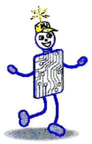


CHAPTER 3: Number Systems

The Architecture of Computer Hardware and Systems Software & Networking:

An Information Technology Approach

5th Edition, Irv Englander

PowerPoint slides authored by Angela Clark, University of South Alabama

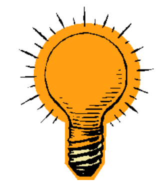
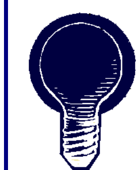
PowerPoint slides for the 4th edition were authored by Wilson Wong, Bentley University

Counting and Arithmetic

- Decimal or base 10 number system
  - Origin: counting on the fingers
  - "Digit" from the Latin word *digitus* meaning "finger"
- *Base*: the number of different digits, including zero, in the number system
  - Example: Base 10 has 10 digits, 0 through 9
- *Binary* or *base 2*
  - *Bit* (short for *binary digit*): 2 digits, 0 and 1
- *Octal* or *base 8:* 8 digits, 0 through 7
- *Hexadecimal or base 16:* 16 digits, 0 through F
- Examples: $10_{10} = A_{16}$; $11_{10} = B_{16}$

Why Binary?

- Early computer design was decimal
  - Mark I and ENIAC
- John von Neumann proposed binary data processing (1945)
  - Simplified computer design
  - Used for both instructions and data
- Natural relationship between on/off switches and calculations using Boolean logic

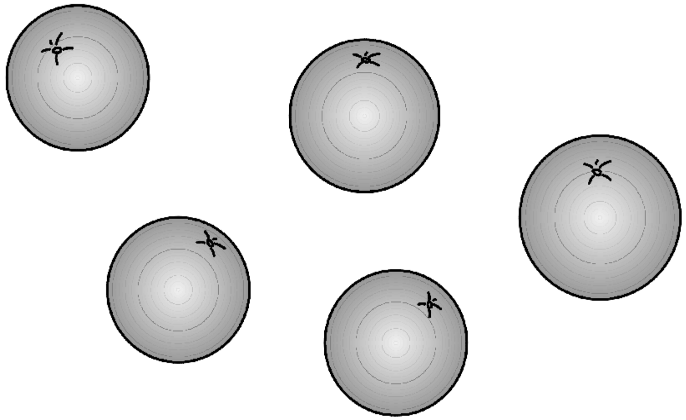
	
On	Off
True	False
Yes	No
1	0

Keeping Track of the Bits

- Bits commonly stored and manipulated in groups
  - 8 bits = 1 *byte*
  - 4 bytes = 1 *word* (in many systems)
- Number of bits used in calculations
  - Affects accuracy of results
  - Limits size and range of numbers manipulated by the computer

Numbers: Physical Representation

- Different numerals, same number of oranges
  - Cave dweller: IIIII
  - Roman: V
  - Arabic: 5
- Different bases, same number of oranges
  - 5_{10}
  - 101_2
  - 12_3



Number Systems

- Roman: position *independent,* simple *additive* system
  - Example: XII is a ten and two ones, i.e. 12
- Modern systems: based on positional notation (place value)
  - Decimal system: based on powers of 10
  - Binary system: based powers of 2
  - Octal system: based on powers of 8
  - Hexadecimal system: based powers of 16

Positional Notation: Base 10

$527_{10} = 5 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$

100's place 10's place 1's place

Place	10^2	10^1	10^0
Value	100	10	1
Evaluate	5 x 100	2 x 10	7 x1
Sum	500	20	7

Positional Notation: Octal

$624_8 = 404_{10}$

64's place   8's place   1's place

Place	8^2	8^1	8^0
Value	64	8	1
Evaluate	6 x 64	2 x 8	4 x 1
Sum for Base 10	384	16	4

Positional Notation: Hexadecimal

$6{,}704_{16} = 26{,}372_{10}$

4,096's place 256's place 16's place 1's place

Place	16^3	16^2	16^1	16^0
Value	4,096	256	16	1
Evaluate	6 x 4,096	7 x 256	0 x 16	4 x 1
Sum for Base 10	24,576	1,792	0	4

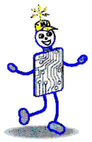
Positional Notation: Binary

$1101\ 0110_2$ = 214_{10}

Place	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Value	128	64	32	16	8	4	2	1
Evaluate	1 x 128	1 x 64	0 x 32	1 x16	0 x 8	1 x 4	1 x 2	0 x 1
Sum for Base 10	128	64	0	16	0	4	2	0

Range of Possible Numbers

- $R = B^K$
  - R = range
  - B = base
  - K = number of digits
- Example #1: Base 10, 2 digits
  - $R = 10^2 =$ 100 different numbers (0…99)
- Example #2: Base 2, 16 digits
  - $R = 2^{16}$ = 65,536 or 64K
  - 16-bit number can represent 65,536 different number values

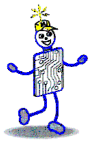


Decimal Range for Bit Widths

Bits	Digits	Range
1	0+	2 (0 and 1)
4	1+	16 (0 to 15)
8	2+	256
10	3	1,024 (1K)
16	4+	65,536 (64K)
20	6	1,048,576 (1M)
32	9+	4,294,967,296 (4G)
64	19+	Approx. 1.6 x 10^{19}
128	38+	Approx. 2.6 x 10^{38}

Different Number Bases

- Base:
  - The number of different symbols required to represent any given number
- The *larger* the base, the *more* symbols are required
  - Base 10: 0,1,2,3,4,5,6,7,8,9
  - Base 2: 0,1
  - Base 8: 0,1,2,3,4,5,6,7
  - Base 16: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F



Number of Symbols vs. Number of Digits

- For a given number, the *larger* the base
  - the *more* symbols required
  - but the *fewer* digits needed
- Example #1:
  - 65_{16} 101_{10} 145_{8} $110\ 0101_{2}$
- Example #2:
  - $11C_{16}$ 284_{10} 434_{8} $1\ 0001\ 1100_{2}$

Counting in Base 2

Binary Number	Equivalent				Decimal Number
	8's (2^3)	4's (2^2)	2's (2^1)	1's (2^0)	
0				0×2^0	0
1				1×2^0	1
10			1×2^1	0×2^0	2
11			1×2^1	1×2^0	3
100		1×2^2			4
101		1×2^2		1×2^0	5
110		1×2^2	1×2^1		6
111		1×2^2	1×2^1	1×2^0	7
1000	1×2^3				8
1001	1×2^3			1×2^0	9
1010	1×2^3		1×2^1		10

Base 10 Addition Table

$3_{10} + 6_{10} = 9_{10}$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13

Base 8 Addition Table

$3_8 + 6_8 = 11_8$

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

(no 8 or 9, of course)

Base 10 Multiplication Table

$3_{10} \times 6_{10} = 18_{10}$

x	0	1	2	3	4	5	6	7	8	9
0				0						
1		1	2	3	4	5	6	7	8	9
2		2	4	6	8	10	12	14	16	18
3		3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5		5	10	15	20	25	30	35	40	45
6		6	12	18	24	30	36	42	48	54
7		7	14	21	28	35	42	49	56	63

Base 8 Multiplication Table

$3_8 \times 6_8 = 22_8$

x	0	1	2	3	4	5	6	7
0				0				
1		1	2	3	4	5	6	7
2		2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4		4	10	14	20	24	30	34
5		5	12	17	24	31	36	43
6		6	14	22	30	36	44	52
7		7	16	25	34	43	52	61

Addition

Base	Problem	Largest Single Digit
Decimal	6 +3	9
Octal	6 +1	7
Hexadecimal	6 +9	F
Binary	1 +0	1

Addition

Base	Problem	Carry	Answer
Decimal	6 +4	Carry the 10	10
Octal	6 +2	Carry the 8	10
Hexadecimal	6 +A	Carry the 16	10
Binary	1 +1	Carry the 2	10

Binary Arithmetic

1	1	1	1	1			
	1	1	0	1	1	0	1
+			1	0	1	1	0
1	0	0	0	0	0	1	1

Binary Arithmetic

- Addition
  - Boolean using XOR and AND
- Multiplication
  - AND
  - Shift *left*
- Division
  - Subtraction or complement
  - Shift *right*

+	0	1
0	0	1
1	1	10

x	0	1
0	0	0
1	0	1

Binary Arithmetic: Boolean Logic

- Boolean logic without performing arithmetic
  - *EXCLUSIVE-OR*
    - Output is “1” only if either input, but ***not*** both inputs, is a “1”
  - *AND (carry bit)*
    - Output is “1” if and only both inputs are a “1”

	1	1	1	1	1			
		1	1	0	1	1	0	1
+				1	0	1	1	0
	1	0	0	0	0	0	1	1

Binary Multiplication

- Boolean logic without performing arithmetic
  - *AND (carry bit)*
    - Output is "1" if and only both inputs are a "1"
  - *Shift*
    - Shifting a number in any base ***left*** one digit ***multiplies*** its value by the base
    - Shifting a number in any base ***right*** one digit ***divides*** its value by the base
    - Examples:
      - 10_{10} shift ***left*** = 100_{10}
      - 10_{10} shift ***right*** = 1_{10}
      - 10_2 shift ***left*** = 100_2
      - 10_2 shift ***right*** = 1_2

Binary Multiplication

			1	**1**	**0**	**1**	
				1	**0**	**1**	
			1	**1**	**0**	**1**	1's place
					0		2's place
	1	**1**	**0**	**1**			4's place (bits ***shifted*** to line up with 4's place of multiplier)
1	**0**	**0**	**0**	**0**	**0**	**1**	Result (***AND***)

Converting from Base 10

- Powers Table

Power / Base	8	7	6	5	4	3	2	1	0
2	256	128	64	32	16	8	4	2	1
8				32,768	4,096	512	64	8	1
16					65,536	4,096	256	16	1

From Base 10 to Base 2

$42_{10} = 101010_2$

Power / Base	6	5	4	3	2	1	0
2	64	32	16	8	4	2	1
		1	0	1	0	1	0
Integer		42/32 = 1	10/16 = 0	10/8 = 1	2/4 = 0	2/2 = 1	0/1 = 0
Remainder		10	10	2	2	0	0

From Base 10 to Base 2

Base 10 42

Remainder

Quotient

2) 42 (0 Least significant bit

2) 21 (1

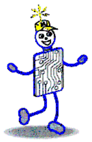
2) 10 (0

2) 5 (1

2) 2 (0

2) 1 Most significant bit

Base 2 101010



From Base 10 to Base 16

$5{,}735_{10} = 1667_{16}$

Power / Base	4	3	2	1	0
16	65,536	4,096	256	16	1
		1	**6**	**6**	**7**
Integer		5,735 / 4,096 = 1	1,639 / 256 = 6	103 /16 = 6	7
Remainder		5,735 - 4,096 = 1,639	1,639 - 1,536 = 103	103 - 96 = 7	

From Base 10 to Base 16

Base 10 8,039

Remainder

Quotient

16)	8,039	(	7	Least significant bit
16)	502	(	6	
16)	31	(	15	
16)	1	(	1	Most significant bit
16)	0			

Base 16 1F67

From Base 8 to Base 10

7263_8 = $3{,}763_{10}$

Power	8^3	8^2	8^1	8^0
	512	64	8	1
	x 7	x 2	x 6	x 3
Sum for Base 10	**3,584**	**128**	**48**	**3**

From Base 8 to Base 10

$7263_8 = 3{,}763_{10}$

7

x 8

56 + **2** = 58

x 8

464 + **6** = 470

x 8

3760 + **3** = **3,763**

From Base 16 to Base 2

- Why hexadecimal?
  - Hex easier to read and write than binary
  - Modern computer operating systems and networks present variety of troubleshooting data in hex format
- The nibble approach

Base 16	1	F	6	7
Base 2	0001	1111	0110	0111

Fractions

- Number point or radix point
  - Decimal point in base 10
  - Binary point in base 2
- No exact relationship between fractional numbers in different number bases
  - Exact conversion may be impossible, such as in the case of 1/3 or 0.3333333…

Decimal Fractions

- Move the number point one place to the right
  - Effect: multiplies the number by the base number
  - Example: 139.0_{10} → 1390_{10}
- Move the number point one place to the left
  - Effect: divides the number by the base number
  - Example: 139.0_{10} → 13.9_{10}

Fractions: Base 10 and Base 2

$.2589_{10} = 2 \times 10^{-1} + 5 \times 10^{-2} + 8 \times 10^{-3} + 9 \times 10^{-4}$

Place	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Value	1/10	1/100	1/1000	1/10000
Evaluate	2 x 1/10	5 x 1/100	8 x 1/1000	9 x1/1000
Sum	.2	.05	.008	.0009

$.101011_2 = 0.671875_{10}$

Place	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
Value	1/2	1/4	1/8	1/16	1/32	1/64
Evaluate	1 x 1/2	0 x 1/4	1x 1/8	0 x 1/16	1 x 1/32	1 x 1/64
Sum	.5		0.125		0.03125	0.015625

Fractions: Base 10 and Base 2

- No general relationship between fractions of types $1/10^k$ and $1/2^k$
  - Therefore a number representable in base 10 may not be representable in base 2
  - But the converse is true: all fractions of the form $1/2^k$ can be represented in base 10
- Fractional conversions from one base to another are stopped
  - If there is a rational solution or
  - When the desired accuracy is attained

Mixed Number Conversion

- Integer and fractional parts must be converted separately
- Radix point: fixed reference for the conversion
  - Digit to the left is a unit digit in every base
  - B^0 is always equal to 1 regardless of the base

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