More Fun with Roots: the Secant Method

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MTH 499-02

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- Implementation and examples

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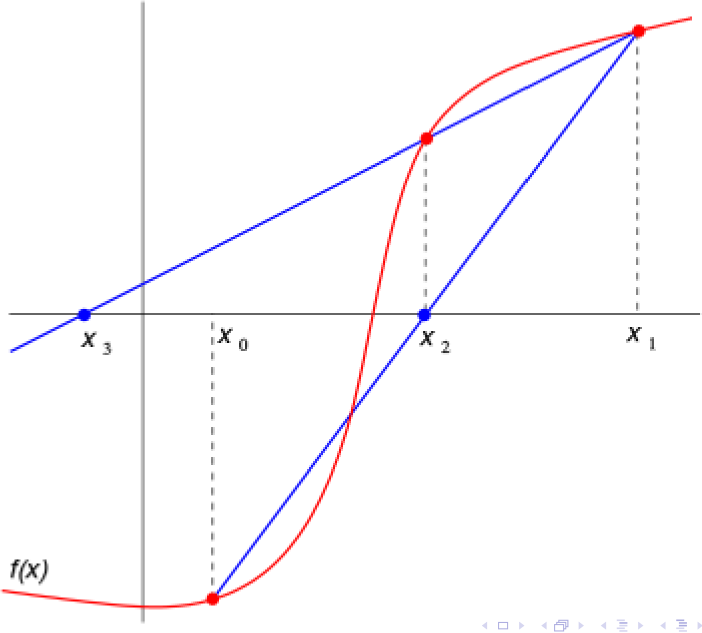
$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$



f(x)
x 3
x 0
x 2
x 1

Secant Method: Pseudocode

function $\textsc{Secant}(f, x_0, x_1, M, \delta, \epsilon)$

  $xvals_0 \leftarrow x_0$; $xvals_1 \leftarrow x_1$

  $fvals_0 \leftarrow f(xvals_0)$; $fvals_1 \leftarrow f(xvals_1)$

  output 0, $xvals_0$, $fvals_0$

  output 1, $xvals_1$, $fvals_1$

  for $k = 2$ **to** M **do**

    if $|fvals_{k-2}| > |fvals_{k-1}|$ **then**

      $xvals_{k-2} \leftrightarrow xvals_{k-1}$; $fvals_{k-2} \leftrightarrow fvals_{k-1}$

    $xvals_k \leftarrow xvals_{k-2} - fvals_{k-2} \cdot \dfrac{xvals_{k-1} - xvals_{k-2}}{fvals_{k-1} - fvals_{k-2}}$

    $xvals_{k-1} \leftarrow xvals_{k-2}$

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    $fvals_k \leftarrow f(xvals_k)$

  return $xvals$, $fvals$

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- Using the definition of the secant method, we have

$$\begin{aligned} e_{n+1} &= x_{n+1} - r \\ &= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} - r. \end{aligned}$$

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- We can substitute $x_n - r = e_n$ from the definition of our error and then simplify to give us

$$e_{n+1} = \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})}.$$

- We can then remove a factor of $e_n e_{n-1}$ and insert a factor of $(x_n - x_{n-1})$ to give

$$e_{n+1} = \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \cdot \left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right] \cdot e_n e_{n-1}. \quad (1)$$

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- Remember this equation (or write it down if you can't memorize this). It's important.
- We wish to bound $f(x_n)$ and $f(x_{n-1})$ in order to estimate our error in terms of a constant. So, first, note that Taylor's theorem gives us

$$\begin{aligned} f(x_n) &= f(r + e_n) \\ &= f(r) + e_n f'(r) + \frac{1}{2} e_n^2 f''(r) + \mathcal{O}(e_n^3). \end{aligned}$$

- We are searching for a zero to the function, so clearly $f(r) = 0$. Thus,

$$\frac{f(x_n)}{e_n} = f'(r) + \frac{1}{2}e_n f''(r) + \mathcal{O}(e_n^2).$$

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- If we change the index from n to $n-1$, we see that

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- Subtracting the previous two equations yields

$$\frac{f(x_n)}{e_n} - \frac{f(x_n)}{e_{n-1}} = \frac{1}{2}(e_n - e_{n-1}) f''(r) + \mathcal{O}(e_{n-1}^2).$$

- Since $x_n - x_{n-1} = e_n - e_{n-1}$, we can divide each side of the previous equation by $x_n - x_{n-1}$ and get

$$\frac{\dfrac{f(x_n)}{e_n} - \dfrac{f(x_n)}{e_{n-1}}}{x_n - x_{n-1}} \approx \frac{1}{2} f''(r).$$

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- Note that the quantity contained within the first set of brackets in Equation (1) can be rewritten as

$$\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \approx \frac{1}{f'(r)}.$$

  - For those who were imprudent, remember that Equation (1) was $e_{n+1} = \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \cdot \left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right] \cdot e_n e_{n-1}$.

- Thus, we have shown that

$$e_{n+1} \approx \frac{1}{2} \cdot \frac{f''(r)}{f'(r)} e_n e_{n-1} = C e_n e_{n-1}.$$

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- Note that this is similar to the error in Newton's method, which was

$$e_{n+1} = \frac{1}{2} \cdot \frac{f''(\xi_n)}{f'(x_n)} e_n^2 \approx \frac{1}{2} \frac{f''(r)}{f'(r)} e_n^2 = C e_n^2.$$

- Code!