Advanced Algorithms for Geo-Information Systems

WiSe 2024/25

Approximation Algorithms

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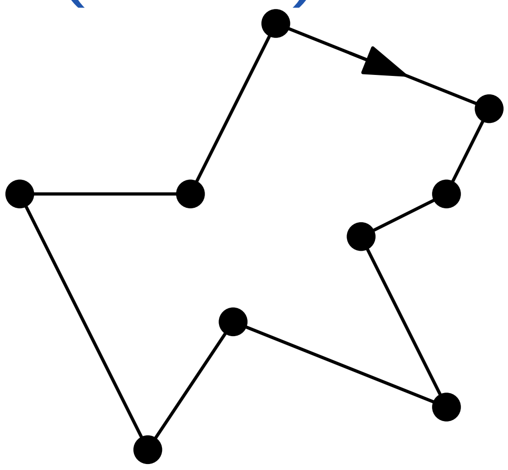
Institut für Geodäsie und Geoinformation
Universität Bonn

Travelling Salesperson Problem (TSP)



Input (optimization variant of TSP):
- n cities
- distances between the cities

Find a round trip of minimum total length visiting all cities

$\mathcal{NP}$-hard!

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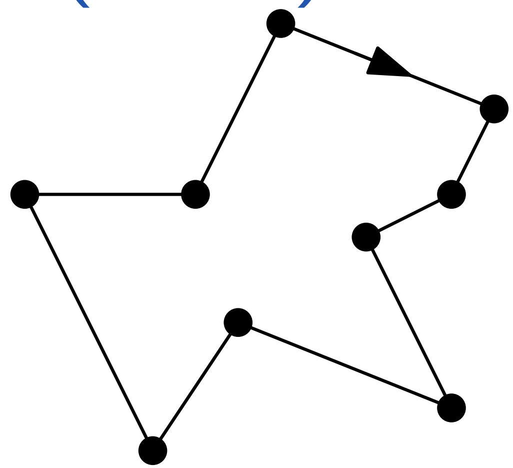

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even if distance is a **metric**, which implies $D(a, c) \leq D(a, b) + D(b, c)$ for all $a, b, c \in S$

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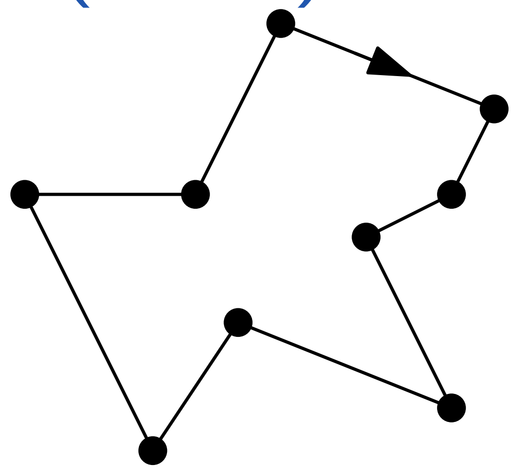

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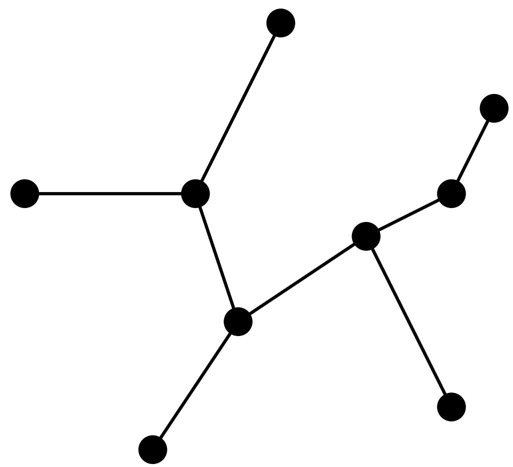

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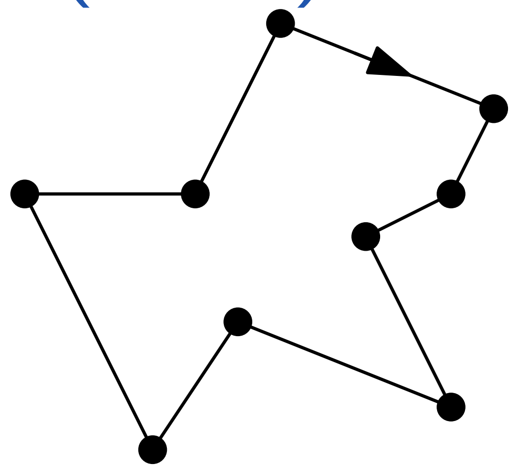

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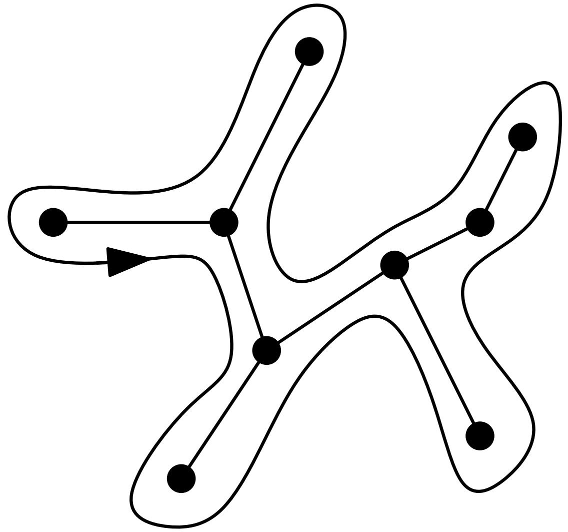

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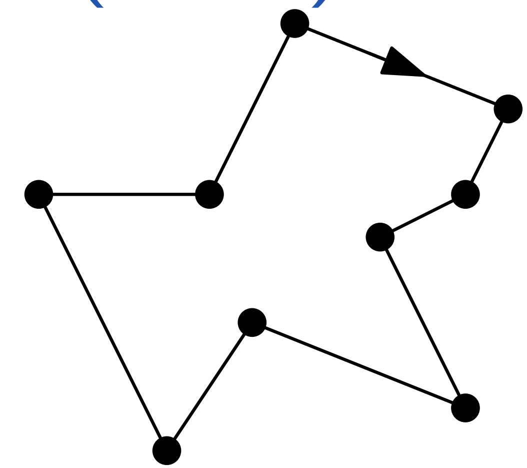

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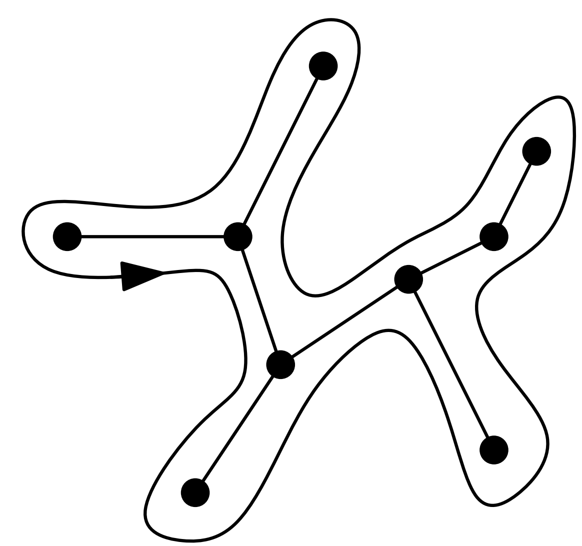

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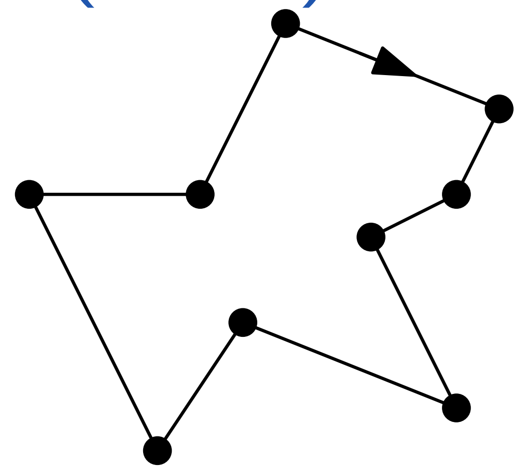

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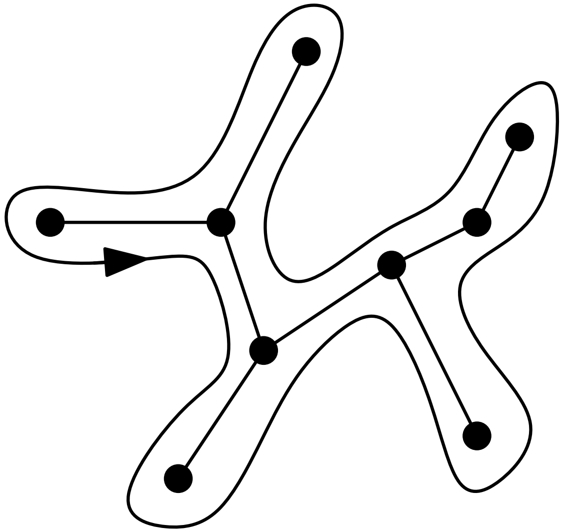
Observation: $w(MST) \leq opt$

Since a round trip (minus one edge) is also a spanning tree, but not necessarily optimal.



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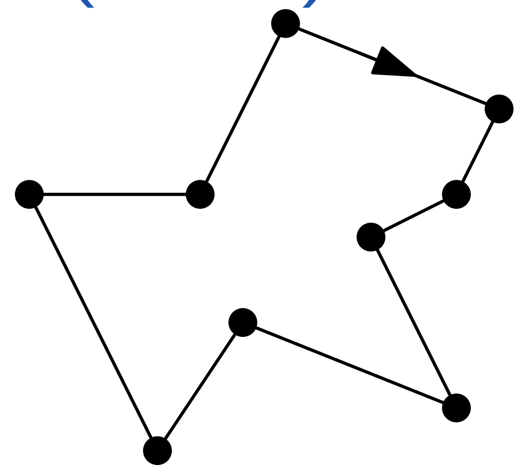

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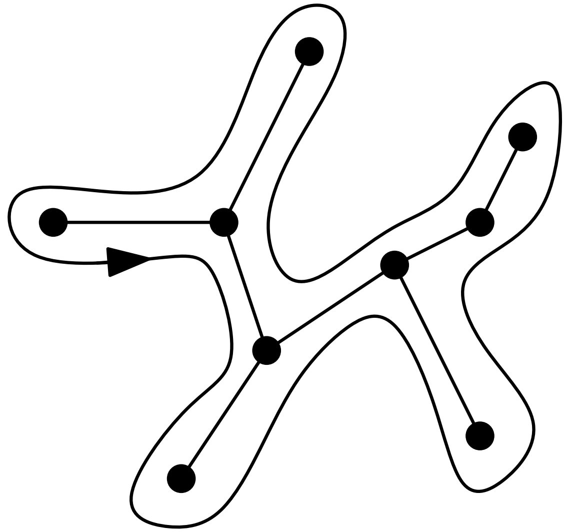

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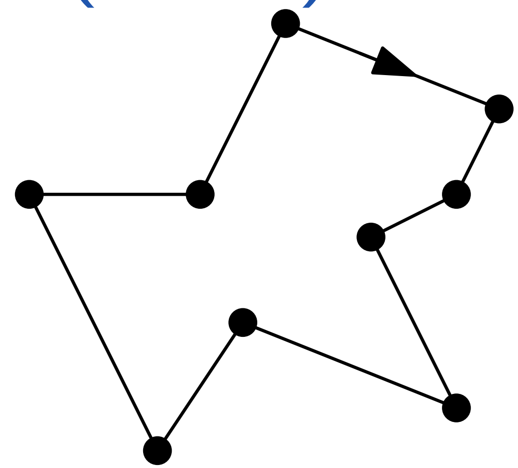

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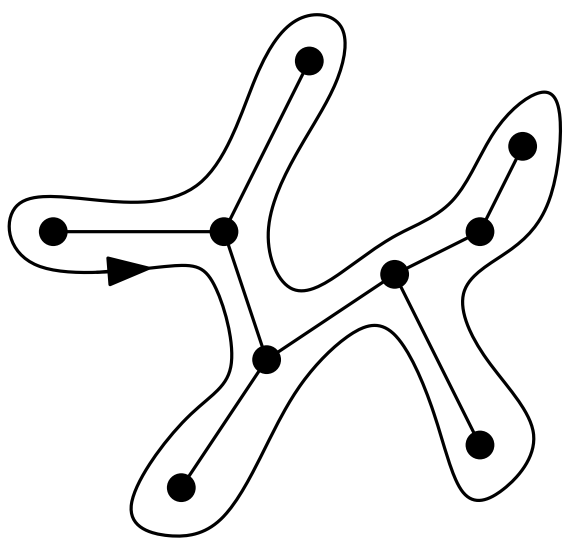

Length of solution is at most twice as long as optimal solution!

Approximation Algorithms

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- A "2-approximation" sounds rather bad, but:
  a) $2 \cdot opt$ is an upper bound. Often the solution of an approximation algorithm is much better than the approximation guarantee.
  b) In contrast to approximation algorithms, heuristics do not provide any guarantee of quality!

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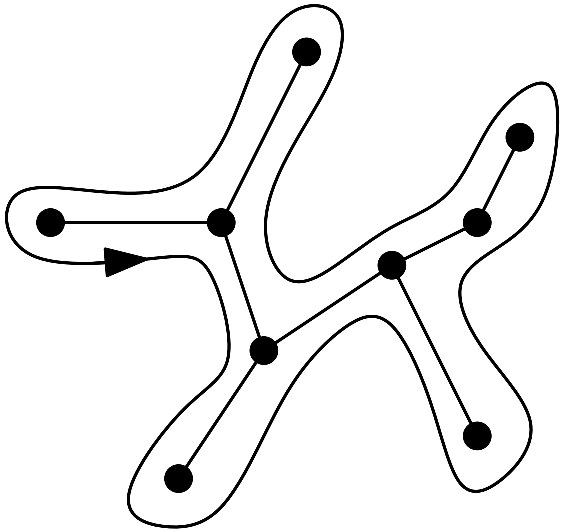

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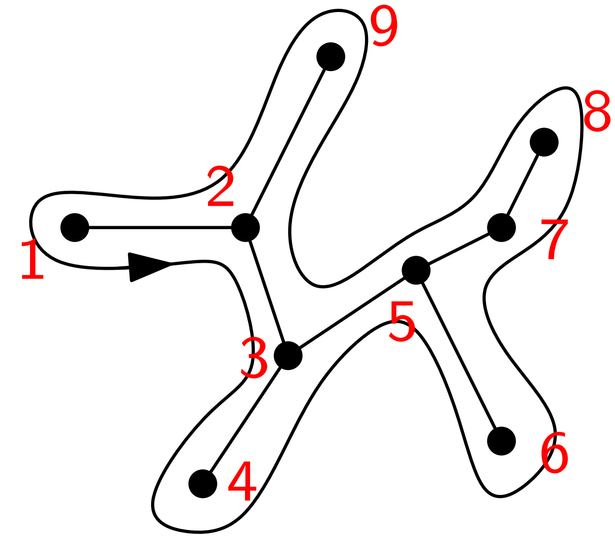

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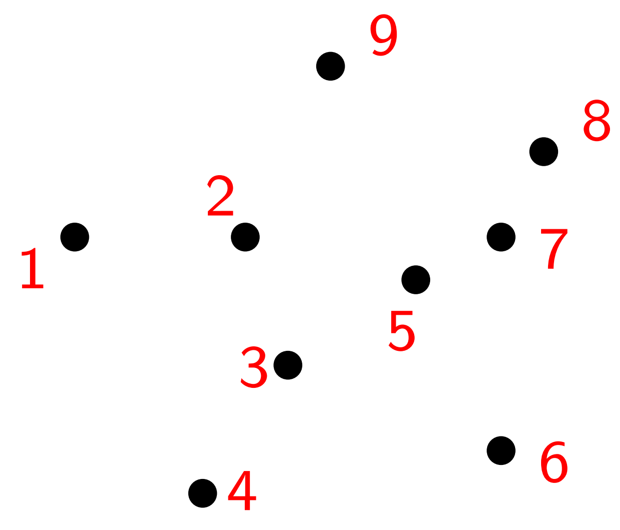

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- Visit the cities in that order – without visiting a city twice.

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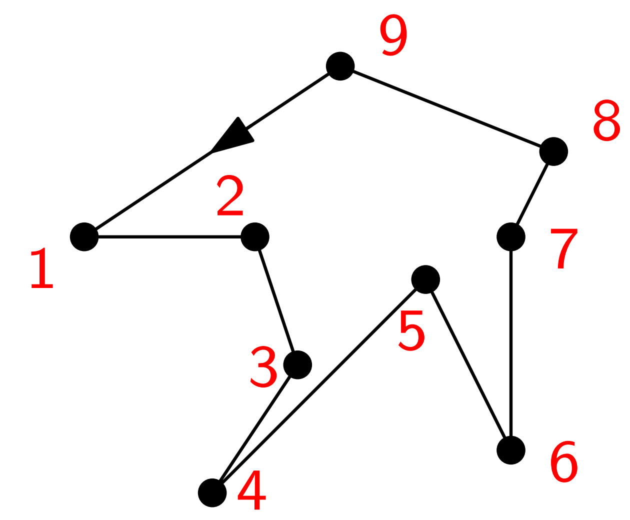

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- Result:
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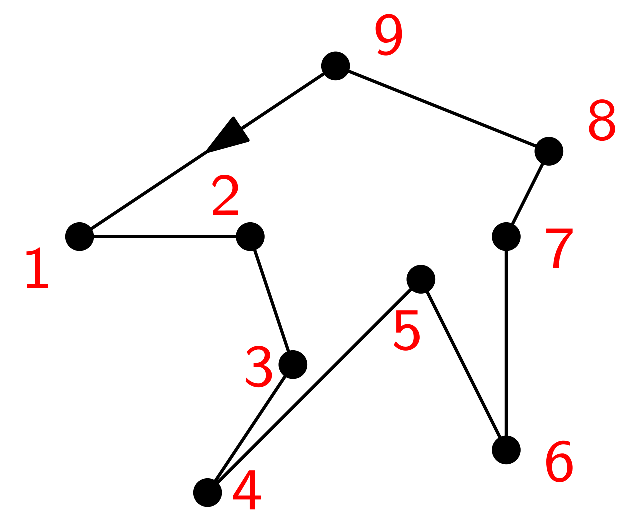

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Is there an algorithm with approximation factor < 2 ?

The Algorithm of Christofides (1976)

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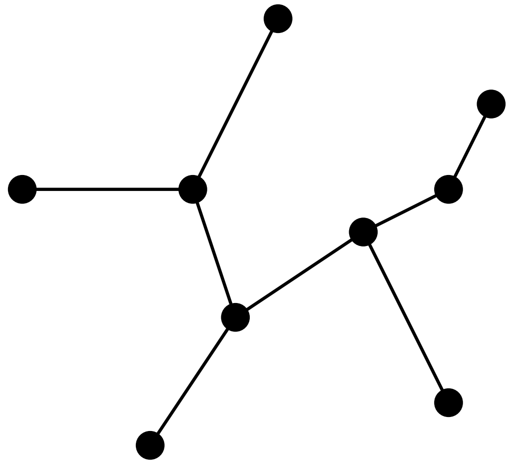

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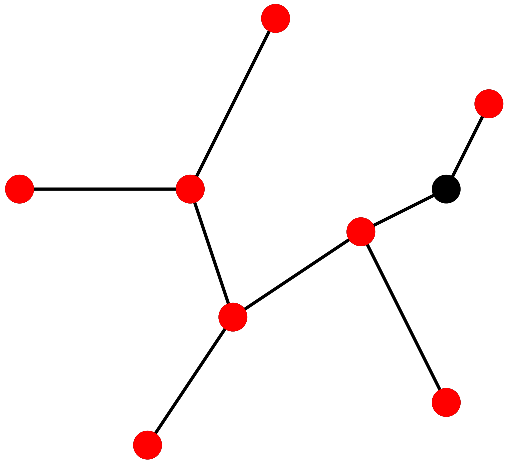
Algorithm for metric TSP:

- Compute minimum spanning tree MST of cities.
- Let V be the set of cities with odd degree in MST.



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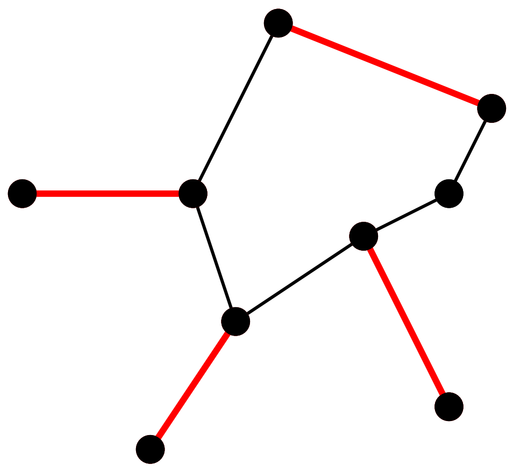

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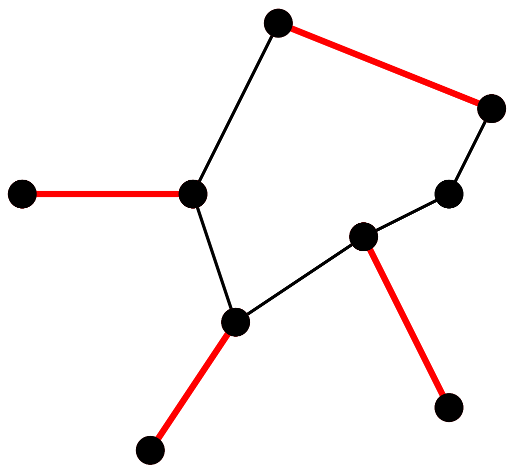

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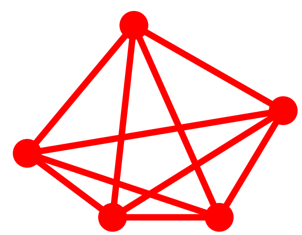
Terminology:

The **complete graph** for a given node set contains an edge connecting each two nodes.



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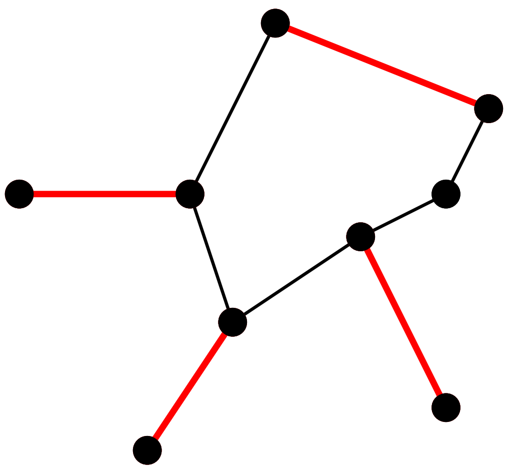

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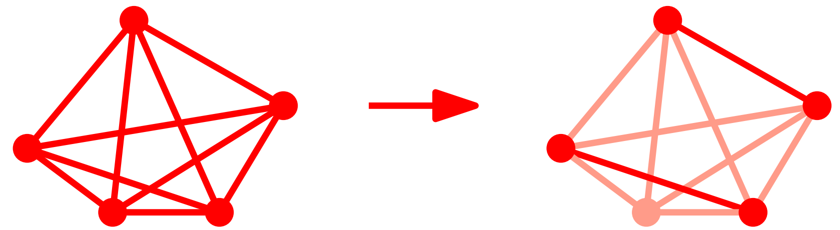
Terminology:

A **matching** of a graph $G = (V, E)$ is a subset of E that contains for each node in V **at most** one incident edge.



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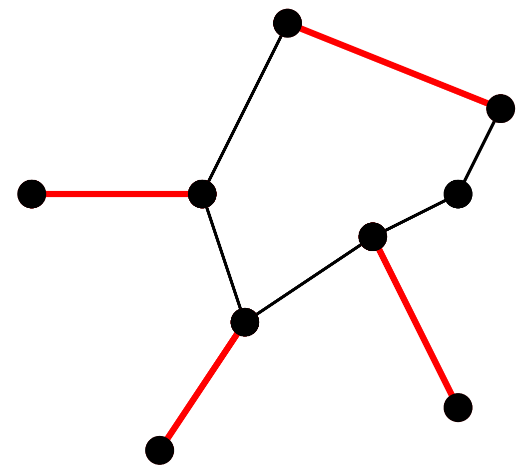
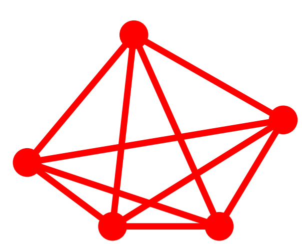


Terminology:

A **perfect matching** of a graph $G = (V, E)$ is a subset of E that contains for each node in V **exactly** one incident edge.



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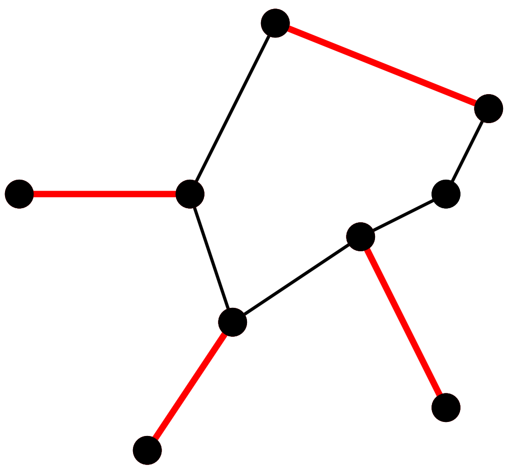

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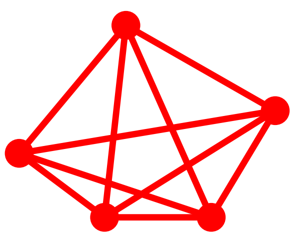

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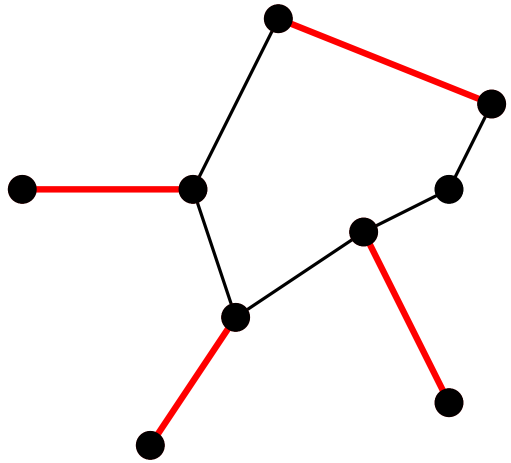

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Given in our case, since $\sum_{v \in V} deg(v)$ is even.

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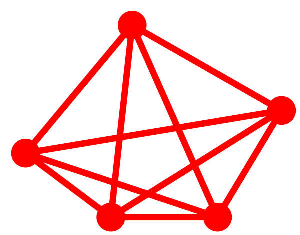

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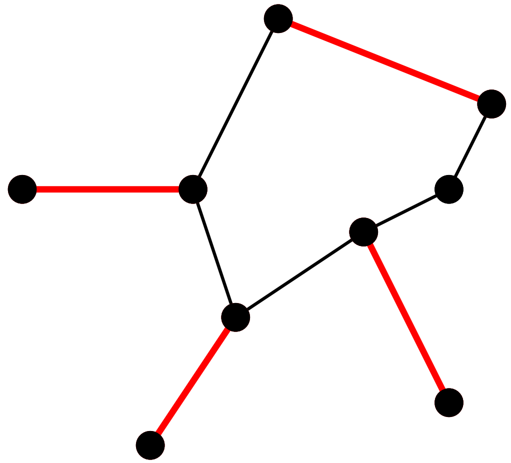

can be computed in $O(|V|^3)$ time

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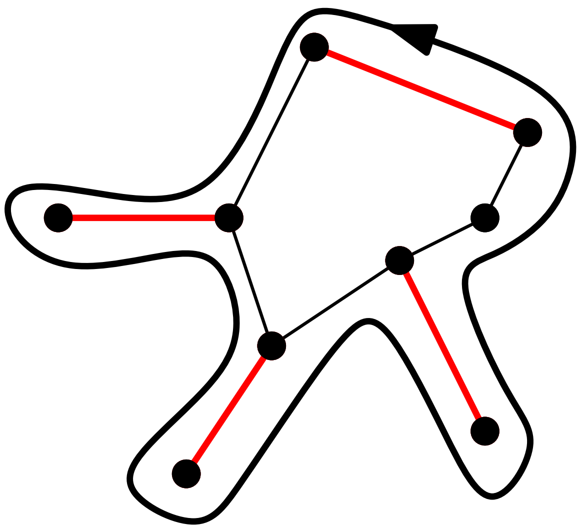

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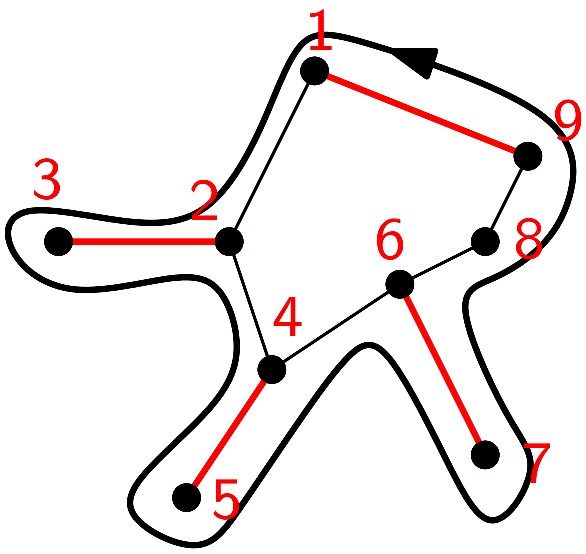

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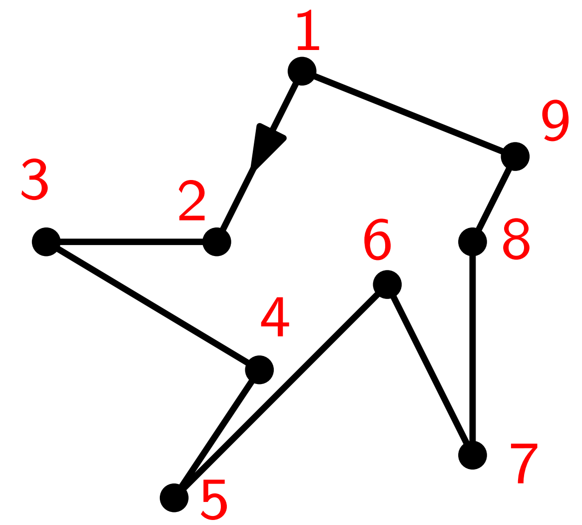

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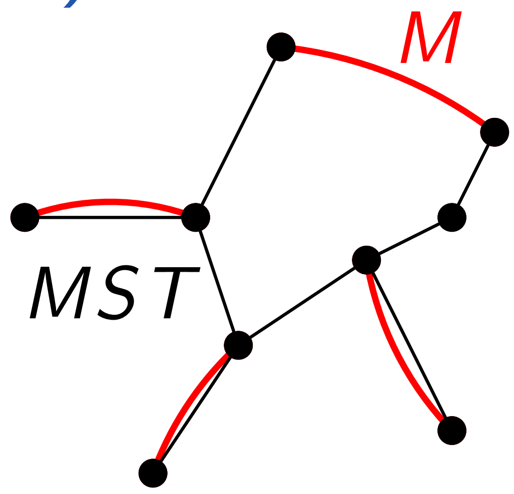

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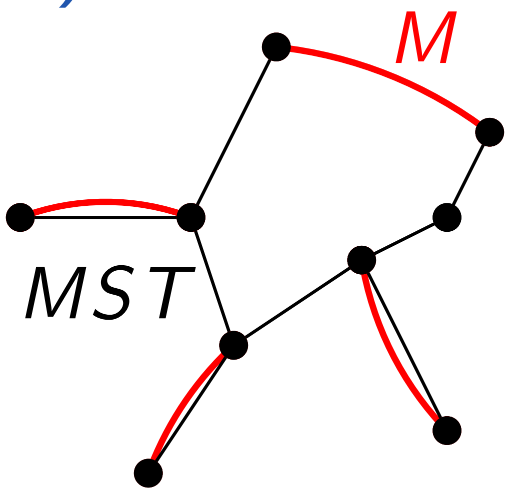

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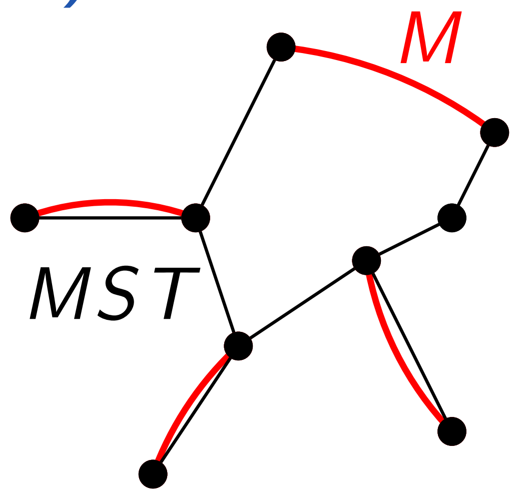

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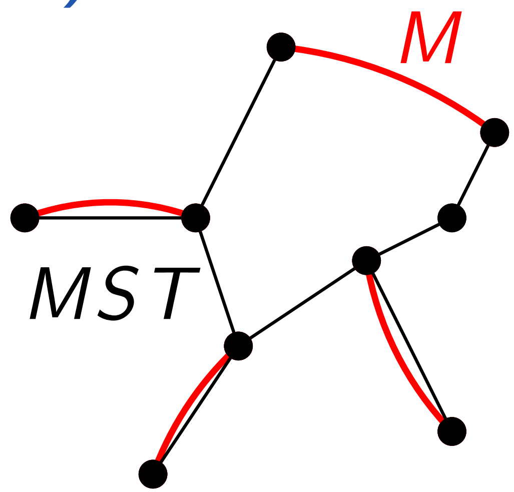

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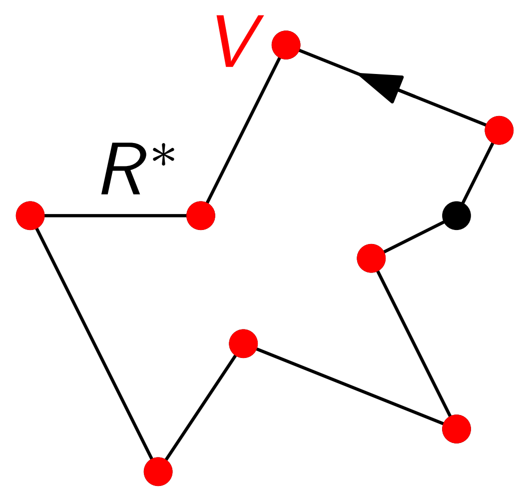

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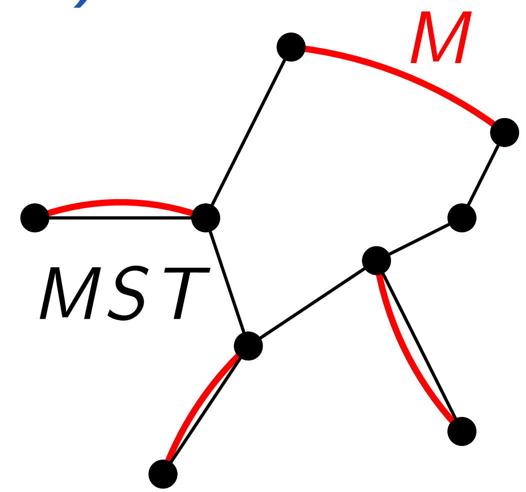

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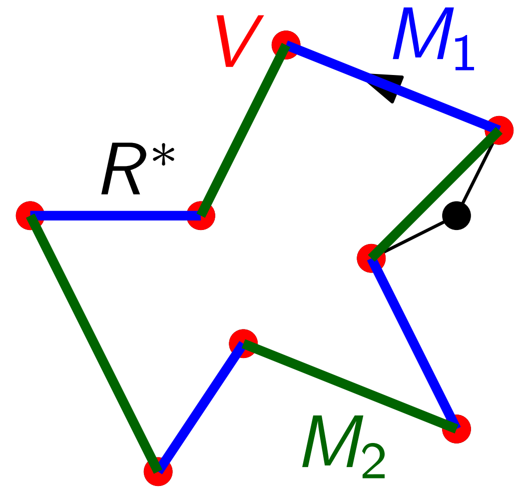
- R^* defines two perfect matchings M_1 and M_2 in the complete graph G for V.



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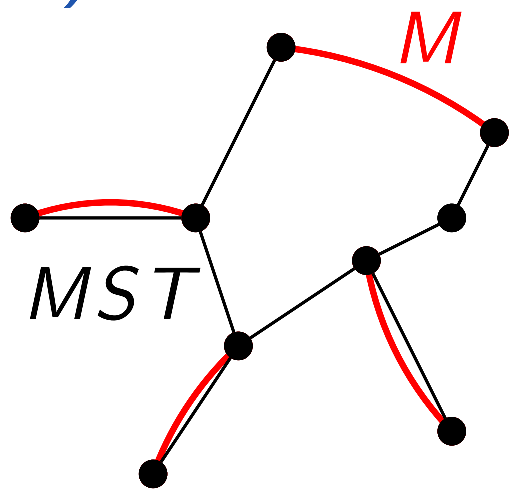

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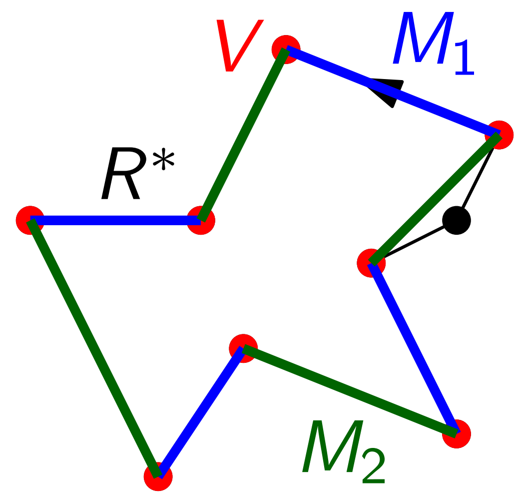
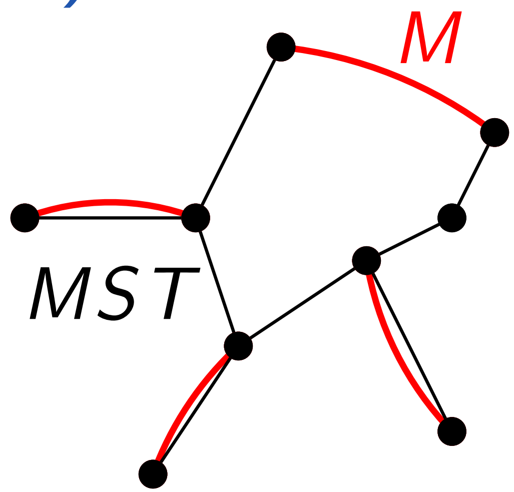


The Algorithm of Christofides (1976)



Theorem: The round trip R returned by the algorithm of Chr. is a 1.5-approximation.

Proof:

It holds $w(R) \leq w(MST) + w(M)$.

Furthermore, it holds $w(MST) \leq opt$. To show: $w(M) \leq \frac{1}{2} opt$

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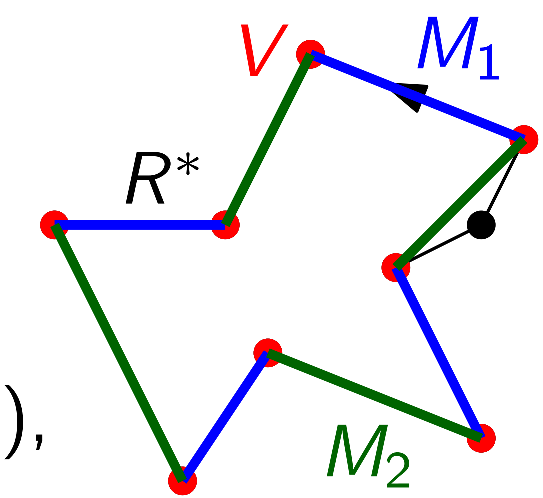

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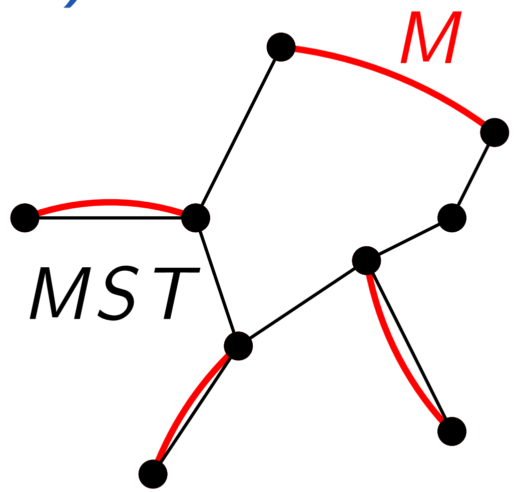

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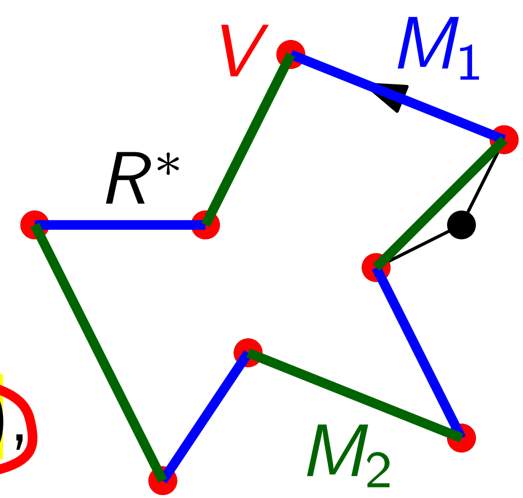

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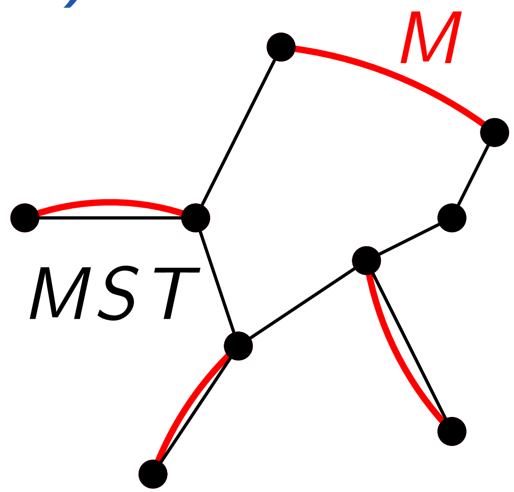

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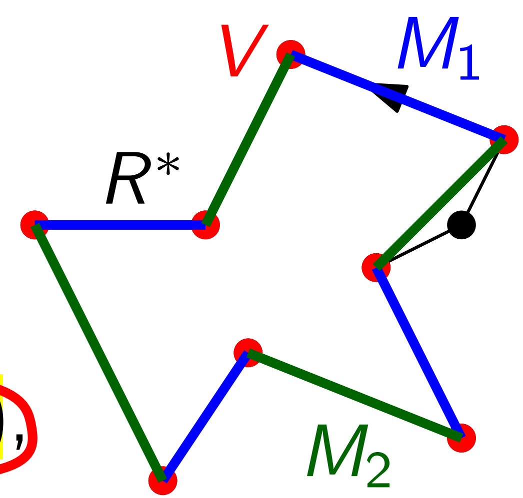

□

Summary

- Approximation algorithms are often used for $\mathcal{NP}$-hard optimization problems.
- Other than ILP-based methods, they are inexact but efficient.
- Other than heuristics, they provide a guarantee of quality.
- Designing approximation algorithms and proving their guarantees is a common algorithmic challenge.