Linear Algebra Worksheet

1. Consider the vectors $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$ $and\ \vec{w} = \begin{bmatrix}1\\0\end{bmatrix}$.

   (a) In the x-y plane mark all nine linear combinations $c\vec{v} + d\vec{w}$, with c = -2,0-2 and d = 0,1,2.

   (b) What shape do all linear combinations $c\vec{v} + d\vec{w}$ fill? A line? The whole plane? Are the vectors $\vec{v}\ and\ \vec{w}$ independent?

2. Consider the vectors $\vec{u} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ $and\ \vec{w} = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

   (a) Can you solve the system $x\vec{u} + y\vec{v} + z\vec{w} = \vec{b}, if\ \vec{b} = \begin{bmatrix}0\\0\\1\end{bmatrix}$?

   (b) What if $\vec{b} = \begin{bmatrix}0\\0\\0\end{bmatrix}$? How many solutions are there?

   (c) Are the vectors $\vec{u}, \vec{v}\ and\ \vec{w}$ dependent or independent?

   (d) Use parts (a) – (c) to decide if $A = \begin{bmatrix}1 & 0 & 2\\0 & -1 & 1\\1 & 1 & 0\end{bmatrix}$ is an invertible matrix or not.

3. Use Gaussian elimination to find all solutions of:

$$1x - 2y + 3z = 3$$

$$2x + y + 8z = -5$$

$$0x + y + 2z = 1$$

4. Consider the linear system for some constants b and g:

$$x - 2y + 3z = 3$$

$$2x + y + bz = -4$$

$$x + 0y + 1z = g$$

(a) What constant b makes the system singular (missing a pivot).

(b) For the value of b found in Part (a), for which values of g, the system has infinitely many solutions?

(c) Find two distinct solutions of the system for that g.

5. Construct a matrix A whose null space contains the vector $\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$, and whose column space contains $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

6. Write the complete solution of the following linear system as $x_p + x_n$:

$$x + 2y - z = 1$$

$$3x + 5y + 2z = 3$$

$$2x + y + 13z = 2$$

7. Consider the vectors $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 7 \\ 9 \\ 1 \\ -1 \end{bmatrix}$. Check whether v_1, v_2, v_3 and v_4 are independent. If they are not, find a basis for the subspace of R^4 spanned by these vectors.

8. Consider the vectors $b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find the projection p of b onto the subspace spanned by a_1 and a_2.

(b) Find the error vector e = b – p and show that it is orthogonal to both a_1 and a_2.

9. Find the line $y = C + Dx$ that best fits the data $(x, y) = \{(0,1), (1,8), (2,8), (3,20)\}$.

10. [Understanding projections and projection matrix] Assume $P = A(A^TA)^{-1}A^T$ is a projection matrix.

(a) [4 pts.] Show that $P^2 = P$ by multiplying $P = A(A^TA)^{-1}A^T$ by itself and canceling.
(b) [3 pts.] Prove (a) geometrically by showing that for any vector b, Pb is the vector in the column space of A closest to b and then use this fact to show $P^2 = P(Pb) = Pb$ for any vector b.
(c) [3 pts.] The matrix P as above projects onto the column space of A. Is $I - P$ a projection matrix? To which subspace does it project onto?

11. Use Gram-Schmidt Process to find an orthogonal basis for the subspace

spanned by a = $\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$, b = $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, c = $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$.

12. Answer each part.

(a) If Q_1 and Q_2 are orthogonal matrices, show that Q_1Q_2 is an orthogonal matrix. [Hint: Use $Q^TQ = I$]

(b) Show that if for orthogonal vectors, q_1, q_2, q_3, if

$$x_1q_1 + x_2q_2 + x_3q_3 = b$$

then for each i, $x_i = q_i = b$. [Hint: Take the dot product of the two sides with each q_i at a time and use orthonormality conditions, $q_i \cdot q_i = 1$ and $q_i \cdot q_j = 0$ if $i \neq j$, to prove the statement].

(c) The vectors $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$, $q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ are orthogonal. Use Part (b), to solve

$$x_1q_1 + x_2q_2 + x_3q_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

13. Let $A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$ and $I + A = \begin{bmatrix}2 & 4\\2 & 4\end{bmatrix}$

(a) [6 pts.] Compute the eigenvalues and eigenvectors of A and A+I.

(b) [4 pts.] Find a relationship between eigenvectors and eigenvalues of A and those of

A+I.

14. Find singular value decomposition (SVD) of $A = \begin{bmatrix}3 & 0\\4 & 5\end{bmatrix}$.

15. Use Gram-Schmidt Process to find and orthogonal basis for the subspace spanned by

$$a = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}, b = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, c = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$