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# Homework 1 Part 1

① Solve the system of equations using elimination and backwards substitution:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0 \end{aligned}$$

**Solution**

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow[R3 = R3 \times \left(-\frac{1}{2}\right)]{R2 = R2 - 2R1} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R3 = R3 - R2]{R1 = R1 - 3R2} \left[\begin{array}{ccc|c} 2 & 0 & -8 & -4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -4 & -4 \end{array}\right] \xrightarrow{R1 = \frac{1}{2}R1 - R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{R2 = R2 - \left(-\frac{3}{4}\right)R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -4 & -4 \end{array}\right] \xrightarrow{R3 = \left(-\frac{1}{4}\right)R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

② Find the rank of the following matrix: $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$

**solution**

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix} \xrightarrow[R3 = R3 - 3R2]{R1 = R1 + R2} \begin{bmatrix} 1 & 3 & 5 & 3 \\ 1 & 2 & 3 & 2 \\ 0 & -5 & -8 & -3 \end{bmatrix} \xrightarrow{R2 = R2 - R1} \begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & -5 & -8 & -3 \end{bmatrix} \xrightarrow{R2 = (-1)R2} \begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix}$$

$$\xrightarrow{R3 = R3 - (-5)R2} \begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow{R3 = \frac{1}{2}R3} \begin{bmatrix} 1 & 3 & 5 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This matrix has 3 pivot columns
Therefore, the rank of the matrix is 3

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③ Construct a matrix A whose column space contains vectors $\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and whose null space contains the vector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

<u>Solution</u>

- Given matrix $A = \begin{bmatrix} 3 & 4 & x_1 \\ 6 & 0 & x_2 \\ 2 & 1 & x_3 \end{bmatrix}$

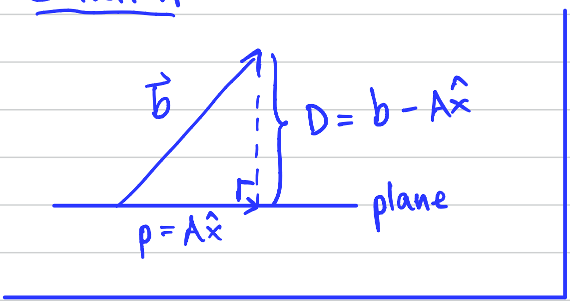
- According to null space contains the vector $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ then,

$$\begin{bmatrix} 3 & 4 & x_1 \\ 6 & 0 & x_2 \\ 2 & 1 & x_3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 + x_1 \\ 12 + x_2 \\ 6 + x_3 \end{bmatrix} \longrightarrow x_1 = -14,\ x_2 = -12,\ x_3 = -6$$

Therefore, a matrix $A = \begin{bmatrix} 3 & 4 & -14 \\ 6 & 0 & -12 \\ 2 & 1 & -6 \end{bmatrix}$ //

④ Find the distance from the vector $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ to the plane spanned by $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

<u>Solution</u>




$D \perp$ plane : $A \cdot D = 0$ ; $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$

$A^T D = 0$

$$A^T(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T A\hat{x} = A^T b$$

$$A^T A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Thus, $\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$$\hat{x}_1 = 1/7, \quad \hat{x}_2 = 5/7$$

$$D = b - A\hat{x}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/7 \\ 5/7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 4/7 \\ -9/7 \\ 6/7 \end{bmatrix}$$

$$D = \begin{bmatrix} 24/7 \\ 16/7 \\ 8/7 \end{bmatrix}$$

$$\|D\| = \sqrt{\left(\frac{24}{7}\right)^2 + \left(\frac{16}{7}\right)^2 + \left(\frac{8}{7}\right)^2} = \frac{8\sqrt{14}}{7} \text{ //}$$

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(5) Compute the following matrix-vector multiplication as:
a. Linear combination of columns.
b. Dot product of rows.

$$\begin{bmatrix} 2 & 1 & 3 \\ 7 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Solution

a. $$\begin{bmatrix} 2 & 1 & 3 \\ 7 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & 1 & 3 \\ 7 & 1 & 0 \\ 3 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} (2)(2)+(1)(4)+(3)(1) \\ (7)(2)+(1)(4)+(0)(1) \\ (3)(2)+(5)(4)+(9)(1) \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}$$

(6) Find the value of k for which the matrix has:-
a. Dependent columns
b. Independent columns

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 4 & 8 & k \end{bmatrix}$$

Solution

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 4 & 8 & k \end{bmatrix} \xrightarrow[R3=R3-4R1]{R2=R2-2R1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 0 & -4 & k-8 \end{bmatrix} \xrightarrow{R2=\left(-\frac{1}{3}\right)R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -4 & k-8 \end{bmatrix} \xrightarrow{R3=R3-(-4)R2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & k-4 \end{bmatrix}$$

a. The matrix has dependent columns when $Ax=0$ has a nonzero solution.
Therefore, in this case the value of k is 4 $(k=4)$.

b. The matrix has independent columns when the rank is equal to the number of pivots, and no free variables, and only $x=0$ is in the nullspace.
Therefore, in this case the value of k is any number except 4 $(k \neq 4)$.

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⑦ Use Gram-Schmidt process to find an orthogonal basis for the subspace spanned by $\vec{a} = \begin{bmatrix}1\\0\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}0\\1\\2\end{bmatrix}$, $\vec{c} = \begin{bmatrix}1\\1\\3\end{bmatrix}$

solution

$$A = a = \begin{bmatrix}1\\0\\3\end{bmatrix}$$

$$B = b - \frac{A^Tb}{A^TA}A = \begin{bmatrix}0\\1\\2\end{bmatrix} - \frac{\begin{bmatrix}1&0&3\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}}{\begin{bmatrix}1&0&3\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}}\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\0\\3\end{bmatrix} = \begin{bmatrix}-1/3\\1\\1\end{bmatrix}$$

$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B = \begin{bmatrix}1\\1\\3\end{bmatrix} - \frac{\begin{bmatrix}1&0&3\end{bmatrix}\begin{bmatrix}1\\1\\3\end{bmatrix}}{\begin{bmatrix}1&0&3\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}}\begin{bmatrix}1\\0\\3\end{bmatrix} - \frac{\begin{bmatrix}0&1&2\end{bmatrix}\begin{bmatrix}1\\1\\3\end{bmatrix}}{\begin{bmatrix}0&1&2\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix}}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$C = \begin{bmatrix}1\\1\\3\end{bmatrix} - 1\begin{bmatrix}1\\0\\3\end{bmatrix} - \frac{7}{5}\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}0\\-2/5\\-14/5\end{bmatrix}$$

$$q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{1^2+0^2+3^2}}\begin{bmatrix}1\\0\\3\end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\0\\3\end{bmatrix} \,/\!/$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{(-\frac{1}{3})^2+1^2+1^2}}\begin{bmatrix}-1/3\\1\\1\end{bmatrix} = \frac{3}{\sqrt{19}}\begin{bmatrix}-1/3\\1\\1\end{bmatrix} \,/\!/$$

$$q_3 = \frac{C}{\|C\|} = \frac{1}{\sqrt{0^2+(-\frac{2}{5})^2+(-\frac{14}{5})^2}}\begin{bmatrix}0\\-2/5\\-14/5\end{bmatrix} = \frac{1}{\sqrt{8}}\begin{bmatrix}0\\-2/5\\-14/5\end{bmatrix} \,/\!/$$