Discrete-Time Random Signals

Is a random process → sequence of random variables

Consists of infinite ensemble of discrete-time sequences.

Statistical Properties of Random Variable

Probability Distribution Function – probability that random variable X takes a value in range $-\infty$ to α

$$P_x(\alpha) = \text{Probability}[X \le \alpha]$$

probability density function

$$p_x(\alpha) = \frac{\partial P_x(x)}{\partial \alpha}$$

If X can assume continuous range of values

⇒ probability distribution function can also be written as

$$P_x(\alpha) = \int_{-\infty}^{\alpha} p_x(u)\,du$$

- Mean – $m_x = E(x) = \int_{-\infty}^{\infty} \alpha\, p_x(\alpha)\,d\alpha$ "expected value"

- mean-square value $E(x^2) = \int_{-\infty}^{\infty} \alpha^2 p_x(\alpha)\,d\alpha$

- variance - $\sigma_x^2 = E\left([X-m_x][X-m_x]^*\right)$

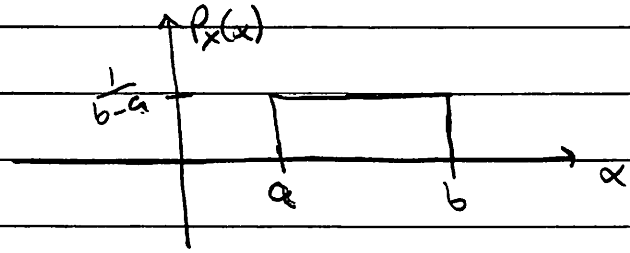
$$= \int_{-\infty}^{\infty} (\alpha-m_x)(\alpha-m_x)^* p_x(\alpha)\,d\alpha$$

$$= E(X^2) - |m_x|^2$$

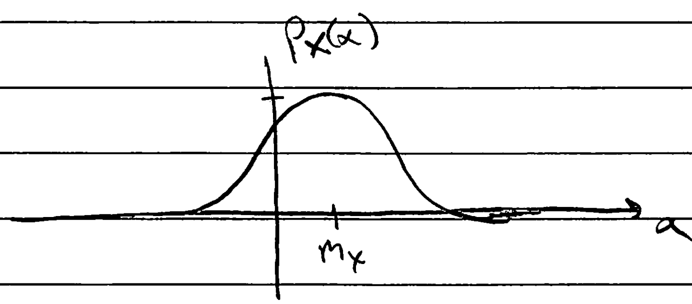
- standard deviation - $\sqrt{\sigma^2}$

- Two common probability density functions

- uniform $p_x(x) = \begin{cases} \frac{1}{b-a}, & a \le \alpha \le b \\ 0, & \text{otherwise} \end{cases}$



- Gaussian $p_x(\alpha) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-(\alpha-m_x)^2/2\sigma_x^2}$



- uniform distributed <u>real</u> random variable X

$$m_x = \frac{1}{b-a}\int_a^b \alpha\, d\alpha = \frac{1}{b-a}\,\frac{\alpha^2}{2}\Big|_a^b = \frac{1}{b-a}\left(\frac{b^2}{2}-\frac{a^2}{2}\right)$$

$$= \frac{1}{b-a}\cdot\frac{1}{2}(b-a)(b+a) = \frac{b+a}{2}$$

$$E(X^2) = \frac{1}{b-a}\int_a^b \alpha^2\, d\alpha = \frac{b^2+a^2+ab}{3}$$

$$\sigma_x^2 = E(x^2) - |m_x|^2 = \frac{b^2+a^2+ab}{3} - \left|\frac{b+a}{2}\right|^2$$

$$= \frac{b^2+a^2+ab}{3} - \left(\frac{b^2+a^2+2ab}{4}\right) = \frac{b^2+a^2-2ab}{12}$$

$$= \frac{(b-a)^2}{12}$$

- Two random variables X and Y

- Joint statistical properties are of interest

- Joint probability density function

$$p_{xy}(\alpha,\beta) = \frac{\partial^2 P_{xy}(\alpha,\beta)}{\partial\alpha\,\partial\beta}$$

- joint probability distribution function.

$$P_{xy}(\alpha,\beta) = \int_{-\infty}^{\alpha}\int_{-\infty}^{\beta} p_{xy}(u,v)\,du\,dv$$

- Joint statistical properties of X & Y are described by cross-correlation & cross-covariance

$$\phi_{xy} = E(XY^*) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha\beta^*\, p_{x,y}(\alpha,\beta)\,d\alpha\,d\beta$$

$$\gamma_{xy} = E\left([X-m_x][Y-m_y]^*\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\alpha-m_x)(\beta-m_y)^* p_{xy}(\alpha,\beta)\,d\alpha\,d\beta$$

$$= \phi_{xy} - m_x m_y^*$$

X & Y are uncorrelated if $E(XY) = E(X)E(Y)$
or linearly independent

statistically independent if $P_{xy}(\alpha,\beta) = P_x(\alpha)\,P_y(\beta)$

Note: If stat indep. then linearly indep

- Statistical properties of random signal

$x[n]$ is a random variable

$$m_{x[n]} = E(x[n]) = \int_{-\infty}^{\infty} \alpha\, p_{x[n]}(\alpha; n)\, d\alpha$$

$$E(x[n]^2) = \int_{-\infty}^{\infty} \alpha^2 p_{x[n]}(\alpha; n)\, d\alpha$$

$$\sigma^2_{x[n]} = E\left(\{x[n] - m_{x[n]}\}^2\right) = E(x[n]^2) - (m_{x[n]})^2$$

Functions of time index n. $\Rightarrow$ also a sequence

$$\phi_{xx}[m,n] = E(x[m]\, x^*[n]) \qquad \text{autocorrelation}$$

autocovariance $$\gamma_{xx}[m,n] = E\left((x[m] - m_{x[m]})(x[n] - m_{x[n]})^*\right)$$

$$= \phi_{xx}[m,n] - m_{x[m]}\, m^*_{x[n]}$$

cross correlation $$\phi_{xy}[m,n] = E(x[m]\, y^*[n])$$

cross-covariance $$\gamma_{xy}[m,n] = E\left((x[m] - m_{x[m]})(y[n] - m_{y[n]})^*\right)$$

$$= \phi_{xy}[m,n] - m_{x[m]}\, m^*_{y[n]}$$

In general, statistical properties of random discrete-time signal $\{X[n]\}$ is time-varying.

- Wide-sense stationary - mean $E(X[n])$ is constant with time index n, and autocorrelation and autocovariance depend only on difference in time indices m and n.

$$m_x = E(x[n]) \text{ for all } n$$

$$\phi_{xx}(\ell) = \phi_{xx}[n+\ell, n]$$

$$\gamma_{xx}[\ell] = \gamma_{xx}[n+\ell, n]$$

For WSS random process - $E(|x[n]|^2) = \phi_{xx}[0]$

$$\sigma_x^2 = \gamma_{xx}[0] = \phi_{xx}[0] - |m_x|^2$$

Cross correlation - $\phi_{xy}[\ell] = E(x[n+\ell]\, Y^*[n])$

$$\gamma_{xy}[\ell] = E\left((x[n+\ell] - m_x)(Y[n] - m_y)^* \right)$$

$$= \phi_{xy}[\ell] - m_x m_y^*$$

- Symmetry properties

$$\phi_{xx}[-\ell] = \phi_{xx}^*[\ell] \qquad \phi_{xy}[-\ell] = \phi_{xy}^*[\ell]$$

$$\gamma_{xx}[-\ell] = \gamma_{xx}^*[\ell] \qquad \gamma_{xy}[-\ell] = \gamma_{xy}^*[\ell]$$

- Power of random signal

Deterministic signal - $P_x = \lim\limits_{K\to\infty} \frac{1}{2K+1} \sum\limits_{n=-K}^{K} |x[n]|^2$

random signal - $P_x = E\left(\lim\limits_{N\to\infty} \frac{1}{2N+1} \sum\limits_{n=-N}^{N} |x[n]|^2 \right)$

or $P_x = \lim\limits_{N\to\infty} \frac{1}{2N+1} \sum\limits_{n=-N}^{N} E\left(|x[n]|^2\right)$

For WSS signal - $P_x = E\left(|x[n]|^2\right)$

$$\Rightarrow P_x = \phi_{xx}[0] = \sigma_x^2 + |m_x|^2$$

- Ergodic Signal

Finite portion of single realization of random signal is available -

Q: How do we estimate statistical properties of ensemble?

ergodicity - A stationary random signal is an ergodic signal if all its properties can be estimated from a single realization of sufficient length.

$\Rightarrow$ time averages = ensemble averages

$$m_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x[n]$$

$$\sigma_x^2 = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} (x[n]-m_x)^2$$

Estimates $\Rightarrow$

$$\hat{m}_x = \frac{1}{M+1} \sum_{n=0}^{M} x[n]$$

$$\hat{\sigma}_x^2 = \frac{1}{M+1} \sum_{n=0}^{M} (x[n]-m_x)^2$$

- DTFT representations.

$$\Phi_{xx}(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} \phi_{xx}[\ell] e^{-j\omega \ell} \qquad |\omega| < \pi$$

"power spectrum" $= P_{xx}(\omega)$ → real valued function of ω.

Power spectrum exists if autocorrelation is absolutely summable.

Likewise –

$$\phi_{xx}[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(\omega) e^{j\omega \ell} \, d\omega$$

$\phi_{xx}[0]$ = average power in random signal $x[n]$

If $\{x[n]\}$ is real random signal ⇒ $P_{xx}(\omega) = P_{xx}(-\omega)$

Cross power spectrum –

$$\Phi_{xy}(e^{j\omega}) = P_{xy}(\omega) = \sum_{\ell=-\infty}^{\infty} \phi_{xy}[\ell] e^{-j\omega \ell} \qquad |\omega| < \pi$$

- White Noise

$X[m]$ and $X[n]$ are uncorrelated when $m \neq n$.

$$\phi_{xx}[\ell] = \sigma_x^2 \delta[\ell] + m_x^2$$

$$P_{xx}(\omega) = \sigma_x^2 + 2\pi m_x^2 \delta(\omega) \quad |\omega| < \pi$$