CSE 152 Section 3
Review of Filters and Frequencies

October 19, 2018

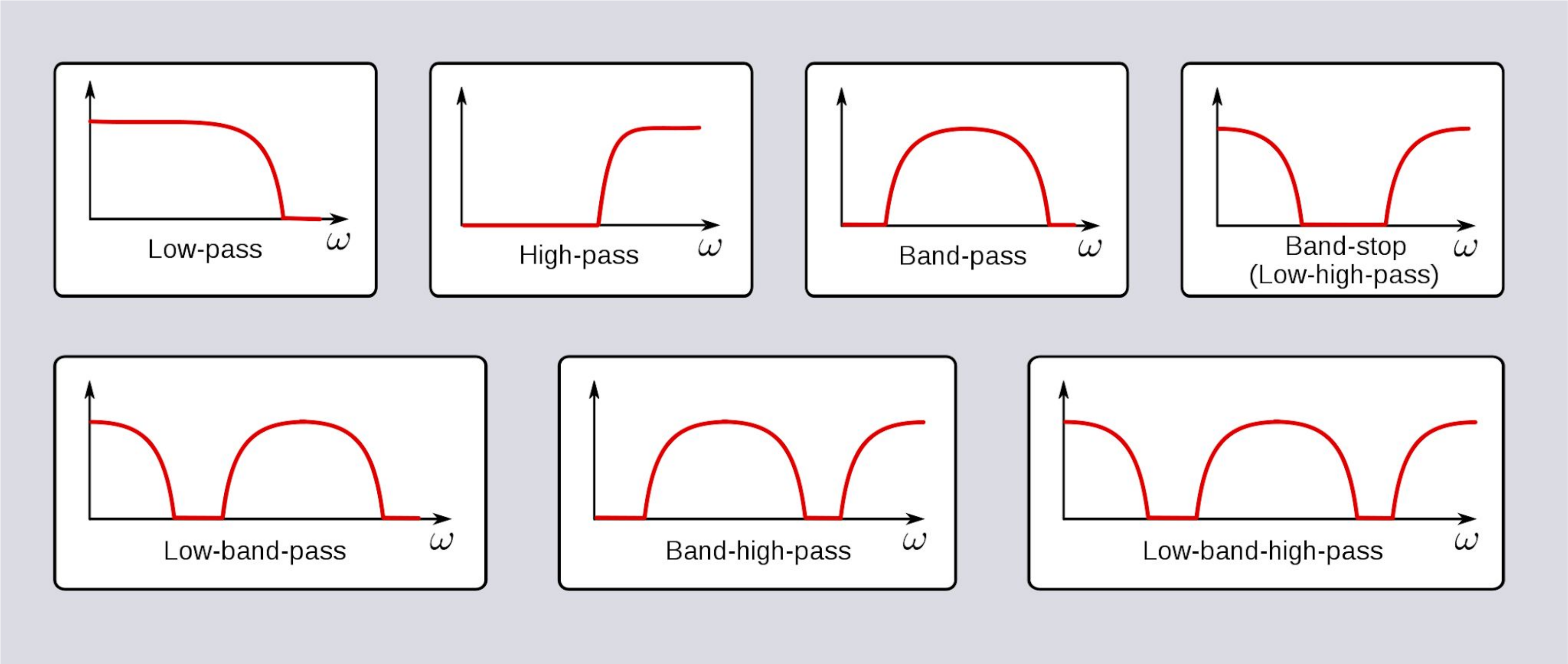
Owen Jow

What is a filter?

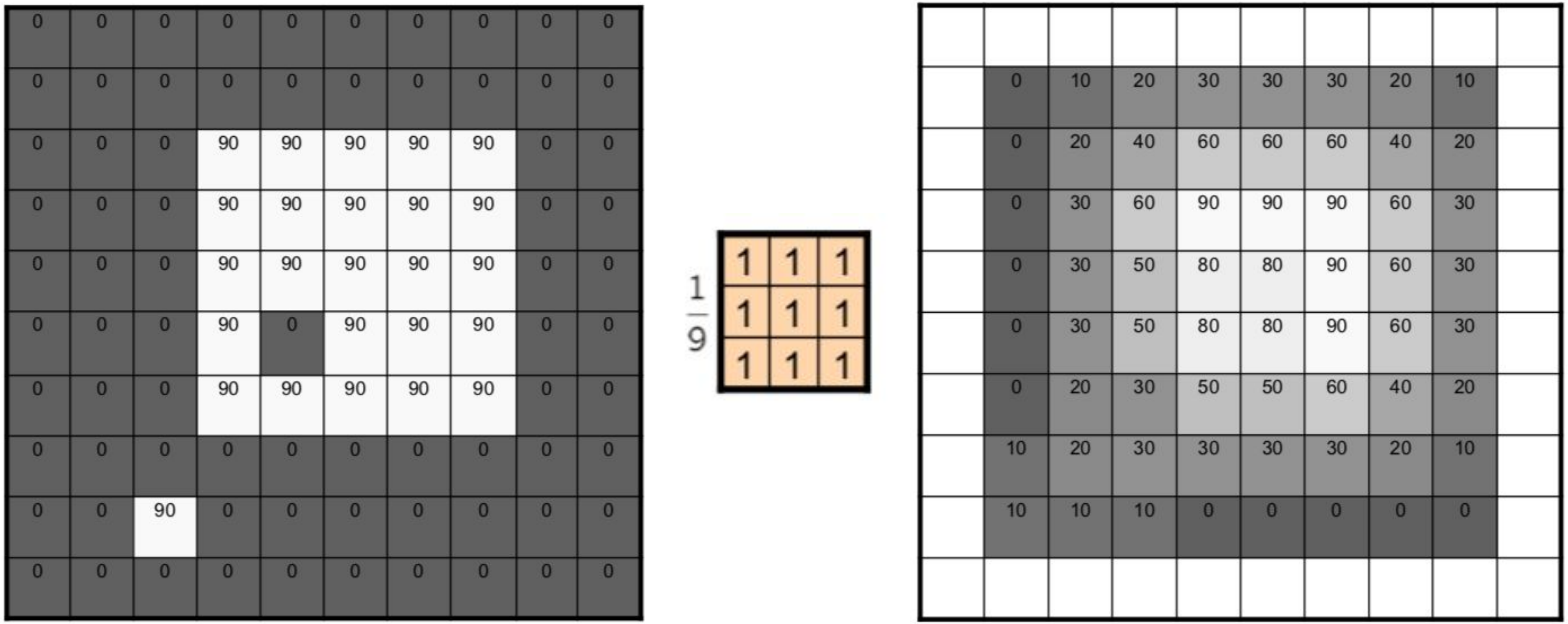
What comes to mind?

A device that lets only some of its inputs through

A process that lets only some of its inputs through



A process that lets only some [scalings] of its inputs through



source: Steven Seitz

and thereby transforms content

source: TapSmart

What does a linear filter do to an image?

Hint: why is it called a *linear* filter?

Linear Filter (An Image Processing View)

Replace each pixel with a **linear combination** of values in its neighborhood.

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Correlation

$$\underbrace{h(x,y)}_{\text{output}} = \sum_{k,l} \underbrace{f(k,l)}_{\text{filter}} \underbrace{I(x+k, y+l)}_{\text{image}}$$

- not commutative or associative
- (preferably) use for measuring similarity

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Convolution

$$\underbrace{h(x,y)}_{\text{output}} = \sum_{k,l} \underbrace{f(k,l)}_{\text{filter}} \underbrace{I(x-k,y-l)}_{\text{image}}$$

- commutative and associative
- **(preferably) use for filtering**

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Replace each pixel with a **linear combination** of values in its neighborhood.

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- **“flip filter horizontally and vertically”**

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Convolution

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- "flip filter horizontally and vertically"
- **denoted h = f * I**

Linear Filter (An Image Processing View)

Correlation and **convolution** are both valid approaches to linear filtering.

Correlation

$$\underbrace{h(x,y)}_{\text{output}} = \sum_{k,l} \underbrace{f(k,l)}_{\text{filter}} \underbrace{I(x+k,y+l)}_{\text{image}}$$

- not commutative or associative
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- (preferably) use for filtering
- “flip filter horizontally and vertically”
- **denoted h = f * I**

Why do we care about associativity?

Why do we care about associativity?

Associativity means that $\mathbf{f * (g * I) = (f * g) * I}$*.*
If we want to apply multiple filters, we can pre-convolve them and use (/reuse) them as a single filter!

Properties of Linear Filters

- **They are linear.**

  $f * (\alpha I + J) = \alpha(f * I) + f * J$

  meaning they obey the superposition principle, shown here

Properties of Linear Filters

- **They are linear.**

  $f * (\alpha I + J) = \alpha(f * I) + f * J$

- **They are shift-invariant.**

  $f * \text{shifted}(I) = \text{shifted}(f * I)$

  “we can shift the image to the left by one pixel, then filter – or we can filter, then shift the result to the left by one pixel”

Properties of Linear Filters

- **They are linear.**

  `f * (αI + J) = α(f * I) + f * J`

- **They are shift-invariant.**

  `f * shifted(I) = shifted(f * I)`

  "we apply the same computation to all of the neighborhoods"

Filter Break

sharpening via
unsharp filtering

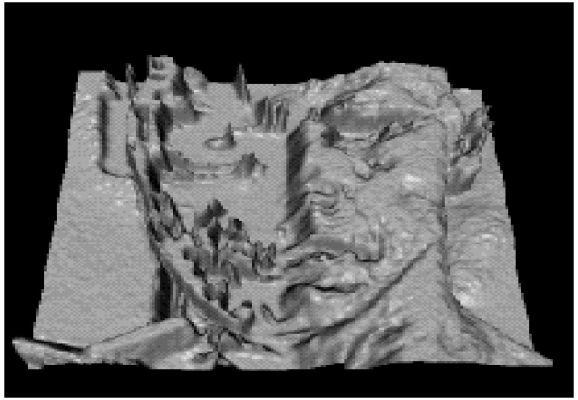
Filter Break

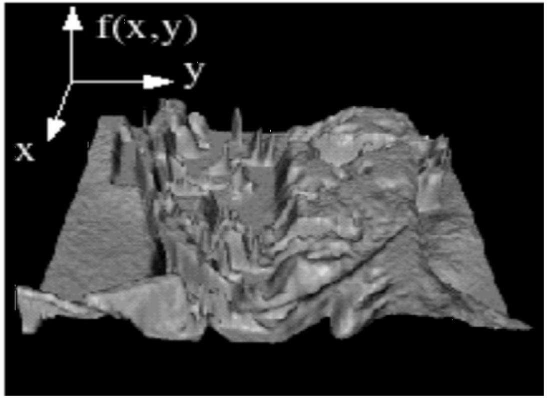
denoising via
median filtering

(nonlinear)

An image is a function f(x, y)

It is a mapping from pixel locations $\in \mathbb{R}^2$ to intensities $\in \mathbb{R}$.

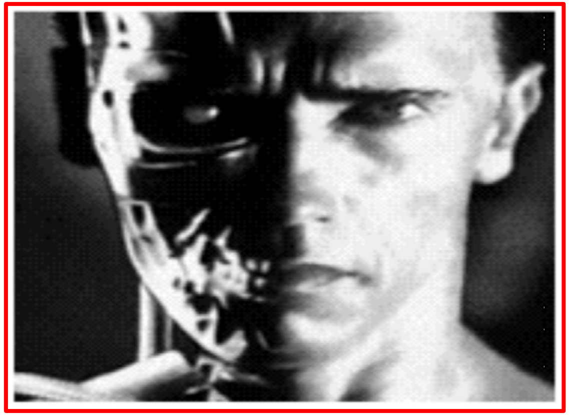


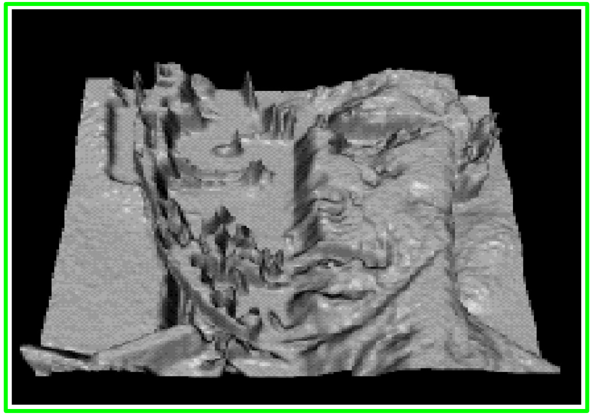


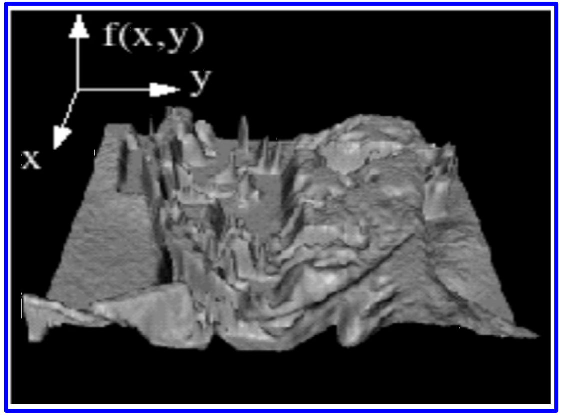
source: Seitz, Szeliski

An image is a function f(x, y)

A color image is a mapping from pixel locations $\in \mathbb{R}^2$ to RGB intensities $\in \mathbb{R}^3$.



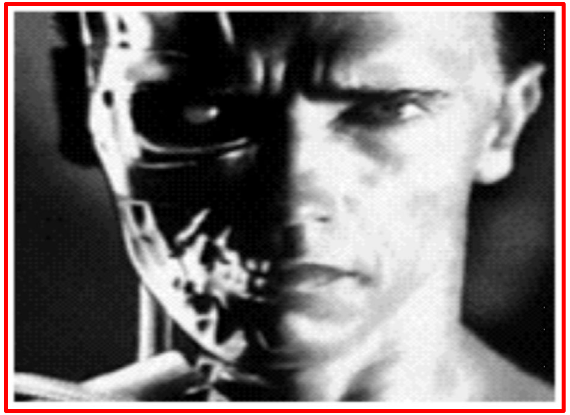


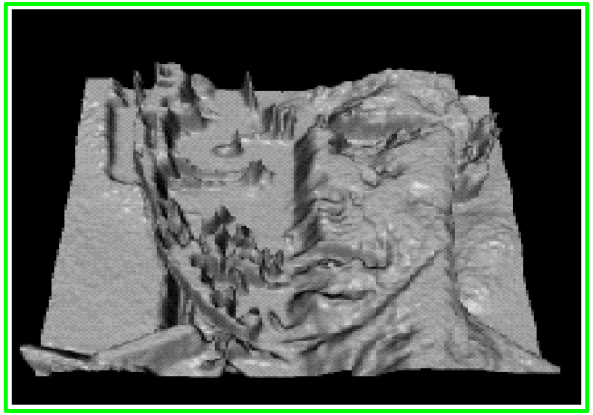


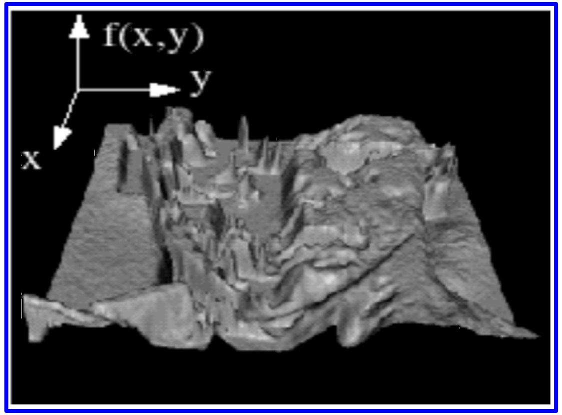
source: Seitz, Szeliski

An image is a function f(x, y)

In the case of digital images, we discretely sample an underlying continuous function.



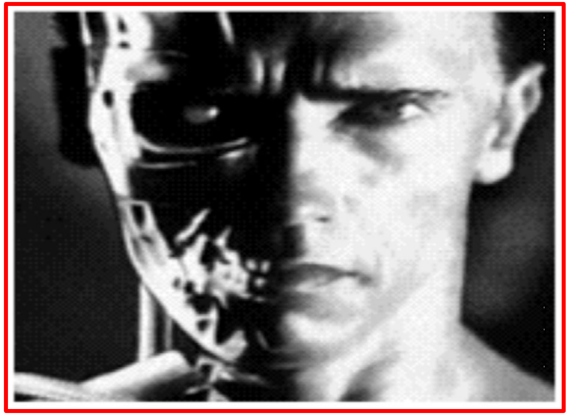


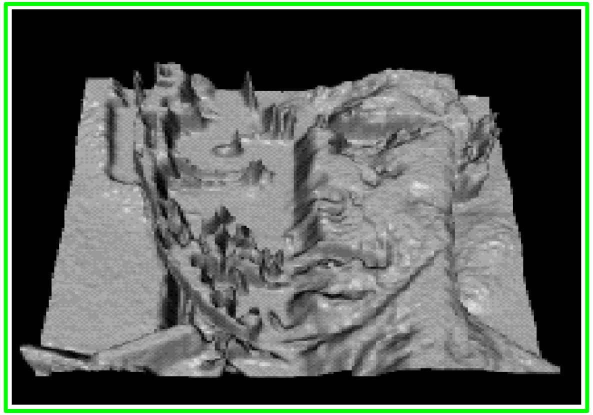


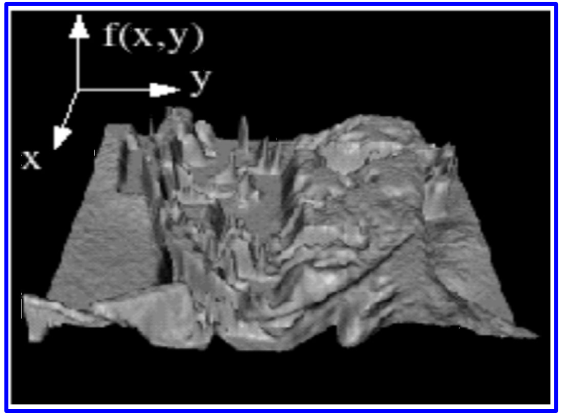
source: Seitz, Szeliski

An image is a signal f(x, y)

In the case of digital images, we discretely sample an underlying continuous function.







source: Seitz, Szeliski

So what we're really doing...

...is **signal processing**

image source: Daniel Sierra

A digital image is a discrete 2D signal (function) (vector).

Traditionally, we think of them as they exist in the spatial domain.

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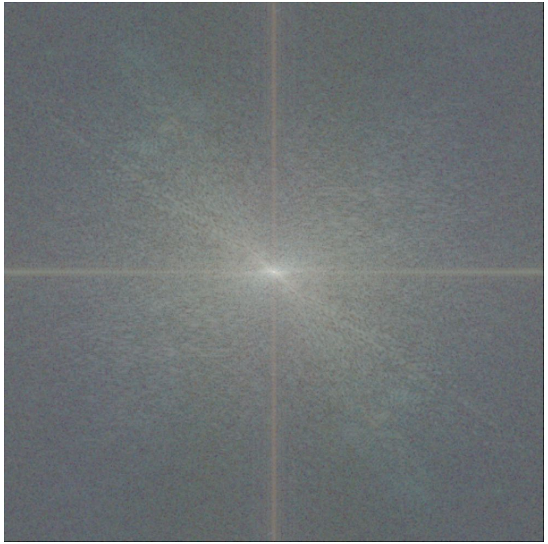
Traditionally, we think of them as they exist in the spatial domain.

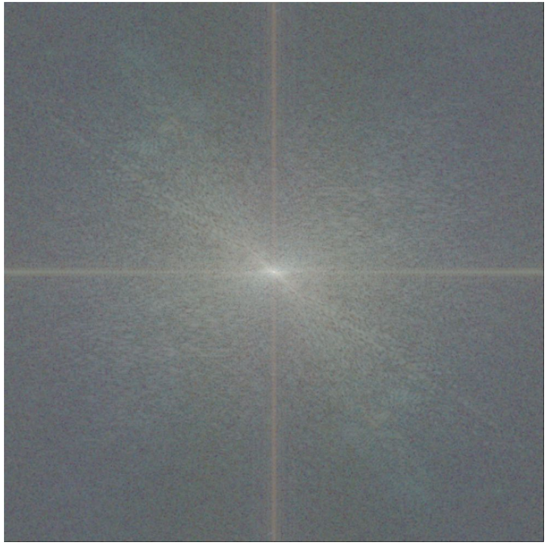
But signal processing gives us a new way to think about things...



A digital image is a discrete 2D signal (function) (vector).

Spatial Domain

Frequency Domain



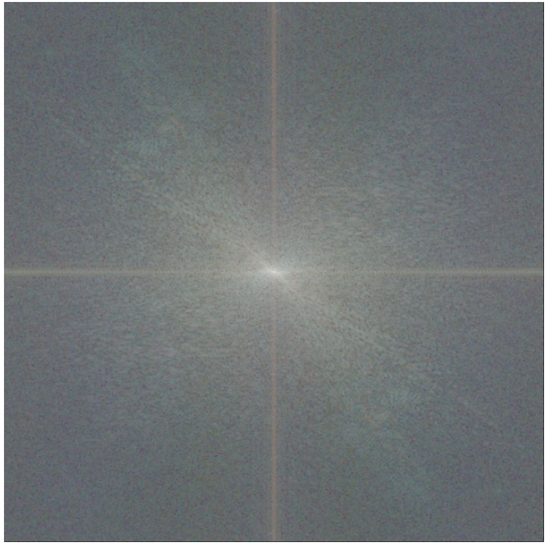
A digital image is a discrete 2D signal (function) (vector).

Spatial Domain

Frequency Domain

Fourier Transform

Inverse Fourier Transform



A digital image is a discrete 2D signal (function) (vector).

Spatial Domain

$$f(x, y)$$

x: distance (px) in horizontal direction
y: distance (px) in vertical direction

f(x, y): intensity at (x, y)

$$f(x, y) \underset{\text{Inverse Fourier Transform}}{\overset{\text{Fourier Transform}}{\rightleftarrows}} F(u, v)$$

Frequency Domain

$$F(u, v)$$

u: frequency (cycles/px) in horizontal direction
v: frequency (cycles/px) in vertical direction

F(u, v): magnitude of frequency (u, v)

1D case (scan line)

Spatial Domain

Frequency Domain

$$f(x) \underset{\text{Inverse Fourier Transform}}{\overset{\text{Fourier Transform}}{\rightleftarrows}} F(u)$$

x: distance (px) in horizontal direction
f(x): intensity at pixel x on scan line

u: frequency (cycles/px)
F(u): magnitude of frequency u

1D case (time-varying signal)

Spatial Domain | **Frequency Domain**

$$f(t) \underset{\text{Inverse Fourier Transform}}{\overset{\text{Fourier Transform}}{\rightleftarrows}} F(\omega)$$

(1D Discrete) Fourier Transform

A discrete Fourier transform (DFT) turns a function into a weighted sum of sines and cosines.

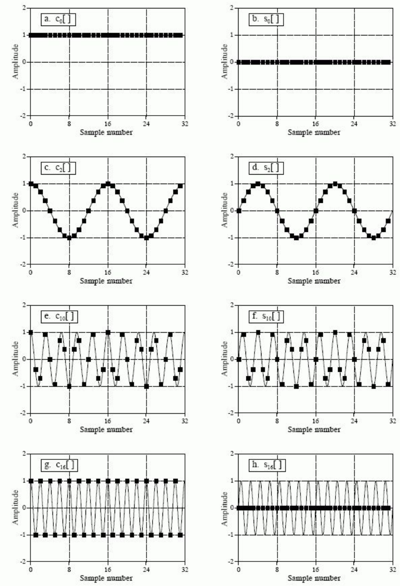


FIGURE 8-5
DFT basis functions. A 32 point DFT has 17 discrete cosine waves and 17 discrete sine waves for its basis functions. Eight of these are shown in this figure. These are discrete signals; the continuous lines are shown in these graphs only to help the reader's eye follow the waveforms.

(1D Discrete) Fourier Transform

A Fourier transform is a **change of basis** into a basis of sine and cosine functions.

$$\cos(2\pi kt/N)$$

$$\sin(2\pi kt/N)$$

If the signal contains **N** samples, the basis will contain **N** sine/cosine functions with different frequencies.

$$\begin{aligned}
F(k) &= \sum_{t=0}^{N-1} f(t) e^{(-2\pi kt/N)i} \\
&= \sum_{t=0}^{N-1} f(t) \left[\cos(2\pi kt/N) - i\sin(2\pi kt/N)\right] \\
&= \sum_{t=0}^{N-1} f(t) \cos(2\pi kt/N) - i \sum_{t=0}^{N-1} f(t) \sin(2\pi kt/N)
\end{aligned}$$

$$\boxed{0 \leq k \leq N-1}$$

(1D Discrete) Fourier Transform

F(k) is a complex number from which we can obtain the magnitude (amplitude) of frequency **k** in the Fourier decomposition.

We can think of the output of our Fourier transform as a **magnitude** for each **frequency**.

$$\cos(2\pi kt/N)$$

$$\sin(2\pi kt/N)$$

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$$\boxed{0 \leq k \leq N-1}$$

Incidentally

$$A\sin(2\pi kt + \varphi)$$

A: amplitude, magnitude, strength, “how much”
k, 2πk: frequency, cycles per pixel or second
φ: phase, shift, “where” the sinusoid is

Incidentally

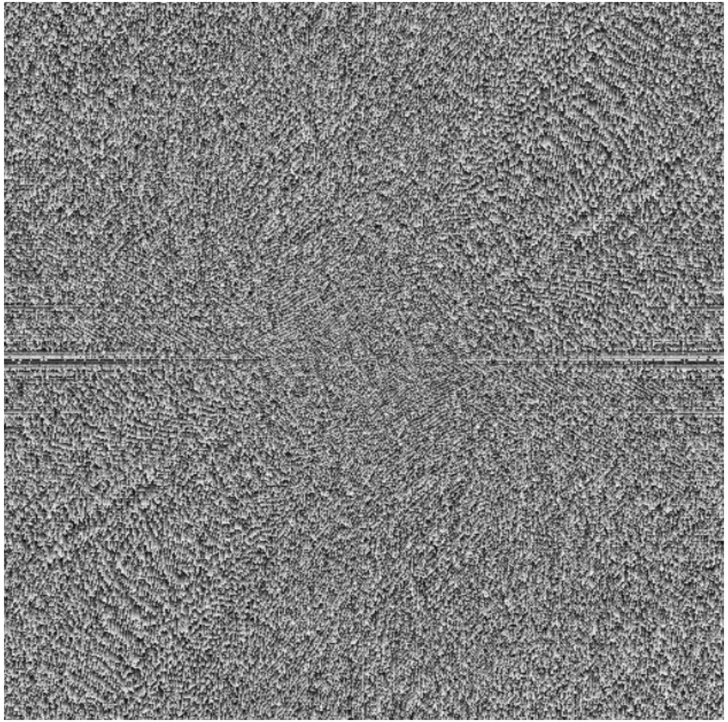
$$A\sin(\omega t) + B\cos(\omega t) = C\sin(\omega t + \varphi)$$

adding a sine and cosine of the same frequency
gives a phase-shifted sine of that frequency

total amplitude is sqrt(A^2 + B^2)
phase shift is arctan(A / B)

(1D Discrete) Fourier Transform

We can also get phase information out of a Fourier transform. But we won't talk about phase much because it isn't very helpful for interpretability.



In Summary: The 1D Fourier Transform

converts a signal `f(t)` into the frequencies that compose it.

$$F(\omega)$$

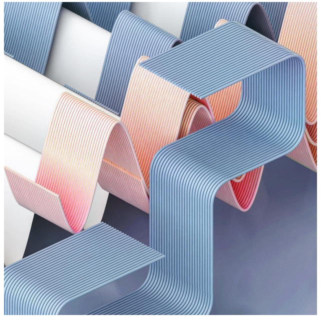
"*what is the strength of the frequency-ω sinusoid in the decomposition of* `f(t)`?"

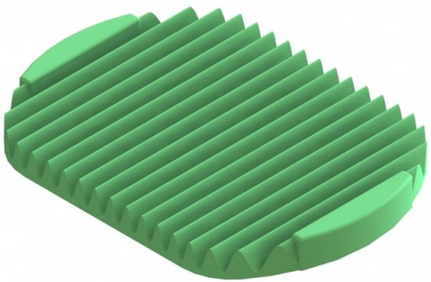
2D DFT

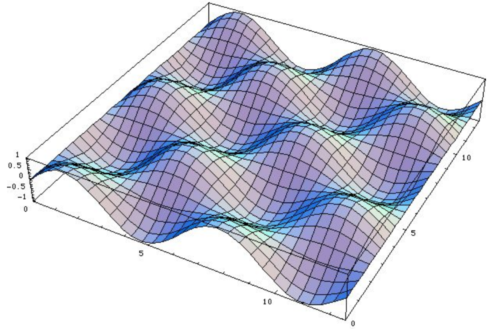
- The 2D DFT is analogous to the 1D DFT; just add another dimension to the input.

$$F(u, v)$$

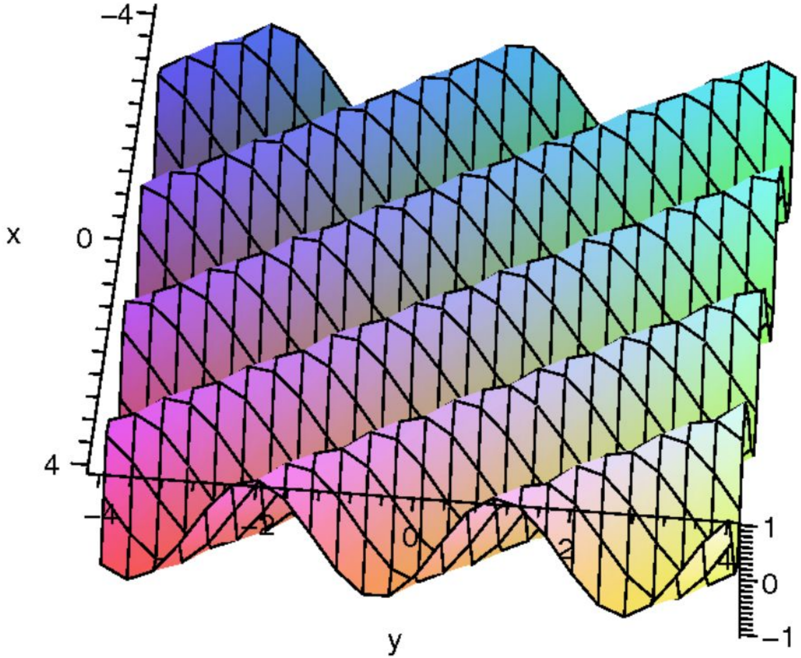
- The main difference is that the sines/cosines can now be *oriented* in 2D.
The orientation depends on the values of u and v passed in. If you draw out the vector (u, v), its direction is the sinusoid's orientation and its length *corresponds* to the sinusoid's frequency.

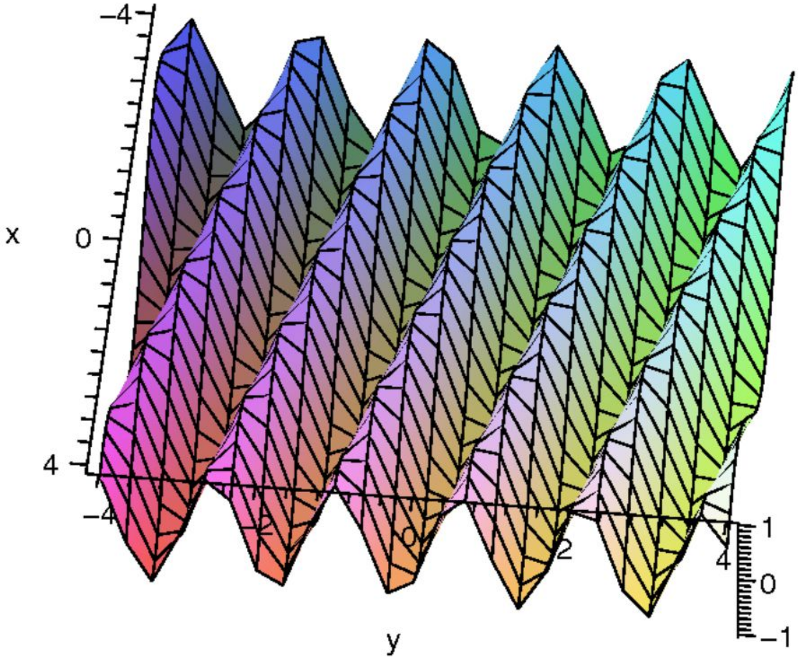






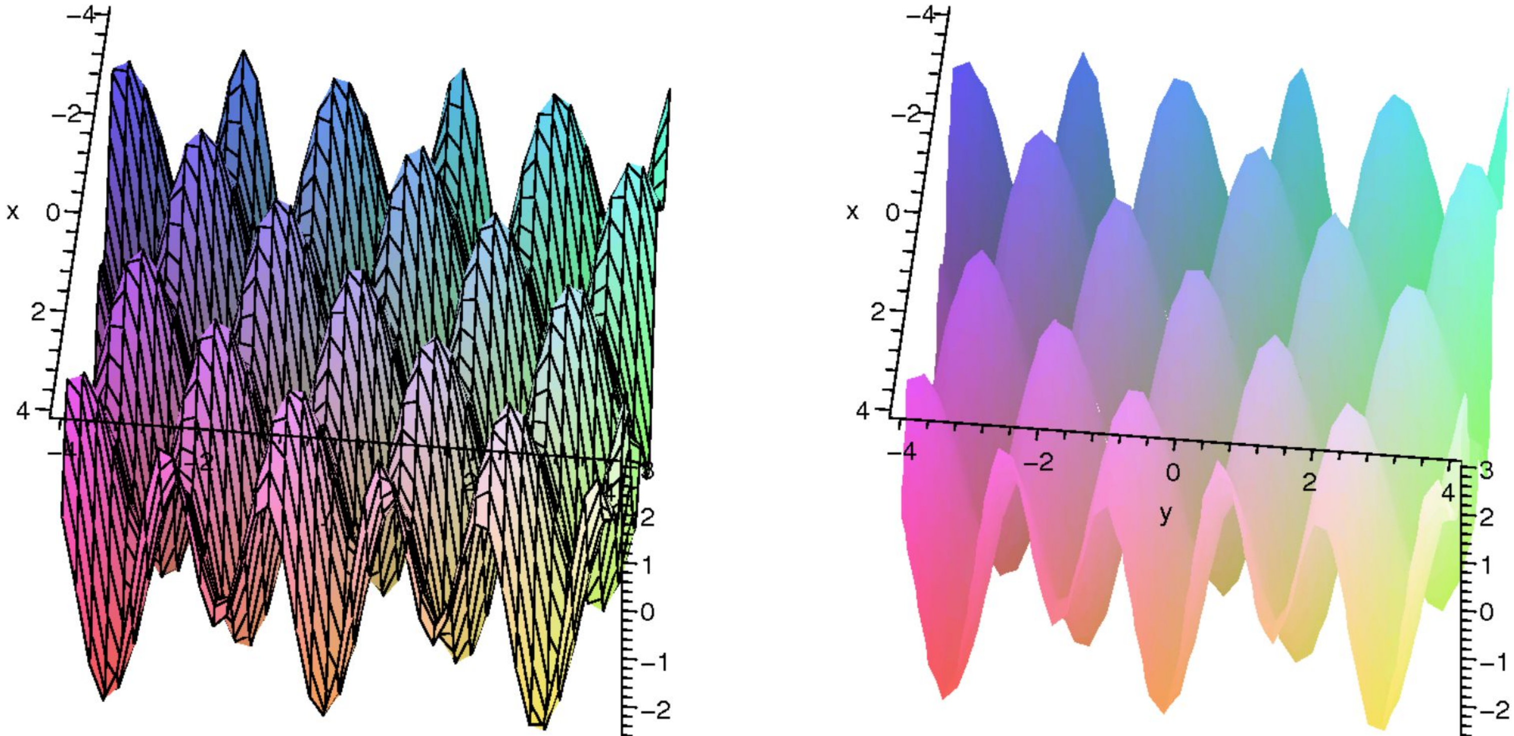
2D Basis Functions





source: Václav Hlaváč

Sum of 2D Basis Functions



source: Václav Hlaváč

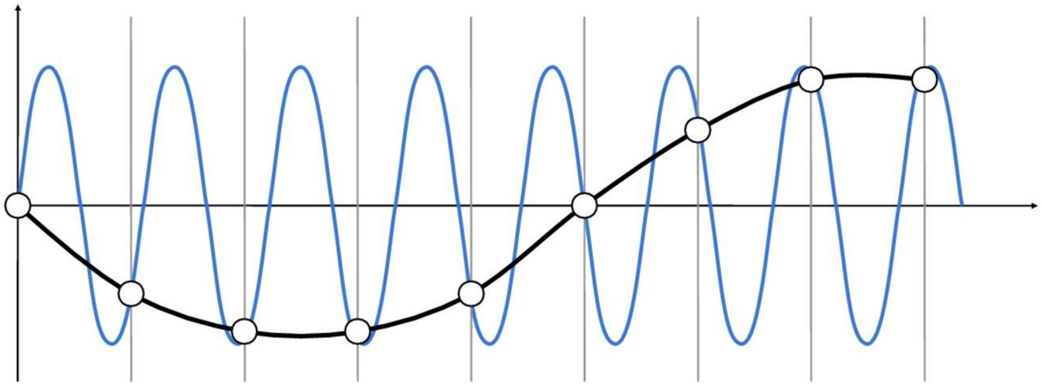
Cycles per pixel?

Cycles per pixel?

We are thinking about **spatial frequency**. The basis sinusoids appear as oriented, repeating stripes. The number of pixels it takes to move along a sinusoid from some intensity back to the same intensity is 1 / (the frequency).

Nyquist frequency

To avoid aliasing, the maximum frequency (*bandwidth*) we can have in a signal is ½ of the number of samples.



Aliasing

source: Ren Ng

Nyquist frequency

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In other words, the maximum frequency we can have in an image is 0.5 cycles per pixel. What does this mean?

Nyquist frequency

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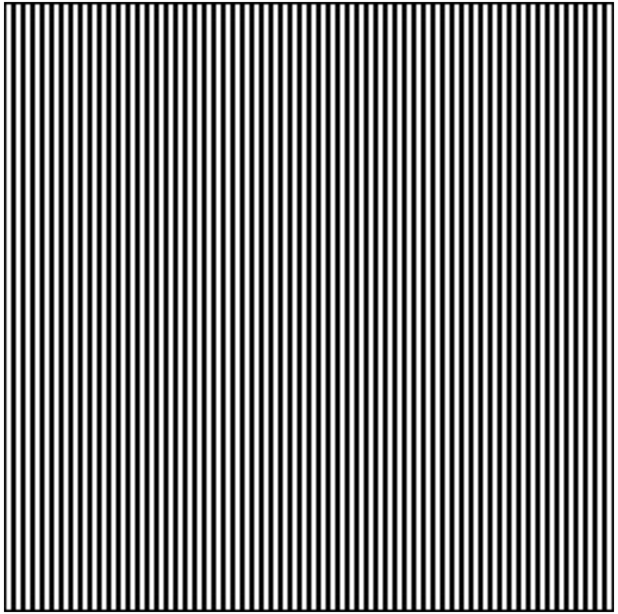
In other words, the maximum frequency we can have in an image is 0.5 cycles per pixel. What does this mean?

at max 1 stripe per pixel extent
0.5 cycles per pixel → intensity alternates between low and high every pixel

Half of the Nyquist frequency



- stripe width 2px
- period 4px
- frequency 0.25 cycles/px

High vs. low frequencies

High frequency means a signal is changing quickly over its domain.

- In the previous visualization, the pixel values were changing very quickly from left to right, and the frequency was almost at its maximum (the Nyquist frequency).

In images,

- **high frequencies correspond to...**

?

High vs. low frequencies

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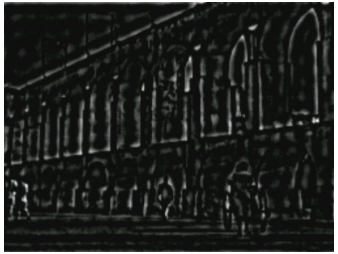
In images,

- high frequencies correspond to **rapid/sharp changes in intensity (edges)**



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In images,

- high frequencies correspond to **rapid/sharp changes in intensity (edges)**
- **low frequencies correspond to...**

?

High vs. low frequencies

High frequency means a signal is changing quickly over its domain.

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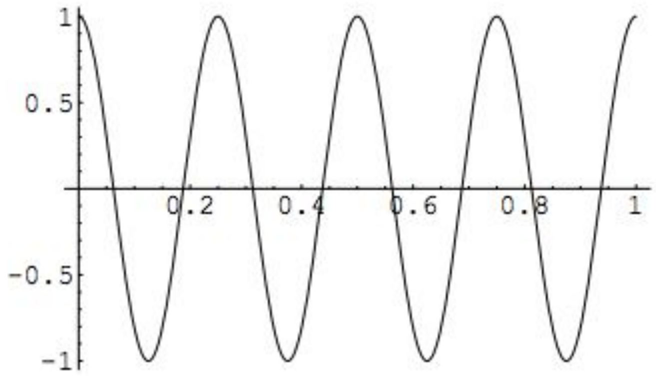
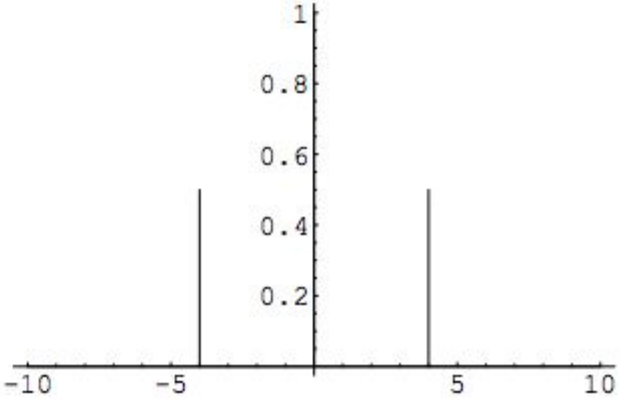
In images,

- high frequencies correspond to rapid/sharp changes in intensity (edges)
- low frequencies correspond to **smooth/slow changes in intensity (blurred/smoothed regions)**

Frequency Domain Images (1D) `understanding the frequency domain`

For a 1D signal, we visualize the frequency domain as a 2D plot
of **frequency** on the horizontal axis and **magnitude** on the vertical axis.

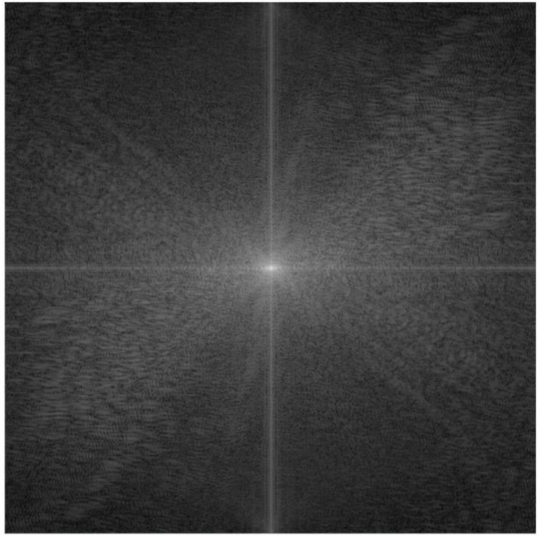




source: Bryan Morse

Frequency Domain Images (2D) `understanding the frequency domain`

For a 2D signal, we can think of the frequency domain as a 3D plot with **oriented frequency** on the xy-plane and **magnitude** on the z-axis.

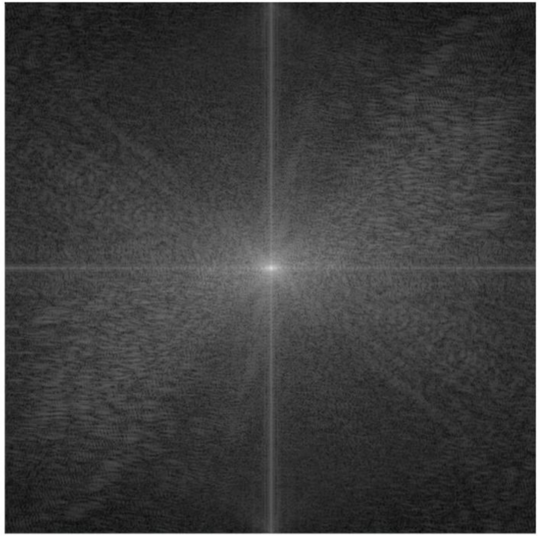


unlike a standard image,
the origin of the plot is in the center

source: Ren Ng

Frequency Domain Images (2D) `understanding the frequency domain`

However, we pretty much always view magnitude as brightness
(i.e. as part of an image), instead of plotting it on a z-axis in 3D.

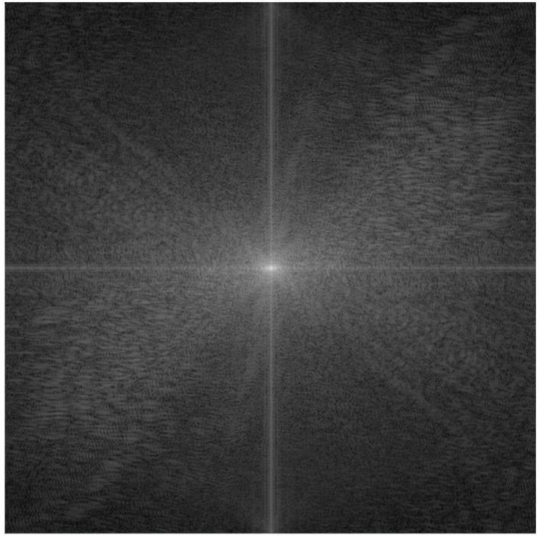


higher magnitude
means **higher brightness**
means **closer to white**

source: Ren Ng

Frequency Domain Images (2D) `understanding the frequency domain`

The spatial domain and frequency domain images are the same size, i.e. the number of frequencies is equivalent to the number of pixels.

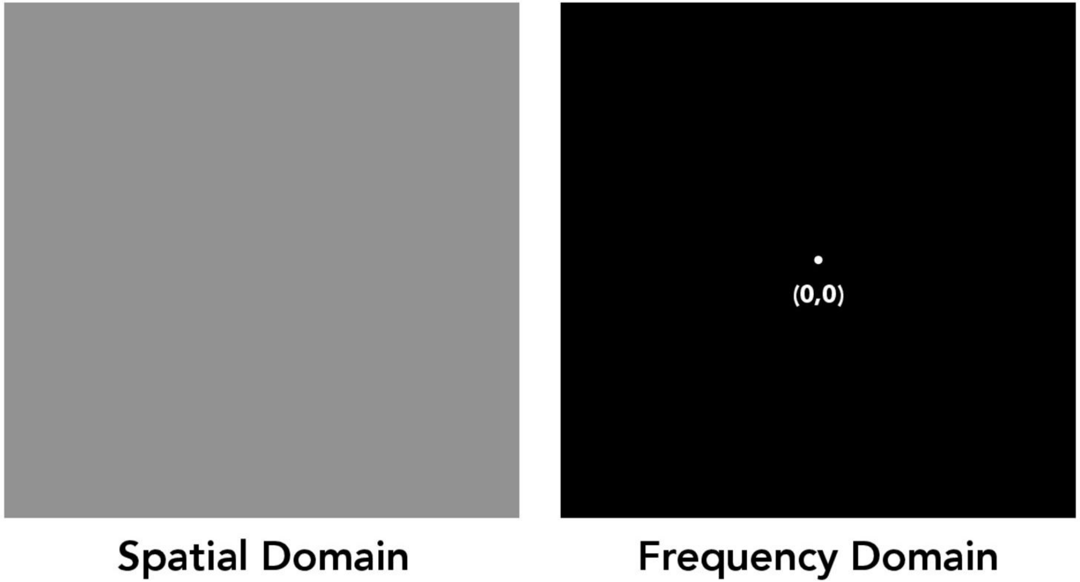


note that frequencies always range from -0.5 to 0.5, so what changes is the step between successive frequencies

source: Ren Ng

Frequency Domain Images (2D) `understanding the frequency domain`

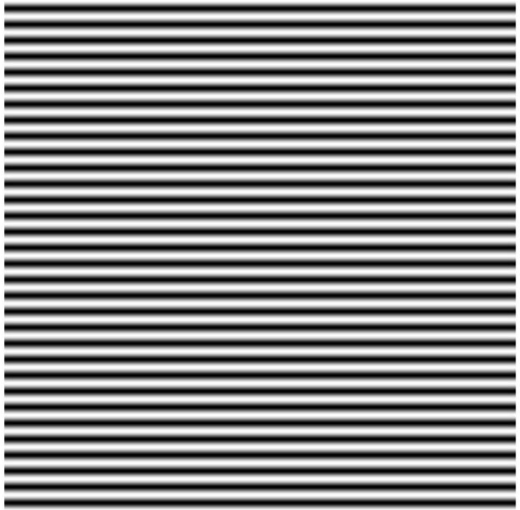
zero frequency (constant; average value in image)



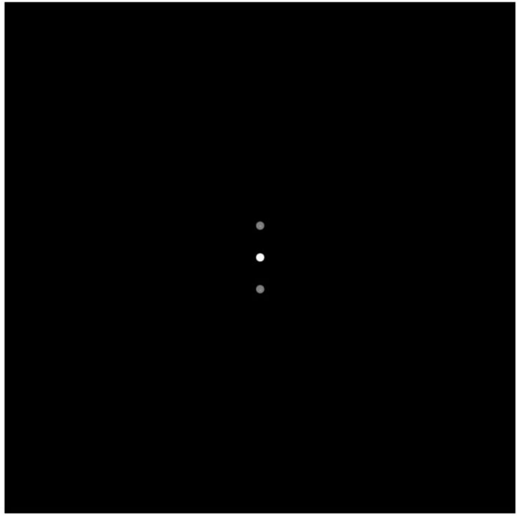
source: Ren Ng

Frequency Domain Images (2D) `understanding the frequency domain`

$$\sin(2\pi/16)y$$



Spatial Domain



Frequency Domain

source: Ren Ng

Interpreting Frequency Visualizations

Let’s say we’re interested in the point **(u, v)**.

- Draw an arrow from the origin to **(u, v)**. This is a vector.

There are three pieces of information we can obtain.

1. The direction of vector **(u, v)** gives the **direction of the sinusoid**.

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1. The direction of vector **(u, v)** gives the **direction of the sinusoid**.
2. The length of vector **(u, v)** gives (note: is not directly equal to) the **frequency of the sinusoid**.

Interpreting Frequency Visualizations

Let’s say we’re interested in the point **(u, v)**.

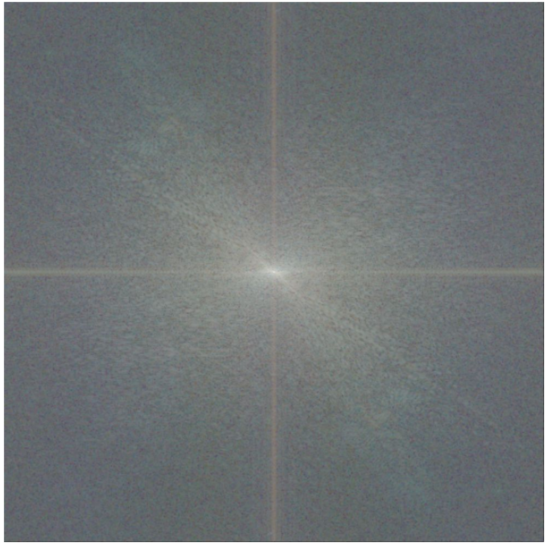
- Draw an arrow from the origin to **(u, v)**. This is a vector.

There are three pieces of information we can obtain.

1. The direction of vector **(u, v)** gives the **direction of the sinusoid**.
2. The length of vector **(u, v)** gives (note: is not directly equal to) the **frequency of the sinusoid**.
3. The brightness at point **(u, v)** gives the **magnitude of the sinusoid** (contrast from low to high).

The + Artifact

The DFT does its computation for an image that is tiled infinitely, meaning we (usually) end up with high-frequency edges where the top/bottom and left/right of the tiled images meet.



Convolution Theorem

Convolution in the spatial domain is equivalent to point-by-point multiplication in the frequency domain.

and what do we use convolution for?

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FILTERING

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and what do we use convolution for?

FILTERING

So we have a choice.

- we can filter by sliding window in the spatial domain `(convolve)`

  - or -

- we can filter by multiplication in the frequency domain `(fftconvolve)`
  (multiply frequency version of the image by the frequency version of the convolution filter)

Why would we want to filter in the frequency domain?

Efficiency.

Note: we don't get the element-wise multiplication completely for free.
We also have to perform Fourier transforms ***to*** *and* ***from*** *the frequency domain.*

For large arrays, it's faster to go to the frequency domain and filter there.
For smaller arrays, it's faster to stay in the spatial domain and do a gridded convolution.

Why would we want to filter in the frequency domain?

Interpretability.

We can look at convolution as an operation over a grid of numbers,
or as modifying the frequencies of an image. It is often intuitive to think in terms of frequencies.

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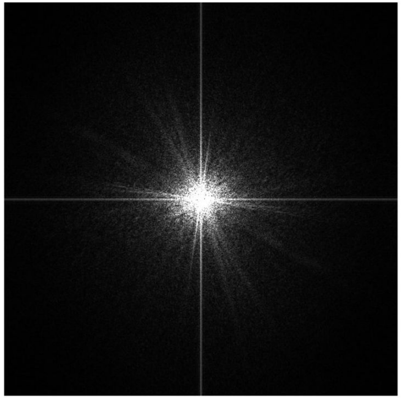

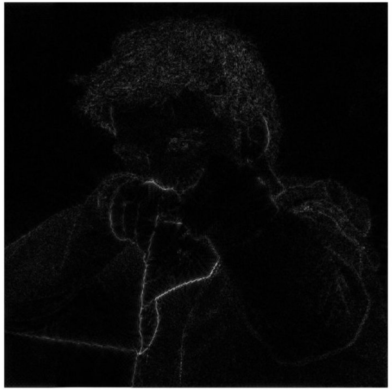

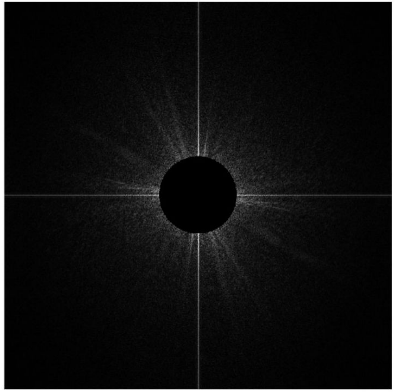

```
high-pass filtering:
pass only the high frequencies
```

source: Ren Ng

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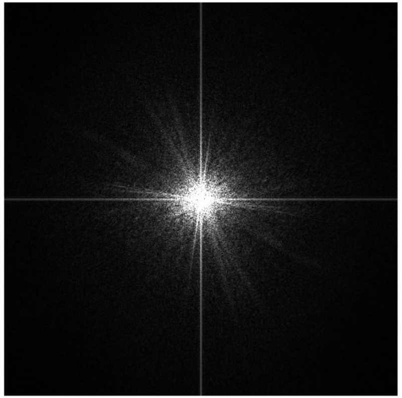

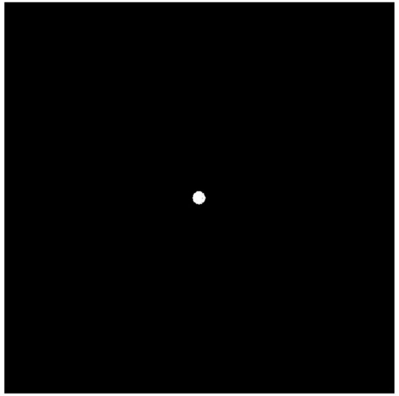

`easy to see smoothness in frequency domain!`

source: Ren Ng

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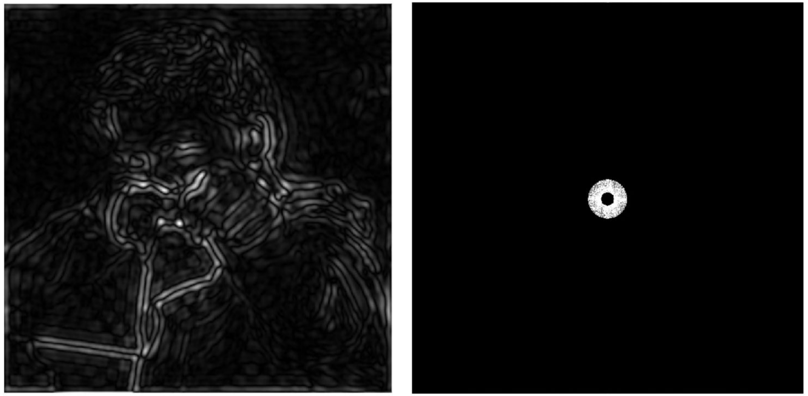

`band-pass filter: filters out both high and low frequencies, looks like a band`

source: Ren Ng

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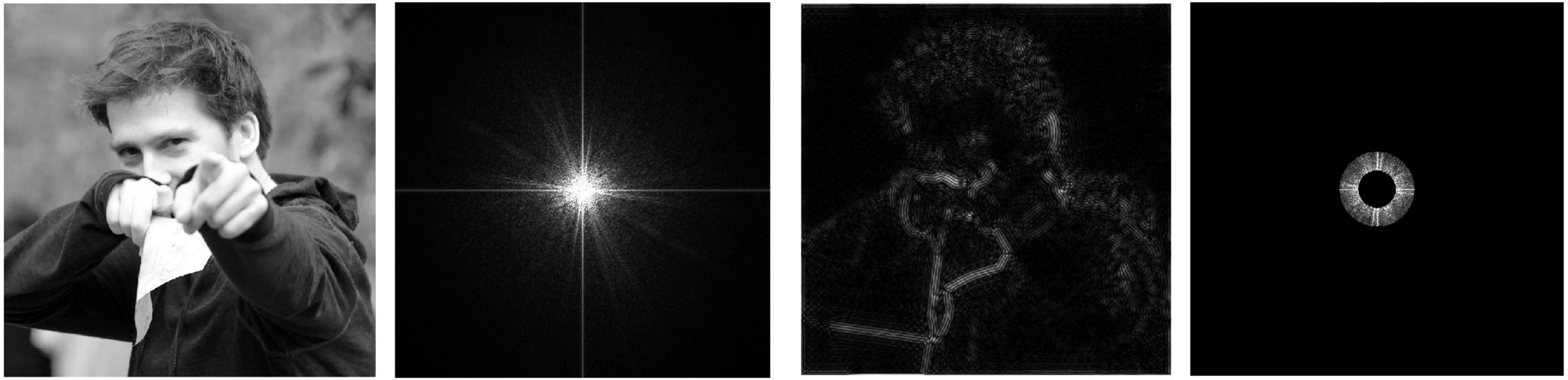

source: Ren Ng

(Extra) Basic Convolution Practice

Convolve the signal **[1, 2, 3, 4, 5]** with the filter **[1, 2, 3]**.

(No need to pad.)

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[10

1x3 + 2x2 + 3x1

(Extra) Basic Convolution Practice

Convolve the signal **[1, 2, 3, 4, 5]** with the filter **[1, 2, 3]**.

(No need to pad.)

[10, 16

2x3 + 3x2 + 4x1

(Extra) Basic Convolution Practice

Convolve the signal **[1, 2, 3, 4, 5]** with the filter **[1, 2, 3]**.

(No need to pad.)

[10, 16, 22]

3x3 + 4x2 + 5x1

The Edge Case

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

source: Steven Seitz

Padding Options

cv2.**filter2D**(src, ddepth, kernel[, dst[, anchor[, delta[, **borderType**]]]])

How to extrapolate pixels over the border?

```
/*
 Various border types, image boundaries are denoted with '|'

 * BORDER_REPLICATE:     aaaaaa|abcdefgh|hhhhhhh
 * BORDER_REFLECT:       fedcba|abcdefgh|hgfedcb
 * BORDER_REFLECT_101:   gfedcb|abcdefgh|gfedcba
 * BORDER_WRAP:          cdefgh|abcdefgh|abcdefg
 * BORDER_CONSTANT:      iiiiii|abcdefgh|iiiiiii  with some specified 'i'
 */
```

Padding Options

scipy.signal.**fftconvolve**(in1, in2, **mode='full'**)

Do we even want to extrapolate?

mode : *str {'full', 'valid', 'same'}, optional*

A string indicating the size of the output:

`full`

The output is the full discrete linear convolution of the inputs. (Default)

`valid`

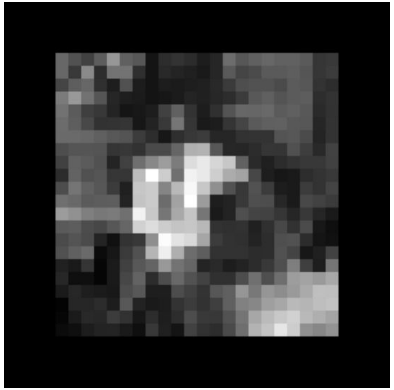
The output consists only of those elements that do not rely on the zero-padding.

`same`

The output is the same size as *in1*, centered with respect to the 'full' output.

Padding Options

- **Constant**
  Pretend that everything outside the image is some specified constant (commonly zero).



source: Richard Szeliski

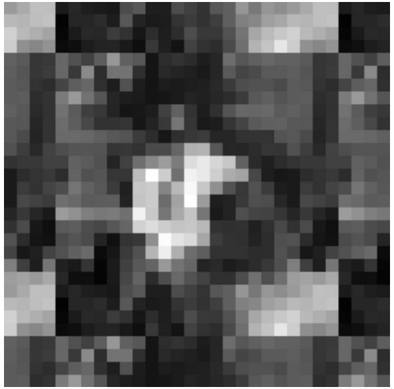
Padding Options

- **Constant**
- **Replicate**
  Pretend that everything outside the image is whatever's already on the edge.

source: Richard Szeliski

Padding Options

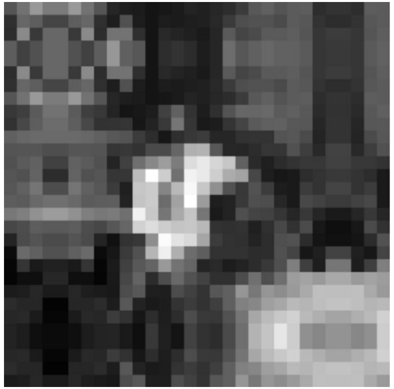
- **Constant**
- **Replicate**
- **Wrap**
  Pretend that the image is tiled indefinitely, i.e. “loop over the image.”



source: Richard Szeliski

Padding Options

- **Constant**
- **Replicate**
- **Wrap**
- **Mirror**
  Reflect the image across its edges.



source: Richard Szeliski

Padding Options

- Constant
- Replicate
- Wrap
- Mirror
- ...

Additional Readings

Sobel filter

- https://stackoverflow.com/questions/17078131/why-sobel-operator-looks-that-way

Fourier domain images

- http://cns-alumni.bu.edu/~slehar/fourier/fourier.html
- https://www.cs.toronto.edu/~guerzhoy/320/lec/FreqDomain.pdf

Circular paths

- https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/