Analytic Signal Generation—Tips and Traps

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***Abstract*—In this correspondence we discuss methods to produce the discrete analytic signal from a discrete real-valued signal. Such an analytic signal is complex and contains only positive frequencies. Its projection onto the real axis is the same as the original signal. Our use stems from instantaneous-frequency estimation and time-frequency signal analysis problems. For these problems the negative frequency component of real signals causes unwanted interference. The task of designing a filter to produce an approximation to the ideal analytic signal is not as simple as its formulation might suggest. Our result is that the direct methods of zeroing the negative frequencies, or using Hilbert transform filters, have undesirable defects. We present an alternative which is similar to the "quadrature" filters used in modem designs.**

I. Introduction: The Problem

Considerable work has been reported discussing time–frequency distributions [1]–[4] and instantaneous frequency (IF) estimation [5]. The methods detailed in these works all assume the signal is analytic [1]. An analytic signal $z(t)$ is, by definition, a signal with no negative frequency components with Fourier transform $Z(f) = \mathcal{F}[z(t)] = 0$ for $f < 0$. For an analytic signal $z(t)$ associated with a real signal $s(t)$ (i.e., the *complex envelope* of the signal $s(t)$), then $s(t) = \Re[z(t)]$.

Most practical signals are purely real, a condition that implies a corresponding negative frequency component for every positive frequency component, and so are not analytic ($S(f) = \mathcal{F}[s(t)] = S^*(-f)$, for real $s(t)$).

A method is required which accurately generates analytic signals from real signals and so suppresses the negative frequency components while preserving the positive components.

The paper reviews the methods currently used to generate analytic signals, and emphasizes their weaknesses. Examples of the methods are compared with the new method. Practical issues regarding the new method, such as parameter selection, are discussed in the final section.

II. The Direct Solution

From the statement of the problem, the direct solution is to set the negative frequency components to zero by first transforming the signal into the frequency domain. The desired complex signal is then produced by an inverse discrete Fourier transform (DFT) [1]. There are problems with this approach.

The first problem is that the whole signal must be operated on at once, which makes this approach inapplicable for real-time applications, or those that use large data sets. It is not possible to perform this operation on nonoverlapping segments of the signal, the discontinuities at the ends of each segment will introduce negative frequencies. In the time domain, the effect is the same as performing circular convolution with a (modulated) sinc function that is the same length as the signal itself. This has the drawbacks that simple-minded filter design introduce, such as poor suppression of side lobes and poor approximation to an ideal frequency response. Even if the whole signal is available, any analysis that assumes zero values outside the data record may introduce negative frequency effects.

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III. The FIR Hilbert Filter Implementation

Oppenheim and Schafer [7] explain that the analytic signal is related to the real signal by the relation $z(n) = s(n) + j\mathcal{H}[s(n)]$, where $z(n)$ is the analytic signal, $s(n)$ is the real signal, $\mathcal{H}[\cdot]$ is the Hilbert transform operator, and $j^2 = -1$.

This formulation can be used to build an approximation to an analytic signal generator by designing an FIR (Finite Impulse Response) filter approximation to the Hilbert transform operator [8]. The analytic signal can be computed by adding the appropriately time-shifted real signal to the imaginary part generated by the Hilbert filter. Since an ideal Hilbert filter impulse response extends infinitely in both directions, approximations are necessary. We discuss some relevant aspects of these approximations below.

An FIR filter can be designed by appropriately windowing the ideal impulse response:

$$h(n) = \begin{cases} \dfrac{2}{\pi}\dfrac{\sin^2(\pi n/2)}{n}, & n \neq 0 \\ 0, & n = 0. \end{cases} \tag{1}$$

The window selected will have an effect on the response of the filter, and the performance may not be as good as that of a filter designed using an "optimal" technique.

The Parks-McClellan algorithm is the most popular technique for optimal (in the mini-max or equiripple or H_∞ sense) filter design, and it can be used to design Hilbert filters [7]. There are a few practical traps, though. Firstly, a general note about the FIR Hilbert filter approach. If a Type III (even impulse response length) FIR Hilbert filter is designed, then the group delay will not be a whole number of samples. The real signal will need to be re-sampled at the points halfway between the existing samples before it can be added to the generated imaginary part. This problem does not occur for type IV (odd length) Hilbert filters, but these must be designed to be band-pass, rather than high-pass. The authors have also experienced problems with convergence problems in Remez exchange-based implementations of Parks-McClellan algorithms for type IV Hilbert filters of more than 127 taps that prevented the successful design of this type of filter.

It should also be noted that if this method is used, the real part of the signal will be unfiltered, while the imaginary part has been band-pass filtered. The consequence of this is that the resulting complex signal will not be analytic for frequencies outside the pass-band (near $f = 0$ and $f = f_s/2$), where significant components can exist in the real part. This problem is avoided in the new approach presented below.

IV. The Complex Filter Approach

A. The Algorithm

The first step in the filter design approach we present here is to design a low-pass filter of the correct bandwidth with an "optimum" method, such as the Parks-McClellan algorithm. The

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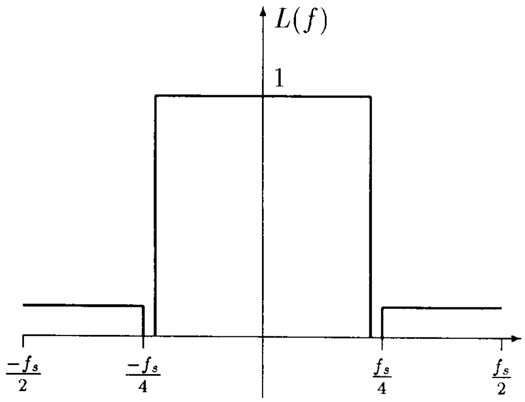

Fig. 1. Diagram of bands to specify to filter design program. Note relationship to $\pm f_s/4$.

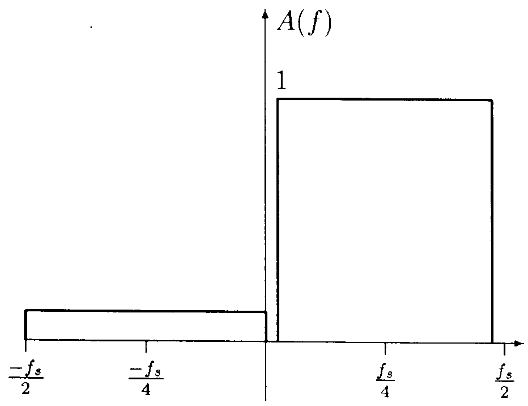

Fig. 2. Diagram of filter after modulation by $f_s/4$. Note relationship of stop band to negative frequencies.

bandwidth must be such that the stop band extends from $\pm f_s/4$ (where f_s is the sampling frequency), not the end of the pass band. This ensures that there will be a minimum of signal energy passed in the negative frequency band. Call the transfer function of this filter "$L(f)$", and its impulse response "$l(n)$". See Fig. 1 for clarification.

These low-pass filter coefficients are modulated by a complex exponential of frequency $f_s/4$, so that the stop band covers the whole of the negative frequency spectrum: $a(n) = e^{-j2\pi n\frac{f_s}{4}}\cdot l(n)$, $A(f) = L(f-\frac{f_s}{4})$. See Fig. 2 for clarification. The analytic signal is computed directly from the real input signal with the resulting complex FIR filter: $z(n) = s(n) * a(n)$.

This method is related to the "quadrature mirror filters" used extensively by modem designers (see 6.4 of [9]). Note that with this method both even and odd length filters can be produced. Both produce output signals with the desired property of purely positive frequency content. The even length filter will have a noninteger group delay, but both real and imaginary parts suffer the same time shift. This will result in a corresponding re-sampling of the real part, but this effect can be easily accounted for in some applications that use the analytic signal.

In comparison to the direct FIR Hilbert filter approach, the same band-pass filter has been applied to both real and imaginary components of the result, and so there are no signal frequencies that can produce a nonanalytic result.

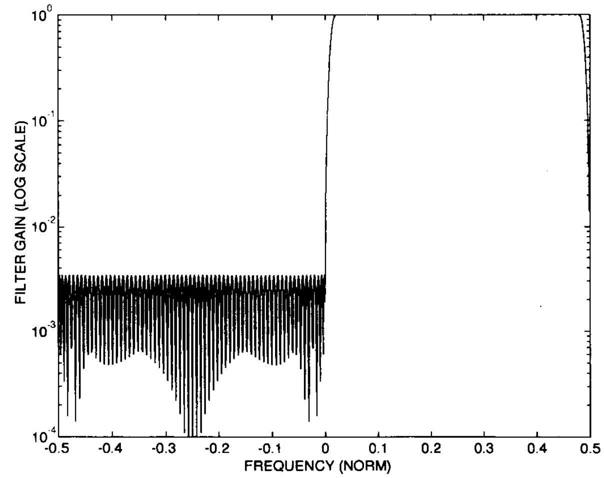

Fig. 3. Log-linear plot of frequency response of 128 tap filter. Note that stop band covers all of the negative frequencies.

B. Filter Length Selection

The length chosen for the filter will affect the accuracy of the approximation to the analytic signal in the same way that it affects the implementation of low-pass FIR filters: more filter taps allow greater signal suppression in the stop band and narrower transition bands. For an analytic signal generating filter, this corresponds to the level of nonsuppressed negative frequency components and the possible positions of the 3 dB points of the pass band, respectively. It is desirable that the passband cutoff frequencies be as close to 0 and $f_s/2$ as possible. The nature of FIR filters is such that it is not possible to improve one of these parameters (passband cutoff or stopband level) without compromising the other, for a given filter length. A tradeoff must be made. A longer filter will allow a better approximation to the ideal filter response, the most obvious artifact of which is the closeness of the approach of the pass band to the frequencies 0 and $f_s/2$. Signals with very low frequency will always be a problem for analytic signal generators, because the required 90° phase shift requires a delay of the imaginary component that approaches infinity as $f \rightarrow 0$.

C. Results

We use a 128-sample filter designed by this method. The filter is implemented as a complex convolution between the signal and the impulse response, using an overlap/add FFT algorithm. The complex result thus occurs as a direct consequence of the convolution process and no extra additions are required.

The response has 0.17% ripple between $f = 0.024$ and $f = 0.478$. The pass band covers 95.3% of the positive axis (3 dB points at $f = 0.0127$ and $f = 0.4893$). The stop band has a maximum gain of 0.17%, (55 dB below the pass band,) and does not exceed this value inside the band $f = -0.5$ and $f = 0$. (All frequency measurements here are normalized to the sampling frequency). A plot of the frequency response of the filter appears in Fig. 3.

Plots that show how stop band gain is related to pass band edge position for filters of length 32, 64, and 128 samples are shown in Fig. 4.

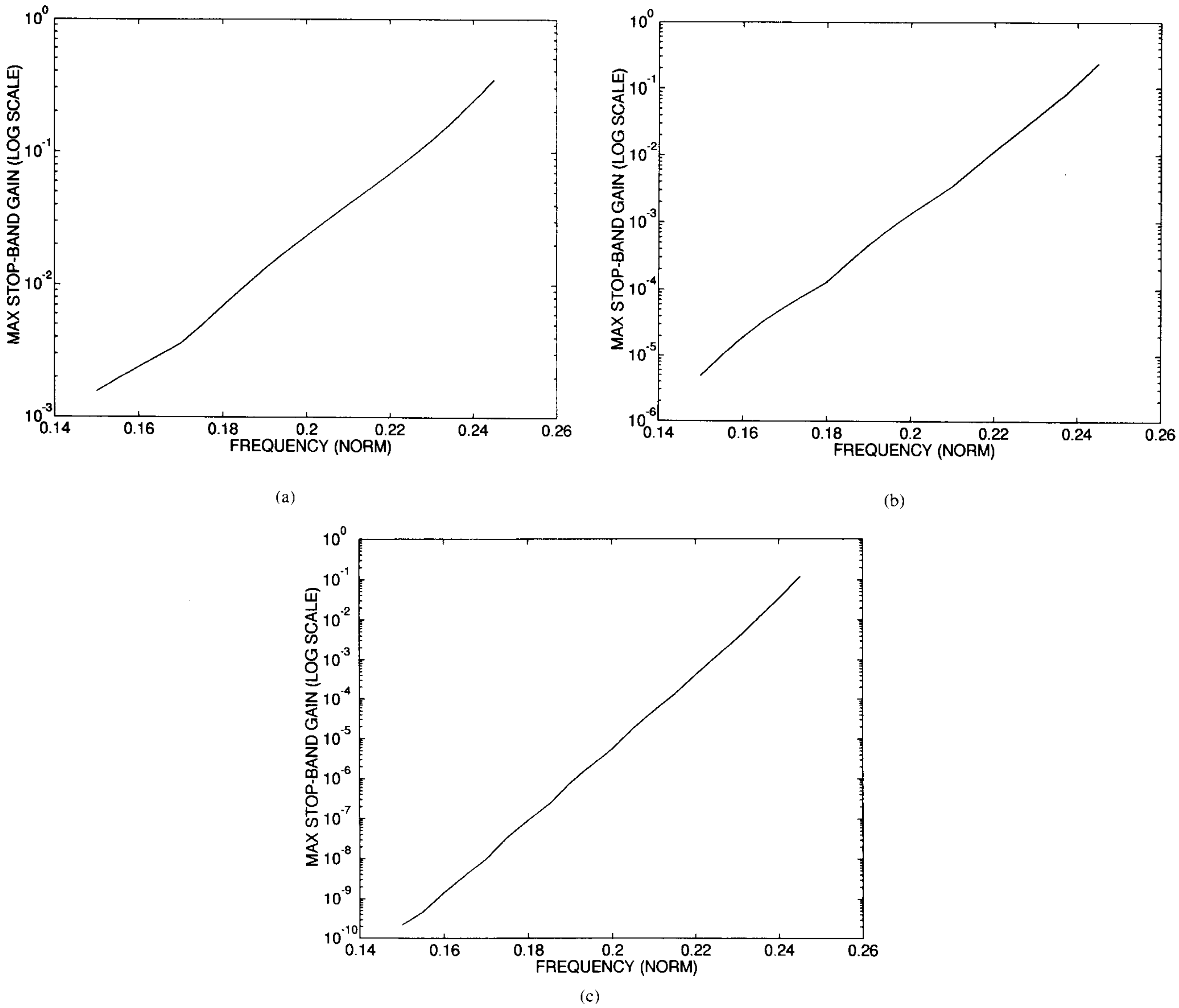

Fig. 4. (a) Maximum stop band gain versus pass band edge frequency: 32 tap filters. (b) Maximum stop band gain versus pass band edge frequency: 64 tap filters. (c) Maximum stop band gain versus pass band edge frequency: 128 tap filters.

A practical comparison of the three methods discussed in this paper is presented in Fig. 5. These show the central-finite-difference (CFD) instantaneous frequency estimate [5] based on an analytic signal computed by each of the methods. The signal under analysis is a simple linear frequency modulated (FM) waveform, with a frequency sweep from $f = 0$ to $f = f_s/2$ in 100 samples. The dotted line is the "ideal" instantaneous frequency, based on the the CFD estimate of the IF of a complex exponential with the same frequency law. The large spike at the beginning of the DFT result represents the frequency content of the initial step in the signal, as the cos( ) function started with a phase of zero.

Both of the FIR-based approaches are much better than the DFT-based approach, but it is not easy to discriminate between these two. This is because the Hilbert filter approach presented is the maximum length realizable using the Parks-McClellan FIR filter design package available (the "`remez`" function in the "Matlab" system [10]): 127 taps. For comparison purposes, the complex filter is the same length. In practice, the complex filter could be made much larger, and therefore be a better approximation, while the Hilbert filter would require a different design procedure to achieve this (due to the convergence problems listed earlier.)

V. CONCLUSION

Problems involved in the generation of a discrete analytic signal from a given discrete real signal have been discussed. This has included a discussion of analytic signals and their properties, and some definitions from which analytic signals might be created. We have explained why the simple methods of implementation have undesirable properties.

We have presented a new complex FIR filter design procedure that produces an approximation to the analytic signal that does not suffer from the problems of the popular methods. It is based on modulating a low-pass filter design. This solution is simple to design and build, and has general applicability.

These conclusions are summarized in Table I.

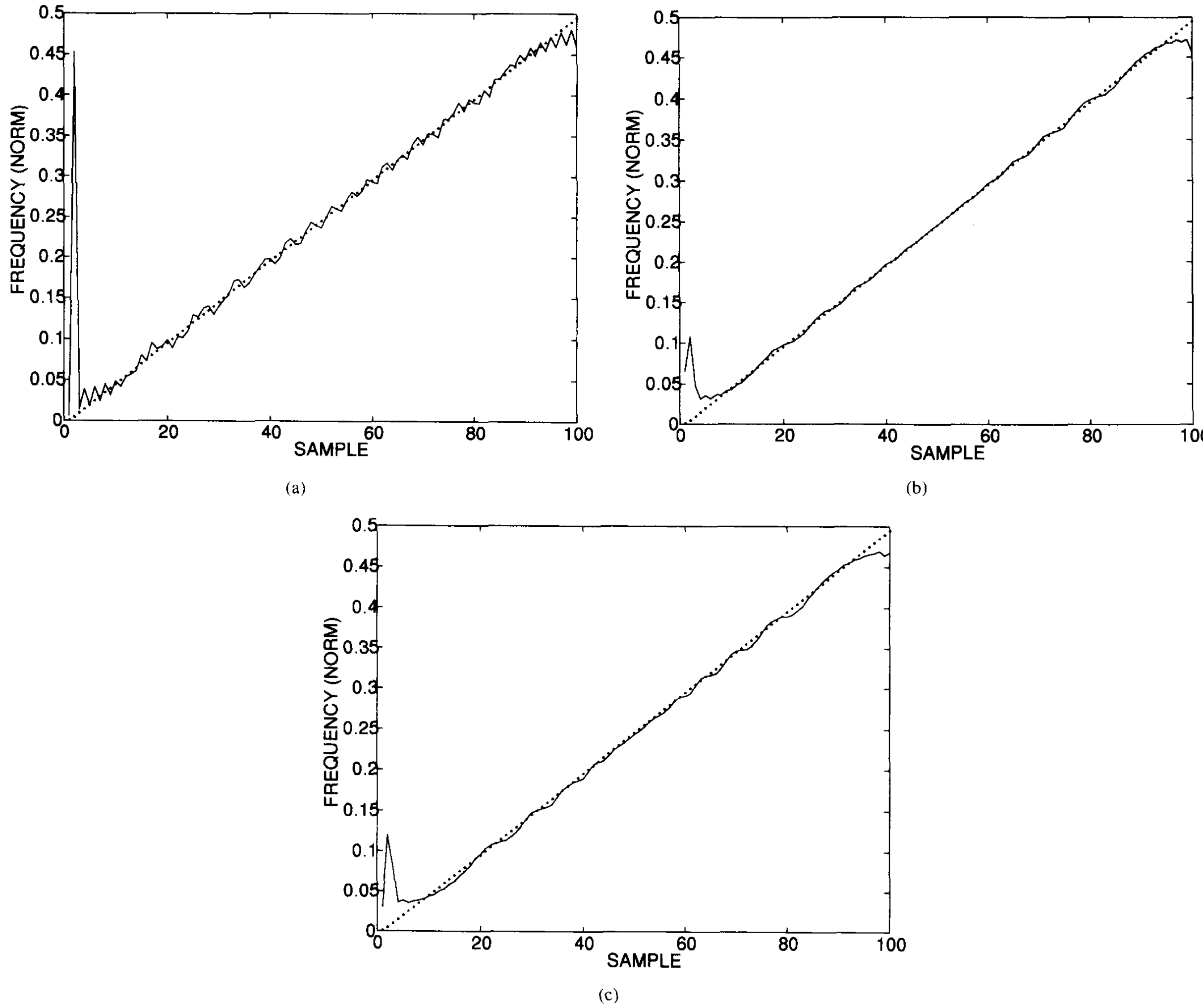

Fig. 5. (a) IF estimate of a linear FM signal based on an analytic signal approximation computed using the discrete Fourier transform approach. (b) IF estimate of a linear FM signal based on an analytic signal approximation computed using an FIR Hilbert filter. (c) IF estimate of a linear FM signal based on an analytic signal approximation computed using the complex filter described in this correspondence.

TABLE I

	Accuracy of Approx.	Realizability
Zero neg. freq.	poor	finite length: yes continuous: no
FIR Hilbert filt.	moderate	sometimes
Complex filter	good	always

An analytic signal makes determination of the instantaneous frequency of a mono-component signal (see [1]) using phase differentiation techniques possible. It makes the analysis of a signal in time and frequency using members of Cohen's class of distributions [11] worthwhile, by preventing the cross-terms that would be generated between the positive and negative frequency components of the real signal. While modern time-frequency distributions do suppress cross terms, this is less successful for signals that are close together in time or frequency, such as low-frequency signals.

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On the Design of Pole-Zero Approximations Using a Logarithmic Error Measure

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***Abstract*—For obtaining a pole-zero approximation of a linear, discrete-time system, a new method is presented which minimizes the squared difference between the log-magnitude spectrum of the system and that of the approximation. Using an iterative procedure, a locally optimal solution is found for the poles and zeros of the system approximation.**

I. Introduction

In approximating the frequency response of a linear, discrete-time system, an auto-regressive moving average (ARMA), or IIR, model is often used. The ARMA model parameters are usually selected such that they minimize an error criterion between the system and the model. For many applications, such as speech analysis/synthesis filters and synthesis of head-related transfer functions (HRTF's) [17]–[19], error criteria based on log-magnitude spectrum differences between the system and the system approximation have shown to be more subjectively meaningful than error criteria based on magnitude or magnitude-squared spectrum differences [2], [11]. The approximation problem based on log-magnitude spectrum differences, however, leads to a nonlinear problem when estimating the ARMA parameters. Many algorithms have been proposed as solutions to this problem [7]–[10], in which the ARMA model coefficients are determined.

In this correspondence, we present an iterative algorithm based on quasi-Newton gradient search methods to find the poles and zeros of a minimum phase, and locally optimal, solution to the ARMA approximation problem using an error criterion based on log-magnitude spectrum differences. No additional factorization is required to implement the solution as a cascade of biquadratic sections, which has been shown to reduce the effects of filter coefficient quantization and round-off noise (see Ch. 7 of [16]). Although other methods may produce similar solutions with fewer computations compared to the algorithm presented here, many times there is a trade-off between additional computations and a significant reduction in the ARMA approximation error.

Definitions and assumptions are stated in Section II, and in Section III we present the algorithm used to find the poles and zeros for the ARMA model. Results from application of the algorithm to system approximation problems, along with comparisons to results from other algorithms, are presented in Section IV. In Section V we present conclusions regarding the use of the new algorithm.

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II. Definitions and Assumptions

In developing the proposed algorithm for system approximation, we consider real, linear, discrete-time systems, d, with Z-transform $D(z)$, and discrete-time Fourier transform $D(\theta) = D(z)|_{z=e^{j\theta}}$. The ARMA model h used to approximate d is defined in the Z-transform domain as

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}} \tag{2.1}$$

with $a_0 = 1$ and $b_0 \neq 0$, or equivalently

$$H(z) = K_h \frac{\prod_{i=1}^{M} (1 - \lambda_i z^{-1})}{\prod_{i=1}^{N} (1 - \gamma_i z^{-1})} \tag{2.2}$$

which is the pole-zero form we will use to develop the proposed algorithm. An advantage of the pole-zero form over the coefficient form in (2.1) is that the pole-zero form allows for direct implementation of the ARMA model as a cascade of biquadratic equations.

Given a desired frequency response $D(\theta)$, its ARMA approximation can be determined by minimizing the following logarithmic error measure:

$$p_{\log}(d, h) = \frac{1}{L} \sum_{i=0}^{L-1} (\ln |D(\theta_i)| - \ln |H(\theta_i)|)^2. \tag{2.3}$$

The symmetry property of $p_{\log}(d, h)$, i.e., $p_{\log}(d, h) = p_{\log}(h, d)$, implies that equal proportional errors from relatively low and relatively high energy spectral regions in $D(\theta)$ are weighted equivalently. This is in contrast to approximating d by minimizing the weighted least squares error measure [15]

$$p_{\text{ls}}(d, h) = \frac{1}{L} \sum_{i=0}^{L-1} \frac{1}{|D(\theta_i)|^2} (|D(\theta_i)| - |H(\theta_i)|)^2 \tag{2.4}$$

the discrete version of the Itakura-Saito error measure proposed by El-Jaroudi and Makhoul [5], or the LPC error measure [3], [4]. Use of these error measures usually results in a better fit to regions where $|D(\theta)| > |H(\theta)|$ compared to regions where $|D(\theta)| < |H(\theta)|$ [3] since they include the term $\frac{|D(\theta)|}{|H(\theta)|}$. There are, however, system approximation problems where matching both spectral peaks and spectral nulls in $|D(\theta)|$ is important, and for which the symmetry property is desirable [18], [19]. In addition, the approximation problem based on these other error measures also leads to a nonlinear problem when estimating ARMA model parameters.