LINEAR ALGEBRA

LECTURE 8: ADVANCED EIGENDECOMPOSITION

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
  Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- <u>Advanced eigendecomposition</u>
- Singular value decomposition

Symmetric Matrix (1 of 9)

- A **symmetric matrix** is a matrix *A* such that $A^T = A$.
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.

Symmetric Matrix (2 of 9)

- **Theorem 1:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
- **Proof:** Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that correspond to distinct eigenvalues, say, λ_1 and λ_2.
- To show that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, compute

$$\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 \quad \text{Since } \mathbf{v}_1 \text{ is an eigenvector}$$

$$= (\mathbf{v}_1^T A^T)\mathbf{v}_2 = \mathbf{v}_1^T (A\mathbf{v}_2) \quad \text{Since } A^T = A$$

$$= \mathbf{v}_1^T (\lambda_2 \mathbf{v}_2) \quad \text{Since } \mathbf{v}_2 \text{ is an eigenvector}$$

$$= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$$

Symmetric Matrix (3 of 9)

- Hence $(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$
- But $\lambda_1 - \lambda_2 \neq 0$, so $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$
- An $n \times n$ matrix *A* is said to be **orthogonally diagonalizable** if there are an orthogonal matrix *P* (with $P^{-1} = P^T$) and a diagonal matrix *D* such that

$$A = PDP^T = PDP^{-1} \qquad (1)$$

- Such a diagonalization requires *n* linearly independent and orthonormal eigenvectors.
- When is this possible?
- If *A* is orthogonally diagonalizable as in (1), then

$$A^T = (PDP^T)^T = P^{TT} D^T P^T = PDP^T = A$$

Spectral Decomposition (1 of 4)

- Suppose $A = PDP^{-1}$, where the columns of *P* are orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of *A* and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are in the diagonal matrix *D*.

- Then, since $P^{-1} = P^T$,

$$A = PDP^T = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

Spectral Decomposition (2 of 4)

- Using the column-row expansion of a product, we can write

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \qquad (2)$$

- This representation of *A* is called a **spectral decomposition** of *A* because it breaks up *A* into pieces determined by the spectrum (eigenvalues) of *A*.
- Each term in (2) is an $n \times n$ matrix of rank 1.
- For example, every column of $\lambda_1 \mathbf{u}_1 \mathbf{u}_1^T$ is a multiple of $\mathbf{u}_1$.
- Each matrix $\mathbf{u}_j \mathbf{u}_j^T$ is a **projection matrix** in the sense that for each **x** in $\mathbb{R}^n$, the vector $\left(\mathbf{u}_j \mathbf{u}_j^T\right)\mathbf{x}$ is the orthogonal projection of **x** onto the subspace spanned by $\mathbf{u}_j$.

Spectral Decomposition (3 of 4)

- **Example 4:** Construct a spectral decomposition of the matrix *A* that has the orthogonal diagonalization

$$A=\begin{bmatrix}7 & 2\\ 2 & 4\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}8 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}$$

- **Solution:** Denote the columns of *P* by $\mathbf{u}_1$ and $\mathbf{u}_2$.
- Then

$$A=8\mathbf{u}_1\mathbf{u}_1^T+3\mathbf{u}_2\mathbf{u}_2^T$$

Spectral Decomposition (4 of 4)

- To verify the decomposition of A, compute

$$\mathbf{u}_1\mathbf{u}_1^T = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{4}{5} \end{bmatrix}$$

and

$$8\mathbf{u}_1\mathbf{u}_1^T + 3\mathbf{u}_2\mathbf{u}_2^T = \begin{bmatrix} \frac{32}{5} & \frac{16}{5} \\ \frac{16}{5} & \frac{8}{5} \end{bmatrix} + \begin{bmatrix} \frac{3}{5} & \frac{-6}{5} \\ \frac{-6}{5} & \frac{12}{5} \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = A$$

The Spectral Theorem (1 of 2)

- The set if eigenvalues of a matrix A is sometimes called the spectrum of A, and the following description of the eigenvalues is called a spectral theorem.
- **Theorem 3:** The Spectral Theorem for Symmetric Matrices
- An $n \times n$ symmetric matrix A has the following properties:

  a. A has n real eigenvalues, counting multiplicities.

The Spectral Theorem (2 of 2)

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. *A* is orthogonally diagonalizable.