**Ver. 1.0.0**

**MATLAB® Program for Product Operator Formalism of Spin-1/2**

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**Purpose**

This program is designed to handle the product operator formalism of spin-1/2 using MATLAB. The program can manipulate various types of operators and this ability provides a rich environment of the calculations to users. The program will be helpful for educational use, for example, showing how to calculate product operators and explaining how pulse sequence components (Hahn-echo, INEPT etc.) work. Also, the program can be used to calculate evolutions of density operators under a pulse sequence including phase cycling.

**Requirement**

MATLAB and MATLAB Symbolic Math Toolbox are required to run this program. It is tested under the MATLAB R2020b environment. Basic knowledge of MATLAB programming and Symbolic toolbox is required.

**Limitation**

This program can only handle weakly-coupled spin-1/2 systems. The calculation speed of the program is slower than that of a similar program written for *Mathematica* (P. Güntert , 2006.).

**Introduction**

Before describing details, I will show a couple of examples demonstrating the ability and flexibility of this program.

The first example is how to create spin operators in the workspace. The command

>> PO.create({'I' 'S'})

creates parameters, Ix, Iy, Iz, Ip, Im, Ia, Ib, Sx, Sy, Sz, Sp, Sm, Sa, Sb, and hE. They can be used as spin angular momentum operators (*I*x, *I*y, *I*z, …) , lowering/raising operators (*I*+, *I*-, …), polarization operators (*I*α, *I*β, …) and unity operator (1/2*E*) of the *I* – *S* spin system. They care called **PO objects**.

Almost any calculations using product operators can be obtained by using these **PO objects**. For example, if you like to calculate [*I*x, *I*y] = *I*x*I*y – *I*y*I*x,

>> rho = Ix\*Iy – Iy\*Ix

rho =

PO with properties:

txt: 'Iz\*1i'

spin\_label: {'I' 'S'}

basis: 'xyz'

disp: 1

axis: [3 0]

coef: [1×1 sym]

Ncoef: [1×1 sym]

sqn: [1×1 sym]

M:

[1i/2, 0, 0, 0]

[ 0, 1i/2, 0, 0]

[ 0, 0, -1i/2, 0]

[ 0, 0, 0, -1i/2]

coherence:

[aa, 0, 0, 0]

[ 0, ab, 0, 0]

[ 0, 0, ba, 0]

[ 0, 0, 0, bb]

The result is stored in a new **PO object**, rho. A **PO object** stores several types of information to describe the current product operator(s). They are called **PO properties**. As you can see a **PO property** named **txt**, the result is Iz\*1i that corresponds to the well-known equation [*I*x, *I*y] = i\**I*z.

Another example is that if you like to get a matrix representation of 2*I*x*S*x – 2*I*y*S*y,

>> rho = 2\*Ix\*Sx – 2\*Iy\*Sy;

Then, the matrix representation can be accessed via the property **M** by

>> rhoMatrix = rho.M

[ 0, 0, 0, 1]

[ 0, 0, 0, 0]

[ 0, 0, 0, 0]

[ 1, 0, 0, 0]

As you can see, it is a pure DQ state. If you like to express rho using the shift operators *I*+, *I*-, *S*+ and *S*-,

>> rho\_pmz = xyz2pmz(rho);

>> rho\_pmz.txt

ans =

'IpSp + ImSm'

You can check how NMR interactions affect a spin state using provided functions called **PO methods**. For example, if you like to see how *I*z evolves under a 90°y pulse followed by the chemical shift interaction,

>> rho = pulse(Iz,'I','y',pi/2).cs('I',o1\*t);

Pulse: I 90y

Ix

CS: I o1\*t

Ix\*cos(o1\*t) + Iy\*sin(o1\*t)

The program has a flexibility to change the number of spins and spin labels. For example, if you like to create a *I*1-*I*2-*I*3 spin system,

>> PO.create({'I1' 'I2' 'I3'})

then I1x, I1y, …, I3a, I3b, and hE will be created.

The program provides various functions to construct and manipulate product operators that will be helpful for understanding of the ideas of NMR spectroscopy.

**How to Use the Program**

**1. Creating an Initial State of a System (Class Constructor)**

An initial state of a system, i.e., a density operator at the beginning, can be created in two different ways using **PO** methods.

The first method is to use preset spin operators to construct a density operator by their combinations. A **PO** method **PO.create(spin\_label\_cell)** creates spin operators with the labels defined in a cell array **spin\_label\_cell**. For example, in the case of a I-S two spin system,

>> PO.create({'I' 'S'})

creates Ix, Iy, Iz, Ip, Im, Sx, Sy, Sz, Sp, Sm, and hE in the workspace. It is possible to create a desired density operator by using operators ‘\*’, ‘+’, ‘-‘ and ‘/’ with coefficients in the system. Commonly used symbolic class coefficients are also created when **PO.create()** is executed (see the explanation for **PO.symcoef()**). As an example, to create ρ = *I*x\*cosθ + 2*I*y*S*z\*sinθ,

>> rho = Ix\*cos(q) + 2\*Iy\*Sz\*sin(q);

There are a couple of rules for the use of the ‘\*’, ‘+’, ‘-‘ and ‘/’ operators with **PO objects**. Firstly, these operators should be used between objects with the same number of spin types. Rules for each operator are shown below.

-‘\*’ operator can be used to calculate

1. **obj1**\***obj2**. **obj1** and **obj2** are **PO objects**. If the **basis** properties of these objects are different, the rules below are applied to determine the basis of the returned object.

Old bases New basis

('xyz', 'pmz') 🡺 'pmz'

('xyz', 'pol') 🡺 'pol'

('pmz', 'pol') 🡺 'pol'

where 'xyz' is for the Cartesian operator basis (*I*x, *I*y, *I*z), 'pmz' is for the lowering/raising operator basis (*I*+, *I*-, *I*z, E) and the polarization operator basis (*I*α, *I*β, b, *I*+, *I*-). These rules mean that the 'xyz' basis is overloaded by the 'pmz' or 'pol' bases and the 'pmz' basis is overloaded by the 'pol' basis. If you like to get a result with a different basis, use a basis-conversion method such as **pmz2xyz()**, **pol2xyz()** or **pol2pmz()**. For example, to obtain the result of Ix\*Ia with the 'xyz' basis instead of the 'pol' basis,

>> Ix\*pol2xyz(Ia)

2. **a\*obj** or **obj\*a**. **obj** is a **PO object** and **a** can be a double, sym, or char class. For example, expressions 2\*Ix, sym(2)\*Ix and '2'\*Ix are equivalent.

**- ‘+’** operator can be used to calculate

1. **obj1 + obj2**. **obj1** and **obj2** are **PO objects**. If the **basis** properties of these objects are different, the same rules for the '\*' operator are applied.

2. **obj** + **a** or **a + obj**. **obj** is a **PO object** and **a** can be a double, sym, or char class. This calculation means an addition of **a\*E** to **obj**.

**- ‘-’** operator can be used to calculate

1. **obj1 - obj2**. **obj1** and **obj2** are **PO objects**. If the **basis** properties of these objects are different, the same rules for the '\*' operator are applied.

2. **obj** - **a** or **a - obj**. **obj** is a **PO object** and **a** can be a double, sym, or char class. This calculation means **obj - a\*E** or **a\*E - obj**, respectively.

**- ‘/’** operator can be used to calculate

**obj/a**. **obj** is a **PO object** and **a** can be a double, sym, or char class. Note that a **PO object** cannot be a divisor, i.e., **a/obj** or **obj1/obj2** can not be calculated.

The second method is the use of the class constructor **PO()**. As an example, if you like to construct ρ = *I*1x\*cosθ + 2*I*1y*I*2z\*sinθ,

>> syms q;

>> rho = PO(2, {'Ix' 'I1yI2z'},{cos(q) sin(q)},{'I1' 'I2'});

The general syntax of the constructor is

**obj = PO(spin\_no, sp\_cell, coef\_cell, spin\_label\_cell)**

where **obj** is a **PO** **object**, **spin\_no** is the number of spin types in the system, **sp\_cell** is a cell array for product operators (without 2*N*s – 1 coefficients, e.g., 2 in 2*I*z*S*z, 4 in 4*I*x*S*y*K*z), **coef\_cell** is a cell array for coefficients (coefficients can be given by double, char or sym class. They should not also include the 2*N*s – 1 coefficients) and **spin\_label\_cell** is a cell array for spin labels that will be used as the property **spin\_label**.

There are some examples showing how to use the **PO** constructor.

ρ = *I*x + *S*y

>> rho = PO(2, {'Ix' 'Sy'});

If only the first two parameters are given, the third, **coef\_cell**, is automatically set as {1 1} and the fourth, **spin\_label\_cell**, is set as {'I' 'S'} that is from the 1st and 2nd components of the default cell array {'I' 'S' 'K' 'L' 'M'}. If **spin\_no** is given as 3, {'I' 'S' 'K'} is used instead of {'I' 'S'}. Note that if **spin\_label\_cell** is not given explicitly, spin types used for **sp\_cell** should be selected from the first **spin\_no** components of {'I' 'S' 'K' 'L' 'M'}.

ρ = a\**I*1x + b\**I*2y

>> rho = PO(2, {'I1x' 'I2y'},{'a' 'b'},{'I1' 'I2'});

Since I1 and I2 are used instead of I and S, it is necessary to input {'I1' 'I2'} as **spin\_label\_cell.** To assign the coefficients a and b, text inputs can be used in **coef\_cell**. Note that the format of each label should be one letter or two characters with a letter and a number. For example, 'SP' or 'I10' is not allowed to use as a label.

ρ = *I*x\*cosθ + *I*y\*sinθ

>> syms q;

>> rho = PO(1,{'Ix' 'Iy'},{cos(q) sin(q)});

The coefficients cosθ and sinθ can be described by the symbolic class. In this case, create q as a symbolic class and give cos(q) and sin(q) to **coef\_cell**.

ρ = *I*1x in a 3-spin system (*I*1, *I*2 and *I*3)

>> rho = PO(3, {'I1x'},{1},{'I1' 'I2' 'I3'});

The number of total spin types for further calculations should be set by the constructor, i.e., it cannot be changed later. **spin\_label\_cell** should be set for this particular 3-spin system.

ρ = *I­*1*x* + 4*I­*1x *I­*2y *I­*3z

>> rho = PO(3, {'I1x' 'I1xI2yI3z'},{1 1},{'I1' 'I2' 'I3'});

Both **sp\_cell** and **coef\_cell** should not include the 2*N*s - 1 coefficient, 4 in 4*I­*1x *I­*2y *I­*3z.

ρ = *I*+*S*-*K*z

>> rho = PO(3, {'IpSmKz'});

**PO** methods can handle the raising/lowering operator basis (**pmz** basis). In this basis, '**p**', '**m**' and '**z**' can be used where '**p**' and '**m**' are the raising and lowering operators, respectively. Note that it is not allowed to use different bases in a single **PO** object. In the case of the **pmz** basis, **Ncoef** is set as 1. The values for the **axis** property are 4 for '**p**' and 5 for '**m**'. To convert the basis from **xyz** to **pmz**, **xyz2pmz()** can be used. Reversely, **pmz2xyz()** is for the conversion from **pmz** to **xyz** basis.

ρ = *I*α*S*β

>> rho = PO(2, {'IaSb'});

**PO** methods can handle the polarization operator basis (**pol** basis). In this basis, '**a**', '**b**' , '**p**' and '**m**' can be used where '**a**' and '**b**' are the polarization operators. In the case of the **pol** basis, **Ncoef** is set as 1. The values for the **axis** property are 6 for '**a**' and 7 for '**b**'. To convert the basis from **xyz** to **pol**, **xyz2pol()** can be used. Reversely, **pol2xyz()** is for the conversion from **pol** to **xyz** basis.

Note that **PO()** cannot construct a product operator of same spin-type operators, i.e., ρ = *I*x\**I*y.

As a special case,

>> rho = PO(1,{'1'});

creates a **1/2E** operator with rho.axis = [0], rho.coef = [1], rho.Ncoef = [1/2] and rho.txt = '1/2E' and rho.M = [1/2 0; 0 1/2]. Actually, any types of characters that is not defined in **spin\_label** are considered as **1/2E** operator.

The first method, **PO.create()**, will be helpful for demonstrations of calculations of product operators in teaching classes. The second method, **PO()**, will be useful in a script describing a pulse sequence.

The characters of a **PO object** are stored as **PO properties**. Useful properties for users are '**txt**' that shows a text output of the operators, '**M**' that shows a matrix representation of the operators, and '**coherence**' that shows populations and coherences in the matrix. The values of the **PO** **properties** can be obtained by a syntax **obj.PropertyName**. For example, to get a matrix representation,

>> rho\_matrix = rho.M;

**2. Applying NMR Interactions to a System**

There are **PO methods** for NMR interactions i.e., RF pulse, chemical shift and *J*-coupling to apply a spin system.

**RF Pulses**

**Single Pulse**

The method to apply a single pulse is

**obj = pulse(obj, sp, ph, q)** or

**obj = obj.pulse(sp, ph, q)** .

where **obj** is a **PO object**, **sp** is a type of spin for the pulse, **ph** is a quadrature phase, and **q** is a flip angle in radian.

**sp** can be characters ('I', 'S', 'I1' or 'I2' etc. defined in **spin\_label**) or the number of the order of the spins in **spin\_label** (1 for 'I', 2 for 'S' in {'I' 'S'} etc.). **ph** can be characters such as 'x', 'X' or '-y' or numbers 0, 1, 2 or 3 for x, y, -x or -y, respectively. **q** can be double or symbolic class, such as pi/2 (double) or syms q (symbolic).

Examples

>> rho = PO(1, {'Iz'});

>> rho = pulse(rho, 'I', 'x', pi/2);

or equivalently,

>> rho = rho.pulse(1, 0, pi/2);

**Simultaneous Pulses**

The method to apply simultaneous pulses is

**obj = simpulse(obj, sp\_cell, ph\_cell, q\_cell)** or

**obj = obj.simpulse(sp\_cell, ph\_cell, q\_cell)** .

**sp\_cell**, **ph\_cell** and **q\_cell** are cell arrays corresponding to **sp**, **ph** and **q** for **pulse()**, respectively.

>> rho = PO(2, {'Iz' 'Sz'});

>> rho = simpulse(rho, {'I' 'S'}, {'x' 'y'}, {pi/2 pi/2});

It applies a 90x pulse to I and 90y to S.

Equivalently,

>> rho = simpulse(rho, {1 2}, {0 1}, {pi/2 pi/2});

The wildcard character **'\*'** can be used for **sp\_cell** to make a command simpler.

Let's assume a 5-spin system, *I*1, *I*2, *I*3, *S*4 and *S*5.

>> rho = PO(4,{'I1z' 'I2z' 'I3z' 'S4z' 'S5z'},{sym(1) sym(1) sym(1) sym(1) sym(1)},{'I1' 'I2' 'I3' 'S4' 'S5'});

If applying a 90x pulse to all *I* spins, the wildcard character can be used as

>> rho = simpulse(rho, {'I\*'},{'x'},{pi/2});

If applying 90x pulses to all spins,

>> rho = simpulse(rho, {'\*'},{'x'},{pi/2});

If applying a 90x pulse to all *I* spins and a 180y pulse to *S* spins,

>> rho = simpulse(rho, {'I\*' 'S\*'},{'x' 'y'},{pi/2 pi});

**Pulses with Phase Shift**

The method to apply a single pulse with a phase shift is

**obj = pulse\_phshift(obj, sp, ph, q)** or

**obj = obj.pulse\_phshift(sp, ph, q)** .

The difference from **pulse()** is that **ph** is an arbitrary phase in radian. **ph** can be a double or symbolic class.

Accordingly, simultaneous pulses with phase shifts can be applied by

**obj = simpulse\_phshift(obj, sp\_cell, ph\_cell, q\_cell)** or

**obj = obj.simpulse\_phshift(sp\_cell, ph\_cell, q\_cell)** .

**Chemical Shift**

The method to apply a chemical shift evolution is

**obj = cs(obj, sp, q)** or

**obj = obj.cs(sp, q)** ,

where **sp** is a type of the spin and **q** is a rotation angle in radian. The formats of **sp** and **q** are same as the ones used for **pulse()**.

It is possible to obtain chemical shift evolutions of multiple spins with

**obj = simcs(obj, sp\_cell, q\_cell)** or

**obj = obj.simcs(sp\_cell, q\_cell)**

where **sp\_cell** and **q\_cell** have the same formats as the one used for **simpulse()**. The wildcard character **'\*'** also can be used for **sp\_cell** same as **simpulse()**.

>> syms oI oS t

>> rho = PO(3, {'I1x' 'I2x' 'S3x'},{1 1 1},{'I1' 'I2' 'S3'});

>> rho = cs(rho,'I1',oI\*t);

>> rho = cs(rho,'I2',oI\*t);

>> rho = cs(rho,'S3',oS\*t);

or equivalently,

>> rho = simcs(rho,{1 2 3},{oI\*t oI\*t oS\*t});

or

>> rho = simcs(rho,{'I\*' 'S3'},{oI\*t oS\*t});

***J*-coupling**

The method to apply a *J*-coupling evolution is

**obj = jc(obj, sp, q)** or

**obj = obj.jc(sp, q)** ,

where **sp** is labels of a spin pair and **q** is a rotation angle in radian.

**sp** can be characters 'IS', 'I1I3' etc. or a 1 x 2 vector showing the index of spins such as [1 2] or [1 3].

>> rho = PO(2,{'Ix'});

>> syms J12 t

>> rho = jc(rho,'IS',pi\*J12\*t);

It is possible to obtain *J*-coupling evolutions of multiple spin-pairs by

**obj = simjc(obj, sp\_cell, q\_cell)** or

**obj = obj.simjc(sp\_cell, q\_cell)** ,

where **sp\_cell** is a cell array of spin pairs. Note that the wildcard character **'\*'** cannot be used for **sp\_cell** in this method. The spin pairs must be explicitly given in **sp\_cell**.

>> rho = PO(3,{'I1x' 'I2x'},{1 1},{'I1' 'I2' 'S3'});

>> syms J13 J23 t

>> rho = simjc(rho,{'I1S3' 'I2S3'},{pi\*J13\*t pi\*J23\*t});

**Pulse Field Gradients**

The method to apply a pulse field gradient is

**obj = pfg(obj, G, gamma\_cell)** or **obj = obj.pfg(G, gamma\_cell)**.

**G** is a strength of the gradient field and **gamma\_cell** is a cell array to store gyromagnetic ratio of spins in the system.

**G** can be double or symbolic class. Components of **gamma\_cell** can be also double or symbolic class.

>> syms G gH gC

>> rho = PO(3,{'I1x' 'I2x' 'S3x'},{1 1 1},{'I1' 'I2' 'S3'});

>> rho = pfg(rho, G, {gH gH gC});

Internally, angles are calculated from **G** and **gamma\_cell** as their products and they are used as input parameters of **simcs()**. In the case above, the angles are {G\*gH G\*gH G\*gC}. Note that a length of the gradient pulse is not considered in the calculation. If necessary, involve a time constant into **G** (e.g. G\*t). This method was inspired by a reference (Güntert, 2006).

**Independence of Methods from Basis Type**

The methods shown above can be applied to a **PO** object with any basis type. The basis type of the returned object is same as the one of the input object.

For example,

>> PO.create({'I' 'S'})

>> pulse(xyz2pmz(Iz),'I','y',pi/2);

Pulse: I 90y

Ip\*1/2 + Im\*1/2

>> pulse(xyz2pol(Iz),'I','y',pi/2);

Pulse: I 90y

IpSa\*1/2 + IpSb\*1/2 + ImSa\*1/2 + ImSb\*1/2

**Applying Multiple PO Methods in One Line**

You can apply multiple methods in one line using the dot **'.'** as a separator between the methods,

>> rho = rho.pulse('I','y',pi/2).cs('I',q).jc('IS',pi\*J12\*t);

that is equivalent to

>> rho = rho.pulse('I','y',pi/2);

>> rho = rho.cs('I',q);

>> rho = rho.jc('IS',pi\*J12\*t);

Note that the order to be applied to the system is left to right (**pulse()** => **cs()** => **jc()**) but not right to left (**jc()** => **cs()** => **pulse()**).

**3. Utilities**

There are additional methods as utilities. They can be used, for example, when a pulse sequence is simulated. Some of them are called **static methods**, i.e., they are called by **PO.MethodName()**.

There are methods for the basis-conversion among the 'xyz', 'pmz' and 'pol' bases. For example, the methods between the 'xyz' and 'pmz' bases are

**obj\_pmz = xyz2pmz(obj\_xyz)** or

**obj\_pmz = obj\_xyz.xyz2pmz()**

and

**obj\_xyz = pmz2xyz(obj\_pmz)**

**obj\_xyz = obj\_pmz.pmz2xyz()**

that convert ρ between Cartesian operator basis (**obj\_xyz)** and raising/lowering operator basis (**obj\_pmz**) (Güntert, 2006). **xyz2pol()** and **pol2xyz()** are used for the conversion between 'xyz' and 'pol', and **pmz2pol()** and **pol2pmz()** are used for the conversion between 'pmz' and 'pol'.

**PO.symcoef(spin\_label\_cell)**

creates commonly used symbolic constants based on the information of **spin\_label\_cell**. For example,

>> PO.symcoef({'I' 'S'})

creates oI, oS, o1, o2, JIS, J12, gI, gS, g1, g2 systematically from {'I' 'S'} in addition to a, b, d, t1, t, q, B and G.

**obj3 = commutator(obj1, obj2)** or

**obj3 = obj1.commutator(obj2)**

calculates the commutation relation of **obj1** and **obj2**, i.e., **obj3** = [**obj1**, **obj2**] = **obj1**\***obj2** – **obj2**\***obj1** where **obj1** , and **obj2** and **obj3** are **PO objects**.

**phout = PO.phmod(phx, ii)**

outputs a phase value, **phout**, from a phase table (vector), **phx**. If **ii** is smaller or equal to length(phx), phout = phx(ii) otherwise phout = phx(mod(ii,length(phx)). This method can be used in cases where there are phase tables with different lengths and phase-cycle them together.

Example

ph1 = [0 1 2 3]; % 4 steps

ph2 = [0 1 2 3 2 3 0 1]; % 8 steps

To phase-cycle them together, 8 steps are necessary. In such case, **PO.phmod(ph1,ii)** returns

ii = 1 🡺 0, ii = 2 🡺 1, ii = 3 🡺 2, ii = 4 🡺 3, ii = 5 🡺 0, ii = 6 🡺 1, ii = 3 🡺 2, ii = 8 🡺 3

while **PO.phmod(ph2,ii)** returns

ii = 1 🡺 0, ii = 2 🡺 1, ii = 3 🡺 2, ii = 4 🡺 3, ii = 5 🡺 2, ii = 6 🡺 3, ii = 0 🡺 2, ii = 8 🡺 1 .

**dispPOtxt(obj)** or

**obj.dispPOtxt()**

displays the **txt** property of **obj** to the Command Window.

**dispPO(obj)** or

**obj.dispPO()**

displays terms in **obj** in the following manner.

ID Product-Operator Coefficient

For example, in the case of ρ = Ix\*cosθ + Iy\* sinθ

>> syms q; rho = PO(1,{'Ix' 'Iy'},{cos(q) sin(q)});

>> dispPO(rho)

1 Ix cos(q)

2 Iy sin(q)

**id\_vec = findcoef (obj, coef\_in\_cell)** or

**id\_vec = obj. findcoef (coef\_in\_cell)**

finds particular terms that includes the coefficients defined in **coef\_in\_cell** and returns index numbers for these terms as **id\_vec**.

In the case of the example above,

>> id\_vec = rho.findcoef({cos(q)});

returns id\_vec = 1.

**obj = delPO(obj, id\_in)** or

**obj = obj.delPO(id\_in)**

Delete particular terms from **obj** using the information given by **id\_in**.

If **id\_in** is a vector including numbers, these numbers are indexes for terms to be deleted. The ID number of each term can be found by **obj.dispPO()** or can be obtained from **obj.findcoef()**. The example is

>> rho = PO(3,{'Ix' 'Sx' 'Kx'});

>> rho.dispPO()

1 Ix 1

2 Sx 1

3 Kx 1

>> rho = delPO(rho,[1 2])

>> rho.dispPO()

1 Kx 1

If **id\_in** is a cell array with characters describing product operators, terms for these operators are deleted.

There are some examples.

>> rho = delPO(rho,{'Ix'}); % delete Ix term

>> rho = delPO(rho,{'IzSz'}) % delete 2IzSz term

>> rho = delPO(rho,{'Ix' 'IzSz'})% delete Ix and 2IzSz term

The wildcard character **'\*'** can be used to choose all three phases.

>> rho = delPO(rho,{'IxS\*'})% delete 2IxSx, 2IxSy and 2IxSz if they exist

>> rho = delPO(rho,{'I\*S\*' 'I\*S\*K\*'})% delete any terms including 2I\*S\* and 4I\*S\*K\*

**obj = selPO(obj, id\_in)** or

**obj = obj.delPO(id\_in)**

Select particular terms from **obj** using the information given by **id\_in**. The format of **id\_in** is same as in **delPO()**.

**[a0\_V,rho\_V] = SigAmp(obj, sp\_cell, phR)** or

**[a0\_V,rho\_V] = obj.SigAmp(sp\_cell, phR)**

Calculation of signal amplitudes *a* corresponding to (-1) quantum coherences in the equation

*a*[-] = 2\*i\*ρ[-](0) \*exp(-i\*φrec)

in *Spin Dynamics*, p. 288 (eq. 11.48).

For example, in the case of homonuclear 2-spin system, the equation is

[*a*[-β] *a*[-α] *a*[β-] *a*[α-]] = 2\*i\*[ρ[-β](0) ρ[-α](0) ρ[β-](0) ρ[α-](0)]\*exp(-i\*φrec)

shown in p. 379.

Related topics: *Spin Dynamics* (2nd Ed.), p.262, p. 287, p. 371, p.379, pp.608-610.

**sp** describes spin types to be observed (e.g., 'I', 'IS', 'I1I2', [1], [1 2]). **phR** is a quadrature receiver phase (e.g. 'x', 'y', 0, 1).

**a0\_V** is a vector corresponding to 2\*i\*[ρ[-β…](0) ρ[-α…](0) ρ[β-…](0) ρ[α-…](0) …] \*exp(-i\*φrec).

**rho\_V** is a vector describing which component of **a0\_V** comes from which coherence in the density operator (e.g., 'ma' for '-α', 'bm' for 'β-').

**dispProp(obj, PropertyName)** or

**obj.dispProp(PropertyName)**

displays a property **PropertyName** of **obj** to the Command Window. This method is useful to check the protected properties which is invisible from the Command Window.

**Technical Details**

**Design of the Program**

The code is written with the object-oriented programming (OOP) style. In manner of the OOP, parameters for characterizing product operators and functions for NMR interactions are called **properties** and **methods** of a **class** named **PO**, respectively. A **PO class object** stores the **PO class properties**, and the **object** is processed by the **PO class methods**. By designing **properties** and **methods** properly, **PO class objects** can be handled in manner of the product operator formalism.

**PO Class Properties**

Any product operator can be described by three characteristic properties, spin types, axis labels (x, y or z) and a coefficient. For example, in the case of -4*I*x*S*z*K*z\*cosθ, the axis labels are x, z and z for the 1st (*I*), 2nd (*S*) and 3rd (*K*) spins, respectively, and the coefficient is -cosθ. Note that the coefficient “4” is related to the number of the active spin types in the system (in this case the number (*N*s) is 3 for *I*, *S* and *K*, and 4 can be obtained from 2*N*s – 1) and thus it is not considered as an independent coefficient. In the **PO class properties**, information on the axis labels for the spins and the coefficients are stored as **axis** and **coef**, respectively.

**axis**: This property stores axis labels of product operators. It is a *M* x *N* matrix where *M* is the number of product operators in the system and *N* is the number of spin types in the system. Column positions of the matrix correspond to the spin types. Each component in the matrix has a value of 0, 1, 2 or 3 corresponding to 1-, x-, y- or z-operators, respectively, in the case of the Cartesian basis. For example, the **axis** property of *I*1x + *I*2x + *I*2z (*M* = 3) in the *I*1-*I*2 system (*N* = 2) is [1 0; 0 1; 0 3].

**coef**: This property stores coefficients including signs for product operators. It is a *M* x 1 vector. Note that the 2Ns – 1 coefficient at the beginning of a product operator is stored in another property, **Ncoef**.

There are additional properties in the **PO** class.

**spin\_label**: This property defines labels of spin types in a system such as, I,S,K, … or I1, I2, I3, … etc. The default labels are I, S, K, L, M.

**Ncoef**: This property stores the 2*N*s - 1 coefficients for product operators in the current system. It is automatically calculated from the **axis** property.

**txt**: This property stores a text output of the current system. It is automatically generated.

**M**: This property stores a matrix representation of the current system. It is automatically generated.

**basis**: This property stores a type of the operator basis of the current system. There are three types of bases, ‘xyz’ for the Cartesian operator basis (Ix, Iy, Iz and 1/2E), ‘pmz’ for the lowering/raising operator basis (Ip, Im, Iz and 1/2sE) and ‘pol’ for the polarization operator basis (Ia, Ib, Ip, Im).

**sqn**: This property stores a spin quantum number. As a default, it stores sym(1/2) for spin-1/2.

**disp**: This property stores values of 1 or 0 to control the output display of the applied method and the calculated result on the command window. The default value is 1 for display 'ON'.

**coherence**: This property is a 2*N* x 2*N* matrix displaying populations of spin states on the diagonal and coherences between states on the off-diagonal. **a** and **b** mean |α> and |β> states, respectively, and **m** and **p** mean coherences of |α> => |β> and |β> => |α>, respectively.

**Description of a Product Operator in the Program**

In the program, a product operator is described as a 1 x *N* vector where *N* is the number of spin types in a system. The axis label of the product operator is described as 1 for x, 2 for y, and 3 for z, and 0 for 1/2*E*. For example, *I*x and 2*I*z*S*z in the I-S system are described as [1 0] and [3 3], respectively. The intrinsic coefficient 2*N*s - 1 (e.g., 2 for 2*I*z*S*z) is calculated automatically and is stored separately. In the case of multiple terms, 1 x *N* vectors are combined to a *M* x *N* matrix where *M* is the number of the terms. A coefficient of each term is stored separately.

The **axis** property is beneficial for some important calculations in this program. One is a multiplication of operators. This calculation can be handled as an addition of the **axis** properties.

For example, **axis** properties of *I*z, *S*y, *K*x and their product *I*z\* *S*y\* *K*x are

Iz : [3 0 0]

Sy : [0 2 0]

Kx : [0 0 1]

Iz\*Sy\*Kz: [3 2 1]

As you can see, the **axis** property of *I*z\* *S*y\* *K*x is the sum of the three vectors (i.e., the **axis** properties) of the three operators.

As a note, there is an exception for a multiplication of the same spin type. For example, if you consider the product of *I*z, *I*x*S*y and *K*x,

Iz : [3 0 0]

IxSy : [1 2 0]

Kx : [0 0 1]

Iz\*Ix\*Sy\*Kz: [4 2 1]

In this case, a special calculation is necessary for the *I*-spin because the **axis** value obtained by the addition is not correct. The **axial** value of *I*z\**I*x should be 2 because *I*z\**I*x = i/2\**I*y. In the code, there is a branch to calculate an appropriate **axis** value for this type of cases.

Another benefit is that **axis** values of two operators can be used as indexes of a matrix that describes the cyclic commutations of the two operators. The details are explained in the next section.

**Cyclic Commutation Rules**

If operators *A*, *B*, and *C* shows [*A*, *B*] = i*C*, [*B*, *C*] = i*A* and [*C*, *A*] = i*B* (cyclic commutation) then

exp(-iθ*A*) *B* exp(iθ*A*) = *B* cosθ + *C* sinθ,

exp(-iθ*B*) *C* exp(iθ*B*) = *C* cosθ + *A* sinθ, and

exp(-iθ*C*) *A* exp(iθ*C*) = *A* cosθ + *B* sinθ

It is known that the spin angular momentum operators *I*x, *I*y, *I*z are in the cyclic commutation ([*I*z, *I*x] = i*I*y) and the formula above can be used to describe an evolution of a density operator under a Hamiltonian. For example, a density operator ρ(0) = *I*x evolves under a chemical shift Hamiltonian *H* = ω*I*z during a time period of t as

ρ(t) = exp(-i*H*t) ρ(0) exp(i*H*t) = exp(-iωt*I*z) *I*x exp(iωt*I*z) = *I*x cos(ωt) + *I*y sin(ωt)

**Use of the master table to accelerate calculation**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  | B |  |
|  |  |  | x | y | z |
|  |  |  | 1 | 2 | 3 |
|  | x | 1 | **0** | **3** | **-2** |
| A | y | 2 | **-3** | **0** | **1** |
|  | z | 3 | **2** | **1** | **0** |
|  |  |  |  | C |  |

The cyclic commutation rules can be summarized as a table (master table) using the equation exp(-iθ*A*) *B* exp(iθ*A*) = *B* cosθ + *C* sinθ above and *I*x, *I*y and *I*z. For example, in the case of exp(-iθ*I*z) *I*x exp(iθ*I*z) = *I*x cosθ + *I*y sinθ, the axis numbers of the *A* and *B* positions are 3 (z) and 1 (x), respectively. Then the axis number for *C* is the 3rd-row, 1st-column component in the table, i.e., 2 meaning *I*y. If the value for *C* is 0 for given *A* and *B*, then *A* and *B* are not in the cyclic commutation (e.g., *A* = *B* = *I*x). If the value for *C* is negative, then -*C* sinθ is used instead of +*C* sinθ.

This is the basic idea of the calculation in the code. This process does not handle a matrix calculation (e.g., expm(-1i\*q\*Iz.M)\**I*x.M\*expm(1i\*q\*Iz.M) ) which usually has a high calculation cost.

**When can the master table be used for the calculation?**

If the two rules below are satisfied, an evolution of ρ under *H* can be calculated with using the master table. Otherwise ρ does not evolve under *H*.

**Rule 1**. There should be at least one spin type matching between *H* and ρ.

AND

**Rule 2**. Only one spin type in the matching spin types has different axis labels between *H* and ρ.

These rules can be used for spin-1/2 with *H* that is a product of up to two spin operators (e.g. *H* = 2*I*z*S*z but not like 4*I*z*S*z*K*z).

The rule 1 is obvious but how about the rule 2. Here is the analysis. Suppose *H* = 2*I*a*S*b and ρ = 8*I*b*S*b*KL* where *K*, *L* are spin operators which are different types each other in addition to *I* and *S*. Suppose [*I*a, *I*b] = i*I*c in the cyclic commutation. There are two spin-types matching between *H* and ρ, thus satisfying the rule 1, and only one of them (*I* spin) has different labels between *H* and ρ, thus satisfying the rule 2.

Then [*H*, ρ] = 2*I*a*S*b 8*I*b*S*b*KL* – 8*I*b*S*b*KL* 2*I*a*S*b = 16 *I*a*I*b*S*b2*KL* - 16*I*b*I*a*S*b2*KL*

= 16 (*I*a*I*b – *I*b*I*a)*S*b2*KL* = 4(*I*a*I*b – *I*b*I*a)*KL* = i4*I*c*KL*. Note that *S*b2 = 1/4*E* for spin-1/2.

[4*I*c*KL*, *H*] = 4*I*c*KL* 2*I*a*S*b - 2*I*a*S*b 4*I*c*KL* = 8*I*c*I*a*S*b*KL* - 8*I*a*I*c*S*b*KL* = 8(*I*c*I*a – *I*a*I*c)*S*b*KL* = i8*I*b*S*b*KL* = iρ.

[ρ, 4*I*c*KL*] = 8*I*b*S*b*KL* 4*I*c*KL* - 4*I*c*KL* 8*I*b*S*b*KL* = 32*I*b*I*c*S*b*K*2*L*2 – 32*I*c*I*b*S*b *K*2*L*2 = 2(*I*b*I*c – *I*c*I*b) *S*b = i2*I*a*S*b = i*H*. Note that *K*2 = *L*2 = 1/4*E* for spin-1/2.

Thus, *H* and ρ are in the cyclic commutation.

What if the rule 2 is not satisfied? In the case of *H* = 2*I*b*S*b and ρ = 8*I*b*S*b*KL*,

[*H*, ρ] = 2*I*b*S*b 8*I*b*S*b*KL* – 8*I*b*S*b*KL* 2*I*b*S*b = 16 *I*b2*S*b2*KL* - 16*I*b2*S*b2*KL* = 0

thus, *H* and ρ are not in the cyclic commutation.

In the case of *H* = 2*I*a*S*b and ρ = 8*I*b*S*c*KL*, [*H*, ρ] is calculated as 0 from [2*I*a*S*b, 2*I*b*S*c] = 0 that can be obtained from the matrix representation.

[*H*, ρ] = 2*I*a*S*b 8*I*b*S*c*KL* - 8*I*b*S*c*KL* 2*I*a*S*b = 2*I*a*S*b 2*I*b*S*c 4*KL* - 2*I*b*S*c 4*KL* 2*I*a*S*b = 2*I*a*S*b 2*I*b*S*c 4*KL* - 2*I*b*S*c 2*I*a*S*b 4*KL* = [2*I*a*S*b, 2*I*b*S*c]\*4*KL* = 0 (Read *Spin Dynamics* (2nd Ed.), p.403, Eq. 15.24 and p. 407, Note 3).

Examples for these rules

ρ = *I*y and *H* = *I*z

Both ρ and *H* include *I*-spin (Rule 1: yes) and they have different axis labels (*I*y vs. *I*z) (Rule 2: yes). 🡺 Master Table: Yes

ρ = *S*y and *H* = *I*z

ρ and *H* don't have a same type of spin (Rule 1: No) 🡺 Master Table: No

ρ = *I*y and *H* = *I*y

Both ρ and *H* include *I*-spin (Rule 1: yes) but they have the same axis label (*I*y) (Rule 2: no). 🡺 Master Table: no

ρ = 2*I*z*S*y and *H* = *I*z

Both ρ and *H* include I-spin (Rule 1: yes) but *I*-spins have the same axis label (*I*z) (Rule 2: no). 🡺 Master Table: no

ρ = 2*I*z*S*y and *H* = 2*I*z*S*z

Both ρ and *H* include *I*- and *S*-type product operators (Rule 1: yes) and only *S*-spins have different axis labels (*S*y vs. *S*z) (Rule 2: yes). 🡺 Master Table: yes

ρ = 2*I*x*S*x and *H* = *I*z

Both ρ and *H* include *I*-spin (Rule 1: yes) and they have different axis labels (*I*x vs. *I*z) (Rule 2: yes). 🡺 Master Table: Yes

ρ = 2*I*x*S*x and *H* = 2*I*z*S*z

Both ρ and *H* include *I*- and *S*-type product operators (Rule 1: yes) but both spin types have different axis labels (*I*x vs. *I*z and *S*x vs. *S*z) (Rule 2: no). 🡺 Master Table: No

**Special cases for *J*-coupling evolutions**

According to *Spin Dynamics* (2nd Ed.), p. 483, there are four cases where a product operator does not evolve under a *J*-coupling Hamiltonian *I*jz*I*kz.

Case 1. If both spin *I*j and *I*k are missing in the product operator.

Case 2. If only one spin *I*j or *I*k is present, and that spin carries a z label.

Case 3. If both spins *I*j and *I*k are present, but both spins carry a z label.

Case 4. If both spins *I*j and *I*k are present, but neither spin carries a z label.

These all four cases are excluded by the two rules above.

Case 1. Rule 1 is not satisfied.

Case 2. Rule 1 is satisfied but Rule 2 is not satisfied.

Case 3. Rule 1 is satisfied but Rule 2 is not satisfied.

Case 4. Rule 1 is satisfied but Rule 2 is not satisfied.

**Implementation of the two rules in the programing code**

The cyclic commutation using the two rules above with the master table is calculated in the **PO** method, **UrhoUinv()**. In this method, the two rules are evaluated as shown below.

type\_mask\_vec = (rho\_axis.\*H\_axis)~=0;

% Check how many spin types get matched, matched: 1, unmatched: 0

% Ex. rho\_axis = [1 0 0] and H\_axis = [3 3 0] 🡺 type\_mask\_vec = [1 0 0];

axis\_diff\_vec = rho\_axis ~= H\_axis;

% Check the difference of the direction of each spin type, unmatched: 1, matched: 0

% Ex. rho\_axis = [1 0 0] and H\_axis = [3 3 0] 🡺 axis\_diff\_vec = [1 1 0];

axis\_mask\_vec = type\_mask\_vec.\*axis\_diff\_vec;

% Comparing the spin-type matching and spin-label unmatching.

% Ex. type\_mask\_vec = [1 0 0] and axis\_diff\_vec = [1 1 0] 🡺 axis\_mask\_vec = [1 0 0]

axis\_mask = sum(axis\_mask\_vec);

axis\_mask becomes 1 ONLY when the two rules are satisfied.

If axis\_mask is 1, then prepare

H\_axis = [h1 h2 h3 …] corresponding to *A* and

rho\_axis = [r1 r2 r3 …] corresponding to *B*

to calculate a new product operator

axis\_tmp = [a1 a2 a3 …] corresponding to *C*.

For each n (n = 1, 2, 3, …), the steps below are calculated.

- if both rn and hn are not 0, an takes an absolute value of the [hn, rn] component of the master table. If the value from the master table is negative, the sign of the coefficient is inverted.

- if rn is not 0 but hn is 0, an is same as rn. This is for cases such as

ρ = 2*I*z*S*z ([r1 r2] = [3 3]) and *H* = *I*y ([h1 h2] = [2 0]) 🡺 *C*: 2*I*x*S*z ([a1 a2] = [a1 r2] = [1 3])

- if rn is 0 but hn is not 0, an is same as hn. This is for cases such as

ρ = *I*x ([r1 r2] = [1 0]) and *H* = 2*I*z*S*z ([h1 h2] = [3 3]) 🡺 *C*: 2*I*yS*z* ([a1 a2] = [a1 h2] = [2 3])

**Tips for MATLAB Symbolic Math Toolbox**

In many cases, calculated coefficients by Symbolic Math Toolbox have simplified and readable expressions. However, it may be necessary to rewrite the coefficients in another expression or to organize them with some certain terms. There are helpful Symbolic Math Toolbox functions for those operations. The functions below are some selected ones. Please read the MATLAB documentations for details.

**S = simplify(expr, v)**

performs algebraic simplification of **expr** with **v** steps. Inside the code, **simplify()** is used to simplify **obj.coef**. Each component of **obj.coef** is a result of the 10-step simplification. The number of the steps can be changed by a **PO** method **obj = set\_SimplifySteps(obj, new\_v)** to **new\_v**. A smaller value makes a calculation faster, but the output result may not be simplified enough. Reversely, a larger value may provide a more simplified expression, but it makes the calculation longer.

**R = rewrite(expr, target)**

This function is helpful to rewrite any trigonometric function in terms of the exponential function by specifying the target 'exp'.

>> syms q; coef = cos(q) + 1i\*sin(q); coefnew = rewrite(coef,'exp')

coefnew = exp(q\*1i)

**[C, T] = coeffs(p, vars)**

This function is helpful to organize terms in **p** with respect to **vars**. For example, if you like to separate **coef** above to terms with cos(q) from terms with sin(q),

>> [C,T] = coeffs(coef, [cos(q) sin(q)])

C = [1, 1i]

T = [cos(q), sin(q)]

**References**

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