- Causality & stability

Impulse response of causal LTI system is zero for $n < 0$.

- A pole inside unit circle contributes an exponentially decaying term to impulse response.

- A pole outside unit circle contributes an exponentially increasing term $\Rightarrow$ unstable

- A pole on the unit circle contributes a complex sinusoid.
$\Rightarrow$ unstable or $\sum_{n=-\infty}^{\infty} |h[n]| \not< \infty$

Note: If a system is stable, and causal, $h[n]$ is absolutely summable

$$\sum_{n=0}^{\infty} |h[n]| < \infty,$$ and the DTFT exists.

Therefore, the ROC for $H(z)$ must include the unit circle.

<u>Example</u> Investment Computation

$$y[n] - \rho\, y[n-1] = x[n]$$

where $\rho = 1 + r/100$

r = interest rate/period

Take z-transform

$$Y(z) - \rho z^{-1} Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\rho z^{-1}} = \frac{z}{z-\rho}$$

pole at $z = \rho = 1 + r/100$

zero at $z = 0$

$\Rightarrow$ System cannot be stable and causal

- Inverse systems $-$ $h^{inv}[n] * h[n] = \delta[n]$

$$H^{inv}(z) \cdot H(z) = 1 \quad \Rightarrow \quad H^{inv}(z) = \frac{1}{H(z)}$$

Note: The zeros of $H(z)$ are poles of $H^{inv}(z)$
The poles of $H(z)$ are zeros of $H^{inv}(z)$

A stable/causal $H^{inv}(z)$ is used to reverse the distortion introduced by $H(z)$.

Note: A stable and causal $H^{inv}(z)$ only exists if all the zeros of $H(z)$ are inside unit circle.

<u>Example</u> Multipath Communications Channel

$$y[n] = x[n] + a\,x[n-1]$$

$$H(z) = 1 + az^{-1} \quad \Longrightarrow \quad \begin{array}{l} \text{zero at } z = -a \\ \text{pole at } z = 0 \end{array}$$

so,

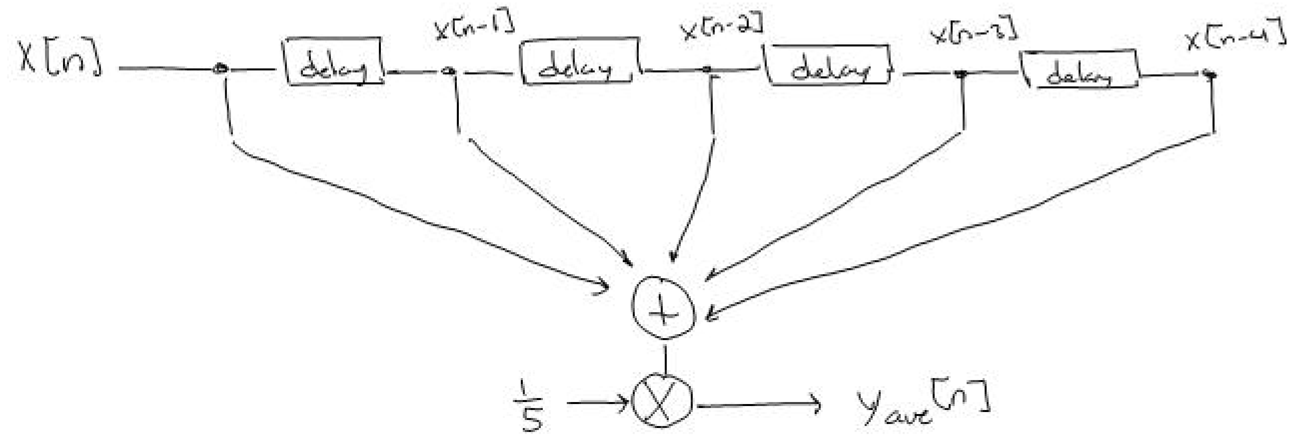
$$H^{inv}(z) = \frac{1}{1+az^{-1}} \qquad |a| \text{ must be } < 1$$

- Introduction to FIR filters
  - use only current and past input samples, and no previous output samples to compute current output sample.

  "nonrecursive"

Averaging filter - $y_{ave}[n] = \frac{1}{5} \sum_{k=n-4}^{n} x[k]$

$$= \frac{1}{5}\left[x[n-4] + x[n-3] + x[n-2] + x[n-1] + x[n] \right]$$



Each tap could be multiplied by different coefficent.

Two factors affect FIR filters' frequency response:

1. number of taps
2. specific coefficient values

Remember convolution - can be used to find filter output

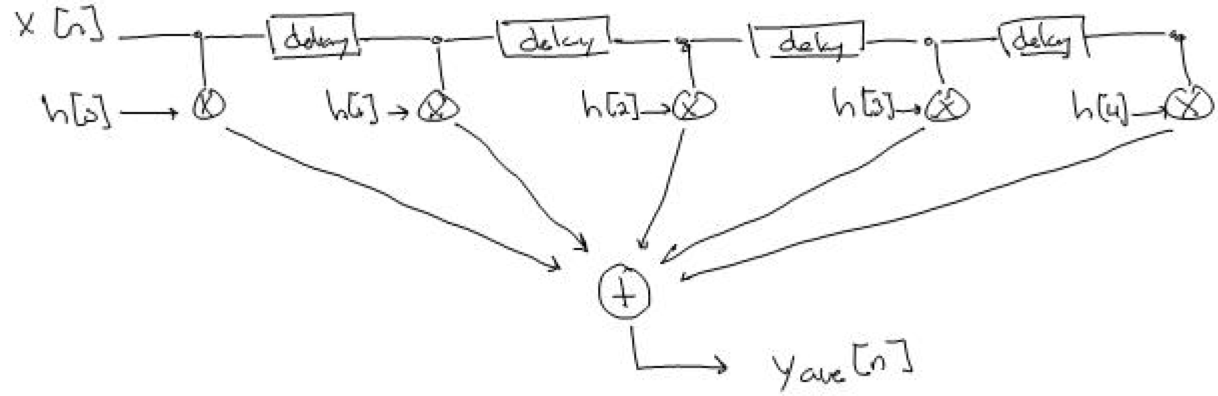
$$y_{ave}[n] = h[4]\,x[n-4] + h[3]\,x[n-3] + h[2]\,x[n-2] + h[1]\,x[n-1] + h[0]\,x[n]$$

$$= \sum_{k=0}^{4} h[k]\,x[n-k] = h[n] * x[n]$$

$$\xleftrightarrow{\text{DTFT}} H[k]\,X[k]$$

Demo - fir-demo.m

What happens if we change the coefficients?



Three things to notice -

1 - different coeff. give different freq. magnitude response.

2 - Sudden changes in coeff. values causes sidelobes in freq. response.

3 - If we minimize sudden changes in coeff. we reduce side lobes, but increase mainlobe of lowpass filter.

Note: Adaptive filters have coeff. that change with time.

Q: How many mults & adds does FIR filter have?

mults = N where N = # of taps

Adds = N

- FIR group delay - $\tau_g = -\frac{\Delta\Phi(\omega)}{\Delta f}$ = slope of arg{H}

$$= -\frac{d\,\theta(\omega)}{d\omega}$$

For FIR filters group delay is constant over passband if coeff. are symmetrical = All freq. components are delayed by equal amount. "No phase distortion".

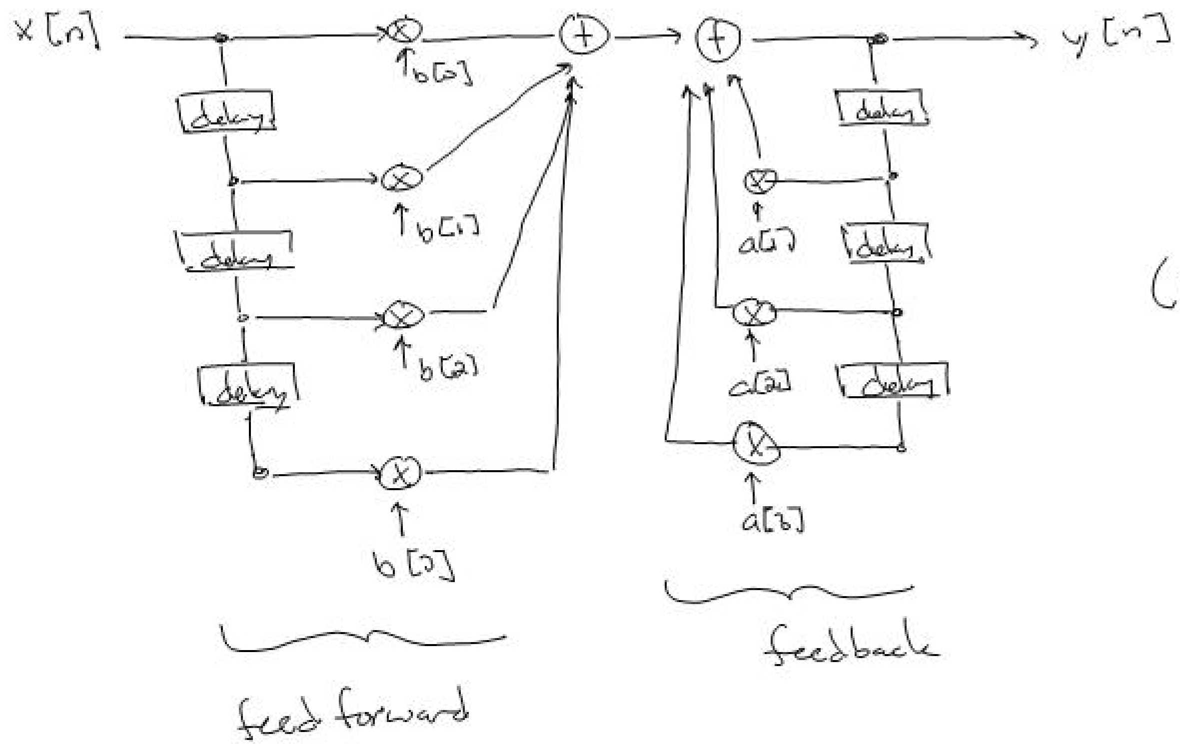
Delay through FIR filter w/ S taps is $G = \frac{(S-1)T_s}{2}$ where T_s is sample period.

- Introduction to IIR filters.
  Practical IIR filters always require feedback. Each filter output depends on previous input samples and previous output samples (has memory).

IIR characteristics -
- have more complicated structure
- harder to design
- do not have linear phase response

Q: Why use them?
A: They are more efficient than FIR filters. - require fewer mults and adds per output sample.

IIR structure - 4 Tap



(Direct Form I)

Bad News! Cannot determine $a[k]$ & $b[k]$ coeff. from impulse response.

Good News! We know how to use the z-transform!

- Using z-transform to analyze IIR filters

$$x[n] \longrightarrow \boxed{\text{delay}} \longrightarrow y[n] = x[n-1]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n-1]\, z^{-n}$$

Let $k = n-1$

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-(k-1)} = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k} z^{-1} = z^{-1} \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

$$= z^{-1} X(z)$$

so, $x[n] \longrightarrow \boxed{\text{delay}} \longrightarrow y[n]$

$\Rightarrow$ $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n]$

We can write the IIR transfer function $H(z)$ by inspecting the IIR filter structure, or difference equation.

From $H(z)$ we can determine freq. Response & stability

From the previous IIR structure -

$$y[n] = b_0 x[n] + b_1 [n-1] + b_2 x[n-2] + \dots + b_N x[n-N] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_M y[n-M]$$

so,

$$Y(z) = b_0 X(z) + b_1 X(z) z^{-1} + b_2 X(z) z^{-2} + \dots + b_N X(z) z^{-N} + a_1 Y(z) z^{-1} + a_2 Y(z) z^{-2} + \dots + a_M Y(z) z^{-M}$$

$$Y(z) = X(z) \sum_{k=0}^{N} b_k z^{-k} + Y(z) \sum_{k=1}^{M} a_k z^{-k}$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 - \sum_{k=1}^{M} a_k z^{-k}}$$

Freq. Response is $\left. H(z) \right|_{z=e^{j\omega}} = H(e^{j\omega}) = \dfrac{\sum_{k=0}^{N} b_k e^{-jk\omega}}{1 - \sum_{k=0}^{M} a_k e^{-jk\omega}}$