Optimization Problems

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November 8, 2022

In the Box Factory

A box factory can create cardboard templates for folding boxes with any given shape and size, but they want to make sure they're optimizing the volume that their brand of boxes can hold per unit of material they spend on the boxes' frames.

1. The first boxes they make are made from simple $8 * 8\text{cm}^2$ square of cardboard, which they cut square corners into to fold it into a box. What is the optimal size of the square cutouts, and what will the resulting volume of the optimal box be?

$V = (8-2x)(8-2x)x = 4x^3 - 32x^2 + 64x$

$V'(x) = 12x^2 - 64x + 64 \; , \; x \in (0,4)$

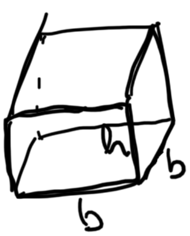
$12x^2 - 64x + 64 = 0$

$3x^2 - 16x + 16 = 0$

$x = 4/3$

$V\left(\frac{4}{3}\right) = \frac{1024}{27} \text{cm}^3$

2. For one of their clients, the boxes are made of uniform material that costs $\$0.50/\text{cm}^2$, cut into box-shaped templates with a square base and arbitrary sides. The company wants to make a \$12 box - what are the optimal dimensions, and the resulting volume, of the ideal box for this price?



$C(b,h) = 0.5(4hb + 2b^2) = 12 \to h = (12-b^2)/2b$

$V(b,h) = hb^2 \to V(b) = \frac{12-b^2}{2b} \cdot b^2 = \frac{12-b^2}{2} b = \frac{12b-b^3}{2}$

$V'(b) = \frac{1}{2}(12 - 3b^2) = 0$

$b = 2, \; h = 2, \; V = hb^2 = 8 \text{ cm}^3$

3. A new client asks that they print designs around the four sides of the box, but not the square bases of the top or bottom. The material with the printed designs costs twice as much per unit - $\$1/\text{cm}^2$ - than the plain material used for the top and bottom (and all six sides of the box in problem 2). What are the new optimal dimensions, and the resulting volume, of the best box that can now be produced for \$12?

$C(b,h) = 1 \times 4hb + 0.5 \times 2b^2 = 4hb + b^2 = 12 \to h = \frac{12-b^2}{4b}$

$V(b,h) = hb^2, \; V(b) = \frac{12-b^2}{4b} \cdot b^2 = \frac{12b-b^3}{4}$

$V'(b) = \frac{1}{4}(12 - 3b^2)$

$b = 2, \; h = 1, \; V(2,1) = 4 \text{ cm}^3$

4. How are the results of problems 2 and 3 related? What might this tell us about the "ideal" square box and its shape?

5. A bold new engineer thinks that they might be able to get more useful area out of their boxes by making them out of a new shape. The engineer provides a few shapes he thinks might be useful in the form of an equation for their volume, V, and surface area, S:

Modified Cubic Box (corners trimmed):
- $V(x,t) = x^3 - 4t^3$
- $S(x,t) = 6x^2 - 12t^2$
- *NOTE:* $2t \leq x$ for the box to be valid.

Sphere:
- $V(r) = \frac{4}{3}\pi r^3$
- $S(r) = 4\pi r^2$

Cylinder:
- $V(r,h) = \pi r^2 h$
- $S(r,h) = 2\pi r * (r+h)$

(a) For each of the new shapes, calculate the dimensions (as given in the volume/surface area functions) and volume of the best \$12 'box',assuming that these boxes are produced for the uniform price \$0.50/cm^2 of material.

(b) Which box shape is optimal - at least, optimal in *volume for the price*? Will the optimal box for the job might change as the budget increases from \$12 up towards infinity - why or why not?

Modified Cubic Box

$C(x,t) = 0.5 \times S(x,t) = 0.5 \times (6x^2 - 12t^2) = 3x^2 - 6t^2 = 12$

$t = \sqrt{\frac{x^2}{2} - 2}$, $x \in [2, 2\sqrt{2}]$ since $2\sqrt{\frac{x^2}{2} - 2} \leq x$

$V(x) = x^3 - 4\left(\frac{x^2}{2} - 2\right)^{3/2}$, $V'(x) = 3x^2 - 6x\sqrt{\frac{x^2}{2} - 2}$

$3x^2 - 6x\sqrt{\frac{x^2}{2} - 2} = 0$

$9x^4 = 36x^2\left(\frac{x^2}{2} - 2\right) \rightarrow 9x^4 - 18x^4 + 72x^2 = 0$

$72x^2 - 9x^4 = 0$

$x = 2\sqrt{2}$

$V(2\sqrt{2}) = (2\sqrt{2})^3 - 4\cdot\left(\frac{(2\sqrt{2})^2}{2} - 2\right)^{3/2}$

$= 11.43$

Sphere

$C(r) = 2\pi r^2 = 12 \rightarrow r = \sqrt{\frac{6}{\pi}}$, $V\left(\sqrt{\frac{6}{\pi}}\right) \approx 11{,}056$

Cylinder

$C(r,h) = \pi r \times (r+h) = 12$, $h = \frac{12}{\pi r} - r$

$V(r) = \pi r^2 \cdot \left(\frac{12}{\pi r} - r\right) = 12r - \pi r^3$

$V'(r) = 12 - 3\pi r^2 = 0$

$r = \sqrt{\frac{4}{\pi}} \approx 1.128$, $h \approx 2.691$, $V = 9.027\text{cm}^3$