SVD

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Finding V and Σ

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T$$

$$A^T A = V\Sigma^T U^T U\Sigma V^T$$

$$A^T A = V\Sigma^2 V^T$$

$$A^T A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$$\begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = 0$$

$$(25-\lambda)(25-\lambda) - 225 = 0$$

$$\lambda^2 - 50\lambda + 400 = 0$$

$$(\lambda - 10)(\lambda - 40) = 0$$

$$\lambda_1 = 40,\ \lambda_2 = 10$$

① $\lambda_1 = 40$

$$(A^T A - 40I)V_1 = 0$$

$$\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{unit vector}} V_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

② $\lambda_2 = 10$

$$(A^T A - 10I)V_2 = 0$$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{unit vector}} V_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

Finding U

$$Av_1 = \sigma_1 u_1$$

$$u_1 = \frac{1}{\sqrt{40}}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{40}}\begin{bmatrix} -4/\sqrt{2} \\ -8/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$Av_2 = \sigma_2 u_2$$

$$u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 4/\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \sqrt{2}\sqrt{5}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix} -2 & 2 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = A \checkmark$$