6. 포아송분포에 대한 베이지안 추론

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강의 목표

- 포아송모형에 대한 베이지안 추론
- Parameter Estimation (모수 추정)
    - Point Estimation (점추정)
    - Credible Interval (구간추정)
- Prediction (예측)

Poisson Distribution

- Probability mass function for Poisson with rate θ"

$$f(X = x \mid \theta) = \frac{\theta^x e^{-x}}{x!}.$$

- Suppose $x_1, ..., x_n$ have $Poi(\theta)$. Then the likelihood is

$$f(x_1, x_2, ..., x_n \mid \theta) = \prod_{i=1}^{n} f(x \mid \theta) \propto \theta^{\sum x_i} e^{-n\theta}.$$

- Sufficient Statistics: $\sum X_i$.

How to Choose Appropriate Prior

- Uniform:

$$\pi(\theta) \propto 1.$$

- Posterior Dist:

$$\pi(\theta \mid x_1, ..., x_n) \propto \theta^{\sum x_i} e^{-n\theta}.$$

- Gamma distribution with $\sum x_i + 1$ and n.

How to Choose Appropriate Prior

- Prior:

$$\pi(\theta) \propto \theta^{a}.$$

- Posterior Dist:

$$\pi(\theta \mid x_1, ..., x_n) \propto \theta^{\sum x_i + a} e^{-n\theta}.$$

- Gamma distribution with $\sum x_i + a + 1$ and n.

How to Choose Appropriate Prior

- Prior Dist:

$$\pi(\theta) \propto e^{-b\theta}.$$

- Posterior Dist:

$$\pi(\theta \mid x_1, ..., x_n) \propto \theta^{\sum x_i} e^{-(n+b)\theta}.$$

- Gamma distribution with $\sum x_i + 1$ and $n + b$.

How to Choose Appropriate Prior

- Prior Dist:

$$\pi(\theta) \propto \theta^{a+1} e^{-b\theta},$$

  that is, $\theta \sim Gamma(a, b)$.

- Posterior Dist:

$$\pi(\theta \mid x_1, ..., x_n) \propto \theta^{\sum x_i + a + 1} e^{-(n+b)\theta}.$$

- Gamma distribution with $\sum x_i + a$ and $n + b$.
- If the prior distribution is gamma, the posterior is gamma (conjugate).

The Gamma/Poisson Bayesian Model

- Posterior Mean:
$$\hat{\lambda} = \frac{\sum x_i + a}{n + b}.$$
- It can be decomposed:
$$\hat{\lambda} = \left(\frac{n}{n+b}\right)\left(\frac{\sum x_i}{n}\right) + \left(\frac{b}{n+b}\right)\left(\frac{a}{b}\right).$$
- The data get weighted more heavily as $n \to \infty$.
- a: prior guess of the number of events
- b: prior sample size

Bayesian Learning

- We can use the Bayesian approach to update our information about the parameter(s) of interest **sequentially** as new data become available.
- Suppose we formulate a prior for our parameter θ and observe a random sample x_1.
- Then the posterior is

$$\pi(\theta \mid x_1) \propto \pi(\theta) L(\theta \mid x_1)$$

- Then we observe a new (independent) sample x_2.

Bayesian Learning

- We can use our previous posterior as the new prior and derive a new posterior:

$$\begin{aligned}\pi(\theta \mid x_1, x_2) &\propto p(\theta)L(\theta \mid x_1, x_2)\\ &\propto p(\theta)L(\theta \mid x_1)L(\theta \mid x_2)\\ &\propto p(\theta \mid x_1)L(\theta \mid x_2)\end{aligned}$$

- Note this is the same posterior we would have obtained when x_1 and x_2 arrived at the same time.
- This "sequential updating" process can continue indefinitely in the Bayesian setup.

Traffic Example

- 두 도시에서 차량통행량 등 주위의 교통환경이 비슷한 교차로를 하나씩 선택하여 매주 발생한 교통사고 건 수를 1년 동안 조사하였다.
- 첫 번째 도시에서는 직진 후 좌회전 신호를 사용하고 두 번째 도시에서는 좌회전 후 직진 신호를 사용한다.
- 교통사고 건수는 독립적으로 포아송 분포를 따른다고 가정한다.
- 교통통제 등의 이유로 조사를 할 수 없었던 기간을 제외하고 다음 표와 같은 조사 결과를 얻었다.

Traffic Example

교통사고 건수	0	1	2	3	4	5	6	7
City 1의 사고 건수	7	14	13	8	4	2	2	0
City 2의 사고 건수	4	13	15	6	2	2	3	1

$$n_1 = 50, \quad \sum x_{1i} = 102, \quad \bar{x}_1 = 2.04$$

$$n_2 = 46, \quad \sum x_{2i} = 104, \quad \bar{x}_1 = 2.26$$

Traffic Example

- 두 도시의 실제 평균 교통사고 건수 θ_1과 θ_2에 대하여 $Gamma(2,1)$의 사전분포를 가정하자.
- 그렇다면 다음과 같은 posterior distribution을 찾을 수 있다.

$$\pi(\theta \mid n_1 = 50, \sum x_{1i} = 102) \sim Gamma(2 + 102, 1 + 50),$$

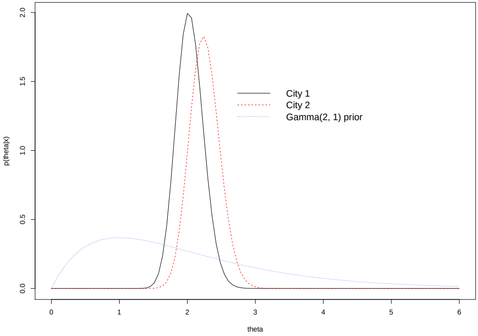
$$\pi(\theta \mid n_1 = 46, \sum x_{1i} = 104) \sim Gamma(2 + 104, 1 + 46).$$

Traffic Example

```
x1 = rep(c(0, 1, 2, 3, 4, 5, 6), c(7, 14, 13, 8, 4, 2, 2))
x2 = rep(c(0, 1, 2, 3, 4, 5, 6, 7), c(4, 13, 15, 6, 2, 2, 3, 1))
a =2; b = 1
n1 = length(x1); s1 = sum(x1)
n2 = length(x2); s2 = sum(x2)
postmean.theta1 = (a+s1)/(b+n1)
postmean.theta2 = (a+s2)/(b+n2)
### plot the posterior
par(mfrow=c(1, 1))
theta <- seq(0, 6, length=100)
plot(theta, dgamma(theta, a+s1, b+n1), type="l", xlab="theta", ylab="p(theta|x)")
lines(theta, dgamma(theta, a+s2, b+n2), lty=2, col = "red")
lines(theta, dgamma(theta, a, b), lty=3, col = "blue")
legend ( 2.5, 1.5, legend=c (paste ("City 1"), paste("City 2"),
paste("Gamma(2, 1) prior")), cex = 1.3, lty=c(1, 2, 3), col=c(1, 2, 4),
bty="n")
```

Traffic Example

- City 1의 사고 발생 건수가 City 2에 비해 작다.
- 사후 분포의 분산이 사전 분포의 분산보다 작다.
- Likelihood의 영향으로 사후 분포들이 구간 $(1.5, 3)$이외에는 매우 비슷한다.



Prediction Distribution

- Poisson - Gamma Prediction distribution:

$$
\begin{aligned}
& f(x_{n+1} \mid x_1, \ldots, x_n) \\
= \quad & \int f(x_{n+1} \mid \theta, x_1, \ldots, x_n) \pi(\theta \mid x_1, \ldots, x_n) d\theta \\
= \quad & \int f(x_{n+1} \mid \theta) \pi(\theta \mid x_1, \ldots, x_n) d\theta \\
= \quad & \int \frac{\theta^{x_{n+1}} e^{-\theta}}{x_{n+1}!} \times \frac{(b+n)^{a+\sum x_i}}{\Gamma(a+\sum x_i)} \theta^{a+\sum x_i - 1} e^{-(b+n)\theta} d\theta \\
= \quad & \frac{(b+n)^{a+\sum x_i}}{x_{n+1}! \Gamma(a+\sum x_i)} \int \theta^{a+\sum x_i + x_{n+1} - 1} e^{-(b+n+1)\theta} d\theta \\
= \quad & \frac{(b+n)^{a+\sum x_i}}{x_{n+1}! \Gamma(a+\sum x_i)} \times \frac{\Gamma(a+\sum x_i + x_{n+1})}{(b+n+1)^{a+\sum x_i + x_{n+1}}} \\
\propto \quad & \frac{1}{x_{n+1}!} \times \frac{\Gamma(a+\sum x_i + x_{n+1})}{(b+n+1)^{a+\sum x_i + x_{n+1}}}
\end{aligned}
$$

Prediction Distribution

$$
\begin{aligned}
&\propto \binom{a+\sum x_i + x_{n+1} - 1}{a+\sum x_i - 1}\left(\frac{b+n}{b+n+1}\right)^{a+\sum x_i}\left(\frac{1}{b+n+1}\right)^{x_{n+1}} \\
&= \binom{a+\sum x_i + x_{n+1} - 1}{x_{n+1}}\left(\frac{1}{b+n+1}\right)^{x_{n+1}}\left(\frac{b+n}{b+n+1}\right)^{a+\sum x_i}.
\end{aligned}
$$

Prediction Distribution

$$\Pr(X = x) = \binom{x+r-1}{x} p^x (1-p)^r \quad \text{for } x = 0, 1, 2, \ldots$$

Hence the prediction distribution is

$$NB\left(a + \sum_{i=1}^{n} x_i, \frac{1}{b+n+1}\right).$$

Prediction Distribution

- 예측 기대치:

$$\mathbb{E}(X_{n+1} \mid x_1, x_2, \ldots, x_n) = \frac{a + \sum x_i}{b + n}.$$

- 예측기대치는 사후기대치와 동일.
- 예측 분산:

$$\mathrm{Var}(X_{n+1} \mid x_1, x_2, \ldots, x_n) = \frac{a + \sum x_i}{(b + n)^2}(b + n + 1).$$

- 예측분산은 사후분산에 $b + n + 1$ 곱한 만큼 크다.

Prediction Expectation

- Expectation: using the fact $\mathbb{E}(X_{n+1} \mid \theta) = \theta$.

$$\begin{aligned} \mathbb{E}(X_{n+1} \mid x_1, \ldots, x_n) &= \mathbb{E}(\mathbb{E}(X_{n+1} \mid \theta, x_1, \ldots, x_n) \mid x_1, \ldots, x_n) \\ &= \mathbb{E}(\theta \mid x_1, \ldots, x_n). \end{aligned}$$

- Hence, the expected prediction is the same as the posterior expectation.

Prediction Variance

- Variance: using the fact $\mathrm{Var}(X_{n+1} \mid \theta) = \theta$,

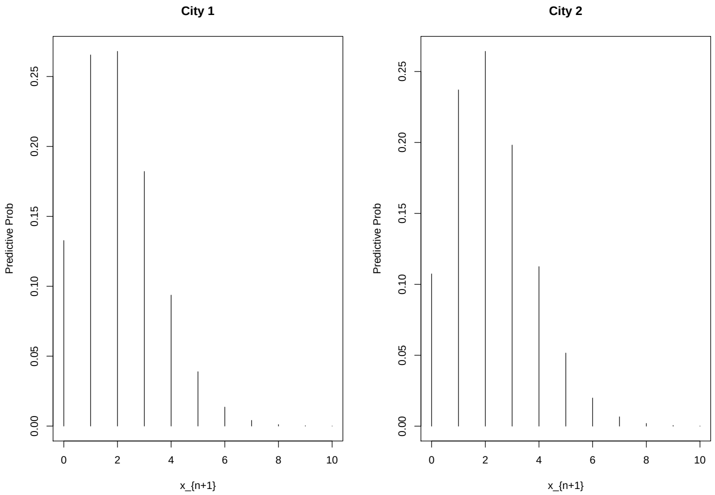
$$\begin{aligned}
\mathrm{Var}(X_{n+1} \mid x_1, \ldots, x_n) &= \mathrm{Var}(\mathbb{E}(X_{n+1} \mid \theta, x_1, \ldots, x_n) \mid x_1, \ldots, x_n) \\
&\quad + \mathbb{E}(\mathrm{Var}(X_{n+1} \mid \theta, x_1, \ldots, x_n) \mid x_1, \ldots, x_n) \\
&= \mathrm{Var}(\theta \mid x_1, \ldots, x_n) + \mathbb{E}(\theta \mid x_1, \ldots, x_n).
\end{aligned}$$

- Hence, the variance of prediction distribution is larger than the the variance of the posterior expectation.

Predictive Probability

```
> ## Ch 6
> #predictive distribution of X_{n+1}
> x1=c(rep(0,7),rep(1,14),rep(2,13),rep(3,8),rep(4,4),rep(5,2),
+      rep(6,2))
> x2=c(rep(0,4),rep(1,13),rep(2,15),rep(3,6),rep(4,2),rep(5,2),
+      rep(6,3),rep(7,1) )
> a =2;b = 1
> n1 = length(x1); s1 = sum(x1)
> n2 = length(x2); s2 = sum(x2)
> x = seq(0,10)
> par(mfrow=c(1, 2))
> plot(x,dnbinom(x,size=a+s1,prob=(b+n1)/(b+n1+1)), xlab="x_{n+1}",
+      ylab="Predictive Prob", type="h", main="City 1")
> plot(x,dnbinom(x,size=a+s2,prob=(b+n2)/(b+n2+1)), xlab="x_{n+1}",
+      ylab="Predictive Prob" ,type="h" , main="City 2" )
```

Predictive Probability



Monte Carlo Method for Traffic Example

- 두 도시의 실제 평균 교통사고 건수 θ_1과 θ_2에 대하여 다음과 같은 posterior distribution을 찾았다.

$$\pi(\theta_1 \mid n_1 = 50, \sum x_{1i} = 102) \sim Gamma(2 + 102, 1 + 50),$$

$$\pi(\theta_2 \mid n_2 = 46, \sum x_{2i} = 104) \sim Gamma(2 + 104, 1 + 46).$$

- Gamma 분포는 이미 알려져 있지만, 관련 statistics는 여전히 찾기 어렵다.

Monte Carlo Method for Traffic Example

- 이 예제에서 주요 목적은 θ_1과 θ_2 차이가 얼마인지 찾는데에 있다. (i.e., $\theta_1 - \theta_2$).
    - Posterior expectation for $\theta_1 - \theta_2$ given data.
    - Posterior variance for $\theta_1 - \theta_2$ given data.
- Monte Carlo Method를 통해 두 parameter들의 차이에 관련된 통계량을 찾을 수 있다

Monte Carlo Method for Traffic Example

- For notational convenience, let $\eta = \theta_1 - \theta_2$.

```
a =2 ; b = 1
n1 = 50; s1 = 102 ; n2 = 46; s2 = 104 ;
nsim = 30000
theta1.sim = rgamma(nsim,a+s1,b+n1)
theta2.sim = rgamma(nsim,a+s2,b+n2)
eta=theta1.sim - theta2.sim
mean(eta)
[1] -0.2155787
var( eta)
[1] 0.08880491
```

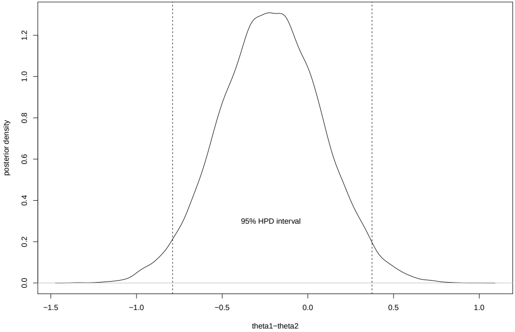
Monte Carlo Method for Traffic Example

Note that the true values are given by

- $\mathrm{E}(\eta \mid x_1, \ldots, x_n) = -0.2161$,
- $\mathrm{Var}(\eta \mid x_1, \ldots, x_n) = 0.0879$.

Monte Carlo Method for Traffic Example

```
HPD=HPDsample(eta)
par(mfrow=c(1,1))
plot(density(eta), type="l", xlab= "theta1-theta2",
ylab="posterior density", main="")
abline( v= HPD, lty=2)
text(mean(eta),0.3, "95% HPD interval" )
```



Monte Carlo Method for Prediction

- In general, the prediction distribution of $X_{n+1} \mid x_1, \ldots, x_n$ has more complicated form than the posterior.
- Monte Carlo Method is helpful to estimate $\mathrm{E}(X_{n+1} \mid x_1, \ldots, x_n)$ and $\mathrm{Var}(X_{n+1} \mid x_1, \ldots, x_n)$.
- Recall that

$$f(X_{n+1} \mid x_1, \ldots, x_n) = \mathrm{E}^{\pi}(f(X_{n+1} \mid \theta) \mid x_1, \ldots, x_n).$$

- Here we do not discuss random sampling from the prediction distribution directly.
- How to calculate $\mathrm{E}(X_{n+1} \mid x_1, \ldots, x_n)$?

First Method: Naive approach

$$
\begin{aligned}
\mathrm{E}(X_{n+1} \mid x_1, \ldots, x_n) &= \iint x_{n+1} f(x_{n+1}, \theta \mid x_{1:n}) d\theta dx_{n+1} \\
&= \iint x_{n+1} f(x_{n+1} \mid \theta, x_{1:n}) \pi(\theta \mid x_{1:n}) d\theta dx_{n+1} \\
&= \iint x_{n+1} f(x_{n+1} \mid \theta) \pi(\theta \mid x_{1:n}) d\theta dx_{n+1}
\end{aligned}
$$

First Method: Naive approach

1. Sample θ_i's from the posterior:

$$\theta_i \overset{iid}{\sim} \pi(\theta \mid x_1, ..., x_n), i = 1, \dots, N.$$

2. Sample $X_{n+1,i}$ from $f(x_{n+1} \mid \theta_i)$:

$$x_{n+1,i} \overset{ind}{\sim} f(x_{n+1} \mid \theta_i), i = 1, \dots, N.$$

3. Calculate

$$\widehat{\mathrm{E}}(X_{n+1} \mid x_1, ..., x_n) = \frac{1}{N} \sum_{i=1}^{N} x_{n+1,i}.$$

Rao-Blackwell Theorem

- Let $\widehat{\theta}$ be an estimator of θ with $\mathrm{E}(\widehat{\theta}^2) < \infty$. Suppose T is a sufficient statistic for θ and $\theta^* = \mathrm{E}(\widehat{\theta} \mid T)$. Then

$$\mathrm{E}(\theta^* - \theta)^2 \quad \leq \quad \mathrm{E}(\widehat{\theta} - \theta)^2.$$

- (Key idea)

$$\mathrm{Var}\big[\mathrm{E}(X \mid Y)\big] \quad \leq \quad \mathrm{Var}(X)$$

Second Method: Rao-Blackwellization

$$
\begin{aligned}
\mathrm{E}(X_{n+1} \mid x_1, \ldots, x_n) &= \iint x_{n+1} f(x_{n+1}, \theta \mid x_{1:n}) d\theta dx_{n+1} \\
&= \int x_{n+1} \int f(x_{n+1} \mid \theta) \pi(\theta \mid x_{1:n}) d\theta dx_{n+1} \\
&= \int x_{n+1} \mathrm{E}^{\pi}\Big[f(x_{n+1} \mid \theta) \mid x_{1:n} \Big] dx_{n+1}
\end{aligned}
$$

Second Method: Rao-Blackwellization

1. Sample θ_i's from the posterior:

$$\theta_i \overset{iid}{\sim} \pi(\theta \mid x_1, ..., x_n), i = 1, \ldots, N.$$

2. Estimate the conditional density of X_{n+1} given $x_1, \ldots, x_n$:

$$\hat{f}(x_{n+1} \mid x_1, ..., x_n) = \frac{1}{N} \sum_{i=1}^{N} f(x_{n+1} \mid \theta_i).$$

- Estimation

i. $\widehat{\mathrm{E}}(X_{n+1} \mid x_1, ..., x_n) = \sum_{\text{all } x_{n+1}} x_{n+1} \hat{f}(x_{n+1} \mid x_1, ..., x_n)$

ii. $\widehat{\mathrm{Var}}(X_{n+1} \mid x_1, ..., x_n) = \sum_{\text{all } x_{n+1}} x_{n+1}^2 \hat{f}(x_{n+1} \mid x_1, ..., x_n) - \left(\sum_{\text{all } x_{n+1}} x_{n+1} \hat{f}(x_{n+1} \mid x_1, ..., x_n)\right)^2.$

Second Method: Rao-Blackwellization

- In the Monte Carlo context, replacing a naive estimator with its conditional expectation is called Rao-Blackwellization.
- In general, Rao-Blackwell approach gives more accurate results.

Example: Rao-Blackwellization

- Consider computing $P(X > Y)$, where (X, Y) follows a standard bivariate normal with correlation ρ.
- Naive: simulate

$$(X_i, Y_i) \overset{iid}{\sim} N_2\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

  and let

$$P\widehat{(X > Y)} = \frac{1}{N}\sum_{i=1}^{N} I(X_i > Y_i).$$

- Note that $X \mid Y = y \sim N(\rho y, 1 - \rho^2)$. Let $h(y) = P(X > y \mid Y = y) = 1 - \Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}y\right)$.
- Rao-blackwell: simulate $Y_i \sim N(0, 1)$ and let

$$P\widehat{(X > Y)}^* = \frac{1}{N}\sum_{i=1}^{N} h(Y_i).$$

Example: Rao-Blackwellization

```
library(mvtnorm)
set.seed(12)
N = 10000
rho = 0.7
X = matrix(0, nrow=N, ncol=2)
Cov = matrix(c(1,rho, rho,1), 2,2)

# 1. Naive
X = rmvnorm(N, mean=c(0,0), sigma=Cov)
naive = mean(X[,1] > X[,2])
# 2. R-B
Y = rnorm(N)
RB = mean(1 - pnorm(sqrt((1-rho)/(1+rho))*Y))

naive; RB
[1] 0.5035  0.4997608
```