Lecture Note 9, 19 November 2014

Transportation Problems

Example 1. Powerco Formulation

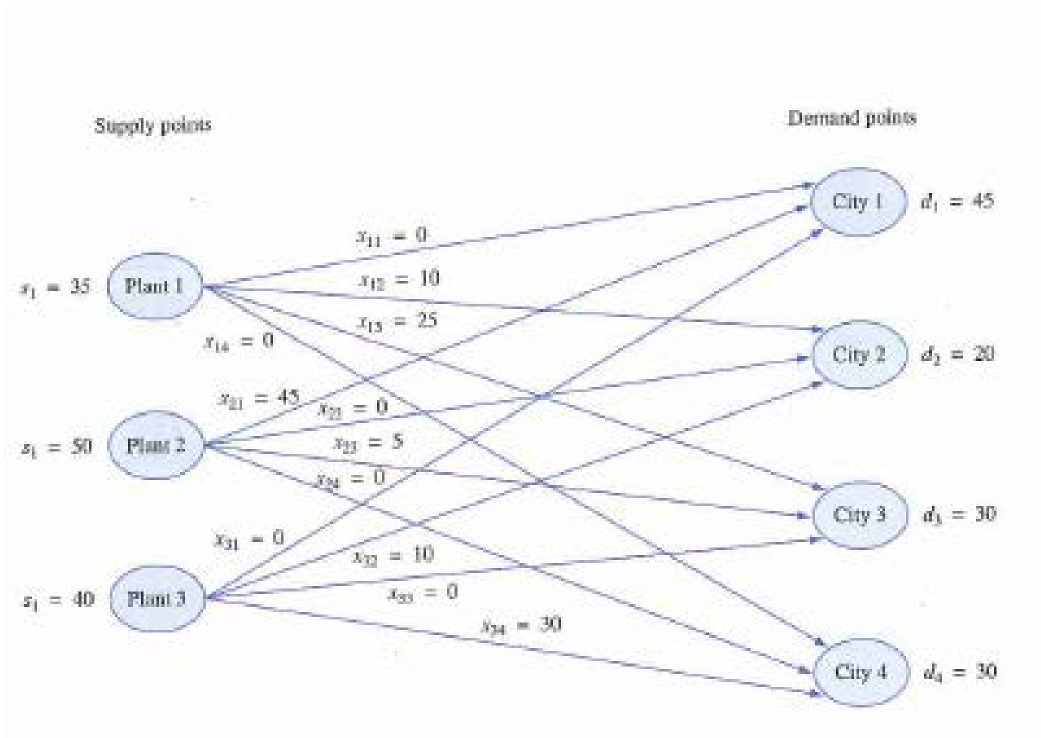
Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity: plant 1 – 35 million; plant 2 – 50 million; plant 3 – 40 million. The peak power demands in these cities, which occur at the same time (2 p.m.), are as follows (in kwh): city 1 – 45 million; city 2 – 20 million; city 3 – 30 million; city 4 – 30 million. The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel. Formulate an LP to mimize the cost of meeting each city's peak power demand.

Table 1. Shipping Costs, Supply, and Demand for Powerco

From\To	City 1	City 2	City 3	City 4	Supply (million kwh)
Plant 1	$8	$6	$10	$9	35
Plant 2	$9	$12	$13	$7	50
Plant 3	$14	$9	$16	$5	40
Demand (million kwh)	45	20	30	30	

$$
\begin{aligned}
\min z = {} & 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} \\
& + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} \le 35 \qquad \text{(Supply constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} \le 50 \\
& x_{31} + x_{32} + x_{33} + x_{34} \le 40
\end{aligned}
$$

$$
\begin{aligned}
x_{11} + x_{21} + x_{31} &\geq 45 \quad \text{(Demand constraints)}\\
x_{12} + x_{22} + x_{32} &\geq 20\\
x_{13} + x_{23} + x_{33} &\geq 30\\
x_{14} + x_{24} + x_{34} &\geq 30\\
x_{ij} \geq 0 \quad (i = 1, 2, 3;\ j = 1, 2, 3, 4&
\end{aligned}
$$



General Description of a Transportation Problem

In general, a transportation problem is specified by the following information:

1. A set of m ***supply points*** from which a good is shipped. Supply point i can supply at most s_i units. In the Powerco example, $m = 3$, $s_1 = 35, s_2 = 50, s_3 = 40$.

2. A set of n ***demand points*** to which the good is shipped. Demand point j must receive at least d_j units of the shipped good. In the Powerco example, $n=4$, $d_1=45, d_2=20, d_3=30, d_4=30$.
3. Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}. In the Powerco example, $c_{12}=6$.

Let

$$x_{ij} = \text{number of units shipped from supply point } i \text{ to demand point } j$$

then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} \le s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)}$$

$$\sum_{i=1}^{i=m} x_{ij} \ge d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)}$$

$$x_{ij} \ge 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$$

If $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$, then total supply equals total demand, and the problem is said to be a **balanced transportation problem**. In a balanced transportation, all the constraints must be binding, i.e.

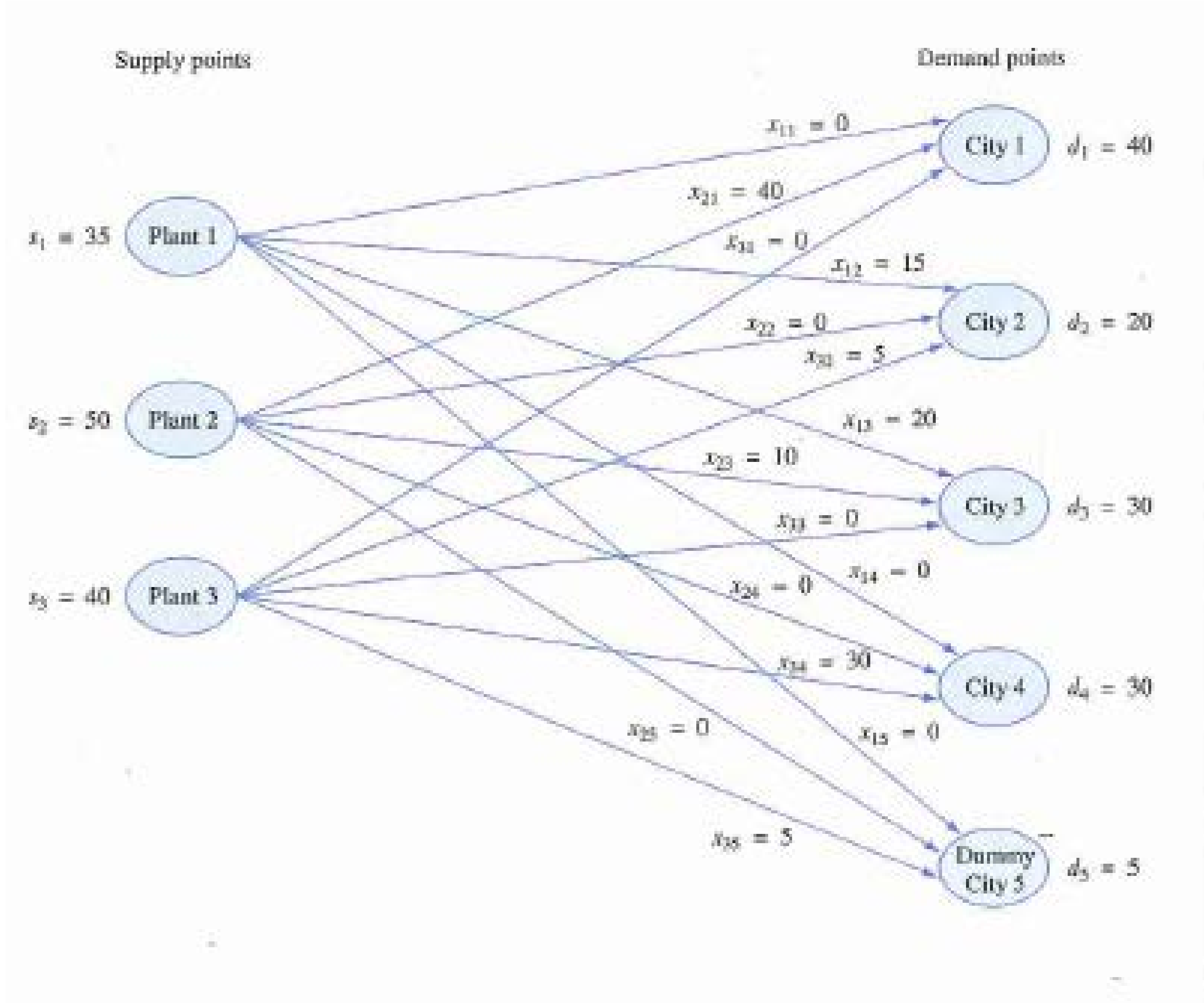
$$
\begin{aligned}
&\min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} c_{ij} x_{ij} \\
&\text{s.t.} \quad \sum_{j=1}^{j=n} x_{ij} = s_i \quad (i = 1, 2, \ldots, m) \qquad \text{(Supply constraints)} \\
&\qquad\quad \sum_{i=1}^{i=m} x_{ij} = d_j \quad (j = 1, 2, \ldots, n) \qquad \text{(Demand constraints)} \\
&x_{ij} \geq 0 \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)
\end{aligned}
$$

We will see that is relatively simple to find a basic feasible solution for a balanced transportation. Also, simplex pivots for these problems do not involve multiplication and reduce to additions and subtractions. For these reasons, it is desirable to formulate a transportation problem as a balanced transportation problem.

Balanced a Transportation Problem

i. If Total Supply Exceeds Total Demand

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Because shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. Shipments to the dummy demand point indicate unused supply capacity. To understand the use of a dummy demand point, suppose that in the Powerco problem, the demand for city 1 were reduced to 40 million kwh. To balance the Powerco problem, we would add a dummy demand point with a demand of 5 million kwh. The shipping cost per million kwh from each plant to the dummy is 0.



A transportation problem is specified by the supply, the demand, and the shipping costs, so the relevant data can be summarized in a **transportation tableau**. The square, or **cell**, in row i and column j of a transportation tableau corresponds to the variable x_{ij}. If x_{ij} is a basic variable, it value appears in the cell, otherwise it does not.

					Supply
	$\boxed{c_{11}}$	$\boxed{c_{12}}$	$\cdots$	$\boxed{c_{1n}}$	s_1
	$\boxed{c_{21}}$	$\boxed{c_{22}}$	$\cdots$	$\boxed{c_{2n}}$	s_2
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$\boxed{c_{m1}}$	$\boxed{c_{m2}}$	$\cdots$	$\boxed{c_{mn}}$	s_m
Demand	d_1	d_2	$\cdots$	d_n	

	City 1	City 2	City 3	City 4	Supply
Plant 1	$\boxed{8}$	10 $\boxed{6}$	25 $\boxed{10}$	$\boxed{9}$	35
Plant 2	45 $\boxed{9}$	$\boxed{12}$	5 $\boxed{13}$	$\boxed{7}$	50
Plant 3	$\boxed{14}$	10 $\boxed{9}$	$\boxed{16}$	30 $\boxed{5}$	40
Demand	45	20	30	30	

The tableau format implicitly express the supply and demand constraints through the fact that the sum of the variables in row i must equal s_i and the sum of the variables in column j must equal d_j.

ii. If Total Supply Is Less Than Total Demand

If a transportation problem has a total supply that is ***strictly*** less than total demand, then the problem has no feasible solution. For example, if plant 1 had only 30 million kwh of capacity, then a total of only 120 million kwh would be available. This amount of power would be insufficient to meet the total demand of 125 million kwh, and the Powerco problem would no longer have a feasible solution.

When total supply is less than total demand, it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, a penalty is often associated with unmet demand. Next example illustrates how such a situation can yield a balanced transportation problem.

Example 2. Handling Shortages

Two reservoirs are available to supply the water needs of three cities. Each reservoir can supply up to 50 million gallons of water per day. Each city would like to receive 40 million gallons per day. For each million gallons per day of unmet demand, there is a penalty. At city 1, the penalty is $20; at city 2, the penalty is $22; and at city 3, the penalty is $23. The cost of transporting 1 million gallons of water from each reservoir to each city is shown in Table 2. Formulate a balanced transportation problem that can be used to minimize the sum of shortage and transport costs.

Table 2. Shipping Costs for Reservoir

From\To	City 1	City 2	City 3
Reservoir 1	$7	$8	$10
Reservoir 2	$9	$7	$8

Let

$x_{ij} =$ number of units of water supplied from reservoir i to city j

	City 1	City 2	City 3	Supply
Reservoir 1	7 20	8 30	10	50
Reservoir 2	9	7 10	8 40	50
Dummy (shortage)	20 20	22	23	20
Demand	40	40	40	

Solving a Transportation Problem

Finding Basic Feasible Solutions (bfs) for Transportation Problems

Consider a balanced transportation problem with m supply points and n demand points. Such a problem contains $m+n$ equality constraints. It is difficulty to find a bfs if all of an LP's constraints are equalities. However, due to the special structure of a balanced transportation problem makes it easy for us to find a bfs.

There are three methods commonly used to find a bfs to a balanced transportation problem; before describing these methods, we make the following important observation. *If a set of values for the x_{ij}'s satisfies all but one of the constraints of a balanced transportation problem, then the values for the x_{ij}'s will automatically satisfy the other constraints*.

The preceding discussion shows that when we solve a balanced transportation problem, we may omit from consideration any one of the problem's constraints and solve an LP having $m+n-1$ constraints. We (arbitrarily) assume that the first supply constraint is omitted from consideration.

Northwest Corner Method

To find a bfs by the northwest corner method, we always work on the upper left (or northwest) corner of the transportation tableau and set the corresponding x_{ij} as large as possible.

				5
				1
				3
2	4	2	1	

2				3
				1
				3
×	4	2	1	

2	3			×
				1
				3
×	1	2	1	

2	3			×
	1			×
				3
×	0	2	1	

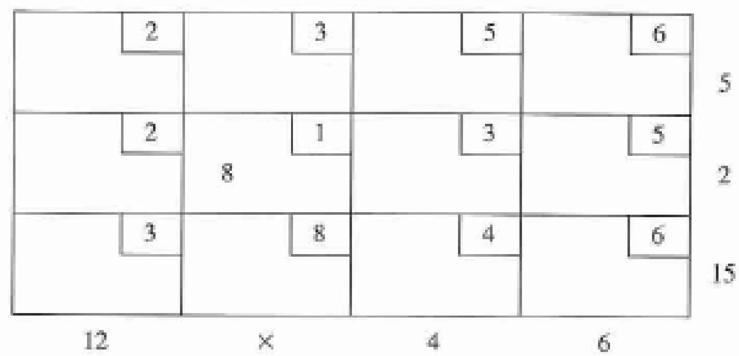
2	3			×
	1			×
	0			3
×	×	2	1	

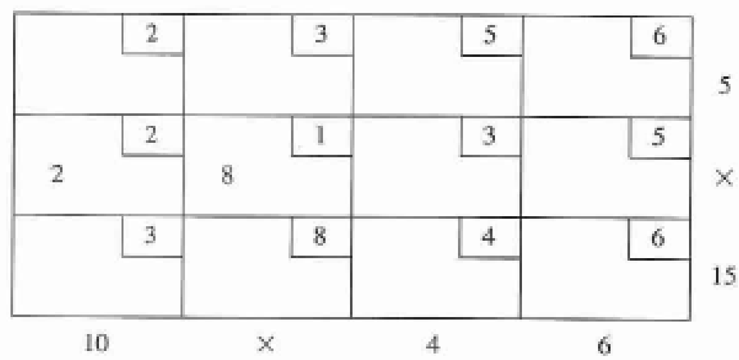
2	3			×
	1			×
	0	2		1
×	×	×	1	

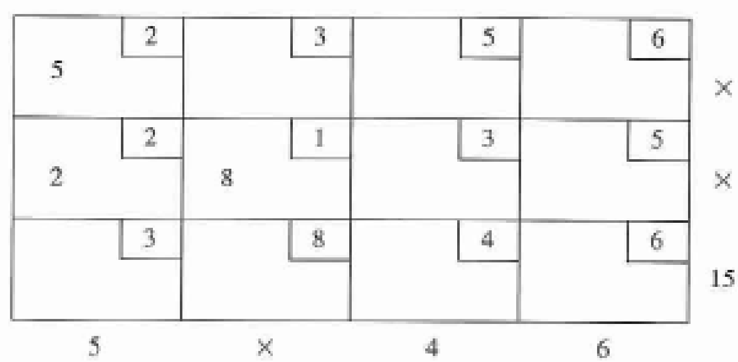
Minimum-Cost Method

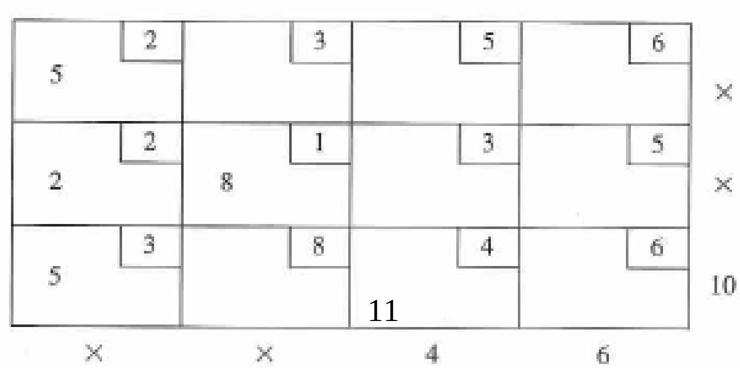
The northwest corner method does not utilize shipping costs, so it can yield an initial bfs that a very high shipping cost. Then determining an optimal solution may require several pivots. The minimum-cost method uses the shipping costs in an effort to produce a bfs that has a lower total cost. Hopefully, fewer pivots will then be required to find the problem's optimal solution.

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	


2 3 5 6
5
2 1 3 5
8
2
3 8 4 6
15
12 × 4 6


2 3 5 6
5
2 1 3 5
2 8
×
3 8 4 6
15
10 × 4 6


2 3 5 6
5
×
2 1 3 5
2 8
×
3 8 4 6
15
5 × 4 6


2 3 5 6
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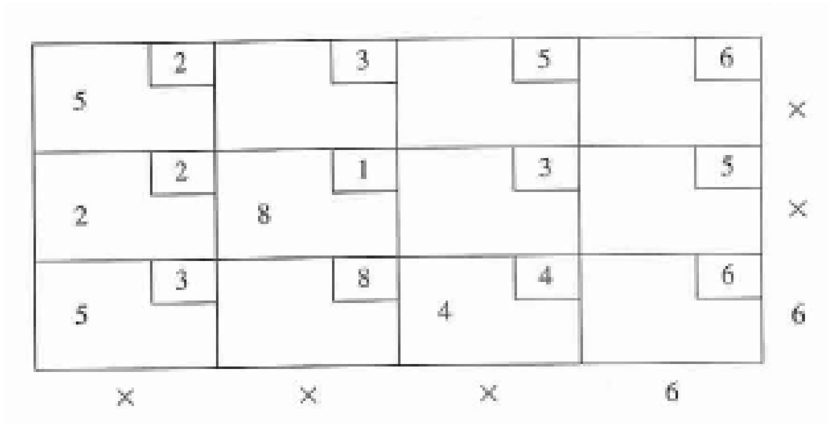
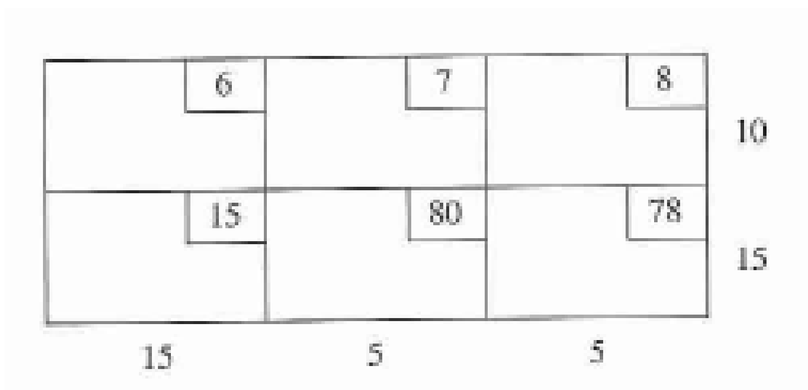
×
2 1 3 5
2 8
×
3 8 4 6
5
10
× × 4 6



Drawback of Minimum-Cost Method



Vogel's Method

Begin by computing for each row (and column) a "penalty" equal to the difference between the two smallest costs in the row (column). Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest shipping costs. As described in the northwest corner and minimum-cost methods, make this variable as large as possible, cross out a row or column, and change the supply or demand associated with basic variable. Now recomputed new penalties (using only cells that do not lie in a crossed-out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with variable, and cross out the variable's row and column. A bfs has now been obtained.

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	10	7 − 6 = 1
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 − 15 = 63
Demand	15	5	5		
Column Penalty	15 − 6 = 9	80 − 7 = 73	78 − 8 = 70		

				Supply	Row Penalty
	$\boxed{6}$	5 $\boxed{7}$	$\boxed{8}$	5	8 − 6 = 2
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 − 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

				Supply	Row Penalty
	$\boxed{6}$	5 $\boxed{7}$	5 $\boxed{8}$	0	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

				Supply	Row Penalty
	0 $\boxed{6}$	5 $\boxed{7}$	5 $\boxed{8}$	×	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

6 0	7 5	8 5	10
15 15	80	78	15
15	5	5	

Observe that Vogel’s method avoids the costly shipments associated with x_{22} and x_{23}. This is because the high shipping costs resulted in large penalties that caused Vogel’s method to choose other variables to satisfy the second and third demand constraints.

Of these three methds we have discussed for finding a bfs, the northwest corner method requires the least effort and Vogel’s method requires the most effort. Extensive research has shown, however, that when Vogel’s method is used to find an initials bfs, it usually takes substantially fewer pivots than if the other two methods had been used. For this reason, the northwest corner and minimum-cost methods are rarely used to find a basic feasible solution to a large transportation problem.

The Transportation Simplex Method

How to Pivot in a Transportation Problem

35				35
10	20	20		50
		10	30	40
45	20	30	30	

35 – 20			0 + 20	35
10 + 20	20	20 – 20 (nonbasic)		50
		10 + 20	30 – 20	40
45	20	30	30	

The preceding illustration of the pivoting procedure makes it clear that each pivot in a transportation problem involves only additions and subtractions. Using this fact, we can show that if all the supplies and demands for transportation problems are integers, then the transportation problem will have an optimal solution in which all the variables are integers. Begin by observing that, by the northwest corner method, we can find a bfs in which each variable is an integer. Each pivot involves only additions subtractions, so each bfs obtained by performing the simplex algorithm (including the optimal solution) will assign all variables integer values. The fact that a transportation problem with integer supplies and demands has an optimal integer solution is useful, because it ensures that we need not wory about whether the Divisibility Assumption is justified.

Solving the Powerco Problem by Using Transportation Simplex Method

35 [8]	[6]	[10]	[9]	35
10 [9]	20 [12]	20 [13]	[7]	50
[14]	[9]	10 [16]	30 [5]	40
45	20	30	30	

35 [8]	[6]	[10]	[9]	35
10 [9]	20 [12]	20 [13]	[7]	50
[14]	[9]	10 [16]	30 [5]	40
45	20	30	30	

$v_j =$	8	11	12	7	
$u_i = 0$	35 [8]	[6]	[10]	[9]	35
1	10 [9]	10 [12]	30 [13]	[7]	50
−2	[14]	10 [9]	[16]	30 [5]	40
	45	20	30	30	

$v_j =$	8	6	12	2	
$u_i = 0$	$\boxed{8}$ 25	$\boxed{6}$ 10	$\boxed{10}$	$\boxed{9}$	35
1	$\boxed{9}$ 20	$\boxed{12}$	$\boxed{13}$ 30	$\boxed{7}$	50
3	$\boxed{14}$	$\boxed{9}$ 10	$\boxed{16}$	$\boxed{5}$ 30	40
	45	20	30	30	

$v_j =$	6	6	10	2	
$u_i = 0$	$\boxed{8}$	$\boxed{6}$ 10	$\boxed{10}$ 25	$\boxed{9}$	35
3	$\boxed{9}$ 45	$\boxed{12}$	$\boxed{13}$ 5	$\boxed{7}$	50
3	$\boxed{14}$	$\boxed{9}$ 10	$\boxed{16}$	$\boxed{5}$ 30	40
	45	20	30	30	

Summary of the Transportation Simplex Method

Step 1. If the problem is unbalanced, balance it.

Step 2. Use one of the methods described to find the bfs.

Step 3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1\ u_2\ ...u_m\ v_1\ v_2\ ...v_n\ ]$ for the current bfs.

Step 4. If $u_i + v_j - c_{ij} \le 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, then we enter the variable the most positive $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure. This yields a new bfs.

Step 5. Using the new bfs, return to Steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4'.

Step 4'. If $u_i + v_j - c_{ij} \ge 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure described earlier.

Assignment Problems

Example 3. Machineco has four machines and four jobs to be completed. Each machine must be assigned to complete one job. The time required to set up each machine for completing each job is shown in the table below:

Setup Times for Machineco

	Time (Hours)			
Machine	Job 1	Job 2	Job 3	Job 4
1	14	5	8	7
2	2	12	6	5
3	7	8	3	9
4	2	4	6	10

Machineco wants to minimize the total setup time needed to complete the four jobs. Use linear programming to solve this problem.

Solution Machineco must determine which machine should be assigned to each job. We define (for $i, j = 1, 2, 3, 4$)

$$x_{ij} = 1 \text{ if machine } i \text{ is assigned to meet the demands of job } j$$
$$x_{ij} = 0 \text{ if machine } i \text{ is not assigned to meet the demands of job } j$$

Then Machineco's problem may be formulated as

$$\begin{aligned}
\min z = {} & 14x_{11} + 5x_{12} + 8x_{13} + 7x_{14} + 2x_{21} + 12x_{22} + 6x_{23} + 5x_{24} \\
& + 7x_{31} + 8x_{32} + 3x_{33} + 9x_{34} + 2x_{41} + 4x_{42} + 6x_{43} + 10x_{44} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} + x_{14} = 1 \qquad \text{(Machine constraints)} \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
& x_{11} + x_{21} + x_{31} + x_{41} = 1 \qquad \text{(Job constraints)} \\
& x_{12} + x_{22} + x_{32} + x_{42} = 1 \\
& x_{13} + x_{23} + x_{33} + x_{43} = 1 \\
& x_{14} + x_{24} + x_{34} + x_{44} = 1 \\
& x_{ij} = 0 \quad \text{or} \quad x_{ij} = 1
\end{aligned} \tag{13}$$

Transportation Simplex is inefficient in solving assignment problems; instead, an efficient method, namely, the Hungarian Method can be used to them efficiently.

Hungarian Method

Step 1. Find the minimum element in each row of the $m \times m$ cost matrix. Construct a new matrix by subtracting from each cost the minimum cost in its row. For this new matrix, find the minimum cost in each column. Construct a new matrix (called the reduced cost matrix) by subtracting from each cost the minimum cost in its column.

Step 2. Draw the minimum number of lines (horizontal, vertical, or both) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, then an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, then proceed to step 3.

Step 3. Find the smallest nonzero element (call its value k) in the reduced cost matrix that is uncovered by the lines drawn in step 2. Now subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to step 2.

Remark:

1. To solve an assignment problem in which the goal is maximize the objective function, multiply the profits matrix by -1 and solve the problem as a minimization problem.
2. If the number of rows and columns in the cost matrix are unequal, then the assignment problem is **unbalanced**. The Hungarian method may yield an incorrect solution if the problem is unbalanced. Thus, any assignment problem should be balanced (by the addition of one or more dummy points) before it is solved by the Hungarian Method.

				Row Minimum
14	5	8	7	5
2	12	6	5	2
7	8	3	9	3
2	4	6	10	2

9	0	3	2
0	10	4	3
4	5	0	6
0	2	4	8
Column Minimum	0	0	2

9	0	3	0
0	10	4	1
4	5	0	4
0	2	4	6

10	0	3	0
0	9	3	0
5	5	0	4
0	1	3	5

Optimal solution: $x_{12} = 1, x_{24} = 1, x_{33} = 1, x_{41} = 1$

Reference:

Winston, W. L. and Venkataramanan, M. (2003) "Introduction To Mathematical Programming", 4^{th} Edition, Brooks/Cole- Thomson Learning.