Monte-Carlo method III
(Other variance reduction techniques)

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Control variates

Suppose that $\theta(=\int h(x)f(x)dx = E(h(X)))$ is the parameter of interest. The key idea is

$$\theta = E(h(X) - c(Y)) + E(c(Y))$$

where $E(c(Y))$ is a known quantity. Then,

$$\hat{\theta}_c = \left(\frac{1}{n}\sum_{i=1}^{n}(h(X_i) - c(Y_i))\right) + E(c(Y)).$$

Q) When the variance of $\hat{\theta}_c$ is reduced ?

In practice, we consider a coefficient α in the previous equation, that is

$$\hat{\theta}_c = \frac{1}{n}\sum_{i=1}^{n}(h(X_i) - \alpha(c(Y_i) - E(c(Y_1))).$$

Then, the questions is:

- Find the optimal α^* minimizing the variance of $\widehat{\theta}_c$.

If there are several control variates, consider a linear combination of them.

$$\hat{\theta}_c = \frac{1}{n}\sum_{i=1}^{n}(h(X_i) - \sum_j \beta_j(c_j(Y_i) - E(c_j(Y_1)))).$$

Note that β_j are unknown quantities.

Q) Which β_j should we use ?

Q) How can we get such β_j ?

Example: Ripley (1987)

$$\theta = \int_0^2 \frac{1}{\pi(1+x^2)} dx$$

Q) Suggest a Monte-Carlo estimator for θ.

Q)For this problem, Ripley(1987) proposed U^2 and U^4 as control variates where $U \sim U(0,2)$. Construct a Monte-Carlo estimator for θ using these control variates.

R-program example

```
MC1<-function(n){
        obs<-runif(n,0,2)
        theta<-mean(2/(pi*(1+obs^2)))
        return(theta)
}
```

```
MC.control<-function(n){
        obs<-runif(n,0,2)
        obs2<-obs^2
        obs4<-obs^4

        response<-2/(pi*(1+obs^2))

        RES.lm<-lm(response~obs2+obs4)
        b0<-coefficients(RES.lm)[1]
        b1<-coefficients(RES.lm)[2]
        b2<-coefficients(RES.lm)[3]

        theta<-mean(response-b1*obs2-b2*obs4)+b1*4/3+b2*16/5
        return(theta)
}
```

Antithetic variates

Suppose that X_i^* and X_i have the same distribution. Consider the following Monte-Carlo estimator:

$$\hat{\theta}_a = \frac{1}{n} \sum_{i}^{n/2} (h(X_i) + h(X_i^*))$$

Q)$E(\hat{\theta}_a)$? $Var(\hat{\theta}_a)$?

Q) When can we achieve larger reduction of $Var(\hat{\theta}_a)$?

Q) How can we obtain such X_i^* ?

Example: Ripley (1987)

$$\theta = \int_0^2 \frac{1}{\pi(1+x^2)} dx$$

Q) Suggest a Monte-Carlo estimator for θ using antithetic variates.

R-program example

```
MC.anti<-function(n){
        n.half<-floor(n/2)
        obs<-runif(n.half,0,2)
        theta<-sum(2/(pi*(1+obs^2))+2/(pi*(1+(2-obs)^2)))/n
        return(theta)
}
```