||_{2} || \cdot ||_{\infty}}{x}$$

$$y$$

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$$\frac{\| \| \cdot \|_{1} \| \cdot \|_{2} \| \cdot \|_{\infty}}{x 6}$$

$$y$$

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$$||x||_1 = 2 + 1 + 3 = 6$$

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$$\frac{|| || \cdot ||_{1} || \cdot ||_{2} || \cdot ||_{\infty}}{x || 6 3.74}$$

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$$||x||_2 = \sqrt{2^2 + 1^2 + 3^2} = 3.74$$

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$$||x||_{\infty} = 3$$

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	$ \cdot _1$	$ \cdot _2$	$ \cdot _{\infty}$
x	6	3.74	3
y	14	8.83	7
z	13	9.54	9

Matrix Norms

Definition

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• Subordinate to vector norm $\|\cdot\|_2$ is

$$||A||_2 = \sup_{\|x\|_2 = 1} \{||Ax||_2\}$$
$$= \sqrt{\rho(A^T A)},$$

where $\rho(A^TA)$ is spectral radius of A^TA , largest eigenvalue of A^TA .



• Consider matrix
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

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- $\bullet \ (A^T A) = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$
- For A^TA , eigenvalues are $\sigma_1^2=6.8541$ and $\sigma_2^2=0.1459$.
- So, $||A||_2 = \sqrt{6.8541} \approx 2.618$.

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$$B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

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• $||B||_2$ is simply the maximum absolute row sum, which is 9.

Theorem (Subordinate Matrix Norm)

If $\|\cdot\|$ is any norm in \mathbb{R}^n , then the equation

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defines a norm on the linear space of all $n \times n$ matrices.

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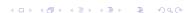
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Other Results

- $\bullet ||Ax|| \le ||A|| ||x|| \qquad (x \in \mathbb{R}^n)$
- $||AB|| \le ||A|| ||B||$
- ||I|| = 1

Results

• Let Ax = b be a linear system and let B be a perturbation of A such that $B\tilde{x} = b$. Then the following holds:

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \|I-BA\|$$

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- $\mathcal{K}(A)$ large, A ill-conditioned, else A well-conditioned
- $\mathcal{K}(A) > 0$



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- Question: why useful?
- Answer: $b, r, \mathcal{K}(A)$ known, so can estimate the relative error in x

Error and Conditino Number Example

Consider Ax = b perturbed to $A\tilde{x} = \tilde{b}$ where

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}, \quad b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix} \quad \tilde{b} = \begin{bmatrix} 1998.99 \\ 1997.01 \end{bmatrix}$$
$$\|A\|_{\infty} = \|A^{-1}\|_{\infty} = 1999, \quad \mathcal{K}(A) \approx 3.996 \times 10^{6}$$

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Compute

$$\frac{\|x - \tilde{x}\|_{\infty}}{\|x\|_{\infty}} \le \mathcal{K}(A) \frac{\|b - \tilde{b}\|_{\infty}}{\|b\|_{\infty}}$$