LINEAR ALGEBRA

LECTURE 1: ELEMENTS IN LINEAR ALGEBRA

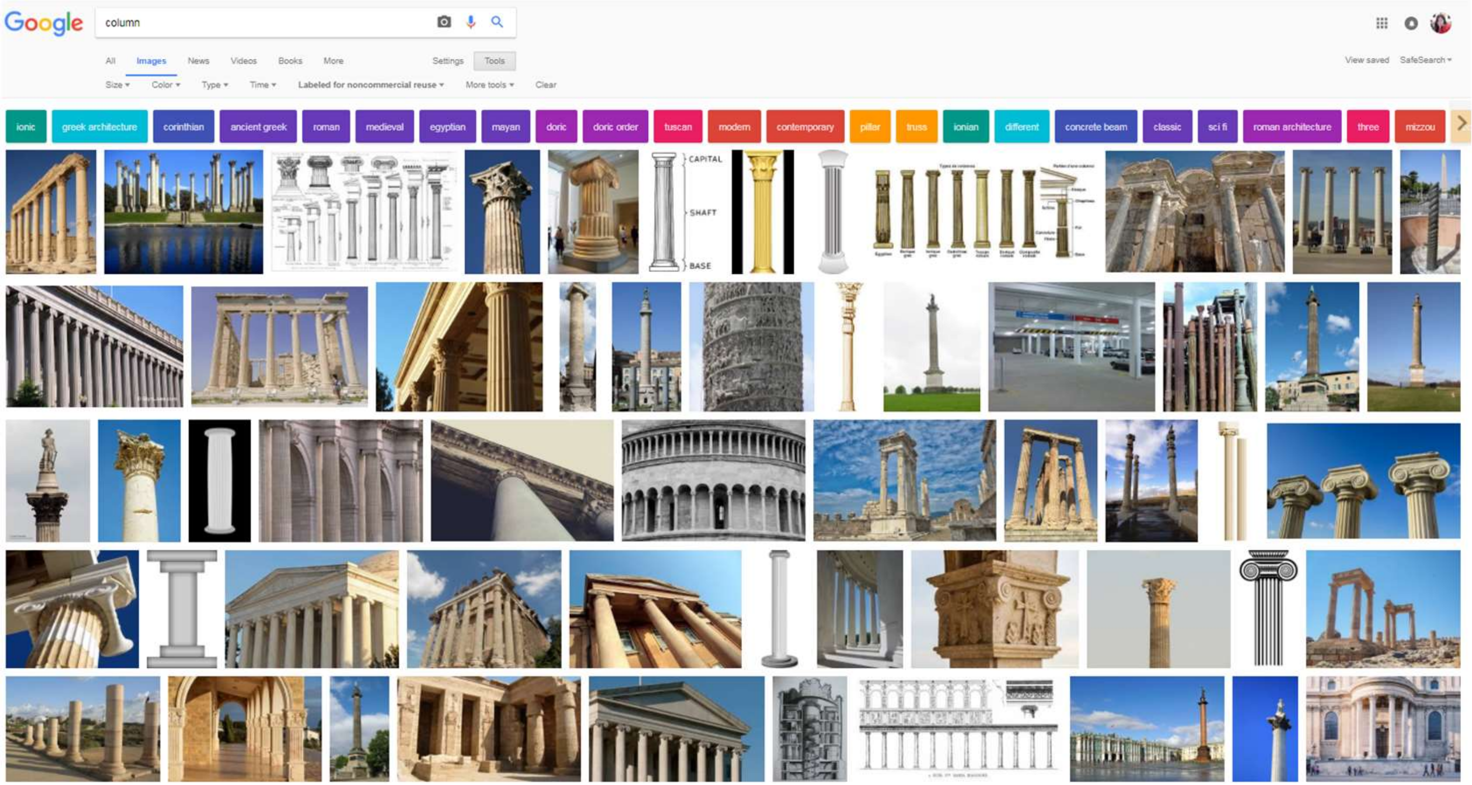
Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Advanced eigendecomposition
- Singular value decomposition

Scalar, Vector, and Matrix

- Scalar: a single number $s \in \mathbb{R}$ (lower case), e.g., 3.8
- Vector: an ordered list of numbers, e.g. $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ (boldface, lower-case), e.g., $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^3$
- Matrix: a two-dimensional array of numbers, e.g. $A = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ (capital letter)
  - Matrix size: 3×2 means 3 rows and 2 columns
  - Row vector: a horizontal vector
  - Column vector: a vertical vector

Column is Vertical Vector
(Don't be Confused!)



Column Vector and Row Vector

- A vector of n-dimension is usually a column vector, i.e., a mat
  rix
  of the size $n \times 1$
  - $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n = \mathbb{R}^{n\times 1}$
- Thus, a row vector is usually written as its transpose, i.e.,
  - $\mathbf{x}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1 \quad x_2 \quad \cdots \quad x_n] \in \mathbb{R}^{1\times n}$

Matrix Notations

- $A \in \mathbb{R}^{n \times n}$: Square matrix (#rows = #columns)
  - e.g., $B = \begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$
- $A \in \mathbb{R}^{m \times n}$: Rectangular matrix (possible: #rows ≠ #columns)
  - e.g., $A = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$
- A^T: Transpose of matrix (mirroring across the main diagonal)
  - e.g., $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 4 & 2 \end{bmatrix}$
- A_{ij} : (i, j)-th component of A, e.g., $A_{2,1} = 3$
- $A_{i,:}$: i-th row vector of A, e.g., $A_{2,:} = [3 \quad 4]$
- $A_{:,j}$: j-th column vector of A, e.g., $A_{:,2} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

Vector/Matrix Additions and Multiplications

- $C = A + B$: Element-wise addition, i.e., $C_{ij} = A_{ij} + B_{ij}$
  - A, B, C should have the same size, i.e., $A, B, C \in \mathbb{R}^{m \times n}$
- $c\mathbf{a}$, cA : Scalar multiple of vector/matrix
  - e.g., $2\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, $2\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 12 \\ 6 & 8 \\ 10 & 4 \end{bmatrix}$
- $C = AB$: Matrix-matrix multiplication, i.e., $C_{ij} = \sum_k A_{i,k} B_{k,j}$
  - e.g., $\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 1 \\ 9 & -3 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = [14]$, $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 10 \end{bmatrix}$

  Size: $(3 \times 2)(2 \times 2) = 3 \times 2$, $(1 \times 3)(3 \times 1) = 1 \times 1$, $(3 \times 1)(1 \times 2) = 3 \times 2$

Matrix multiplication is NOT commutative

- $AB \neq BA$: Matrix multiplication is NOT commutative.
- e.g., Given $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 5}$, AB is defined, but BA is n ot even defined.
- What if BA is defined, e.g., $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 2}$? Still, the sizes of $AB \in \mathbb{R}^{2\times 2}$and $BA \in \mathbb{R}^{3\times 3}$ does not match, so $AB \neq BA$.
- What if the sizes of AB and BA match, e.g., $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$? Still in this case, generally, $AB \neq BA$.
- E.g., $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$, $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

Other Properties

- $A(B+C) = AB + AC$: Distributive
- $A(BC) = (AB)C$: Associative
- $(AB)^T = B^T A^T$: Property of transpose