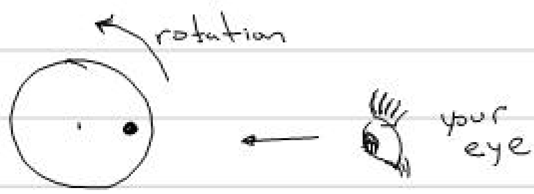
- Processing Complex Signals

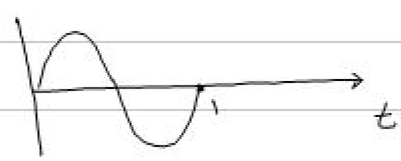
Much of the power of digital signal processing lies in the ability to form and manipulate complex signals.

Consider the real signal $x(t) = \cos(\Omega t)$. It is possible to determine the frequency of this signal by counting the number of zero crossings of the signal in a given second. However, from the zero crossings it is impossible to distinguish if the frequency is positive or negative.

Imagine a wheel with a handle as shown below

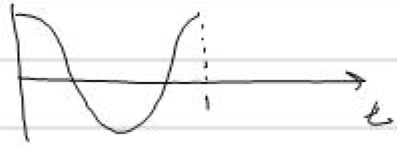


You are looking at the wheel handle from the right side and plotting the handle displacement as a function of time. The plot would look like this:



$\sin(\Omega t)$

If you looked at the wheel from the bottom of the page the plot of displacement vs. time would look like



$\cos(\Omega t)$

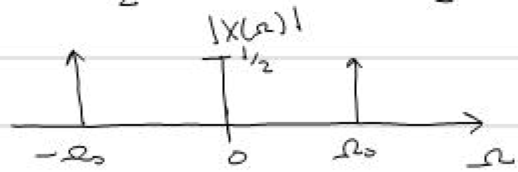
We call the first viewpoint the real axis and the second the imaginary axis. This gives the signal representation for the wheel handle displacement as

$$z(t) = \cos(\Omega t) + j \sin(\Omega t)$$

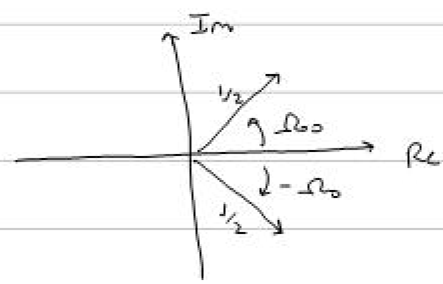
By looking at the wheel from both directions we can determine the direction of rotation. For a possitive rotation (counterclockwise) the cosine decreased as the sine increases. For a negative rotation (clockwise), the sine would go negative as the cosine decreased.

Since $\cos(\Omega t)$ and $\sin(\Omega t)$ are real functions, they must have both positive and negative components (frequency). The frequency spectrum of $\cos(\Omega_0 t)$ is

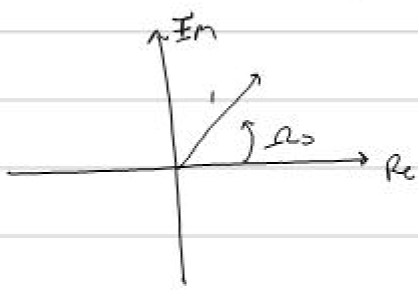
$$X(\Omega) = \frac{1}{2}\delta(\Omega - \Omega_0) + \frac{1}{2}\delta(\Omega + \Omega_0)$$



We can use the polar form to plot a vector on the complex plane.



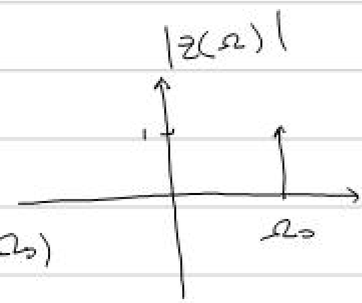
Now consider the complex signal $z(t) = \cos(\Omega_0 t) + j\sin(\Omega_0 t)$. The vector can be plotted as



spectrum

$\Longrightarrow$

$Z(\Omega) = \delta(\Omega - \Omega_0)$

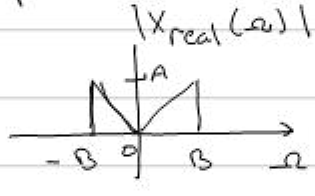


Note, the spectrum only has possitive frequencies, not both. The frequency spectrum of a complex signal does not have the symmetrical properties that a real signal spectrum does.

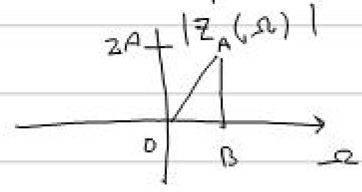
- Analytic Signal — a complex signal having only possitive or only negative frequencies.

  - Properties of Continuous-time Analytic signal $z(t)$

    - Spectrum is one-sided and 2x amplitude.



Mag. spectrum of real signal



Mag. spectrum of Analytic signal

- Can recover original signal (real signal) by

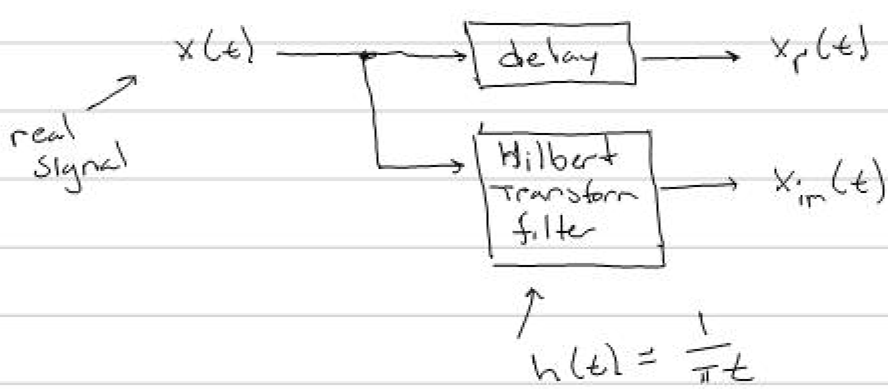
$$x(t) = Re\{z(t)\}$$

- Real and imaginary parts of analytic signal are orthogonal

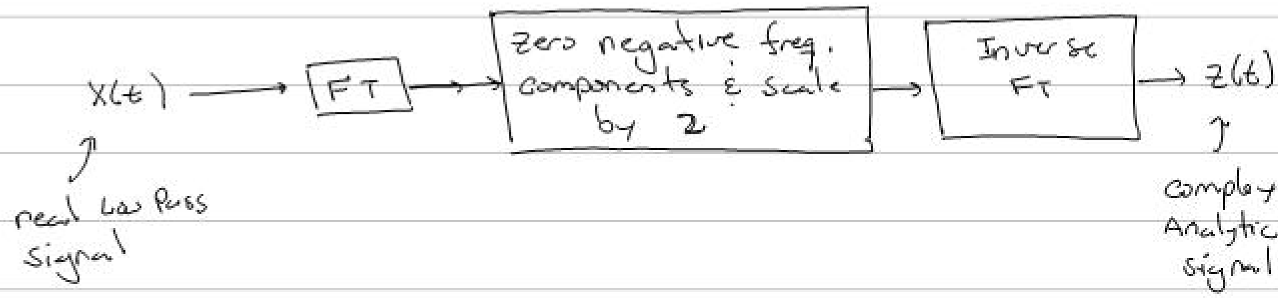
$$\int_{-\infty}^{\infty} Re\{z(t)\}\, Im\{z(t)\}\, dt = 0$$

- How do we create analytic signal from a real lowpass signal?

  - Time-domain approach



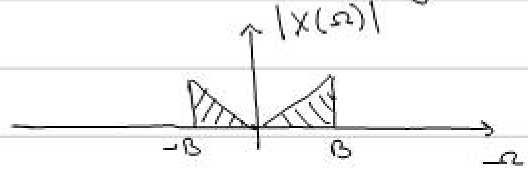
- Frequency domain approach



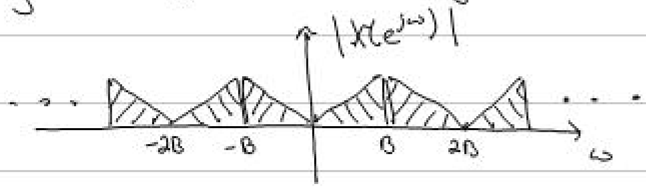
Note: In this case $z(t)$ is also a low pass signal.

- Signal cases - There are three general signal cases

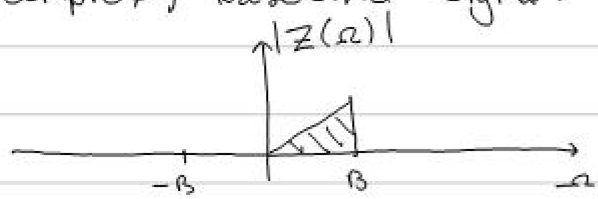
<u>case #1</u> Real Bandlimited Baseband Signal



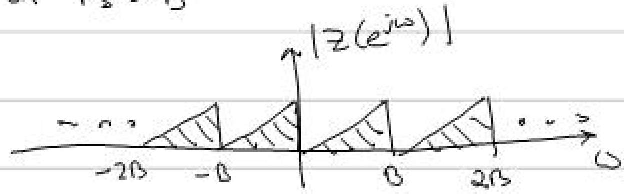
After sampling at $F_s = 2B$ we get a periodic spectrum



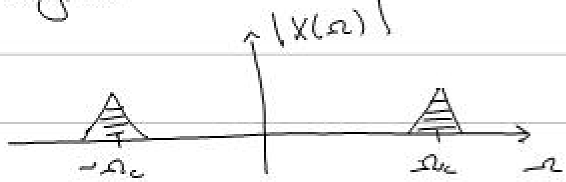
<u>case #2</u> Analytic (complex) Baseband Signal



After sampling at $F_s = B$



<u>case #3</u> Bandpass signal

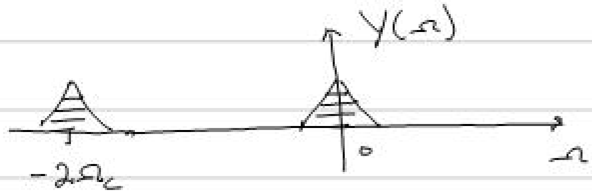


Q: How can we make this a complex signal?

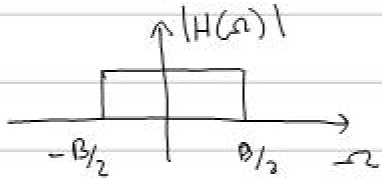
A: complex demodulation

First, demodulate to baseband $\Rightarrow$ $x(t)\exp(-j\Omega_c t)$

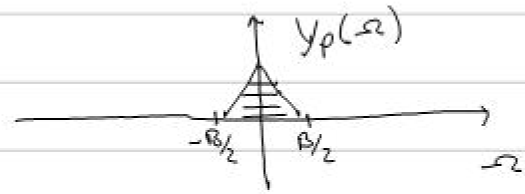
$Y(\Omega) = X(\Omega + \Omega_c)$



Then, low pass filter $\Rightarrow$ $Y_p(\Omega) = X(\Omega + \Omega_c)\cdot H(\Omega)$



$\Longrightarrow$



• Implementation

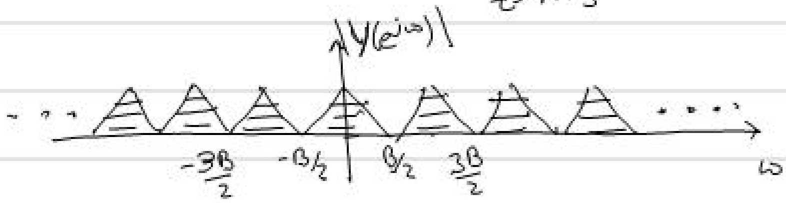
$x(t) \longrightarrow \otimes \longrightarrow \boxed{\text{LPF}} \longrightarrow y_p(t)$

real signal

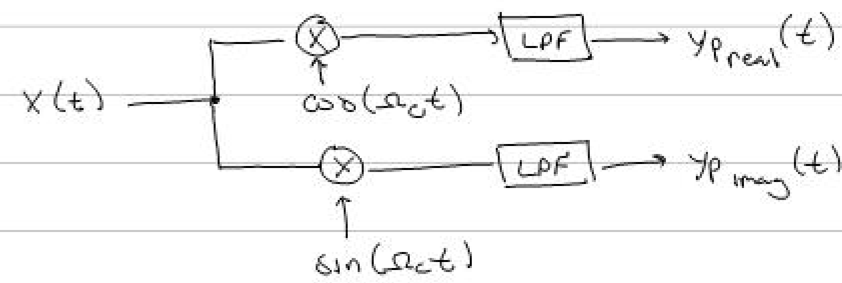
$\exp(-j\Omega_c t)$

complex signal

After sampling $\Rightarrow$ $y_p[n] = y_p(t)\Big|_{t=nT_s}$ $\quad F_s = B$



Another implementation of complex demodulation



$$y_p(t) = y_{p_{real}}(t) + j\, y_{p_{imag}}(t)$$

- The Hilbert Transform filter.

Consider the complex analytic signal $y[n] = x[n] + j\hat{x}[n]$ where both $x[n]$ and $\hat{x}[n]$ are real. The FT of $y[n]$ is

$$Y(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega})$$

Both $X(e^{j\omega})$ and $\hat{X}(e^{j\omega})$ are conjugate symmetric since $x[n]$ and $\hat{x}[n]$ are real.

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad \& \quad \hat{X}(e^{j\omega}) = \hat{X}^*(e^{-j\omega})$$

Hence, $X(e^{j\omega})$ equals the even part of $Y(e^{j\omega})$ and $j\hat{X}(e^{j\omega})$ equals the odd part of $Y(e^{j\omega})$.

For an analytic signal $Y(e^{j\omega}) = 0$ for $-\pi \le \omega < 0$, so

$$Y(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}), & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

Thus, analytic signal can be created by zeroing negative frequencies and scaling positive frequencies by 2.

$$H(e^{j\omega}) = \begin{cases} 2, & 0 \le \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

- Discrete-time Hilbert Transformer

$$\hat{X}(e^{j\omega}) = \frac{1}{2j}\left[Y(e^{j\omega}) - Y^*(e^{-j\omega}) \right]$$

$Y(e^{-j\omega}) = 0$ for $0 \le \omega < \pi$ and

$Y(e^{j\omega}) = 0$ for $-\pi \le \omega < 0$

So,
$$\hat{X}(e^{j\omega}) = \begin{cases} -j\, X(e^{j\omega}), & 0 \le \omega < \pi \\ j\, X(e^{j\omega}), & -\pi \le \omega < 0 \end{cases}$$

from using $X(e^{j\omega}) = \underbrace{\frac{1}{2}\left[Y(e^{j\omega}) + Y^*(e^{-j\omega}) \right]}_{\text{even part}}$

Thus, imaginary part of analytic signal can be generated by passing its real part through a linear discrete-time system, with the freq. response

$$H_{HT}(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega < \pi \\ +j, & -\pi \le \omega < 0 \end{cases}$$

This is called the Hilbert transformer. Note that

$|H_{HT}(e^{j\omega})| = 1$ for all frequencies and has a $-90°$ phase-shift for negative freqs and a $+90°$ phase-shift for possitive freqs. The impulse response is

$$h_{HT}[n] = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{2}{\pi n}, & \text{for } n \text{ odd} \end{cases}$$

Notice: This ideal Hilbert transformer is two-sided infinite length, and is therefore unrealizable.
We need approximations!