- Output of any LTI system is the convolution of the input with the system impulse response.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$y[n] = x[n] * h[n] = \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]}_{\text{by commutative property of convolution}}$$

- Involves 4 steps:

1. $h[k]$ is time-reversed (reflected about origin) and shifted in time by n to form $h[n-k]$.
2. $x[k]$ and $h[n-k]$ are multiplied together for all values of k with n fixed.
3. All product outputs from step 2 are summed to produce a single output sample.
4. Steps 1 to 3 are repeated for all n values to produce output sequence $y[n]$.

<u>Example</u>

$$x[n] = \delta[n] + 2[n-2] \qquad h[n] = \delta[n-1]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

1st form is easy

$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k-1] = x[k-1]$$

$$= \delta[n-1] + 2[n-3]$$

Used property $\quad x[n] * \delta[n-n_0] = x[n-n_0]$

time-shifted

- Important convolution properties to know when using special functions.

a) $$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$$

b) $$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k-n_0] = x[n-n_0]$$

c) $$x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

d) $$x[n] * u[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

- Matlab convolution demo

- Notation Review

(CT) $x(t) = \cos(\Omega t + \phi)$, $\Omega = 2\pi F$

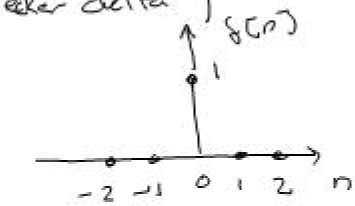
(DT) $x[n] = \cos(\omega n + \phi)$, $\omega = 2\pi f$

$\omega = \frac{\Omega}{F_s}$ and $f = \frac{F}{F_s}$ Normalized by sample frequency

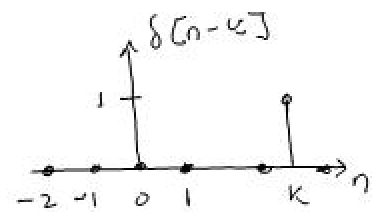
- Special Function Review (DT only)

- Delta Function (or unit impulse, kronecker delta)

$$\delta[n] = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases}$$

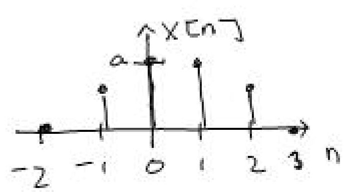


$$\delta[n-k] = \begin{cases} 1 & , n = k \\ 0 & , n \neq k \end{cases}$$

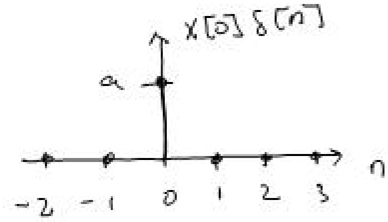


Properties -

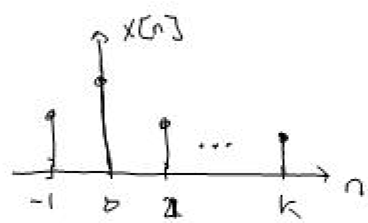
- $x[n]\,\delta[n] = x[0]\,\delta[n]$



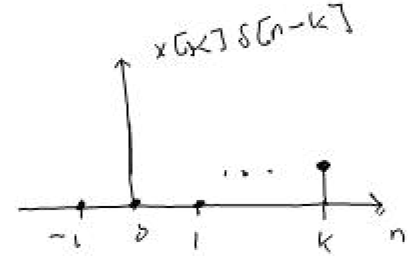
$\Longrightarrow$ $x[n]\,\delta[n]$ $\Longrightarrow$



- $x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$

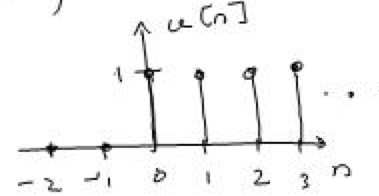


$\Longrightarrow$ $x[n]\,\delta[n-k]$ $\Longrightarrow$

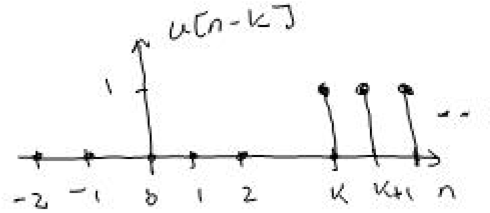


- unit step function (or step function)

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



$$u[n-k] = \begin{cases} 1, & n \geq k \\ 0, & n < k \end{cases}$$



- Relationship between $\delta[n]$ and $u[n]$

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

- Other important properties

$$\delta[an] = \delta[n] \qquad \text{for } |a| > 0$$

$$\sum_{k=n_1}^{n_2} x[n]\,\delta[n-n_0] = \begin{cases} x[n_0], & \text{if } n_1 \leq n_0 \leq n_2 \\ 0, & \text{otherwise} \end{cases}$$

Correlation

- Used to compare a reference signal with one or more signals. Provides a measure of the similarity between signals.

Important! Correlation is very important to understand. Used in communications, radar, sonar, ... everything!

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-\ell] \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

"lag"

or

$$r_{yx}[\ell] = \sum_{n=-\infty}^{\infty} y[n]\, x[n-\ell]$$

"Cross Correlation"

Note Property – $r_{yx}[\ell] = r_{xy}[-\ell]$

- Autocorrelation –

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-\ell]$$

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x$$

and

$$r_{xx}[\ell] = r_{xx}[-\ell]$$

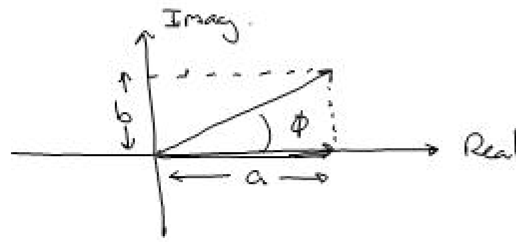
Even Function

- Looks like convolution

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, y[-(\ell-n)] = x[\ell] * y[-\ell]$$

- Complex valued samples and Complex Signals

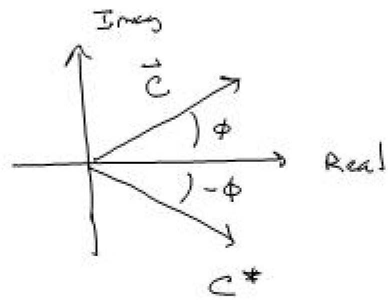
- Complex samples $\quad c = a + jb \quad$ where $j^2 = -1$



$$a = |c| \cos\phi$$
$$b = |c| \sin\phi$$
$$|c| = \sqrt{a^2 + b^2} \quad \text{"Magnitude"}$$
$$\phi = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{"phase"}$$

$$\Rightarrow \quad c = |c| \left(\cos\phi + j \sin\phi \right)$$

Q: What does c^* look like?



$$c^* = a - jb$$
$$\Rightarrow \quad c^* = |c| \left(\cos\phi - j \sin\phi \right)$$

- Remember that $e^{j\phi} = \cos\phi + j\sin\phi$

so, $\quad c = |c| e^{j\phi}$

- Complex signals

$$x[n] = x_{re}[n] + j\, x_{im}[n]$$

$$|x[n]| = \sqrt{x_{re}^2[n] + x_{im}^2[n]} \quad \text{"Magnitude"}$$

$$\phi[n] = \tan^{-1}\left(\frac{x_{im}[n]}{x_{re}[n]} \right) \quad \text{"phase"}$$

- Polar Form

$$x[n] = |x[n]| \, e^{j\phi[n]}$$

magnitude ($|x[n]|$), phase ($\phi[n]$)

Likewise, $x^*[n] = x_{re}[n] - j\,x_{im}[n]$

- Extracting real & imag parts of a signal

$$\mathrm{Re}\{x[n]\} = \frac{x[n] + x^*[n]}{2} = x_{re}[n]$$

$$\mathrm{Im}\{x[n]\} = \frac{x[n] - x^*[n]}{2j} = x_{Im}[n]$$

Example Let $x[n] = e^{j\omega n}$

$$\mathrm{Re}\{e^{j\omega n}\} = \frac{e^{j\omega n} + e^{-j\omega n}}{2} = \cos(\omega n)$$

$$\mathrm{Im}\{e^{j\omega n}\} = \frac{e^{j\omega n} - e^{-j\omega n}}{2j} = \sin(\omega n)$$

- 4-way decomposition of complex signals

$$x[n] = \underbrace{x^e_{re}[n]}_{\text{real \& even}} + \underbrace{x^o_{re}[n]}_{\text{real \& odd}} + j\,\underbrace{x^e_{Im}[n]}_{\text{imag \& even}} + j\,\underbrace{x^o_{In}[n]}_{\text{imag \& odd}}$$

Even $x^e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$

odd $x^o[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$

- Arbitrary Signal & LTI systems

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

superposition — scales — delayed impulses — delayed impulse responses

We can consider $\delta[n-k]$ as the signal basis and $x[k]$ as weights, or scale factors.

Q: What if we use a complex exponential as the basis signal instead of the impulse function?

- Remember $$Ae^{j(\omega n+\phi)} = \underbrace{A\cos(\omega n+\phi)}_{\text{real}} + j\underbrace{A\sin(\omega n+\phi)}_{\text{imag}}$$

Lets probe an LTI system with a complex exponential

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Let $x[n] = e^{j\omega n}$

$$\Rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h[k]\,e^{j\omega n}e^{-j\omega k}$$

$$= e^{j\omega n}\underbrace{\sum_{k=-\infty}^{\infty} h[k]\,e^{-j\omega k}}$$

Call this $H(e^{j\omega})$

What happened -

$$e^{j\omega n} \longrightarrow \boxed{h[n]} \longrightarrow H(e^{j\omega})\,e^{j\omega n}$$

input — Scaling (complex) — Input reproduced

- DTFT

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}$$

or Let $k \Rightarrow n$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} \qquad \text{(DTFT)}$$

This is a Fourier series expansion of the periodic function $H(e^{j\omega})$.

- Using a complex exponential as the input to an LTI system turned convolution into multiplication.

$$y[n] = x[n] * h[n] = e^{j\omega n} * h[n] = H(e^{j\omega})\, e^{j\omega n}$$

"Frequency Response" or "spectrum"

The spectrum can be computed for any input discrete time sequence.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Example Let $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$$

Spectrum of an impulse is a constant.

- Polar Form

$$H(e^{j\omega}) = \left| H(e^{j\omega}) \right| e^{j\phi(\omega)}$$

where

"magnitude spectrum" $\left| H(e^{j\omega}) \right| = \left(\mathrm{Re}^2\{H(e^{j\omega})\} + \mathrm{Im}^2\{H(e^{j\omega})\} \right)^{\frac{1}{2}}$

"phase spectrum" $\phi(\omega) = \tan^{-1}\left(\dfrac{\mathrm{Im}\{H(e^{j\omega})\}}{\mathrm{Re}\{H(e^{j\omega})\}} \right)$

- Remember: $H(e^{j\omega})$ is a continuous-time complex valued function of the real variable ω.

- Inverse DTFT or IDTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Only need to compute integral over frequency range of 2π. Notice structure -

$$\int_{-\pi}^{\pi} \underset{\uparrow}{X(e^{j\omega})} \underbrace{\frac{1}{2\pi} e^{j\omega n} d\omega}_{\text{basis}}$$

supperposition weights basis

$x[n] \longrightarrow X(e^{j\omega})$ called "Analysis"

$X(e^{j\omega}) \longrightarrow x[n]$ called "Synthesis".

- Continuous-time Fourier Transform (CTFT)

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

note: frequencies are not bounded!

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$x_a(t) \xleftrightarrow{CTFT} X_a(j\Omega) \qquad -\infty < \Omega < \infty$$

- Polar Form

$$X(j\Omega) = |X(j\Omega)|\, e^{j\phi_a(\Omega)}$$

- When does the CTFT exist?
  - Dirichlet Conditions:

    a.) Signal has finite number of discontinuities & finite number of maxima and minima in any finite interval.

    b.) Signal is absolutely summable $\int_{-\infty}^{\infty} |x_a(t)|\, dt < \infty$

<u>Example</u> $x_a(t) = \delta(t)$

$$X_a(j\Omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\Omega t}\, dt = 1$$

- DTFT properties

$X(e^{j\omega})$ is a complex valued function of the real variable ω.

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j\, X_{im}(e^{j\omega})$$

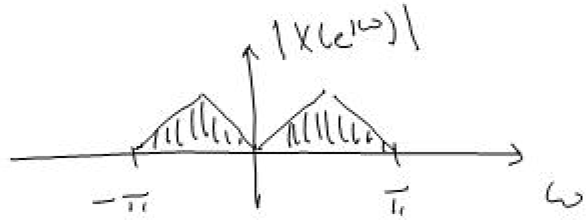
$X(e^{j\omega})$ is conjugate symmetric if

$$\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\} \qquad \text{Even}$$

$$\mathrm{Im}\{X(e^{j\omega})\} = -\,\mathrm{Im}\{X(e^{-j\omega})\} \qquad \text{Odd}$$

For real signals -

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$



- Parseval's Relation for finite energy CT signals

$$\int_{-\infty}^{\infty} |x_a(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2\, d\Omega$$

$|X_a(j\Omega)|^2$ = magnitude spectrum squared

= "Energy Density Spectrum"