Lab 4 - Curvatures

Geometry Processing (GPR)

1 Curvature from a Monge patch

When we have a point cloud or triangle mesh we compute the Hessian at a point by locally adjusting a function. This is known as a *Monge patch*.

For a given point $\mathbf{p}$, its normal $\mathbf{n}$, and its neighbors $\mathcal{P} = \{p_i\}_{1 \leq i \leq m}$, we need to fit a quadratic function:

$$w(u, v) = au^2 + buv + cv^2 + du + ev + f = \mathbf{q}^T \mathbf{s}$$

$$\mathbf{q} = \begin{pmatrix} u^2 & uv & v^2 & u & v & 1 \end{pmatrix}^T, \qquad \mathbf{s} = \begin{pmatrix} a & b & c & d & e & f \end{pmatrix}^T$$

First we need to transform the neighbors to a coordinate system $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ aligned with the normal $\mathbf{n}$. The origin will be point $\mathbf{p}$ itself.

$$\begin{aligned} \mathbf{w} &= -\mathbf{n} \\ \mathbf{u} &= \mathbf{OX} \times \mathbf{w} \\ \mathbf{v} &= \mathbf{w} \times \mathbf{u} \end{aligned}$$

We transform all the neighbors to this system from $\mathbf{p_i}$ to (u_i, v_i, w_i):

$$\begin{aligned} u_i &= < \mathbf{u}, \mathbf{p_i} - \mathbf{p} > \\ v_i &= < \mathbf{v}, \mathbf{p_i} - \mathbf{p} > \\ w_i &= < \mathbf{w}, \mathbf{p_i} - \mathbf{p} > \end{aligned}$$

Then we fit the function using least squares:

$$
\begin{aligned}
E(\mathcal{P}, w(u,v)) &= \sum_{i=1}^{m}(w(u,v) - w_i)^2 = \sum_{i=1}^{m}(\mathbf{q_i}^T\mathbf{s} - w_i)^2 \\
&= \sum_{i=1}^{m}[(\mathbf{q_i}^T\mathbf{s})^2 + w_i^2 - 2w_i(\mathbf{q_i}^T\mathbf{s})] \\
&= \mathbf{s}^T(\sum_{i=1}^{m}\mathbf{q_i}\mathbf{q_i}^T)\mathbf{s} + \sum_{i=1}^{m} w_i^2 - 2\mathbf{s}^T\sum_{i=1}^{m} w_i\mathbf{q_i}
\end{aligned}
$$

$$
\min_w E(\mathcal{P}, w) \quad \rightarrow \quad \nabla E(\mathcal{P}, w) = 0 \iff 2(\sum_{i=1}^{m}\mathbf{q_i}\mathbf{q_i}^T)\mathbf{s} - 2\sum_{i=1}^{m} w_i\mathbf{q_i} = 0 \iff
$$

$$
(\sum_{i=1}^{m}\mathbf{q_i}\mathbf{q_i}^T)\mathbf{s} = \sum_{i=1}^{m} w_i\mathbf{q_i} \iff \mathbf{As} = \mathbf{b}
$$

Then the Hessian $\mathbf{H_w}$ is simple to extract from the function $w(u,v)$:

$$
\mathbf{H_w} = \begin{pmatrix} \frac{\partial^2 w}{\partial u^2} & \frac{\partial^2 w}{\partial v \partial u} \\ \frac{\partial^2 w}{\partial u \partial v} & \frac{\partial^2 w}{\partial v^2} \end{pmatrix} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}
$$