Probability Theory 1

CS 556

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Probability

- Probability is a branch of mathematics focusing on numerical descriptions of how likely an event is to occur.
- A probabilistic model is a mathematical description of an uncertain situation.

Experiment

- An experiment is any activity or process whose outcome is subject to uncertainty.
- Examples:
    - Tossing a coins once or several times
    - Selecting a card from a deck of cards
    - Obtaining blood types from a group of people

Sample Space

- The sample space of an experiment, denoted by Ω, is the set of all possible outcomes of that experiment.
- Examples:
  - The sample space of tossing a coin is {H, T}.
  - The sample space of tossing two coins is {HH, HT, TH, TT}
  - The sample space of rolling a dice is {1, 2, 3, 4, 5, 6}

Event

- An event denoted by A is any sunset of outcomes contained in the sample space Ω.

- An event is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

- Consider the experiment of tossing two coins.

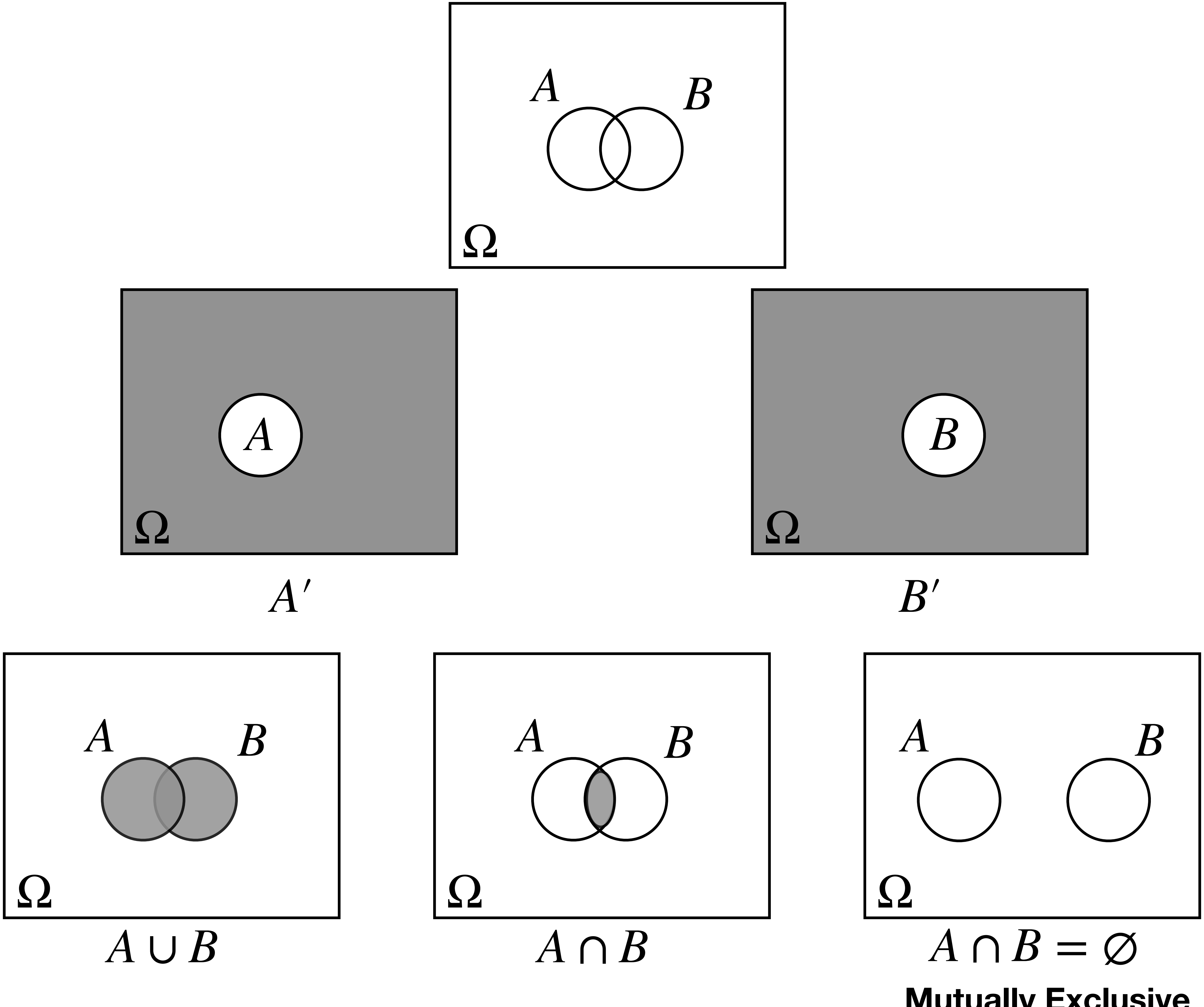
$$\Omega = \{HH, HT, TH, TT\}, \quad A_1 = \{HH\}, \quad A_2 = \{HH, TT\}$$

Complement, Union, Intersection

- The complement of an event A, denoted by A', is the set of all outcomes in the sample space Ω that are not contained in A.

- The union of two events A and B, denoted by $A \cup B$, is the event consisting of all the outcomes in at least one of the events.

- The intersection of two events A and B, denoted by $A \cap B$, is the event consisting of all outcomes that are both in A and B.


A
B
Ω
A
Ω
B
Ω
A′
B′
A
B
Ω
A ∪ B
A
B
Ω
A ∩ B
A
B
Ω
A ∩ B = ∅
Mutually Exclusive

Probability law

- The probability law assigns an event A a nonnegative number $P(A)$, called the probability of A that specifies a precise measure of the chance that A with occur.

The Axioms of Probability

- (Non negativity) For every event A, $P(A) \geq 0$.
- (Normalization) Probability of the entire sample space Ω is equal to 1, $P(\Omega) = 1$.
- (Additivity) If A and B are two disjoint events, then the probability of their union satisfies: $P(A \cup B) = P(A) + P(B)$.

Example

Consider an experiment involving three coin tosses. The outcomes will be a 3-long string of heads or tails. We assume that each possible outcome has the same probability 1/8. What is the probability of getting exactly two heads?

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$A = \{exactly\ 2\ heads\ occurs\} = \{HHT, HTH, THH\}$$

$$\begin{aligned} P(A) &= P(\{HHT, HTH, THH\}) \\ &= P(HHT) + P(HTH) + P(THH) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= \frac{3}{8} \end{aligned}$$

Example

Consider the experiment of rolling a pair of 4-sided fair dice. Compute the following probabilities:

- P({the sum of the rolls is even})
- P({the sum of the rolls is odd})
- P({the first roll is equal to the second})
- P({the first roll is larger than the second})
- P({at least one roll is equal to 4})

Example

Rolling a pair of 4-sided dice

		Dice 2			
		1	2	3	4
Dice 1	1	2	3	4	5
	2	3	4	5	6
	3	4	5	6	7
	4	5	6	7	8

- P({the sum of the rolls is even}) = 8/16
- P({the sum of the rolls is odd}) = 8/16
- P({the first roll is equal to the second}) = 4/16
- P({the first roll is larger than the second}) = 6/16
- P({at least one roll is equal to 4}) = 7/16

More probability properties

- For any event A, $P(A) \leq 1$.

- For any event A, $P(A) + P(A') = 1$.

- For any two events A and B :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example

In a city, 60% of the households get Internet service from a local cable company, 80% get television service from that company, and 50% get both services from that company. If a household is randomly selected what is the probability that it gets at least one of these two services from the company?

A = {gets Internet service}, *B* = {get Television service}

P({subscribes to at least one of the two services}) = $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.8 - 0.5 = 0.9$$

Counting Techniques

When all the outcomes of an experiment are equally likely, the task of computing the probability reduces to counting. When *N* denotes the number of outcomes in the sample space and *N(A)* represents the number of outcomes contained in event *A*, then $P(A) = N(A)/N$.

[**Product Rule**] If the first element of an ordered pair can be selected in ***x*** ways and for each of these ***x*** ways the second element can be selected in ***y*** ways, then the number of pairs is ***xy***.

A **permutation** is an ordered subset: $P_{k,n} = \dfrac{n!}{(n-k)!}$.

A **combination** is an unordered subset: $C_{k,n} = \dfrac{P_{k,n}}{k!} = \dfrac{n!}{k!(n-k)!}$.

Example

A YouTube playlist contains 100 songs, 10 out of which are by Bon Jovi. Suppose the songs are played in random order. What is the probability that the first Bon Jovi's song heard is the fifth song played?

$$P(1st\ Jovi's\ is\ the\ 5th\ song\ played) = \frac{P_{4,90} \times 10}{P_{5,100}} = 0.0679$$

Statistical Independence

Two events *A* and *B* are said to be independent, if and only if $P(A \cap B) = P(A)P(B)$ that is, when knowing the value of A does not give us additional information for the value of *B*.

Conditional Probability

- For any two events A and B with P(B)>0, the conditional probability of A given that B has occurred is defined by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- When two events are not independent, knowing one allows better estimate of the other (e.g. outside temperature, season).

- If the events are independent, then $P(A \mid B) = P(A)$.

Example

We toss a fair coin three successive times. We wish to find the condition probability P(A|B) where A and B are the events A={more heads that tails come up} and B = {1st toss is a head}.

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$B = \{HHH, HHT, HTH, HTT\},\ P(B) = \frac{4}{8}$$

$$A \cap B = \{HHH, HHT, HTH\},\ P(A \cap B) = \frac{3}{8}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{4}$$

Law of Total Probability

Let $A_1, \cdots, A_k$ be mutually exclusive and exhaustive events.
Then for any other event B .

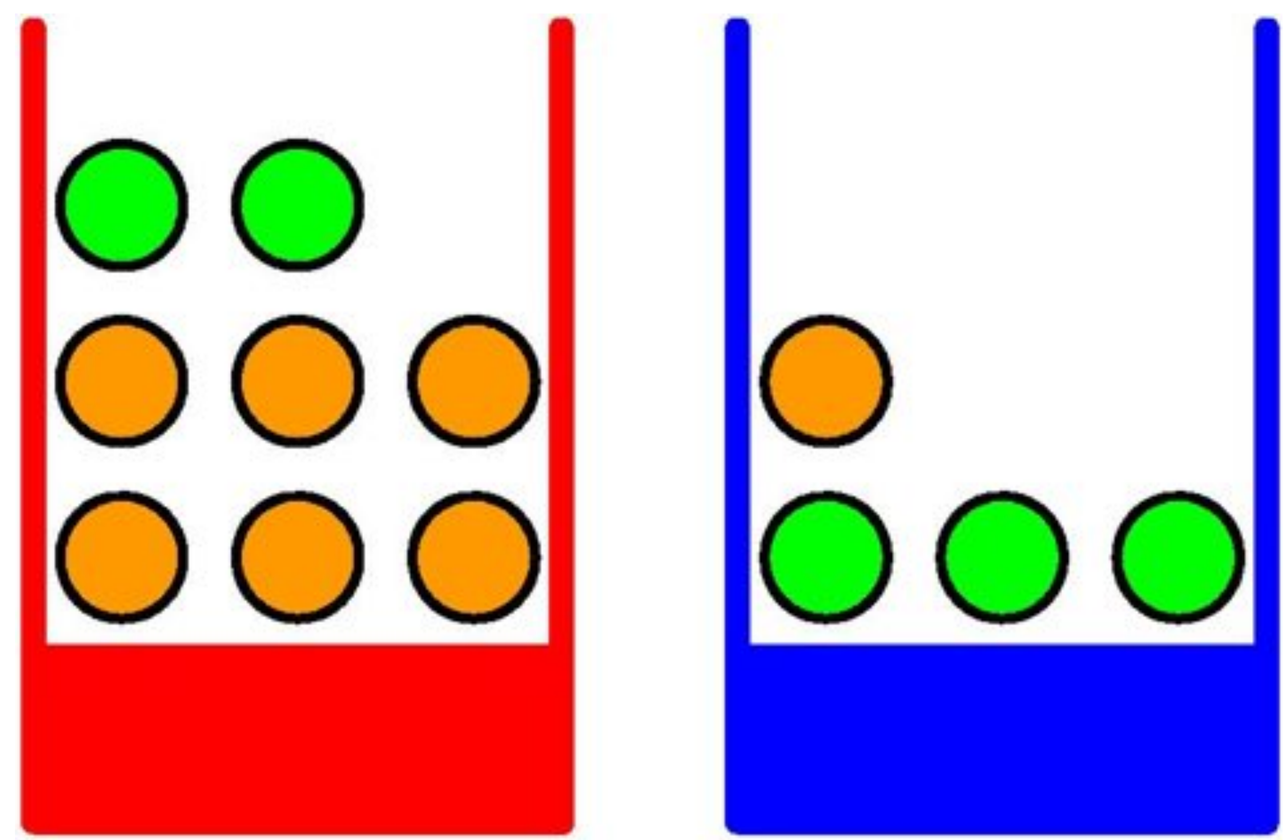
$$P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k)$$

$$= \sum_{j=1}^{k} P(B \mid A_j)P(A_j)$$

Example - Fruits

We have two boxes: one red and one blue. The red box has 2 apples and 6 oranges. The blue box has 3 apples and 1 orange. We pick red box 40% of the time and blue box 60% of the time, then pick one item of fruit

Define the events
B color of box (r or b)
F identity of fruit (a or o)



C.M. Bishop, “Pattern Recognition and Machine Learning”, 2006

Example Fruits

$$P(B = r) = 4/10,\ P(B = b) = 6/10$$

$$P(B = r) + P(B = b) = 1$$

$$P(F = a \,|\, B = r) = 1/4$$
$$P(F = o \,|\, B = r) = 3/4$$
$$P(F = a \,|\, B = b) = 3/4$$
$$P(F = o \,|\, B = b) = 1/4$$

$$P(F = a) = P(F = a \,|\, B = r)P(B = r) + P(F = a \,|\, B = b)P(B = b)$$

$$= 1/4 \times 4/10 + 3/4 \times 6/10 = 11/20$$

Bayes Rule

Let $A_1, \cdots, A_k$ be a collection of k mutually exclusive and exhaustive events with prior probabilities $P(A_i)(i = 1, \cdots, k)$. Then for any other event B for which $P(B) > 0$, the posterior probability of A_j given than B has occurred is:

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i)P(A_i)}, j = 1, \cdots, k$$

Bayes Rule on the Fruits Example

Suppose we have selected a orange. Which box did it come from?

$$P(B = r \mid F = o) = \frac{P(F = o \mid B = r) \times P(B = r)}{P(F = o)}$$

$$= \frac{3/4 \times 4/10}{9/20}$$

$$= \frac{2}{3}$$

Exercise 1

A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% have Covid. Let F stands for an event of a child being sick with the flu and C stands for an event of a child having Covid. Assume there are no other maladies in the neighborhood. A well-known symptom of Covid is fever (the event we denote by T). Assume $P(T|C) = 0.95$. However, occasionally children with flu also have fever with $P(T|F) = 0.08$. Upon examination the child, the doctor finds fever. What is the probability that the child has Covid?

Exercise 2

In a study, doctors were asked what are the odds of breast cancer would be in a women who was initially thought to take a 1% risk of cancer but who ended up with a positive mammogram result . A mammogram accurately classifies about 80% of cancerous tumors and 90% of benign tumors. What are the chances this patient has cancer?

Exercise 3

Suppose we have 3 cards identical in form except that both sides of the first card are colored red (RR), both sides of the second card are colored black (BB), and one side of the third card is colored red and the other is colored black (RB). The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Exercise 4

It is estimated that 50% of the emails are spam. Our spam filtering software can detect 90% of spam emails, and the probability of a false positive is 5%. If an email is detected as spam, then what is the probability that it is in fact a non-spam email?

Exercise 5

Two different teams named A and B are asked to separately design a new product within a month. From past experience we know that:

1. The probability that team A is successful is 2/3.
2. The probability that team B is successful is 1/2
3. The probability that at least one team is successful is 3/4.

Assuming that exactly one successful design is produced, what is the probability that it was designed by team B?

Exercise 6

You enter a chess tournament where you probability of winning a game is 0.3 against half the players (call them type 1). 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent.

- What is the probability of winning?
- Assuming that you win. What is the probability that you had an opponent of type 1?

Exercise 7

We are given three coins and are told that two of the coins are fair and the third coin is biased, landing heads with probability 2/3. We are not told which of the three coins is biased. We permute the coins randomly, and then flip each of the coins. The first and second coins come up heads, and the third comes up tails. What is the probability that the first coin is the biased one?

Exercise 8

We draw eight cards at random from an ordinary deck of 52 cards. Given that three of them are spades, what is the probability that the remaining five are also spades?

Thank you!