Lecture 7, 29 October 2014

Linear Programming (LP)

Motivation

1. Production Model

A plastic factory produces three products. The production of each type of product has to go through 2 or 3 operations as shown below:

Time Required Per Unit Production (Mins)				
Operations	Product 1	Product 2	Product 3	Operations Capacity (Mins/Day)
1	1	2	1	430
2	3	0	2	460
3	1	4	0	420
Profit ($)/Unit	3	2	5	

Problem: Determine the production schedule such that the total profit of production will be maximized.

2. Feed Mix Model

To formulate a broiler diet, and the daily requirement is 100 kg. The diet must contain

i. at least 0.8% but not more than 1.2% calcium;
ii. at least 22% protein;
iii. at most 5% crude fiber.

The main ingredients used include limestone, corn and soybean meal. The nutritive content of these ingredients is as follow:

Unit Content Per Unit (kg) of Ingredient				
Ingredient	Calcium	Protein	Fiber	Cost per kg
Limestone	0.380	0	0	0.0164
Corn	0.001	0.09	0.02	0.0463
Soybean	0.002	0.5	0.08	0.1250

Problem: Determine the feed mix method that minimizes the daily cost of producing 100 kg diet while satisfying the nutritive requirements.

Mathematical Formulations of the Problems:

1. Let x_j be the number of units of product j to be produced each day for $j = 1, 2, 3$.

$$\text{Maximize} \quad z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 430 \\ 3x_1 \quad\quad + 2x_3 &\le 460 \\ x_1 + 4x_2 \quad\quad &\le 420 \\ x_j &\ge 0 \quad , \quad j = 1, 2, 3 \end{aligned}$$

2. Let x_1, x_2, and x_3 be the amount of limestone, corn and soybean meal used in producing 100 units (kg) of feed mix.

$$\text{Minimize} \quad z = 0.0164x_1 + 0.0463x_2 + 0.1250x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &= 100 \\ 0.380x_1 + 0.001x_2 + 0.002x_3 &\ge 0.008*100 \\ 0.380x_1 + 0.001x_2 + 0.002x_3 &\le 0.012*100 \\ 0.09x_2 \quad + 0.50x_3 &\ge 0.22*100 \\ 0.02x_2 \quad + 0.08x_3 &\le 0.05*100 \\ x_j &\ge 0 \quad , \quad j = 1, 2, 3 \end{aligned}$$

Both above are Linear Programming (LP) Formulations.

Linear Programming was conceived in 1947 by ***George B. Dantzig*** *while he was the head of the Air Force Statistical Control's Combat Analysis Branch at the Pentagon. The military referred to its plans for training, supplying and deploying combat units as "programs". When Dantzig analyzed Air Force planning problems, he realized* *that they could be formulated as a system of linear inequalities – hence his original name for the technique, "programming in a linear structure", which was later shortened to "linear programming".*

Terms:

1. Decision Variables (i.e. x_j's)
2. Objective Function (i.e. Max or Min z)
3. Constraints (e.g. $x_1 + 2x_2 + x_3 \leq 430$)
4. Sign Restriction (i.e. $x_j \geq 0 \quad , \quad j = 1, 2, 3$)

Feasible Region

Solve
$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\leq 3 \\ x_2 - x_1 &\leq 1 \end{aligned}$$

The objective function is a plane or hyper-plane above the feasible region. Simplex Method will be used to solve the LP problems. But for now, let's look at a problem with two variables only, which can be solved graphically.

Graphical Solution to a Linear Programming (Maximization) Problem

Example 1. Beaver Creek Pottery Company is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colors. The two primary resources used by the company are special pottery clay and skilled labor. Given these limited resources, the company desires to know how many bowls and mugs to produce each day in order to maximize profit. This is generally referred to as a *product mix* problem type. The two products have the following resource requirements for production and profit per item produced:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (LB./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day for production. Determine the number of bowls and number of mugs to produce per day, such that the daily profit of the company will be maximized.

Decision Variables:

$$x_1 = \text{number of bowls to produce}$$
$$x_2 = \text{number of mugs to produce}$$

Objective Function:

$$\text{Maximize} \quad z = 40x_1 + 50x_2$$

Constraints:

$$x_1 + 2x_2 \le 40$$
$$4x_1 + 3x_2 \le 120$$

Sign Restrictions:

$$x_j \ge 0, \quad j = 1,2$$

The Linear Programming model for Beaver Creek Pottery Company is:

Maximized $z = 40x_1 + 50x_2$

subject to

$$x_1 + 2x_2 \le 40$$
$$4x_1 + 3x_2 \le 120$$
$$x_j \ge 0, \quad j = 1,2$$

Graphical Solution to Minimization Problem

Example 2. Moore's Meatpacking Company produces a hot dog mixture in 1,000-pound batches. The mixture contains two ingredients – chicken and beef. The cost per pound of each of these ingredients is as follows:

Ingredient	**Cost/lb.**
Chicken	$3
Beef	$5

Each batch has the following recipe requirements:

i. At least 500 pounds of chicken
ii. At least 200 pounds of beef

The ratio of chicken to beef must be at least 2 to 1. The company wants to know the optimal mixture of ingredients that will minimize cost. Formulate a linear programming model for this problem.

Solution:

Define

$x_1 =$ number of pounds of chicken
$x_2 =$ number of pounds of beef

Objective:

$$\text{Min} \quad z = 3x_1 + 5x_2$$

where $z =$ cost per 1000-lb batch
$3x_1 =$ cost of chicken
$5x_2 =$ cost of beef

Constraints:

$$x_1 + x_2 = 1000$$
$$x_1 \ge 500$$
$$x_2 \ge 200$$
$$x_1 - 2x_2 \ge 0$$
$$x_1, x_2 \ge 0$$

Therefore, the LP model:

$$\text{Min} \quad z = 3x_1 + 5x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 + x_2 &= 1000 \\ x_1 &\geq 500 \\ x_2 &\geq 200 \\ x_1 - 2x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Example 3. Solve the following linear programming model graphically:

$$\text{Min} \quad z = 8x_1 + 6x_2$$

$$\begin{aligned} \text{subject to} \quad 4x_1 + 2x_2 &\geq 20 \\ -6x_1 + 4x_2 &\leq 12 \\ x_1 + x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

Finding the Dual of an LP

Associated with any LP is another LP, called the **dual**. Knowing the relation between an LP and its dual is important because it gives us interesting economic insights. When taking the dual of a given LP, we refer to the given LP as the **primal**. If the primal is a max problem, then the dual will be a min problem, and vice versa. For convenience, we define the variables for the max problem to be $z, x_1, x_2, ..., x_n$ and the variable for the min problem to be $w, y_1, y_2, ..., y_m$. We begin by explaining how to find the dual of a max problem in which all variables are required to be non-negative and all constraints are "$\le$" constraints (called a **normal max problem**). A normal max problem may be written as:

$$
\begin{aligned}
\text{Max} \quad & z = c_1x_1 + c_2x_2 + ... + c_nx_n \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \le b_2 \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \le b_m \\
& x_j \ge 0, \quad j = 1, 2, ..., n
\end{aligned}
\qquad (*)
$$

The dual of a normal max problem is defined to be:

$$
\begin{aligned}
\text{Min} \quad & w = b_1y_1 + b_2y_2 + ... + b_my_m \\
\text{s.t.} \quad & a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m \ge c_1 \\
& a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m \ge c_2 \\
& \quad \vdots \\
& a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m \ge c_n \\
& y_i \ge 0, \quad i = 1, 2, ..., m
\end{aligned}
\qquad (**)
$$

A min problem such as (**) that has all "$\ge$" constraints and all variables non-negative is called a **normal min problem**. If the primal is a normal min problem such as (**), the dual of (**) is (*).

Example 4 (Dakota Problem). The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given in the following table.

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for \$60, a table for \$30, and a chair for \$20. Dakota believes that demands for desks and chairs are unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Define

$x_1 =$ number of desks produced

$x_2 =$ number of tables produced

$x_3 =$ number of chairs produced

The LP model is:

$$\text{Max} \quad z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned} \text{s.t.} \quad 8x_1 + 6x_2 + x_3 &\le 48 \quad \text{(Lumber constraint)} \\ 4x_1 + 2x_2 + 1.5x_3 &\le 20 \quad \text{(Finishing constraint)} \\ 2x_1 + 1.5x_2 + 0.5x_3 &\le 8 \quad \text{(Carpentry constraint)} \\ x_2 &\le 5 \quad \text{(Limitation on table demand)} \end{aligned}$$

$$x_j \ge 0, \quad j = 1, 2, 3$$

The corresponding dual is:

$$\text{Min} \quad w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned} \text{s.t.} \quad 8y_1 + 4y_2 + 2y_3 &\ge 60 \\ 6y_1 + 2y_2 + 1.5y_3 &\ge 30 \end{aligned}$$

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$
$$y_i \geq 0, \quad i = 1, 2, 3$$

Example 5 (Diet Problem). My diet requires that the food I eat come from one of the four "basic food groups" (chocolate cake, ice-cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 50 cents, each scoop of chocolate ice cream costs 20 cents, each bottle of cola costs 30 cents, and each piece of pineapple cheesecake costs 80 cents. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in the following table. Formulate a linear programming model that can used to satisfy my daily nutritional requirements at minimum cost.

Type of Food	Calories	Chocolate (oz)	Sugar (oz)	Fat (oz)	Cost (cents)
Brownie	400	3	2	2	50
Chocolate ice cream (1 scoop)	200	2	2	4	20
Cola (1 bottle)	150	0	4	1	30
Pineapple Cheesecake	500	0	4	5	80

Define

$y_1 =$ number of brownies eaten daily
$y_2 =$ number of scoops of chocolate ice cream eaten daily
$y_3 =$ number of cola drunk daily
$y_4 =$ number pineapple cheesecake eaten daily

The LP Model:

$$\text{Min} \quad z = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{aligned}
\text{s.t.} \quad & 400y_1 + 200y_2 + 150y_3 + 500y_4 \geq 500 && \text{(Calorie constraint)} \\
& 3y_1 + 2y_2 \geq 6 && \text{(Chocolate constraint)} \\
& 2y_1 + 2y_2 + 4y_3 + 4y_4 \geq 10 && \text{(Sugar constraint)} \\
& 2y_1 + 4y_2 + y_3 + 5y_4 \geq 8 && \text{(Fat constraint)}
\end{aligned}$$

$$y_j \geq 0, \quad j = 1, 2, 3, 4$$

The dual:

$$\text{Max} \quad z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{aligned} \text{s.t.} \quad 400x_1 + 3x_2 + 2x_3 + 2x_4 &\le 50 \\ 200x_1 + 2x_2 + 2x_3 + 4x_4 &\le 20 \\ 150x_1 \quad + 4x_3 + x_4 &\le 30 \\ 500x_1 \quad + 4x_3 + 5x_4 &\le 80 \end{aligned}$$

$$x_j \ge 0, \quad j = 1, 2, 3, 4$$

Economic Interpretation of the Dual Problem

i. Interpreting the Dual of a Max Problem

The dual of the Dakota problem is:

$$\text{Min} \quad w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned} \text{s.t.} \quad 8y_1 + 4y_2 + 2y_3 &\ge 60 && \text{(Desk constraint)} \\ 6y_1 + 2y_2 + 1.5y_3 &\ge 30 && \text{(Table constraint)} \\ y_1 + 1.5y_2 + 0.5y_3 &\ge 20 && \text{(Chair constraint)} \\ y_i \ge 0, \quad i &= 1, 2, 3 \end{aligned}$$

y_1 is associated with lumber
y_2 is associated with finishing hours
y_3 is associated with carpentry hours

Resource	Desk	Table	Chair	Amount of Resource Available
Lumber (board ft)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	0.5	8
Selling Price ($)	60	30	20	

Now, suppose an entrepreneur wants to purchase all of Dakota's resources. The entrepreneur must determine the price he or she is willing to pay for a unit of each Dakota's resources. We define,

y_1 = price paid for 1 board ft of lumber

y_2 = price paid for 1 finishing hour
y_3 = price paid for 1 carpentry hour

Question: In setting resource prices, what constraints does the entrepreneur face?

Answer: Resource prices must be set high enough to induce Dakota to see. For example, the entrepreneur must offer Dakota at least \$60 for a combination of resources that includes 8 board feet of lumber, 4 finishing hours, and 2 carpentry hours, because Dakota could use these resources to produce a desk that can be sold for \$60. The entrepreneur is offering $8y_1 + 4y_2 + 2y_3$ for the resources used to produce a desk, so he or she must choose y_1, y_2 and y_3 to satisfy

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

Similarly, the resource prices must be set high enough such that

$$6y_1 + 2y_2 + 1.5y_3 \geq 30$$
$$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$

are satisfied. Put everything together, we see that the solution of the dual does yield prices for lumber, finishing hours, and carpentry hours. In summary, when the primal is a normal max problem, the dual variables are related to the value of the resources available to the decision maker. For this reason, the dual variables are often referred to as **resource shadow prices**.

ii. Interpreting the Dual of a Min Problem

To interpret the dual of a min problem, we consider the dual of the diet problem.

$$\text{Max} \quad z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{aligned}
\text{s.t.} \quad 400x_1 + 3x_2 + 2x_3 + 2x_4 &\leq 50 && \text{(Brownie constraint)} \\
200x_1 + 2x_2 + 2x_3 + 4x_4 &\leq 20 && \text{(Ice cream constraint)} \\
150x_1 \qquad + 4x_3 + x_4 &\leq 30 && \text{(Soda constraint)} \\
500x_1 \qquad + 4x_3 + 5x_4 &\leq 80 && \text{(Cheesecake constraint)}
\end{aligned}$$

$$x_j \geq 0, \quad j = 1, 2, 3, 4$$

Type of Food	Calories	Chocolate (oz)	Sugar (oz)	Fat (oz)	Cost/Price (cents)
Brownie	400	3	2	2	50
Chocolate ice cream (1 scoop)	200	2	2	4	20
Cola (1 bottle)	150	0	4	1	30
Pineapple Cheesecake	500	0	4	5	80
Requirements	500	6	10	8	

Suppose Susan is a "nutrient" salesperson who sells calories, chocolate, sugar, and fat. Susan needs to determine

$x_1 =$ price per calorie to charge dieter
$x_2 =$ price per ounce of chocolate to charge dieter
$x_3 =$ price per ounce of sugar to charge dieter
$x_4 =$ price per ounce of fat to charge dieter

Susan wants to maximize her revenue from selling the dieter the daily ration of required nutrients. However, Susan must set prices low enough so that it will be in the dieter's economic interest to purchase all nutrients from her. For example, by purchasing a brownie for 50 cents, the dieter can obtain 400 calories, 3 oz of chocolate, 2 oz of sugar, and 2 oz of fat. Therefore, Susan cannot charge more than 50 cents for this combination of nutrients. This leads to the following (brownie) constraint:

$$400x_1 + 3x_2 + 2x_3 + 2x_4 \leq 50$$

Similar reasoning yields the other constraints.

Thus, x_1 would be the price for 1 calorie, x_2 would be the price for 1 oz of chocolate, and so on.

Reference:

Winston, W. L., Venkataramanan M., “Introduction to Mathematical Programming”, 4th edition, Brooks/Cole- Thomson Learning, 2003.