CS556 Assignment 1 Answer Guide

Question 1

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 0 \end{bmatrix}$$

Convert to augmented matrix for ease of representation:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

Convert to row-echelon form:

R2 = R2-2R1

R3 = R3+2R2

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \qquad \text{Note: Pivots are circled.}$$

Perform backwards substitution:

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \rightarrow \boxed{z = 1} \\ &\boxed{y = 1} \\ &\boxed{x = 2} \end{aligned}$$

Question 2

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 0 & -5 & -8 & -3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$rk(A) = \boxed{3}$

Question 3

$$A = \begin{bmatrix} 3 & 4 & x \\ 6 & 0 & y \\ 2 & 1 & z \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} 3 & 4 & x \\ 6 & 0 & y \\ 2 & 1 & z \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 6 + 8 + x = 0 &\implies x = -14 \\ 12 + y = 0 &\implies y = -12 \\ 4 + 2 + z = 0 &\implies z = -6 \end{aligned}$$

$$A = \boxed{\begin{bmatrix} 3 & 4 & -14 \\ 6 & 0 & -12 \\ 2 & 1 & -6 \end{bmatrix}}$$

Note: There are many possible unique values for the final answer; this is just perhaps the most commonly expected. Acceptable solutions may rearrange the columns from this answer and/or change either of the two columns based on the column space by a scalar in the first step, for instance.

Question 4

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad \Lambda^\top \Lambda = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \quad (\Lambda^\top \Lambda)^{-1} = \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\vec{p} = \Lambda(\Lambda^\top \Lambda)^{-1} \Lambda^\top \vec{b} = \frac{1}{7} \begin{bmatrix} 4 \\ -9 \\ 6 \end{bmatrix}$$

$$\vec{c} = \vec{b} - \vec{p} = \frac{8}{7} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad ||\vec{c}|| = \boxed{\frac{8}{7}\sqrt{14}}$$

Question 5

a.) $2 \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}}$

b.) $\begin{bmatrix} (2\times 2)+(1\times 4)+(3\times 1) \\ (7\times 2)+(1\times 4)+(0\times 1) \\ (3\times 2)+(5\times 4)+(9\times 1) \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ 18 \\ 35 \end{bmatrix}}$

Question 6

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 4 & 8 & k \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 4 & 8 & k \end{bmatrix} \xrightarrow{R_3 = R_3 - 4R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -3 & -3 \\ 0 & -4 & k-8 \end{bmatrix}$$

$$\xrightarrow{R_2 = \frac{-1}{3} R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -4 & k-8 \end{bmatrix} \xrightarrow{R_3 = R_3 + 4R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & k-4 \end{bmatrix}$$

a.) For dependent columns, the matrix needs to be singular (must have a missing pivot); therefore, solve for $k - 4 = 0 \implies \boxed{k = 4}$

b.) For independent columns, the matrix must be non-singular (must have 3 pivots); therefore, k may be any value that isn't 4: $\boxed{k \neq 4}$.

Question 7

$$A = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad B = \frac{1}{5}\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \quad C = \frac{1}{7}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\boxed{q_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad q_2 = \frac{1}{\sqrt{35}}\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \quad q_3 = \frac{1}{\sqrt{14}}\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}$$