- Lowpass FIR filters

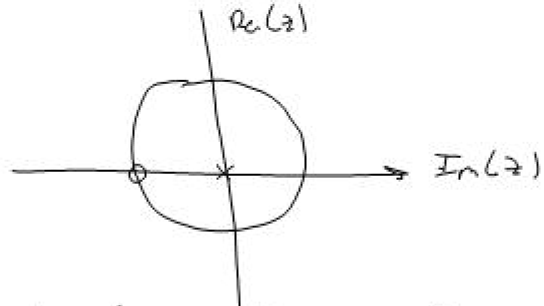
simple MA filter – $h[n] = \begin{cases} 1/M & , 0 \le n \le M-1 \\ 0 & , \text{otherwise} \end{cases}$

Let $M = 2$

$$H_0(z) = \sum_{n=0}^{1} \tfrac{1}{2} z^{-n} = \frac{1}{2}\left(1 + z^{-1}\right) = \frac{z+1}{2z}$$

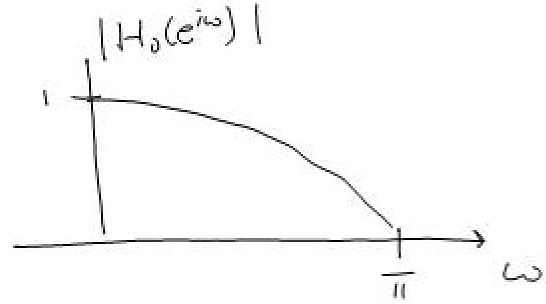
pole at $z = 0$

zero at $z = -1$



Pole vector magnitude is unity for all ω. Zero vector magnitude decreased as ω approaches π.

$|H_0(e^{j0})| = 1$, $|H_0(e^{j\pi})| = 0$



$$H_0(e^{j\omega}) = e^{-j\omega/2} \cos(\omega/2)$$

What is cutoff freq? $|H_0(e^{j\omega})| = \frac{1}{\sqrt{2}} |H_0(e^{j0})|$

Use dB –

$$20 \log_{10} |H_0(e^{j0})| - 20 \log_{10} \sqrt{2} \cong -3.0\,\text{dB}$$

$$|H_0(e^{j\omega_c})|^2 = \cos^2\left(\frac{\omega_c}{2}\right) = \frac{1}{2}$$

$\Rightarrow \omega_c = \pi/2$ power is halved at $\pi/2$

- Highpass FIR filters

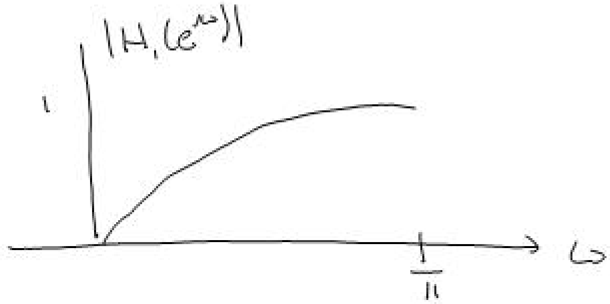
  Replace z w/ z^{-1} in MA filter

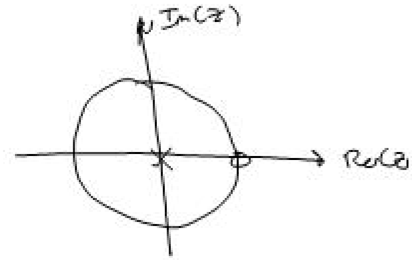
$$H_1(z) = \frac{1}{2}(1 - z^{-1}) = \frac{z-1}{2}$$

zero at $z = 1$
pole at $z = 0$

Freq. Response -

$$H_1(e^{j\omega}) = j e^{-j\omega/2} \sin(\omega/2)$$





- Inverse z-transform

  Recover DT sequence from its z-transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

Two useful methods to avoid contour integration -

1. Partial Fraction Expansion

   For rational functions of z, simple poles

Assume $\cancel{p > q}$ $N > M$

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\Rightarrow X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad \Rightarrow \quad A_k = X(z)(1 - d_k z^{-1})\Big|_{z = d_k}$$

<u>Example</u> $X(z) = \dfrac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$ $|z| > \frac{1}{2}$ $\Rightarrow$ right sided sequence.

$$X(z) = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$A_1 = (1-\tfrac{1}{4}z^{-1})\,X(z)\Big|_{z=1/4} = \frac{(1-1/4\,z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1/4} = -1$$

$$A_2 = (1-\tfrac{1}{2}z^{-1})\,X(z)\Big|_{z=1/2} = \frac{(1-1/2\,z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1/2} = 2$$

so, $X(z) = \dfrac{-1}{(1-1/4\,z^{-1})} + \dfrac{2}{(1-\frac{1}{2}z^{-1})}$ $\Rightarrow$ $x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$

2- Power Series Expansion - The z-transform is a power series expansion:-

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\,z^{-n} = \ldots x[-2]\,z^{2} + x[-1]\,z^{1} + x[0] + x[1]\,z^{-1} + \ldots$$

Find power series expansion form of $X(z)$ and then find DT sequence by inspection.

<u>Exampl</u> $X(z) = \dfrac{z^{-1}}{(1-z^{-1})^2}$ $|z| > 1$ use binomial expansion.

Binomial Expansion $\quad (1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots$

$\alpha = -2 \;,\; x = -z^{-1}$

$$\Rightarrow \frac{1}{(1-z^{-1})^2} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \ldots$$

$$\Rightarrow \frac{z}{(1-z^{-1})^2} = z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \ldots$$

so, $\quad x[n] = 1\delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + \ldots$

$$x[n] = \begin{cases} n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$

3- Contour Integration

Cauchy's Integral Theorem - If C is a closed contour that encircles the origin in a counter-clockwise direction -

$$\frac{1}{2\pi j} \oint_C z^{-k}\, dz = \begin{cases} 1 \\ 0 \end{cases}$$

- Linear phase systems

An LTI system has linear phase if

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{-j\alpha\omega} \qquad \alpha \text{ is real valued}$$

Linear phase systems have constant group delay

$$\tau_g = -\frac{d\,\phi(\omega)}{d\omega} = -\frac{d(\alpha\omega)}{d\omega} = \alpha$$

Note: In order for causal system w/ rational transfer function to have linear phase, the impulse response must be finite in length.

$\Rightarrow$ IIR filters cannot have linear phase!

- 4 types of linear phase FIR systems.

Type I - symmetrical impulse response, and N is even

$$h[n] = h[N-n] \qquad 0 \le n \le N$$

. The delay through the filter is $N/2$

Type II - symmetrical impulse response, and N is odd

. delay is $N/2$

Type III - anti symmetrical impulse response, and N is even

$$h[n] = -h[N-n] \qquad 0 \le n \le N$$

. delay is $N/2$

Type IV - anti symmetrical impulse response, and N is odd.

- delay is $N/2$

- Locations of zeros for FIR Linear phase systems.

Type I & II - $H(z) = \sum_{n=0}^{N} h[N-n] z^{-n} = \sum_{k=0}^{N} h[N-n] z^{-n}$

Let $m = N-n$ $= \sum_{m=0}^{N} h[m] z^{-N+m} = z^{-N} \sum_{m=0}^{N} h[m] z^{m} = z^{-N} H(z^{-1})$

If z_0 is a zero, $H(z_0) = z_0^{-N} H(z_0^{-1}) = 0$

so, z_0^{-1} must be a zero.

Note; For $h[n]$ w/ real-valued coeff., the zeros occur in complex conjugate pairs. Hence, a zero at $z = z_0$ is associated with a set of four zeros given by $z_0, z_0^*, z_0^{-1}, z_0^{-1*}$.

Important case - zero at $z = -1$

If N is even $H(-1) = (-1)^{-N} H(-1) = H(-1)$

If N is odd $H(-1) = (-1) H(-1)$ so $H(-1)$ must be zero - always!

For symmetric $h[n]$ with odd N, system must have zero at $z = -1$. Freq Resp. is constrained to $\emptyset$ at $\omega = \pi$. Do not use for highpass filter.

Type III & IV - $H(z) = -z^{-N} H(z^{-1})$

Both $z = 1$ & $z = -1$ are special cases.

If $z = 1$ $H(1) = -H(1)$ $\Rightarrow$ must have zero at $z = 1$ for both N even and odd

If $z = -1$ $H(-1) = (-1)^{-N} H(-1)$ $\Rightarrow$ $z = -1$ must be zero for N even