CS559-B Homework 2

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Problem 1 (30pt): [Perceptron Algorithm]

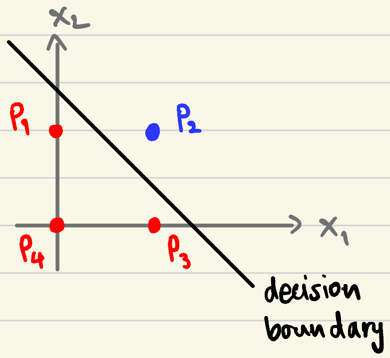
In this problem, we learn the linear discriminant function for boolean NAND function. Suppose we have two dimensional $x = (x_1, x_2)$, x_1 and x_2 can be either 0 (false) or 1 (true). The boolean NAND function is defined as: $f(x_1, x_2) = x_1$ **NAND** x_2. Specifically, $f(0,0) =$ **true**, $f(1,0) =$ **true**, $f(0,1) =$ **true**, and $f(1,1) =$ **false** where **false** can be treated as *positive class* and **true** can be treated as *negative class*. You can think of this function as having 4 points on the 2D plane (x_1 being the horizontal axis and x_2 being the vertical axis): $P_1 = (0,1), P_2 = (1,1), P_3 = (1,0), P_4 = (0,0)$, P_2 in *positive class* and P_1, P_3, P_4 in *negative class*.

(1) [5pt] For boolean NAND function, is the negative class and positive class linearly separable?

(2) [25pt] Is it possible to apply the **perceptron algorithm** to obtain the linear decision boundary that correctly classify both the positive and negative classes? If so, write down the updation steps and the obtained linear decision boundary. (You may assume the initial decision boundary is $x_1 + x_2 - \frac{1}{2} = 0$, and sweep the 4 points in clockwise order, i.e., $(P_1, P_2, P_3, P_4, P_1, P_2, \dots)$, note that you **can not** write down the arbitrary linear boundary without updation steps.)

Solution

(1)



From these four points in the question, I can plot them on the graph as shown at the left side. We can see that we can draw a linear decision boundary that completely separates the two classes.

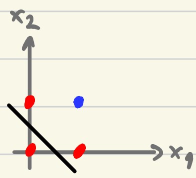
Therefore, for boolean NAND function, the negative class and positive class are linearly separable.

(2) Yes, it is possible to apply the perceptron algorithm to obtain the linear decision boundary that correctly classify both the positive and negative classes because if this algorithm incorrectly predicts the classes, it updates the weights and bias. It will repeatly perform its operation until the two classes are classified.

Updation steps

Step 1 : Initialize w (weight) and b (bias)

Let $\eta = 1, W_1 = 1, W_2 = 1, b = -\frac{1}{2} \rightarrow$



Step 2 : Loop over the training data $(P_1, P_2, P_3, P_4, P_1, P_2, P_3, P_4, \dots)$

From, $\hat{y} = \begin{cases} 1, & \text{if } w^Tx + b > 0 \\ -1, & \text{if } w^Tx + b < 0 \end{cases}$ and $\hat{y}_i = \text{sign}(w^Tx + b)$

($\hat{y}$: predicted value)

(2) continue..

if $t_i(w^T x_i) > 0 \rightsquigarrow$ correctly classified

$t_i(w^T x_i) \le 0 \rightarrow$ misclassified $\rightarrow$ mistake on positive $\rightarrow w' = w + x$, $b' = b + \eta$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\, \hookrightarrow$ mistake on negative $\rightarrow w' = w - x$, $b = b - \eta$

the first point $\longrightarrow P_1(0,1)$: $w = \begin{bmatrix}1\\1\end{bmatrix}$, $b = -\frac{1}{2}$, $t_1 = -1$

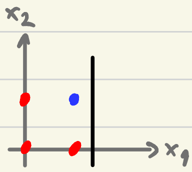
Step 3 : Compute the predicted output ($\hat{y}$)

$$\hat{y}_1 = w^T x + b = (-1)\left(\begin{bmatrix}1\\1\end{bmatrix}[0\ 1] - \frac{1}{2}\right) = (-1)\left(\frac{1}{2}\right) = -\frac{1}{2} < 0 \text{ (misclassified)}$$

Step 4 : update the weight and bias

$w' = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \rightarrow$

$b' = -\frac{1}{2} - 1 = -\frac{3}{2}$

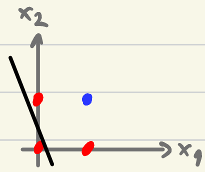


Step 5 : Repeat step 2 – 4 until the two classes are correctly classified

$P_2(1,1)$: $w = \begin{bmatrix}1\\0\end{bmatrix}$, $b = -\frac{3}{2}$ $\rightarrow$ $\hat{y}_2 = (1)\left(\begin{bmatrix}1\\0\end{bmatrix}[1\ 1] - \frac{3}{2}\right)$ $\longrightarrow$ $w' = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} \rightarrow$

$t_2 = 1$ $\quad\quad = (1)\left(-\frac{1}{2}\right)$ $\quad\quad b' = -\frac{3}{2} + 1 = -\frac{1}{2}$

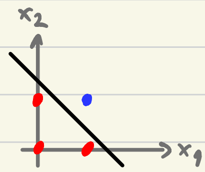
$\quad\quad = -\frac{1}{2} < 0$ (misclassified)



$P_3(1,0)$: $w = \begin{bmatrix}2\\1\end{bmatrix}$, $b = -\frac{1}{2}$ $\longrightarrow$ $\hat{y}_3 = (-1)\left(\begin{bmatrix}2\\1\end{bmatrix}[1\ 0] - \frac{1}{2}\right)$ $\longrightarrow$ $w' = \begin{bmatrix}2\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \rightarrow$

$t_3 = -1$ $\quad\quad = (-1)\left(\frac{3}{2}\right)$ $\quad\quad b' = -\frac{1}{2} - 1 = -\frac{3}{2}$

$\quad\quad = -\frac{3}{2} < 0$ (misclassified)



$P_4(0,0)$: $w = \begin{bmatrix}1\\1\end{bmatrix}$, $b = -\frac{3}{2}$ $\rightarrow$ $\hat{y}_4 = (-1)\left(\begin{bmatrix}1\\1\end{bmatrix}[0\ 0] - \frac{3}{2}\right)$ $\longrightarrow$ Not update w and b

$t_4 = -1$ $\quad\quad = (-1)\left(-\frac{3}{2}\right)$

$\quad\quad = \frac{3}{2} > 0$ (correctly classified)