DTFT $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

spectrum (complex valued) ↑ input sequence ↑ basis signals

IDTFT $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

CTFT $$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)\, \underbrace{e^{-j\Omega t}\, dt}$$

spectrum (complex valued) ↑ input ↑ basis function

ICTFT $$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega t}\, d\Omega$$

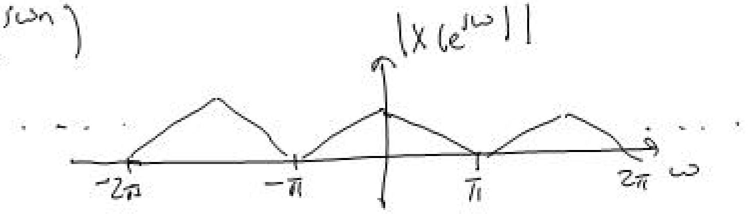
Unlike the CTFT, the DTFT is a periodic function in ω with a period 2π.

$$X(e^{j(\omega + k2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, e^{-j2\pi k n}$$

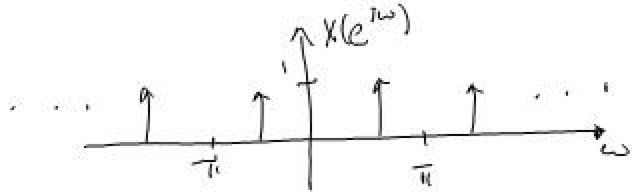
$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{since } e^{j2\pi k n} = 1 \text{ for all } k, n$$

$$= X(e^{j\omega n})$$



Example DTFT of $x[n] = e^{j\omega_0 n}$

Use table 3.3 $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$



"periodic Impulse train"

IDTFT of $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega-\omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega_0 n}$$

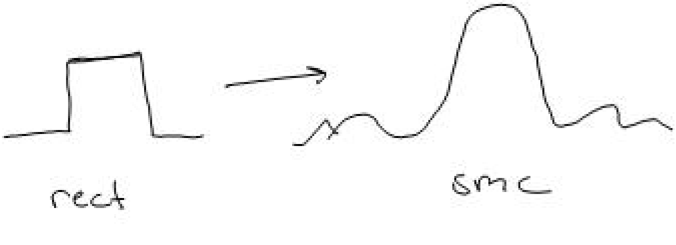
Example CTFT of $x_a(t) = \frac{\sin(t)}{\pi t}$, show that

$$X_a(\Omega) = \begin{cases} 1, & |\Omega| \le 1 \\ 0, & |\Omega| > 1 \end{cases}$$

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} e^{j\Omega t}\, d\Omega = \frac{1}{2\pi j t}\left(e^{jt} - e^{-jt}\right)$$

$$= \frac{\sin(t)}{\pi t}$$



- DTFT Theorems

  - Linearity - $a x_1[n] + b x_2[n] \xleftrightarrow{DTFT} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$

  - Time-Reversal - $x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$

  - Time-shifting - $x[n-n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$

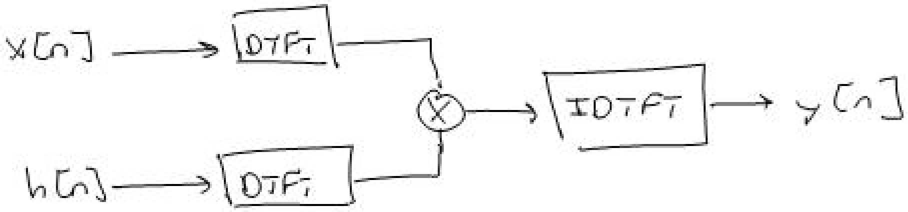
    Note: does not change magnitude spectrum, only phase spectrum.

  - Frequency-shifting - $e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

  - Convolution -

$$x[n] * h[n] \xleftrightarrow{DTFT} X(e^{j\omega}) H(e^{j\omega})$$

"Fast Filtering"



- Proof of Convolution theorem

$$y[n] = \sum_{k=-\infty}^{\infty} g[k]\, h[n-k]$$

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} g[k]\, h[n-k] \right) e^{-j\omega n}$$

sub $m = n-k$ into first sum

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g[k]\, h[m]\, e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} g[k] \left(\sum_{m=-\infty}^{\infty} h[m]\, e^{-j\omega m} \right) e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} g[k]\, H(e^{j\omega})\, e^{-j\omega k}$$

$$= H(e^{j\omega}) \sum_{k=-\infty}^{\infty} g[k]\, e^{-j\omega k}$$

$$= H(e^{j\omega})\, G(e^{j\omega})$$

Vector Spaces

if $\vec{v}, \vec{w} \in V$ ← vector space

then, $a\vec{v} + b\vec{w} \in V$

A linear combination of vectors also gives a vector. Likewise, a linear combination of signals also gives a signal.

Concepts that carry over from vectors to signals:

Length, Angle, orthogonality, etc.

- Inner Product

$$\vec{g} = [g_1, g_2, g_3, \cdots, g_n]^T$$

$$\vec{f} = [f_1, f_2, f_3, \cdots, f_n]^T$$

$$\langle \vec{g}, \vec{f} \rangle = \vec{g}^T \vec{f} = \sum_{i=1}^{n} g_i f_i = \vec{g} \cdot \vec{f}$$

Two vectors are orthogonal if $\langle \vec{g}, \vec{f} \rangle = 0$

Other Properties:

$$\langle c\vec{f}, \vec{g} \rangle = c \langle \vec{f}, \vec{g} \rangle$$

$$\langle \vec{f} + \vec{g}, \vec{h} \rangle = \langle \vec{f}, \vec{h} \rangle + \langle \vec{g}, \vec{h} \rangle$$

$$\langle \vec{f}, \vec{g} \rangle = \langle \vec{g}, \vec{f} \rangle$$ for all real valued vectors

- Euclidean Norm –

$$\|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

- Distance – Given $\vec{x}$ & $\vec{y}$ $\quad d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$

- Projection – project vector $\vec{f}$ onto $\vec{x}$

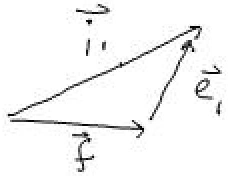
$$\text{proj}(\vec{f},\vec{x}) = \frac{\langle \vec{f}, \vec{x} \rangle}{\langle \vec{x}, \vec{x} \rangle} = \frac{\langle \vec{f}, \vec{x} \rangle}{\|\vec{x}\|^2}$$

- Approximating vectors using other vectors.
  - Suppose we have a vector $\vec{i}_1$ and we want to find best approximation to vector $\vec{f}$ using $\vec{i}_1$.

$$\vec{f} = \overset{\text{scale}}{c_1}\, \vec{i}_1 + \vec{e}_1 \longleftarrow \text{error}$$

The best approximation in a minimum norm sense is found when the length of $\vec{e}_1$ is minimized.

Notice that $\vec{e}_1$ is minimum when $\vec{i}_1$ & $\vec{e}_1$ are orthogonal, or $\perp$.



$$\vec{i}_1 \perp \vec{e}_1 \Rightarrow \langle \vec{i}_1, \vec{e}_1 \rangle = 0$$

$$\Rightarrow \langle \vec{f} - c_1 \vec{i}_1, \vec{i}_1 \rangle = 0 = \langle \vec{f}, \vec{i}_1 \rangle - c_1 \langle \vec{i}_1, i_1 \rangle$$

$$c_1 = \frac{\langle \vec{f}, i_1 \rangle}{\langle \vec{i}_1, i_1 \rangle} = \frac{\langle \vec{f}, \vec{i}_1 \rangle}{\|\vec{i}_1\|^2}$$

- What if we have multiple vectors to approximate $\vec{f}$ with?

$$\vec{f} = \sum_{k=1}^{n} c_k \vec{i}_k + \vec{e}_k$$

  - It seems that if we take n large enough, we could represent any vector. In fact, it gets better...

- Function spaces - infinite dimensional spaces

$$f(t) = \sum_{k=1}^{n} c_k \, i_k(t)$$

function to approximate

supperposition

scales

basis functions

We can treat functions just like vectors in a vector space.

- Inner Product - $\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)\, g(t)\, dt$

- Note: Two functions are orthogonal if $\langle f(t), g(t) \rangle = 0$

- Projection - $\text{proj}\left(f(t), g(t) \right) = \dfrac{\langle f(t), g(t) \rangle}{\langle g(t), g(t) \rangle}$

- Length (Euclidean norm) - $\|f(t)\| = \langle f(t), f(t) \rangle^{1/2} = \left(\int f^2(t)\, dt \right)^{1/2}$

- Distance - $d\left(f(t), g(t)\right) = \langle f(t) - g(t), f(t) - g(t) \rangle^{1/2}$

Note: The kinds of functions that can be approximated without error depends upon the kinds of basis functions used.

- Just like we did for vectors, the coefficents for

$$f(t) = \sum_{k=1}^{n} c_k i_k(t) + e_k(t)$$

can be solved for by minimizing the error. This will give,

$$c_k = \frac{\langle f(t), i_k(t) \rangle}{\langle i_k(t), i_k(t) \rangle}$$

Example Suppose we have the function

$$f(t) = \begin{cases} 1 & 0 \le t \le \pi \\ -1 & \pi < t \le 2\pi \end{cases}$$

Approximate f(t) using only $i_1(t) = \sin(t)$

$$f(t) \approx c \sin(t)$$

The best coefficent in a minimum error sense is given by

$$c = \frac{\int_0^{2\pi} f(t) \sin(t)\,dt}{\int_0^{2\pi} \sin^2(t)\,dt} = \frac{1}{\pi}\left[\int_0^{\pi} \sin(t)\,dt - \int_{\pi}^{2\pi} \sin(t)\,dt\right]$$

$$c = \frac{1}{\pi}\left[-\cos(t)\Big|_0^{\pi} + \cos(t)\Big|_{\pi}^{2\pi}\right] = \frac{4}{\pi}$$

So, $f(t) \approx \frac{4}{\pi} \sin(t)$

- Looks like Fourier series!

Uses periodic basis functions -

$$i_0(t) = \cos(0t) = 1$$
$$i_1(t) = \cos(\omega_0 t + \phi_1)$$
$$i_2(t) = \cos(2\omega_0 t + \phi_2)$$
$$\vdots$$
$$i_n(t) = \cos(n\omega_0 t + \phi_n)$$

Thus,

$$f(t) \approx c_0 + \sum_{k=1}^{n} c_k \cos(\underbrace{k\omega_0 t}_{k^{\text{th}} \text{ harmonic of } \omega_0} + \phi_k)$$

Two parameter knobs to adjust → c_k & ϕ_k

Often want to use both sin & cos → apply trig identity

$$f(t) \approx a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

Used -

$$c_k \cos(k\omega_0 t + \phi_k) = c_k \cos(\phi_k)\cos(k\omega_0 t) - c_k \sin(\phi_k)\sin(k\omega_0 t)$$

and $a_k = c_k \cos(\phi_k)$

$b_k = c_k \sin(\phi_k)$

Fourier Series Demo -

- DFT / DTFT relationship

$$(\text{DTFT}) \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\text{If} \quad 0 \le n \le N-1 \implies X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Lets uniformly sample the continuous function $X(e^{j\omega})$ at N equally spaced frequencies on ω-axis between $0 \le k \le N-1$.

$$(\text{DFT}) \quad X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Note: For time-domain sequences w/ finite values, the DFT always exists.

- Matlab functions - fft (x) or fft(x,N) analysis

  ifft(X) or ifft(X,N) synthesis

- Basis sequences for DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad 0 \le k \le N-1$$

$$\varphi[k,n] = e^{j\frac{2\pi kn}{N}} \qquad (\text{basis sequences})$$

In general $X[k]$ is complex valued, even when $x[n]$ is real.

- Other Notation - $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$ where $W_N = e^{-j\frac{2\pi}{N}}$

- Inverse DFT, or IDFT

$$x[n] = \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi kn}{N}} = \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

- Energy Density Spectrum

Total energy of finite-energy sequence x[n] is -

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n] = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \right)^*$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega})\, e^{-j\omega n}\, d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega}) \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, d\omega$$

$$\boxed{= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega})\, X(e^{j\omega})\, d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2}$$

Let $S_{xx}(e^{j\omega}) = \left| X(e^{j\omega}) \right|^2$ called "Energy Density Spectrum"

- Autocorrelation ⟶ Energy Density spectrum (Linkage)

Recall - (autocorrelation) $r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[n-\ell]$

or, $r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]\, x[-(\ell-n)] = x[\ell] * x[-\ell]$

Now, use convolution theorem -

$$x[\ell] * x[-\ell] \xrightarrow{DFT} X(e^{j\omega})\, X(e^{-j\omega}) = \left| X(e^{j\omega}) \right|^2$$ for real sequences.

Therefore,

$$S_{xx}(e^{j\omega}) = \left| X(e^{j\omega}) \right|^2 = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell]\, e^{-j\omega\ell}$$

or

$$S_{xx}(e^{j\omega}) \xleftrightarrow{DFT} r_{xx}[\ell]$$

" Wiener - Khintchine Theorem "

(Very Important!)

- Frequency Domain Characterization of LTI System

Recall Time-domain - $y[n] = x[n] * h[n]$

Now we can look at frequency domain.

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$ by convolution theorem of DFT

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$ Frequency Response