Linear Algebra Part 1

CS 556

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Linear Algebra

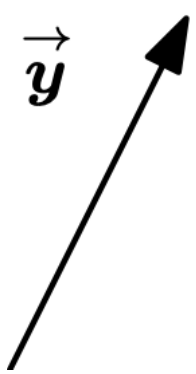
- Linear Algebra is the study of vectors and certain rules to manipulate vectors.
- We represent numerical data as vectors

Vectors

- An algebraic vector is ordered list of elements, where the number of elements determine the dimensionally of the vector.

  Examples $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, $\mathbf{w} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

- A geometric vector is a straight line with some length and some direction.

  Example: 

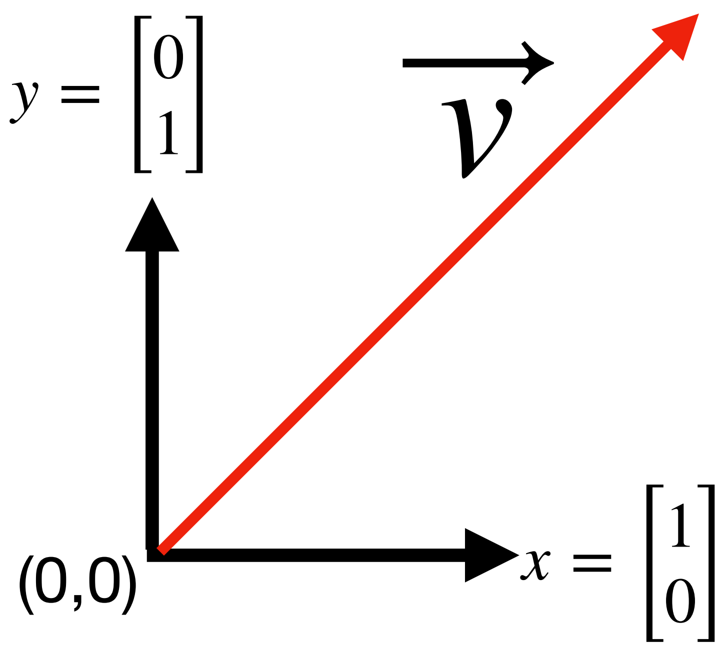
Vectors

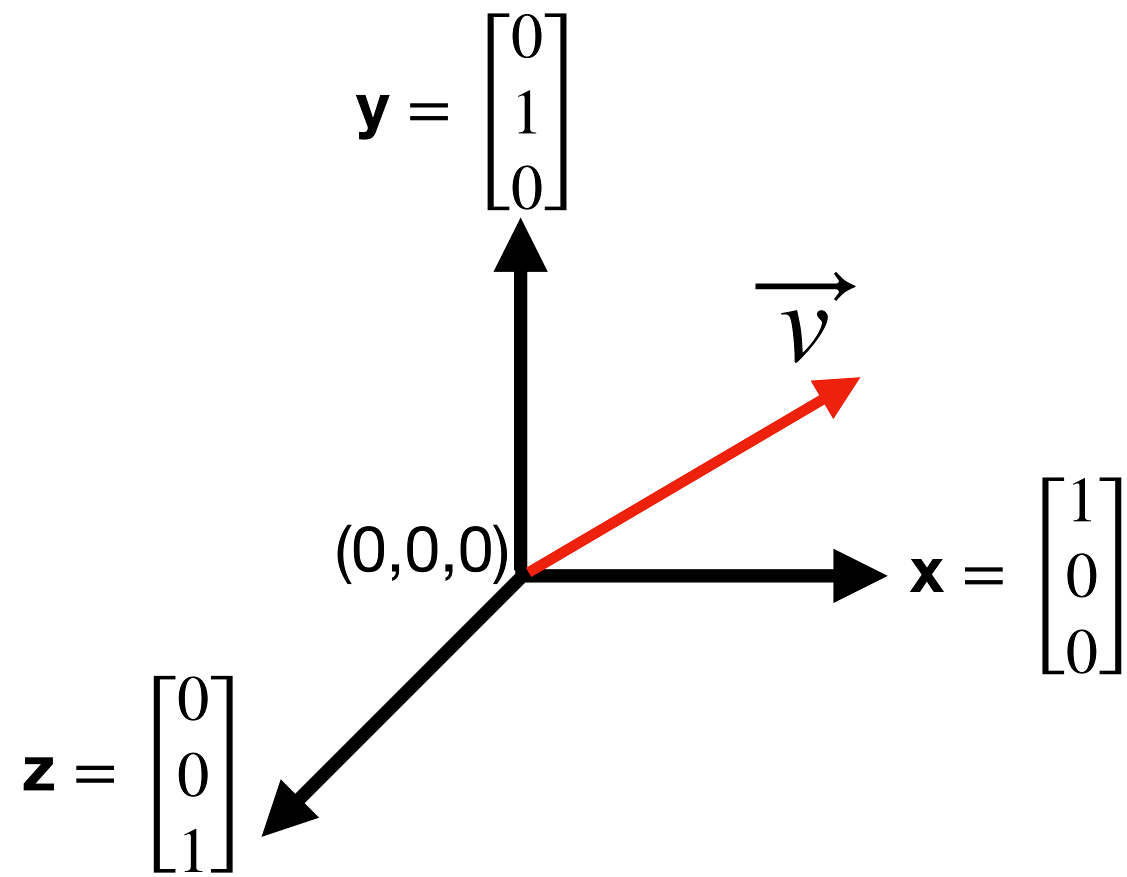
- Vectors - tuples *n* of real numbers $\mathbb{R}^n$

3 bedrooms
2 bathrooms
2 floors,
Year 2008
Price 150K

$$\longrightarrow \quad a = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2008 \\ 150000 \end{bmatrix}$$

How to express vectors?





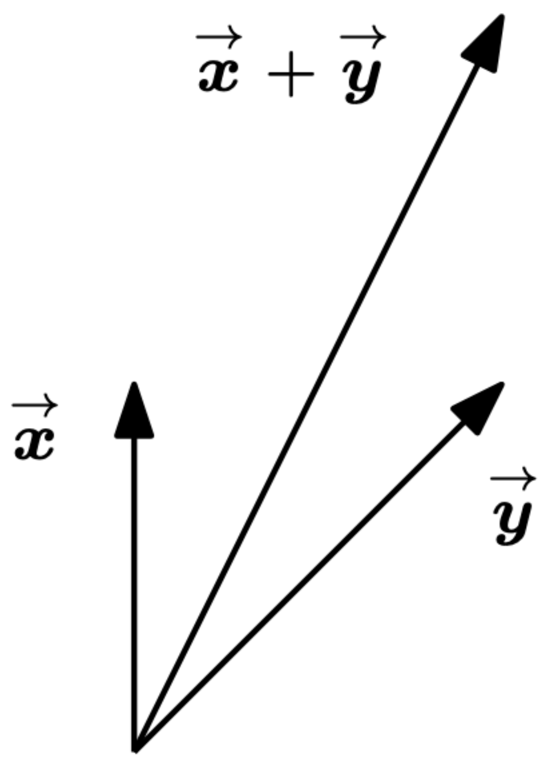
$$\mathbf{v} = ax + by = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{v} = ax + by + cz = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

Vectors Operations

- Addition
- Scalar Multiplication

Addition

- Add elements across corresponding dimensions.
- Put the tail of one vector at the head of the other vector.



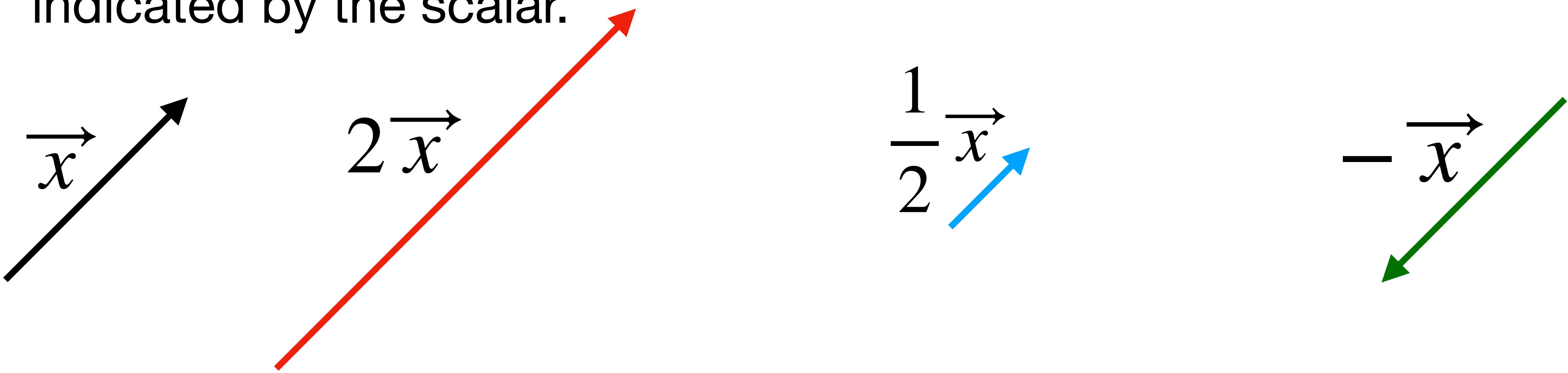
$$\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \ \mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\mathbf{x} - \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Scalar Multiplication

- Scalar: A number, represented by a lower case greek letter such as α, β, λ
- Algebraic: $\lambda\mathbf{x}$ multiply each element of the vector by the scalar
- Geometric: Stretch or shrink the vector by the amount indicated by the scalar.

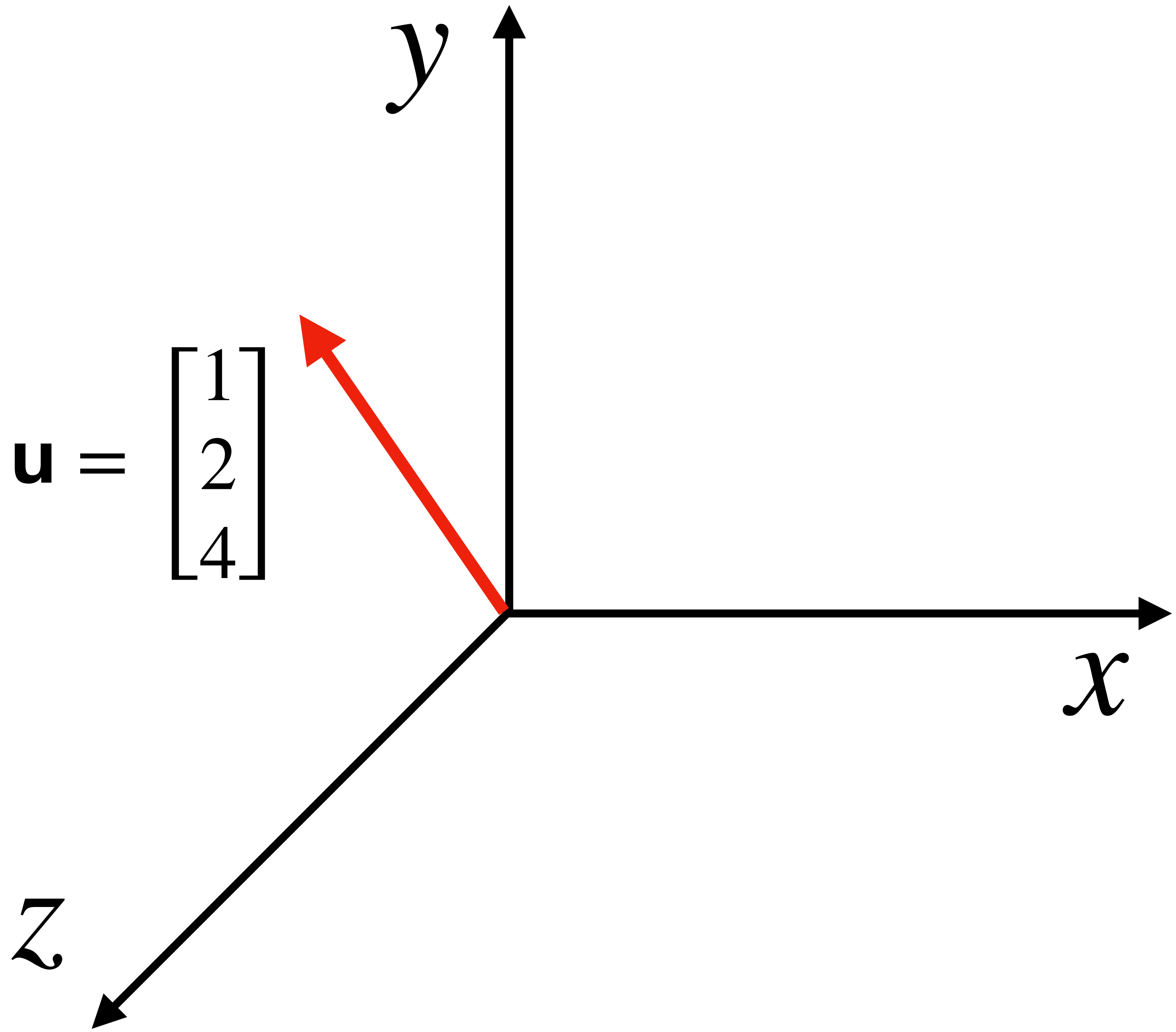


$x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $2x = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $\frac{1}{2}x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $-x = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

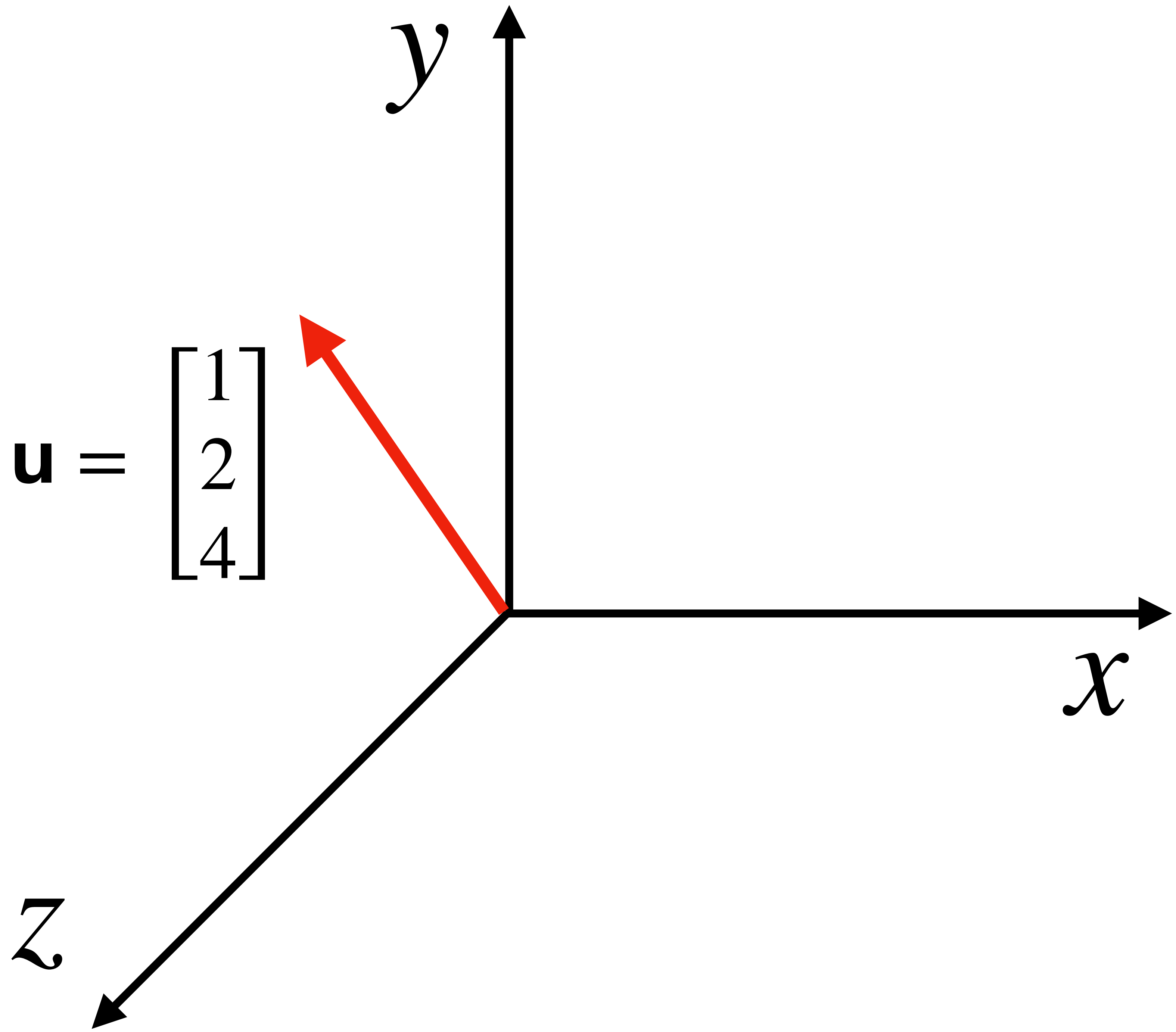
Linear Combinations

- Linear combinations of vectors are created by combining addition with scalar multiplication.

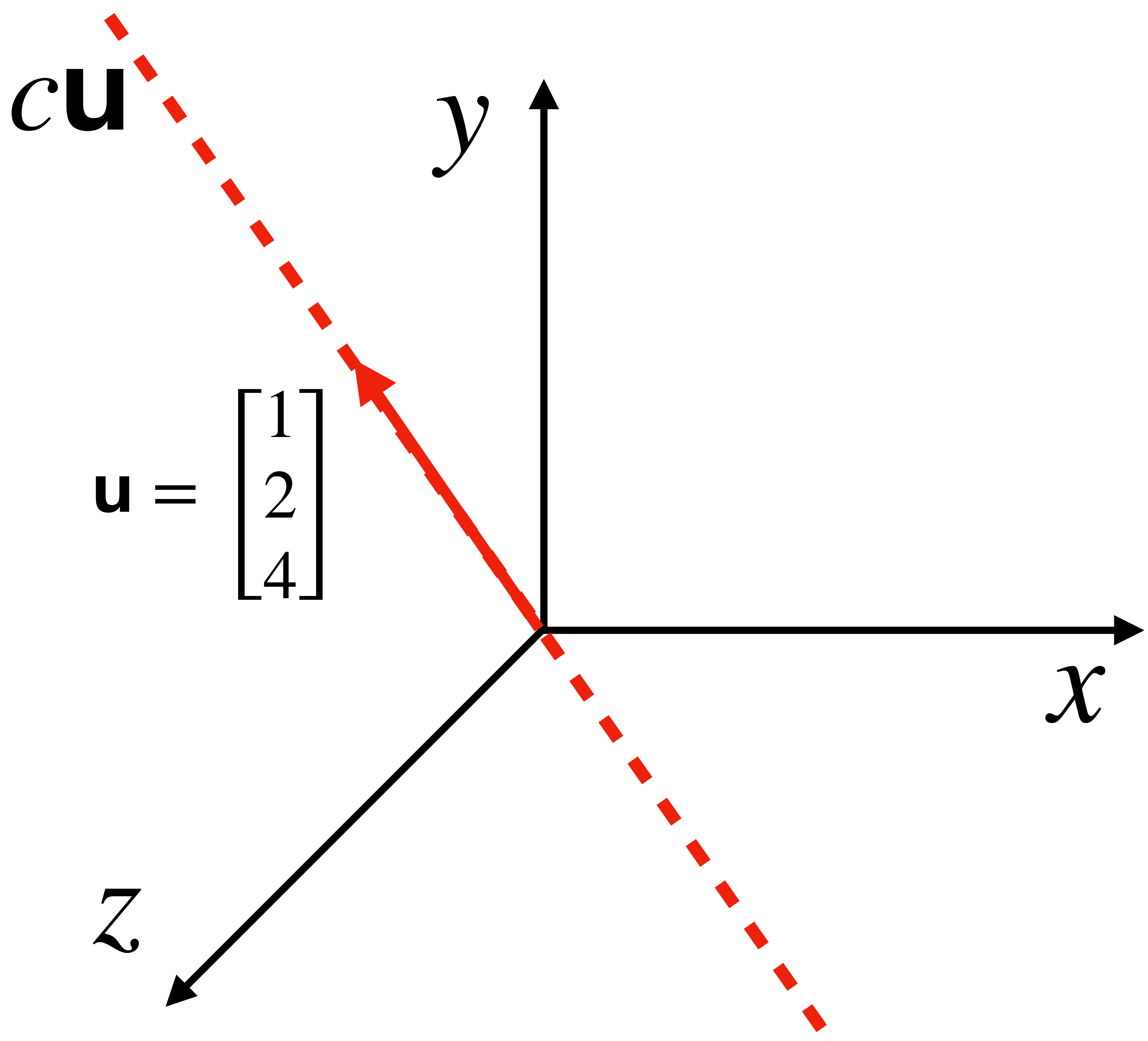
- For instance, assume we have two vectors **v** and **w** and *c* and *d* are two scalars. The sum of $c\mathbf{v}$ and $d\mathbf{w}$ is a linear combination $c\mathbf{v} + d\mathbf{w}$.



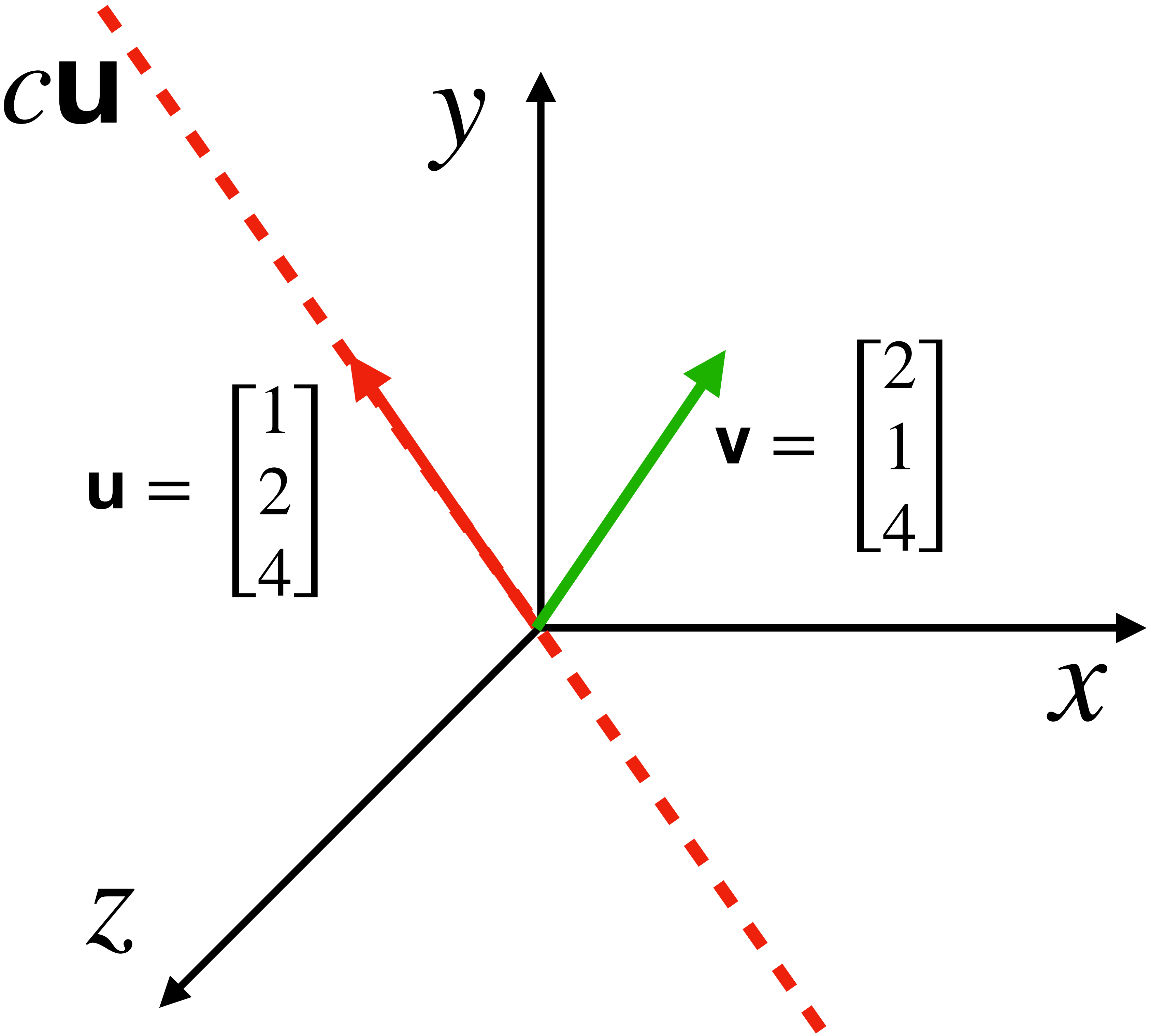
y
$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$
x
z



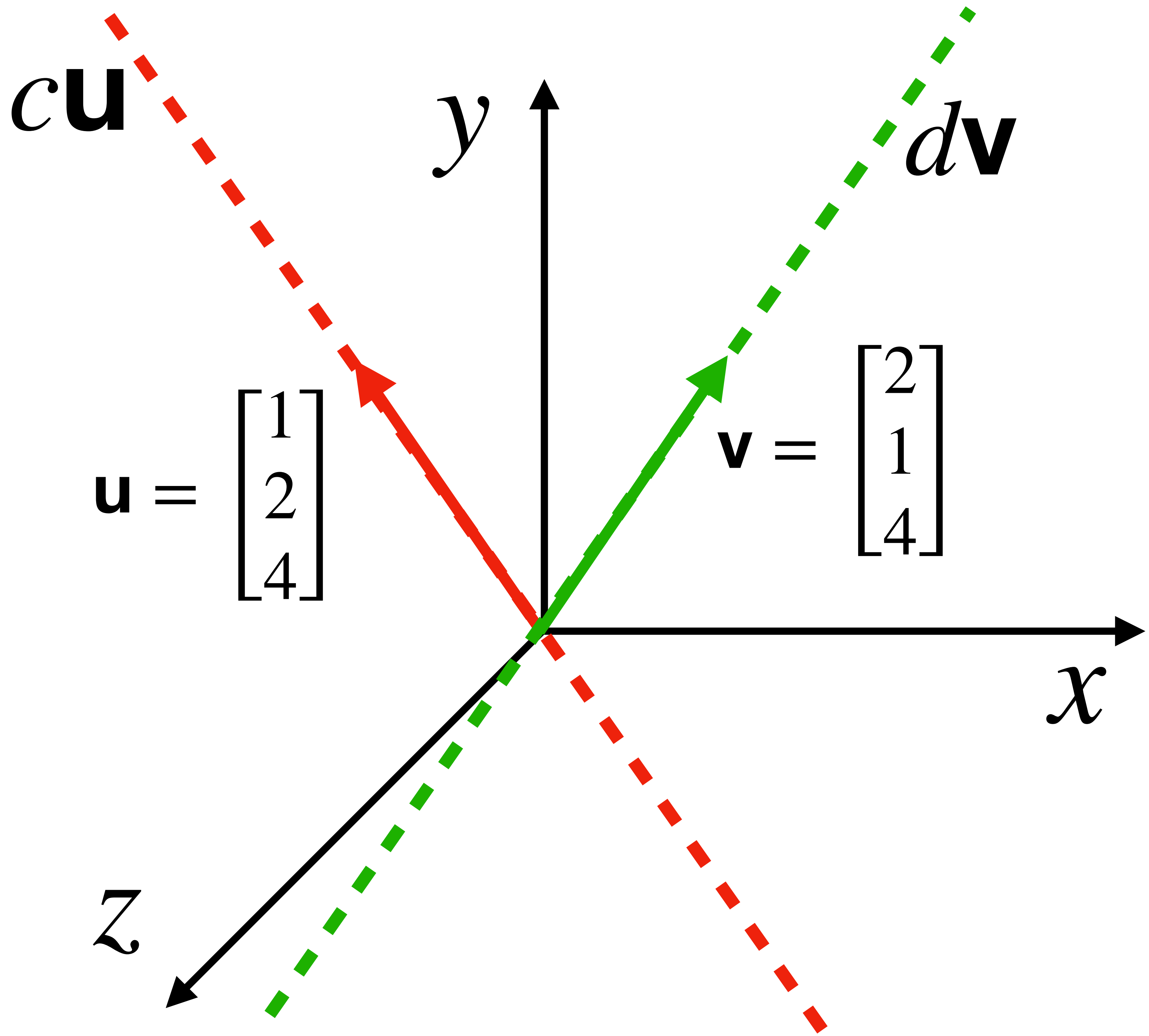
For one vector ***u***, the only linear combinations are the multiples *c**u***.



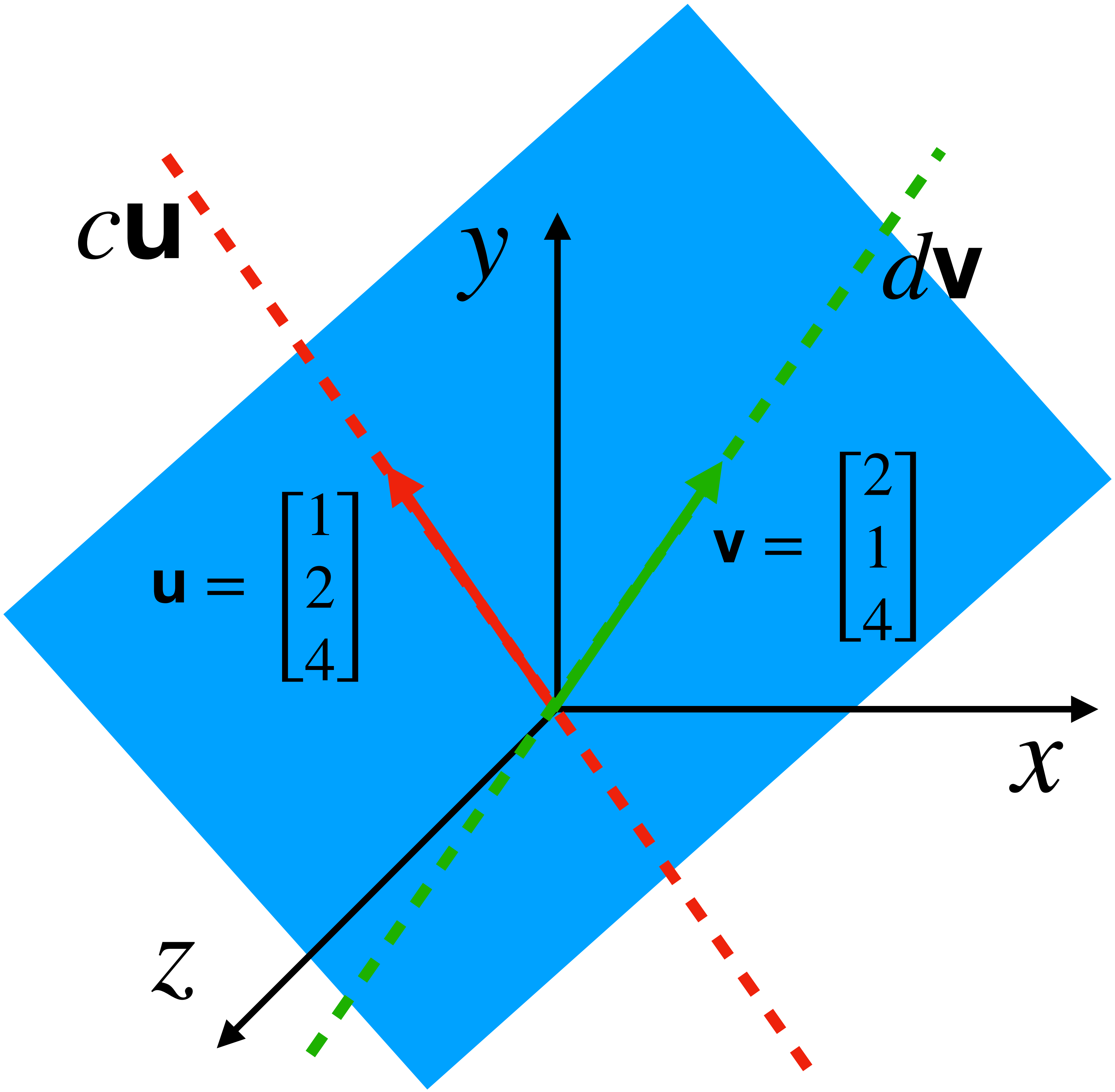
For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$.
The combinations of $c\boldsymbol{u}$ fill a line through (0,0,0).



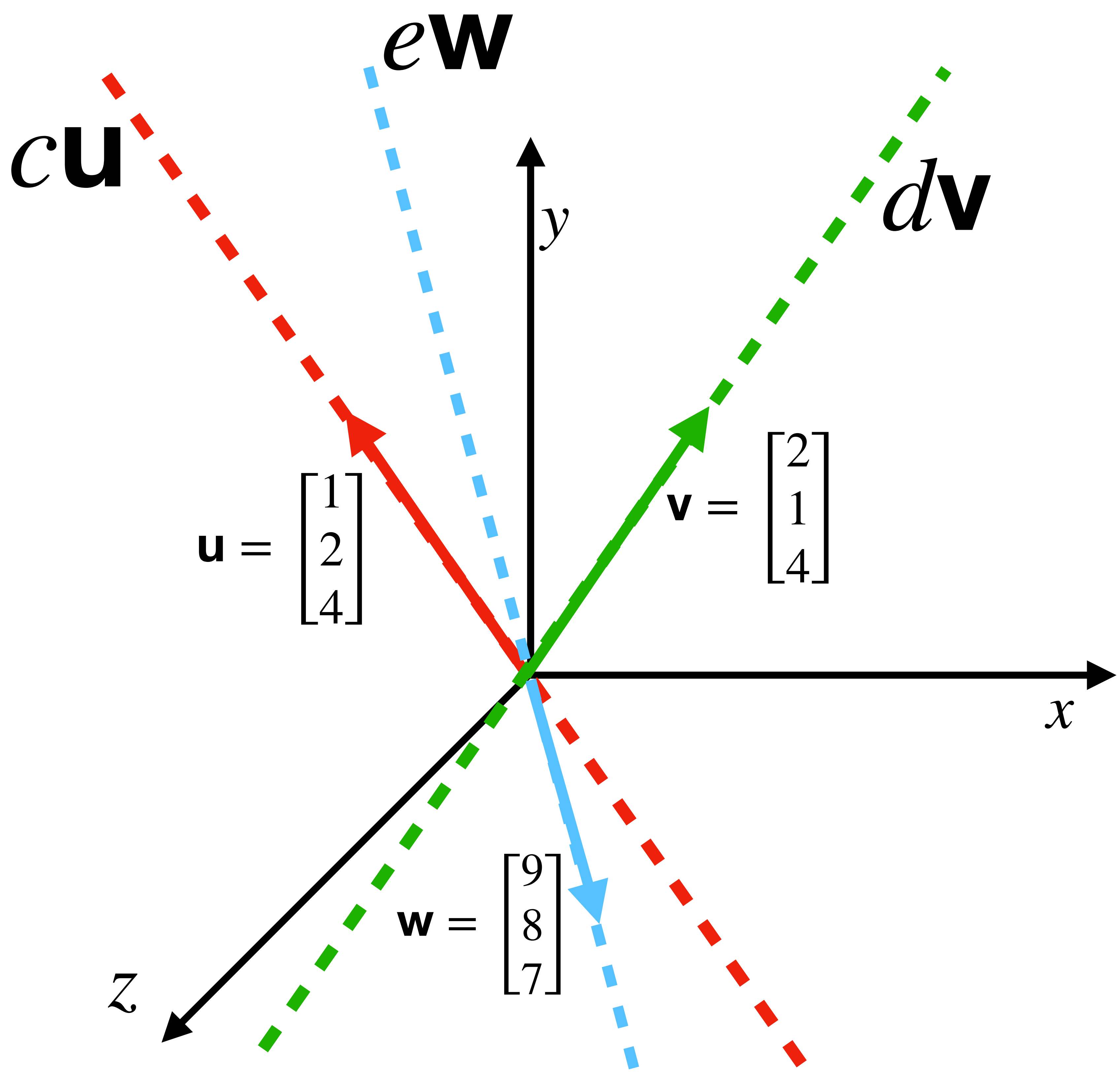
$c\mathbf{u}$
y
$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$
$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$
x
z



For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ the linear combinations are $c\boldsymbol{u} + d\boldsymbol{v}$.



For two vectors ***u*** and ***v*** the linear combinations are *c**u*** + *d**v***.
The combinations *c**u*** + *d**v*** of two typical nonzero vectors fill a plane through (0,0,0).



For three vectors ***u, v*** and ***w*** the linear combinations are *c**u*** + *d**v*** + *e**w***.
The combinations *c**u*** + *d**v*** + *e**w*** of three typical non-zero vectors fill three dimensional space.

Example

- Describe the plane in $\mathbb{R}^3$ that is filled by the linear combinations of $v = (1,1,0)$ and $w = (0,1,1)$.

$$Combinations\ cv + dw = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix} fill\ the\ plane$$

- Find a vector that is not a combination of v and w.

$$\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

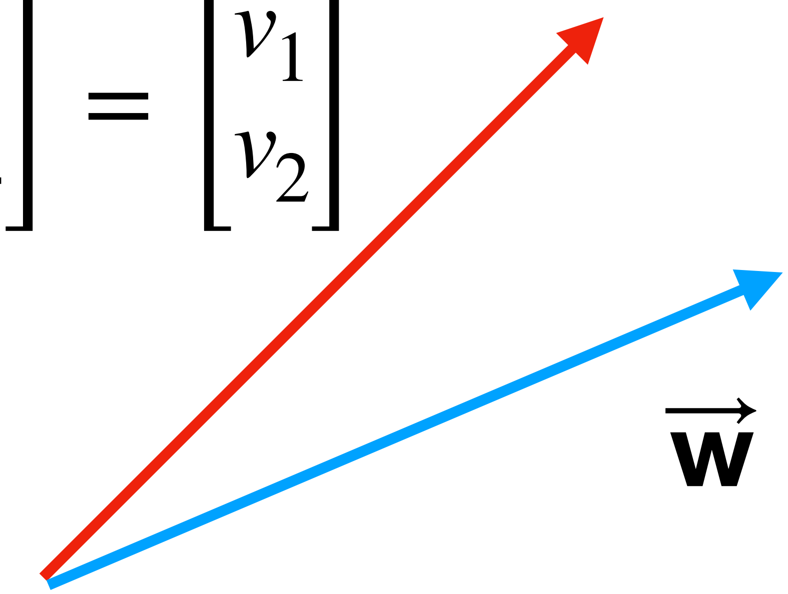
Dot Product

Dot product or **inner product** is an algebraic operation that takes two equal-length sequences of numbers and returns a single number.

$$v \cdot w = < v, w > = v^T w = \sum_{i=1}^{n} v_i w_i$$

Dot Product - Example

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



$$\vec{\mathbf{w}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$v \cdot w = v_1 w_1 + v_2 w_2 = 3x4 + 4x3 = 24$$

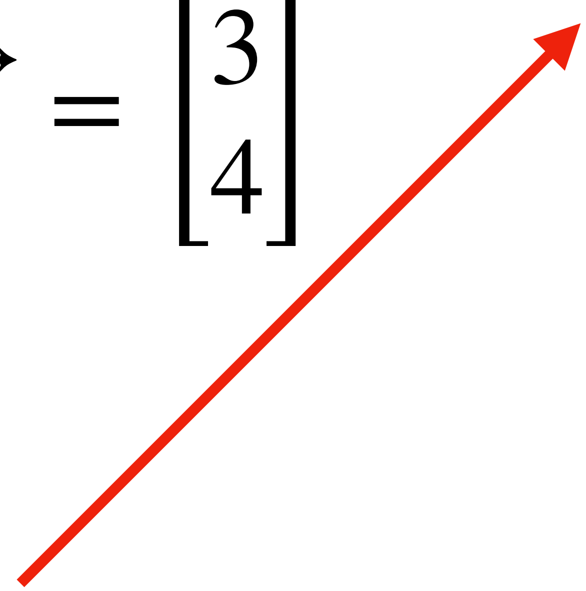
Properties of Dot Product

- Distributive $u^T(v + w) = u^T v + u^T w$
- Not Associative: $u^T(v^T w) \neq (u^T v)^T w$
- Commutative: $u^T v = v^T u$

Vector Length/Magnitude/Norm

$$||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

Unit Vectors

Unit Vector: Vector with length of 1

$$\mu \mathbf{v} \ s.t. \ ||\mu \mathbf{v}|| = 1$$

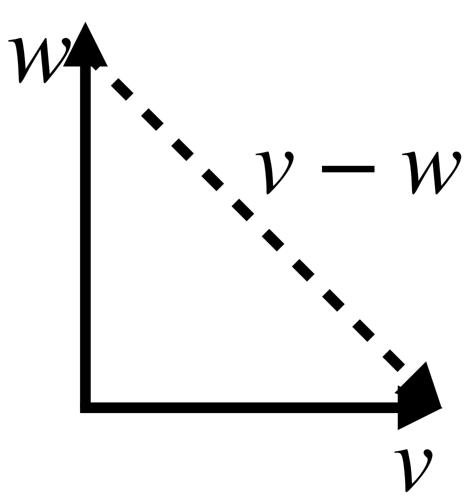
How to choose μ?

$$\mu = \frac{1}{||\mathbf{v}||}$$

Angle between two vectors

The dot product is $v \cdot w = 0$ when v is perpendicular to w.

Proof: When v and w are perpendicular, they form the sides of a right triangle. The hypotenuse is $v - w$.



$$||v||^2 + ||w||^2 = ||v - w||^2$$

$$(v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (v_1 - w_1)^2 + (v_2 - w_2)^2$$

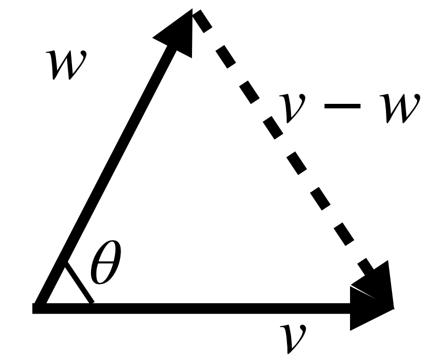
$$-2v_1w_1 - 2v_2w_2 = 0$$

$$v_1w_1 + v_2w_2 = 0$$

$$v \cdot w = 0$$

■

Cosine Formula for Dot Product



Let v, w be two non-zero vectors in $\mathbb{R}^n$, then:

$$v \cdot w = v^T w = ||v||\ ||w||\ cos(\theta)$$

Proof:

$$||v-w||^2 = ||v||^2 + ||w||^2 - 2||v||\ ||w|| cos(\theta)$$ **Cosine Law**

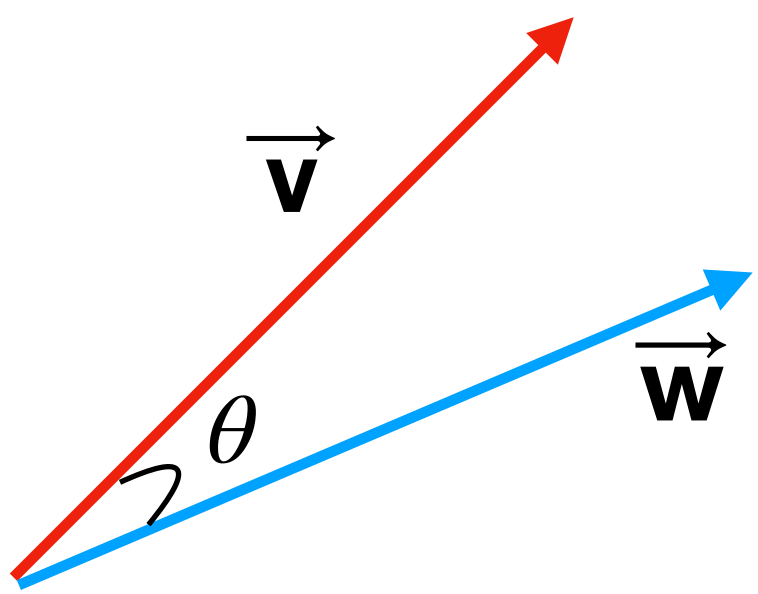
$$||v-w||^2 = (v-w) \cdot (v-w) = v \cdot v - 2(v \cdot w) + w \cdot w$$
$$= ||v||^2 - 2(v \cdot w) + ||w||^2$$

$$||v||^2 - 2(v \cdot w) + ||w||^2 = ||v||^2 + ||w||^2 - 2||v||\ ||w|| cos(\theta)$$

$$v \cdot w = ||v||\ ||w||\ cos(\theta) \qquad \blacksquare$$

Dot Product

Cosine of the angle between the vectors scaled by the product of the lengths of these vectors.



$$v \cdot w = v^T w = ||v||\ ||w||\ cos(\theta)$$

Matrices

With $m, n \in \mathbb{N}$, a real-valued (m, n) *matrix* A is a *m***n*-tuple of elements which is ordered according to a rectangle scheme consisting of *m* rows and *n* columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \ a_{ij} \in \mathbb{R}$$

Matrix Addition

The sum of two matrices $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{mxn}$

is defined as the element wise sum, i.e.

$$\mathbf{A+B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix} \in \mathbb{R}^{mxn}$$

Matrix Scalar Multiplication

$$\delta \mathbf{A} = \begin{bmatrix} \delta a_{11} & \delta a_{12} & \dots & \delta a_{1n} \\ \delta a_{21} & \delta a_{22} & \dots & \delta a_{2n} \\ \vdots & \vdots & & \vdots \\ \delta a_{m1} & \delta a_{m2} & \dots & \delta a_{mn} \end{bmatrix} \in \mathbb{R}^{mxn}$$

Diagonal Matrix

In $\mathbb{R}^{nxn}$ we define the diagonal matrix as the nxn matrix containing numbers on the diagonal and 0 elsewhere.

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix} \in \mathbb{R}_{3x3}$$

Identity Matrix

In $\mathbb{R}^{nxn}$ we define the identity matrix as the nxn matrix containing 1 on the diagonal and 0 elsewhere.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}_{nxn}$$

Transpose

For $A \in \mathbb{R}^{mxn}$ the matrix $B \in \mathbb{R}^{nxm}$ with $b_{ij} = a_{ji}$ is called the transpose of A. We write $B = A^T$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Symmetric Matrix

A matrix where elements are mirrored around the diagonal.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix} \in \mathbb{R}_{3x3}$$

If the elements are mirrored around the diagonal with a flipped sign, the matrix is called Skew-symmetric.

Symmetric: $A = A^T$

Skew-symmetric: $A = -A^T$

Matrix Trace

Sum of elements in the diagonal of the matrix in a square matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & 2 & 4 \\ 2 & \mathbf{5} & 6 \\ 4 & 6 & \mathbf{8} \end{bmatrix}$$

$$trace(A) = \sum_{i=1}^{3} A_{ii} = 16$$

Matrix-Vector Multiplication

Three vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

The linear combinations in $\mathbb{R}^3$ are $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \longleftarrow$$

Matrix times vector
Linear combination of columns in the matrix.

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b \longleftarrow$$

Which combinations of *u, v, w* produces a particular vector *b*?

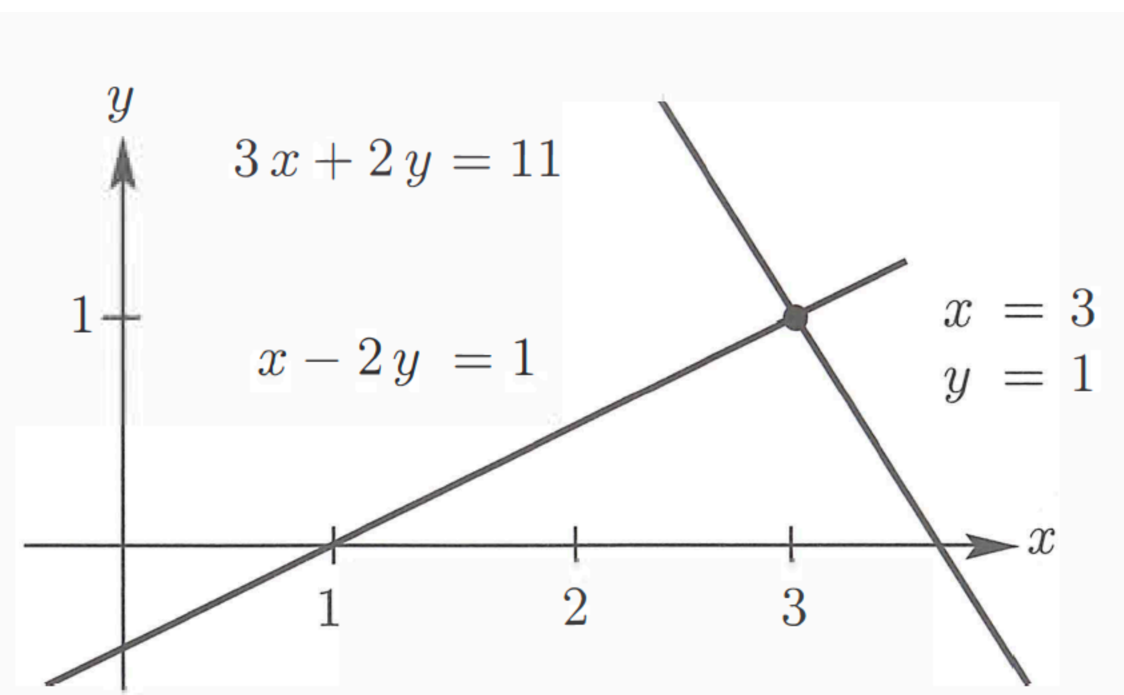
Solving Linear Equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

Column Picture

$$\mathbf{x}\begin{bmatrix}1\\3\end{bmatrix} + \mathbf{y}\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

Row Picture



Matrix Equation

$$Ax = b$$

$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

3 equations in 3 unknowns

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

Row Picture

The row picture shows three planes meeting at a single point

Column Picture

$$\mathbf{x}\begin{bmatrix}1\\2\\6\end{bmatrix} + \mathbf{y}\begin{bmatrix}2\\5\\-3\end{bmatrix} + \mathbf{z}\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Matrix Equation

$$Ax = b$$

$$\begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Ax comes from dot products, each row times the column *x*.

$$\mathbf{Ax} = \begin{bmatrix}(row\ 1)\cdot x\\(row\ 2)\cdot x\\(row\ 3)\cdot x\end{bmatrix}$$

Ax is a combination of column vectors.

$$Ax = x(col\ 1) + y(col\ 2) + z(col\ 3)$$

Systems of Linear Equations SLE

$$x + y + 3z = 15$$
$$x + 2y + 4z = 21$$
$$x + y + 2z = 13$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

How to solve SLE?

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

Key to solving SLE these elementary transformation, that keep the solution set the same, but that transform the equation system into a simpler form:

- Exchange of two rows
- Multiplication of a row with a constant
- Addition of two rows

Gaussian Elimination

Performs elementary transformation to bring a system of linear equation into reduced row-echelon form.

A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix
- Looking at nonzero rows only, the first nonzero number (aka pivot) is always strictly to the right of the pivot above it.

An matrix is in reduced row echelon form if:

- It is in row-echelon form
- Every pivot is 1
- The pivot is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

$$\begin{array}{l} \\ \textbf{R2 - R1} \\ \textbf{R3 - R1} \end{array} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

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$$\begin{array}{r} \\ \\ \textbf{-R3} \end{array} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ -2 \end{bmatrix}$$

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$$\begin{array}{r} \textbf{R1 - 3R3} \\ \textbf{R2 - R3} \\ \\ \end{array}\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ 2 \end{bmatrix}$$

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$$\begin{array}{r} \\ \\ \textbf{-R3} \end{array} \begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}15\\ 6\\ -2\end{bmatrix}$$

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$$\begin{array}{r} \textbf{R1 - 3R3} \\ \textbf{R2 - R3} \\ \\ \end{array} \begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}15\\ 6\\ 2\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}9\\ 4\\ 2\end{bmatrix}$$

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$$\begin{array}{r} \textbf{R1 - R2} \\ \\ \\ \end{array}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 4\\ 2\end{bmatrix}$$

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REDUCED ROW-ECHELON FORM

Matrix Multiplication

For matrices $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{nxk}$ the elements c_{ij} of the product $C = AB \in \mathbb{R}^{mxk}$ are computed as:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \qquad i = 1,\ldots,m,\ j = 1,\ldots,k$$

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{AB} = \begin{bmatrix} 2 & 7 \\ 2 & 9 \end{bmatrix}$$

Matrix Multiplication

- Each column of AB is a combination of the columns of A.

Matrix *A* times every column of *B*

$$A[b_1\ b_2\ \ \dots\ \ b_k] = [Ab_1\ Ab_2\ \ \dots\ \ Ab_k]$$

- Every row of AB is a combination of the rows of B.

Every row of *A* times matrix *B*

$$[a_{1i}\ a_{2i}\ \ \dots a_{ni}]B = [row\ \ i\ \ of\ AB]$$

Multiplication Properties

- Associativity

$$\forall A \in \mathbb{R}^{mxn},\ B \in \mathbb{R}^{nxp},\ C \in \mathbb{R}^{pxq}:\ \ (AB)C = A(BC)$$

- Distributivity

$$\forall A, B \in \mathbb{R}^{mxn},\ C, D \in \mathbb{R}^{nxp}:$$
$$(A+B)C = AC + BC,\ A(C+D) = AC + AD$$

- Multiplication with the identity matrix

$$\forall A \in \mathbb{R}^{mxn}:\ \ I_m A = A I_n = A$$

Inverse Matrix

Consider a square matrix $A \in \mathbb{R}^{nxn}$. Let matrix $B \in \mathbb{R}^{nxn}$ have the property that $AB = I_n = BA$. B is called the inverse of A and denoted by A^{-1}. Not every matrix A possesses an inverse A^{-1}.

If this inverse exists, matrix A is called regular/invertible/nonsingular, otherwise singular/noninvertible.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & -4 \end{bmatrix}$$

Invertibility Test

- The algorithm to test invertibility is elimination: Matrix A_{nxn} must have n(non-zero) pivots.
- The algebra test for invertibility is the determinant of $A : det(A)$ must be non zero.
- The equation that test for invertibility is $Ax = 0$: where $x = 0$ must be the only solution.

Gauss-Jordan Elimination

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

"Elementary Row Operations"

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Determinant

- The determinant of a square matrix $A \in \mathbb{R}^{nxn}$ is a function that maps A into a real number.

- Notation $det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

- Only for square matrix

Basic Properties of Determinants

- The determinant of the identity matrix is 1.
- The determinant changes sign when two rows are exchanged.
- The determinant is a linear function of each row separately.

$$\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$$

Derived Properties of Determinants

- If two rows of matrix A are equal, then det(A) = 0.

- Subtracting a multiple of one row from another row leaves the determinant of matrix A unchanged.

- A matrix with a row of zeros has det(A) = 0.

- If A is triangular then det(A) is the product of diagonal entries.

- If A is singular, then det(A) = 0. If A is invertible, then det(A) ≠ 0.

- $det(AB) = det(A)det(B)$

- $det(A^T) = det(A)$

Laplace Expansion

Consider a matrix $A \in \mathbb{R}^{nxn}$. Then for all $j = 1,\ldots,n$:

1. Expansion along column j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j}\ a_{kj}\ det(A_{k,j})\,.$$

2. Expansion along row j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j}\ a_{jk}\ det(A_{j,k})\,.$$

Here $A \in \mathbb{R}^{(n-1)x(n-1)}$ is the sub matrix of A

that we can obtain when deleting row k and column j.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(A) = (-1)^{1+1} \cdot 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{1+2} \cdot 2 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{1+3} \cdot 3 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1(1-0) - 2(3-0) + 3(0-0) = -5$$

Determinant Applications

Geometry

- Area/Volume of shape specified by coordinates in the matrix

Determinant Applications

Geometry

- Area/Volume of shape specified by coordinates in the matrix

Matrix Inverse

- Divide by determinant. No inverse of matrix if determinant is zero

Computing Inverses

- The **minors matrix**: a matrix of determinants
- The **cofactors matrix**: the minors matrix element-wise multiplied by a grid of alternating +1 and -1.
- The **adjugate matrix**: the transpose of the cofactors matrix
- The **inverse matrix**: the adjugate matrix divided by the determinant

Inverse for a 2x2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Inverse for a 3x3 Matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}$$

$$\mathbf{Minors} = \begin{bmatrix} -7 & -2 & +4 \\ +7 & +1 & -5 \\ +6 & +1 & -4 \end{bmatrix}$$

$$\mathbf{Cofactors} = \begin{bmatrix} -7 & +2 & +4 \\ -7 & +1 & +5 \\ +6 & -1 & -4 \end{bmatrix}$$

$$\mathbf{Adjugate} = \begin{bmatrix} -7 & -7 & +6 \\ +2 & +1 & -1 \\ +4 & +5 & -4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & -4 \end{bmatrix}$$