Problem Solution 1 (a) The sample space consists of all ways of drawing 7 elements out of a 52-element set, so it contains $\binom{52}{7}$ possible outcomes. Let us count those outcomes that involve exactly 3 aces. We are free to select any 3 out of the 4 aces, and any 4 out of the 48 remaining cards, for a total of $\binom{4}{3}\binom{48}{4}$ choices. Thus,

$$\mathbf{P}(7 \text{ cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}.$$

(b) Proceeding similar to part (a), we obtain

$$\mathbf{P}(7 \text{ cards include exactly 2 kings}) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}.$$

(c) If A and B stand for the events in parts (a) and (b), respectively, we are looking for $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. The event $A \cap B$ (having exactly 3 aces and exactly 2 kings) can occur by choosing 3 out of the 4 available aces, 2 out of the 4 available kings, and 2 more cards out of the remaining 44. Thus, this event consists of $\binom{4}{3}\binom{4}{2}\binom{44}{2}$ distinct outcomes. Hence,

$$\mathbf{P}(7 \text{ cards include 3 aces and/or 2 kings}) = \frac{\binom{4}{3}\binom{48}{4} + \binom{4}{2}\binom{48}{5} - \binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}.$$

Problem Solution 2 Since Bob tosses one more coin that Alice, it is impossible that they toss both the same number of heads and the same number of tails. So Bob tosses either more heads than Alice or more tails than Alice (but not both). Since the coins are fair, these events are equally likely by symmetry, so both events have probability 1/2.

An alternative solution is to argue that if Alice and Bob are tied after $2n$ tosses, they are equally likely to win. If they are not tied, then their scores differ by at least 2, and toss $2n+1$ will not change the final outcome. This argument may also be expressed algebraically by using the total probability theorem. Let B be the event that Bob tosses more heads. Let X be the event that after each has tossed n of their coins, Bob has more heads than Alice, let Y be the event that under the same conditions, Alice has more heads than Bob, and let Z be the event that they have the same number of heads. Since the coins are fair, we have $\mathbf{P}(X) = \mathbf{P}(Y)$, and also $\mathbf{P}(Z) = 1 - \mathbf{P}(X) - \mathbf{P}(Y)$. Furthermore, we see that

$$\mathbf{P}(B \mid X) = 1, \qquad \mathbf{P}(B \mid Y) = 0, \qquad \mathbf{P}(B \mid Z) = \frac{1}{2}.$$

Now we have, using the total probability theorem,

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(X) \cdot \mathbf{P}(B \mid X) + \mathbf{P}(Y) \cdot \mathbf{P}(B \mid Y) + \mathbf{P}(Z) \cdot \mathbf{P}(B \mid Z) \\
&= \mathbf{P}(X) + \frac{1}{2} \cdot \mathbf{P}(Z) \\
&= \frac{1}{2} \cdot \big(\mathbf{P}(X) + \mathbf{P}(Y) + \mathbf{P}(Z)\big) \\
&= \frac{1}{2}.
\end{aligned}$$

as required.

Problem Solution 3 In this problem, there is a tendency to reason that since the opposite face is either heads or tails, the desired probability is $1/2$. This is, however, wrong, because given that heads came up, it is more likely that the two-headed coin was chosen. The correct reasoning is to calculate the conditional probability

$$\begin{aligned} p &= \mathbf{P}(\text{two-headed coin was chosen} \,|\, \text{heads came up}) \\ &= \frac{\mathbf{P}(\text{two-headed coin was chosen and heads came up})}{\mathbf{P}(\text{heads came up})}. \end{aligned}$$

We have

$$\mathbf{P}(\text{two-headed coin was chosen and heads came up}) = \frac{1}{3},$$

$$\mathbf{P}(\text{heads came up}) = \frac{1}{2},$$

so by taking the ratio of the above two probabilities, we obtain $p = 2/3$. Thus, the probability that the opposite face is tails is $1 - p = 1/3$.

Problem Solution 4 We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{m+1}{m+n+1} p_i + \frac{m}{m+n+1}(1 - p_i) = \frac{1}{m+n+1} p_i + \frac{m}{m+n+1},$$

starting with the initial condition $p_1 = m/(m+n)$. Thus, we have

$$p_2 = \frac{1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} = \frac{m}{m+n}.$$

More generally, this calculation shows that if $p_{i-1} = m/(m+n)$, then $p_i = m/(m+n)$. Thus, we obtain $p_i = m/(m+n)$ for all i.

Problem Solution 5 (a) Let A denote the event that the city experiences a black-out. Since the power plants fail independent of each other, we have

$$\mathbf{P}(A) = \prod_{i=1}^{n} p_i.$$

(b) There will be a black-out if either all n or any $n-1$ power plants fail. These two events are disjoint, so we can calculate the probability $\mathbf{P}(A)$ of a black-out by adding their probabilities:

$$\mathbf{P}(A) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} \left((1 - p_i) \prod_{j \neq i} p_j \right).$$

Here, $(1 - p_i) \prod_{j \neq i} p_j$ is the probability that $n - 1$ plants have failed and plant i is the one that has not failed.