- see windowing_demo.m

- Laplace Transform - Tool for solving linear diff. Eq. (continuous time)

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

general solution to linear diff eq.

causal systems

$$s = \sigma + j\omega$$

Laplace transform is a continuous function, where the complex value of the function for a particular value of s is a correlation of

$$\langle f(t), e^{-st} \rangle$$

damped exponential w/ freq. = ω

$$e^{-st} = e^{-(\sigma + j\omega)t} = e^{-\sigma t} e^{-j\omega t} = \frac{e^{-j\omega t}}{e^{\sigma t}}$$

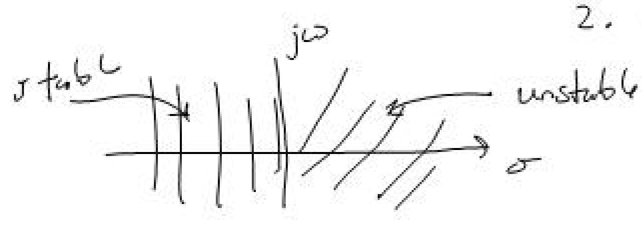
$\Rightarrow$ $e^{-j\omega t}$ is a unity mag. phasor rotating clockwise at ω.

$e^{\sigma t}$ is a real value, if $\sigma > 0$ it gets larger w/ time.

$$H(s) = \frac{Y(s)}{X(s)}$$ "Transfer Function"

$|H(s)|$ Tells us: 1. Is system stable?

2. What is freq. Response?



- z-Transform (discrete-time)

$$x[n] \longrightarrow \boxed{LTI}_{h[n]} \longrightarrow y[n]$$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Let $x[n] = z^n$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

$$= z^n H(z)$$

Transfer Function

– Polar Form

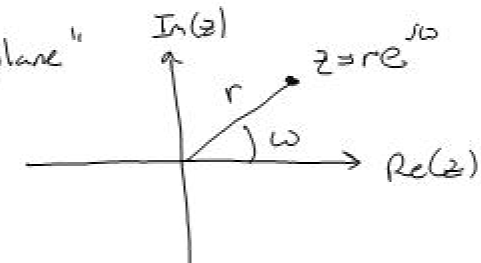
$$H(z) = |H(z)|\, e^{j\phi(z)}$$

Therefore, system output is

$$y[n] = z^n\, |H(z)|\, e^{j\phi(z)}$$

Use $z = re^{j\omega}$ = general complex number

"z-plane"



Compare input $x[n] = z^n$ with output:

system modifies amplitude of input by $|H(re^{j\omega})|$ and shifts phase by $\phi(re^{j\omega})$.

- z-transform of arbitrary signal $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \text{where } z = Re(z) + j\, Im(z)$$

↑ CT complex variable.

Inverse z-transform is defined ~~as~~ as

$$x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$$

- Why use z-transform?

Recall - system is stable only if impulse response is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

and, DTFT only exists for stable systems.

<u>Cool fact #1</u> - z-transform can be used to analyze unstable LTI systems!!

$h[n] \xrightarrow{DTFT} H(e^{j\omega})$ — "Frequency Response"

$h[n] \xrightarrow{z} H(z)$ — "Transfer Function"

z-transform

- Another look at z-transforms

  Let $z = re^{j\omega}$

$$X(z) = X(re^{j\omega}) = \underbrace{\sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n}}$$

This is DTFT of $x[n]r^{-n}$

Notice that for $r=1$, the z-transform reduces to DTFT.

- Convergence - when does $X(z)$ converge?

  - For a given sequence, the set R of values of z for which its z-transform converges is called the <u>Region of Convergence</u>, or ROC

$$\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]\, r^{-n}| < \infty$$

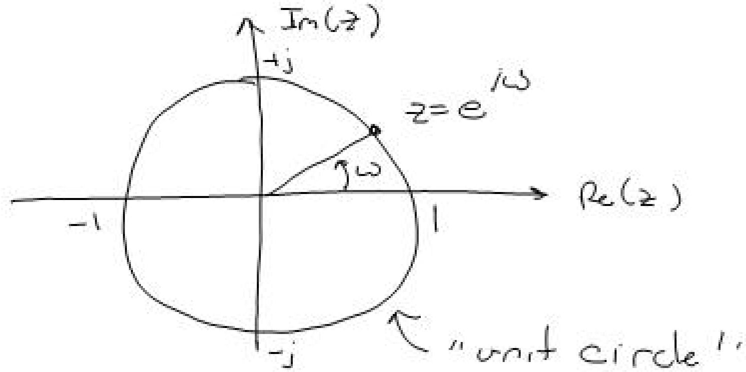
Recall, DTFT required absolute summability of $x[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty .$$

Now, we use restricted values of r to ensure that $x[n]r^{-n}$ is absolutely summable.

<u>Cool Fact #2</u> - We can work with signals that do not have a DTFT!

- z-plane - complex plane for plotting $z = re^{j\omega}$



Note: $X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$ when $r=1$

DTFT is z-transform evaluated on unit circle.

<u>Example</u> $x[n] = \begin{cases} 1, & n=-1 \\ 2, & n=0 \\ -1, & n=1 \\ 1, & n=2 \\ 0, & \text{otherwise} \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = z^{+1} + 2 - z^{-1} + z^{-2}$$

What is DTFT?

$$X(e^{j\omega}) = e^{j\omega} + 2 - e^{-j\omega} + e^{-2j\omega}$$

Comment: That was easy!

- Poles and Zeros

Common form of z-transform output -

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$X(z) = \frac{\tilde{b} \prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \qquad \text{where } \tilde{b} = b_0/a_0$$

$c_k \longrightarrow$ roots of numerator, called "zeros"

$d_k \longrightarrow$ roots of denominator, called "poles"

<u>Example</u>: $x[n] = \alpha^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z} \right)^n$$

← geometric series

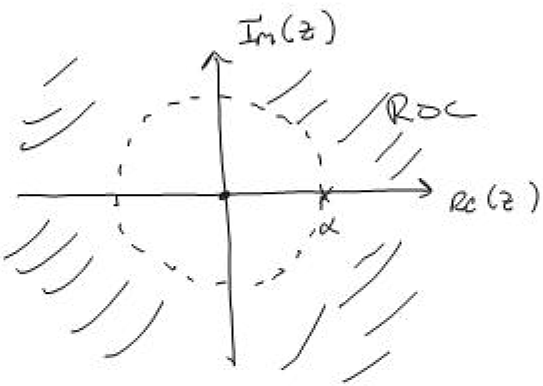
$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha|$$

$$= \frac{z}{z-\alpha} \qquad |z| > |\alpha|$$

Notice: Pole at $z = \alpha$

zero at $z = 0$

ROC is $|z| > |\alpha|$



- Rules of Thumb – system = filter

  If poles are located inside the unit circle, the filter is stable. If any pole is located outside the unit circle, the filter is unstable.

- Common geometric series, β is complex number.

$$\sum_{n=0}^{M-1} \beta^n = \begin{cases} \dfrac{1-\beta^M}{1-\beta}, & \beta \neq 1 \\ M, & \beta = 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta}, \quad |\beta| < 1$$

$$\sum_{n=0}^{\infty} n\beta^n = \frac{\beta}{(1-\beta)^2}, \quad |\beta| < 1$$

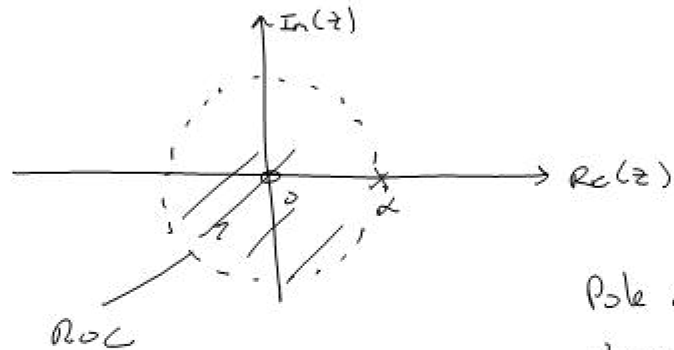
Example $x[n] = -\alpha^n u[-n-1]$ anticausal exponential sequence

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1]\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n \qquad \text{Let } n=-k$$

$$= -\sum_{-k=-\infty}^{+1} \left(\frac{\alpha}{z}\right)^{-k} = -\sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k = 1-\sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k$$

$$= 1 - \frac{1}{1-z\alpha^{-1}} = \frac{z}{z-\alpha} \qquad |z| < |\alpha|$$



Pole at $z=\alpha$
zero at $z=0$

Note: We must know ROC to determine the correct inverse z-transform.

- Properties of ROC

  - Cannot contain any poles.
  - The ROC for a finite-duration signal includes entire z-plane, except possibly $z=0$ or $|z|=\infty$, or both.
  - ROC for right-sided sequences $(M \le n < \infty)$ is exterior to circle outside furthest pole.
  - ROC for left-sided sequences $(-\infty \le n < M)$ is interior to circle inside nearest pole.

- Properties of z-transform

  - Linearity $\quad ax[n] + by[n] \xrightarrow{z} aX(z) + bY(z)$

    w/ ROC = $R_x \cap R_y$

    "Intersection"

  - Time-reversal $\quad x[-n] \xrightarrow{z} X(\frac{1}{z})$

    w/ ROC = $\frac{1}{R_x}$

  - Time shift $\quad x[n-n_o] \xrightarrow{z} z^{-n_o} X(z)$

    w/ ROC = R_x except possibly at $z=0$ or $|z| = \infty$

    note: mult. by z^{-n_o} introduces pole of order n_o at $z=0$ if $n_o > 0$.

  - Convolution $\quad x[n] * y[n] \xrightarrow{z} X(z)Y(z) \quad$ ROC $\approx R_x \cap R_y$

  - Mult. by exponential $\quad \alpha^n x[n] \xrightarrow{z} X(\frac{z}{\alpha}) \quad$ ROC = $|\alpha| R_x$

  - Differentiation $\quad n\,x[n] \xrightarrow{z} -z \frac{d}{dz} X(z) \quad$ ROC = R_x

- Parseval's Relation.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi j} \oint X(z) X(z^{-1}) z^{-1} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

- FIR Digital Filter

$$y[n] = \sum_{k=N_1}^{N_2} h[k]\, x[n-k]$$

$h[n]$ is of finite length $N_1 \leq n \leq N_2$

Transfer function is

$$H(z) = \sum_{n=N_1}^{N_2} h[n] z^{-n}$$

"Causal" FIR Filter if $0 \leq N_1 \leq N_2$

Note: All poles of $H(z)$ are at the origin in z-plane

$\Rightarrow$ ROC is entire z-plane, excluding $z=0$.

<u>Example</u>: MA Filter.

$$H(z) = \frac{1}{M} \sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M(z^{M-1}(z-1))}$$

Notice -

M zeros on unit circle.

$M-1^{th}$ order pole at $z=0$

$\Rightarrow$ Pole at $z=1$ cancels w/ zero at $z=1$
"pole-zero cancellation".