Derivatives

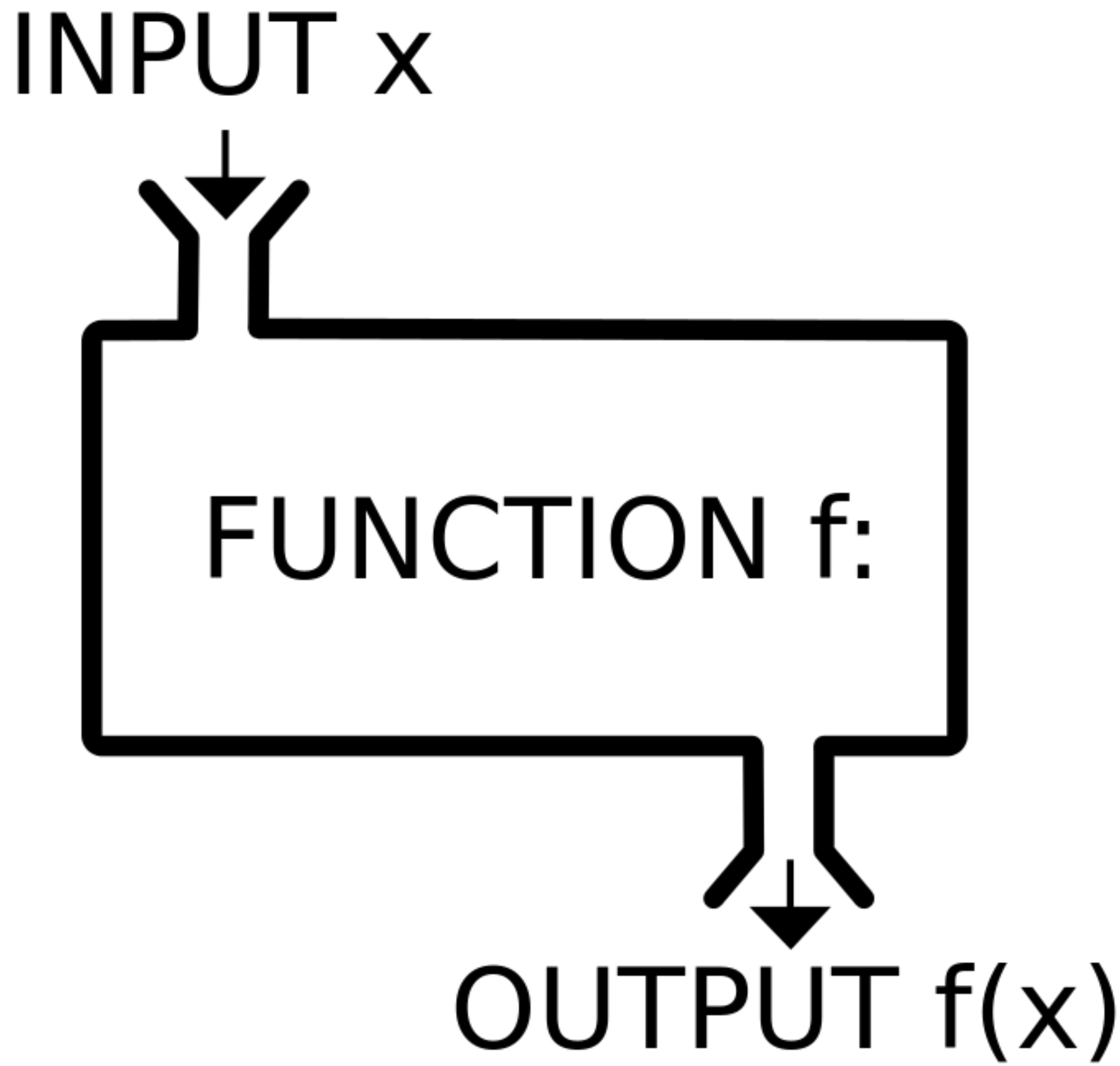
CS 556

Calculus

- Calculus is the branch of mathematics that deals with the finding and properties of derivatives and integrals of functions

- Calculus was developed independently by Newton and Leibniz.

Functions

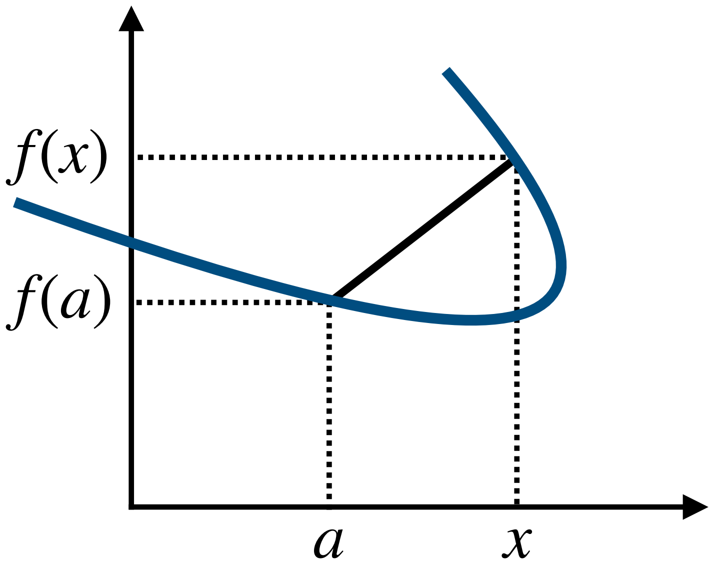


https://en.wikipedia.org/wiki/Function_(mathematics)#/media/File:Function_machine2.svg

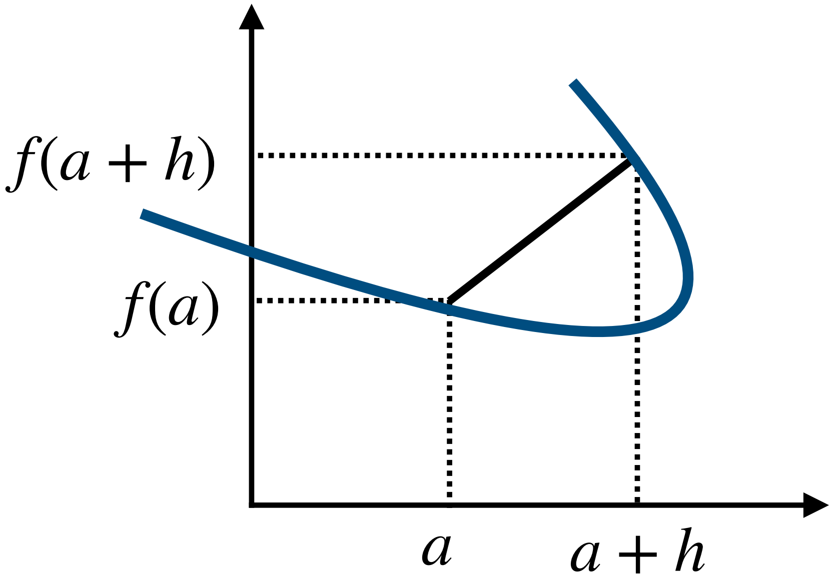
Slope

The slope describes the direction and steepness of a function

$$slope = m = \frac{rise}{run} = \frac{f(x) - f(a)}{x - a}$$

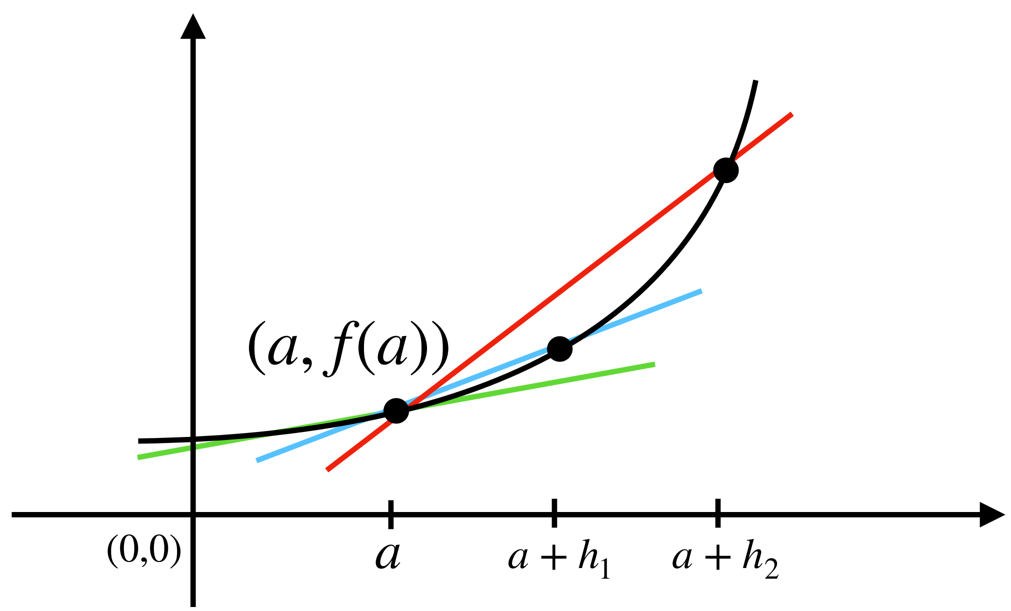


$$slope = m = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$



Rate of Change

The slope of the tangent line at x is the rate of change of the function at x.



$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example

What is the slope and the equation of the line tanget to $f(x) = x^2$ at $x = 3$?

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - x^2)}{h} = \lim_{h \to 0} (2x + h)$$

$$= 2x = 6$$

Slope of the tangent line at x = 3 is 6. From the line equation y = mx + b, we can find that the slope equation is y = 6x - 9.

Derivatives

Let $f(x)$ be a function defined in an open interval containing a. The derivative of a function $f(x)$ at a, denoted by $f'(x)$, is defined by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let f be a function. The derivative function, denoted by f', is the function whose domain consists of those values of x such that

the limit $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

A function $f(x)$ is said to be differentiable at $x = a$ if $f'(a)$ exists.

Example

Find the derivative of f(x) = 3x + 1 at any point.

$$f'(x) = \lim_{h \to 0} \frac{(3(x+h)+1) - 3x - 1}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h}$$

$$= 3$$

Example

Find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(x+h-x)}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{2\sqrt{x}}$$

Differentiation Rules

Constant Rule

If $f(x) = c$, then $f'(x) = 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

Power Rule

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

From $(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$

We have: $(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + nxh^{h-1} + h^n$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \cdots + nxh^{h-1} + h^n}{h}$$

$$= \lim_{h \to 0} (nx^{n-1} + \binom{n}{2} x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}) = nx^{n-1}$$

Example

If $f(x) = x^3$ then $f'(x) = 3x^2$

Sum Rule

The derivative of the sum of a function ***f*** and a function ***g*** is the same as the sum of the derivative of ***f*** and the derivative of ***g***.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Difference Rule

The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Constant Multiple Rule

The derivative of a constant c multiplied by a function f(x) is the same as the constant multiplied by the derivative

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x)$$

Example

If $f(x) = 2x^5 + 7$ then

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(2x^5 + 7) \\
&= \frac{d}{dx}(2x^5) + \frac{d}{dx}7 \\
&= 2\frac{d}{dx}(x^5) + 0 \\
&= 10x^4 .
\end{aligned}
$$

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

Example:

$$\frac{d}{dx}((3x+1)x^2) = 3x^2 + 2x(3x+1))$$

Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

Example

$$a(x) = \frac{5x^2}{4x + 3}$$

$$a'(x) = \frac{10x(4x + 3) - 20x^2}{(4x + 3)^2}$$

$$= \frac{20x^2 + 30x}{(4x + 3)^2}$$

Combining differentiation rules

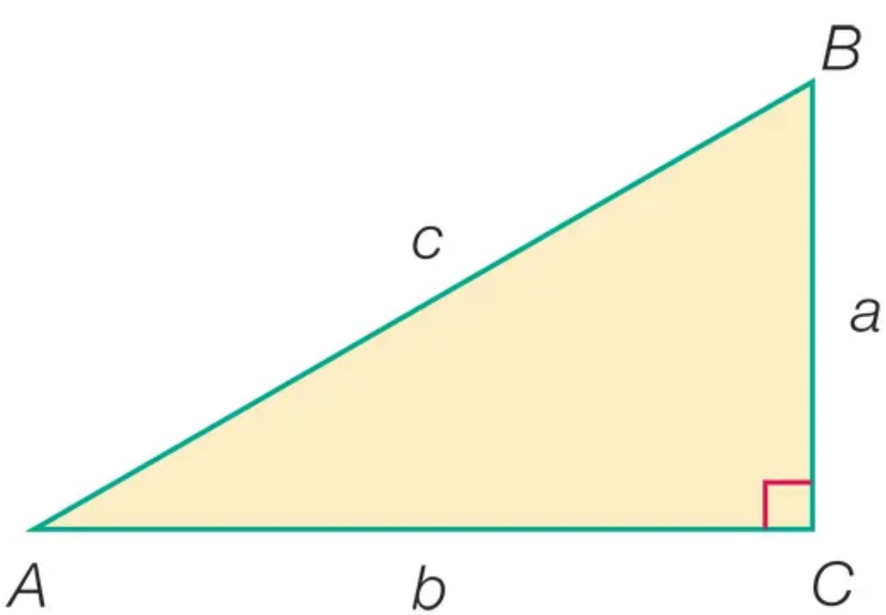
$$f(x) = x^3 + 3x^2 - 1$$

$$f'(x) = 3x^2 + 6x$$

Trigonometric Functions

$$\frac{d}{dx}(sinx) = cosx \qquad \frac{d}{dx}(cosx) = -\,sinx \qquad \frac{d}{dx}(tanx) = sec^2x$$

$$\frac{d}{dx}(cotx) = -\,csc^2x \qquad \frac{d}{dx}(secx) = secxtanx \qquad \frac{d}{dx}(cscx) = -\,cscxcotx$$



$$\sin A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\cos A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad \sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\tan A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \cot A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$$

Example

$$f(x) = cosx + sinx$$

$$f'(x) = -sinx + cosx$$

Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$ the derivative of the composite function $h(x) = f(g(x))$ is given by $h'(x) = f'(g(x))g'(x)$.

Example:

$$h(x) = \frac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$$

$$h'(x) = -2(3x^2+1)^{-3}(6x)$$

Exercises

Find the derivative of the following functions:

$$h(x) = cos(5x^2)$$

$$h(x) = (2x + 1)^5(3x - 2)^7$$

Exponential Functions

Let $f(x) = e^x$ be the natural exponential function.

Then $f'(x) = e^x$. In general, $\frac{d}{dx}(e^{g(x)}) = e^{g(x)}g'(x)$.

$$\frac{d}{dx}(b^{g(x)}) = b^{g(x)}g'(x)ln(b)$$

Example:

$$f(x) = e^{sin(2x)}$$

$$f'(x) = e^{sin(2x)}\frac{d}{dx}(sin(2x))$$

$$f'(x) = 2e^{sin(2x)}cos(2x)$$

Logarithmic Functions

Let $f(x) = ln(x)$ be the natural logarithmic function.

Then $f'(x) = \frac{1}{x}$. In general $\frac{d}{dx}(ln(g(x)) = \frac{1}{g(x)}g'(x)$.

$$\frac{d}{dx}(log_b g(x)) = \frac{g'(x)}{g(x)ln(b)}$$

Logarithmic Differentiation

- Let $h(x) = f(x)^{g(x)}$.
- To differentiate $y = h(x)$ take the natural logarithm of both sides of the equation $lny = ln(h(x))$.
- Expand $ln(h(x))$ as much as possible.
- Differentiate both sides of the equation.One the left we will have $\frac{1}{y}\frac{dy}{dx}$.
- Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
- Replace y by $h(x)$.

Exercises

Find the derivative of the following functions:

$$y = (2x^4 + 1)^{cosx}$$

$$y = \frac{x\sqrt{2x + 1}}{e^x sin^3 x}$$

Partial Derivatives

The partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant.

The partial derivative of a function $f(x, y, z, \ldots)$ with respect to variable x is denoted as $\frac{\partial f}{\partial x}$.

Example

$$f(x, y) = x^2 + xy - x$$

$$\frac{\partial f}{\partial x} = 2x + y - 1$$

$$\frac{\partial f}{\partial y} = x$$

Numerical Differentiation

Numerical differentiation is the process of finding the numerical value of a derivative of a given function at a given point.

Three approximations to the derivative $f'(a)$ are;

- The one-sided (forward) difference $\dfrac{f(a+h)-f(a)}{h}$
- The one-sided (backward) difference $\dfrac{f(a)-f(a-h)}{h}$
- The central difference $\dfrac{f(a+h)-f(a-h)}{2h}$

Example

The distance x of a runner from a fixed point is measured in meters at intervals of half of a second. The data obtained are:

T	0.0	0.5	1.0	1.5	2.0
X	0.00	3.65	6.80	9.00	12.15

Use central differences to approximate the runner's velocity at $t = 0.5$ s and $t = 1.25$ s.

T	0.0	0.5	1.0	1.5	2.0
X	0.00	3.65	6.80	9.00	12.15

$$f'(0.5) = \frac{f(0.5 + 0.5) - f(0.5 - 0.5)}{(2 * 0.5)}$$

$$= \frac{f(1.0) - f(0.0)}{1.0} = 6.8m/s$$

$$f'(1.25) = \frac{f(1.25 + 0.25) - f(1.25 - 0.25)}{(2 * 0.25)}$$

$$= \frac{f(1.5) - f(1.0)}{0.5} = \frac{9.0 - 6.8}{0.5} = 4.4m/s$$

Thank you!