Linear Algebra Worksheet

1. Consider the vectors $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}1\\0\end{bmatrix}$.

(a) In the x-y plane mark all nine linear combinations $c\vec{v} + d\vec{w}$, with c = -2,0,2 and d = 0,1,2.

(b) What shape do all linear combinations $c\vec{v} + d\vec{w}$ fill? A line? The whole plane? Are the vectors $\vec{v}$ and $\vec{w}$ independent?

a)

	c = -2	c = 0	c = 2
d = 0	(-2,-4)	(0,0)	(2,4)
d = 1	(-1,-4)	(1,0)	(3,4)
d = 2	(0,-4)	(2,0)	(4,4)

b) All linear combination $c\vec{v} + d\vec{w}$ fill the whole xy-plane since they define distict lines and are independent.

2. Consider the vectors $\vec{u} = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}2\\1\\0\end{bmatrix}$.

(a) Can you solve the system $x\vec{u} + y\vec{v} + z\vec{w} = \vec{b}, if\ \vec{b} = \begin{bmatrix}0\\0\\1\end{bmatrix}$?

(b) What if $\vec{b} = \begin{bmatrix}0\\0\\0\end{bmatrix}$? How many solutions are there?

(c) Are the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ dependent or independent?

(d) Use parts (a) – (c) to decide if $A = \begin{bmatrix}1 & 0 & 2\\0 & -1 & 1\\1 & 1 & 0\end{bmatrix}$ is an invertible matrix or not.

a) $x\vec{u} + y\vec{v} + z\vec{w} = x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}0\\-1\\1\end{bmatrix} + z\begin{bmatrix}2\\1\\0\end{bmatrix} = \vec{b}$

$\begin{bmatrix}x+2z\\-y+z\\x+y\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow$ Unique solution $x=2, y=-1, z=-1$

b) $\begin{bmatrix}x+2z\\-y+z\\x+y\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow$ Unique solution $x=y=z=0$

c) They are independent, since the solutions to the above systems are unique.

d) Invertible. Even thought, we have not yet given a complete proof of the following fact, we have noticed so far that:

$\vec{u}, \vec{v}, \vec{w}$ are independent $\Leftrightarrow A\vec{x}=\vec{b}$ has unique solution for any $\vec{b} \Leftrightarrow$ A is invertible.

3. Use Gaussian elimination to find all solutions of:

$$1x - 2y + 3z = 3$$

$$2x + y + 8z = -5$$

$$0x + y + 2z = 1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & 8 & -5 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 = -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & 2 & -11 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 2 & -11 \end{array}\right]$$

$$\xrightarrow{R_3 = -5R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -8 & -16 \end{array}\right] \xrightarrow{R_3 = R_3/-8} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ Row echelon form

$$\begin{cases} z = 2 \\ y + 2z = 1 \\ x - 2y + 3z = 3 \end{cases} \rightarrow \begin{cases} z = 2 \\ y = -3 \\ x = -9 \end{cases} \qquad \text{Solution} = \begin{bmatrix} -9 \\ -3 \\ 2 \end{bmatrix}$$

Or continue elimination until RREF

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R1 = R1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 5 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 7 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \text{Solution} = \begin{bmatrix} -9 \\ -3 \\ 2 \end{bmatrix}$$

4. Consider the linear system for some constants b and g:

$$x - 2y + 3z = 3$$

$$2x + y + bz = -4$$

$$x + 0y + 1z = g$$

(a) What constant b makes the system singular (missing a pivot).

(b) For the value of b found in Part (a), for which values of g, the system has infinitely many solutions?

(c) Find two distinct solutions of the system for that g.

a) $$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & b & -4 \\ 1 & 0 & 1 & g \end{array}\right] \xrightarrow[\text{② } R_3 = R_3 - R_1]{\text{① } R_2 = R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & b-6 & -10 \\ 0 & 2 & -2 & g-3 \end{array}\right] \xrightarrow[\text{② } R_2 = R_2/2]{\text{① } R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -1 & (g-3)/2 \\ 0 & 5 & b-6 & -10 \end{array}\right] \xrightarrow{R_3 = R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -1 & (g-3)/2 \\ 0 & 0 & b-1 & -5(g+1)/2 \end{array}\right]$$

To miss a pivot we need $b = 1$.

b) With $b=1$ the system becomes $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -1 & (g-3)/2 \\ 0 & 0 & 0 & -5(g+1)/2 \end{array}\right]$,

which is consistent only if $-5(g+1)/2 = 0 \rightarrow g = -1$.

For $g=-1$, the system will have infinitely many solutions.

$x = 3 + 2y - 3z$

$y = z - 2$

c) In the last system,

set $z = 0$, then $y = -2$ and $x = -1$

set $z = 1$, then $y = -1$ and $x = -2$

5. Construct a matrix A whose null space contains the vector $\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$, and whose column space contains $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

Matrix A could be the matrix:

$$A = \begin{bmatrix} 1 & 2 & v_1 \\ 2 & 3 & v_2 \\ 4 & 0 & v_3 \end{bmatrix} \text{ where } \begin{bmatrix} 1 & 2 & v_1 \\ 2 & 3 & v_2 \\ 4 & 0 & v_3 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The latter equation gives:

$$-7 + 8 + v_1 = 0 \rightarrow v_1 = -1$$

$$-14 + 12 + v_2 = 0 \rightarrow v_2 = 2$$

$$-28 + v_3 = 0 \rightarrow v_3 = 28$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 4 & 0 & 28 \end{bmatrix}$$

Note that there is no such unique A, for example one could place the vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ on a different column. The result will be a different matrix.

6. Write the complete solution of the following linear system as $x_p + x_n$:

$$x + 2y - z = 1$$

$$3x + 5y + 2z = 3$$

$$2x + y + 13z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 5 & 2 & 3 \\ 2 & 1 & 13 & 2 \end{array}\right] \xrightarrow[\text{(2) } R_3 = R_3 - 2R_1]{\text{(1) } R_2 = R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 5 & 0 \\ 0 & -3 & 15 & 0 \end{array}\right] \xrightarrow{R_2 = R_2/-1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -5 & 0 \\ 0 & -3 & 15 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

<u>Xparticular</u>

$z = 0$

$y = 0$

$x = 1$

$$X_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

<u>Xnullspace</u>

$z = 1$

$y = 5$

$x = -9$

$$X_s = \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix} \qquad X_n = c \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$

$$X = X_p + X_n = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -9 \\ 5 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$

7. Consider the vectors $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 7 \\ 9 \\ 1 \\ -1 \end{bmatrix}$. Check whether v_1, v_2, v_3 and v_4 are independent. If they are not, find a basis for the subspace of R^4 spanned by these vectors.

We form the matrix $A = [v_1\ v_2\ v_3\ v_4]$ and row reduce.

$$A = \begin{bmatrix} 2 & 0 & -1 & 7 \\ 1 & 3 & 0 & 9 \\ 1 & 0 & 2 & 1 \\ -1 & 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column doesn't have a pivot, the vectors are dependent.

The first 3 columns (pivot columns) of A are independent. So v_1, v_2, v_3 from the original matrix A form a basis of the span.

8. Consider the vectors b = $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, a_1 = $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, a_2 = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find the projection p of b onto the subspace spanned by a_1 and a_2.

(b) Find the error vector e = b – p and show that it is orthogonal to both a_1 and a_2.

a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

The projection matrix $P = A(A^TA)^{-1}A^T$

$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$ $(A^TA)^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$

$P = A(A^TA)^{-1}A^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $p = Pb = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$

b) $e = b - p$

$e = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Clearly $e \cdot a_1 = e \cdot a_2 = 0$ ✓

9. Find the line $y = C + Dx$ that best fits the data $(x, y) = \{(0,1), (1,8), (2,8), (3,20)\}$.

If the line $y = C + Dx$ passes through all these points, then C and D satisfy the following equation:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_{b}$$

Since the system $Ax = b$ is unsolvable, we find the best approximation to it by solving

$$A^T A \hat{x} = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 37 \\ 84 \end{bmatrix}$$

Solve

$$\left[\begin{array}{cc|c} 4 & 6 & 37 \\ 6 & 14 & 84 \end{array}\right] \to \hat{C} = \frac{7}{10} \text{ and } \hat{D} = \frac{57}{10}$$

The least squares line is $y = \frac{7}{10} + \frac{57}{10}x$

10. [Understanding projections and projection matrix] Assume $P = A(A^T A)^{-1}A^T$ is a projection matrix.

(a) Show that $P^2 = P$ by multiplying $P = A(A^T A)^{-1}A^T$ by itself and canceling.

(b) Prove (a) geometrically by showing that for any vector b, Pb is the vector in the column space of A closest to b and then use this fact to show $P^2 = P(Pb) = Pb$ for any vector b.

(c) The matrix P as above projects onto the column space of A. Is $I - P$ a projection matrix? To which subspace does it project onto?

a) $P^2 = (A(A^TA)^{-1}A^T)(A(A^TA)^{-1}A^T)$

$= A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$

$= A(A^TA)^{-1}A^T$

b) Firstly, the error vector $e = b - Pb$ is orthogonal to $C(A)$, so Pb is the closest point in $C(A)$ to b. Since b was an arbitrary vector the above statement holds on Pb, that is, $P(Pb)$ is the closest point in $C(A)$ to Pb. But Pb is already in $C(A)$. So, $P(Pb) = Pb$ for all vectors b.

c) Matrix $I-P$ projects onto the orthogonal complement of $C(A)$, since for any b, $(I-P)b = b - Pb = e$, which we know is orthogonal to $C(A)$.

11. Use Gram-Schmidt Process to find an orthogonal basis for the subspace spanned by a = $\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$, b = $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, c = $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$.

- $A = a = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}$
- $B = b - \dfrac{A \cdot b}{A \cdot A} A = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \dfrac{2}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} = \dfrac{2}{3}\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}$
- $C = c - \dfrac{A \cdot c}{A \cdot A} A - \dfrac{B \cdot c}{B \cdot B} B = \begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \dfrac{3}{6}\begin{bmatrix}1\\2\\0\\1\end{bmatrix} - 0 = \dfrac{1}{2}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$

Therefore,

$$q_1 = \frac{A}{\|A\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\0\\1\end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}$$

$$q_3 = \frac{C}{\|C\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

12. Answer each part.

(a) If Q_1 and Q_2 are orthogonal matrices, show that Q_1Q_2 is an orthogonal matrix. [Hint: Use $Q^TQ = I$]

(b) Show that if for orthogonal vectors, q_1, q_2, q_3, if

$$x_1q_1 + x_2q_2 + x_3q_3 = b$$

then for each i, $x_i = q_i \bullet b$. [Hint: Take the dot product of the two sides with each q_i at a time and use orthonormality conditions, $q_i \cdot q_i = 1$ and $q_i \cdot q_j = 0$ if $i \neq j$, to prove the statement].

(c) The vectors $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal. Use Part (b), to solve

$$x_1q_1 + x_2q_2 + x_3q_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

a) Q is orthogonal iff $Q^TQ = I$

$(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2 = I$

b) $x_1q_1 + x_2q_2 + x_3q_3 = b$

$\Rightarrow q_1\cdot(x_1q_1 + x_2q_2 + x_3q_3) = q_1\cdot b$

$\Rightarrow x_1 q_1\cdot q_1 + x_2 q_1\cdot q_2 + x_3 q_1\cdot q_3 = q_1\cdot b$

$\Rightarrow x_1 + 0 + 0 = q_1\cdot b$

$x_1 = q_1 b$

Same can be done with q_2 and q_3, proving the statement

c) $x_1 = q_1\cdot b = -\frac{1}{\sqrt{2}}$, $x_2 = q_2\cdot b = \frac{1}{\sqrt{6}}$, $x_3 = q_3\cdot b = \frac{4}{\sqrt{3}}$

13. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and $I + A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

(a) [6 pts.] Compute the eigenvalues and eigenvectors of A and A+I.

(b) [4 pts.] Find a relationship between eigenvectors and eigenvalues of A and those of

A+I.

a) • $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$

Eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$

For $\lambda_1 = 5 \rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1 \rightarrow x_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

• Similarly for $A + I \rightarrow \beta_1 = 6, \quad \beta_2 = 0$

$y_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

b) λ is an eigenvalue of A with corresponding eigenvector x iff $1+\lambda$ is an eigenvalue of I+A with the corresponding eigenvector x.

14. Find singular value decomposition (SVD) of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.

- Done in class

15. Use Gram-Schmidt Process to find and orthogonal basis for the subspace spanned by

$$a = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Same as question 11.