- Understanding the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}} \quad , \quad 0 \le k \le N-1$$

N input samples $\longrightarrow$ N equally spaced frequency samples

$$X[k] = \sum_{n=0}^{N-1} x[n]\left[\cos\left(\frac{2\pi k n}{N}\right) - j\sin\left(\frac{2\pi k n}{N}\right)\right]$$

~~Remember: $x(t)\big|_{nT_s} = x(nT_s) = x[n]$~~

~~so $X[k] = \sum_{n=0}^{N-1} x(nT_s)\left[\cos\left(\frac{2\pi k n}{N}\right) - j\sin\left(\frac{2\pi k n T_s}{N}\right)\right]$~~

Example: $N = 4$ $\qquad X[k] = \sum_{n=0}^{3} x(n)\left[\cos\left(\frac{2\pi k n}{4}\right) - j\sin\left(\frac{2\pi k n}{4}\right)\right]$

$k=0$; $\quad X[0] = x(0)\cos(2\pi\cdot 0\cdot 0\cdot 1/4) - j x(0)\sin(2\pi\cdot 0\cdot 0\cdot 1/4)$
$\qquad + x(1)\cos(2\pi\cdot 0\cdot 1\cdot 1/4) - j x(1)\sin(2\pi\cdot 0\cdot 1\cdot 1/4)$
$\qquad + x(2)\cos(2\pi\cdot 0\cdot 2\cdot 1/4) - j x(2)\sin(2\pi\cdot 0\cdot 2\cdot 1/4)$
$\qquad + x(3)\cos(2\pi\cdot 0\cdot 3\cdot 1/4) - j x(3)\sin(2\pi\cdot 0\cdot 3\cdot 1/4)$

$k=1$; $\quad X[1] = x(0)\cos(2\pi\cdot 1\cdot 0\cdot 1/4) - j x(0)\sin(2\pi\cdot 1\cdot 0\cdot 1/4)$
$\qquad + x(1)\cos(2\pi\cdot 1\cdot 1\cdot 1/4) - j x(1)\sin(2\pi\cdot 1\cdot 1\cdot 1/4)$
$\qquad + x(2)\cos(2\pi\cdot 1\cdot 2\cdot 1/4) - j x(2)\sin(2\pi\cdot 1\cdot 2\cdot 1/4)$
$\qquad + x(3)\cos(2\pi\cdot 1\cdot 3\cdot 1/4) - j x(3)\sin(2\pi\cdot 1\cdot 3\cdot 1/4)$

$k=2$; $\quad \ldots\ldots$ etc, etc

Note: Exact frequencies of different sinusoids depends on f_s and N.

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \Rightarrow \omega = \frac{\Omega}{f_s}$$

↑ sample freq.

so
$$\Omega = \frac{2\pi k f_s}{N} = 2\pi f_{analysis}$$

$$\Rightarrow f_{analysis} = \frac{k f_s}{N}$$

Example: If $f_s = 500 \frac{\text{samples}}{\text{sec}}$, $N = 16$, $k = 1$

$$f_{analysis} = \frac{500}{16} = 31.25\ Hz$$

$$f_0 = 0 \cdot 31.25Hz = 0$$
$$f_1 = 1 \cdot 31.25Hz = 31.25Hz$$
$$f_2 = 2 \cdot 31.25Hz = 62.5\ Hz$$
$$f_3 = 3 \cdot 31.25Hz = 468.75\ Hz$$

so, $X[0] \xrightarrow{\text{specifies}}$ magnitude & phase of "DC" component

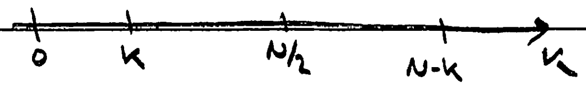
$X[1] \rightarrow$ magnitude & phase of 31.25 Hz component

etc, etc.

Note: DFT accepts complex & real inputs

When input is real - complex output for $k=1$ to $k=N/2-1$ are redundant with outputs for $k > N/2$

$$X[k] = X[N-k]$$



$$X[k] = |X[k]|\, e^{jX_\phi[k]} = |X[N-k]|\, e^{-jX_\phi[N-k]}$$

phase angles are opposite in sign

So, when $x[n]$ is real-valued $\quad X[k] = X^*[N-n]$

$\Rightarrow$ real part is even

$\Rightarrow$ odd part is odd

- DFT Magnitudes

real-valued sinewave input of amp. A_0 w/ integral number of cycles over N input samples -

$$\text{DFT output magnitude} = A_0 N/2$$

Important if using fixed-point hardware.

complex-valued sinusoid of magnitude A_0 (i.e. $A_0 e^{j2\pi f t}$) with integral number of cycles over N input samples —

$$\text{DFT output magnitude} = A_0 N$$

Often see. $\quad X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi n k/N}$

Q: How do we increase the resolution (ability to seperate components at closely spaced frequencies) of the DFT output?

A: Make $f_{analysis}$ smaller — $f_{analysis} = \frac{k f_s}{N}$

Make N larger (i.e. use more data)!

Note: Zero-padding does not improve resolution, it improves precision (via interpolation).

Rule of Thumb — To realize F_{res} Hz resolution, you must collect $1/F_{res}$ seconds of non-zero time samples.

- DFT processing gain — detecting signal energy embedded in noise.

- DFT output bin has $\frac{\sin(x)}{x}$ amplitude response.
- Can think of DFT output bin as bandpass filter with center freq. of $\frac{k f_s}{N}$
- Output mag. of DFT increases as N increases.
- DFT output bin main-lobe decreases as N increases.

Demo —

- Fast Fourier Transform

$$\text{DFT - } \quad X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nm}{N}}$$

Q: How many mults and adds?
A: N^2 mults, N adds

FFT is a fast method for computing DFT

$$\text{FFT} \propto \frac{N}{2} \cdot \log_2 N \quad \text{mults}$$

Example: $N = 512$

DFT requires $(512)^2 = 262{,}144$

FFT requires $\simeq 13{,}107$

20 x difference.

$N = 8192 \Rightarrow$ 1000 x difference.

$N = 2{,}097{,}152 \Rightarrow$ If FFT took 10 seconds DFT takes 3 weeks.