Probability Worksheet

1. Given than P(A,B) = 0.3 and P(B) = 0.4, find P(A|B)

$P(A \cap B) = 0.3$
$P(B) = 0.4$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

2. Knowing P(B|A) = 0.8, P(A) = 0.3 and P(B) = 0.9, calculate P(A|B)

$P(B|A) = 0.8$
$P(A) = 0.3$
$P(B) = 0.9$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$= \frac{0.8 \times 0.3}{0.9} \approx 0.27$$

3. A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighbourhood have the flu, while the other 10% have Covid. Let F stands for an event of a child being sick with the flu and C stands for an event of a child having Covid. Assume there are no other maladies in the neighbourhood. A well-known symptom of Covid is fever(the event we denote by T). Assume P(T|C) = 0.95. However, occasionally children with flu also have fever with P(T|F) = 0.08. Upon examination the child, the doctor finds fever. What is the probability that the child has Covid?

Events
F = Having Flu
C = Having Covid
T = Have Fever

$P(F) = 0.9$
$P(C) = 0.1$
$P(T|C) = 0.95$
$P(T|F) = 0.08$

$$P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{P(T|C) \times P(C)}{P(T|C) \times P(C) + P(T|F) P(F)}$$

$$= \frac{0.95 \times 0.1}{0.95 \times 0.1 + 0.08 \times 0.9} = \frac{0.095}{0.095 + 0.072} = \frac{0.095}{0.167}$$

$$\approx 0.57$$

4. In a study, doctors were asked what are the odds of breast cancer would be in a women who was initially thought to take a 1% risk of cancer but who ended up with a positive mammogram result . A mammogram accurately classifies about 80% of cancerous tumors and 90% of benign tumors. What are the chances this patient has cancer?

Events

+ = Positive Mammogram

− = Negative Mammogram

M = Cancer

B = No Cancer

$P(+|M) = 0.8$

$P(-|B) = 0.9$

$P(M) = 0.01$

$$P(M|+) = \frac{P(+|M) \cdot P(M)}{P(+)} = \frac{P(+|M)\,P(M)}{P(+|M) \times P(M) + P(+|B) \cdot P(B)}$$

$$= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.1 \times 0.99} = \frac{0.008}{0.107} \approx 0.075$$

5. Suppose we have 3 cards identical in form except that both sides of the first card are coloured red (RR), both sides of the second card are coloured black (BB), and one side of the third card is coloured red and the other is coloured black (RB). The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is coloured red, what is the probability that the other side is coloured black?

$P(RR) = 1/3$

$P(BB) = 1/3$

$P(RB) = 1/3$

$P(R|RR) = 1$

$P(R|BB) = 0$

$P(R|RB) = 1/2$

$$P(RB|R) = \frac{P(R|RB) \times P(RB)}{P(R|RB)P(RB) + P(R|RR) \cdot P(RR)}$$

$$= \frac{1/2 \times 1/3}{1/2 \times 1.3 + 1 \times 1/3}$$

$$= \frac{1/6}{1/6 + 2/6}$$

$$= \frac{1/6}{3/6}$$

$$= 1/3$$

6. It is estimated that 50% of the emails are spam. Our spam filtering software can detect 90% of spam emails, and the probability of a false positive is 5%. If an email is detected as spam, then what is the probability that it is in fact a non-spam email?

Events

$S \Rightarrow$ Email is spam

$S' \Rightarrow$ Email is not spam

$+ \rightarrow$ Email is flagged as spam

$P(S) = 0.5$

$P(S') = 0.5$

$P(+|S) = 0.9$

$P(+|S') = 0.05$

$$P(S'|+) = \frac{P(+|S')\,P(S')}{P(+|S')P(S') + P(+|S).P(S)}$$

$$= \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.9 \times 0.5} = \frac{0.025}{0.475} \simeq 0.053$$

7. Two different teams named A and B are asked to separately design a new product within a month. From past experience we know that:
    1. The probability that team A is successful is 2/3.
    2. The probability that team B is successful is 1/2
    3. The probability that at least one team is successful is 3/4

    Assuming that exactly one successful design is produced, what is the probability that it was designed by team B?

4 possible outcomes

SS = Both teams succeed

SF = A succeeds, B fails

FS = A fails, B succeeds

FF = Both teams fails

1 $P(SS) + P(SF) = 2/3$

2 $P(SS) + P(FS) = 1/2$

3 $P(SS) + P(SF) + P(FS) = 3/4$

$\{SF, FS\} \rightarrow$ events exactly one successful design is produced

$$P(FS \mid \{SF, FS\}) = \frac{P(FS \cap \{SF, FS\})}{P(\{SF, FS\})}$$

$$= \frac{P(FS)}{P(SF) + P(FS)}$$

$$= \frac{1/12}{1/4 + 1/12} = \frac{1}{4}$$

8. You enter a chess tournament where you probability of winning a game is 0.3 against half the players (call them type 1). 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent.

- What is the probability of winning?
- Assuming that you win. What is the probability that you had an opponent of type 1?

A, B, C = Players from different types

W = Event of winning the game

$P(A) = 0.5, \quad P(B) = 0.25, \quad P(C) = 0.25$

$P(W|A) = 0.3, \quad P(W|B) = 0.4, \quad P(W|C) = 0.5$

1 $P(W) = P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C)$

$= 0.3 \times 0.5 + 0.4 \times 0.25 + 0.5 \times 0.25 = 0.375$

2 $P(A|W) = \dfrac{P(W|A) \cdot P(A)}{P(W)}$

$= \dfrac{0.3 \times 0.5}{0.375} = 0.4$

9. We are given three coins and are told that two of the coins are fair and the third coin is biased, landing heads with probability 2/3. We are not told which of the three coins is biased. We permute the coins randomly, and then flip each of the coins. The first and second coins come up heads, and the third comes up tails. What is the probability that the first coin is the biased one?

3 coins, 2 fair and 1 biesed

B_i = Event that i is biesed

$P(B_1) = P(B_2) = P(B_3) = 1/3$

$P(HHT|B_1) = 2/3 \times 1/2 \times 1/2 = 1/6$

$P(HHT|B_2) = 1/2 \times 2/3 \times 1/2 = 1/6$

$P(HHT|B_3) = 1/2 \times 1/2 \times 1/3 = 1/12$

$P(B_1|HHT) = \dfrac{P(HHT|B_1) \cdot P(B_1)}{P(HHT)}$

$= \dfrac{1/6 \times 1/3}{1/3(1/6 + 1/6 + 1/12)} = 0.4$

10. We draw eight cards at random from an ordinary deck of 52 cards. Given that three of them are spades, what is the probability that the remaining five are also spades?

52 cards, draw 8 cards

A = Event that all are spades

B = Event that at least 3 are spades

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \frac{\binom{13}{8}}{\binom{52}{8}}$$

$$P(B) = \frac{\sum_{x=3}^{8} \binom{13}{x}\binom{39}{8-x}}{\binom{52}{8}}$$

$$P(A|B) \simeq 5.44 \times 10^{-6}$$