Gauss and the History of the Fast Fourier Transform

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INTRODUCTION

THE fast Fourier transform (FFT) has become well known as a very efficient algorithm for calculating the discrete Fourier Transform (DFT) of a sequence of N numbers. The DFT is used in many disciplines to obtain the spectrum or frequency content of a signal, and to facilitate the computation of discrete convolution and correlation. Indeed, published work on the FFT algorithm as a means of calculating the DFT, by J. W. Cooley and J. W. Tukey in 1965 [1], was a turning point in digital signal processing and in certain areas of numerical analysis. They showed that the DFT, which was previously thought to require N^2 arithmetic operations, could be calculated by the new FFT algorithm using only $N \log N$ operations. This algorithm had a revolutionary effect on many digital processing methods, and remains the most widely used method of computing Fourier transforms [2].

In their original paper, Cooley and Tukey referred only to I. J. Good's work published in 1958 [3] as having influenced the'r development. However, it was soon discovered there are major differences between the Cooley-Tukey FFT and the algorithm described by Good, which is now commonly referred to as the prime factor algorithm (PFA). Soon after the appearance of the Cooley-Tukey paper, Rudnick [4] demonstrated a similar algorithm, based on the work of Danielson and Lanczos [5] which had appeared in 1942. This discovery prompted an investigation into the history of the FFT algorithm by Cooley, Lewis, and Welch [6]. They discovered that the Danielson-Lanczos paper referred to work by Runge published at the turn of the century [7, 8]. The algorithm developed by Cooley and Tukey clearly had its roots in, though perhaps not a *direct* influence from, the early twentieth century.

In a recently published history of numerical analysis [9], H. H. Goldstine attributes to Carl Friedrich Gauss, the eminent German mathematician, an algorithm similar to the FFT for the computation of the coefficients of a finite Fourier series. Gauss' treatise describing the algorithm was not published in his lifetime; it appeared only in his collected works [10] as an unpublished manuscript. The presumed year of the composition of this treatise is 1805, thereby suggesting that efficient algorithms for evaluating coefficients of Fourier series were developed at least a century earlier than had been thought previously. If this year is accurate, it predates Fourier's 1807 work on harmonic analysis. A second reference to Gauss' algorithm was found in an article in the *Encyklopädie der Mathematischen Wissenschaften* [11], written by H. Burkhardt in 1904. It is interesting to note that Goldstine's and Burkhardt's work went almost as unnoticed as Gauss' work.

Because of Goldstine's discovery, the history of the FFT is again open to question. Is Gauss' method indeed equivalent to a modern FFT algorithm? If so, which type? And why was this work by a great mathematician not known to engineers and physicists even after the publication of Goldstine's book? What influenced Gauss' work and who developed the DFT? How firmly established is the date of writing? To answer these questions and to trace the history of Fourier series coefficient calculation into the eighteenth and nineteenth centuries, we undertook our own historical investigation[1], dealing primarily with original texts and concentrating on Gauss' work. What follows is a summary of our work, with historical references and evidence provided for readers to pursue the history as they wish. Another class of efficient DFT algorithms, called prime factor algorithms, which includes work from Thomas [13], Good [3], Winograd [14], and others, is not included in this investigation.

THE TWENTIETH CENTURY

Cooley, Lewis, and Welch [6] discovered that Danielson and Lanczos referred to the work of Carl David Tolmé Runge (1856–1927) [7, 8] as the inspiration for their algorithm. In these two papers and the book by Runge and König [15], a doubling algorithm is described which computes the Fourier transform of two N-point subsequences to obtain a $2N$-point Fourier transform using approximately N auxiliary operations. This algorithm is not as general as the Cooley-Tukey FFT algorithm because it only allows doubling of the original sequence length, whereas the Cooley-Tukey approach efficiently computes the DFT for any multiple of the original length. The work of Runge also influenced Stumpff, who, in his book on harmonic analysis and periodograms [16], gives a doubling and tripling algorithm for the evaluation of harmonic series. Furthermore, on p. 142 of that book, he suggests a generalization to an arbitrary multiple. All of this historical information was exposed by Cooley, Lewis, and Welch [6] in much greater detail, and is mentioned here to provide

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a background on the knowledge of the history of the FFT circa 1967.

LATE NINETEENTH AND EARLY TWENTIETH CENTURY

One interesting aspect to this historical information is that the work of Runge was well known in the early part of the twentieth century and is even referred to in the popular textbook written by Whittaker and Robinson [17], originally published in 1924. Whittaker and Robinson state that Runge's method had been widely published and cite a paper by Silvanus Phillips Thompson (1851–1916) [18]. Thompson—also a biographer of Sir William Thomson, Lord Kelvin (1824–1907)—was apparently trying to popularize Runge's method in Great Britain [18, 19]. Thompson's second paper [18] does not actually use an FFT method to obtain computational savings, but is interesting because of the discussion included at the end of the paper. This discussion includes comments by George Howard Darwin (1845–1912) (son of the more famous Charles), who claims to have used efficient techniques for the harmonic analysis of tides in 1883 [20] which he attributes to Archibald Smith (1813–1872) in 1874 [21], and to Sir Richard Strachey (1817–1908) in 1884 [22]. In Darwin's paper [20], reference is made to a paper by Joseph David Everett (1831–1904) published in 1860 [23], and credit is also given to Archibald Smith. Everett was working with Lord Kelvin on the harmonic analysis of daily temperature variations, and he gives a method for harmonic analysis using 12 samples. He claims this is an extension of the method used by Lord Kelvin in [24]. Lord Kelvin used a method based on 32 samples which was credited to Archibald Smith, originally published in 1846 [25] and presented in more detail in 1850 [26] and 1855 [27]. Apparently, S. P. Thompson was unaware of the use of efficient harmonic-analysis techniques in this paper.

The British discovery of efficient techniques for harmonic analysis can be reliably traced to Archibald Smith in 1846. Other algorithms had been developed independently by various researchers in the nineteenth century, and are tabulated on pp. 686–687 of [11]. The earliest method referenced in [11] is that of Francesco Carlini (1783–1862) from 1828 [28] for $n = 12$. The only other method which predates Archibald Smith's is that of Peter Andreas Hansen (1795–1874) from 1835 [29] for $n = 64$. Hansen was heavily influenced by Gauss in his astronomical work, but does not mention Gauss in the development of his algorithms for harmonic analysis, for reasons which shall be made clear later.

An important detail that should not be overlooked is that most of the methods preceding Runge were not intended for computing harmonics above the fourth. This was adequate for most applications of harmonic analysis in the nineteenth century, because the measurement quantization was generally on the same order of magnitude as the contributions of the higher-order harmonics. These methods are, therefore, similar to what are now called "pruned" FFT's [30]. Most of these methods were described by computational tables for a fixed number of samples and were not presented as general techniques for computing harmonics for an arbitrary number of samples.

[1]This has resulted in a bibliography of over 2000 entries [12].

EARLY NINETEENTH CENTURY

Based only on the aforementioned evidence, one might conclude that FFT-type methods originated with Francesco Carlini in 1828. However, on p. 249 of *A History of Numerical Analysis from the 16th Through the 19th Century* by Herman H. Goldstine [9], the following footnote appears.

> This fascinating work of Gauss was neglected and was rediscovered by Cooley and Tukey in an important paper in 1965.

This quotation refers to the treatise written by Gauss (1777–1855) entitled "Theoria Interpolationis Methodo Nova Tractata" [10]; this was published posthumously in volume 3 of his collected works in 1866, but was originally written, most likely, in 1805. Goldstine [9] gives, on p. 249–253, an English translation of parts of Gauss' paper related to trigonometric interpolation algorithms, which are Articles 25–28 of the original Latin text. Gauss wrote his important works and his personal mathematical diary in a nineteenth-century version of Latin which is now called neo-Latin. Unfortunately, neo-Latin is difficult for the casual student of classical Latin to translate accurately. Another source of difficulty for the modern reader is the notation adopted by Gauss to describe his method. Examples of this notation are the use of π as the length of a sequence (instead of N), the use of symbols $a, b, c, d, \ldots; a', b', c', d', \ldots; a'', b'', c'', d''$; etc. as the indices of the time series; and the use of capital letters to refer to the value of a function at a point whose index is the corresponding small letter (e.g., $f(a) = A$). Gauss' method was also derived using real trigonometric functions rather than complex exponentials, making it more difficult to relate his method to current FFT techniques.

At this point, there are three questions to be addressed. Was the method used by Gauss in [10] a form of what is now called an FFT? If so, what type and how general was it? Did Gauss realize he had developed a computationally efficient algorithm?

THE BACKGROUND OF GAUSS' WORK

The analysis of a trigonometric series goes back at least to the work of Leonhard Euler (1707–1783) [31–33]. He had no particular application in mind as far as we can tell. He dealt with infinite, cosine-only series, and did not concern himself with convergence issues. The stature of Euler in his own time meant that his work was read by his contemporaries, particularly the French mathematicians Clairaut, d'Alembert, and Lagrange. Alexis-Claude Clairaut (1713–1765) published in 1754 [34] what we currently believe to be the earliest formula for the DFT, but it was restricted to a cosine-only finite Fourier series. Joseph Louis Lagrange (1736–1813) published a DFT-like formula for sine-only series in 1762 [35]. Daniel Bernoulli (1700–1782) expressed the form of a vibrating string as a series of sine and cosine terms with arguments of both time and distance in 1753 [36], which implied that an arbitrary function could be expressed as an infinite sum of cosines. The most authoritative compilation of the early history of trigonometric series is a 536-page article by H. Burkhardt [37].

Clairaut and Lagrange were concerned with orbital mechanics and the problem of determining the details of an orbit from a finite set of observations. Consequently, their data was periodic and they used an interpolation approach to orbit determination: in modern terminology and notation, an even periodic function $f(x)$ which has a period of one is represented as a finite trigonometric series by

$$f(x) = \sum_{k=0}^{N-1} a_k \cos 2\pi kx, \qquad 0 < x \leq 1. \tag{1}$$

The problem is to find the coefficients $\{a_k\}$ from the N values of $f(x)$ for values of $x_n = n/N$ with $n = 0, 1, \ldots, N-1$. By forcing $f(x)$ to equal the observed values at the abscissas $\{x_n\}$, one can easily show that the coefficients $\{a_k\}$ are given by the cosine-only DFT of the observed values of $f(x)$. Gauss knew of the works of Euler and Lagrange [38]: he borrowed their works from the library at Göttingen while a student from 1795 to 1798.[2]

GAUSS' ALGORITHM FOR COMPUTING THE DFT

Gauss extended this work on trigonometric interpolation to periodic functions, which are not necessarily odd or even. This was done while considering the problem of determining the orbit of certain asteroids from sample locations. These functions are expressed by a Fourier series of the form

$$f(x) = \sum_{k=0}^{m} a_k \cos 2\pi kx + \sum_{k=1}^{m} b_k \sin 2\pi kx \tag{2}$$

where $m = (N-1)/2$ for N odd, or $m = N/2$ for N even. Gauss showed in Articles 19–20 of his interpolation treatise that if one were given the values of $f(x_n)$, $x_n = n/N$ ($n = 0, 1, \ldots, N-1$), the coefficients a_k and b_k are given by the now well-known formulas for the DFT [40]. This set of equations is the earliest explicit formula for the general DFT that we have found.

Gauss develops his efficient algorithm by using N_1 (or μ in his notation) equally spaced samples over one period of the signal. This set of N_1 samples is a subset of N total samples, where $N = N_1 N_2$ (or $\pi = \mu\nu$). Gauss computes the finite Fourier series which passes through these samples using m harmonics, where m is as defined in (2). He then assumes that another subset of N_1 equally spaced samples of the signal are measured which are offset from the original set of samples by a fraction, $1/N_2$, of the original sample interval where N_2 is a positive integer. A finite Fourier series with m harmonics is computed which passes through this new set of samples, and it is discovered that these coefficients are quite different from those computed for the original N_1 samples. Gauss realized the problem and proceeded to develop a method for correcting the coefficients he had already calculated, and to determine additional coefficients for the higher-frequency harmonics. Using modern terminology, we would say that the waveform was undersampled, and that therefore the coefficients were in error because of aliasing of the high-frequency harmonics [2].

Gauss' solution to this problem was to measure a total of N_2 sets of N_1 equally spaced samples, which together form an overall set of $N = N_1 N_2$ equally spaced samples. The finite Fourier series for the entire set of N samples is computed by first computing the coefficients for each of the N_2 sets of length N_1, all shifted relative to a common origin, and then computing coefficients of the N_1 series of length N_2 which are formed from the coefficients of corresponding terms in the N_2 sets of coefficients originally computed. A final trigonometric identity is used to convert these coefficients into the finite Fourier series coefficients for the N samples.

In modern terminology, the DFT of the samples of $f(x)$ is defined by

$$C(k) = \sum_{n=0}^{N-1} X(n) W_N^{nk} \tag{3}$$

where, if $f(x)$ has a period of one, $X(n)=f(n/N)$ are the N equally spaced samples, $W_N=e^{-j2\pi/N}$, and $k=0, 1, \ldots, N-1$ are the indices of the Fourier coefficients. This DFT can be rewritten in terms of N_2 sets of N_1 subsamples by the change of index variables [41]

$$n = N_2 n_1 + n_2$$
$$k = k_1 + N_1 k_2$$

for $n_1, k_1 = 0, 1, \ldots, N_1 - 1$ and $n_2, k_2 = 0, 1, \ldots, N_2 - 1$. Each subsequence is a function of n_1; n_2 denotes which subsequence it is. The DFT in (3) becomes

$$C(k_1 + N_1 k_2) = \sum_{n_2=0}^{N_2-1} \left[\sum_{n_1=0}^{N_1-1} X(N_2 n_1 + n_2) W_{N_1}^{n_1 k_1} W_N^{n_2 k_1} \right] W_{N_2}^{n_2 k_2} \tag{4}$$

where the inner sum calculates the N_2 length-N_1 DFT's corrected by W_N, and the outer sum calculates the N_1 length-N_2 DFT's. This is exactly the exponential form of Gauss' algorithm where the W_N term accounts for the shifts from the origin of the N_2 length-N_1 sequences. This is also exactly the FFT algorithm derived by Cooley and Tukey in 1965 [1] where the W_N is called a *twiddle factor* [2], a factor to correct the DFT of the inner sum for the shifted samples of $X(n)$. The equivalence of Gauss' algorithm and the Cooley-Tukey FFT is not obvious due to the notation and trigonometric formulation of Gauss. One can easily verify the results by calculating the inner sum of (4) and comparing the numerical results with the intermediate calculation in Article 28 of [10] after converting from exponential to trigonometric form and correcting for factors of $1/N_1$.

The example of $N = 12$ for the orbit of the asteroid

[2]Dunnington [39] has compiled a list of books borrowed by Gauss at Göttingen by searching the University library records.

DATING OF GAUSS' WORK ON THE FFT
THEORIA INTERPOLATIONIS METHODO NOVA TRACTATA
VOLUME III, *WERKE*

April 30, 1777	Gauss is born in Brunswick.
Sept., 1795	Gauss arrives at Göttingen. Throughout his stay there, he checked out many books from the university library. In particular, he continually read *Miscellanea Taurinensia,* the proceedings of the academy located in Turin. When these proceedings were being published, Lagrange was there and this journal served as his exclusive outlet. In Volume III, Lagrange's DFT (sine only) appears.
Nov. 25, 1796	Diary entry 44[3], which reads *Formula interpolationis elegans*. Translated, this entry means "Elegant formula for interpolation." The editor of the diary connected this with the Lagrange interpolation formula. No specific library books can be readily connected to this entry.
Dec. 1796	Diary entry 46, which reads *Formulae trigonometricae per series expressae*. Translated, this entry means "Trigonometric formulas expressed with series." The editor of the diary made no comment about this entry. On December 16, 1796, Gauss checked out both volumes of Euler's *Opuscula Analytica*. In this work, Euler relates algebraic series for trigonometric functions to the calculation of π. This diary entry probably refers to this kind of series.
Sep. 28, 1798	Gauss returns home to Brunswick after finishing his studies at Göttingen.
Dec. 1804–Dec. 1805	Correspondence between Gauss and Bessel indicates their concern with the interpolation problem. No mention is made, however, of the trigonometric interpolation problem.[4]
May 1805	Publication date of an issue of *Monatliche Correspondenz,* a collection of unreviewed notes containing astronomical observations and information. This note credits Gauss with further measurements of the orbit of the asteroid *Juno,* among which is the eccentricity value 0.254236.[5] This number is used by Gauss in an example in his FFT writings. Thus, the treatise must have been completed after this date.
Nov. 1805	Diary entry 124, which reads *Theoriam interpolationis ulterius excoluimus.* Translated, this entry means "We have worked out further a theory of interpolation." The editor takes this entry to mean that his treatise on interpolation could not have been written before November 1805. He refers to a notebook of Gauss consisting of short mathematical notes *(Mathematische Brouillons),* which was begun in October 1805. Volume 18 of the notebook contains an opening note on interpolation. The editor takes this note to be a first draft of the treatise. However, the collected work of Gauss does not contain this paper.
July 30, 1806	Date attached to a letter sent from Gauss to Bode, in which the eccentricity value for *Juno* of 0.2549441 is given. Presumably, this means that the FFT treatise must have been written prior to this date. This letter appeared in *Monatliche Correspondenz* later in 1806.[6]
Nov. 8, 1808	A letter from Schumacher, a former student of Gauss, to Gauss mentions that Schumacher's mother has a handwritten copy of his work on interpolation.[7] It is unclear whether this letter is referring to *Mathematische Brouillons* or the *Theoria interpolationis*.
June 8, 1816	Schumacher writes Gauss that he has a handwritten version of Gauss' work on interpolation, which he hopes Gauss will publish soon.[7] Thus, Gauss did not keep this work secret, but presumably was not interested in publishing it.

[3]The entries of Gauss' mathematical diary are published with accompanying comments by the editors in volume X.1 of the *Werke*, pp. 488–571.
[4]*Briefwechsel zwischen Gauss und Bessel,* Leipzig, 1880.
[5]Volume VI, *Werke,* p. 262.
[6]Volume VI, *Werke,* p. 279.
[7]Volume X.2, *Werke,* p. 125.

Principal Discoveries of Efficient Methods of Computing the DFT

Researcher(s)	Date	Sequence Lengths	Number of DFT Values	Application
C. F. Gauss [10]	1805	Any composite integer	All	Interpolation of orbits of celestial bodies
F. Carlini [28]	1828	12	—	Harmonic analysis of barometric pressure
A. Smith [25]	1846	4, 8, 16, 32	5 or 9	Correcting deviations in compasses on ships
J. D. Everett [23]	1860	12	5	Modeling underground temperature deviations
C. Runge [7]	1903	$2^n k$	All	Harmonic analysis of functions
K. Stumpff [16]	1939	$2^n k$, $3^n k$	All	Harmonic analysis of functions
Danielson and Lanczos [5]	1942	2^n	All	X-ray diffraction in crystals
L.H. Thomas [13]	1948	Any integer with relatively prime factors	All	Harmonic analysis of functions
I. J. Good [3]	1958	Any integer with relatively prime factors	All	Harmonic analysis of functions
Cooley and Tukey [1]	1965	Any composite integer	All	Harmonic analysis of functions
S. Winograd [14]	1976	Any integer with relatively prime factors	All	Use of complexity theory for harmonic analysis

Pallas was worked out in Article 28 of [10] for $N_1 = 4$, $N_2 = 3$ and for $N_1 = 3$, $N_2 = 4$, and an example was given in Article 41 for $N = 36$ with $N_1 = N_2 = 6$ and for the special case of odd symmetry.

In Article 27, Gauss states that his algorithm can be generalized to the case where N has more than two factors, although no examples are given. This and the observed efficiency are seen in the following translation from Article 27.

> And so for this case, where most of the proposed values of the function X, an integral period of the arrangement, the number is composite and $= \pi = \mu\nu$, in articles 25, 26, we learned that through the division of that period into ν periods of μ terms, it produces, when all values are given, the same satisfactory function, which by the immediate application of the general theory applies to the whole period; truly, that method greatly reduces the tediousness of mechanical calculations, success will teach the one who tries it. Now the work will be no greater than the explanation, of how that division can be extended still further and can be applied to the case where the majority of all proposed values are composed of three or more factors, for example, if the number μ would again be composite, in that case clearly each period of μ terms can be subdivided into many lesser periods.

Gauss did not, however, go on to quantify the computational requirements of his method to obtain the now familiar $N \sum N_i$ or $N \log N$ expression for its computational complexity. From his short excerpt, Gauss clearly developed his procedure because it was computationally efficient and because it could be applied to a select, but interesting, set of sequence lengths. Thus, Gauss' algorithm is as general and powerful as the Cooley-Tukey common-factor algorithm and is, in fact, equivalent to a decimation-in-frequency algorithm adapted to a real data sequence.

The hints used by Gauss' biographers [39, 42] and by us to establish a date for this work are summarized in the accompanying table. From these facts, we infer that Gauss wrote this treatise in October-November 1805. This work predates the 1807 work of Jean Baptiste Joseph Fourier (1768–1830) on representations of functions as infinite harmonic series. Fourier did not publish his results until 1822 [43] because his presentation to the Academy of Sciences in Paris on December 21, 1807 was not well received by Lagrange and was refused publication in the Memoirs of the Academy. One of his earlier manuscripts dates back to 1804–1805 and includes research which he may have started as early as 1802 [44].

The DFT approach to solving the orbital mechanics problem was one of several. Approaches related to Newtonian mechanics gave alternative solutions to the problem, and, in the end, came to be preferred even by Gauss. Mathematicians concerned with orbital mechanics who read his posthumous treatise in 1866 would probably not have found the technique described therein of much interest. Thus, the dated nature of the publication, its publication in Latin, and the lack of notice of Goldstine's and Burkhardt's work contributed to the "loss" of Gauss' FFT technique until now.

CONCLUSION

Although it appears that the discrete Fourier transform should really have been named after Gauss, it is obviously not practical to rename it. However, the term "Gauss-Fourier transform (GFT)" was coined by Hope [45] in 1965, and the term "discrete Gauss transform (DGT)" has also been previously used [46]. T. S. Huang was unknowingly accurate when he satirically remarked in 1971 that the FFT was Gauss' 1001st algorithm [47].

This investigation has demonstrated, once again, the virtuosity of Carl Friedrich Gauss. In addition, it has shown that certain problems can be timeless, but their solution rediscovered again and again. Burkhardt pointed out this algorithm in 1904 and Goldstine suggested the connection between Gauss and the FFT in 1977, but both of these went largely unnoticed, presumably because they were published in books dealing primarily with history. It was shown that various FFT-type algorithms were used in Great Britain and elsewhere in the nineteenth century, but were unrelated to the work of Gauss and were, in fact, not as general or well-formulated as Gauss' work. Almost one-hundred years passed between the publication of Gauss' algorithm and the modern rediscovery of this approach by Cooley and Tukey.

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