- Back to the DFT

$x[n]$ is a length-N discrete-time sequence

(DFT) $$X[k] = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi k}{N}n}}_{= \varphi^*[k,n]} \quad \text{"basis sequence"}$$

Def. <u>orthogonal</u> basis sequences -

$$\frac{1}{N}\sum_{n=0}^{N-1} \varphi[k,n]\,\varphi^*[\ell,n] = \begin{cases} 1, & \ell = k \\ 0, & \ell \neq k \end{cases}$$

(IDFT) $$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] \underbrace{e^{j\frac{2\pi k n}{N}}}_{= \varphi[k,n]}$$

- Important consequence of orthogonal basis -

  - Energy Preservation

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n]\, x^*[n]$$

$$= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\,\varphi[k,n]\right) x^*[n]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\left(\sum_{n=0}^{N-1}\varphi[k,n]\, x^*[n]\right) = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,X^*[k]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 \quad \text{"Parseval's Relation"}$$

- DFT Examples

Example 1 $\qquad x[n] = \begin{cases} 1, & n=0 \\ 0, & 1 \le n \le N-1 \end{cases}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} = 1 \cdot e^{-j\frac{2\pi k \cdot \phi}{N}} = 1$$

Example 2

$$x[n] = \begin{cases} 1, & n=m \qquad 0 \le m \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} = x[m] e^{-j\frac{2\pi k m}{N}} = 1 \cdot e^{-j\frac{2\pi k m}{N}}$$

Example 3 $\qquad x[n] = \cos\left(\frac{2\pi k_0 n}{N}\right) \qquad 0 \le n \le N-1$

$$\Rightarrow \quad x[n] = \frac{1}{2} e^{j\frac{2\pi k_0 n}{N}} + \frac{1}{2} e^{-j\frac{2\pi k_0 n}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}} = \frac{1}{2}\sum_{n=0}^{N-1} e^{j\frac{2\pi k_0 n}{N}} e^{-j\frac{2\pi k n}{N}}$$

$$+ \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi k_0 n}{N}} e^{-j\frac{2\pi k n}{N}}$$

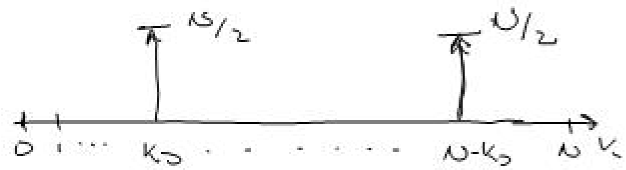
$$= \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n}$$

use property : $\sum_{k=0}^{N-1} e^{j\frac{2\pi n}{N}(k-l)} = \begin{cases} N, & k=l \text{ or } k=N-l \\ 0, & k \ne l \end{cases}$

$$\Rightarrow \quad X[k] = \begin{cases} N/2, & k=k_0 \\ N/2, & k=N-k_0 \\ 0, & \text{otherwise} \end{cases}$$

Example 3 cont.

$$X[k] = \begin{cases} N/2 & , k = k_0 \text{ or } k = N - k_0 \\ 0 & , \text{otherwise} \end{cases}$$



Note: Only one component at $k = k_0$ and $k = N - k_0$ if $\omega_0 = \frac{2\pi k_0}{N}$ where k_0 is an integer.

see leakage_demo.m

• DFT computations

$$X[k] = \sum_{n=0}^{N-1} \underbrace{x[n]\, e^{-j\frac{2\pi k n}{N}}}$$

→ N multiplications (N samples per sequence)

→ N times through

$\Rightarrow$ N^2 multiplications & $N(N-1)$ complex additions

complex

- DFT response to $x[n] = \cos(n\omega_0)$ when $\omega_0 \neq \frac{2\pi k_0}{N}$

$$X[k] = \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n} + \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+k_0)n}$$

Use geometric series $\rightarrow$

$$= \frac{1}{2}\,\frac{1-e^{jN\left(\frac{2\pi}{N}k-\omega_0\right)}}{1-e^{-j\left(\frac{2\pi}{N}k-\omega_0\right)}} + \frac{1}{2}\,\frac{1-e^{-jN\left(\frac{2\pi}{N}k+\omega_0\right)}}{1-e^{-j\left(\frac{2\pi}{N}k+\omega_0\right)}}$$

Factor out exponential $\longrightarrow$

$$= \frac{1}{2}e^{j\left(\frac{N-1}{2}\right)\left(\frac{2\pi}{N}k-\omega_0\right)}\,\frac{\sin\left(\pi k-\frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N}-\frac{\omega_0}{2}\right)}$$

$$+ \frac{1}{2}e^{-j\left(\frac{N-1}{2}\right)\left(\frac{2\pi}{N}k+\omega_0\right)}\,\frac{\sin\left(\pi k+\frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N}+\frac{\omega_0}{2}\right)}$$

Notice: Unless ω_0 is an integer multiple of $\frac{2\pi}{N}$, $X[k]$ is non-zero for each k.

<u>Rule of Thumb</u> - Amplitude response of DFT to real input is $\approx \frac{N}{2}\,\frac{\sin[\pi(k-m))}{\pi(k-m)}$

- DFT samples and frequency relationship -

  normalized frequency associated with index k is

$$\omega_0 = \frac{2\pi k}{N}$$

  <u>Example:</u>

$$N = 32 \quad , k = 11 \quad \Longrightarrow \quad \omega = \frac{11\pi}{16} \text{ radians}$$

$$\text{or} \quad f = \frac{\omega}{2\pi} = \frac{11}{32} \text{ Hz}$$

- Use DFT to approximate DTFT

  Evaluate $X(e^{j\omega})$ at a dense grid of frequencies.

$$\omega_k = \frac{2\pi k}{M} \quad , \quad 0 \le k \le M-1 \quad \text{where } M >> N$$

$$X(e^{j\omega})\Big|_{\omega=\omega_k} = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{M}}$$

Question: How do we get M output samples from N input samples of $x[n]$?

Answer: zero-pad $x[n]$ with $M-N$ zeros before computing DFT.

see zero_pad_demo.m

- Circular shift

$x[n]$, $0 \le n \le N-1$

Let $r = m$ modulo N $= \langle m \rangle_N$

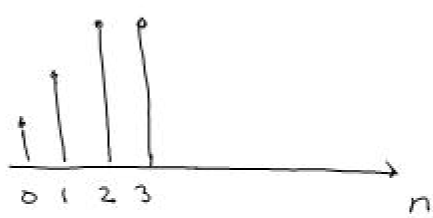
Example $\langle 25 \rangle_7 = 4$ $\quad\quad$ $\langle -16 \rangle_7 = 5$

Circular shift using modulo -

$$x_c[n] = x[\langle n - n_0 \rangle_N] = \begin{cases} x[n-n_0] & , n \ge n_0 \\ x[N+n-n_0] & , n < n_0 \end{cases}$$

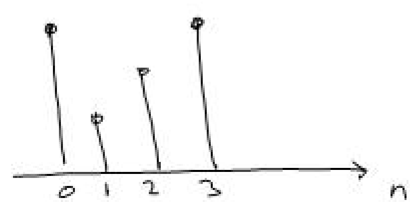
If $n_0 > 0 \longrightarrow$ right shift

If $n_0 < 0 \longrightarrow$ left shift



$x[n]$

$\longrightarrow$



$x[\langle n-1 \rangle_N]$

or

$x[\langle n+3 \rangle_N]$

Frequency domain $\rightarrow$ circular freq. shift

$$X_c[k] = X[\langle k - k_0 \rangle_N] = \begin{cases} X[k-k_0] & , k \ge k_0 \\ X[N+k-k_0] & , k < k_0 \end{cases}$$

- Circular Convolution

Consider two length-N sequences, $x[n]$ & $h[n]$.

Linear convolution - $y[n] = \sum_{k=0}^{N-1} x[k]\, h[n-k]$

for $0 \leq n \leq 2N-2$

Note: Both sequence were zero-padded to length $2N-1$ before convolution computation.

Circular convolution - $y_c[n] = \sum_{k=0}^{N-1} x[k]\, h[\langle n-k \rangle_N]$

$= x[n] \; N \; h[n]$

- Fourier-Domain Filtering

$$x[n] * h[n] \xrightarrow{DTFT} X(e^{j\omega})\, H(e^{j\omega})$$

Comment: If $x[n]$ & $h[n]$ are real-valued, only take real output as filter output $\rightarrow \mathrm{Re}\{y[n]\}$

Simple approach - set $H(e^{j\omega}) = 0$ in band(s) you want to suppress. Set $H(e^{j\omega}) = 1$ in band(s) you want to pass with unity gain.

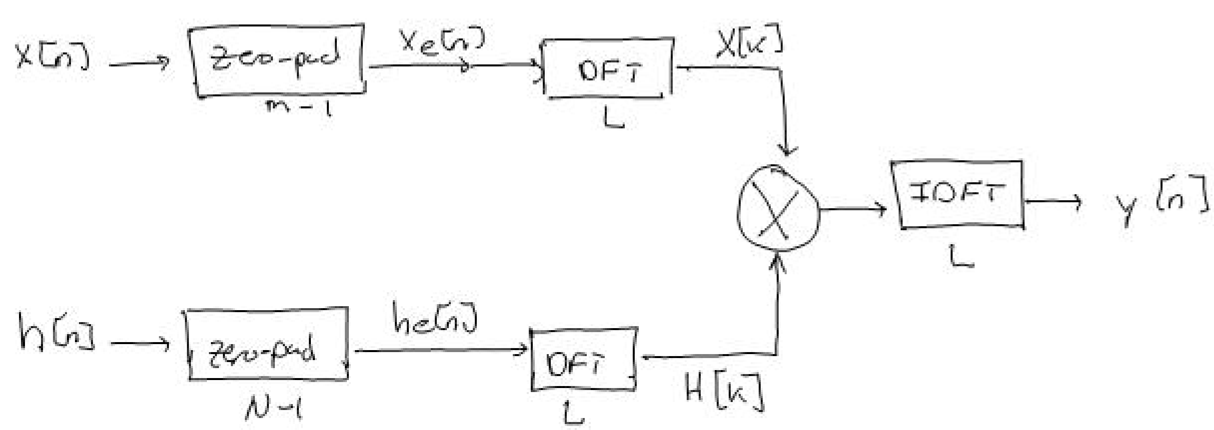
$H(e^{j\omega})$ has zero-phase in this case and will not change the phase of $X(e^{j\omega})$.

- Linear Convolution using DFT

Need to extend each sequence by zero-padding to $N+M-1$ length. Let $L = N+M$

$$x_e[n] = \begin{cases} x[n] & , 0 \le n \le N-1 \\ 0 & , N \le n \le L-1 \end{cases}$$

$$h_e[n] = \begin{cases} h[n] & , 0 \le n \le M-1 \\ 0 & , M \le n \le L-1 \end{cases}$$



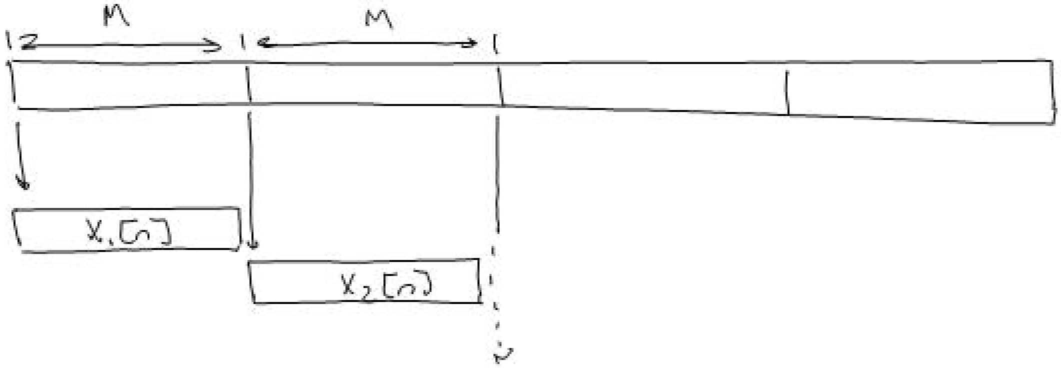
Note: $L \ge N+M-1$ could be used.

- Linear convolution of finite length sequence and infinite length sequence.

Two approaches:

1. Overlap-Add method

Assume $x[n]=0$ for $n<0$, and $x[n]$ length is $\gg L$.
$h[n]$ length is L.

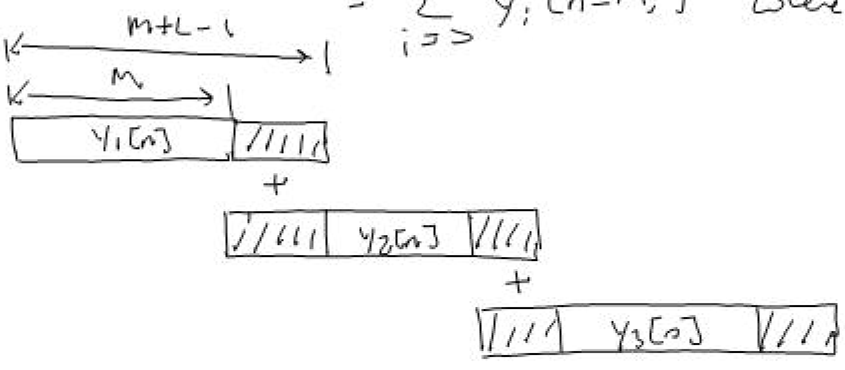


$$x[n] = \sum_{i=0}^{\infty} x_i[n-Mi] \quad \text{where} \quad x_i[n] = \begin{cases} x[n+Mi] & n=0,1,\dots M-1 \\ 0 & ,\text{otherwise} \end{cases}$$

Convolution -

$$y[n] = x[n] * h[n] = \sum_{i=0}^{\infty} x_i[n-Mi] * h[n]$$

$$= \sum_{i=0}^{\infty} y_i[n-Mi] \quad \text{where} \quad y_i[n] = x_i[n] * h[n]$$



etc.

- Application of filters

Distortionless system $\quad x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = C\,x(t-t_0)$

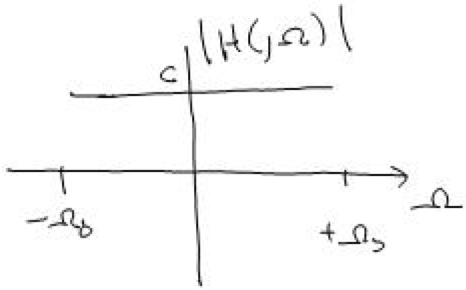
$$Y(j\Omega) = C\,X(j\Omega)\,e^{-j\Omega t_0}$$

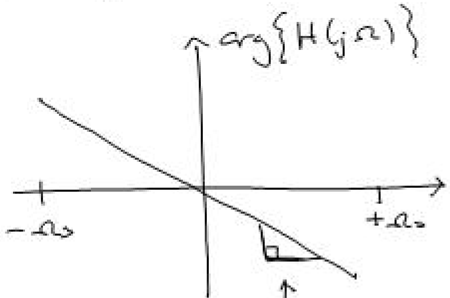
$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = C\,e^{-j\Omega t_0}$$

therefore, $\quad h(t) = C\,\delta(t-t_0)$

Magnitude Spectrum - $\quad |H(j\Omega)| = C$

Phase Spectrum - $\quad \arg\{H(j\Omega)\} = -\Omega t_0$





slope = $-t_0$

Discrete-Time -

$|H(e^{j\omega})| = C \qquad$ constant over passband

$\arg\{H(e^{j\omega})\} = -\omega n_0 \qquad$ Linear phase w/ slope $-n_0$

- Ideal low pass filter

Let $C=1 \Rightarrow H(j\Omega) = \begin{cases} e^{-j\Omega t_0} & , |\Omega| \leq \Omega_c \\ 0 & , |\Omega| > \Omega_c \end{cases}$

Impulse response:

$$h(t) = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega(t-t_0)}\, d\Omega = \frac{\Omega_c}{\pi}\operatorname{sinc}\left(\frac{\Omega_c(t-t_0)}{\pi}\right)$$

Note: Ideal Low Pass filter is nonimplementable because it is noncausal.

In practice we must tolerate an acceptable level of distortion by allowing certain "deviations":

1) Passband magnitude should be between 1 & $1-\epsilon$.

$$1-\epsilon \leq |H(j\Omega)| \leq 1 \quad \text{for } 0 \leq |\Omega| \leq \Omega_p$$

2) $|H(j\Omega)| \leq \delta$ in stopband, or $|\Omega| \geq \Omega_s$

3) Transition bandwidth from passband to stopband has finite width,

$$\Omega_s - \Omega_p > 0$$