- Simple FIR lowpass and highpass filters

$$H_0(z) = \frac{1}{2}(1+z^{-1}) = \frac{z+1}{2z}$$ "Moving Average filter" - lowpass

$$H_1(z) = \frac{1}{2}(1-z^{-1}) = \frac{z-1}{2z}$$ replaced z w/ z^{-1} from lowpass $H_0(z)$

highpass filter due to zero at $z=1$

Q: How can we make these filters better (e.g. make cutoff transition faster)

A: Cascade multiple filters.

$$H_c(z) = H_0(z)\, H_0(z)$$

$$x[n] \longrightarrow \boxed{H_0} \longrightarrow \boxed{H_0} \longrightarrow y[n]$$

- Minimum-phase

Recall inverse system - $h^{inv}[n] * h[n] = \delta[n]$

$$\xrightarrow{Z} H^{inv}(z)\, H(z) = 1$$

$$H^{inv}(z) = \frac{1}{H(z)}$$ $\Rightarrow$ zeros of $H(z)$ become poles of $H^{inv}(z)$ and poles become zeros.

Any system described by a rational transfer function has an inverse. However, we usually want an inverse that is both causal and stable.

use - reverse distortion of $H(z)$ to our signal

$H^{inv}(z)$ is both stable and casaul if all its poles are inside the unit circle.

~~$H^{inv}(z)$ is both stable and causal if all~~
$H(z)$ zeros must be inside unit circle.

<u>Example</u>: $H(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$ ROC $|z| > .5$

$H^{inv}(z) = 1 - \frac{1}{2}z^{-1}$ ROC $|z| > 0$

If all poles and all zeros are inside the unit circle, the system is called "minimum phase".

$H_m(z)$ is a minimum phase system:

- $H_m(z)$ group delay is smaller than group delay for $H(z)$ with same magnitude response.

- system is uniquely defined by $|H(e^{j\omega})|$

- All pass systems

Definition – $|H_{AP}(e^{j\omega})|^2 = 1$ for all ω

Transfer function (General) – has real coefficients

$$H_{AP}(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \quad \text{where}$$

$$D_M(z) = 1 + d_1 z^{-1} + \dots + d_M z^{-M}$$

so, $$H_{AP}(z) H_{AP}(z^{-1}) \Big|_{z=e^{j\omega}} = |H_{AP}(e^{j\omega})|^2$$

$$\Rightarrow \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \cdot \frac{z^{M} D_M(z)}{D_M(-z)} = 1$$

Properties:

1- Causal, stable, real coeff allpass system is lossless

$$\sum_{n=-\infty}^{\infty} y^2[n] = \sum_{n=-\infty}^{\infty} x[n]^2$$

$$x[n] \longrightarrow \boxed{H_{AP}} \longrightarrow y[n]$$

2- Changes in phase from 0 to π is equal to $M\pi$ $\Rightarrow$ group delay is everywhere possitive.

Application – Delay equalizer to correct nonlinear phase response of another system by cascading.

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{H_{AP}(z)} \longrightarrow y[n]$$

output has constant group delay over all freqs. of interest.

Note: Any causal, rational system function can be expressed as

$$H(z) = H_A(z)\, H_{min}(z)$$

Allpass system

min phase system

If we have distortion

$$H_d(z) = H_{dAP}(z)\; H_{dmin}(z)$$

After applying inverse compensation filter $H_c(z) = \dfrac{1}{H_{dmin}(z)}$

$$H_d(z) \cdot H_c(z) = H_{dAP}(z)\, H_{dmin} \frac{1}{H_{dmin}(z)} = H_{dAP}(z)$$

Result is an allpass system.

- Linear-Phase Transfer Function

For Causal LTI system with non-zero phase response, phase distortion can be avoided by

$$y[n] = x[n-D]$$

$$Y(e^{j\omega}) = e^{-j\omega D} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D}$$

Notice: $|H(e^{j\omega})| = e^{-j\omega D} \cdot e^{j\omega D} = 1$

If input $x[n] = Ae^{j\omega n}$

$$y[n] = Ae^{j\omega(n-D)}$$

If we want to pass frequency components of input signal undistorted (both mag. & phase), then $H(z)$ should be linear phase and have unit magnitude response.

Example: $$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_0} & 0 < |\omega| < \omega_c \\ 0 & \omega_c \le |\omega| \le \pi \end{cases}$$

You already know this ...

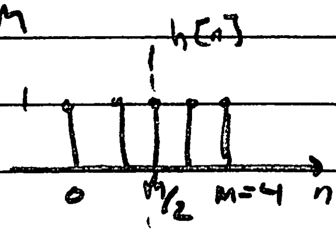
$$h_{LP}[n] = \frac{\sin \omega_c(n-n_0)}{\pi(n-n_0)} \qquad -\infty < n < \infty$$

4 Types of Linear Phase FIR systems

Type I – Symmetric impulse response

$$h[n] = h[M-n] \quad , \quad 0 \le n \le M$$

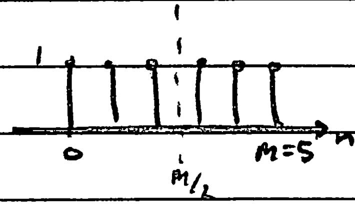
w/ M an even integer.
Delay through filter is $M/2$ (integer)



Type II – Symmetric impulse response

$$h[n] = h[M-n] \quad , \quad 0 \le n \le M$$

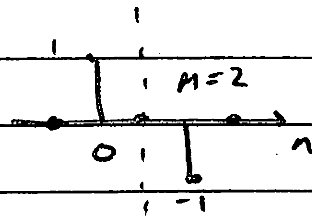
w/ M an odd integer.
Delay is $M/2$ (integer + $1/2$)



Type III – antisymmetric impulse response

$$h[n] = -h[M-n] \quad , \quad 0 \le n \le M$$

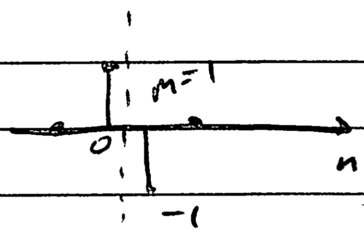
w/ M an even integer.
Delay is $M/2$ (integer)



Type IV – antisymmetric impulse response

$$h[n] = -h[M-n] \quad , \quad 0 \le n \le M$$

w/ M an odd integer.
Delay is $M/2$ (integer + $1/2$)



Locations of zeros for FIR Linear Phase Systems.

$$H(z) = \sum_{n=0}^{M} h[n]\, z^{-n}$$

Type I & Type II -

Let $k = M - n \Rightarrow n = M - k$

$$H(z) = \sum_{n=0}^{M} h[M-n]\, z^{-n} = \sum_{k=M}^{0} h[k]\, z^{k-M}$$

$$= z^{-M} \sum_{k=M}^{0} h[k]\, z^{k} = z^{-M} H(z^{-1})$$

so, if $z_0 = re^{j\phi}$ is a zero of $H(z)$

$$H(z_0) = z_0^{-M} H(z_0^{-1}) = 0$$

$z_0^{-1} = r^{-1} e^{-j\phi}$ is also a zero of $H(z)$

Also, if $h[n]$ is real and z_0 is a zero, $z_0^* = re^{-j\phi}$ will also be a zero of $H(z)$.

so, $(z_0^*)^{-1} = r^{-1} e^{j\phi}$ is also a zero of $H(z)$.

($h[n]$ is real)

Rule: Zeros not on unit circle will be a set of 4 reciprocal zeros

$$(1 - re^{j\phi} z^{-1})(1 - re^{-j\phi} z^{-1})(1 - r^{-1} e^{j\phi} z^{-1})(1 - r^{-1} e^{-j\phi} z^{-1})$$

on unit circle - $(1 - e^{j\phi} z^{-1})(1 - e^{-j\phi} z^{-1})$

Also, $z = \pm 1$ can appear alone.

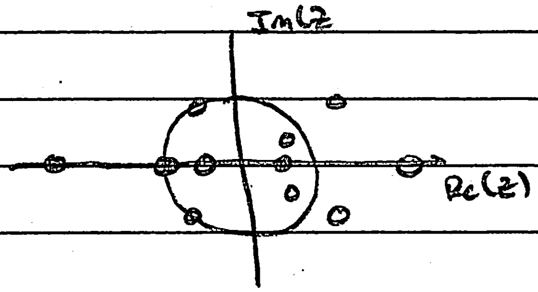
Example: Case of $z=-1$ is a zero

$$H(z) = z^{-M} H(z^{-1})$$

$$H(-1) = (-1)^{-M} H(-1)$$

If M is even - $H(-1) = H(-1)$ "identity"
If M is odd - $H(-1) = -H(-1)$ so, $H(-1)$ must be zero.

So, For Type II systems, $z=-1$ must always be a zero.



Note - Do not use type II for HPF design.

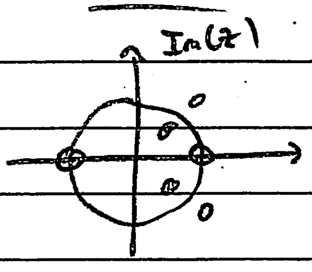
Notice - In antisymmetric case, both $z=1$ and $z=-1$ are interesting. $H(z) = -z^{-M} H(z^{-1})$

Case $z=1$ $\quad H(1) = -H(1)$ $\Rightarrow$ must have zero at $z=1$ always.

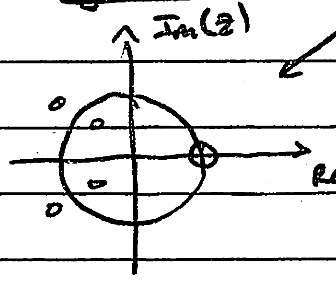
Case $z=-1$ $\quad H(-1) = (-1)^{-M+1} H(-1)$

If M = even $\Rightarrow$ $H(-1) = -H(-1)$ $\Rightarrow$ $z=-1$ must be zero

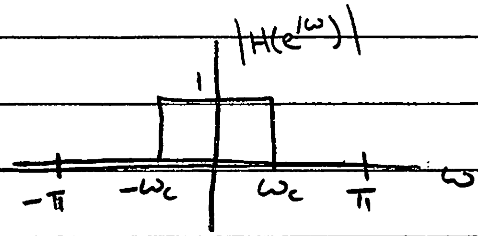
Type III BPF



Type IV HPF



- Digital Filters
  - Ideal Magnitude Response



Low Pass Filter

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega_c| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

iDTFT

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{e^{j\omega n}}{jn}\bigg|_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty, \quad n \neq 0$$

for $n = 0$

$$h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

so,

$$h_{LP}[n] = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

Note: $h_{LP}[n]$ is not absolutely summable, and is non causal.

- Ideal $-\pi/2$ phase shifter

$$H(e^{j\omega}) = \begin{cases} e^{-j\pi/2}, & \omega > 0 \\ e^{j\pi/2}, & \omega < 0 \end{cases}$$

Notice - $e^{-j\pi/2} = \cos(-\pi/2) + j\sin(-\pi/2) = -j$

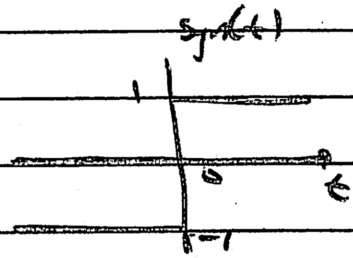
$$e^{j\pi/2} = \cos(\pi/2) + j\sin(\pi/2) = j$$

so, $$H(e^{j\omega}) = \begin{cases} -j, & \omega > 0 \\ +j, & \omega < 0 \end{cases}$$

Let's define $$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

"signum" function $\Rightarrow H(e^{j\omega}) = -j\,\text{sgn}(\omega)$

Can be expressed as $\text{sgn}(t) = 2u(t) - 1$



$$\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$$

$$\text{sgn}(t) \xleftrightarrow{FT} X(\omega)$$

$$\frac{d}{dt}\text{sgn}(t) = 2\delta(t) \xleftrightarrow{FT} j\omega X(\omega) = 2$$

$$\Rightarrow X(\omega) = \frac{2}{j\omega}$$

Duality Property - $x(t) \longleftrightarrow 2\pi x(-\omega)$

$$\frac{2}{jt} \longleftrightarrow 2\pi\, \mathrm{sgn}(-\omega) = -2\pi\, \mathrm{sgn}(\omega)$$

or

$$\frac{1}{\pi t} \longleftrightarrow -j\, \mathrm{sgn}(\omega)$$

$$h(t) = FT^{-1}\{H(e^{j\omega})\} = \frac{1}{\pi t}$$

Q: what is output of phase shifter with arbitrary input?

$$y(t) = x(t) * h(t) = x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{(t-\tau)}\, d\tau}_{\text{"hilbert transform" of } x(t) = \hat{x}(t)}$$

Example Let $x(t) = \cos(\omega_0 t) \longleftrightarrow \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

$$Y(\omega) = X(\omega)H(\omega) = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]\left[-j\, \mathrm{sgn}(\omega)\right]$$
$$= -j\pi\, \mathrm{sgn}(\omega_0)\,\delta(\omega-\omega_0) - j\pi\, \mathrm{sgn}(-\omega_0)\,\delta(\omega+\omega_0)$$
$$= -j\pi\,\delta(\omega-\omega_0) + j\pi\,\delta(\omega+\omega_0)$$
$$\Rightarrow \sin(\omega_0 t) = \cos(\omega t - \pi/2)$$

$1-\alpha z^{-1} = 0$

$z - \alpha = 0$

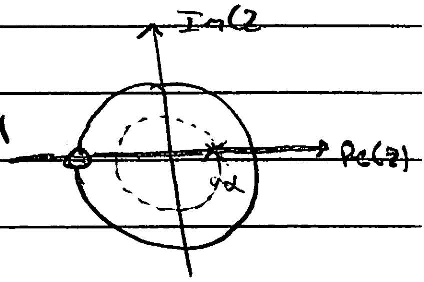
$z = \alpha$

- Simple IIR Digital Filters.

Note: Causal FIR filters have poles at origin - shape of freq. response is determined only from zero locations.

IIR filters allow poles to move inside unit circle - more complex freq. responses can be achieved.

Low Pass IIR digital filter.



$$H_{LP}(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}} \qquad 0<|\alpha|<1$$

$$|H_{LP}(e^{j0})| = \frac{2K}{1-\alpha} \qquad \text{max}$$

$$|H_{LP}(e^{j\pi})| = 0 \qquad \text{min}$$

Usually want DC gain of 0dB $\Rightarrow$ max mag. = 1

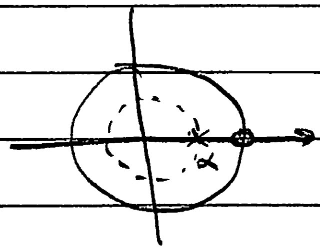
Let $K = \frac{(1-\alpha)}{2}$

$$H_{LP}(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}}$$

Highpass IIR digital filters

$$H_{HP}(z) = \frac{1+\alpha}{2} \cdot \frac{1-z^{-1}}{1-\alpha z^{-1}}$$

α - close to 1 $\longrightarrow$ DC blocker



Bandpass IIR digital filters

$$H_{BP}(z) = \frac{K(1-z^{-2})}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

Higher-Order IIR digital filters -

By cascading simple filters we can get sharper magnitude responses.

$$G(z) = H_1(z)\,H_2(z)$$

$$x[n] \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow y$$

K 1st order lowpass filters -

$$G_{LP}(z) = \left(\frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}}\right)^{K}$$

- All Pass Transfer Function

$$|A(e^{j\omega})|^2 = 1 \quad \text{for all } \omega$$

M-th order transfer function.

$$A_M(z) = \pm \frac{d_M + d_{M-1} z^{-1} + \ldots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

Let $D_M(z) = 1 + d_1 z^{-1} + \ldots + d_{M-1} z^{-M+1} + d_M z^{-M}$

so, $$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

If $z = re^{j\phi}$ is a pole, then $z = \frac{1}{r} e^{-j\phi}$ is a zero.