Discrete optimization: Exercise Set 4

1. **Understanding LP duality (2 Points)**
   So far, we have assumed the primal LP maximization problem to be in standard form, so that we have primal constraints of the form $\boldsymbol{a}_i^\intercal \boldsymbol{x} \leq b_i$, with the associated dual variable $y_i \geq 0$. However, we might not want to change the constraints, and there are ways to determine the dual variables for arbitrary constraints without transforming them to standard form (see ILP lecture notes, for example).

   (a) Prove that for a given primal constraint $\boldsymbol{a}_i^\intercal \boldsymbol{x} \geq b_i$, the associated dual variable is $y_i \leq 0$.

   (b) Prove that for a given primal constraint $\boldsymbol{a}_i^\intercal \boldsymbol{x} = b_i$, the associated dual variable is unconstrained, i.e. $y_i \in \mathbb{R}$.

2. **Modelling a graph problem (1 point)**
   A company owns a number of locations throughout the country for building chemical plants. While the exhaust and run-off from each individual plant would create a relatively low concentration in dangerous chemicals, the combined concentration from multiple plants becomes toxic. This can happen if there is an unfavorable combination of factors such as spatial proximity, a lack of separating structures like mountains, and connections such as wind and water currents.

   The company wants to build as many plants as possible, and hires you as a technical advisor. Your idea is to create the following graph model: Each node corresponds to a factory location. There is an edge between two nodes if there is a risk that the joint pollution from plants built at those locations becomes dangerous, and no edge if they could be operated together.

   (a) Formulate an ILP model that maximizes the number of factories that can be built without creating toxic pollution levels. What well-known graph problem does this correspond to?

   (b) (Ethics discussion, not mandatory) Under what conditions and assumptions...
   - ...is the company acting ethically?
   - ...is it ethical for you to take the job?