LINEAR ALGEBRA

LECTURE 4: LINEAR INDEPENDENCE, SPAN, SUBSPACE

Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- <u>Linear independence, span, and subspace</u>
- Linear transformation
- Least squares
- Eigendecomposition
- Advanced eigendecomposition
- Singular value decomposition

Recall: Linear System

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$

- Or, a vector equation is written as

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when $\mathbf{b} \in \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for $A\mathbf{x} = \mathbf{b}$, when is it unique?
- It is unique when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent.
- Infinitely many solutions exist when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent.

Linear Independence

(Practical) Definition:

- Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, check if $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$ for $j = 1, \ldots, p$, e.g.,

$$\mathbf{v}_j \in \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\} \text{ for some } j = 1, \ldots, p?$$

- If at least one such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly dependent.
- If no such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

(Formal) Definition:

- Consider $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots + x_p\mathbf{v}_p = \mathbf{0}$.
- Obviously, one solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

  which we call a trivial solution.

- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly independent if this is the only solution.
- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent if this system also has other nontrivial solutions, e.g., at least one x_i being nonzero.

Two Definitions are Equivalent

- If $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent, consider a nontrivial solution.
- In the solution, let's denote j as the last index such that $x_j \neq 0$.
- Then, one can write $x_j \mathbf{v}_j = -x_1 \mathbf{v}_1 - \cdots - x_{j-1} \mathbf{v}_{j-1}$,
  and safely divide it by x_j, resulting in

$$\mathbf{v}_j = -\frac{x_1}{x_j}\mathbf{v}_1 - \cdots - \frac{x_{j-1}}{x_j}\mathbf{v}_{j-1} \in \text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\right\}$$

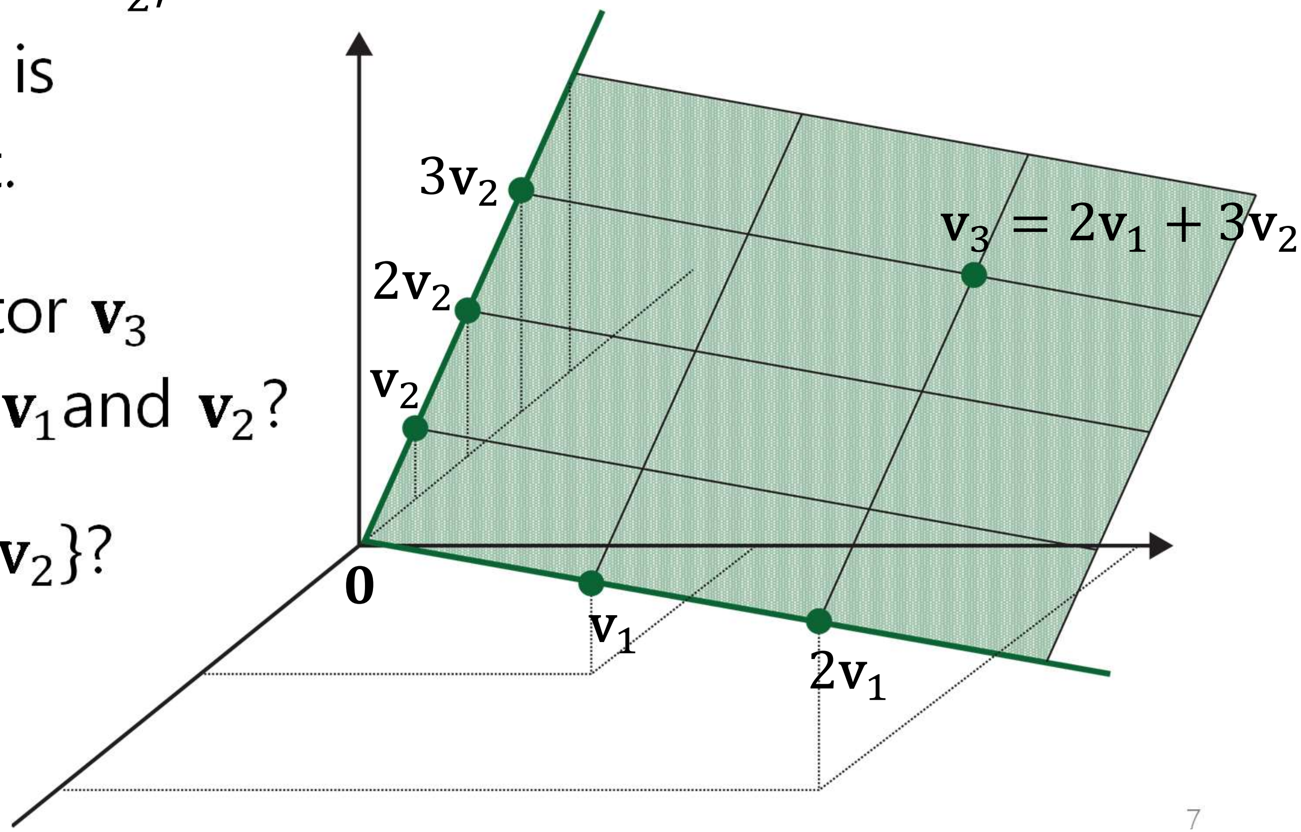
  which means $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors.

Geometric Understanding of Linear Dependence

- Given two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, Suppose Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane on the right.

- When is the third vector $\mathbf{v}_3$ linearly dependent of $\mathbf{v}_1$ and $\mathbf{v}_2$?

- That is, $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$?



Linear Dependence

- A linearly dependent vector does not increase Span!
- If $\mathbf{v}_3 \in$ Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, then

$$\text{Span } \{\mathbf{v}_1,\ \mathbf{v}_2\} = \text{Span } \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\},$$

- Why?
- Suppose $\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$, then the linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_1 + d_1)\mathbf{v}_1 + (c_1 + d_1)\mathbf{v}_2$$

which is also a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Linear Dependence and Linear System Solution

- Also, a linearly dependent set produces multiple possible linear combinations of a given vector.
- Given a vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$, suppose the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, i.e., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{b}$.
- Suppose also $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2$, a linearly dependent case.
- Then, $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_1 + 3\mathbf{v}_2) = 5\mathbf{v}_1 + 5\mathbf{v}_2$, so $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ is another solution. Many other solutions exist.

Linear Dependence and Linear System Solution

- Actually, many more solutions exist.
- e.g., $3\mathbf{v}_1 + 2\mathbf{v}_2 + 1\mathbf{v}_3 = 3\mathbf{v}_1 + 2\mathbf{v}_2 + (2\mathbf{v}_3 - 1\mathbf{v}_3)$
  $= 3\mathbf{v}_1 + 2\mathbf{v}_2 + 2(2\mathbf{v}_1 + 3\mathbf{v}_2) - 1\mathbf{v}_3 = 7\mathbf{v}_1 + 8\mathbf{v}_2 - 1\mathbf{v}_3$,

  thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}$ is another solution.

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when $\mathbf{b} \in \text{Span}\ \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for $A\mathbf{x} = \mathbf{b}$, when is it unique?
- It is unique when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent.
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Span and Subspace

- **Definition**: A **subspace** H is defined as a subset of $\mathbb{R}^n$ closed under linear combination:
    - For any two vectors, $\mathbf{u}_1$,$\mathbf{u}_2 \in H$, and any two scalars c and d, $c\mathbf{u}_1 + d\mathbf{u}_2 \in H$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is always a subspace. Why?
    - $\mathbf{u}_1 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$, $\mathbf{u}_2 = b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p$
    - $c\mathbf{u}_1 + d\mathbf{u}_2 = c(a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p) + d(b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p)$
    $= (ca_1 + db_1)\mathbf{v}_1 + \cdots + (ca_p + db_p)\mathbf{v}_p$
- In fact, a subspace is always represented as Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

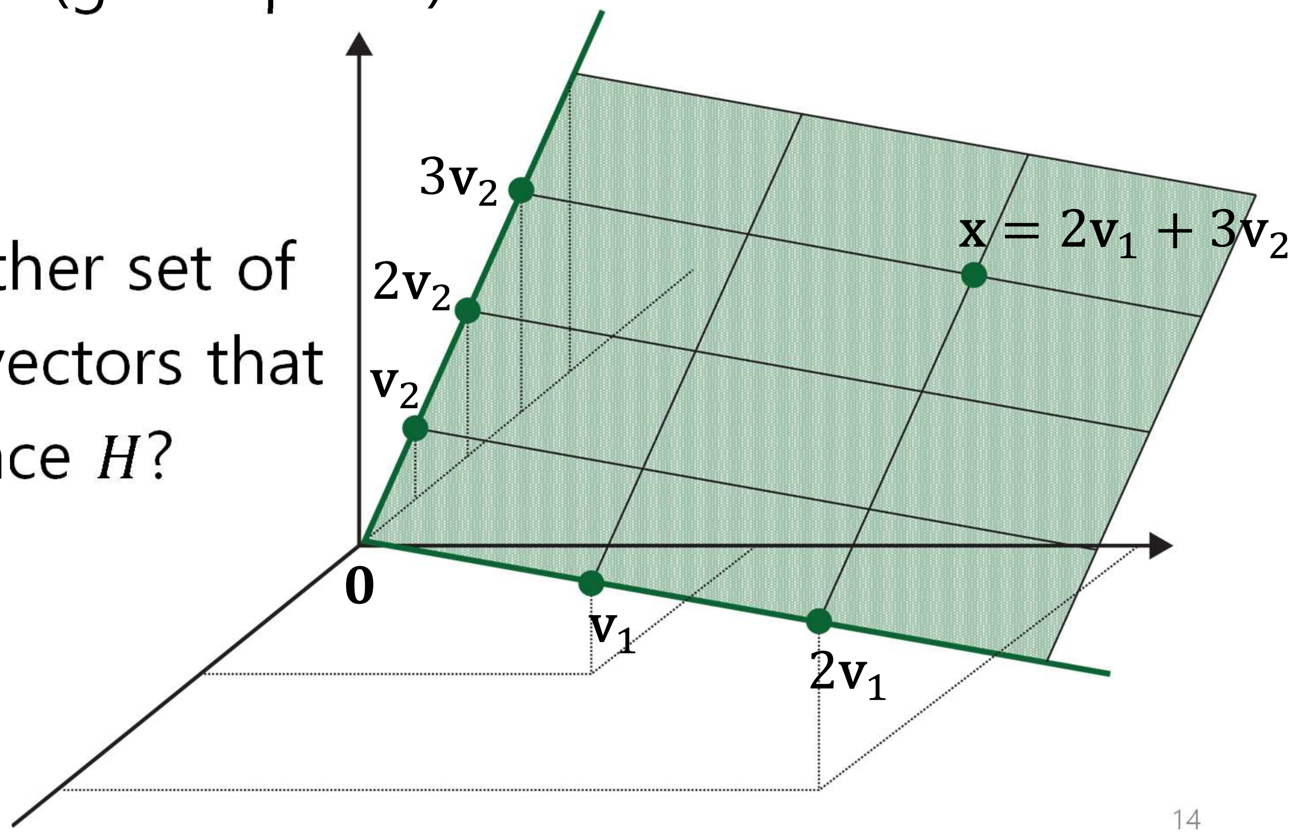
Basis of a Subspace

- **Definition**: A **basis** of a subspace H is a set of vectors that satisfies both of the following:
  - Fully spans the given subspace H
  - Linearly independent (i.e., no redundancy)

- In the previous example, where $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H, but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

Non-Uniqueness of Basis

- Consider a subspace H (green plane).
- Is a basis unique?
- That is, is there any other set of linearly independent vectors that span the same subspace H?



Dimension of Subspace

- What is then unique, given a particular subspace H?
- Even though different bases exist for H, the number of vectors in any basis for H will be unique.
- We call this number as the dimension of H, denoted as $\dim H$.
- In the previous example, the dimension of the plane is 2, meaning any basis for this subspace contains exactly two vectors.

Column Space of Matrix

- **Definition**: The **column space** of a matrix A is the subspace spanned by the columns of A.
  We call the column space of A as Col A.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- What is dim Col A?

Matrix with Linearly Dependent Columns

- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

  i.e., the third column is a linear combination of the first two.

$$\text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right\} \Rightarrow \text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

- What is dim Col A?

Rank of Matrix

- **Definition**: The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A:
  - rank A = dim Col A