- Simple FIR lowpass and highpass filters

$$H_0(z) = \frac{1}{2}(1+z^{-1}) = \frac{z+1}{2z}$$

"Moving Average filter" - lowpass

$$H_1(z) = \frac{1}{2}(1-z^{-1}) = \frac{z-1}{2z}$$

replaced z w/ z^{-1} from lowpass $H_0(z)$

highpass filter due to zero at $z=1$

Q: How can we make these filters better (e.g. make cutoff transition faster)

A: Cascade multiple filters.

$$H_c(z) = H_0(z)\, H_0(z)$$

$$x[n] \longrightarrow \boxed{H_0} \longrightarrow \boxed{H_0} \longrightarrow y[n]$$

- Minimum-phase

Recall inverse system - $h^{inv}[n] * h[n] = \delta[n]$

$$\xrightarrow{\mathcal{Z}} H^{inv}(z)\, H(z) = 1$$

$$H^{inv}(z) = \frac{1}{H(z)}$$ $\Rightarrow$ zeros of $H(z)$ become poles of $H^{inv}(z)$ and poles become zeros.

Any system described by a rational transfer function has an inverse. Ho ver, we usually want an inverse that is both c sal and stable.

Use - reverse distortion f $H(z)$ to our signal.

$H^{inv}(z)$ is both stable and ca ul if all its poles are inside the unit circle.

$H(z)$ zeros must be inside un circle.

<u>Example:</u> $H(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$ ROC $|z| > .5$

$H^{inv}(z) = 1 - \frac{1}{2}z^{-1}$ ROC $|z| > 0$

If all poles and all zeros are inside the unit circle, the system is called "minimum phase".

$H_m(z)$ is a minimum phase system!

- $H_m(z)$ group delay is smaller than group delay for $H(z)$ with same magnitude response.

- system is uniquely defined by $|H(e^{j\omega})|$

- All pass systems

Definition - $|H_{AP}(e^{j\omega})|^2 = 1$ for all ω

Transfer function (General) - has real coefficients

$$H_{AP}(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \quad \text{where}$$

$$D_M(z) = 1 + d_1 z^{-1} + \dots + d_M z^{-M}$$

e, $$H_{AP}(z) H_{AP}(z^{-1}) \Big|_{z=e^{j\omega}} = |H_{AP}(e^{j\omega})|^2$$

$$\Rightarrow \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \cdot \frac{z^{M} D_M(z)}{D_M(-z)} = 1$$

Properties:

1- Causal, stable, real coeff allpass system is lossless

$$\sum_{n=-\infty}^{\infty} y^2[n] = \sum_{n=-\infty}^{\infty} x^2[n]$$

$$x[n] \longrightarrow \boxed{H_{AP}} \longrightarrow y[n]$$

2- Changes in phase from 0 to π is equal to $M\pi$ $\Rightarrow$ group delay is everywhere positive.

Application – Delay equalizer to correct nonlinear phase response of another system by cascading.

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{H_{AP}(z)} \longrightarrow y[n]$$

output has constant group delay over all freqs. of interest.

Note: Any causal, rational system function can be expressed as

$$H(z) = \underbrace{H_A(z)}_{\text{Allpass system}} \underbrace{H_{min}(z)}_{\text{min phase system}}$$

If we have distortion

$$H_d(z) = H_{dAP}(z) \; H_{dmin}(z)$$

After applying inverse compensation filter $H_c(z) = \frac{1}{H_{dmin}(z)}$

$$H_d(z) \cdot H_c(z) = H_{dAP}(z) \, H_{dmin} \frac{1}{H_{dmin}(z)} = H_{dAP}(z)$$

Result is an allpass system.

- Practical Filter Design

  - Lowpass FIR filters

Ideal lowpass filter - $H_{LP}(e^{j\omega}) = \begin{cases} 1 & , |\omega| \le \omega_c \\ 0 & , \omega_c < |\omega| \le \pi \end{cases}$

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} \qquad -\infty < n < \infty$$

$h_{LP}[n]$ is not practical, lets try truncating it to a finite length.

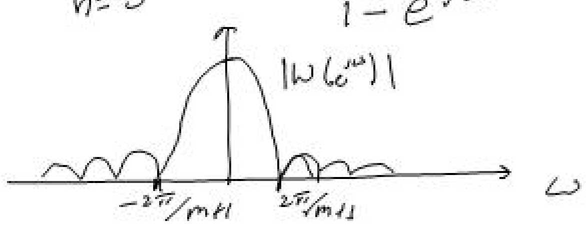
$$h[n] = h_{LP}[n]\, w[n] \qquad \text{where } w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise.} \end{cases}$$

use causal form - $h_{LP}[n] = \begin{cases} \dfrac{\sin(\omega_c(n-M))}{\pi(n-M)} & , 0 \le n \le M \\ 0 & , \text{otherwise} \end{cases}$

Lets take the Fourier Transform of $h_{LP}[n]\, w[n]$

$$H_{LP}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$ "freq. domain convolution"

$$W(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = e^{-j\omega M/2}\frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}$$



This is convolving a rectangular function with a sinc function.

As $M \to \infty$, mainlobe of sinc decreases, sidelobes oscillations occur more rapidly.

- Commonly used window functions

Rectangular $$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hann $$w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Hamming $$w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M), & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

What about linear phase?

General linear phase ideal low pass filter -

$$H_{lp}(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega M/2}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\frac{M}{2}} e^{j\omega n}\, d\omega = \frac{\sin\left(\omega_c (n - M/2)\right)}{\pi (n - M/2)}$$

Notice $h_{LP}[n] = h_{LP}[M-n]$

So, If we use a symmetric window function to truncate ideal $h_{LP}[n]$, the resulting $h[n] = h_{LP}[n] \cdot w[n]$ will be a linear phase filter impulse response.

- Highpass FIR filter

$$h_{HP}[n] = \begin{cases} 1 - \omega_c/\pi & , \text{for } n = 0 \\ -\dfrac{\sin(\omega_c n)}{\pi n} & , \text{for } |n| > 0 \end{cases}$$

- Bandpass FIR filter

$$h_{BP}[n] = \frac{\sin(\omega_{c_2} n)}{\pi n} - \frac{\sin(\omega_{c_1} n)}{\pi n} \quad , |n| > 0$$

You will explore this filter further in your homework problems.

- Hilbert Transformer

$$H_{HT}(e^{j\omega}) = \begin{cases} j & -\pi < \omega < 0 \\ -j & 0 < \omega < \pi \end{cases}$$

$$h_{HT}[n] = \begin{cases} 0 & , n \text{ even} \\ \frac{2}{\pi n} & , n \text{ odd} \end{cases}$$

Used to create analytic signal (e.g. signal with only positive freq. components in spectrum).