LINEAR ALGEBRA

LECTURE 3: LINEAR COMBINATION, VECTOR EQUATION, FOUR VIEWS OF MATRIX MULTIPLICATION

goorm

KAIST AI
Graduate School of AI

DAVIAN
Data and Visual Analytics Lab

Lecture Overview

- Elements in linear algebra
- Linear system
- <u>Linear combination, vector equation,
Four views of matrix multiplication</u>
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Advanced eigendecomposition
- Singular value decomposition

Linear Combinations

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \cdots, c_p$,

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \cdots, \mathbf{v}_p$ with **weights or coefficients** $c_1, \cdots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

From **Matrix** Equation to **Vector** Equation

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\Rightarrow \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$A \qquad \mathbf{x} = \mathbf{b}$$

- A matrix equation can be converted into a vector equation:

$$\Rightarrow \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

Existence of Solution for $A\mathbf{x} = \mathbf{b}$

- Consider its vector equation:

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- When does the solution exist for $A\mathbf{x} = \mathbf{b}$?

Span

- **Definition**: Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is defined as the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.
- That is, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

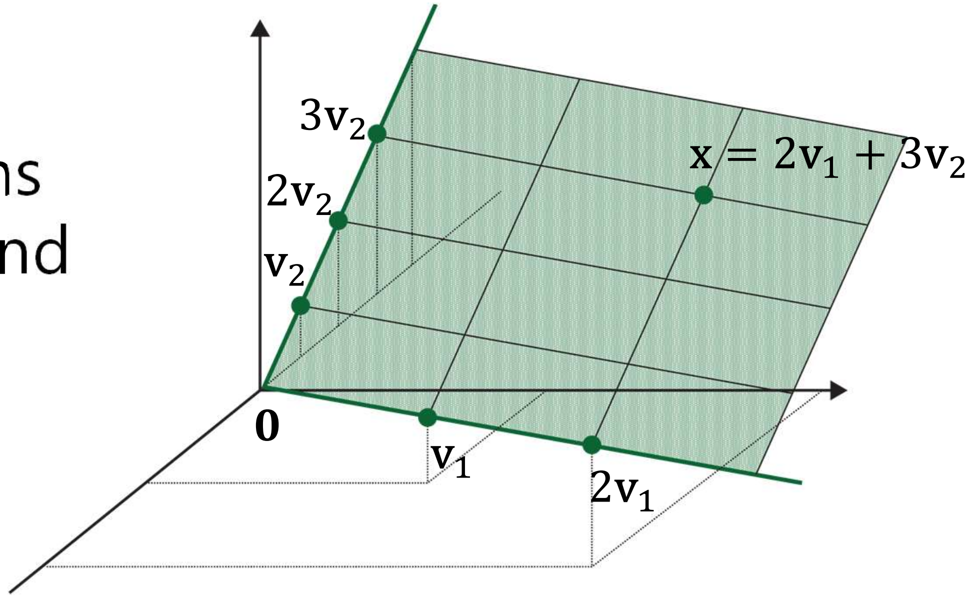
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_p\mathbf{v}_p$$

  with arbitrary scalars $c_1, \cdots, c_p$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is also called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Geometric Description of Span

- If $\mathbf{v}_1$ are $\mathbf{v}_2$ nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}_2$ not a multiple of $\mathbf{v}_1$, then Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{0}$.

- In particular, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{v}_1$ and $\mathbf{0}$ and the line through $\mathbf{v}_2$ and $\mathbf{0}$.



Geometric Interpretation of Vector Equation

- Finding a linear combination of given vectors $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ to be equal to $\mathbf{b}$:

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- The solution exists only when $\mathbf{b} \in \mathrm{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Matrix Multiplications as Linear Combinations of Vectors

- **Recall**: we defined matrix-matrix multiplications as the inner product between the row on the left and the column on the r ight:
  - e.g., $\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 1 \\ 9 & -3 \end{bmatrix}$
- Inspired by the vector equation, we can view $A\mathbf{x}$ as a linear combination of columns of the left matrix:
  - $\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3$

Matrix Multiplications as Column Combinations

- Linear combinations of columns
  - Left matrix: bases, right matrix: coefficients

One column on the right

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}1+\begin{bmatrix}1\\0\\-1\end{bmatrix}2+\begin{bmatrix}0\\1\\1\end{bmatrix}3$$

Multi-columns on the right

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 0\\3 & 1\end{bmatrix}=\begin{bmatrix}x_1 & y_1\\x_2 & y_2\\x_3 & y_3\end{bmatrix}=[\mathbf{x}\ \mathbf{y}]$$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}1+\begin{bmatrix}1\\0\\-1\end{bmatrix}2+\begin{bmatrix}0\\1\\1\end{bmatrix}3$$

$$\mathbf{y}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}(-1)+\begin{bmatrix}1\\0\\-1\end{bmatrix}0+\begin{bmatrix}0\\1\\1\end{bmatrix}1$$

Matrix Multiplications as Row Combinations

- Linear combinations of rows of the right matrix
  - Right matrix: bases, left matrix: coefficients

One row on the left

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & 1\end{bmatrix} = \begin{matrix} 1\times\begin{bmatrix}1 & 1 & 0\end{bmatrix}\\ +2\times\begin{bmatrix}1 & 0 & 1\end{bmatrix}\\ +3\times\begin{bmatrix}1 & -1 & 1\end{bmatrix}\end{matrix}$$

Multiple rows on the left

$$\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & 1\end{bmatrix} = \begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix} = \begin{bmatrix}\mathbf{x}^{\mathrm{T}}\\ \mathbf{y}^{\mathrm{T}}\end{bmatrix}$$

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} = 1\begin{bmatrix}1 & 1 & 0\end{bmatrix} + 2\begin{bmatrix}1 & 0 & 1\end{bmatrix} + 3\begin{bmatrix}1 & -1 & 1\end{bmatrix}$$

$$\mathbf{y}^{\mathrm{T}} = \begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix} = 1\begin{bmatrix}1 & 1 & 0\end{bmatrix} + 0\begin{bmatrix}1 & 0 & 1\end{bmatrix} + (-1)\begin{bmatrix}1 & -1 & 1\end{bmatrix}$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- (Rank-1) outer product

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

- Sum of (Rank-1) outer products

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ -4 & -5 & -6 \\ 4 & 5 & 6 \end{bmatrix}$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- Sum of (Rank-1) outer products is widely used in machine learning
  - Covariance matrix in multivariate Gaussian
  - Gram matrix in style transfer