Problem 1

We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:

(a) The 7 cards include exactly 3 aces.

(b) The 7 cards include exactly 2 kings.

(c) The probability that the 7 cards include exactly 3 aces. or exactly 2 kings, or both.

Thanapoom Phatthanaphan
20011296

Solution

a). $P(7 \text{ cards include } 3 \text{ aces}) = \dfrac{C(\text{Draw 3 aces from 4 aces})C(\text{Draw 4 cards from 48 cards})}{C(\text{Draw 7 cards from 52 cards})}$

$$= \frac{({}^{4}C_3)({}^{48}C_4)}{({}^{52}C_7)}$$

$$= \frac{\left(\frac{4!}{3!\,1!}\right)\left(\frac{48!}{4!\,44!}\right)}{\left(\frac{52!}{7!\,45!}\right)}$$

$P(7 \text{ cards include } 3 \text{ aces}) = \dfrac{9}{1547} = 0.005818$

b) $P(7 \text{ cards include } 2 \text{ kings}) = \dfrac{C(\text{Draw 2 kings from 4 kings})C(\text{Draw 5 cards from 47 cards})}{C(\text{Draw 7 cards from 52 cards})}$

$$= \frac{({}^{4}C_2)({}^{48}C_5)}{({}^{52}C_7)}$$

$$= \frac{\left(\frac{4!}{2!\,2!}\right)\left(\frac{48!}{5!\,43!}\right)}{\left(\frac{52!}{7!\,45!}\right)}$$

$P(7 \text{ cards include } 2 \text{ kings}) = \dfrac{594}{7735} = 0.07679$

c). $P(7 \text{ cards include } 3 \text{ aces or } 2 \text{ kings}) = \dfrac{C(a \cup b)}{C(\text{Draw 7 cards from 52 cards})}$

$$= \frac{({}^{4}C_3)({}^{48}C_4) + ({}^{4}C_2)({}^{48}C_5) - ({}^{4}C_3)({}^{4}C_2)({}^{44}C_2)}{({}^{52}C_7)}$$

$$= \frac{\left(\frac{4!}{3!\,1!}\right)\left(\frac{48!}{4!\,44!}\right) + \left(\frac{4!}{2!\,2!}\right)\left(\frac{48!}{5!\,43!}\right) - \left(\frac{4!}{3!\,1!}\right)\left(\frac{4!}{2!\,2!}\right)\left(\frac{44!}{2!\,42!}\right)}{\left(\frac{52!}{7!\,45!}\right)}$$

$P(7 \text{ cards include } 3 \text{ aces or } 2 \text{ kings}) = 0.08278$

Problem 2

Alice and Bob have 2n+1 coins, each coin with probability of heads equal to 1/2. Bob tosses n+1 coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is 1/2.

Thanapoom Phatthanaphan
20011296

Solution

If Bob and Alice toss n coins each, the probabilities that Bob or Alice has gotten more heads is the same. Thus, I consider another one coin that Bob toss more than Alice

Let A = Bob toss another coin and gets more heads
B = Bob has more heads than Alice after tossed 2n coins
C = Alice has more heads than Bob after tossed 2n coins
D = Bob and Alice have the same number of heads after tossed 2n coins

$P(B) = P(C)$
$P(B) + P(C) + P(D) = 1$
$P(A|B) = 1$
$P(A|C) = 0$
$P(A|D) = \frac{1}{2}$

$$\begin{aligned} P(A) &= P(B)P(A|B) + P(C)P(A|C) + P(D)P(A|D) \\ &= P(B) + \tfrac{1}{2}P(D) \\ &= \tfrac{1}{2}(P(B) + P(C)) + \tfrac{1}{2}P(D) \\ &= \tfrac{1}{2}(P(B) + P(C) + P(D)) \\ &= \tfrac{1}{2}(1) \\ P(A) &= \tfrac{1}{2} \end{aligned}$$

Problem 3

We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

Solution

$$\begin{aligned} P(\text{toss head up}) &= P(\text{toss head up} \mid \text{pick two-heads coin})P(\text{pick two-heads coin}) \\ &\quad + P(\text{toss head up} \mid \text{pick two-tails coin})P(\text{pick two-tails coin}) \\ &\quad + P(\text{toss head up} \mid \text{pick fair coin})P(\text{pick fair coin}) \\ &= (1)(\tfrac{1}{3}) + (0)(\tfrac{1}{3}) + (\tfrac{1}{2})(\tfrac{1}{3}) \\ &= \tfrac{1}{2} \end{aligned}$$

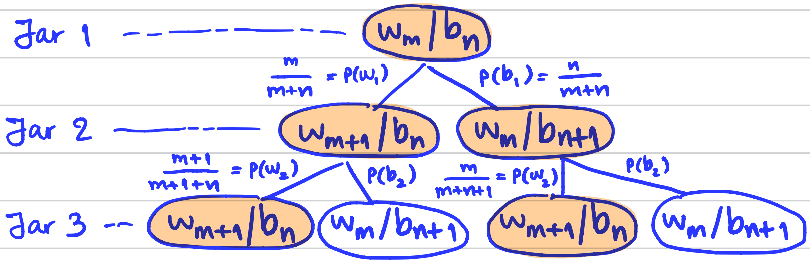
$$P(\text{Pick fair coin and toss head up}) = P(\text{Pick fair coin})P(\text{toss head up}) = (\tfrac{1}{3})(\tfrac{1}{2}) = \tfrac{1}{6}$$

$$P(\text{Pick fair coin} \mid \text{toss head up}) = \frac{P(\text{Pick fair coin and toss head up})}{P(\text{toss head up})} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Thanapoom Phatthanaphan
20011296

Problem 4
Each of k jars contain m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as probability that the first ball is white, i.e. it is m/(m+n).

Solution



Let W_k = pick white ball from Jar k
w_1 = pick white ball from 1^{st} Jar
b_1 = pick black ball from 1^{st} Jar

$$P(W_k) = P(w_1)P(W_k \mid w_1) + P(b_1)P(W_k \mid b_1)$$

$$= \left(\frac{m}{m+n}\right)\left(\frac{m+1}{m+n+1}\right) + \left(\frac{n}{m+n}\right)\left(\frac{m}{m+n+1}\right)$$

$$= \left(\frac{m}{m+n}\right)\left(\frac{m+n+1}{m+n+1}\right)$$

$$P(W_k) = \frac{m}{m+n} = P(w_1)$$

Problem 5
A power utility can supply electricity to a city from n different power plants. Power plant i fails with probability p_i, independent of the others.
(a) Suppose that any one plant can produce enough electricity to supply the entire
city. What is the probability that the city will experience a black-out?
(b) Suppose that two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

Solution

a). P(city black-out) = P(All power plants fail)
$= (p_1)(p_2)(p_3)\dots(p_n)$

b). P(city black-out) = P(all combinations of two plants fail)
$= (p_1p_2) + (p_2p_3) + (p_3p_4) + \dots + (p_{n-1}p_n)$