Matrix Decomposition

CS 556

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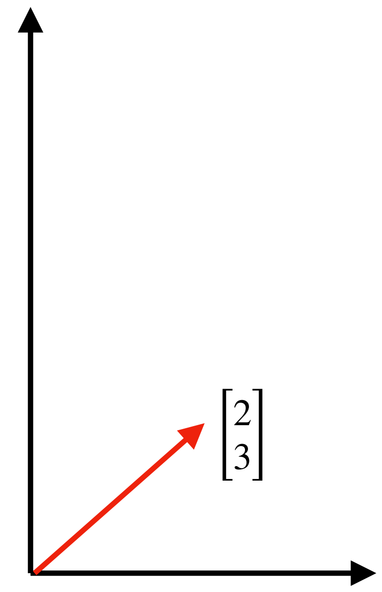
Outline

- Eigen Decomposition
- Singular Value Decomposition (SVD)
- Dimensionality Reduction with SVD
- Movie Recommender System with SVD

2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

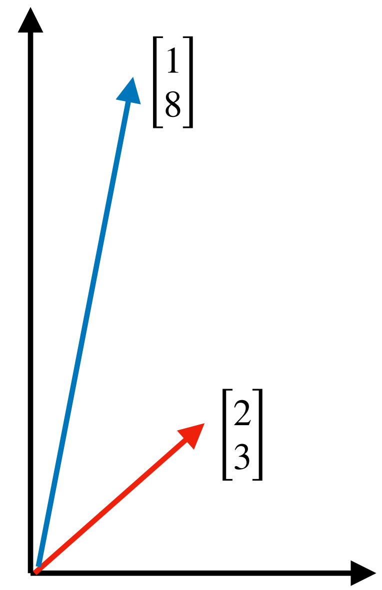
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = ?$$



2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$



2D Transformation Matrix

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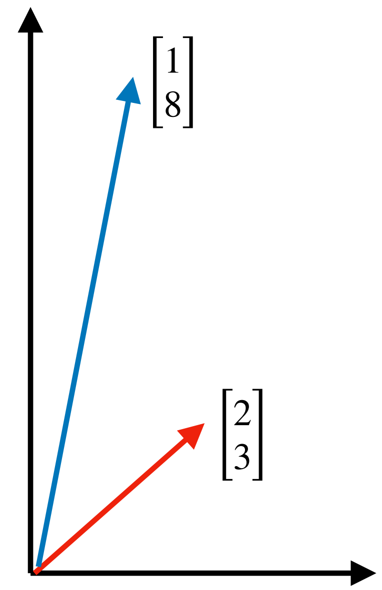


The matrix will transform the vector by rotating and stretching/shortening it

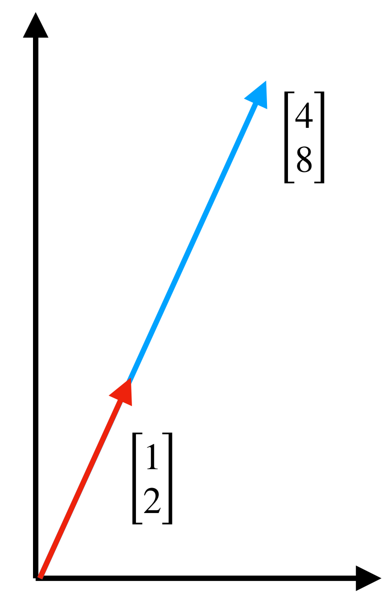
$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \qquad S = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

$Rotation\ by\ \theta$ $\qquad$ $Stretching\ by\ \alpha$

2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

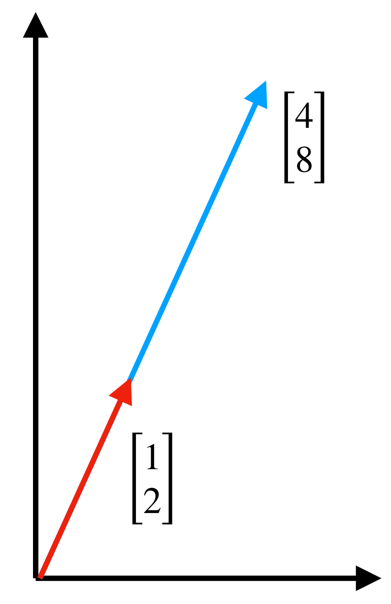


2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$



2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

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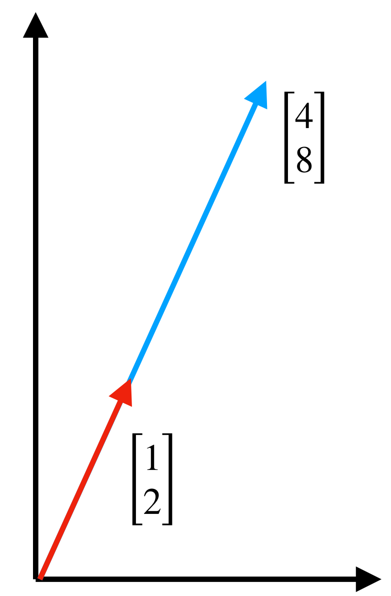
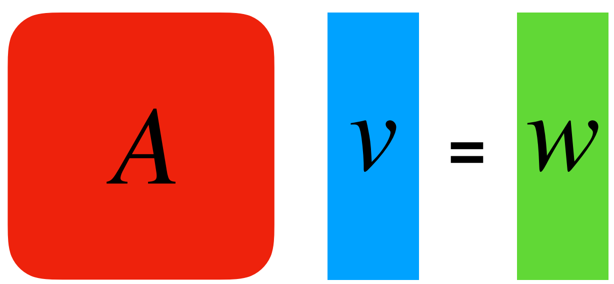
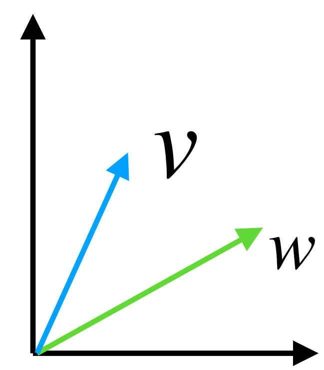
Eigen value

Eigen vector



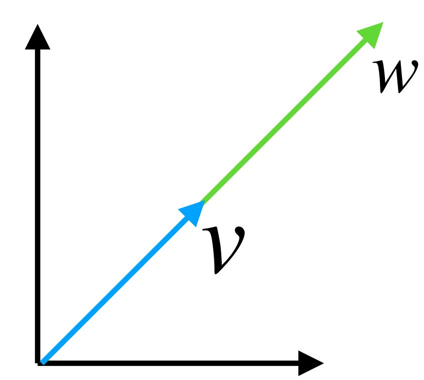
Eigenvectors & Eigenvalues





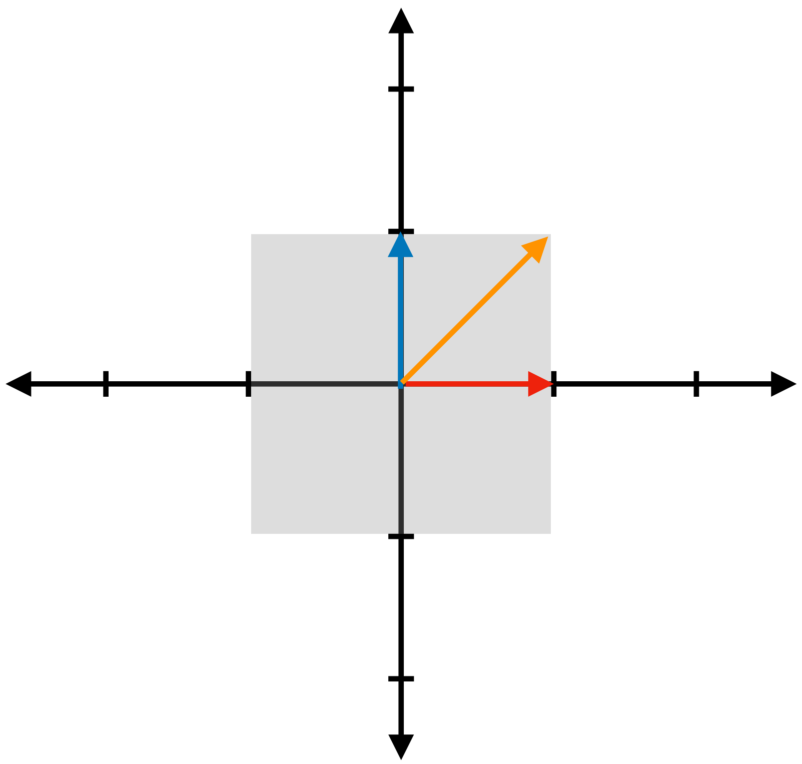
Transformation matrix A is applied to a vector v and outputs a vector w. If w points in the same direction as v (a.k.a. lies on the same 1 dimensional subspace), then v is an eigenvector of matrix A.

$$Av = w$$
$$\lambda v = w$$
$$Av = \lambda v$$



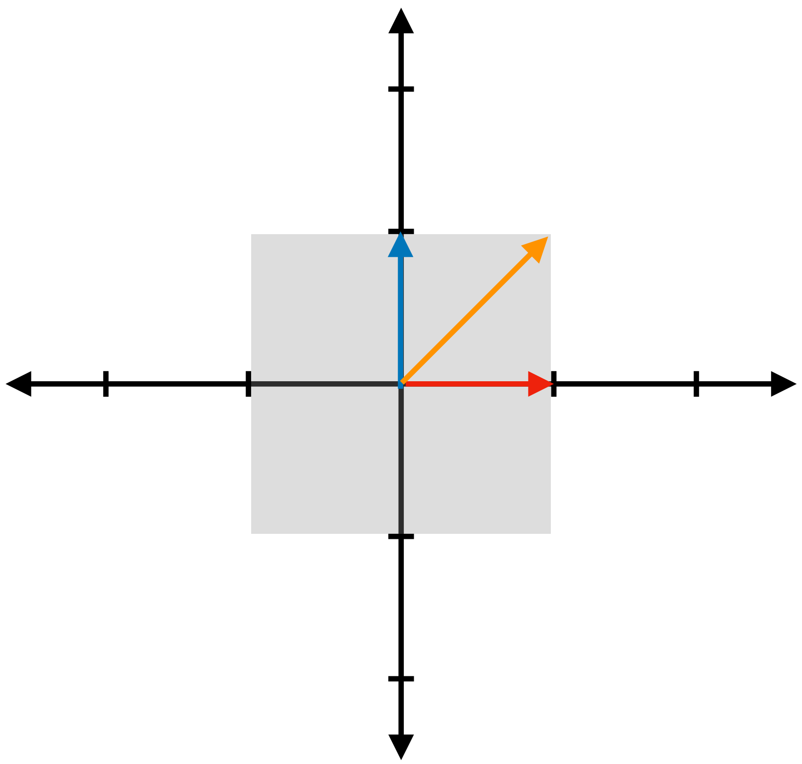
λ is an eigenvalue associated with eigenvector v of A

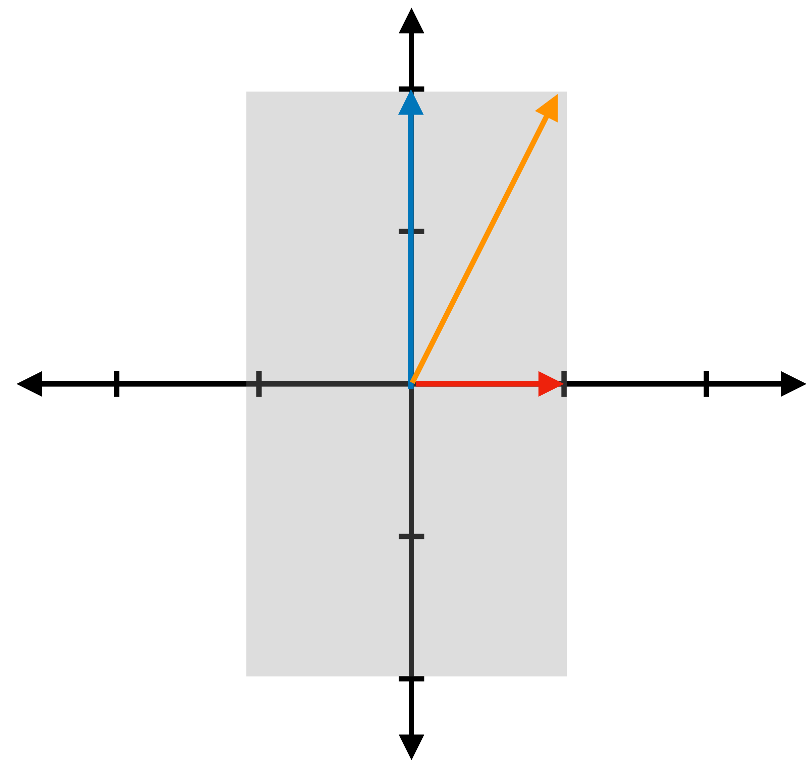
Eigenvectors & Eigenvalues



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigenvectors & Eigenvalues





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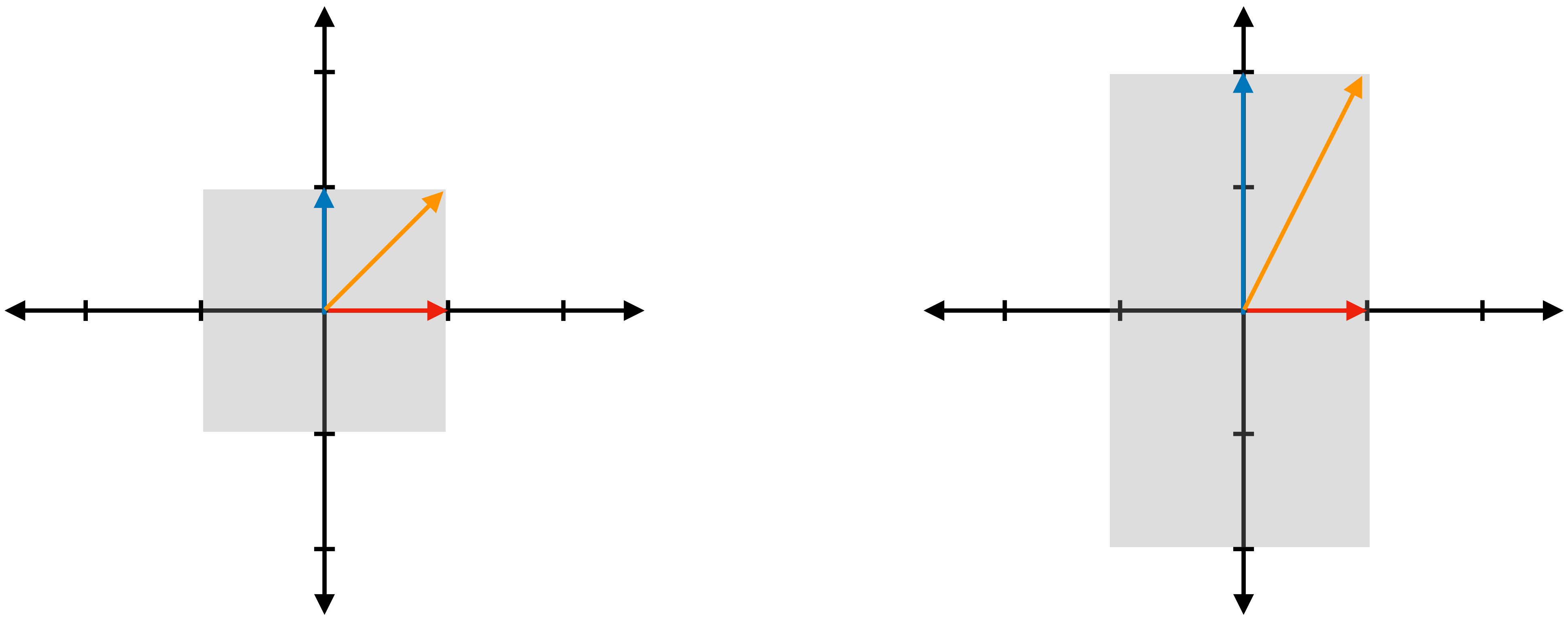
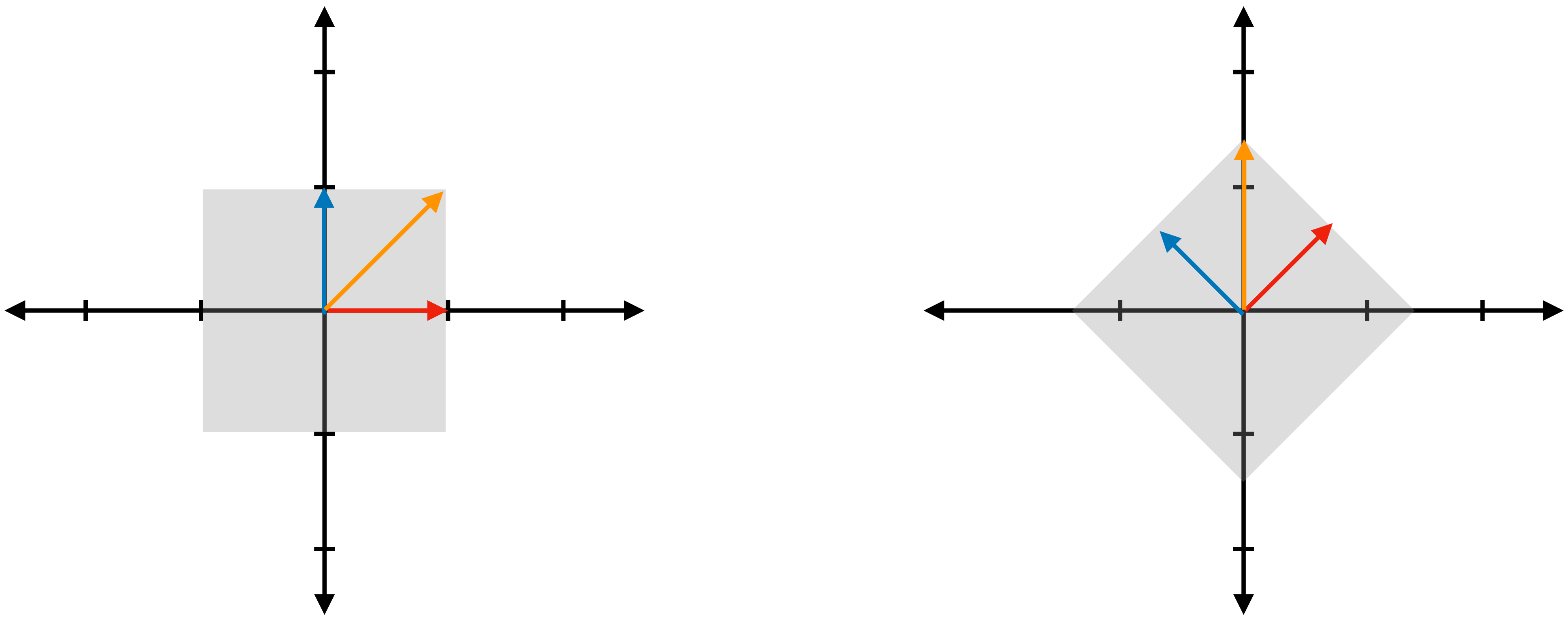
Eigenvectors & Eigenvalues



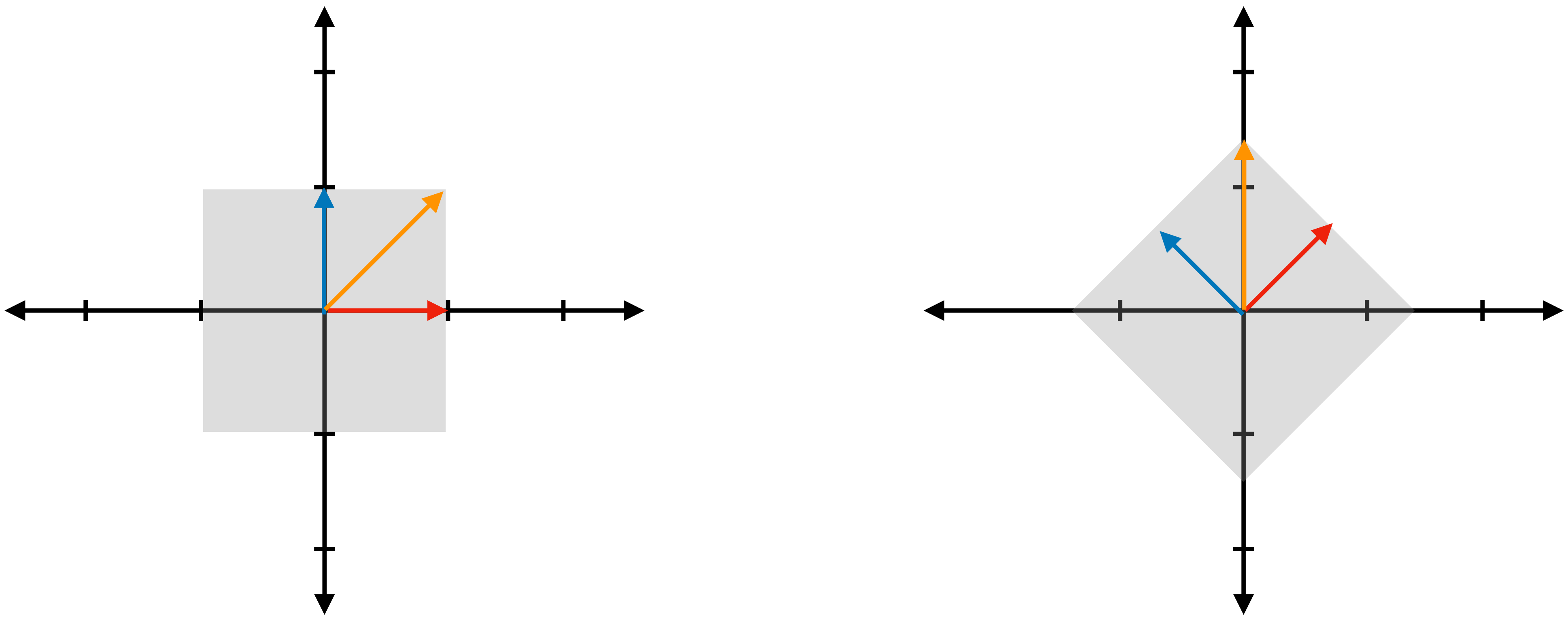
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

2 eigenvectors with values 1 and 2

Eigenvectors & Eigenvalues

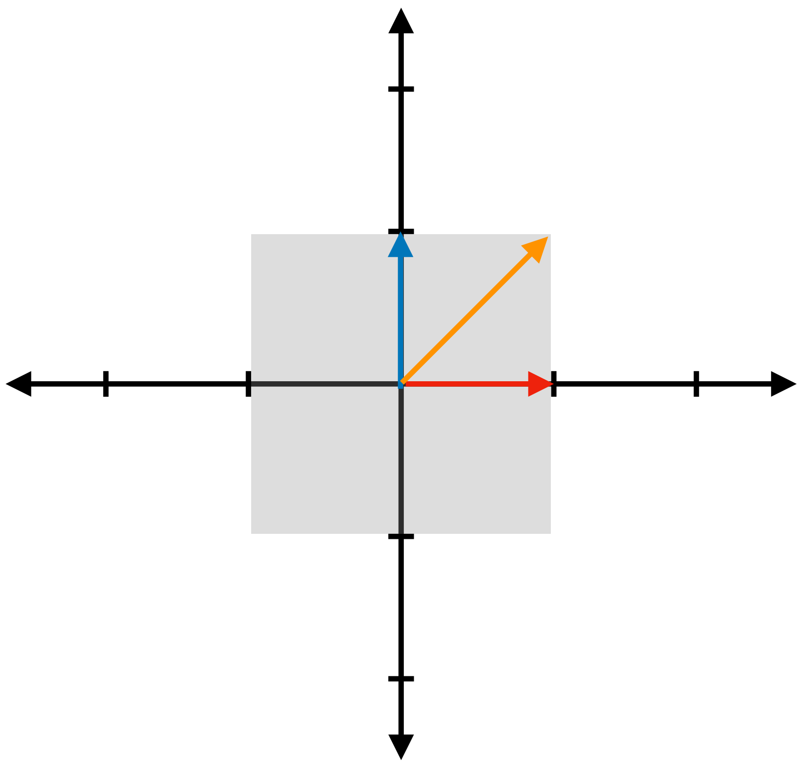


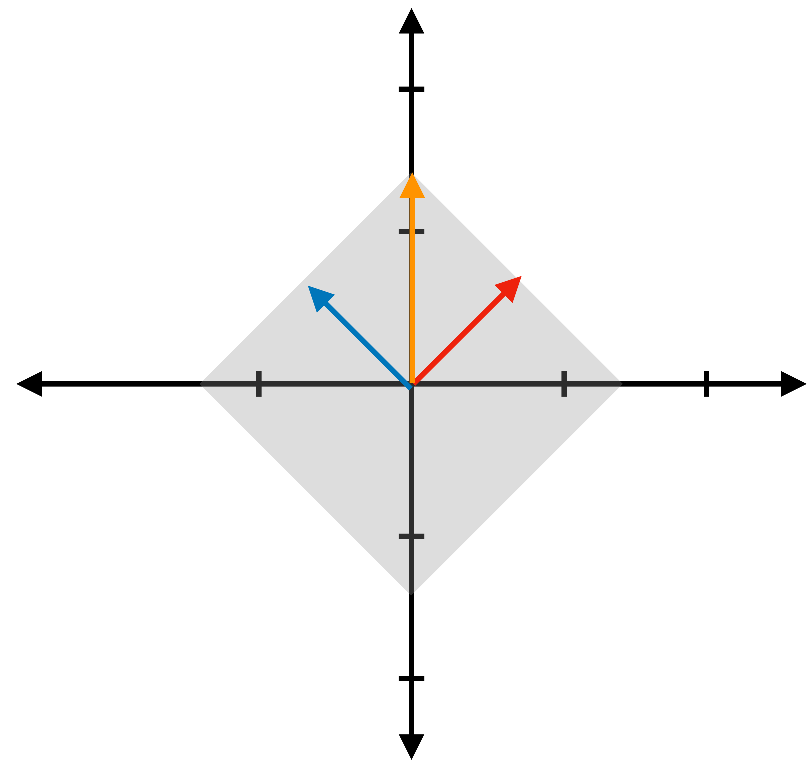
Eigenvectors & Eigenvalues



0 Eigenvectors

Eigenvectors & Eigenvalues





0 Eigenvectors

Check the vectors that lie on the same span after transformation and measure how much they magnitude change

Finding Eigenvalues

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A - \lambda)v = 0$$

Shift the matrix A by λ

$$(A - \lambda I)v = 0$$

It has nontrivial null space.
It must be singular.

$$A - \lambda I$$

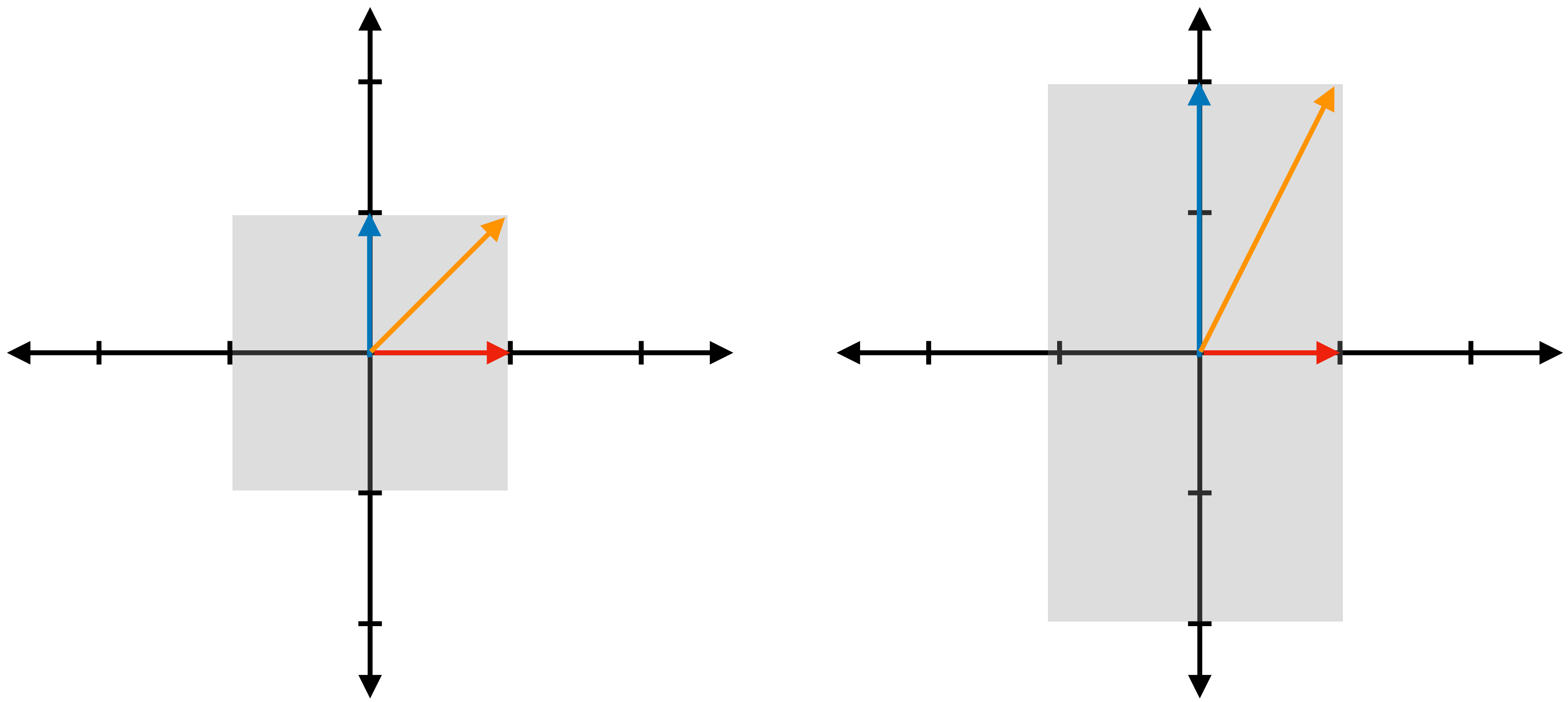
$$det(A - \lambda I) = 0$$

The Determinant of singular matrices is 0.

Finding Eigenvectors

1. Find all eigenvalues λ
2. For each λ , find $v \in N(A - \lambda I)$
   - Find a vector v that is in the null space of the matrix $(A - \lambda I)$

Example 1



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

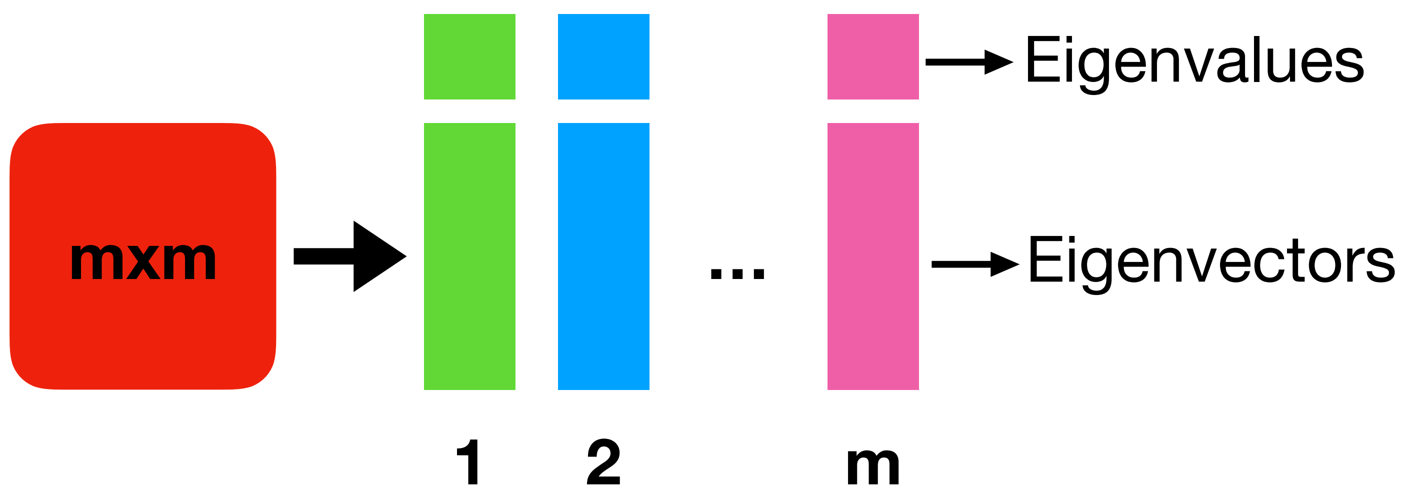
$$det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

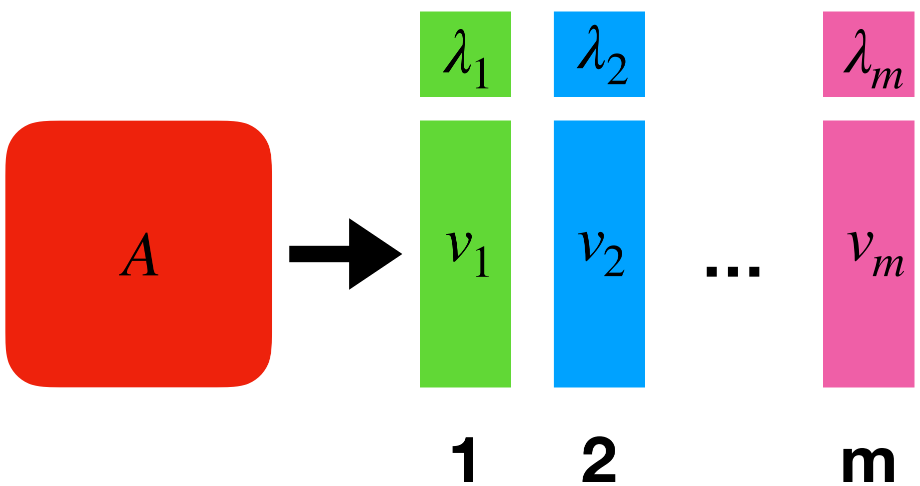
$$\lambda = 2, \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ 0 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigen Decomposition



Eigendecomposition is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors.

Diagonalization



$$
\begin{gathered}
Av_1 = \lambda_1 v_1 \\
Av_2 = \lambda_2 v_2 \\
\vdots \\
Av_m = \lambda_m v_m
\end{gathered}
$$

$$
\begin{aligned}
Av_1 &= \lambda_1 v_1 \\
Av_2 &= \lambda_2 v_2 \\
Av_3 &= \lambda_3 v_3
\end{aligned}
\longrightarrow
\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}
\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
=
\begin{bmatrix} v_{11}\lambda_1 & v_{12}\lambda_2 & v_{13}\lambda_3 \\ v_{21}\lambda_1 & v_{22}\lambda_2 & v_{23}\lambda_3 \\ v_{31}\lambda_1 & v_{32}\lambda_2 & v_{33}\lambda_3 \end{bmatrix}
\longrightarrow AV = V\Lambda
$$

$$
AV = V\Lambda \;\rightarrow\; \boxed{V^{-1}AV = \Lambda} \qquad AV = V\Lambda \;\rightarrow\; \boxed{A = V\Lambda V^{-1}}
$$

Finding a set of basis vectors V such that the original matrix A is diagonal in that basis space, assuming that the columns in V are linearly independent.

Spectral Theorem

Every symmetric matrix has the factorization $S = Q\Lambda Q^{-1}$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q : $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$ with $Q^{-1} = Q^T$.

I.e. Eigenvectors of a real symmetric matrix are always perpendicular.

Proof: Let's assume that x and y are two eigenvectors of S with eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$.

$$Sx = \lambda_1 x,\ Sy = \lambda_2 y$$

$$(Sx)^T y = (\lambda_1 x)^T y,\ x^T S^T y = x^T \lambda_1 y,\ x^T S y = x^T \lambda_1 y$$

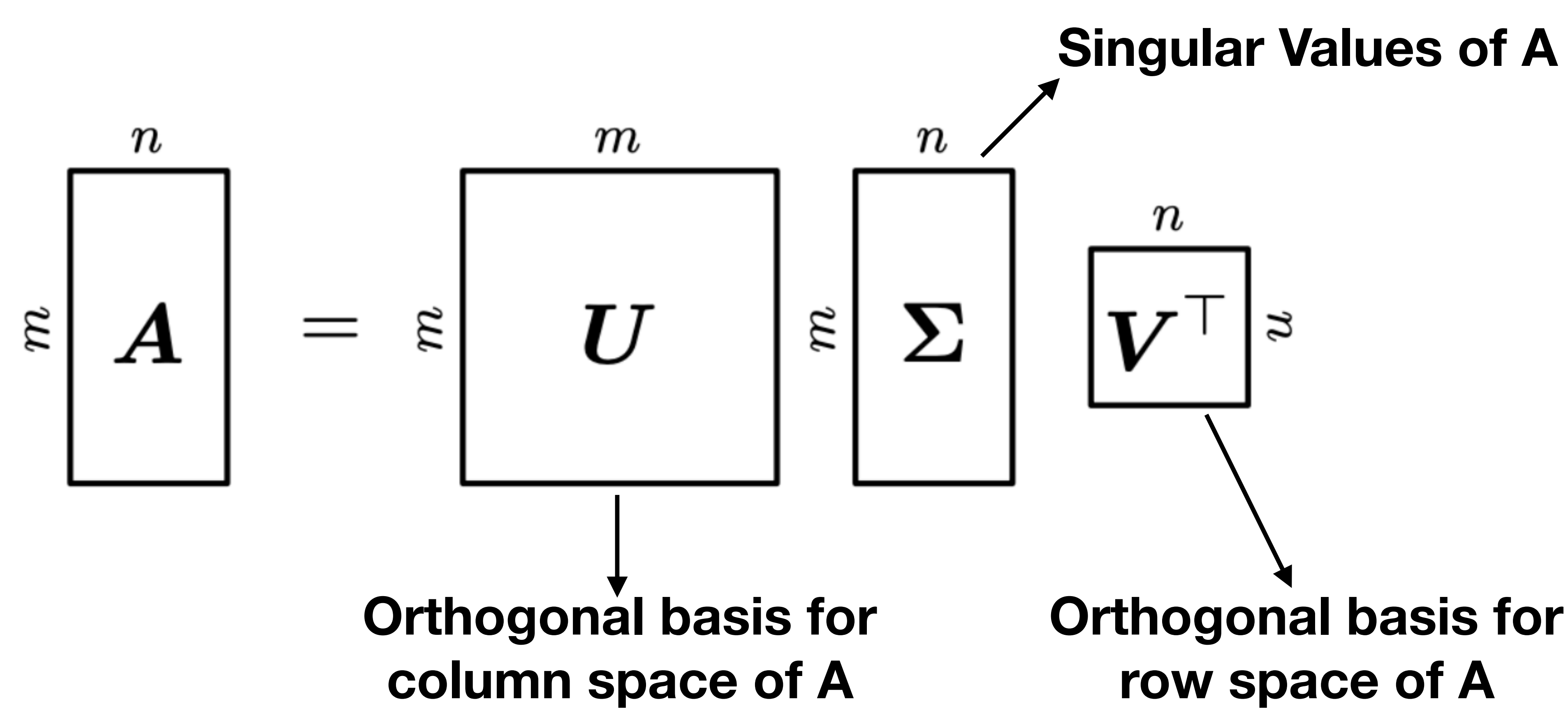
$$x^T \lambda_2 y = x^T \lambda_1 y$$

Since $\lambda_1 \neq \lambda_2$ then $x^T y = 0 \rightarrow x \perp y$.

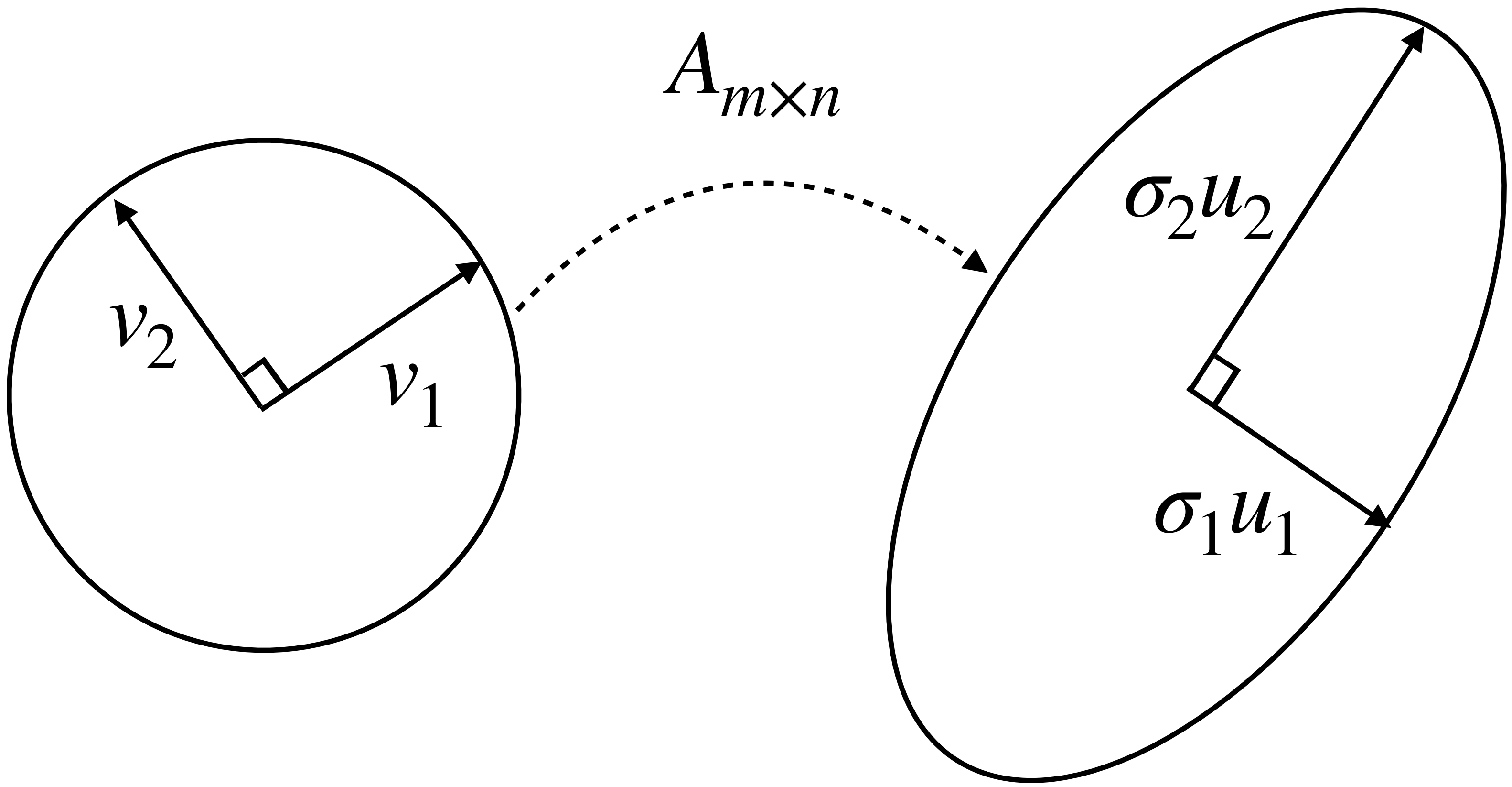
Singular Value Decomposition

SVD

The goal of SVD is to decompose a matrix A as the product of 3 other matrices $A = U\Sigma V^T$, where matrix V and U are orthogonal matrices and Σ is a diagonal matrix.



Formulation



$$\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_r}\} \rightarrow \{\overrightarrow{u_1}, \overrightarrow{u_2}, \ldots, \overrightarrow{u_r}\}$$

$$\overrightarrow{u_1}, \ldots, \overrightarrow{u_r} \;\; orthonormal$$

$$\sigma_1, \ldots, \sigma_r \;\; singular \;\; values$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots \geq \sigma_r \geq 0$$

$$r = rank(A)$$

$$A\overrightarrow{v_1} = \sigma_1 \overrightarrow{u_1}, A\overrightarrow{v_2} = \sigma_2 \overrightarrow{u_2} \rightarrow A\overrightarrow{v_j} = \sigma_j \overrightarrow{u_j}, \forall j = 1,2,\ldots, r$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} & & & \\ v_1 & v_2 & \ldots & v_r \\ & & & \end{bmatrix} = \begin{bmatrix} & & & \\ u_1 & u_2 & \ldots & u_r \\ & & & \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \sigma_r \end{bmatrix}$$

$$AV = U\Sigma$$

$$A = U\Sigma V^{-1} = U\Sigma V^T$$

How to compute SVD?

$$A = U\Sigma V^T$$

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T$$

$$A^T A = V\Sigma^T U^T U\Sigma V^T$$

$$A^T A = V\Sigma^T I\Sigma V^T$$

$$A^T A = V\Sigma^T \Sigma V^T$$

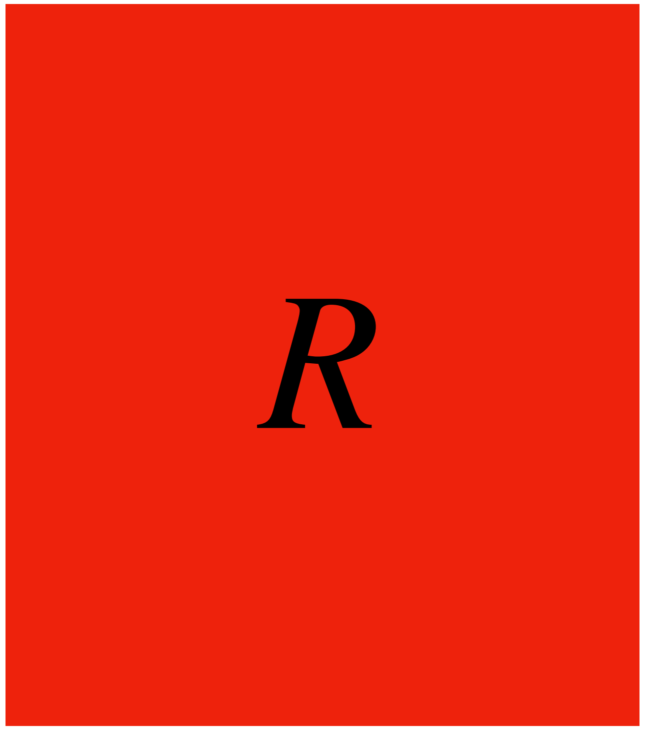
$$A^T A = V\Sigma^2 V^T$$

To find V compute eigendecomposition of $A^T A$ where V will be the eigenvectors of $A^T A$ and Σ^2 are the eigenvalues of $A^T A$.

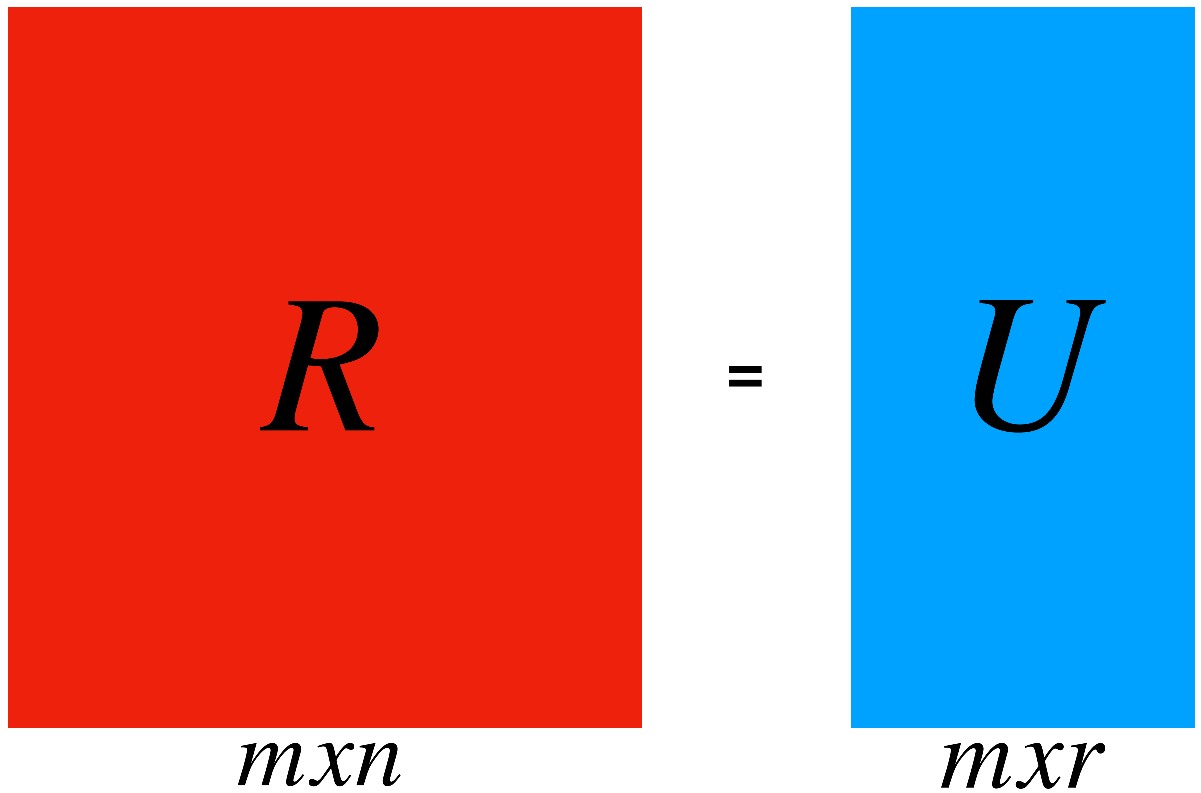
$$AA^T = U\Sigma^2 U^T$$

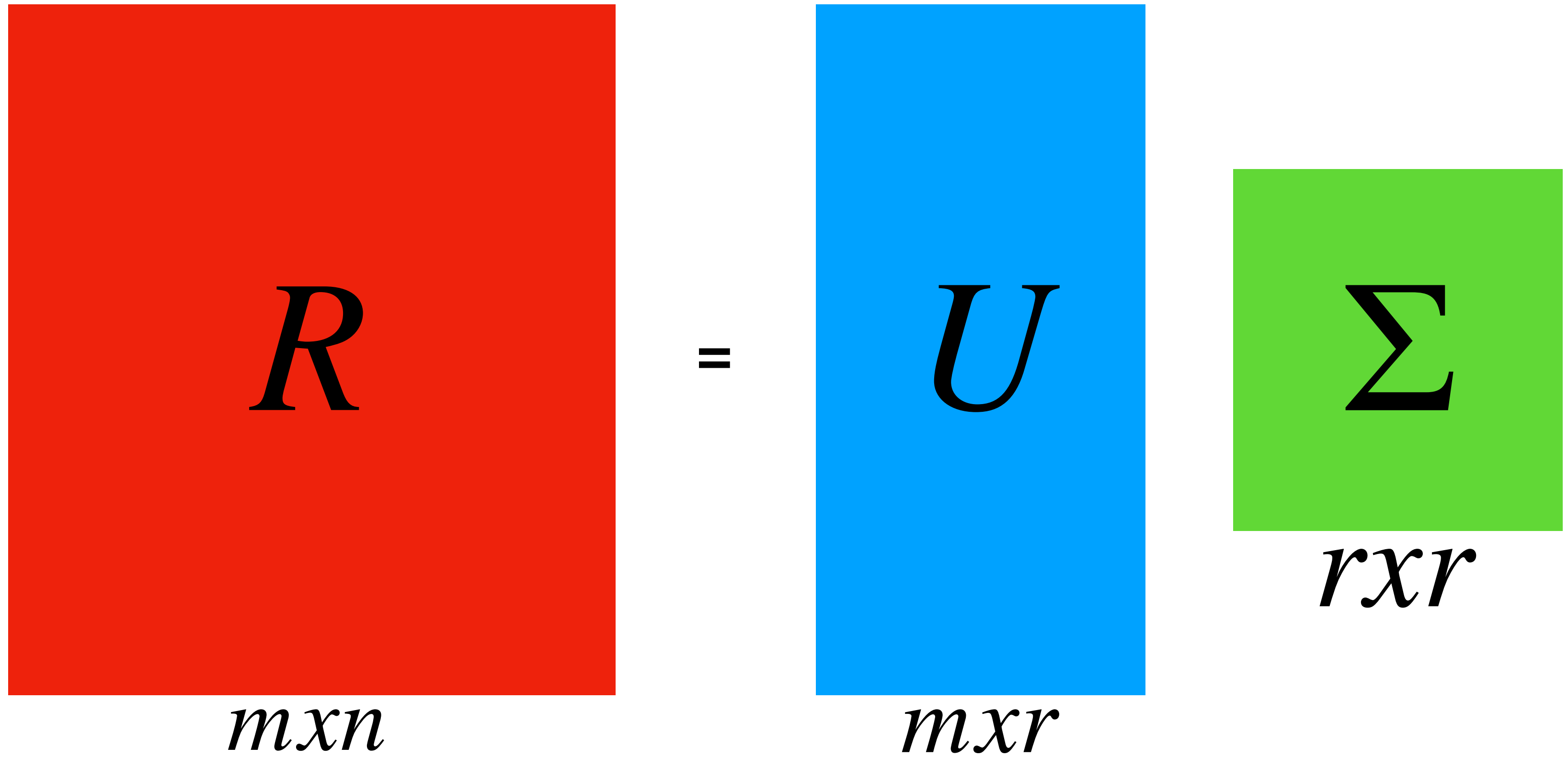
To find U compute eigendecomposition of AA^T where U will be the eigenvectors of AA^T and Σ^2 are the eigenvalues of AA^T.

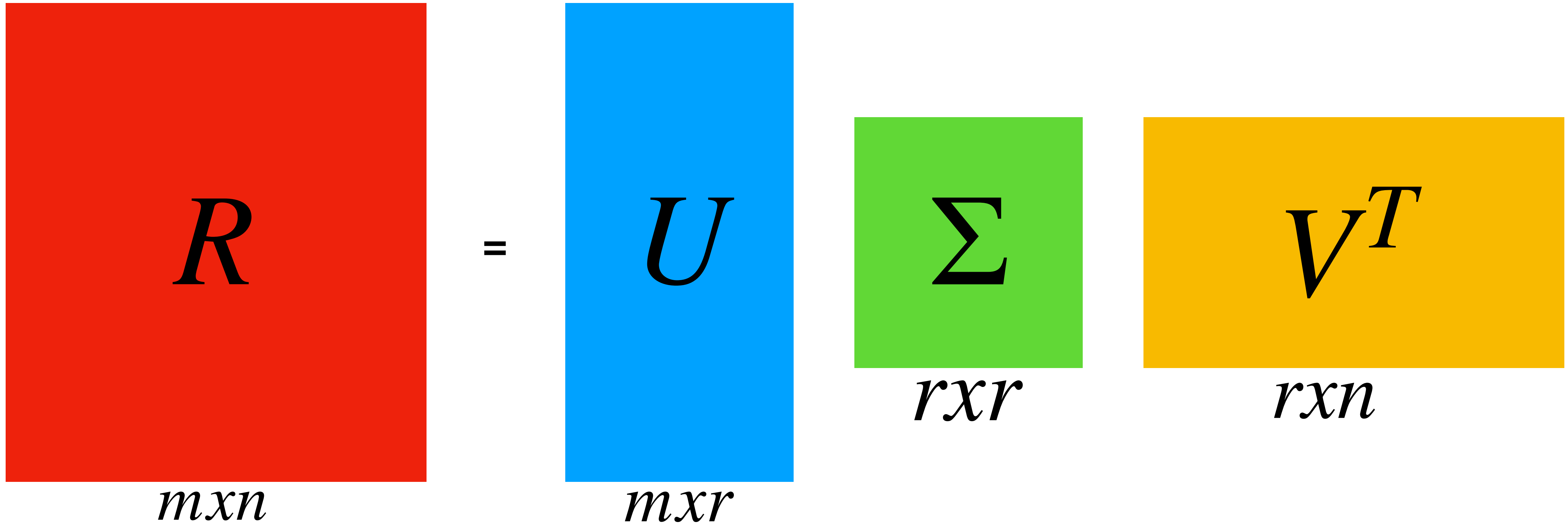
Dimensionality Reduction with SVD



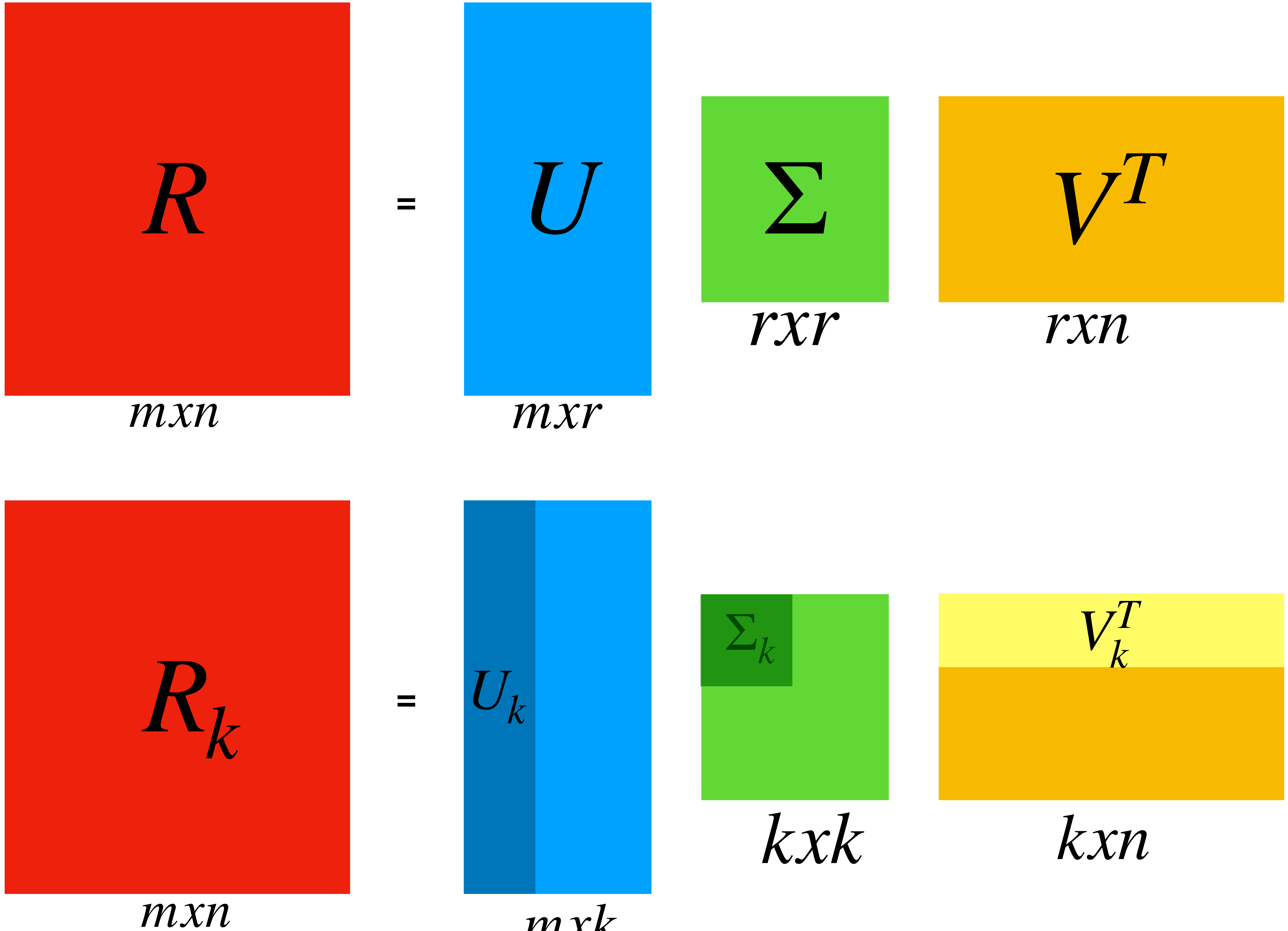
$m x n$


R
=
U
mxn
mxr

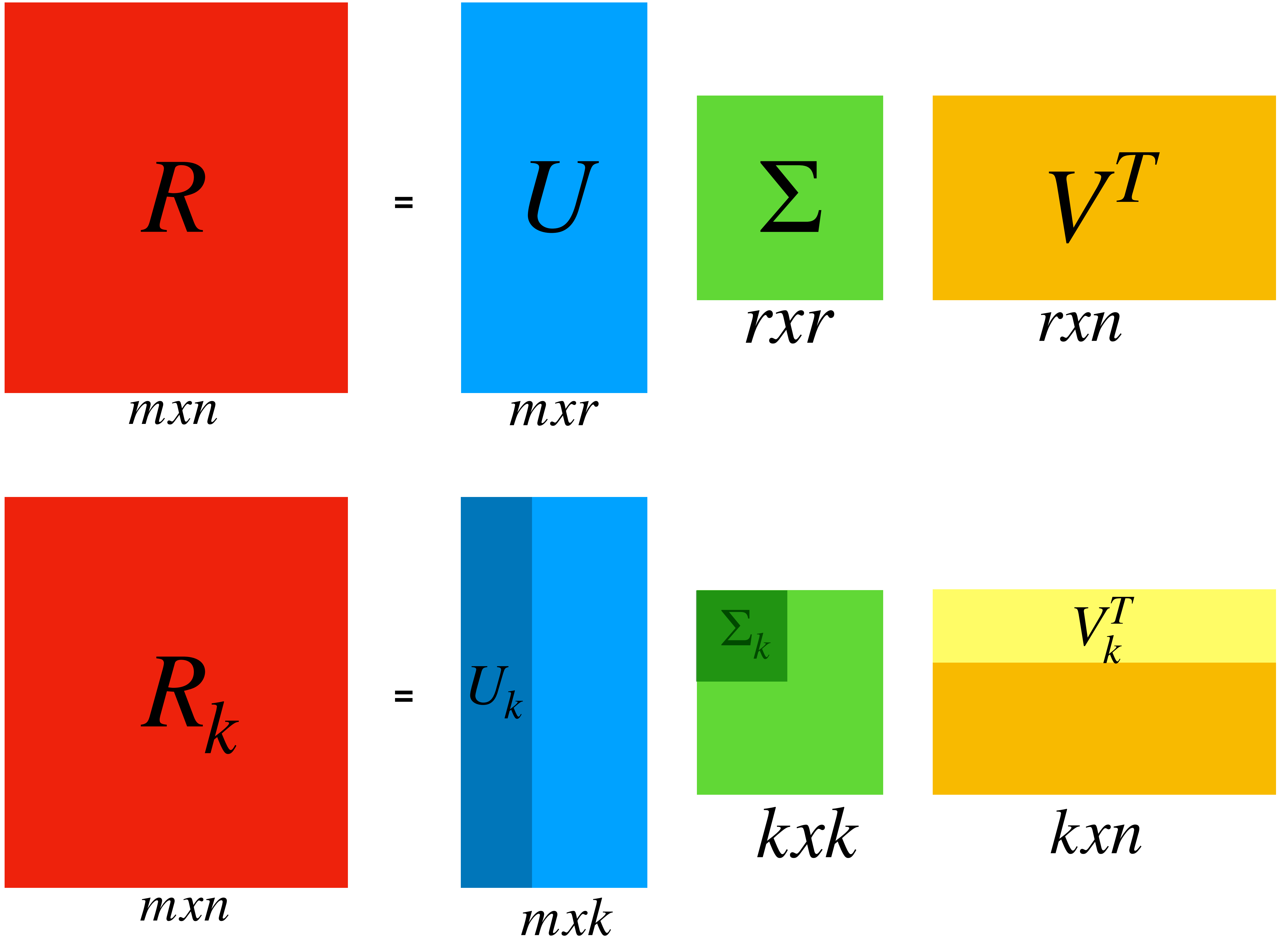

R
=
U
Σ
rxr
mxn
mxr



R
=
U
Σ
V^T
rxr
rxn
mxn
mxr



R
=
U
Σ
V^T
rxr
rxn
mxn
mxr
R_k
=
U_k
Σ_k
V_k^T
kxk
kxn
mxn
mxk



Eckart - Young Theorem 1936 [1]

The reconstruction matrix $R_k = U_k \Sigma_k V_k^T$ is the closest rank-k matrix to R.

Movie Recommendation with SVD

User-Movie Rating

John	5	1	3	5
Tom	?	?	?	2
Alice	4	?	3	?

Image by: Di Noia, Tommaso & Ostuni, Vito. (2015). Recommender Systems and Linked Open Data. 10.1007/978-3-319-21768-0_4.

User Movie Rating

- Overfit representation of user tases

Alice

John

- Computational Complexity, potentially poor results

- Goal: A more compact representation of users tastes and items descriptions

    - Represent our tastes not in terms of the products we like and dislike, but in terms of higher level attributes

How to create these representations?

- Singular Value Decomposition (SVD)
- Intuitive Description: Reduce space to a smaller taste space that is compact and robust

Singular Value Decomposition

$$R = U\Sigma V^T$$

- R *is mxn rating matrix*
- U *is mxk user feature affinity matrix*
- V *is nxk item feature relevance matrix*
- Σ *is kxk diagonal feature weight matrix*
- *Exists for any real* R

Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	-	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

User Feature Matrix

	Drama	Thriller	Action	Fantasy
User 1	0.7	0.2	0.1	0
User 2	0	0.05	0.75	0.2
User m	0.4	0.3	0.2	0.1

Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.4	0	0	0
Thriller	0	0.3	0	0
Action	0	0	0.27	0
Fantasy	0	0	0	0.05

Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	1.0	0	0	0
Captain America	0	0.1	0.7	0.2
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	???	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

User Feature Matrix

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Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.4	0	0	0
Thriller	0	0.3	0	0
Action	0	0	0.27	0
Fantasy	0	0	0	0.05

Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	1.0	0	0	0
Captain America	0	0.1	0.7	0.9
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

$$R(User_1,\ Captain\ America) = U_{User_1} S V^T_{Captain\ America}$$

References

[1] https://citeseer.ist.psu.edu/viewdoc/download;jsessionid=6B65A9A35513A03050594C88798801DB?doi=10.1.1.23.1831&rep=rep1&type=pdf

Thank you!

In-Class Exercise

Find all eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$$

In Class Exercise

Compute SVD for matrix A:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$