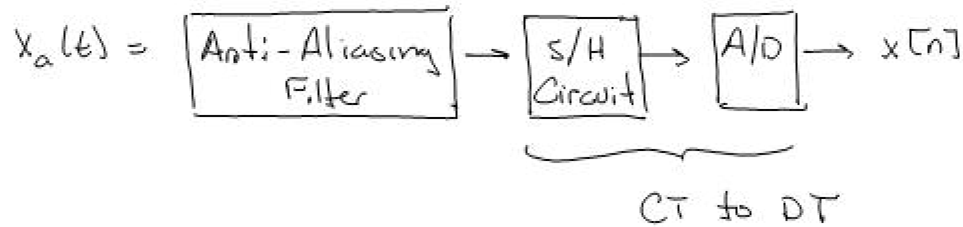
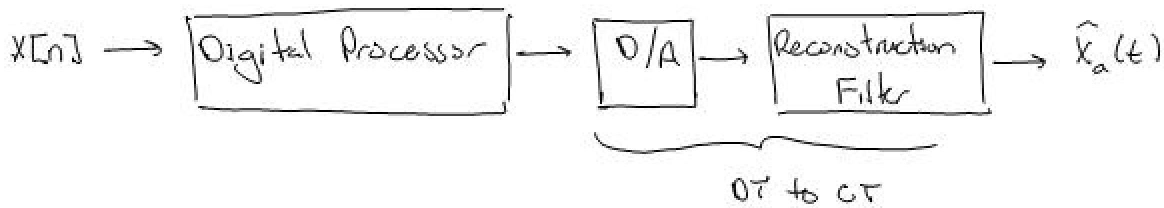
- Processing CT signals





Questions:

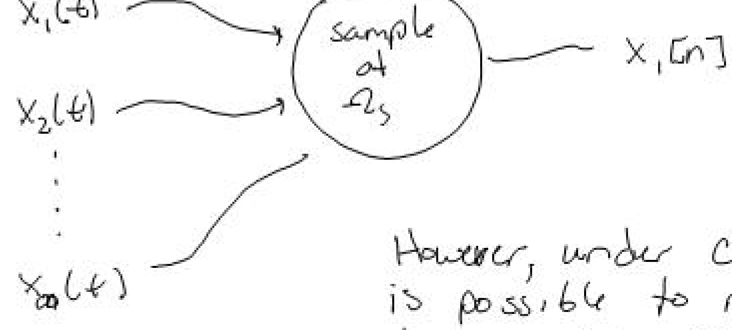
1.) What is the ideal sample rate?

2) What is the ideal anti-aliasing filter?

3) What is the ideal reconstruction filter?

- Sampling CT signals

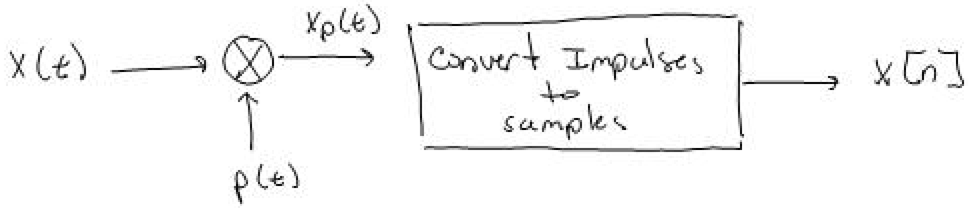
Comment - In general, there exists an infinite number of CT signals, which when sampled at the same rate Ω_s result in the same DT signal.



However, under certain circumstances, it is possible to relate a unique CT signal to a given DT sequence!

And... We can recover, exactly, the CT signal from the DT samples.

- CT signal sampling
  - Multiplication by impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$



$$x_p(t) = x(t)p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

The sample values $x(nT_s)$ weight the delta functions

We can extract the sample values by integration around each impulse..

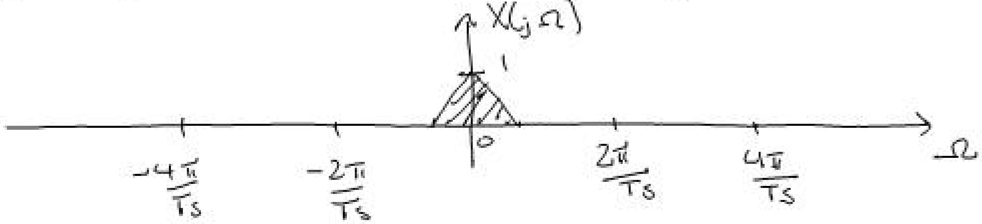
$$\int_{iT_s-\frac{T_s}{2}}^{iT_s+T_s/2} x_p(t)\,dt = \int_{iT_s-\frac{T_s}{2}}^{iT_s+T_s/2} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{iT_s-\frac{T_s}{2}}^{iT_s+T_s/2} \delta(t-nT_s)\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(i-n) = x[i]$$

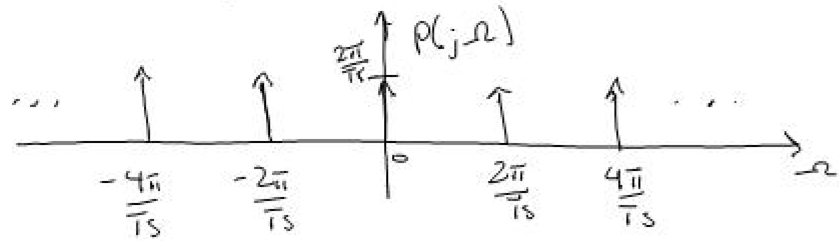
sample value.

Suppose spectrum of $x(t)$ is $X(j\Omega)$ below.



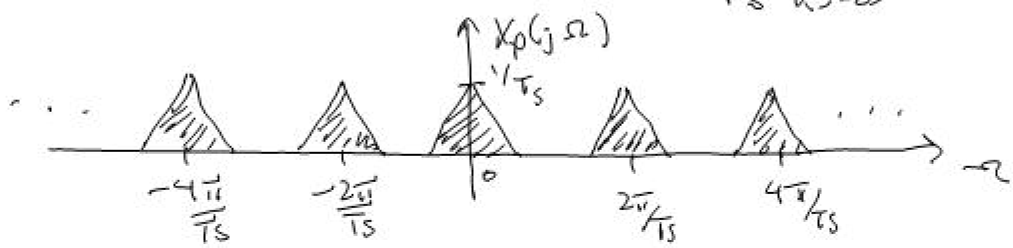
$$p(t) \xrightarrow{CTFT} P(j\Omega) = \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Let $\Omega_s = \frac{2\pi}{T_s}$



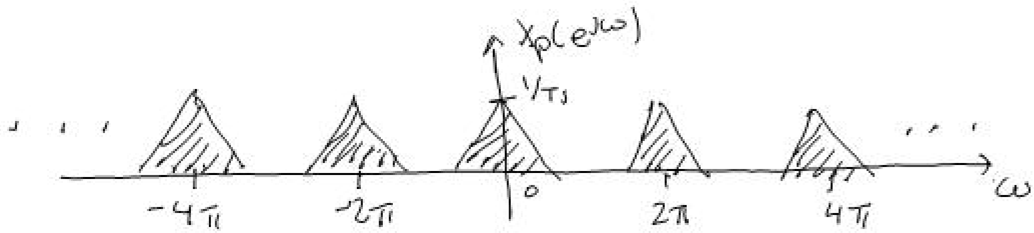
$x_p(t) = x(t)\,p(t) \xrightarrow{CTFT}$ use modulation property

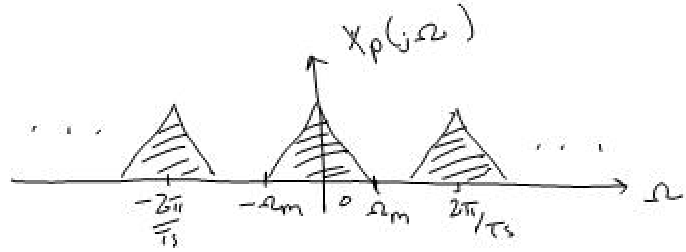
$$X_p(j\Omega) = \frac{1}{2\pi} X(j\Omega) * P(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s)$$



DTFT will look just like CTFT of $x_p(t)$ but frequency axis will be normalized.

$$\omega = \Omega T_s = \frac{\Omega}{F_s}$$





Notice – For no overlap in frequency domain replicas $\Omega_s > 2\Omega_m$. Overlap is called "aliasing". It is a form of distortion and is generally undesirable.

- CT recovery – If $\Omega_s \geq 2\Omega_m$, $x(t)$ can be recovered, exactly, from $x_p(t)$ by passing it through an ideal low pass filter with the following frequency response.

$$H_r(j\Omega) = \begin{cases} T_s, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

where Ω_c is cutoff freq.

Obviously, the cutoff freq. Ω_c should be greater than Ω_m but less that the lowest freq. of the next spectrum image, or replica.

Assume $\Omega_s = \frac{2\pi}{T_s}$, then $\Omega_m < \Omega_c < (\Omega_s - \Omega_m)$

Notice – The ideal reconstruction filter $H_r(j\Omega)$ has a gain of T_s.

- Ideal reconstruction filter impulse response

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)\, e^{j\Omega t} = \frac{T_s}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}\, dt$$

$$= \frac{\sin(\Omega_c t)}{\Omega_c t/2} = \frac{\sin(\pi t/T_s)}{\pi t/T_s} \quad \text{for } \Omega_c = \frac{\pi}{T_s}$$

- Reconstructed CT signal

$$\hat{x}(t) = h_r(t) * x_p(t)$$

$$= \frac{\sin(\pi t/T_s)}{\pi t/T_s} * \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(n - nT_s)$$

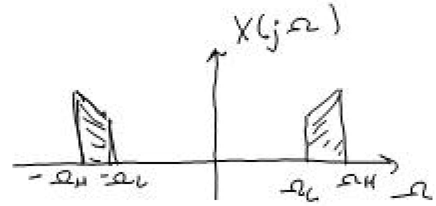
$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin(\pi (t-nT_s)/T_s)}{\pi (t-nT_s)/T_s}$$

Thus, $\hat{x}(t)$ is found by interpolating in between the sample values $x(nT_s)$ using the sinc function.

- Sampling terminology

$\Omega_s \geq 2\Omega_m$	"Nyquist condition"
Ω_m	"Nyquist frequency"
$2\Omega_m$	"Nyquist Rate"
$\Omega_s > 2\Omega_m$	"oversampling"
$\Omega_s < 2\Omega_m$	"under sampling"
$\Omega_s = 2\Omega_m$	"critical sampling"

- Bandpass sampling



$$\Omega_L \le |\Omega| \le \Omega_H$$

$B = \Omega_H - \Omega_L$ "Bandwidth"

Method 1 - Frequency shift to baseband, LP filter, sample at $\Omega_s \ge 2B$.

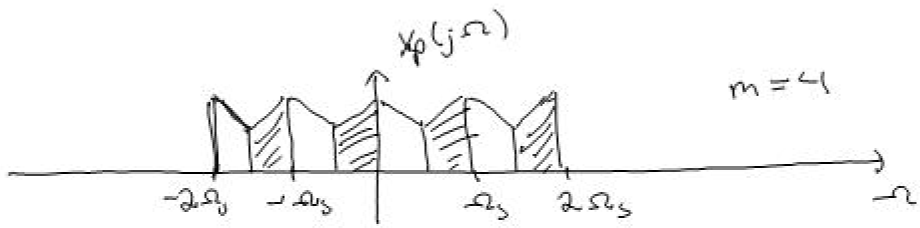
Method 2 - Sample at $\Omega_s \ge 2\Omega_H$.

Method 3 - Bandpass sample

Assume $\Omega_H = mB$ and choose $\Omega_s = 2B = \frac{2\Omega_H}{m}$ where $m > 1$.

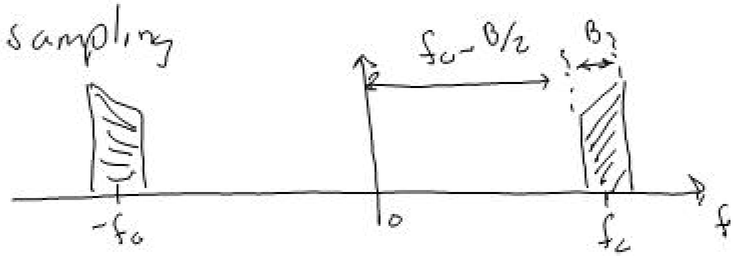
After sampling (multiplication by impulse train) -

$$X_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j(\Omega - 2kB)\right)$$



Q: What if $\Omega_H \ne mB$

- More general bandpass sampling



$$\frac{m f_s}{2} = f_c - B/2$$

$\Rightarrow$ $f_s = \frac{2f_c - B}{m}$ where $m > 0$, but $f_s \ge 2B$!

- Band pass sampling cont.

In fact -
$$f_s \le \frac{2f_c - B}{m}$$

and

$$f_s \ge \frac{2f_c + B}{m+1}$$

$$\Rightarrow \quad \frac{2f_c - B}{m} \ge f_s \ge \frac{2f_c + B}{m+1}$$

- Filtering Concept - Pass certain frequencies of an input signal without distortion and block other frequencies.

$$H(e^{j\omega}) = \left| H(e^{j\omega}) \right| e^{j\phi(\omega)}$$

design this to attenuate or gain certain frequencies.

Low Pass Example -
$$\left| H(e^{j\omega}) \right| \cong \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

If input is $x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n)$ and $\omega_1 < \omega_c$, $\omega_2 > \omega_c$ —

Output is $y[n] \approx A\left| H(e^{j\omega}) \right| \cos(\omega_1 n + \phi(\omega_1))$

- Phase Delay - single frequency component

Input: $x[n] = A \cos(\omega_0 n + \phi)$

Output: $y[n] = A\,|H(e^{j\omega_0})| \cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right)$

$$\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0} \qquad \text{"phase delay"}$$

- Group delay - when input is supperposition of many sinusoidal components, not harmonically related -

Each component has a different phase delay -

$$\tau_g(\omega) = -\frac{d\,\theta(\omega)}{d\omega} \qquad \text{"group delay"}$$