Linear Algebra Part 2

CS 556

Erisa Terolli

Vector Spaces

- The space $\mathbb{R}^n$ consists of all columns vectors **v** with n components.

- We can add any two vectors in $\mathbb{R}^n$, and we can multiply any vector **v** by any scalar c.

- Vector spaces must be closed under addition and multiplication

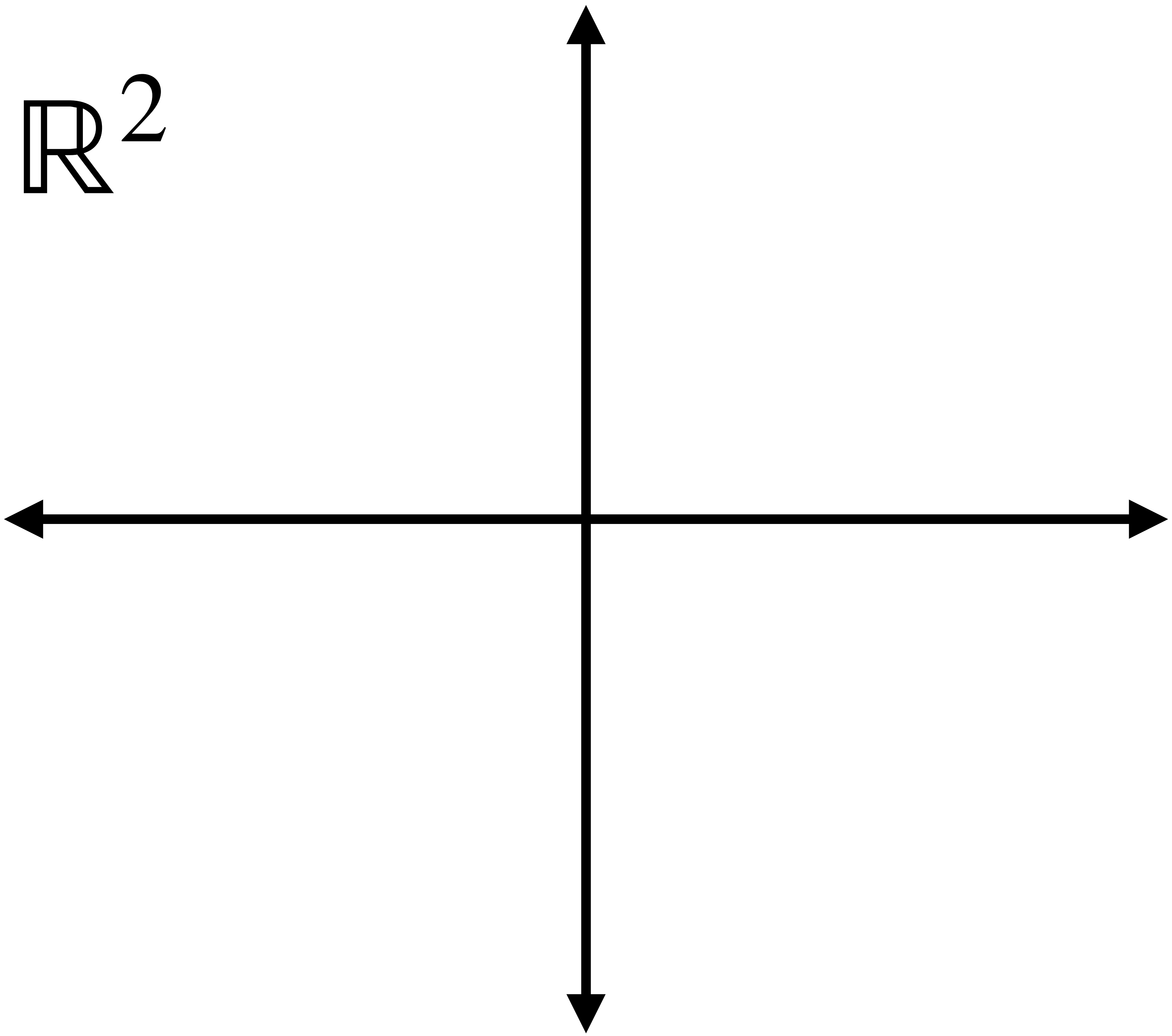
Examples:

- The vector space $\mathbb{R}^2$ is represented by the xy plane. Vector examples in this space: (3, 2), (0, 0) etc.

- The vectors space $\mathbb{R}^3$ is represented by the xyz 3-dimensional space. Vector examples in this space: (1, 2, 3), (0, 0, 0) etc.

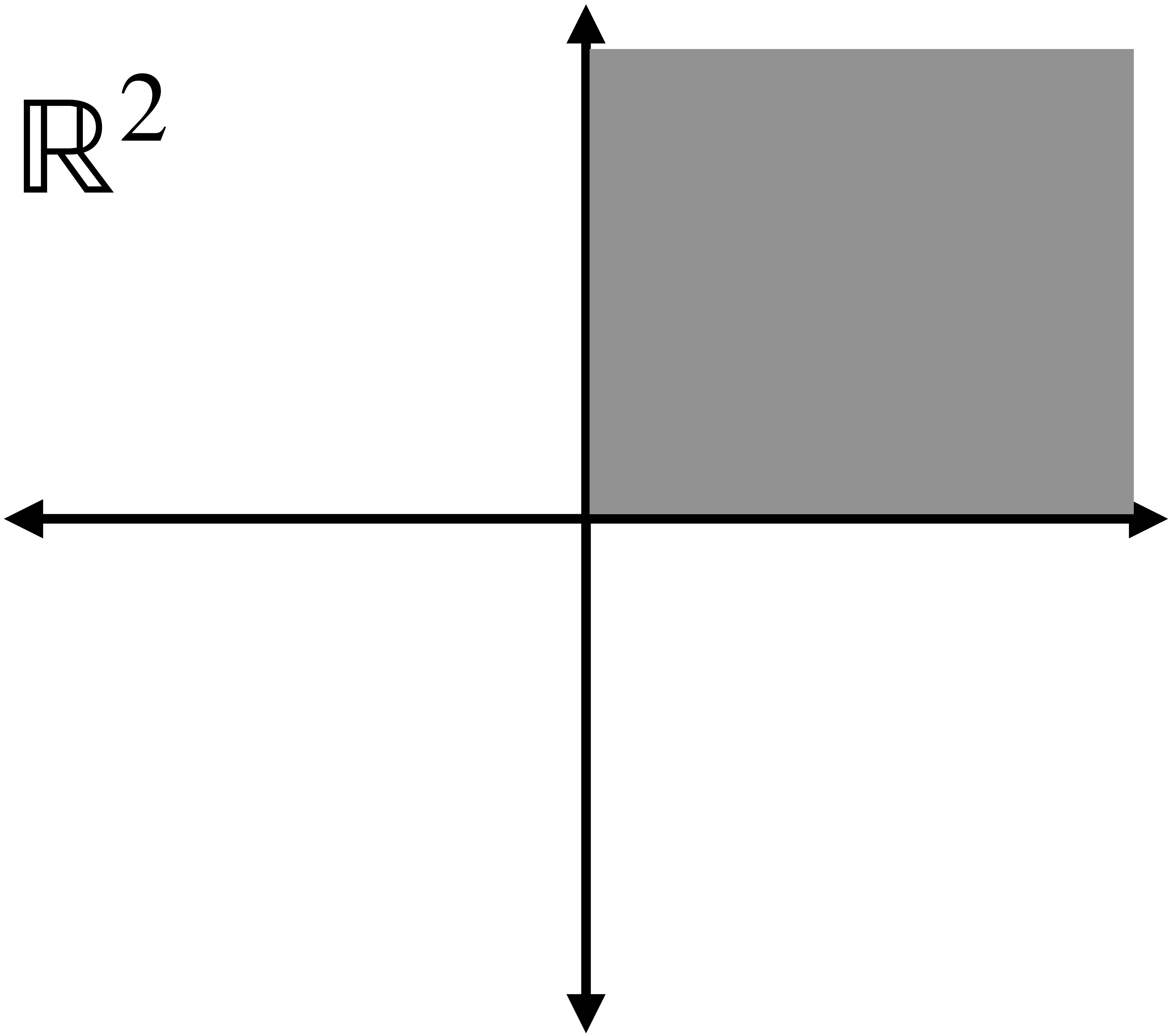
Vector Spaces

$\mathbb{R}^2$



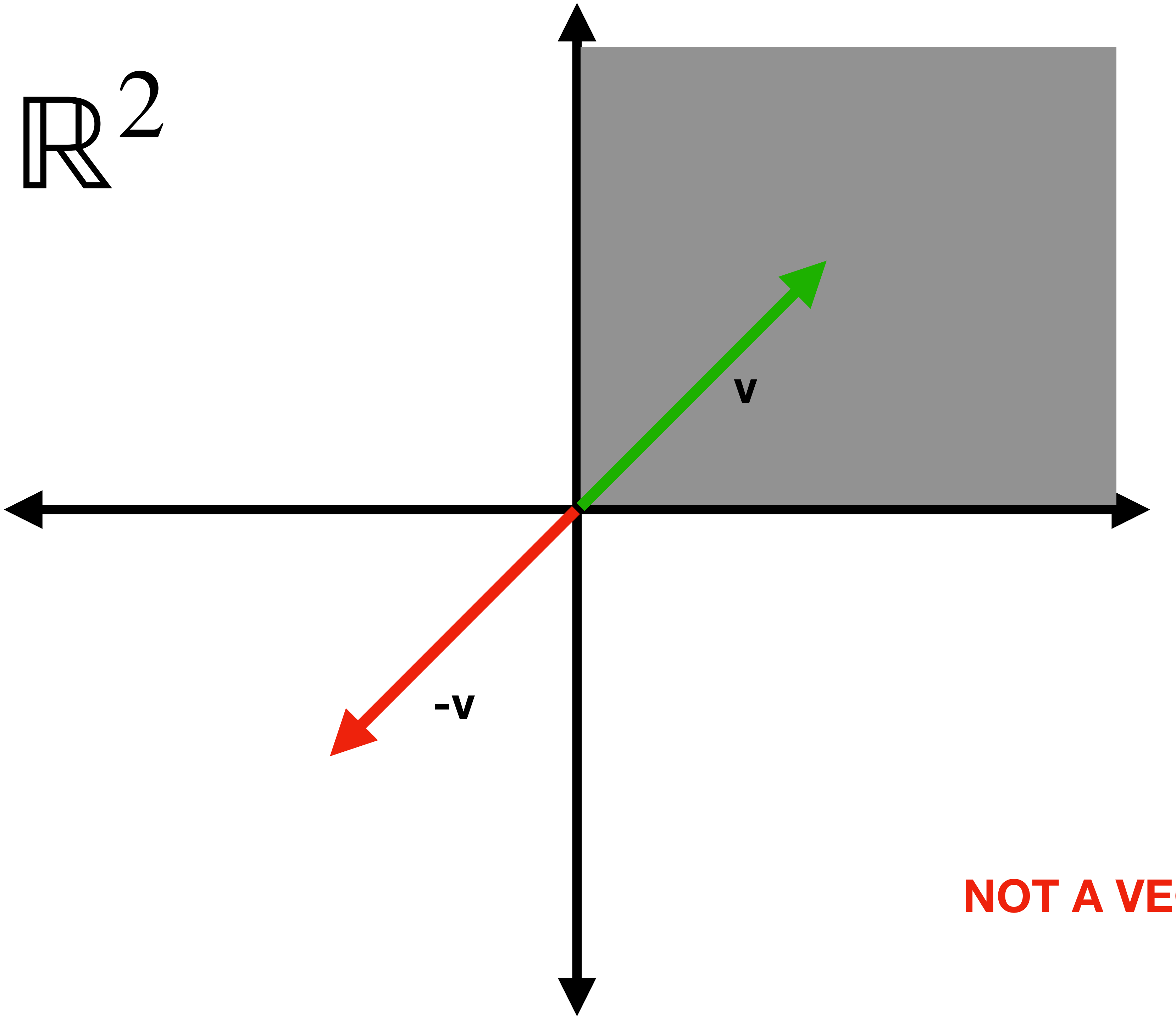
Vector Spaces

$\mathbb{R}^2$



Vector Spaces

$\mathbb{R}^2$



NOT A VECTOR SPACE!!!!

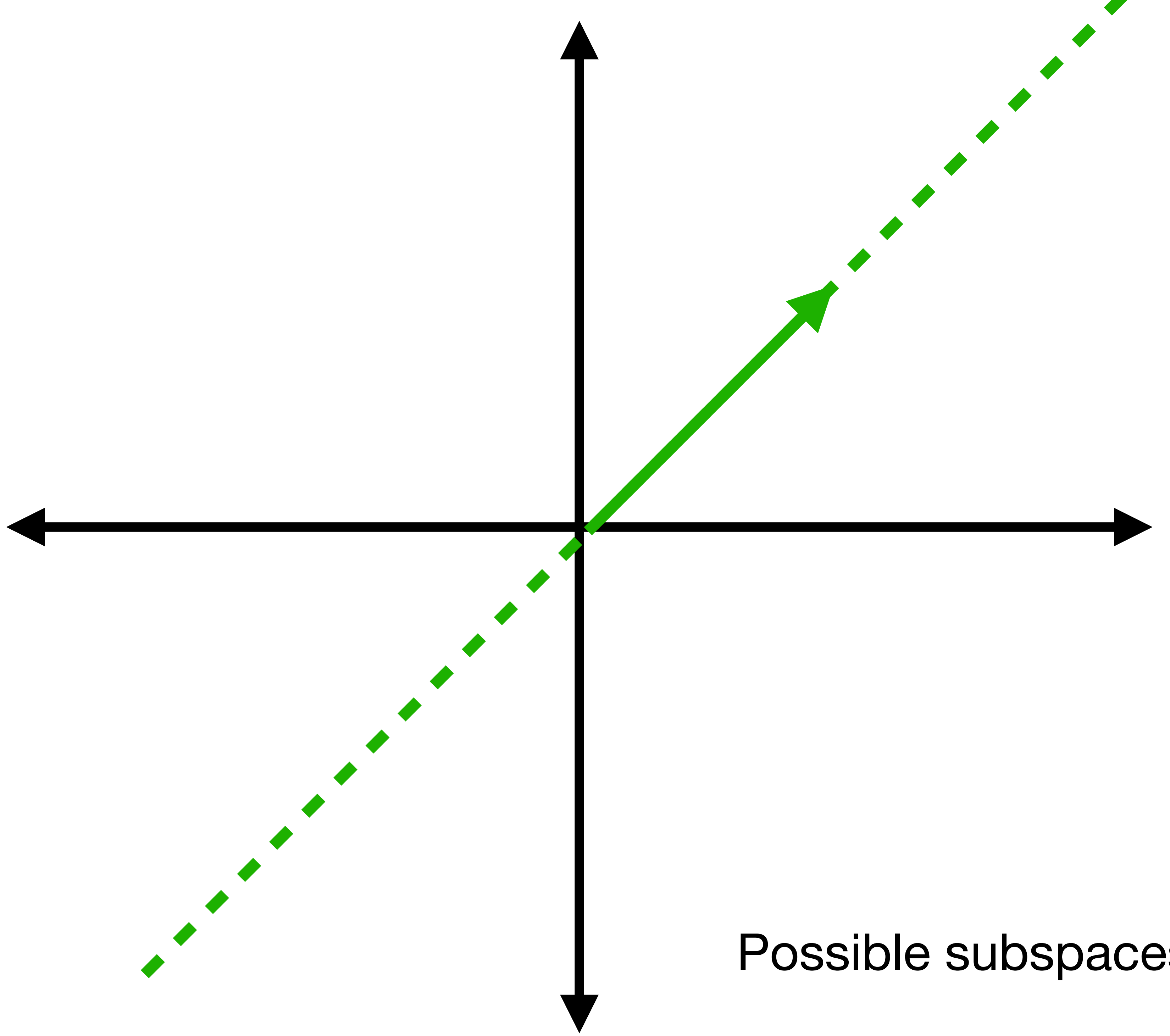
Subspace

A subspace is defined as a set of all vectors that can be created by taking linear combinations of some vectors or a set of vectors.

Formally, a subspace is the set of all vectors that satisfy the following conditions:

- Must be closed under addition and multiplication
- Must contain the zero vector

$$\forall v, w \in \mathbf{V}, \forall \lambda,\ \mu \in \mathbb{R};\ \ \lambda\mathbf{v} + \mu\mathbf{w} \in \mathbf{V}$$



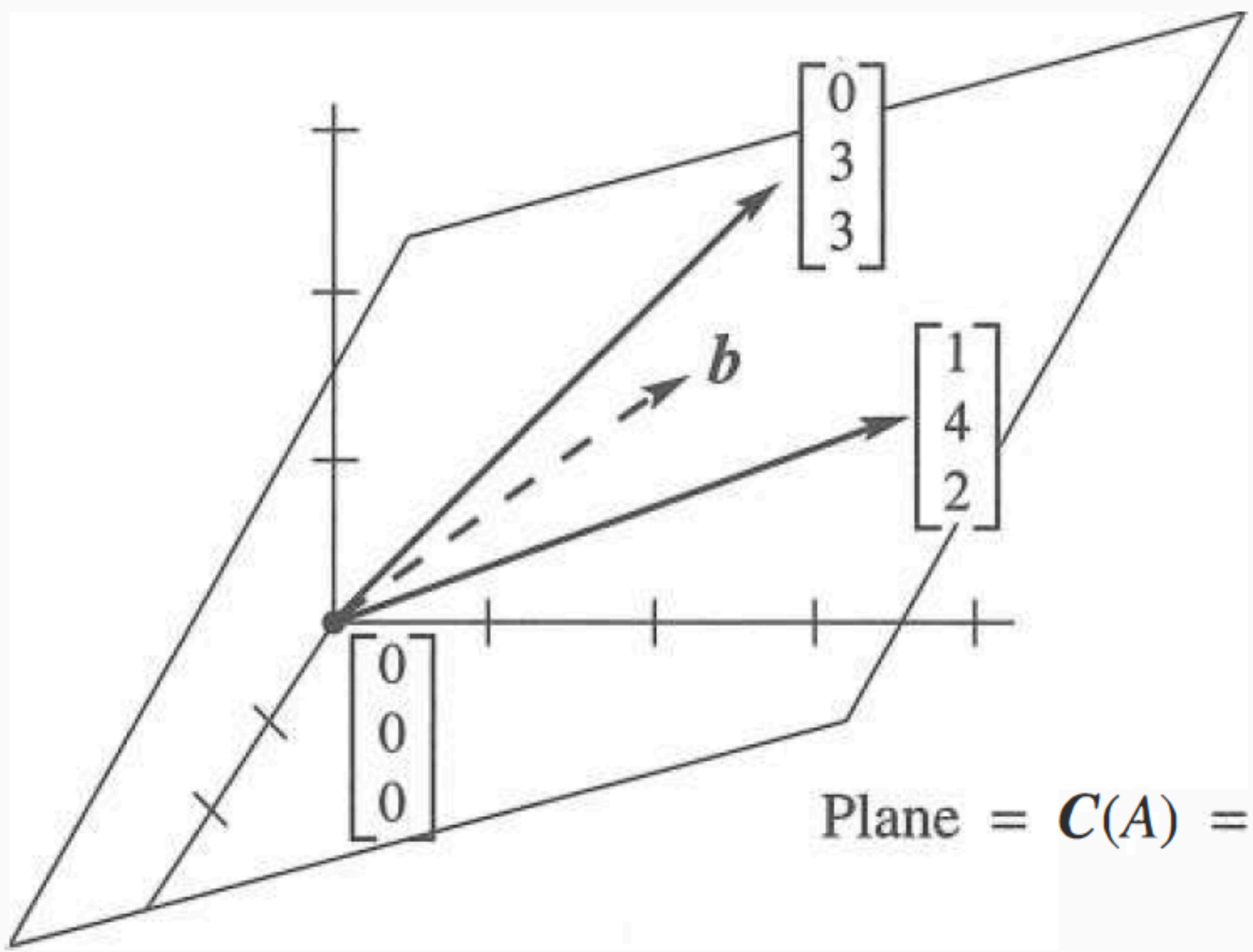
Possible subspaces in $\mathbb{R}^2$

- All of $\mathbb{R}^2$
- Lines that pass through the origin
- The zero vector $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Column Space

- The column space of a matrix A consists of all the linear combinations of the columns of A .
- The combinations are all possible vectors $A\mathbf{x}$, which fill the columns space denoted by $C(A)$.
- The system $A\mathbf{x} = b$ is solvable if and only if b is in the columns space of A.

Column Space Example



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\boldsymbol{b} = .4 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b} \text{ has } \boldsymbol{x} = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$$

Plane $= \boldsymbol{C}(A) =$ all vectors $A\boldsymbol{x}$

Credit: Intro to Linear Algebra. Strang Book

Null Space

- The null space of a matrix A_{mxn} consists of all the solutions to $A\mathbf{x} = 0$.

- The solution vectors x have n components. They are vectors in $\mathbb{R}^n$, so the null space is a subspace of $\mathbb{R}^n$. The columns space $C(A)$ is a subspace of $\mathbb{R}^m$.

$$\mathbf{A} = \begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\end{bmatrix}$$

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 1 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{array}{c} R_1 - R_2 \\ -1R_2 \\ R_3 + 2R_2 \end{array}\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & -1\\ 0 & -2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

REDUCED ROW-ECHELON FORM

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$x + 1 = 0$$
$$y + 1 = 0$$

Pivot Columns **Free Column**

$$\mathbf{s} = \begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$$

Special Solution

$$\mathbf{Z} = c\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$$

Null Space

Complete Solution to Ax = b

- Set all free variables to 0, then solve $Ax = b$ for pivot variables to find $x_{particular}$.
- Find the null space: $x_{null\ space}$.
- The compete solution is:

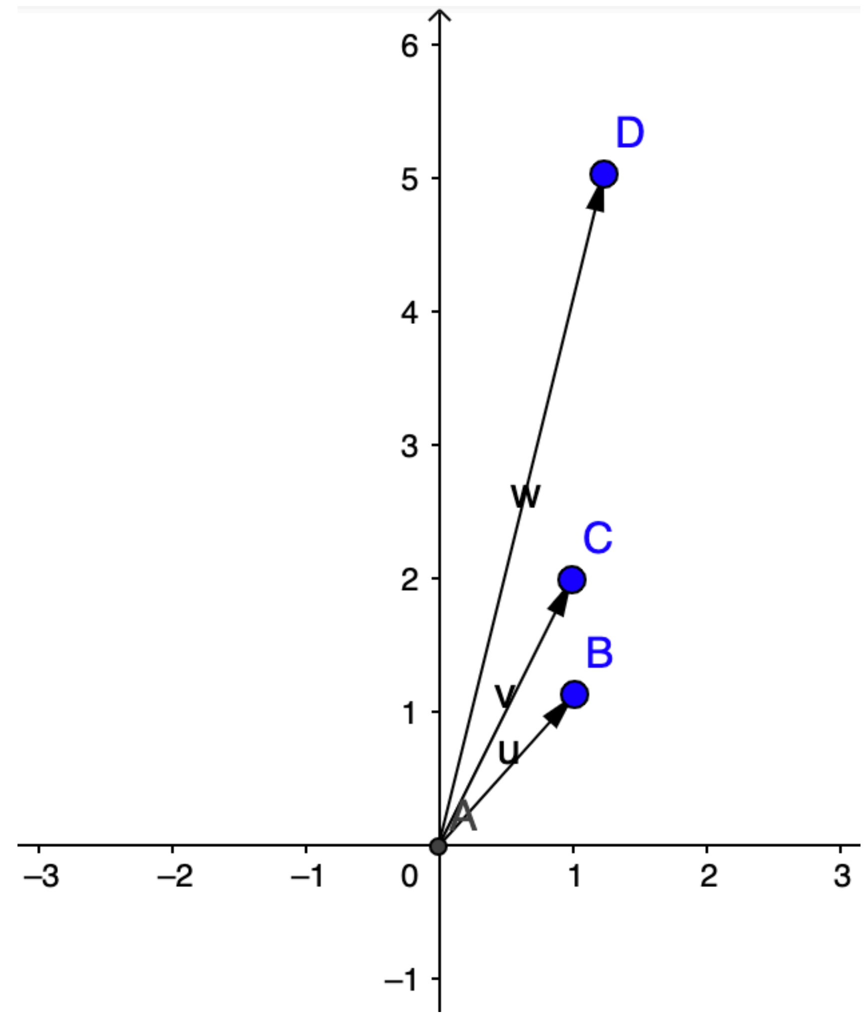
$$x = x_{particular} + x_{null\ space}$$

Matrix Rank

- The rank of a matrix is the number of pivot columns.
- Single number the provides insights into the amount of information that is contained in the matrix.
- Denoted by **r** or **rk(A)** or **rank(A)**
- $r \in \mathbb{N}, s.t.\ 0 \leq r \leq min\{\#cols, \#rows\}$

Matrix Rank

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$$



Perform Gaussian elimination until the matrix is in row-echelon form and then count the number of pivot columns

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightsquigarrow \dots \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad rk(A) = 2$$

Four Fundamental Subspaces

The four fundamental subspaces of A_{mxn} .

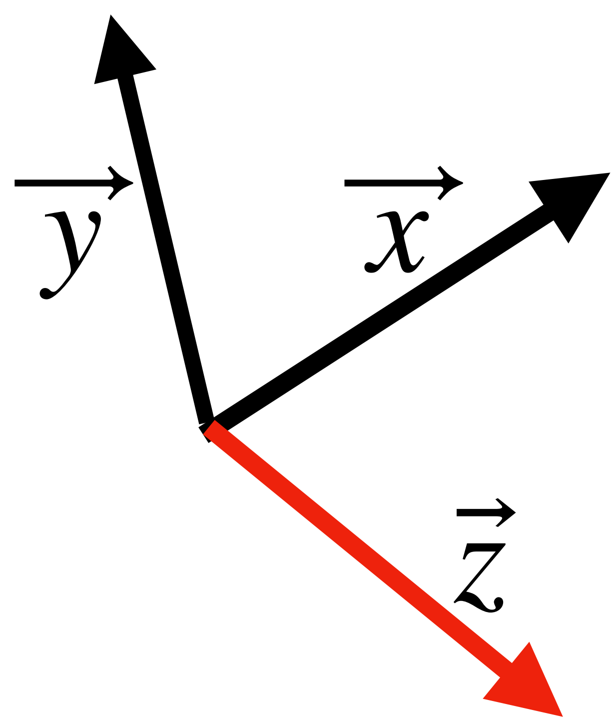
Name	Notation	Note
Column Space	$C(A) \in \mathbb{R}^m$	All combinations of the columns of matrix A.
Null Space	$N(A) \in \mathbb{R}^n$	
Row Space	$C(A^T) \in \mathbb{R}^n$	All combinations of the rows of matrix A.
Left Null Space	$N(A^T) \in \mathbb{R}^m$	

Linear Independence

The sequence of vectors $v_1, \ldots, v_n$ is linearly independent if the only combination that gives the zero vectors is $0v_1 + 0v_2 + \ldots + 0v_n$.

$$x_1v_1 + x_2v_2 + \ldots + x_nv_n = 0$$

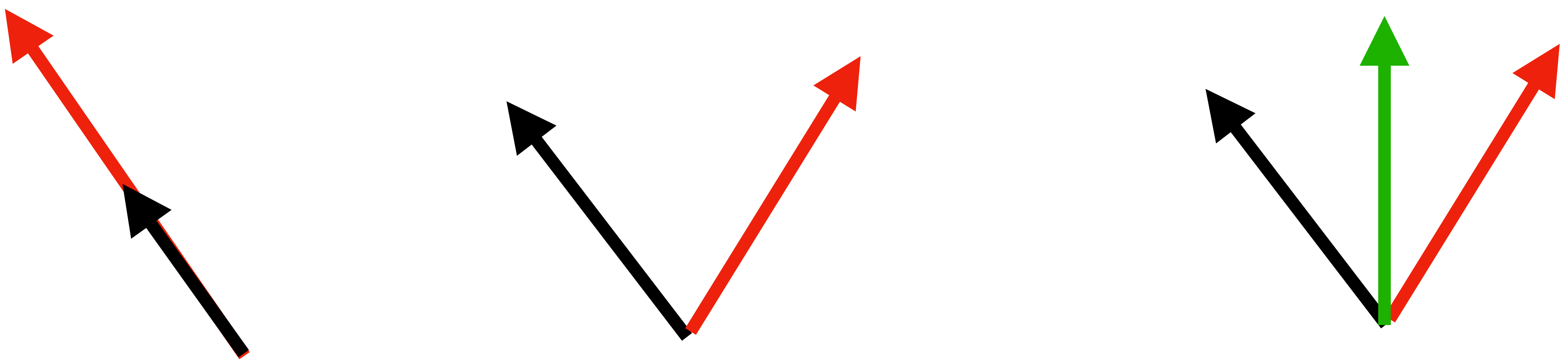
only happens when all x's are zero.



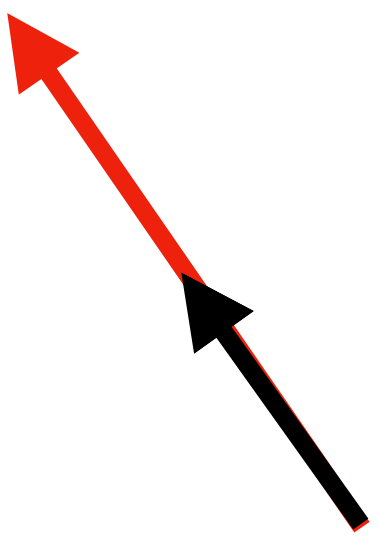
$$z \neq \alpha x + \beta y$$

z can not be express as a linear combination of *x* and *y*

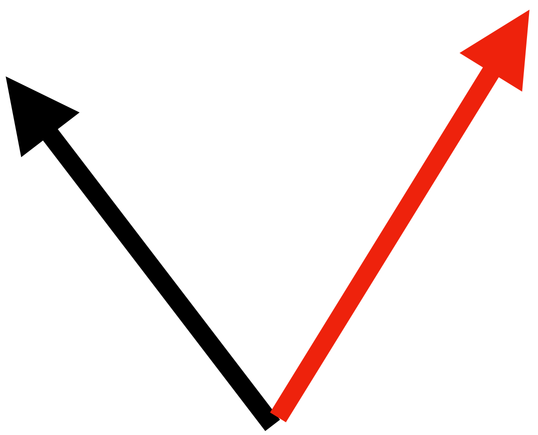
Are these sets of vectors linearly independent?



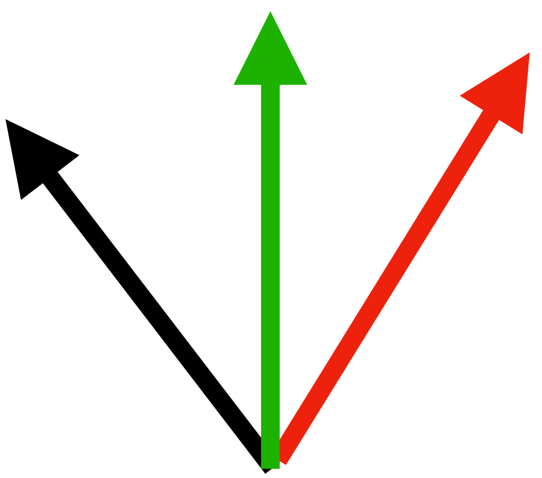
Are these sets of vectors linearly independent?



NO

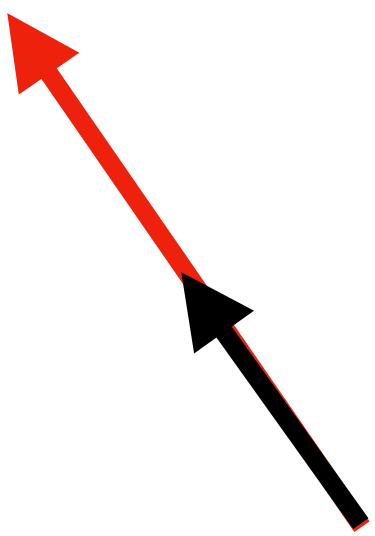


YES

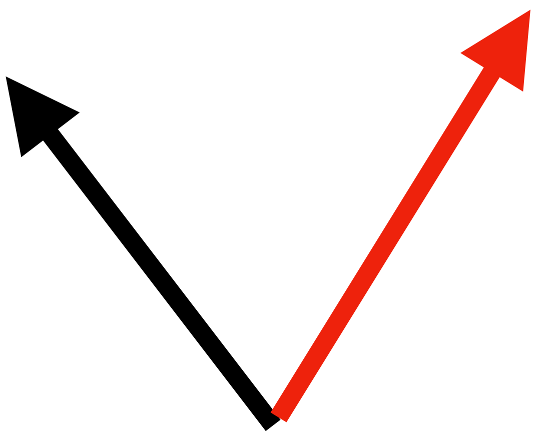


NO

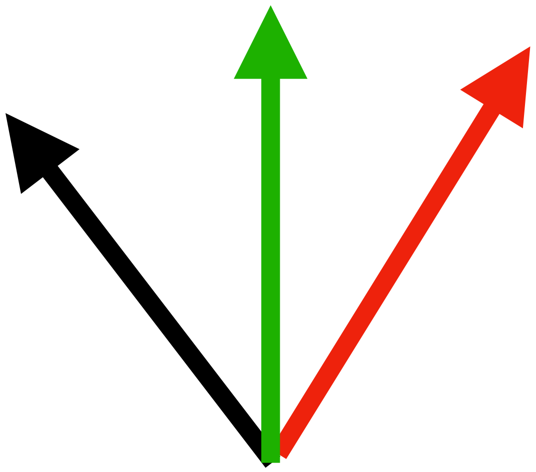
Are these sets of vectors linearly independent?



NO



YES



NO

There are a maximum of N independent vectors in $\mathbb{R}^N$.

Span

Span of a space is defined as all possible linear combinations of all the vectors in that space.

$$span(\{v_1, v_2, \ldots, v_n\}) = \alpha_1 v_1 + \ldots + \alpha_n v_n,\ \alpha_i \in \mathbb{R}$$

Span Example 1

- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full space $\mathbb{R}^2$.
- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ span the full space $\mathbb{R}^2$.
- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ span a line in $\mathbb{R}^2$.

Span Example 2

To determine if a vector $\mathbf{v}$ is in the span of a set S we need to check whether $\mathbf{v}$ can be expressed as a linear combination of vectors in S.

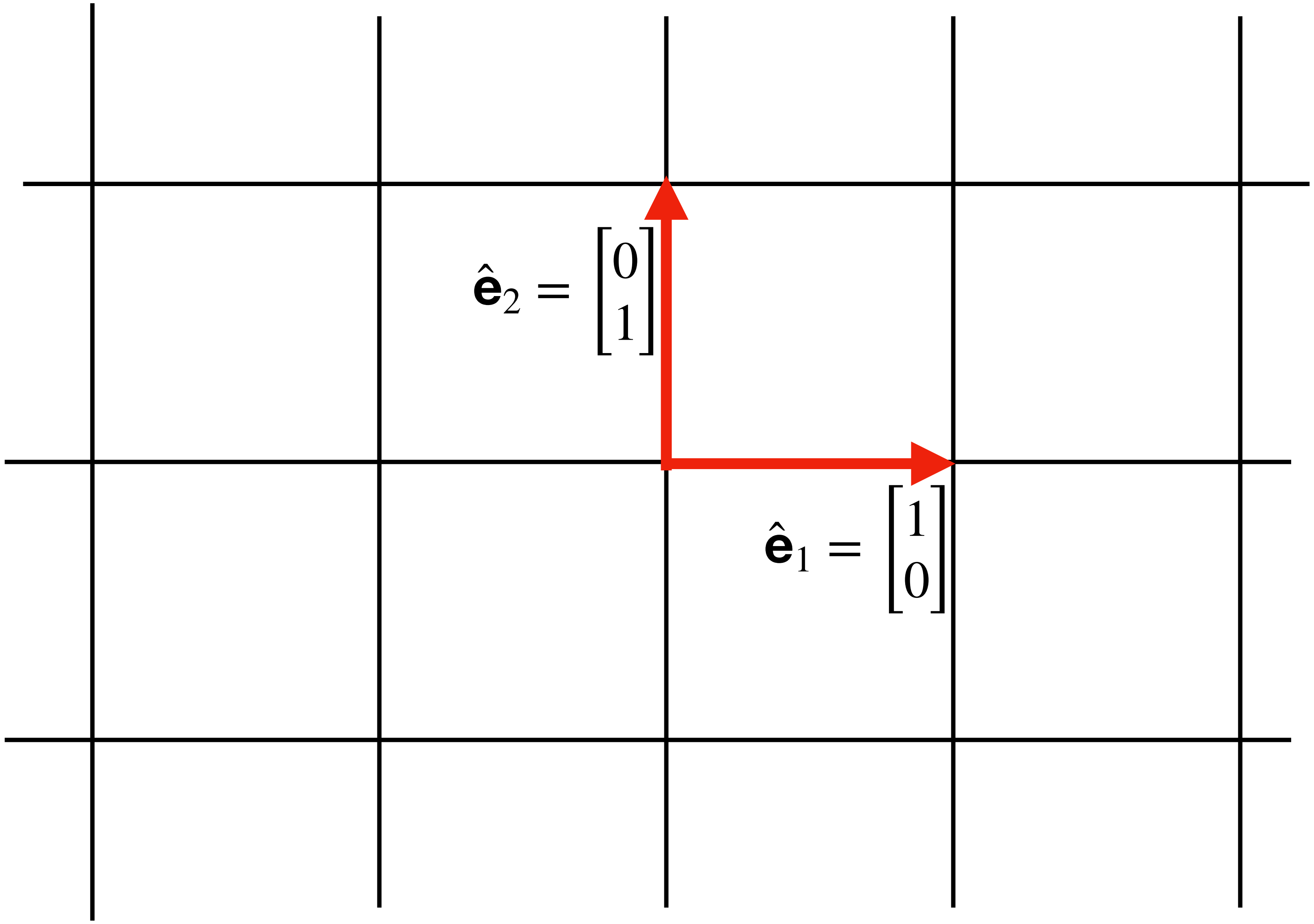
$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \right\}$. Check if $\mathbf{v} \in \mathbf{S}$

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{5}{6} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ Yes, $\mathbf{v} \in \mathbf{S}$

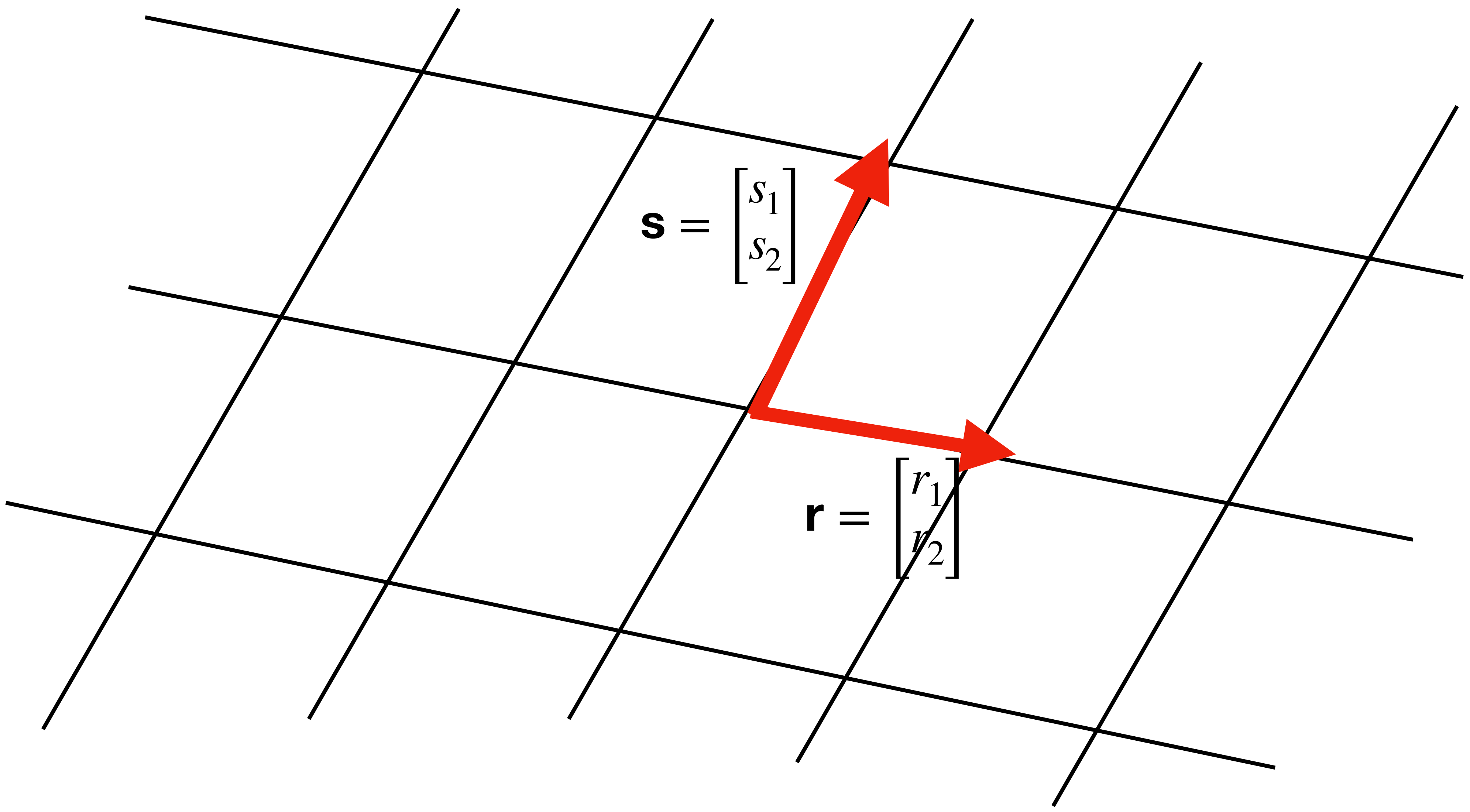
Basis and Dimension

- A basis for a vector space is a sequence of vectors with two properties:
  - The vectors are linearly independent.
  - The vectors span the space.
- Every vector in the space is a unique combination of the basis vectors.
- A space can be spanned by multiple basis.
- The dimension of a space is the number of vectors in every basis.

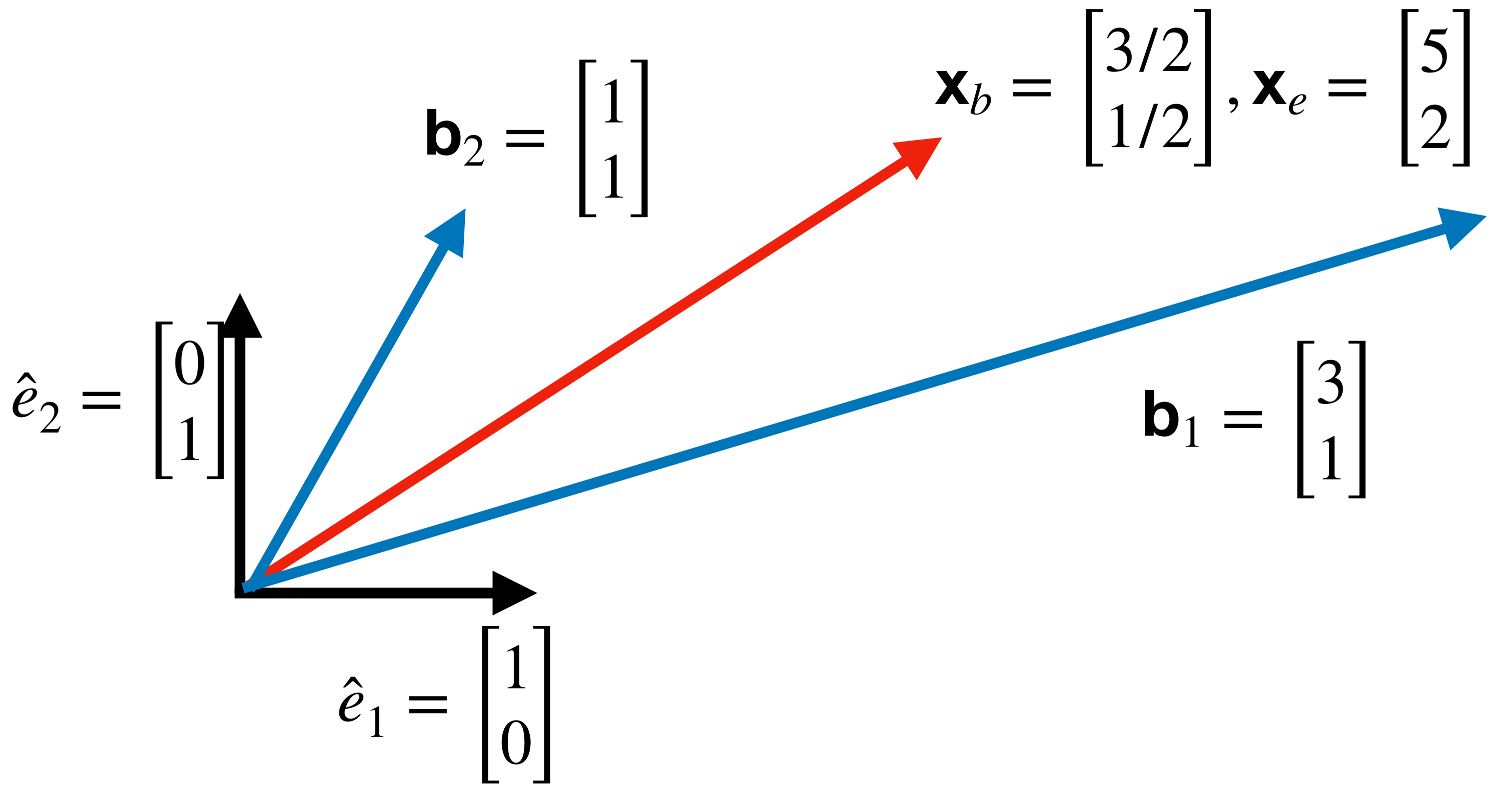
Natural Basis



Another Basis



Changing Basis



$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{x}_b = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \qquad \mathbf{x}_e = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 9/2 + 1/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{x}_e = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad \mathbf{x}_b = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Orthogonality

Orthogonal Vectors

Two vectors v and w are orthogonal when their dot product is 0.

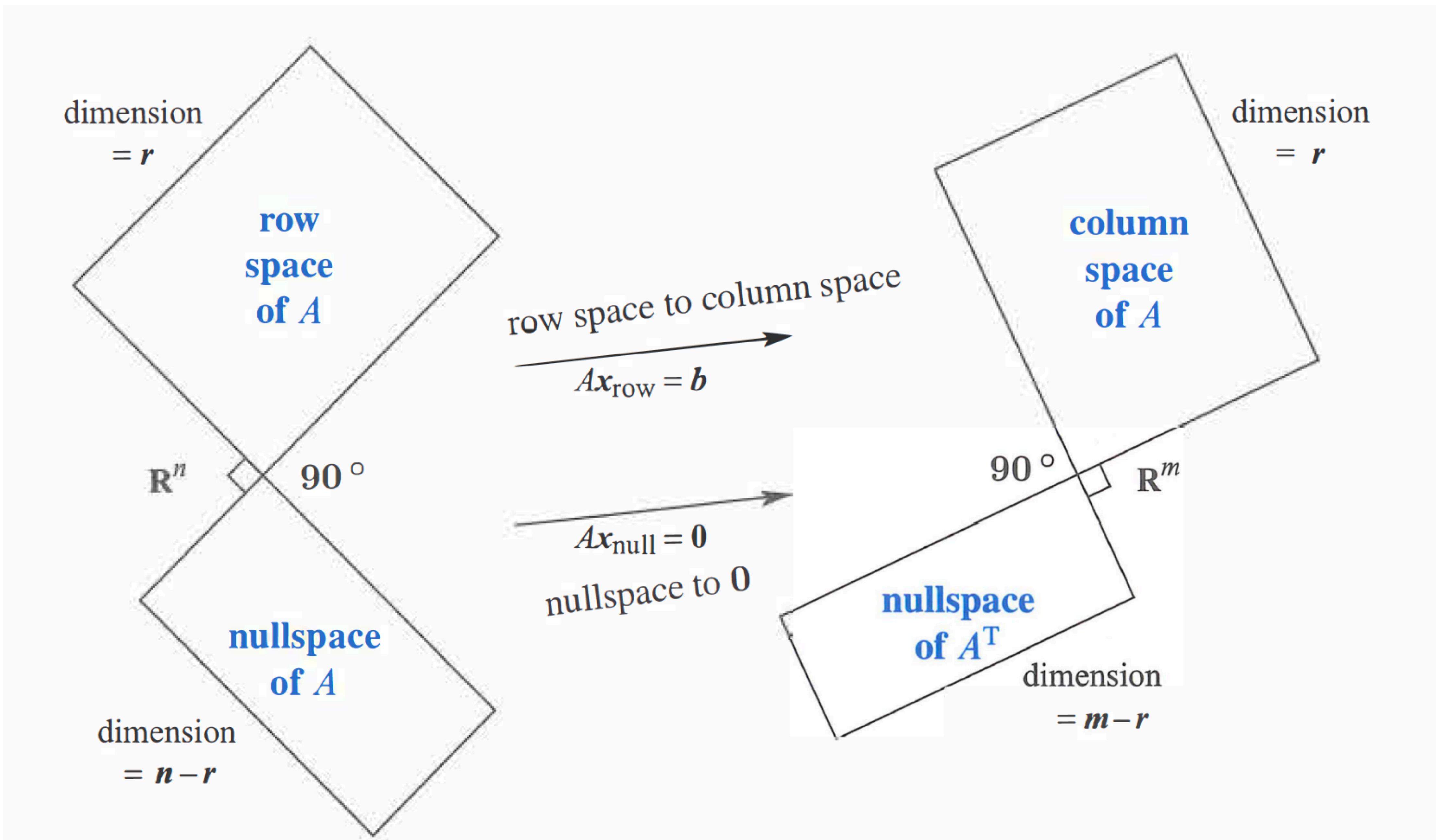
$$v^T w = 0, \qquad ||v||^2 + ||w||^2 = ||v+w||^2$$

Orthogonal Subspaces

Two subspaces V and W of a vector space are orthogonal if every vector $v \in V$ is perpendicular to every vector $w \in W$.

$$v^T w = 0 \; for \; all \; v \in V \; and \; w \in W$$

Orthogonality of the Four Spaces



Orthogonality of the Four Spaces

The null space $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of R^n.

Proof:

$$A\mathbf{x} = \begin{bmatrix} row\ 1 \\ \vdots \\ row\ m \end{bmatrix} [\mathbf{x}] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \begin{matrix} (row\ 1) \cdot \mathbf{x} = 0 \\ \vdots \\ (row\ m) \cdot \mathbf{x} = 0 \end{matrix}$$

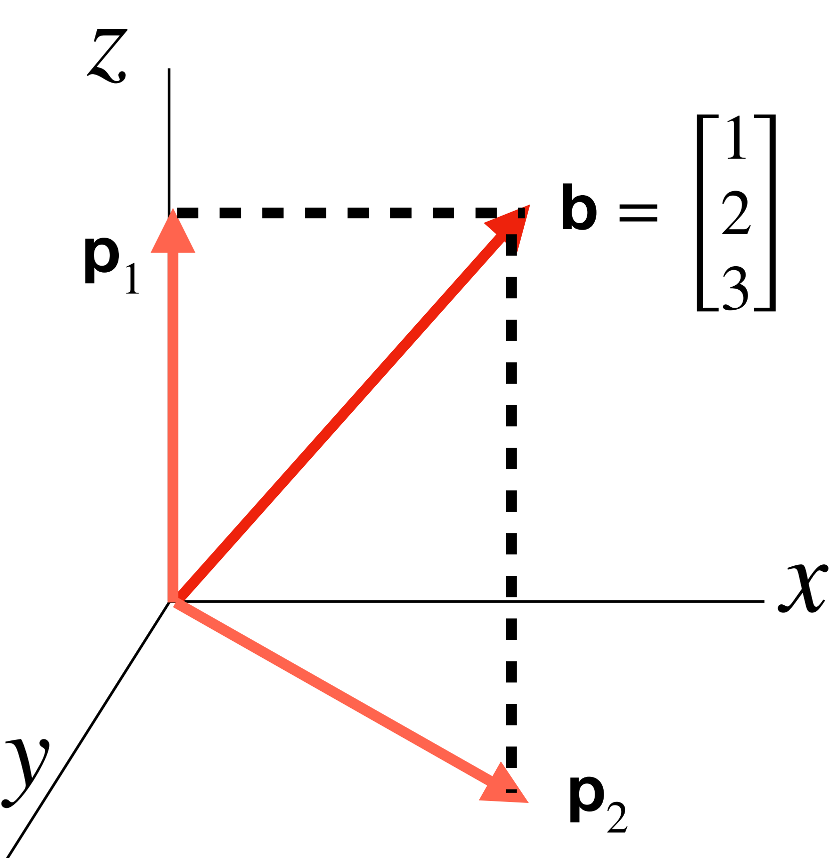
The left null space $N(A^T)$ and the column space $C(A)$ are orthogonal in R^m.

Projections

The projection of b onto a subspace S is the closest vector p in S. For example, when a vector b is projected onto a line, its projection p is the part of b along that line. When b is projected into a plane, p is the part in that plane.

A projection matrix P is a symmetric matrix with $P^2 = P$

The projection of b is Pb.



What is the projection of vector b onto the *z-axis* line and *xy* plane?

$$\mathbf{p}_1 = \begin{bmatrix}0\\0\\3\end{bmatrix} \qquad \mathbf{P}_1 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & \mathbf{1}\end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix} \qquad \mathbf{P}_2 = \begin{bmatrix}\mathbf{1} & 0 & 0\\0 & \mathbf{1} & 0\\0 & 0 & 0\end{bmatrix}$$

$$\mathbf{p}_1 + \mathbf{p}_2 = b,$$

$$\mathbf{P}_1 + \mathbf{P}_2 = I$$

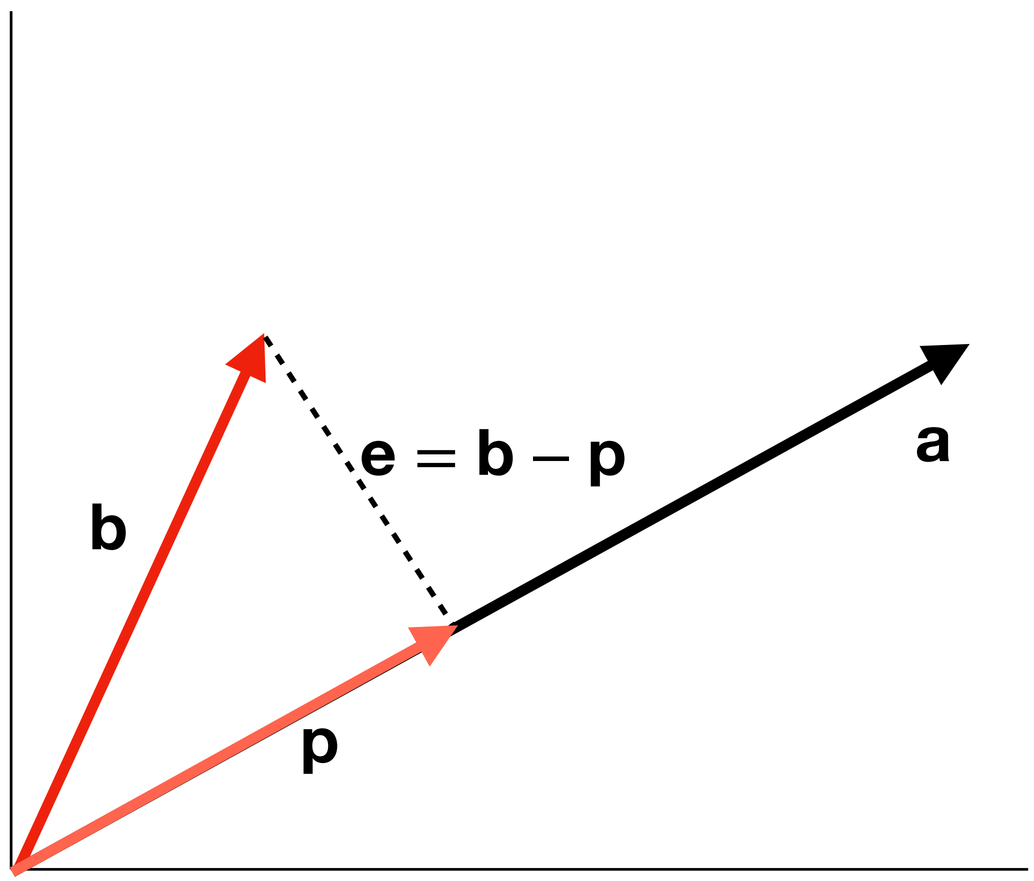
because the line and plane are orthogonal complements.

Why do we need projections?

We need projections to cover cases when $A\mathbf{x} = \mathbf{b}$ does not have any solutions. In these cases we can solve the closest problem $A\hat{\mathbf{x}} = \mathbf{p}$.

If $A\mathbf{x} = \mathbf{b}$ does not have any solutions, $\mathbf{b}$ is not in the column space of A. We can solve $A\hat{\mathbf{x}} = \mathbf{p}$ instead where $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the column space of A.

Projection onto a Line



$$\mathbf{p} = \hat{x}\mathbf{a}, \;\; \mathbf{a} \perp (\mathbf{b} - \mathbf{p})$$

$$\mathbf{a} \cdot (\mathbf{b} - \hat{x}\mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{b} - \hat{x}\mathbf{a} \cdot \mathbf{a} = 0$$

$$\mathbf{a}^T\mathbf{b} - \hat{x}\mathbf{a}^T\mathbf{a} = 0$$

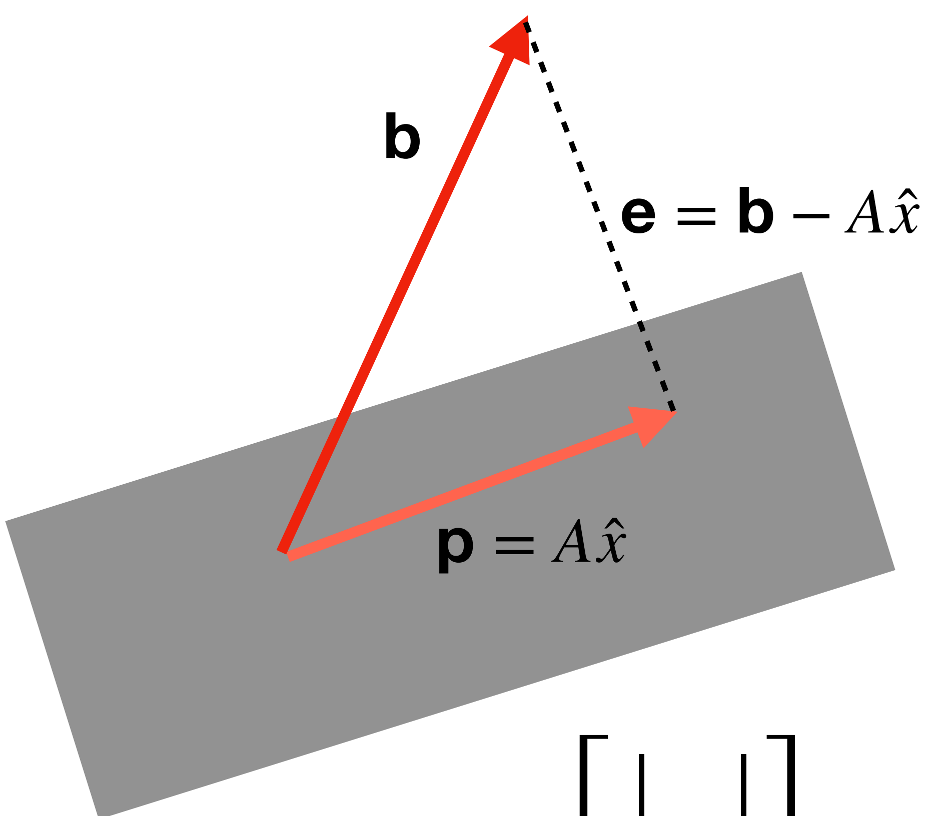
$$\hat{x} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$$

$$\mathbf{p} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}$$

$$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$$

Projection onto a Subspace

Find the projection of $\mathbf{b}$ in $\mathbb{R}^m$ onto the subspace spanned by columns of A.



$$\mathbf{A} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

a_1*, and* a_2 *are basis for plane*

$$\mathbf{p} = A\hat{x}$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(\mathbf{b} - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T(\mathbf{b} - A\hat{x}) = 0$$

$$A^TA\hat{x} = A^T\mathbf{b}$$

$$\hat{x} = (A^TA)^{-1}A^T\mathbf{b}$$

$$\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b}$$

$$\mathbf{P} = A(A^TA)^{-1}A^T$$

$A^T A$ Invertibility

Theorem
If A has linearly independent columns then $A^T A$ is invertible.

Proof:
Show that the the null space of $A^T A$ is only the zero vector.

$$A^T A x = 0$$

$$x^T A^T A x = 0$$

$$(Ax)^T A x = 0$$

$$Ax = 0$$

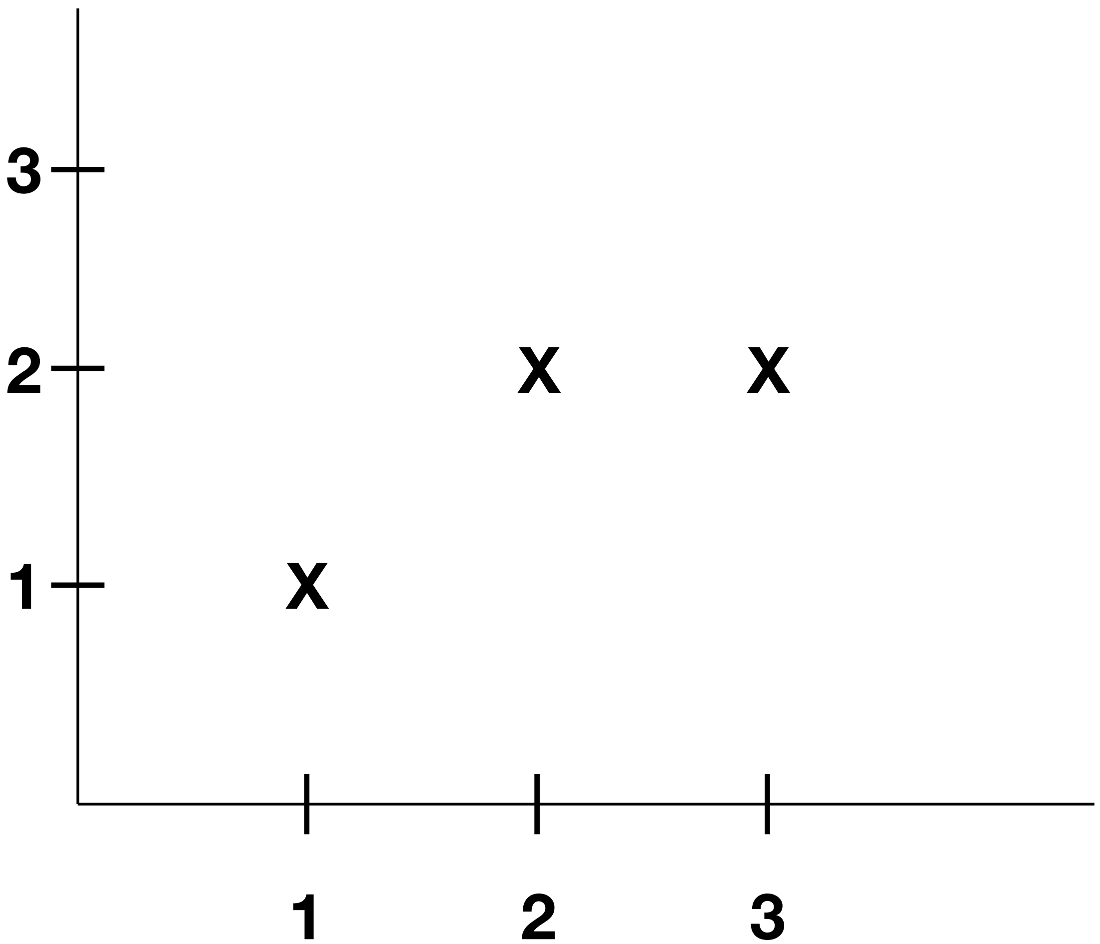
$$x = 0$$

Since columns of A are independent

$\blacksquare$

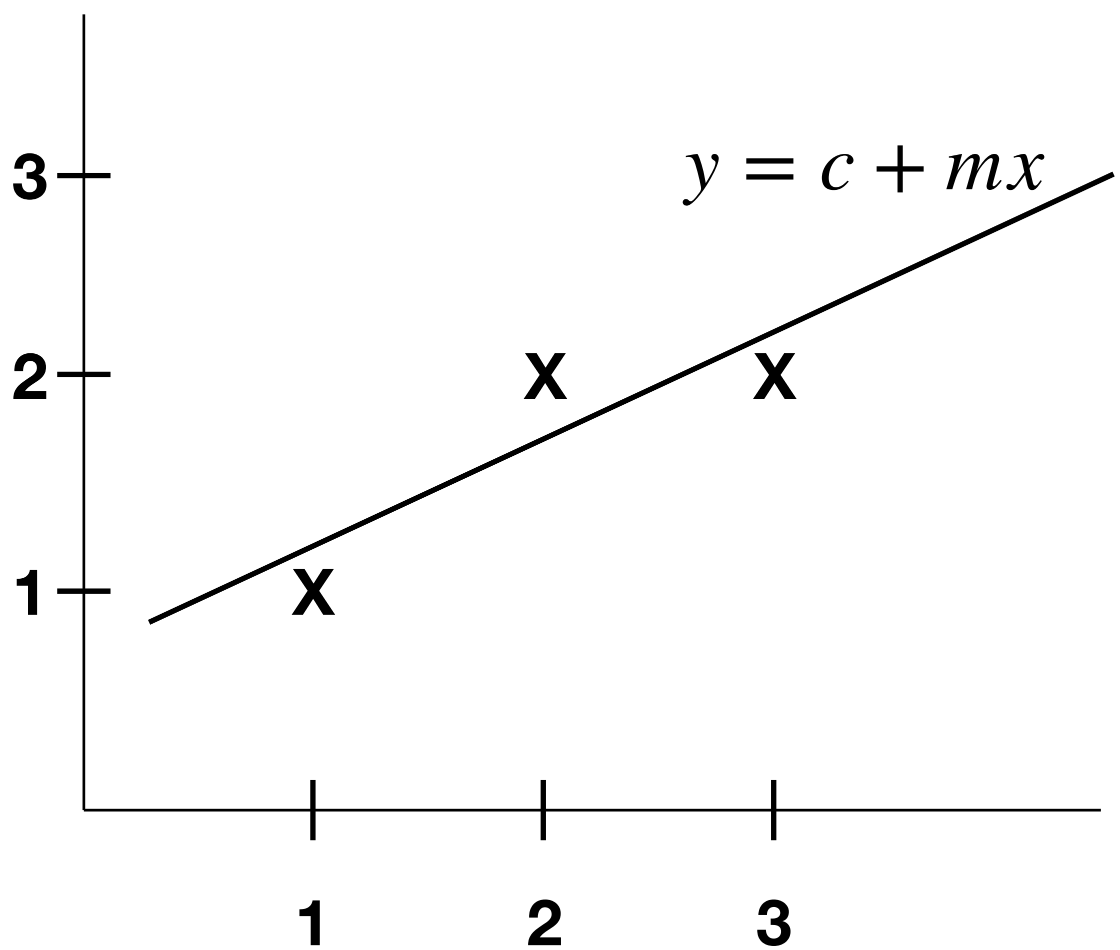
Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



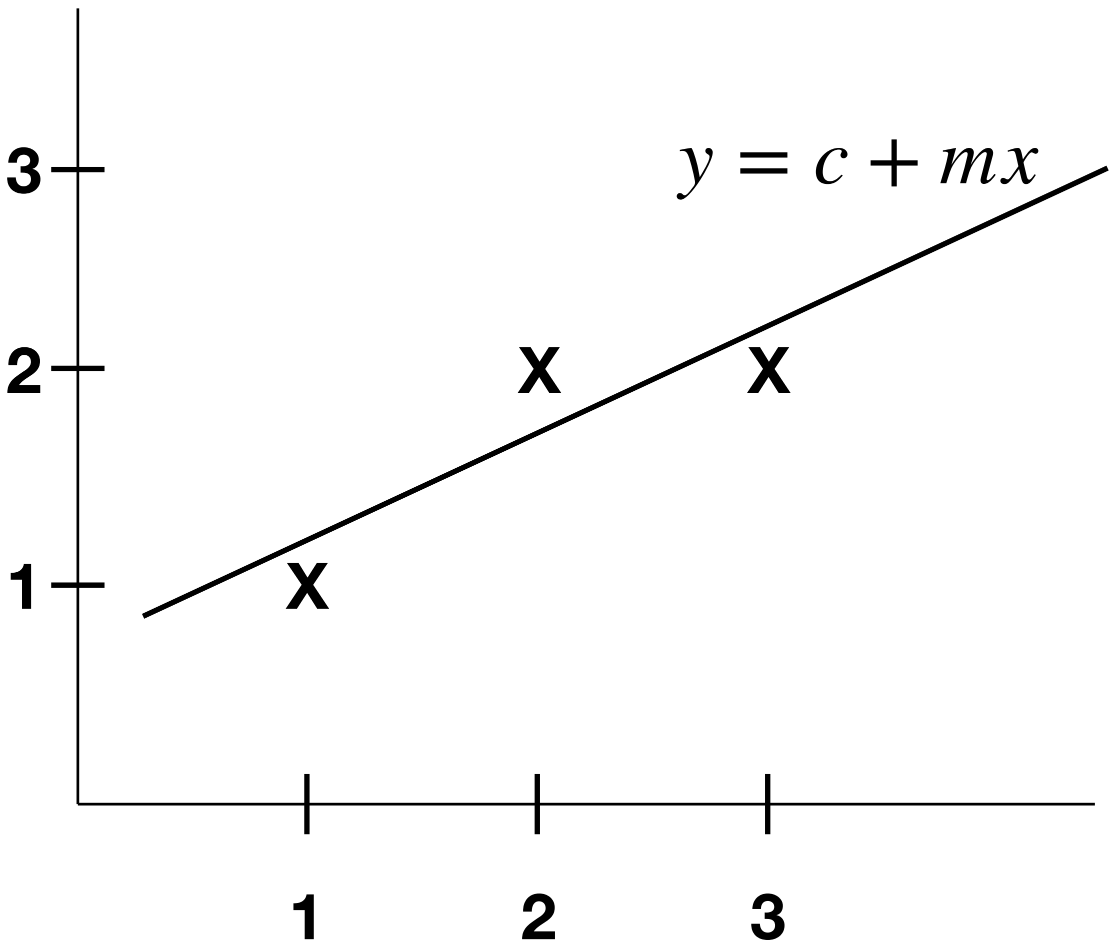
Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



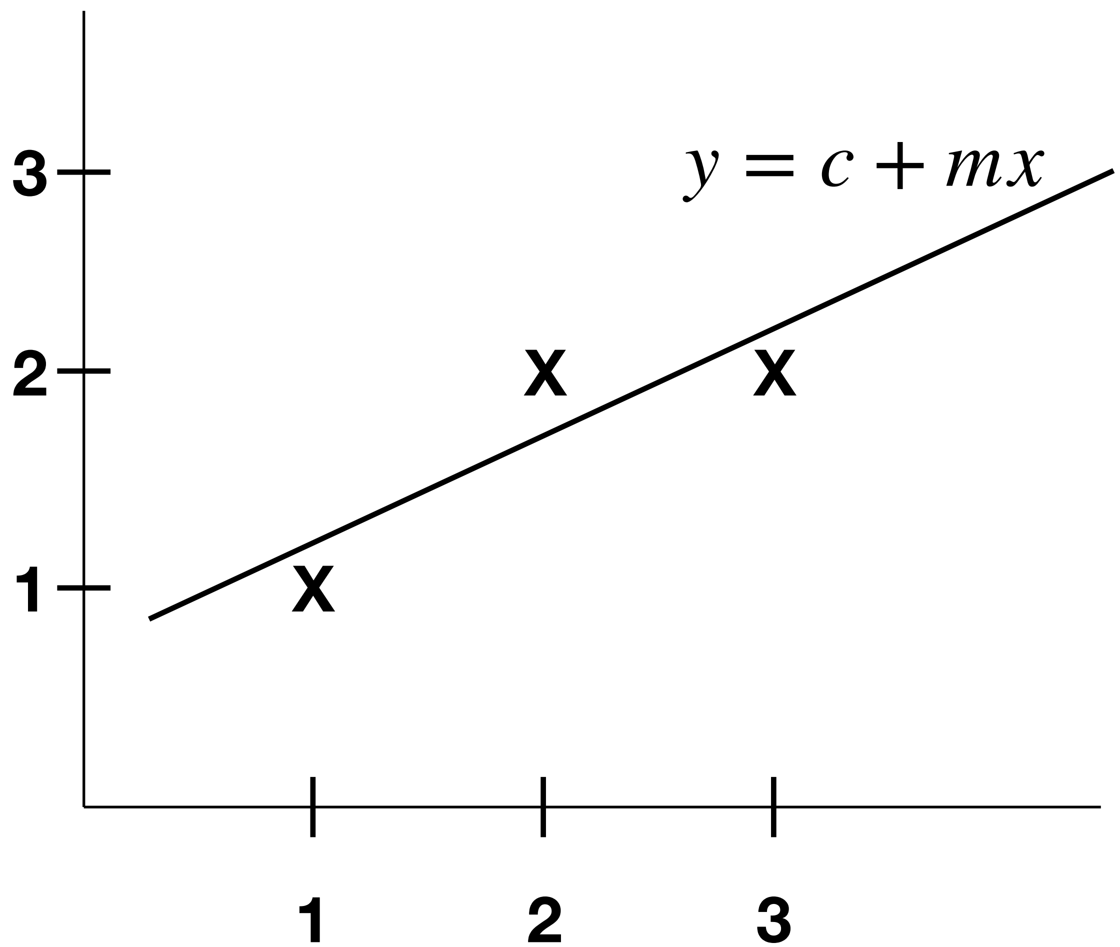
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 equation,
2 unknown
No solution

$$Ax = b$$

Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



Minimize

$$e^2 = ||Ax - \mathbf{b}||^2$$

Find $\hat{x} = \begin{bmatrix} \hat{c} \\ \hat{m} \end{bmatrix}$

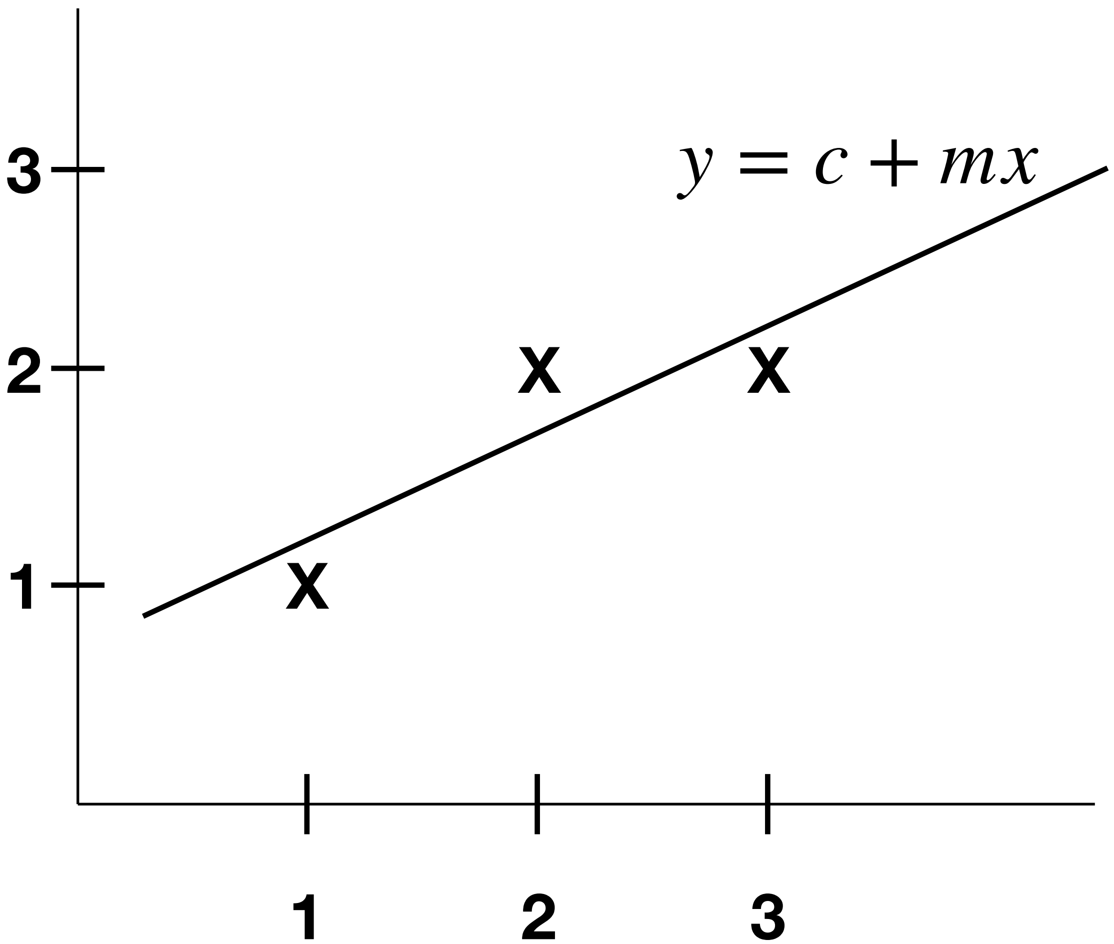
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 equation,
2 unknown
No solution

$$Ax = b$$

Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



Minimize

$$e^2 = ||Ax - \mathbf{b}||^2$$

Find $\hat{x} = \begin{bmatrix} \hat{c} \\ \hat{m} \end{bmatrix}$

$$A^T A \hat{x} = A^T \mathbf{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

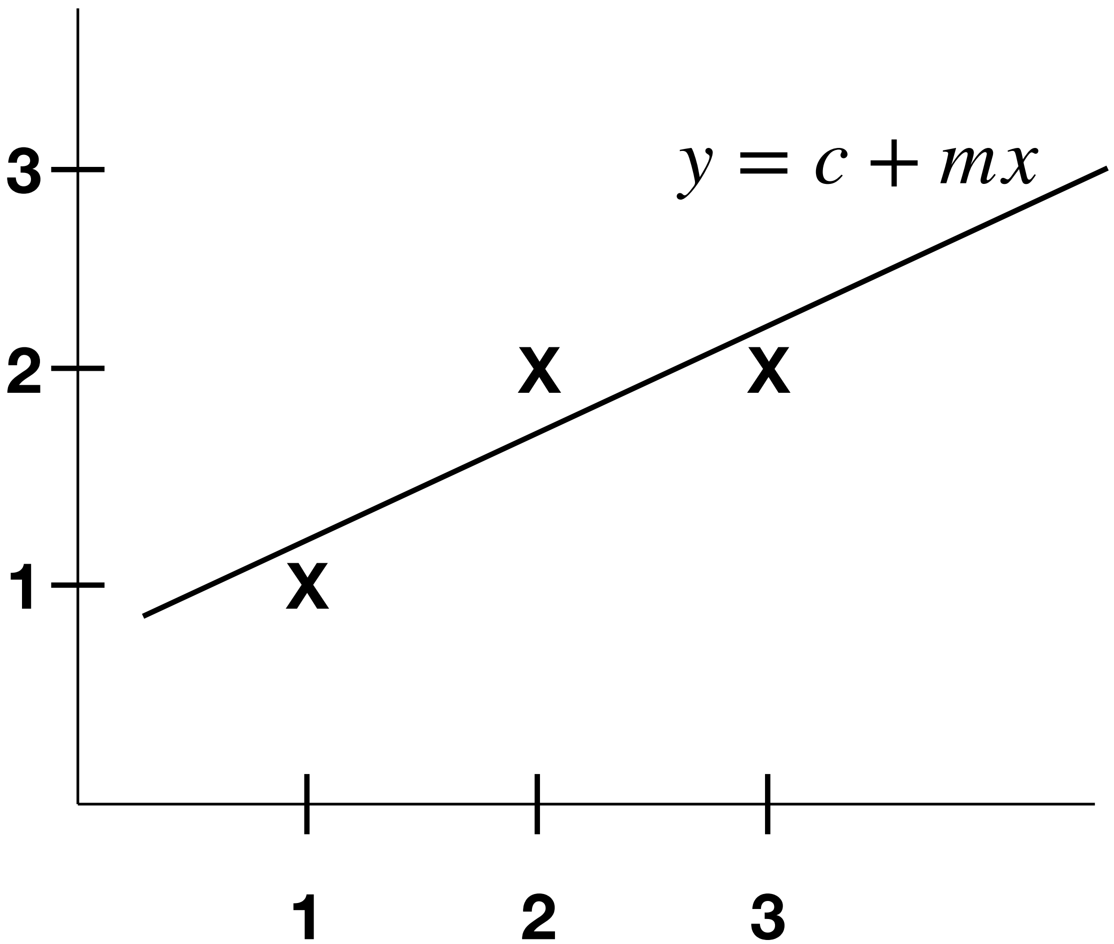
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 equation,
2 unknown
No solution

$$Ax = b$$

Least Squares Approximation

Find the closest line to the points (1,1), (2,2) and (3,2).



$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 equation,
2 unknown
No solution

$$Ax = b$$

Minimize

$$e^2 = ||Ax - \mathbf{b}||^2$$

Find $\hat{x} = \begin{bmatrix} \hat{c} \\ \hat{m} \end{bmatrix}$

$$A^T A\hat{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

$$\begin{cases} 3\hat{c} + 6\hat{m} = 5 \\ 6\hat{c} + 14\hat{m} = 11 \end{cases} \rightarrow \quad \hat{c} = \frac{2}{3}, \quad \hat{m} = \frac{1}{2}$$

Orthonormal Vectors

Vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are **orthonormal** if:

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0 & \textbf{when } i \neq j \text{ (}\textbf{orthogonal vectors}\text{)} \\ 1 & \textbf{when } i = j \text{ (}\textbf{unit vectors: } ||\mathbf{q}_i|| = 1\text{)} \end{cases}$$

A matrix Q with orthonormal columns satisfies $Q^TQ = I$:

$$Q^TQ = \begin{bmatrix} - & \mathbf{q}_1^T & - \\ - & \mathbf{q}_2^T & - \\ - & \mathbf{q}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_n \\ | & | & | \end{bmatrix} = I$$

Examples: $Q_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad Q_2 = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

Permutation **Rotation**

Projections Using Orthonormal Bases

$$\hat{x} = (A^T A)^{-1} A^T \mathbf{b}, \quad \mathbf{P} = A(A^T A)^{-1} A^T$$

Assume that A has orthonormal columns.

$$\hat{x} = (Q^T Q)^{-1} Q^T \mathbf{b}$$

$$\hat{x} = Q^T \mathbf{b}$$

$$\mathbf{P} = Q(Q^T Q)^{-1} Q^T$$

$$\mathbf{P} = Q I^{-1} Q^T$$

$$\mathbf{P} = Q Q^T$$

P satisfies both $P^T = P$ and $P^2 = P$.

Gram-Schmidt Process

For independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, Gram-Schmidt process constructs orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$.

- Start with three independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- Construct three orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as follows:
  - Choose $\mathbf{A} = \mathbf{a}$
  - To construct $\mathbf{B}$, start with $\mathbf{b}$ and subtract its projection along $\mathbf{A}$.

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T\mathbf{b}}{\mathbf{A}^T\mathbf{A}}\mathbf{A}$$

  - To get $\mathbf{C}$, subtract its component in directions $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}^T\mathbf{c}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} - \frac{\mathbf{B}^T\mathbf{c}}{\mathbf{B}^T\mathbf{B}}\mathbf{B}$$

- Produce three orthonormal vectors: $\mathbf{q}_1 = \mathbf{A}/||\mathbf{A}||,\ \mathbf{q}_2 = \mathbf{B}/||\mathbf{B}||, \mathbf{q}_3 = \mathbf{C}/||\mathbf{C}$

Gram-Schmidt Example

$$a = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, c = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}}{\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \frac{8}{8} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} - \frac{24}{8} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + \frac{24}{24} \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thank you!