Are We There Yet?
Iterative Methods for Solving Linear Systems

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Outline

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- Overview of methods of solution: direct, indirect

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- Theory on iterative methods for solving linear systems

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- Description and examples of Richardson, Jacobi, and Gauss-Seidel methods

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  - Example: finding solutions to linear systems (today's topic!)

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    - Note: often the initial vector $x^{(0)}$ is an estimate of the solution or arbitrary ($x = 0$)

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- Thus, if $||I - Q^{-1}A|| < 1$, then $\lim_{k \to \infty} ||x^{(k)} - x|| = 0$

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The spectral radius of a matrix A, $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$, *satisfies*

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Corollary

The iteration formumla

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b$$

will produce a convergent sequence provided that $\rho(I - Q^{-1}A) < 1$.

Method	Q	Iteration Formula: $x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$
Richardson	I	$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$
Jacobi	D	$x^{(k)} = (I - D^{-1}A)x^{(k-1)} + D^{-1}b$
Gauss-Seidel	L	$x^{(k)} = (I - L^{-1}A)x^{(k-1)} + L^{-1}b$

where

- D: diagonal matrix where $d_{ii} = a_{ii}$
- L: lower triangular matrix where $l_{ij} = a_{ij}, i \geq j$