# Are We There Yet? Iterative Methods for Solving Linear Systems

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# Outline

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- Theory on iterative methods for solving linear systems

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- Description and examples of Richardson, Jacobi, and Gauss-Seidel methods

**■** •99€ Iterative Methods for Solving Linear Systems

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  - Example: finding solutions to linear systems (today's topic!)

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where k > 1 denotes the  $k^{\text{th}}$  step in the process

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- Note: often the initial vector  $x^{(0)}$  is an estimate of the solution or arbitrary (x=0)



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Theory On Iterative Methods for Solving Linear Systems Cont.

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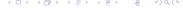
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• Thus, if  $||I - Q^{-1}A|| < 1$ , then  $\lim_{k \to \infty} ||x^{(k)} - x|| = 0$ 



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$$x^{(k)} = Gx^{(k-1)} + c.$$

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#### Corollary

The iteration formumla

$$Qx^{(k)} = (Q - A)x^{(k-1)} + b$$

will produce a convergent sequence provided that  $\rho(I - Q^{-1}A) < 1$ .

Iterative Methods

#### Iterative Methods

Method	Q	Iteration Formula: $x^{(k)} = (I - Q^{-1}A)x^{(k-1)} + Q^{-1}b$
Richardson	I	$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} + r^{(k-1)}$
Jacobi	D	$x^{(k)} = (I - D^{-1}A)x^{(k-1)} + D^{-1}b$
Gauss-Seidel	L	$x^{(k)} = (I - L^{-1}A)x^{(k-1)} + L^{-1}b$

#### where

- D: diagonal matrix where  $d_{ii} = a_{ii}$
- L: lower triangular matrix where  $l_{ij} = a_{ij}, i \geq j$