Numerical linear algebra I

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Vectors

Let $x, y \in R^n$: $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$

Vector operations (with geometrical interpretation)

Scalar multiplication: αx

Addition: $x + y$

Q) How can you deal with “r times c matrix” as a vector ?

The (standard) inner product is defined as

$$< x, y >= x^T y$$

The Euclidean norm of x is defined as

$$||x||_2 = (x^T x)^{1/2}$$

Some properties of Euclidean norm

1. $||\alpha x||_2 = |\alpha| ||x||_2$ for $\alpha \in R^1$
2. $||x + y||_2 \leq ||x||_2 + ||y||_2$
3. $||x||_2 \geq 0$ and $||x||_2 = 0$ only if $x = 0$.

The Cauchy-Schwartz inequality is

$$|x^T y| \leq ||x||_2 ||y||_2$$

Angle between x and y is defined by

$$\theta = \cos^{-1}\left(\frac{x^T y}{||x||_2 ||y||_2}\right)$$

where $0 \leq \theta \leq \pi$.

x and y are orthogonal if

$$x^T y = 0$$

Ex: $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

The projection of a vector x on a vector y is

$$\frac{x^T y}{y^T y} y$$

This definition can be used for constructing perpendicular vectors.

Ex) Construct perpendicular vectors from

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Matrices

Let $R^{m\times n}$ denote the set of $m \times n$ matrices and $A \in R^{m\times n}$.

An element of A is denoted a_{ij} (sometimes, A_{ij}).

Define the followings:

- the transpose of A
- square matrix
- symmetric matrix
- diagonal matrix
- lower triangular matrix
- upper triangular matrix
- orthogonal matrix : norm preserving transformation
- identity matrix
- positive definite matrix

Ex:
$$A = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix},$$
$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Scalar multiplication and addition are well defined on $R^{m\times n}$.

Matrix-vector product: Ax where $A \in R^{m\times n}$ and $x \in R^n$

Matrix-matrix product: AB where $A \in R^{m\times n}$ and $B \in R^{n\times p}$

1) Note that in general $AB \neq BA$.

2) $(AB)^T = B^T A^T$

For $X, Y \in R^{m\times n}$,

$$< X, Y >= tr(X^T Y)$$

Frobenius norm of a matrix $X \in R^{m\times n}$ is $X, Y \in R^{m\times n}$,

$$||X||_F = (tr(X^T X))^{1/2}$$

Positive-definite matrix (PD)

Let A be a symmetric matrix.

A is positive definite if

$$x^T A x > 0$$

Quadratic form

for all $x \neq 0$.

Remark) A is positive definite if and only if every eigenvalue of A is positive.

Remark) The characteristic of PD matrix can be understood easily in terms of its spectral decomposition.

(Symmetric) eigenvalue decomposition

Suppose A is a $n \times n$ symmetric matrix. Then A can be factored as

$$A = Q\Lambda Q^T$$

where $Q \in R^{n\times n}$ is orthogonal, and $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$.

Here, the eigenvalues are ordered decreasingly, i.e. $\lambda_1 \geq \cdots \geq \lambda_n$.

Q) Represent the followings in terms of eigenvalues:

- $tr(A)$
- $det(A)$
- $||A||_F$

"Squareroot of matrix" from the eigenvalue decomposition.

If A is positive definite with eigenvalue decomposition $Q\Lambda Q^T$, the squareroot of A is defined as

$$A^{1/2} = Q\Lambda^{1/2}Q^T.$$

Note that $A^{1/2}A^{1/2} = A$.

Q) What is the inverse matrix of $A^{1/2}$?

Some R commands

- A←matrix(c(1,1.5,1.5,4),2,2)
- eigen(A)

Key references

1) Matrix Cookbook (2012), freely downloadable from Google.

2) Searle, S.R. (1982), Matrix algebra useful for statistics, Wiley series in probability and mathematical statistics.

3) 김병천 편저 (2000), 통계학을 위한 행렬대수학 (개정판), 자유 아카데미

4) (Advanced) Horn, R.A. and Johnson, C.R. (1990), Matrix Analysis, Cambridge University Press.