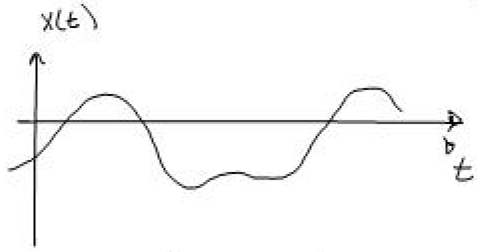
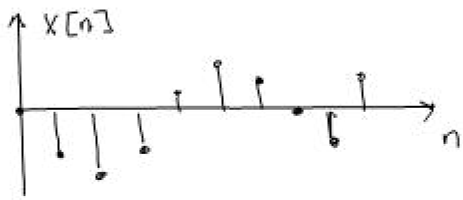
SIGNAL PROCESSING - The science of analyzing time-varying or spatial-varying physical processes.

Continuous-time signals - Waveforms that are continuous in time and have a continuous range of amplitude values.

Discrete-time signals - signals with a discrete independent time variable - We only know the amplitude of the signal at discrete instants in time.



continuous time



discrete-time

Discrete-time signals are represented as sequences of values called samples.

discrete-time sequence $\longrightarrow$ $\{x[n]\}$

discrete-time sample $\longrightarrow$ $x[n]$

where $n \in \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

"All integers"

Book Notation - $\{x[n]\} = \{\ldots, 0.95, -0.2, \underset{\uparrow}{2.17}, 1.1, 0.2, \ldots\}$

$x[0]$

$x[-1] =$

$x[1] =$

Introduction to Sampling

- A discrete-time sequence is formed by uniformly sampling a continuous-time waveform,

$$x_a(t) \longrightarrow \boxed{\text{sampling}} \longrightarrow x[n]$$

Evaluate $x_a(t)$ at integer multiples of the sample period T_s.

$$x_a(t)\Big|_{t=nT_s} = x[n]$$

Example:
$$x_a(t) = \cos(2\pi F_0 t + \phi)$$

$$x_a(t)\Big|_{t=nT_s} = x[n] = \cos(2\pi F_0 nT_s + \phi)$$
$$= \cos\left(2\pi \frac{F_0}{F_s} n + \phi\right)$$

where $F_s = \frac{1}{T_s}$ "sample frequency"

Using $f_0 = \frac{F_0}{F_s}$, called normalized frequency, we can write -

$$x[n] = \cos(2\pi f_0 n + \phi)$$

- The book likes to use angular frequency $\Rightarrow \Omega_0 = 2\pi F_0$

$$x[n] = \cos(\Omega_0 nT_s + \phi) = \cos\left(\frac{\Omega_0}{F_s} n + \phi\right)$$

$$= \cos(\omega_0 n + \phi) \quad \text{where } \omega_0 = \frac{\Omega_0}{F_s}$$

ω_0 is called "normalized angular frequency".

Note - $\omega_0 \left(\frac{\text{radians}}{\text{sample}}\right)$

- In general, discrete-time sinusoids are not unique!

$$\cos\left((\omega_0 + 2\pi K)n\right) = \cos\left(\omega_0 n + 2\pi K n\right)$$

$$= \cos(\omega_0 n)\cos(2\pi K n) - \sin(\omega_0 n)\sin(2\pi K n)$$

$$= \cos(\omega_0 n) \quad \text{for all integers } n, K$$

- The discrete-time frequencies are bounded.

Consider $\cos(\omega_0 n)$ when $\omega_0 = \pi + \epsilon$

$$\cos(\omega_0 n) = \cos\left((\pi+\epsilon)n\right) = \cos\left((\pi+\epsilon)n - 2\pi n\right) \quad \leftarrow K = -1$$

$$= \cos\left(\pi n + \epsilon n - 2\pi n\right) = \cos\left((-\pi+\epsilon)n\right)$$

$$= \cos\left(-(\pi-\epsilon)n\right) = \cos\left((\pi-\epsilon)n\right)$$

$\Longrightarrow$ In discrete-time, a sinusoid with angular frequency $\pi + \epsilon$ is equal to a sinusoid with angular frequency $\pi - \epsilon$!

- If we let $\epsilon \to 0$, we see that π is the highest angular frequency in discrete-time.

$$\omega_0 = \pi = 2\pi f_0 \Longrightarrow f_0 = \frac{1}{2}$$

so,

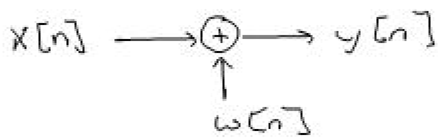
$$-\pi \le \omega_0 \le \pi \quad \text{or} \quad -\frac{1}{2} \le f_0 \le \frac{1}{2}$$

since

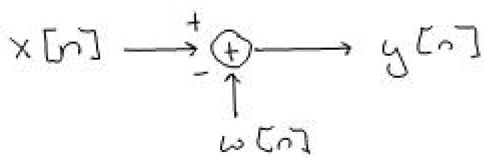
$$f_0 = \frac{F_0}{F_s} \Longrightarrow -\frac{F_s}{2} \le F_0 \le \frac{F_s}{2}$$

Elementary Operations

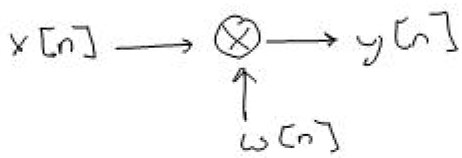
- Addition $\quad y[n] = x[n] + w[n]$



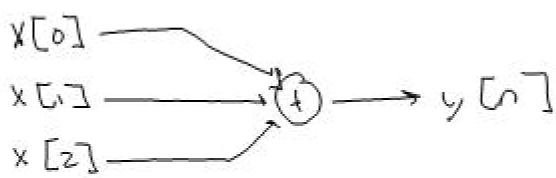
- Subtraction $\quad y[n] = x[n] - w[n]$



- Multiplication $\quad y[n] = x[n] \cdot w[n]$



- Summation $\quad y[n] = \sum_{n=0}^{2} x[n]$



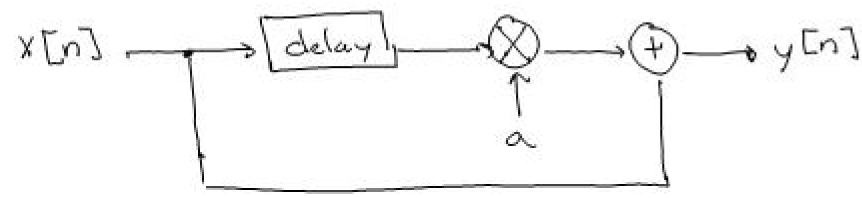
- Unit delay

or

Examples and Demos – Audio Effects

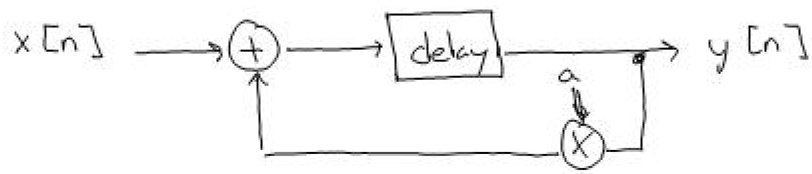
- Simple Delay (Echo)

  Matlab - delay.m



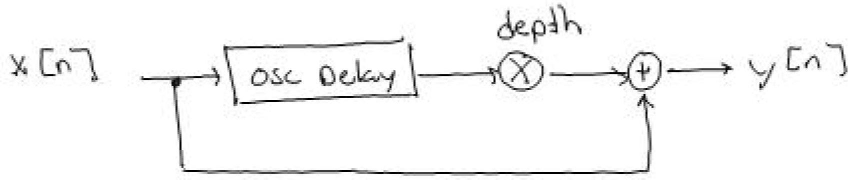
- Multi-Echo Filter

  multidelay.m



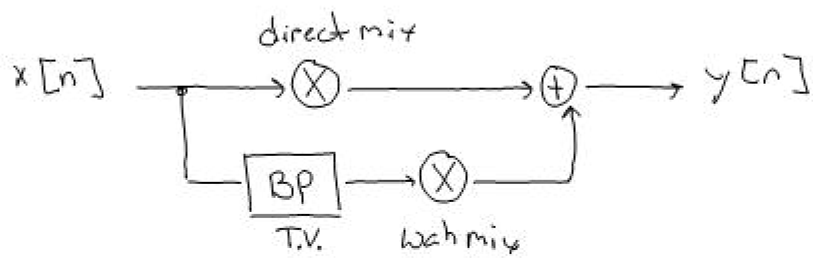
- Flanging

  flanger.m



- Wah-wah

  wahwah.m



- Can you design your own?

Energy and Power signals

- The energy of a signal is defined as

$$(CT) \qquad E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$(DT) \qquad E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \lim_{N\to\infty} \sum_{n=-N}^{n=N} |x[n]|^2$$

A necassary condition for the limits to converge is that the signal decay to zero for large time t, or n.

$$|x(t)| \to 0 \quad \text{as} \quad t \to \infty \qquad \text{or} \quad |x[n]| \to 0 \quad \text{as} \quad |n| \to \infty$$

- A finite length signal (as long as it is bounded) is an energy signal.

- Periodic, infinite length signals have infinite energy. so do infinite constant signals.

- Infinite energy signals have finite power.

$$(CT) \qquad P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \quad < \infty$$

$$(DT) \qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \quad < \infty$$

If the limit exists we call $x(t)$, or $x[n]$, power signals

- The power of a periodic signal can be computed without using limits.

Let $x(t)$ be a periodic signal with period T.

(CT) $$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt < \infty$$

(DT) $$P = \frac{1}{N}\sum_{k=n_0}^{n_0+N-1} |x[n]|^2 < \infty$$

when N is discrete-time period

Special Functions

- CT delta function, or Dirac delta $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} x(t)\,\delta(t-a)\,dt = x(a)$$

other properties -

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad\qquad \delta(at) = \frac{1}{|a|}\delta(t)$$

- CT unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ \text{undefined}, & t = 0 \\ 0, & t < 0 \end{cases}$$

- CT delta function and step function are related

$$\delta(t) = \frac{d}{dt}u(t) \qquad\qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

- DT delta, or Kronecker delta, function and unit step functions are defined by

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases} \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

These two functions are related by

$$\delta[n] = u[n] - u[n-1] \qquad u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

- The discrete-time delta function has the following properties:

$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0] \qquad \delta[an] = \delta[n] \text{ for } |a|>0$$

$$x[n] * \delta[n] = x[n] \qquad \text{(and more..)}$$

- The discrete-time delta function is called the "unit impulse".

Important: Linear-Time-Invariant systems are completely characterized in the time-domain by the output response to a unit impulse input.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$h[n]$ is called the "impulse response"

Demo - Reverb and room impulse response.

reverb.m

Classification of discrete-time systems

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

- Linearity - Involves scaling and addition (superposition)

$$a x_1[n] + b x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow a y_1[n] + b y_2[n]$$

Example: Moving average filter

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l]$$

Let $a x_1[n] + b x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow ?$

solution -

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} [a x_1[n-l] + b x_2[n-l]]$$

$$= \frac{1}{M} \sum_{l=0}^{M-1} a x_1[n-l] + \frac{1}{M} \sum_{l=0}^{M-1} b x_2[n-l]$$

$$= a y_1[n] + b y_2[n]$$

Hence, the system is linear!

- Shift Invariant (Time - Invariant)

  - Response of system to a given signal does not depend when it is applied to system input.

  - Delay or advance of input produces corresponding time shift of output.

$$x[n-n_o] \text{ or } x[n+n_o] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_o] \text{ or } y[n+n_o]$$

<u>Example</u>:

$$y[n] = 5x[n] + 3x[n-3]$$

solution - $y[n-n_o] = 5x[n-n_o] + 3x[n-3-n_o]$

$\Longrightarrow$ Time Invariant

<u>Example</u>: HF communications signal propogation through the ionosphere.

solution - depends on time of day, sun spots, etc.

$\Longrightarrow$ Time Variant

- Causal / Noncausal

  - Causal: Present output depends on present or past input values.

  - Noncausal: Present output depends on future input values.

- Memory / Memoryless

  - memory systems: Present output value depends on future or past input values.

  - memoryless systems: Present output depends only on present input value.

Note: memoryless $\Longrightarrow$ causal

Example:

$$y[n] = x[n] + x^2[n] \qquad \text{memoryless}$$

$$y[n] = \sum_{k=n-5}^{n+5} x[k] \qquad \text{memory}$$

- Stability - BIBO (Bounded Input $\Longrightarrow$ Bounded Output)

$$|x[n]| \le M_x < \infty \iff |y[n]| \le M_y < \infty$$

Important

Stability is garanteed if impulse response is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Let $h[n] = a^n u[n]$

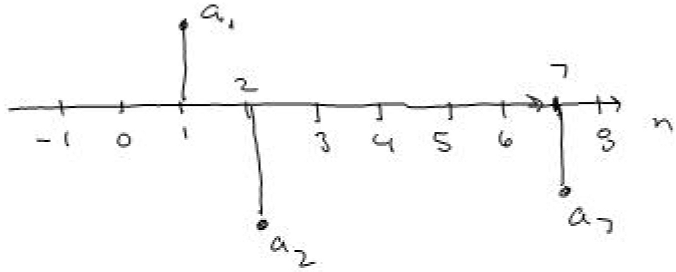
Solution - $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|}$ for $|a| < 1$

System is stable!

Impulse Response

- An arbitrary sequence can be represented as a sum of scaled, time shifted impulses.

$$x[n] = a_1 \delta[n-1] + a_2 \delta[n-2] + a_7 \delta[n-7]$$



More general -

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

- The output of an LTI system due to a unit impulse signal input applied at $t=0$ or $n=0$ is called the impulse response.

    - Completely characterizes the behavior of any LTI system.

- We can determine the output due to an arbitrary input signal by expressing input as weighted superposition of time-shifted pulses.

Example: Accumulator $\quad y[n] = \sum_{k=-\infty}^{n} x[k]$

Impulse response is found by setting $x[n] = \delta[n]$

$$\Longrightarrow \quad h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$

Input-output Relationship

- Arbitrary Input - $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

$$x[n]\,\delta[n-k] \longrightarrow \underset{h[n]}{\boxed{\text{system}}} \longrightarrow x[k]\,h[n-k]$$

so,

$$\sum_{k=-\infty}^{\infty} x[n]\,\delta[n-k] \longrightarrow \underset{h[n]}{\boxed{\text{system}}} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

or

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

Let $k = n-k$ and rewrite as

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]}$$ convolution sum

- Convolution is represented compactly as

$$y[n] = x[n] * h[n]$$

- Convolution properties -
  - commutative $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
  - associative $(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$
  - distributive $x_1[n] * (x_2[n] + x_3[n]) =$

$$x_1[n] * x_2[n] + x_1[n] * x_3[n]$$