polyhedra
a visual approach

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a visual approach

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This volume is dedicated to
Dr. R. Buckminster Fuller
in recognition of his generous inspiration
and encouragement.

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Preface

Polyhedra have been a source of intellectual and aesthetic stimulation since ancient times. Although their practical uses may not always be apparent, their orderliness and their regularities provoke intuitive feelings of significance and seem to demand attention. Unfortunately, the study of polyhedra is usually regarded as a branch of geometry, and most books on the subject have been written for readers with mathematical backgrounds. Readers unsympathetic to such an abstract treatment of the subject are likely to find their interest in polyhedra frustrated. Therefore, as an introduction for the casual but curious reader, this book describes the figures, in simple visual terms, as a series of interrelated shapes. The regular and semiregular convex polyhedra are described and then used as bases for the development of several families of irregular figures, including geodesic polyhedra. The discussion also extends to some of the exciting configurations which can be produced by joining several polyhedra.

For the more mathematically inclined, Appendix 1 includes a discussion of sample solutions to problems in polyhedral geometry. The book also contains many tables of useful geometric data, including several pages of previously unpublished chord factors for geodesic polyhedra (Appendix 2). Since mathematicians and nonmathematicians alike will find models to be invaluable aids to their studies, several model-making techniques are described in Appendix 3.

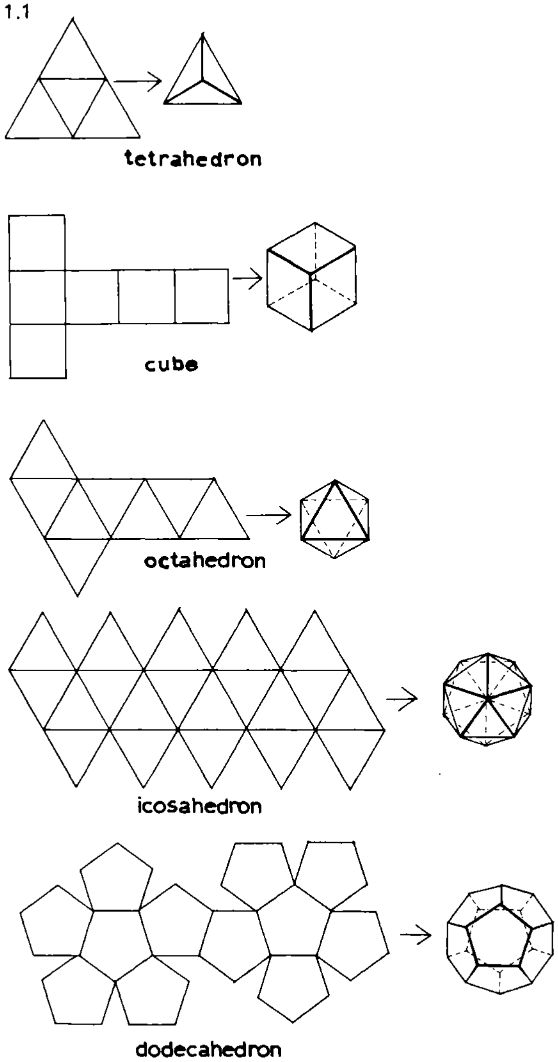
This volume should, therefore, be of interest to readers from many backgrounds. Mathematicians and scientists may find that a visual approach to the subject opens up a new range of possibilities. Artists, architects, designers, and engineers should find it a valuable exercise in three dimensions that will greatly increase their abilities in manipulating space and provide them with a rich vocabulary of three-dimensional forms. And, though no suggestions are made here for the uses of polyhedra, perhaps this book will provide inspiration and information for fertile minds. But it is not intended solely for those with a professional interest in polyhedra – many general readers should find it an uncomplicated introduction to a fascinating subject.

The author gratefully thanks the following for their help and encouragement: Dr. R. Buckminster Fuller; Mike Jerome; Professor A. Douglas Jones; Michael Burton; The Science Research Council of the United Kingdom; Hugh Kenner; Bill Perk and the faculty of the Department of Design, Southern Illinois University at Carbondale.

1. The Platonic Polyhedra

The Platonic polyhedra were not discovered by Plato, but they have been so named because of the studies he and his followers made of them. They are often called the regular polyhedra because they have a high degree of symmetry and order, but in this volume the expression *Platonic polyhedra* is used, to prevent confusion with other regular figures. Many of the polyhedra described in later chapters will be compared with the Platonic polyhedra, so it is important to become familiar with them before moving on to the more complex figures.

1.1



Construction of Models

The best way of becoming familiar with these figures is through building and studying models of them. The accompanying sketches (Diagram 1.1) show a two-dimensional arrangement of faces for each figure, called a net diagram. To construct the models, a similar set of net diagrams should be drawn on a suitable material (suggestions about how to draw the triangular, square, and pentagonal faces are given in Appendix 3). A conveniently-sized set of models will result if the edges of the faces are about 2 inches (5 centimeters) long. Each diagram should be cut out carefully and the edges between faces scored, to facilitate bending. The rest of the edges can then be joined to create the model, care being taken that the same number of faces meet at each vertex of that model. By an alternative method, the faces can be cut out individually and joined one to the other, till the models are complete. It is interesting to note that Albrecht Dürer gave net diagrams for several polyhedra in his book *Unterweisung der Messung Mit dem Zirkel und Richtscheit*, published as long ago as 1525.

Why There Are Five Platonic Polyhedra

It can be seen that each figure is convex, and each has an equal number of similar regular, convex faces meeting at each of its vertices. It is relatively simple to understand that there can be only five such figures, since at least three faces must meet at each vertex to create a three-dimensional form, and the sum of the face angles about that vertex must be less than 360° or else a flat or a concave surface will be developed.

The regular polygon with the fewest sides is the equilateral triangle. Three such faces can be made to meet at a vertex and then a fourth face added, so that three faces meet at each of the figure's four vertices (Diagram 1.2). This figure is called the TETRAHEDRON because it has four faces.

Four equilateral triangles can be made to meet at a vertex and additional faces added to create a figure with four equilateral triangles meeting at each of its six vertices (Diagram 1.3). This figure is called the OCTAHEDRON because it has eight faces.

A figure can be created with five equilateral triangles meeting at each of its twelve vertices (Diagram 1.4). It is called the ICOSAHEDRON because it has twenty faces.

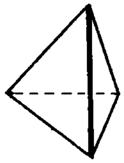
If six equilateral triangles meet at a vertex, as shown in Diagram 1.5, the sum of the face angles about that vertex will be 360°, and the triangles will either lie flat or create concave surfaces. So there can be only three Platonic polyhedra with faces which are equilateral triangles.

The next polygon to consider is the square. A polyhedron can be created with three squares meeting at each of its eight vertices (Diagram 1.6). It is another of the Platonic polyhedra. Popularly known as the CUBE, it is sometimes called a HEXAHEDRON because of its six faces.

A convex polyhedron cannot be formed with four squares meeting at each vertex, as the face angles about each vertex then add up to 360°, as shown in Diagram 1.7.

Then there is the regular pentagon, which has five equal edges and five equal internal angles of 108°. A polyhedron can be formed with three such pentagons meeting at each of its twenty vertices (Diagram 1.8). The resulting polyhedron is called the

1.2



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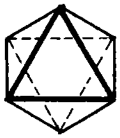

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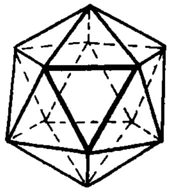

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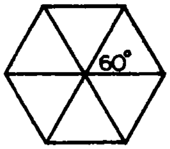

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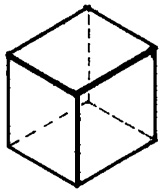

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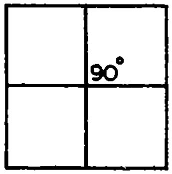

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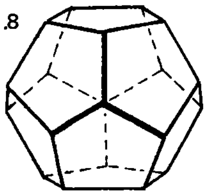
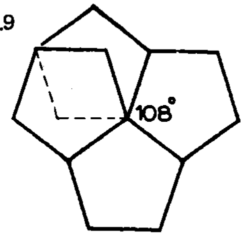


DODECAHEDRON because it has twelve faces. It is often referred to as the PENTAGONAL DODECAHEDRON, to avoid confusion with the rhombic dodecahedron, which is described in Chapter 4.

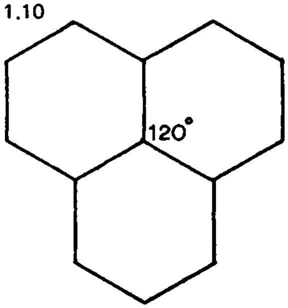
Four pentagons will not fit around a vertex to produce a convex polygon, as the face angles about each vertex add up to more than 360° (Diagram 1.9).

1.9



The next regular convex polygon to consider is the hexagon, but if three hexagons meet at each vertex, as shown in Diagram 1.10, the face angles, which add up to 360°, produce a flat surface. The larger regular polygons have larger face angles, so they cannot be joined to produce regular convex polyhedra.

1.10



Therefore, these five figures are the only Platonic polyhedra: the tetrahedron, the octahedron, the icosahedron, the cube, and the pentagonal dodecahedron.

Some General Characteristics of the Platonic Polyhedra

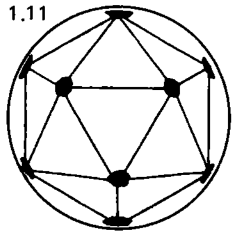
All the faces of the Platonic polyhedra are nonintersecting regular, plane, convex polygons with straight sides. This may seem a complex way of describing the simple triangular, square, and pentagonal faces, but if any of these conditions were relaxed, other polyhedra could be included. In addition, each figure has but one type of face, and the same number of those faces meet at each of its vertices. With the same number of faces, all alike, meeting at each vertex, the sum of the face angles about each vertex of a particular figure will be the same.

Any polyhedron whose faces form "indentations" on its surface is known as a concave polyhedron. Any polyhedron without such indentations is known as a convex polyhedron. It is apparent that all of the Platonic polyhedra are convex. It can also be seen that all pairs of adjacent faces of a particular figure meet at a constant angle. That angle, called the DIHEDRAL ANGLE, is measured on the inside of the figure and is always less than 180° in a convex polyhedron. The fact that the dihedral angle is constant is an important characteristic of a regular polyhedron.

Each Platonic polyhedron can be surrounded by a sphere of an appropriate size so that all its vertices touch that sphere

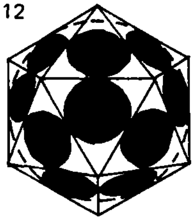
(Diagram 1.11). This sphere is most easily visualized around a model of an icosahedron, since that polyhedron is the most spherical looking of the five polyhedra. A sphere which circumscribes a polygon in this way is referred to as its CIRCUMSCRIBING SPHERE, or CIRCUMSPHERE.

1.11



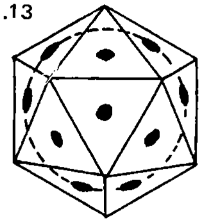
Since the vertices of a Platonic polyhedron touch a circumsphere, its edges will be chords to that sphere. Since the chords are equal in length, their midpoints are a constant distance from the center, so a slightly smaller sphere, called an INTERSPHERE (or MIDSPHERE), can be constructed which touches the midpoint of each edge (Diagram 1.12).

1.12



The face centers of each polyhedron are a constant distance from its vertices and a constant distance from the center of the polyhedron, so a third, smaller sphere, called an INSPHERE, can touch every face (Diagram 1.13). So each Platonic polyhedron has a circumsphere which touches all of its vertices, an intersphere which touches all of its edges, and an insphere which touches all of its faces. Only regular polyhedra have all three spheres.

1.13



The Sum of the Face Angles of a Polyhedron

If all the face angles of any polyhedron with plane faces are added together, it will be found that the sum of those angles is always

$$360° \times V - 720°,$$

where V is the number of vertices. The discovery of this relationship has been attributed to René Descartes (1596-1650) and it holds true for large complex figures as well as smaller ones such as the Platonic polyhedra. For example, the tetrahedron has four vertices, so the sum of its face angles should be

$$360° \times 4 - 720° \ (= 720°),$$

according to the formula. The tetrahedron has four triangular faces, each of which has internal angles which add up to 180°. The sum of all those angles is 720°, as predicted by the formula.

In case it is thought that the formula only applies to a polyhedron with triangular faces, it can be tested against the cube. The cube has eight vertices and, according to the formula, the sum of its face angles should be

$$360° \times 8 - 720° \ (= 2160°).$$

The internal angles of each square face added up to 360°, and, as the cube has six such faces, the total sum of all the face angles is 360° × 6, or 2160°, as predicted.

Though only two examples have been given, the formula works for all polyhedra with plane faces, including the other Platonic polyhedra. It is interesting to note that the constant factor in the equation (720°) is the sum of the face angles of a tetrahedron; the constant terms in many equations for polyhedra relate to the tetrahedron in some way or other. It should also be apparent that the relative size of the 720° will become less and less, in comparison to the value of

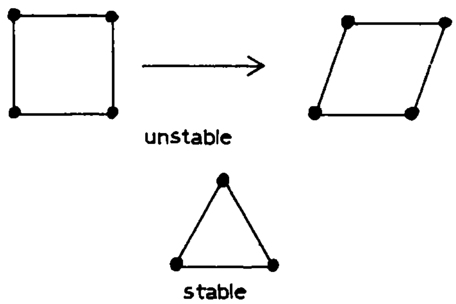
$$360° \times V,$$

as the value of V increases. This implies that generally the more vertices a figure has, the closer the sum of the angles about each vertex comes to 360°, and hence the flatter the arrangements of faces about those vertices are.

Triangulation and the Stability of Polyhedra

If a square or a larger polygon is made from a series of struts which define its edges, and if those struts are connected by flexible joints which allow them to move in relationship to one another, the figure can be distorted as shown in Diagram 1.14. Such distortion can be prevented by using a nonflexible connector between the struts. On the other hand, a triangle made with struts and flexible connections cannot be deformed in this way. So, under those conditions the triangle is a stable configuration, but the other polygons are not.

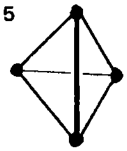
1.14



Polyhedra can be built as frameworks of struts, each strut representing an edge. If the struts are joined with connectors which allow them to move in relationship to one another, some of the frameworks will be stable but others can be deformed. If a framework with flexible joints is to be stable, it must have at least $3J-6$ struts, where J is the number of connectors. This simple expression can be used to predict whether such a framework will be stable; or, if a framework is not stable, the same expression can be used to find out how many struts must be added to brace the figure and make it stable.

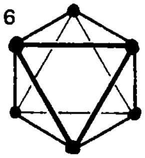
A framework defining the edges of a TETRAHEDRON has a connector at each of its 4 vertices (Diagram 1.15). Substituting that number for J, the figure must have at least 6 struts if it is to be stable. The tetrahedron has 6 edges and hence 6 struts and is stable. If one of the struts were removed, the figure would only have 5 struts and would be unstable.

1.15



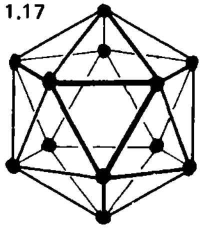
A framework defining the edges of an OCTAHEDRON has a connector at each of its 6 vertices (Diagram 1.16). Substituting the number 6 for J in the expression $3J-6$, this figure would need 12 struts to be stable. Since the octahedron has 12 edges, it has 12 struts and is stable. As before, if one of the struts were removed, the figure would no longer be stable.

1.16



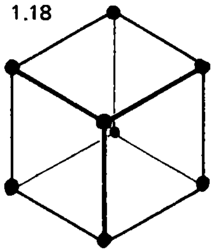
An ICOSAHEDRON has 12 vertices and hence 12 connectors (Diagram 1.17). Substituting that number for J, the figure appears to need 30 struts to be stable. The figure has 30 edges, hence 30 struts, and is stable. Removal of any one of the 30 struts would make the figure locally unstable.

1.17



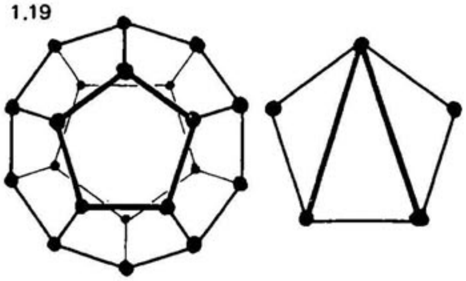
A CUBE will have a connector at each of its 8 vertices, and if that number is substituted for J, it would appear to need 18 struts to be a stable framework. But the cube has 12 edges and hence only 12 struts. If it is made with flexible connectors, it will collapse. If a diagonal is added to each of its six faces, the figure will have 18 struts and be stable. If desired, the 6 diagonals could be arranged to define the 6 edges of a regular tetrahedron as shown in Diagram 1.18.

1.18



As a PENTAGONAL DODECAHEDRON has 20 vertices, it has 20 connectors. Substituting that value for J, it would appear that such a framework would require 54 struts to be stable. But the figure has only 30 edges, and those 30 struts are not enough to make it stable. But if 2 struts are added to each of its 12 pentagonal faces, as shown in Diagram 1.19, the framework will be stable.

1.19

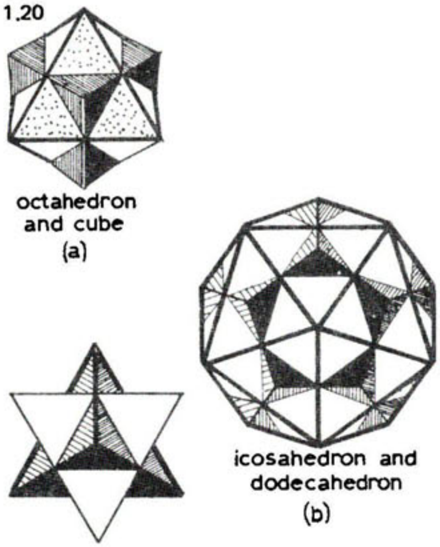


Several things can be learned from the preceding examples. First, to be stable a framework must be triangulated (that is, composed of triangles). The tetrahedron, octahedron, and icosahedron were already triangulated, but the cube and the dodecahedron needed extra struts, added till all their faces were triangulated, before they became stable. If just one strut were to be removed from any of the figures, a nontriangular face would result, and the figure would become unstable about that face. So the triangle is the basis for stability in a framework which has nonrigid connections. Second, the expression $3J-6$ can be used to determine whether a framework with flexible joints will be stable or, if it is not stable, how many struts should be added to make it stable. Finally, the constant factor (6) in the expression $3J-6$ happens to be the number of edges of a tetrahedron.

Duality and the Platonic Polyhedra

The OCTAHEDRON and the CUBE have the same number of edges (12), and each figure has the same number of faces as the other has vertices. A suitably-sized model of each of them can be arranged about a common intersphere so that their edges cross at right angles and touch that sphere at those 12 common points, as shown in Diagram 1.20a. The vertices of each figure lie exactly outside the face centers of the other figure. A similar arrangement can be made with a suitable-sized ICOSAHEDRON and PENTAGONAL DODECAHEDRON, since each of those figures has 30 edges and the same number of faces as the other has vertices (Diagram 1.20b). The remaining Platonic polyhedron, the TETRAHEDRON, has the same number of faces as it has vertices, so two tetrahedra of equal size can be arranged in a similar way to the other pairs of figures (Diagram 1.20c). The 8 vertices of the two tetrahedra define the vertices of a regular cube.

1.20



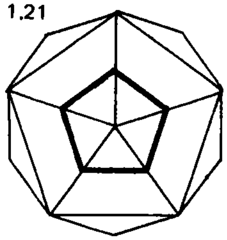
octahedron and cube
(a)

icosahedron and dodecahedron
(b)

two tetrahedra
(c)

This phenomenon is known as duality, and each figure is the dual of the other figure that shares the same intersphere in this way. These relationships of the five Platonic polyhedra were known in ancient Greece, but the full implications of the idea of duality were not appreciated till comparatively recently. The fact that the dual of one regular polyhedron is another regular polyhedron is an important characteristic of a regular polyhedron. It can be attributed to the fact that their vertex figures are sets of congruent regular polygons. (The vertex figure of a regular polyhedron is the polygon formed by joining the midpoints of the edges which meet at a common vertex.) Thus the vertex figure of the icosahedron is a regular pentagon, as shown in Diagram 1.21.

1.21



Introduction to the Golden Proportion

The Fibonacci Series of numbers (1, 1, 2, 3, 5, 8, 13, 21 . . . .) can be produced by adding the last two numbers in the series to give the next one in the series. The next number in the sequence given would be 34 (= 13 + 21), and the subsequent one would be 55 (= 21 + 34). A property of this series is that, if two adjacent numbers are expressed as a ratio (such as 8:13, 13:21, 21:34), the larger the numbers are, the closer the ratio comes to a value of

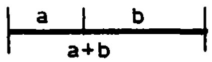
$$1 : \frac{\sqrt{5}+1}{2}$$

(or 1 : 1.6180 to four decimal places). This proportion, known as the Golden Proportion, has often been used as a basis for harmonious relationships in artistic and architectural compositions. In 1509 Luca Pacioli, assisted by his friend Leonardo da Vinci, published a book on the subject entitled *De Divina Proportione*. The Golden Proportion is the basis for many interesting relationships, and it is hardly surprising that the great Renaissance artists and thinkers considered it to be so important.

If a line is divided into two parts so that one part is 1.6180 times longer than the other part, it will be found that the overall length of the two parts is 1.6180 times the length of the longer part (Diagram 1.22). That is,

$$a : b = 1 : 1.6180 = b : a + b.$$

1.22



A rectangle whose longer sides are 1.6180 times the lengths of its shorter sides is often called a GOLDEN RECTANGLE (Diagram 1.23a). A Golden Rectangle can be drawn by constructing a square *ABCD* as shown in Diagram 1.23b and marking the midpoint *E* of the edge *BC*. Centering a pair of compasses at *E*, draw an arc from *D* which intersects *BC* produced at *F*. The fourth vertex of this Golden Rectangle can then be established by extending *AD* and constructing *FG* at right angles to *BF*. If a square is divided from such a rectangle, the sides of the remaining rectangle will be in the Golden Proportion to each other. This process can be repeated to produce a series of smaller and smaller Golden Rectangles (Diagram 1.23c). A spiral can be drawn to pass through the opposite corners of those Golden Rectangles, as in Diagram 1.23d.

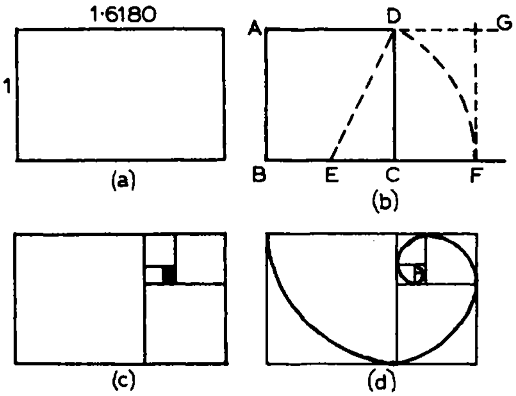
Isosceles triangles whose internal angles are 36°, 72°, 72° or 108°, 36°, 36° have edges whose lengths are in the Golden Proportion (Diagram 1.24). It can be seen that, although the Golden Proportion is an irrational number (1 : 1.6180 . . .), the sizes of the internal angles of those triangles are in simple whole-number relationships to one another.

Since the diagonal of a regular pentagon divides off a 108°, 36°, 36° triangle, the diagonal is 1.618 times the edge length of that pentagon (Diagram 1.25a). This information can be used to construct a regular pentagon, as shown in Diagram 1.25b. First a 36°, 72°, 72° isosceles is constructed with its short side equal to the edge length of the desired pentagon. With the compasses set to the edge length of the pentagon, intersecting arcs are drawn from the vertices of that triangle to define the two final vertices of the pentagon.

As described in Appendix 3, a regular decagon can be constructed inside the circle whose radius is 1.618 times the edge length of that decagon (Diagram 1.26). A regular decagon can be subdivided into ten equal 36°, 72°, 72° triangles whose longer edges are 1.618 times the length of the shorter edges, as shown.

The ten outside edges of a pentagram (a five-pointed star produced by extending the edges of a regular pentagon) are 1.618

1.23



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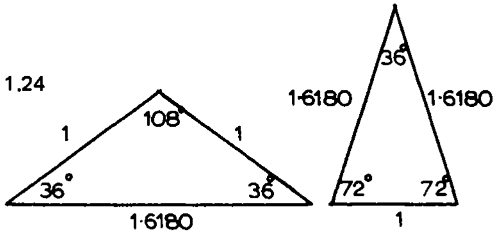

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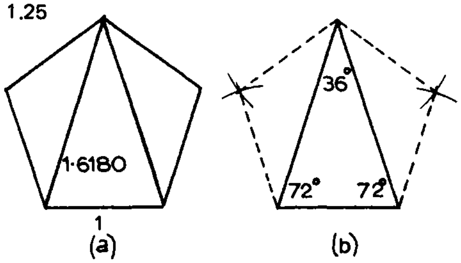

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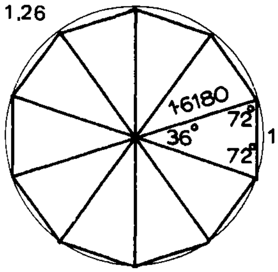
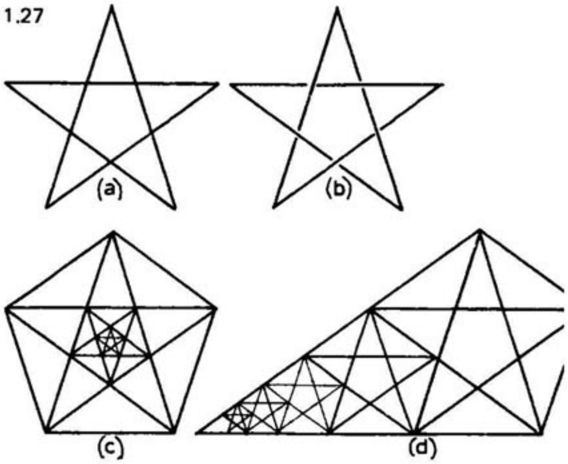


times the length of the edges of the original pentagon (Diagram 1.27a). This figure has often been used as a mystical symbol, since its edges can be followed as a continuous circuit as indicated in Diagram 1.27b. Several compositions can be made with pentagons and pentagrams such as Diagram 1.27c and d.

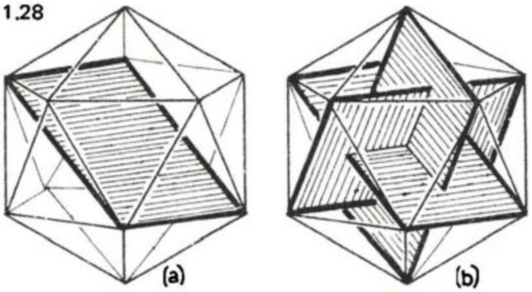
1.27



The Golden Proportion and the Platonic Polyhedra

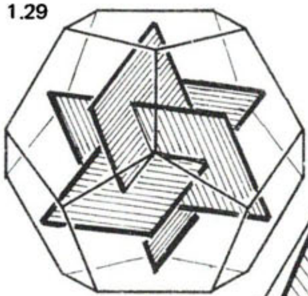
Though the preceding examples are two-dimensional, there are some important three-dimensional relationships involving the Golden Proportion and the Platonic polyhedra. Diagram 1.28a shows a Golden Rectangle arranged inside an icosahedron, the shorter edges following opposite edges of the icosahedron and the longer edges forming diagonals to pentagonal circuits of edges. Three such Golden Rectangles, intersecting at right angles, can be constructed inside the icosahedron as in Diagram 1.28b. An even more complex figure can be constructed with fifteen intersecting Golden Rectangles, each edge of the icosahedron being defined by an edge of a Golden Rectangle.

1.28



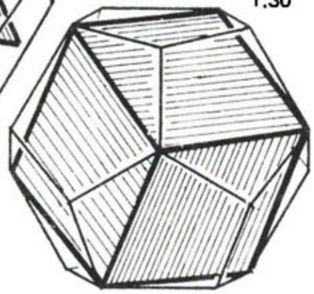
Since the pentagonal dodecahedron is the dual of the icosahedron, similar arrangements of Golden Rectangles can be made within it. In this case (Diagram 1.29) the corners of the rectangles touch the midpoints of faces of the dodecahedron instead of the vertices, as was the case with the icosahedron. The diagram shows three Golden Rectangles, intersecting at right angles, whose twelve corners touch the midpoints of the twelve faces of the dodecahedron.

1.29



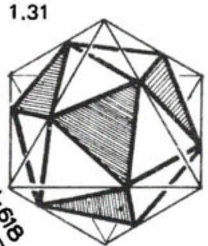
A diagonal can be drawn across each pentagonal face of a dodecahedron to define the twelve edges of a regular cube (Diagram 1.30). Clearly, the edges of the cube will be 1.618 times the length of the edges of the dodecahedron.

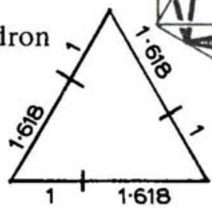
1.30



Perhaps the most unexpected relationship is found by dividing each edge of a regular octahedron in the Golden Proportion, as in Diagram 1.31, left. If those points are joined, a regular icosahedron is described, as shown in Diagram 1.31, right. Note also that the cube and the dodecahedron in Diagram 1.30 share common vertices and that their duals, the octahedron and the icosahedron shown in Diagram 1.31, share common facial planes.

1.31





Sketching the Platonic Polyhedra

It takes a long time to work out vanishing points and the variations of size with distance for perspective drawings of the Platonic Polyhedra. It is much quicker to draw a projection (similar to a plan) where all components remain a constant size regardless of their distance from the viewer. For example, the nearest and furthest faces of the face view of the octahedron in Diagram 1.32 are the same size. Had it been a perspective view, the further triangle would have been smaller than the nearer one. The advantage of projections like these is that they can be measured to establish dimensions. Drawing the Platonic Polyhedra in this way is a useful exercise, as it teaches much about their symmetries. Such drawings can be used as guides when drawing more complex figures. Diagram 1.32 shows each of the Platonic Polyhedra viewed from its face, edge, and vertex. It is worth studying, as it reveals many unexpected symmetries, such as the hexagonal outline of a cube viewed from a vertex and the square outline of a tetrahedron viewed from an edge.

Relationships Between the Five Platonic Polyhedra

Each Platonic polyhedron shares symmetries with the other four Platonic polyhedra and can be arranged inside each of the others in a symmetrical manner. Diagram 1.33 shows each figure arranged symmetrically inside the other figures, the outer figures being shown in outline and the inner ones as solids. (The edge lengths of the enclosed figures vary from example to example.) An understanding of these relationships can be very useful when dealing with more complex polyhedra.

Summary

1. Though they were discovered before his time, these five figures were named Platonic in recognition of the attention Plato and his followers paid them.

2. The Platonic polyhedra are also called regular polyhedra, though care should be taken in using this term, as there are other regular polyhedra.

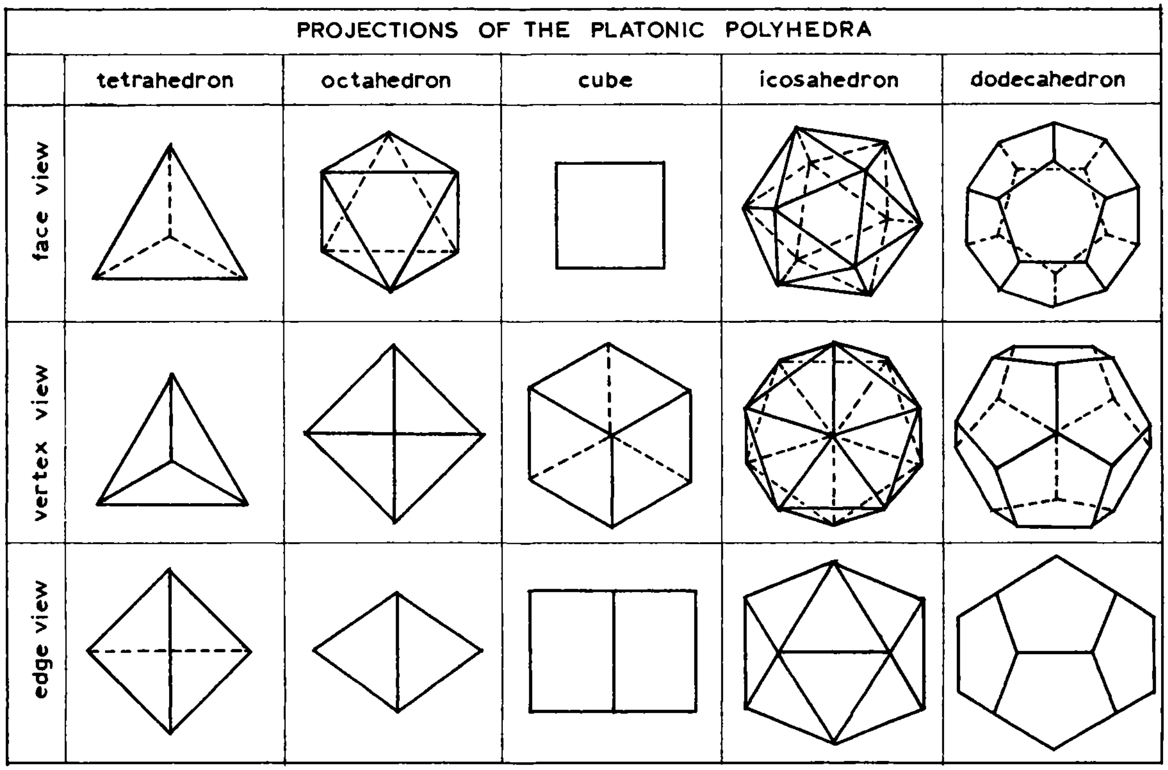
3. The Platonic polyhedra are convex polyhedra and have faces which are nonintersecting regular, plane, convex polygons with straight edges. An equal number of congruent faces meet at each vertex of a particular one of these figures.

4. Each figure has a circumsphere which touches all of its vertices, an intersphere which touches all of its edges, and an insphere which touches all of its faces.

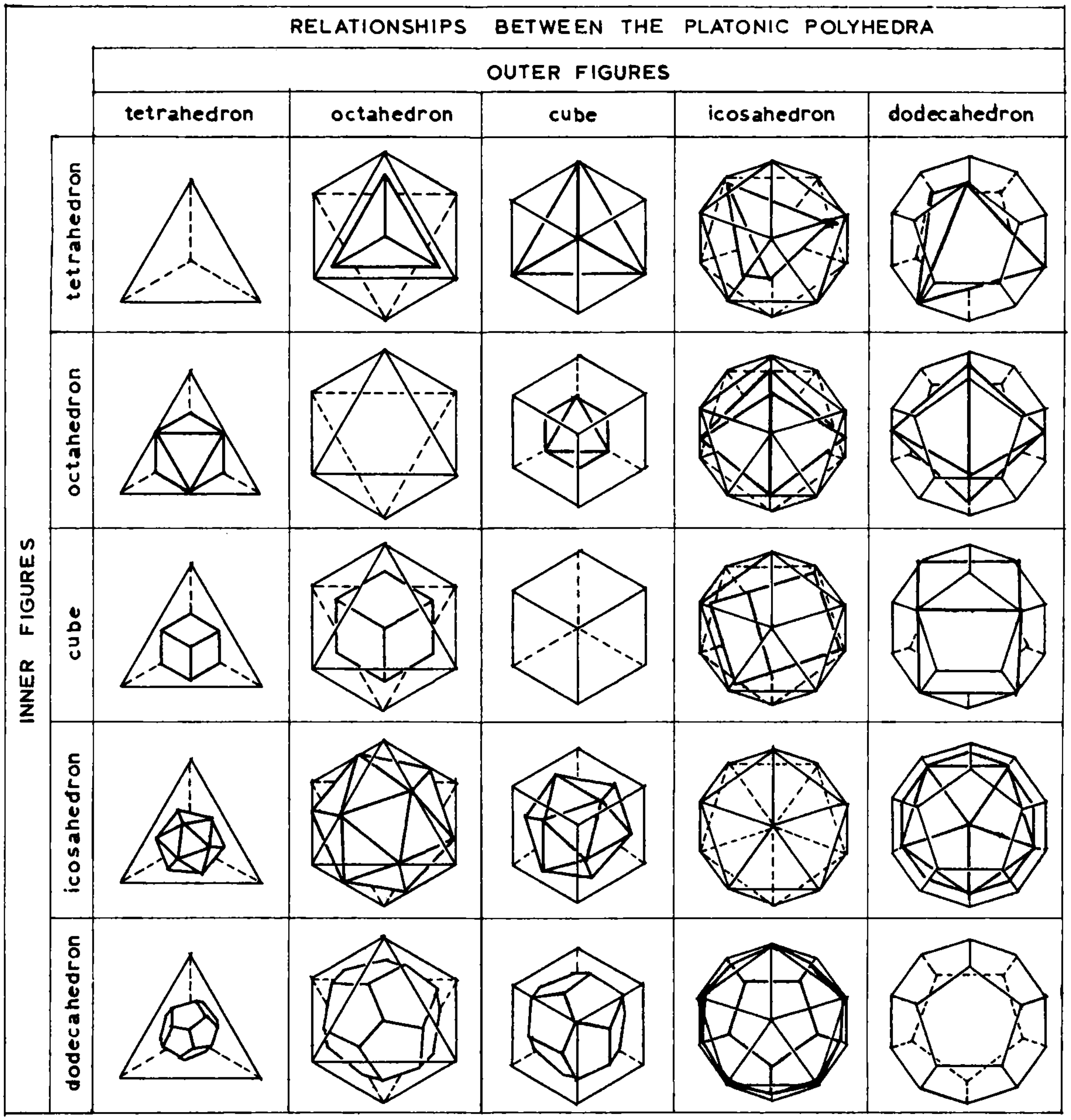
5. The dihedral angle is constant between all the adjacent faces of a particular one of these figures.

6. The vertex figures of each figure are congruent regular polygons; hence, the dual of a Platonic polyhedron is another Platonic polyhedron.

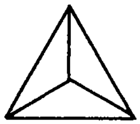
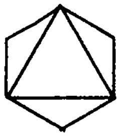
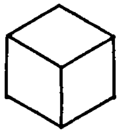
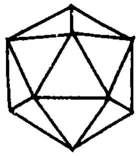
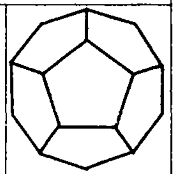
1.32



1.33


RELATIONSHIPS BETWEEN THE PLATONIC POLYHEDRA
OUTER FIGURES
tetrahedron
octahedron
cube
icosahedron
dodecahedron
INNER FIGURES
tetrahedron
octahedron
cube
icosahedron
dodecahedron

1.34

DATA FOR THE PLATONIC POLYHEDRA					
	 tetrahedron	 octahedron	 cube	 icosahedron	 pentagonal dodecahedron
faces	4 triangles	8 triangles	6 squares	20 triangles	12 pentagons
vertices	4	6	8	12	20
edges	6	12	12	30	30
radius of circumsphere	·6124	·7071	·8660	·9511	1·4013
radius of intersphere	·3536	·5000	·7071	·8090	1·3090
radius of insphere	·2041	·4082	·5000	·7558	1·1135
dihedral angle	70°32′	109°28′	90°0′	138°11′	116°34′

Radii of circumspheres, interspheres and inspheres are given (to four decimal places) in terms of the edge lengths of the polyhedra. Dihedral angles are given to the nearest minute.

2. The Archimedean Polyhedra, Facially Regular Prisms, and Facially Regular Antiprisms

The same number of identical regular, convex polygons meet at each vertex of a Platonic polyhedron. If the condition that all the faces should be identical is relaxed to allow similar arrangements of regular, convex polygons of two or more different kinds about each vertex of a polyhedron, the thirteen Archimedean polyhedra, plus an infinite number of facially regular prisms and antiprisms can be produced. The thirteen Archimedean polyhedra were discovered in ancient Greece and were described by Archimedes, but his writings on them were lost, together with the knowledge of the figures. During the Renaissance they were gradually rediscovered and were described by such people as Pierro della Francesca and Albrecht Dürer, though a description of all thirteen figures did not appear until Johannes Kepler's *Harmonices Mundi* was published in 1619.

Construction of Models of the Archimedean Polyhedra

As with the Platonic polyhedra, models of the Archimedean polyhedra are a great help in studying the figures. It is not necessary to build all thirteen; a selection of four or five will suffice for many people. The quickest way of building a model is to cut out the faces individually and join them together, one at a time, till the model is complete. The key to joining them correctly is to count out the right number of faces and to make sure that there is a similar arrangement of faces about each vertex of the model. The number of faces for each figure is given in Diagram 2.15. The arrangements of faces about the vertices are shown in

2.1

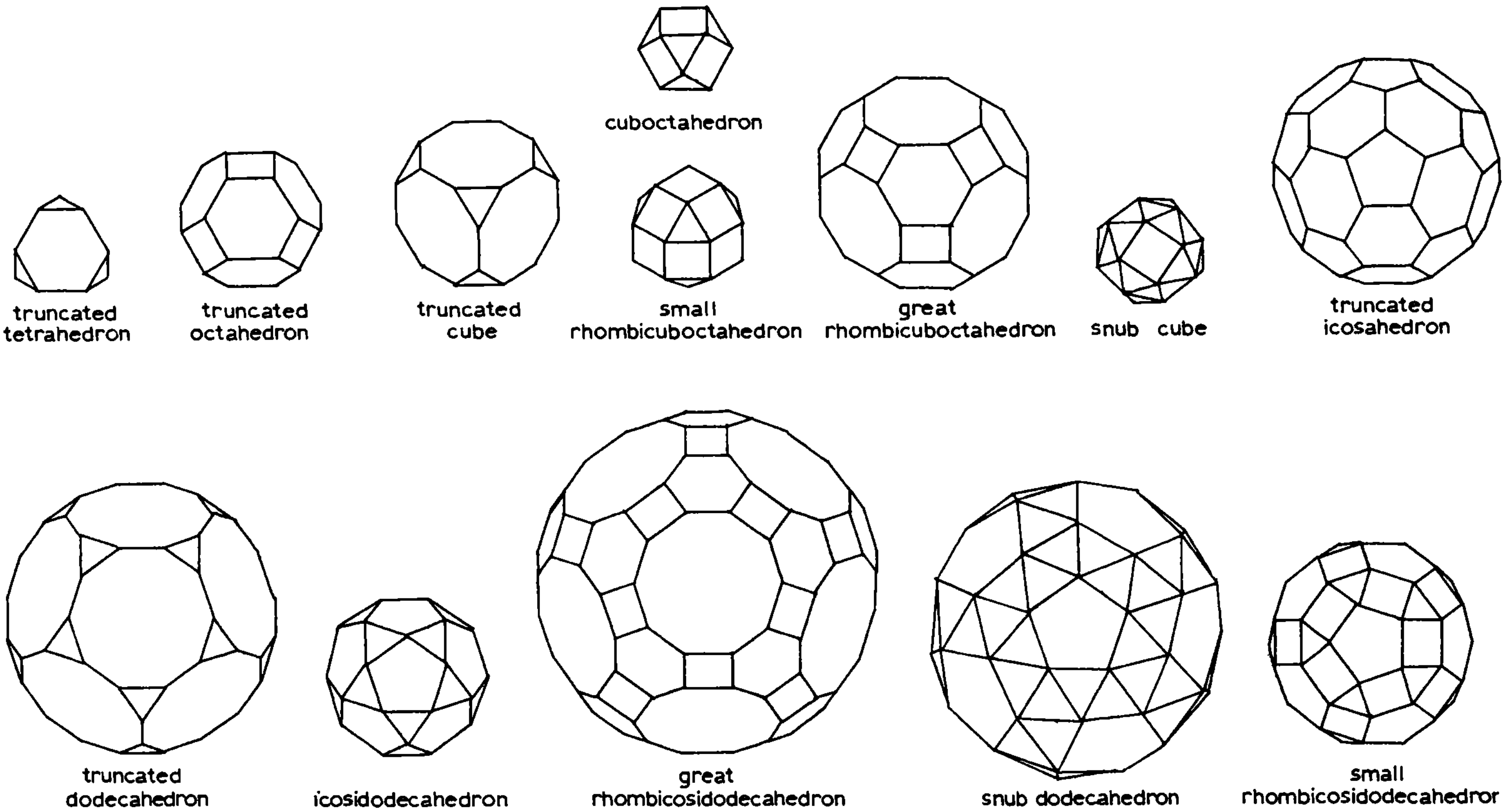


Diagram 2.1. Different colours can be used on different faces to produce very attractive models and to illustrate some of the symmetries of these polyhedra. A conveniently-sized set of models can be made if their edges are about one inch (2.5 centimeters) long. Net diagrams for all of these figures appear in such books as Cundy and Rollett, *Mathematical Models*; Critchlow, *Order in Space*; and Williams, *Natural Structure*, but they tend to take a long time to draw out and the resulting expanse of faces is not easy to handle when joining the edges.

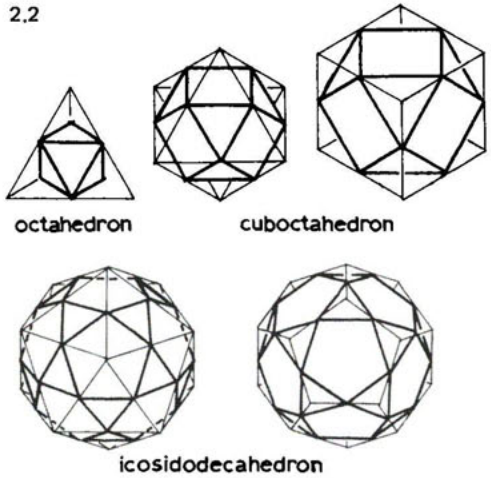
Some Relationships Between the Archimedean Polyhedra and the Platonic Polyhedra

If the midpoints of all edges of a regular tetrahedron are joined with a series of lines, it will be found that those lines define the edges of a regular octahedron, as shown in Diagram 2.2. If the midpoints of a cube or an octahedron are joined in a similar way, a cuboctahedron results (Diagram 2.2). If the midpoints of each edge of an icosahedron or a dodecahedron are joined in a similar way, an icosidodecahedron is defined (Diagram 2.2). Since the octahedron, cuboctahedron, and icosidodecahedron are produced by joining the midpoints of the edges of the Platonic polyhedra and since each midpoint divides that edge into two parts, these three figures are sometimes referred to as TWO-FREQUENCY FIGURES in this volume.

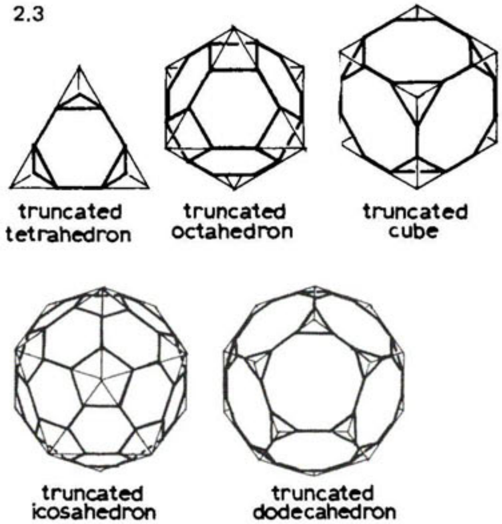
Instead of dividing each edge of the Platonic polyhedra into two parts, each edge could be divided into three parts and the points so established joined with a series of lines as before. The five figures produced by this technique are the truncated tetrahedron, the truncated octahedron, the truncated cube, the truncated icosahedron, and the truncated dodecahedron (Diagram 2.3). Since those figures are derived by dividing each edge of a Platonic polyhedron into three parts, they are referred to here as THREE-FREQUENCY FIGURES.

FOUR-FREQUENCY FIGURES can be produced by joining the midpoints of the edges of the two-frequency figures. Since the cuboctahedron and the icosidodecahedron have rectangular vertex figures, the figures defined originally have rectangular faces together with regular faces of different edge lengths. Those figures must be expanded slightly as shown in Diagram 2.4 to make the rectangles into squares and thus allow all faces to have edges of equal length. The dihedral angles do not change when the figure is expanded in this way.

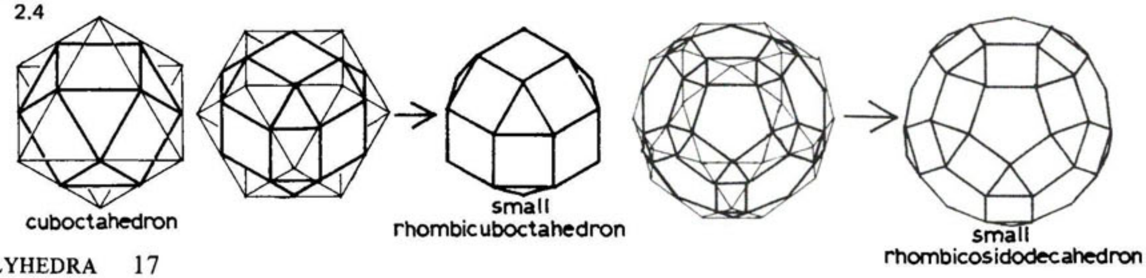
2.2



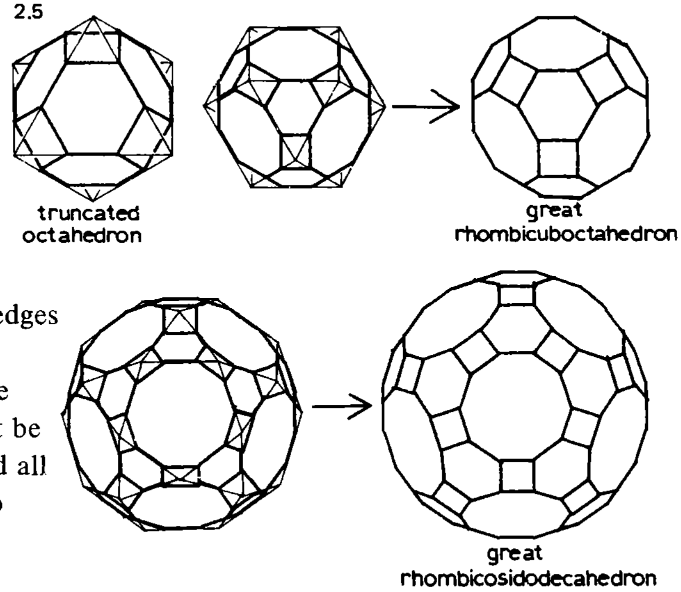
2.3



2.4



2.5

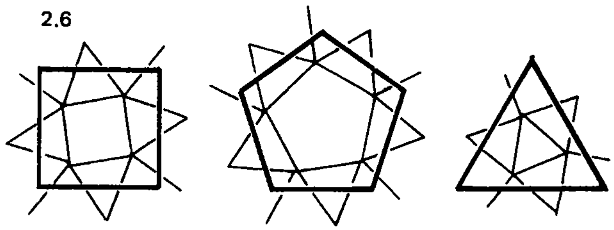


SIX-FREQUENCY FIGURES can be produced by dividing the edges of each of the two-frequency figures into three parts (Diagram 2.5). Since the figures derived from the cuboctahedron and the icosidodecahedron originally have rectangular faces, they must be expanded, as shown, so that the rectangles become squares and all faces become regular, convex polygons. The dihedral angles do not change when the figure is expanded in this way.

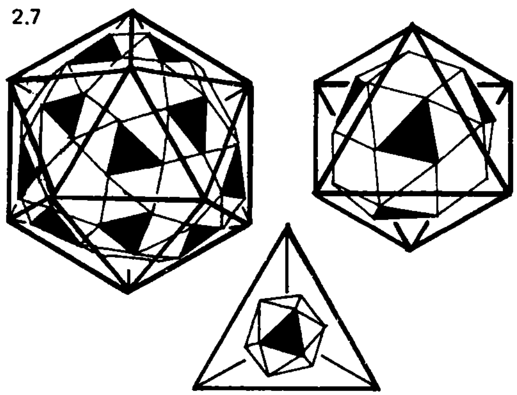
The preceding paragraphs have introduced all but two of the Archimedean polyhedra, the snub cube and the snub dodecahedron. At first it appears strange that there should be snub versions of only two of the five Platonic polyhedra, so it is worth looking at the relationships between the faces of the snub cube and the cube and between the faces of the snub dodecahedron and the dodecahedron by placing each snub figure inside the appropriate Platonic polyhedron (Diagram 2.6). The first two sketches illustrate these two relationships, from which the third sketch – the equivalent relationship for a figure with triangular faces – can be deduced.

As Diagram 2.7 shows, if this pattern is applied to an icosahedron, a snub dodecahedron is produced, and if it is applied to an octahedron, a snub cube results. The greatest surprise comes if the pattern is applied to the tetrahedron, as the icosahedron results. (Kepler noted that the icosahedron could be regarded as a snub tetrahedron.)

2.6



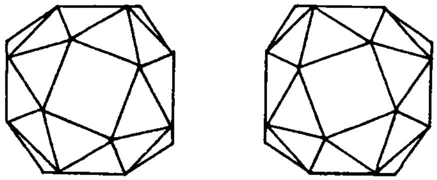
2.7



Two versions of the snub cube and the snub dodecahedron can be built, each the mirror image of the other, as shown by the sketches of the snub cube in Diagram 2.8. The one version cannot be rotated to make it coincide with the other; it will always be a left-handed or right-handed version of the figure. Such figures are called ENANTIOMORPHIC.

The relationships described so far are summarised in Diagram 2.9. The top section of the table shows some of the relationships between Platonic polyhedra, beneath which are listed the figures corresponding to the various frequencies of subdivision. Since the octahedron is a two-frequency tetrahedron, the four- and six-frequency figures derived from the octahedron are repeated as the eight- and twelve-frequency figures derived from the tetrahedron. This table allows some comparisons to be made between the Platonic and Archimedean polyhedra, and it will be referred to in later chapters.

2.8



2.9

SOME RELATIONSHIPS BETWEEN THE PLATONIC AND ARCHIMEDEAN POLYHEDRA					
	duals		self dual	duals	
FREQUENCY	CUBE	OCTAHEDRON	TETRAHEDRON	ICOSAHEDRON	DODECAHEDRON
2	cuboctahedron		octahedron	icosidodecahedron	
3	truncated cube	truncated octahedron	truncated tetrahedron	truncated icosahedron	truncated dodecahedron
4	small rhombicuboctahedron		cuboctahedron	small rhombicosidodecahedron	
6	great rhombicuboctahedron		truncated octahedron	great rhombicosidodecahedron	
8			small rhombicuboctahedron		
12			great rhombicuboctahedron		
SNUB FIGURES	snub cube		icosahedron	snub dodecahedron	

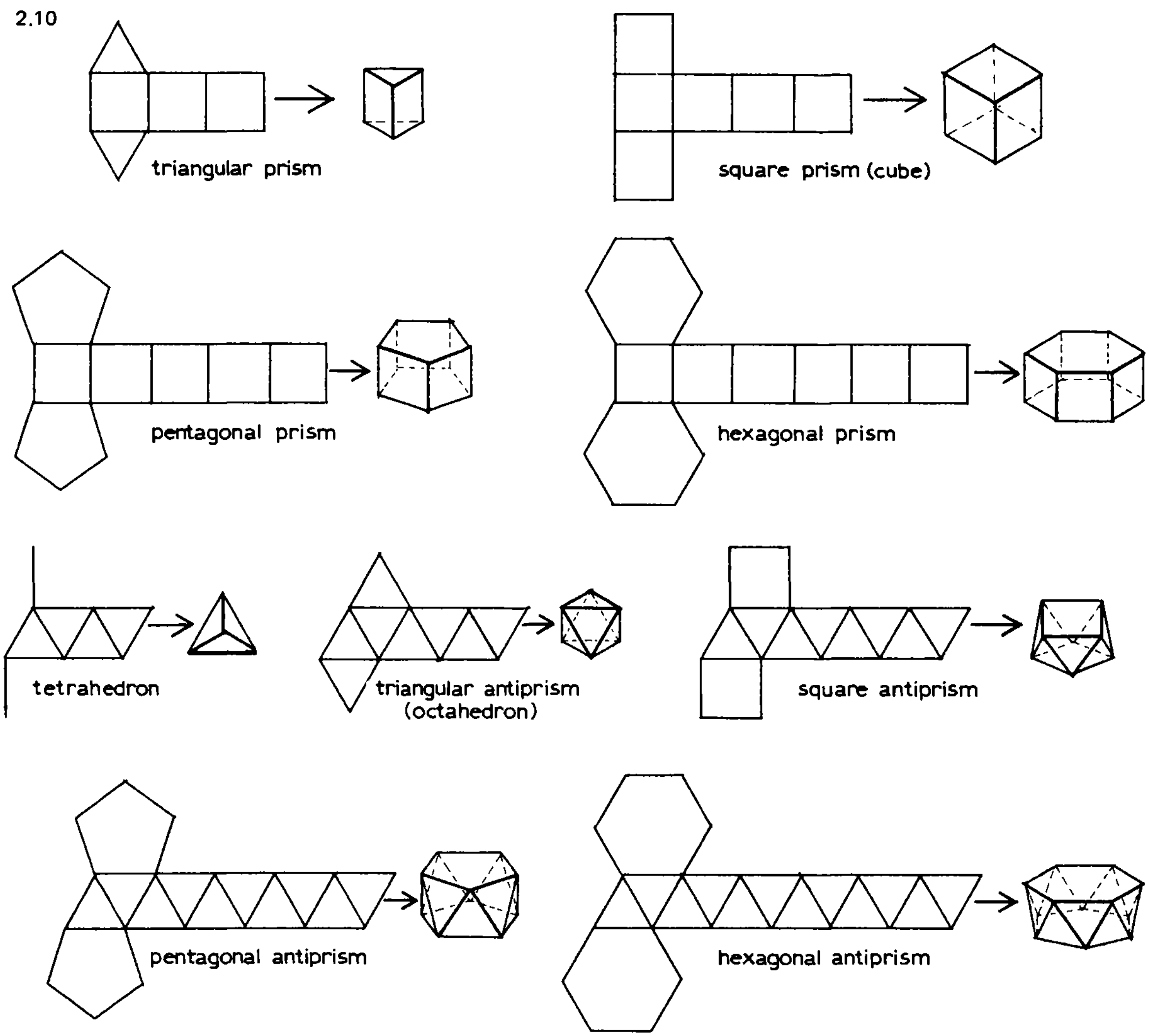
Names of the Archimedean Polyhedra

The names of the Archimedean polyhedra originated in the Latin names used by Kepler. The sheer length of many of them may deter some people, but they are very descriptive and can be easy to remember. The cuboctahedron can be seen to have obvious relationships to both the cube and the octahedron; the icosidodecahedron has similar relationships to both the icosahedron and the dodecahedron. The word *truncated* in such names as *truncated tetrahedron* clearly describes a figure, in this case a tetrahedron, which has had its extremities removed, or truncated. The great rhombicuboctahedron and great rhombicosidodecahedron are sometimes called the truncated cuboctahedron and the truncated icosidodecahedron, respectively. But the former names are used in this volume, as the author feels they give a better description of the figures (the *rhombi* part of the names is explained in Chapter 4). Clearly, the prefixes *great* and *small* differentiate between figures in terms of their sizes. The word *snub* comes from the Latin word *simus* used by Kepler, a literal translation being "snub-nosed," or "flat-nosed."

Description and Construction of Facially Regular Prisms and Facially Regular Antiprisms

A facially regular prism has a circuit of square faces and has opposite ends which are congruent regular polygons; a facially regular antiprism has a circuit of equilateral triangles and has opposite ends which are congruent regular polygons. Both can be seen in Diagram 2.10. Construction of models of these figures can be speeded up by cutting out the circuits of squares or triangles in single strips, instead of as individual polygons. The strips of triangles or squares can be extended and larger polygons fitted to the ends to make an infinite number of figures. These figures can have end faces which are regular heptagons (seven-sided polygons), enneagons (nine-sided polygons), and other polygons not found among the Platonic and Archimedean polyhedra. The CUBE, being the square prism, and the OCTAHEDRON, being the triangular antiprism, are included in Diagram 2.10. Also included is the

2.10

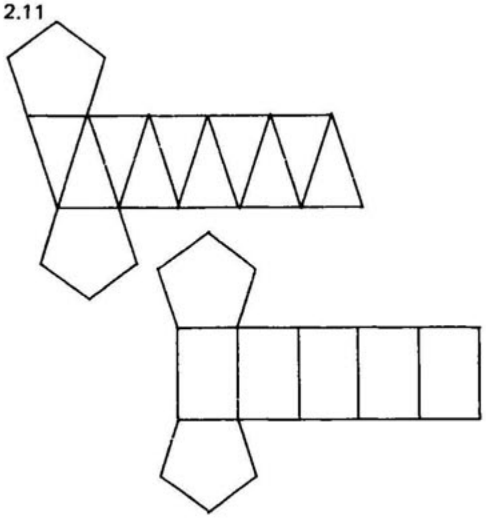


TETRAHEDRON, which can be regarded as an antiprism whose opposite "ends" are two-sided polygons (called digons).

In the diagram it can be seen that the end face of a facially regular prism lies directly over the other end face, whereas the end faces of an antiprism are twisted relative to one another. This characteristic is very useful when joining polyhedra, as is shown in Chapter 5. If a suitable prism is placed between the congruent faces of two polyhedra, it simply spaces them apart; if a suitable antiprism is placed between the congruent faces of two polyhedra, it spaces them apart and rotates the one relative to the other.

Circuits of isosceles triangles can be substituted for the equilateral triangles and circuits of rectangles for the squares, as in Diagram 2.11. Though the triangles and rectangles are not regular faces, a mathematician would regard them as regular prisms and regular antiprisms, so it is important to use the expression *facially regular* for prisms and antiprisms with regular faces.

2.11

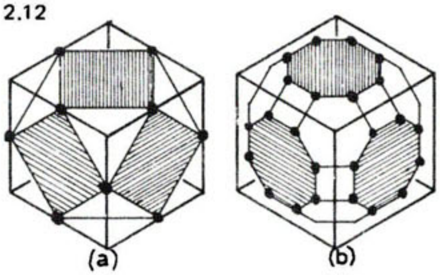


Some General Characteristics of the Archimedean Polyhedra and of the Facially Regular Prisms and Facially Regular Antiprisms

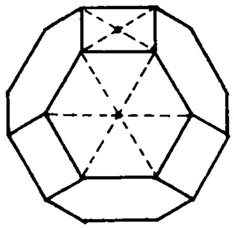
The faces of these figures, just like those of the Platonic polyhedra, are nonintersecting, regular, plane, convex polygons with straight edges. But, whereas all the faces of a Platonic polyhedron are identical, the faces of each of these figures are of two or more types. Since there is an identical arrangement of the faces about each vertex of a particular figure, the sum of the face angles about each of its vertices must be the same.

A Platonic polyhedron has a constant dihedral angle between its adjacent faces. The cuboctahedron, icosidodecahedron, square prism, and triangular antiprism each have constant dihedral angles, but the other figures have two or three different dihedral angles. For example, it is obvious from a model of a truncated cube that the dihedral angle between its adjacent octagonal faces is 90° but that the dihedral angle between any octagon and an adjacent triangle, though constant, is not 90°.

2.12



Each Archimedean polyhedron can be circumscribed by one of the Platonic polyhedra so that all of its vertices lie evenly arranged on the faces or edges of the circumscribing figure, as shown in the sketches of a cuboctahedron and a great rhombicuboctahedron circumscribed by cubes in Diagrams 2.12a and 2.12b. In each case the vertices of the inscribed figure are an equal distance from the face-centers of the circumscribing figure. Since the face-centers of each Platonic polyhedron are a constant distance from the center of the figure itself, the vertices of the inscribed figure must all be a constant distance from it, too. Hence, like a Platonic polyhedron, an Archimedean polyhedron has a circumsphere touching all of its vertices. Since the edges of an Archimedean polyhedron are equal in length, they are also equal chords to its circumsphere, and their midpoints are an equal distance from the center of that circumsphere. Hence, like a Platonic polyhedron, an Archimedean polyhedron can have an intersphere which touches all of its edges. But the face-centers of different types of face are not a constant distance from the vertices or edges of an Archimedean polyhedron, as can be seen with the square and hexagonal faces of the truncated octahedron (Diagram 2.12c). The face-centers of such a figure do not lie a constant distance from the center of the figure, so, unlike with a Platonic polyhedron, no insphere can be constructed to touch every face of an Archimedean polyhedron.


(c)

The vertices of a facially regular prism or antiprism lie on the planes of its two end polygons. They are clearly an equal distance from the center of the polyhedron, so a circumsphere can be arranged to touch all of its vertices, just as with an Archimedean polyhedron. Like the edges of an Archimedean polyhedron, the edges of a facially regular prism or antiprism are equal chords to that circumsphere, so an intersphere can be arranged to touch all of its edges. If the larger facially regular prisms and antiprisms are inspected, it will be obvious that their side faces and end faces cannot touch the surface of a common insphere. The only facially regular prisms and antiprisms which do have inspheres are the three which are Platonic polyhedra – the tetrahedron, the octahedron, and the cube.

The Sum of the Surface Angles

Chapter 1 stated that the sum of the face angles of any convex polyhedron with plane faces is always

$$360^\circ \times V - 720^\circ$$

when V is the number of its vertices. It is interesting to test this formula against one of the Archimedean polyhedron.

The cuboctahedron has twelve vertices, so, according to the formula, the sum of its face angles should be equal to

$$360^\circ \times 12 - 720^\circ, \text{ or } 3600^\circ.$$

Each of the figure's eight triangular faces has an internal angle sum of 180°, making a total of 1440° for the eight faces; each of its six square faces has an internal angle sum of 360°, making a total of 2160° for the six faces. Thus, the total sum of all the face angles, of both the triangular and the square faces, is 3600° (1440° + 2160°). The formula will also work for any other Archimedean polyhedron, facially regular prism, or facially regular antiprism.

Triangulation and Stability

If a framework were constructed with a strut representing each edge of an Archimedean polyhedron, that framework would not be triangulated. Hence, if it were made with flexible connections between the struts, it would distort and ultimately collapse. To make such a framework stiff, additional members would have to be added until all the faces were triangulated. The same is true for the facially regular prisms and antiprisms, with the exception of the tetrahedron and the octahedron, which are already triangulated.

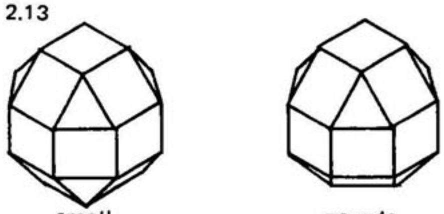
Duality

No Archimedean polyhedron has the same number of vertices as another has faces, so it is impossible to arrange two Archimedean polyhedra about a common intersphere, with the vertices of the one corresponding to the faces of the other, as is possible with the Platonic polyhedra. The reason for this is that the vertex figures of a Platonic polyhedron are identical regular polygons, but the vertex figures of an Archimedean polyhedron are polygons which are identical but not regular, so the faces of its dual are not regular. The vertex figures of a facially regular prism or antiprism are polygons which are similar but not regular, so the faces of its duals are not regular either. (The duals of the figures described in this chapter are described in Chapter 4.)

Variations on the Archimedean Polyhedra

It is possible to join an appropriate number of faces and produce a model which is not one of the Archimedean polyhedra. Usually this is because the faces do not meet in the same order about each vertex, but there is a figure, sometimes called the pseudo rhombicuboctahedron, which is not an Archimedean polyhedron, whose faces follow the same sequence about each vertex. It could be regarded as a small rhombicuboctahedron with a portion containing five squares and four triangles rotated through 45° (Diagram 2.13). Since the symmetries of this figure are different from the thirteen figures, it is not usually counted as an Archimedean polyhedron. Several people have claimed to have discovered this figure in recent years, but it may have been known a long time ago, as Kepler referred to fourteen Archimedean polyhedra in his book *The Six-Cornered Snowflake*, published in 1609.

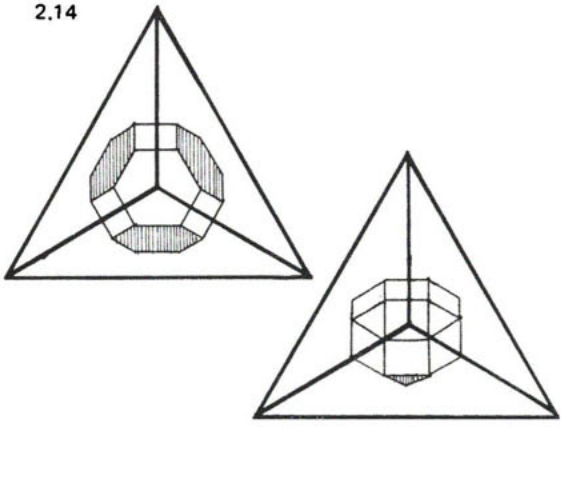
2.13



An Important Characteristic of the Archimedean Polyhedra

Up to this point the Archimedean polyhedra have been described as being convex polyhedra whose faces are nonintersecting regular, plane, convex polygons with straight edges. Furthermore, it has been stated that each figure has a similar arrangement of two or more types of polygon about each of its vertices. However, this description does not differentiate between the Archimedean polyhedra and the facially regular prisms, the facially regular antiprisms, and the pseudo rhombicuboctahedron. The important difference between the Archimedean polyhedra and the other figures is that any Archimedean polyhedron can be circumscribed by a regular tetrahedron so that four of its faces lie on the faces of that tetrahedron, whereas the other figures cannot be circumscribed in that way. The sketches in Diagram 2.14 show, as examples, how four of the hexagonal faces of a truncated octahedron can lie on the faces of an inscribing tetrahedron and that this cannot be done with an inscribed pseudo rhombicuboctahedron. Thus, an Archimedean polyhedron can be defined as a convex polyhedron which has a similar arrangement of nonintersecting, regular, plane, convex polygons of two or more different types about each vertex and which can be circumscribed by a regular tetrahedron so that four of its faces lie on the faces of that tetrahedron. Such a description defines the thirteen Archimedean polyhedra and excludes all other figures.

2.14



DATA FOR THE ARCHIMEDEAN POLYHEDRA											
Figures	triangular faces	square faces	pentagonal faces	hexagonal faces	octagonal faces	decagonal faces	numbers of vertices	numbers of edges	radii of circumspheres	radii of interspheres	dihedral angles
cuboctahedron	8	6	–	–	–	–	12	24	1·0000	·8660	3-4 125°16′
icosidodecahedron	20	–	12	–	–	–	30	60	1·6180	1·5388	3-5 142° 37′
truncated tetrahedron	4	–	–	4	–	–	12	18	1·1727	1·0607	3-6 109°28′ 6-6 70°32′
truncated octahedron	–	6	–	8	–	–	24	36	1·5811	1·5000	4-6 125°16′ 6-6 109°28′
truncated cube	8	–	–	–	6	–	24	36	1·7788	1·7071	3-8 125°16′ 8-8 90°00′
truncated icosahedron	–	–	12	20	–	–	60	90	2·4780	2·4270	5-6 142°37′ 6-6 138°11′
truncated dodecahedron	20	–	–	–	–	12	60	90	2·9695	2·9271	3-10 142° 37′ 10-10 116° 34′
small rhombicuboctahedron	8	18	–	–	–	–	24	48	1·3990	1·3066	3-4 144°44′ 4-4 135° 00′
small rhombicosidodecahedron	20	30	12	–	–	–	60	120	2·2330	2·1763	3-4 159°06′ 4-5 148°17′
great rhombicuboctahedron	–	12	–	8	6	–	48	72	2·3176	2·2630	4-6 144°44′ 4-8 135°00′ 6-8 125°16′
great rhombicosidodecahedron	–	30	–	20	–	12	120	180	3·8024	3·7693	4-6 159°06′ 4-10 148°17′ 6-10 142°37′
snub cube	32	6	–	–	–	–	24	60	1·3437	1·2472	3-3 153°14′ 3-4 142°59′
snub dodecahedron	80	–	12	–	–	–	60	150	2·1557	2·0969	3-3 164° 11′ 3-5 152° 56′

Radii of circumspheres and interspheres are given (to four decimal places) in terms of the edge lengths.

Dihedral angles are given to the nearest minute

3-4 identifies the dihedral angle between triangular and square faces,

3-5 the dihedral angle between triangular and pentagonal faces, and so on.

2.16

DATA FOR SOME FACIALLY REGULAR PRISMS AND ANTIPRISMS						
FIGURES	FACES	numbers of vertices	numbers of edges	radii of circumspheres	radii of interspheres	DIHEDRAL ANGLES
triangular prism	3 squares 2 triangles	6	9	·7638	·5774	3-4 90° 0′ 4-4 60° 0′
square prism (cube)	4 squares 2 squares	8	12	·8660	·7071	4-4 90° 0′ 4-4 90° 0′
pentagonal prism	5 squares 2 pentagons	10	15	·9867	·8507	5-4 90° 0′ 4-4 108° 0′
hexagonal prism	6 squares 2 hexagons	12	18	1·1180	1·0000	6-4 90° 0′ 4-4 120° 0′
heptagonal prism	7 squares 2 heptagons	14	21	1·2550	1·1524	7-4 90° 0′ 4-4 128° 34′
tetrahedron	4 triangles (2 edges)	4	6	·6125	·3536	3-3 70° 32′ (3-2 125° 16′)
triangular antiprism (octahedron)	6 triangles 2 triangles	6	12	·7071	·5000	3-3 109° 28′ 3-3 109° 28′
square antiprism	8 triangles 2 squares	8	16	·8227	·6533	3-3 127° 33′ 3-4 103° 50′
pentagonal antiprism	10 triangles 2 pentagons	10	20	·9511	·8090	3-3 138° 11′ 3-5 100° 49′
hexagonal antiprism	12 triangles 2 hexagons	12	24	1·0877	·9659	3-3 145° 13′ 3-6 98° 54′

Radii of circumspheres and interspheres are given (to four decimal places) in terms of the edge lengths.

Dihedral angles are given to the nearest minute.

3-4 identifies the dihedral angle between triangular and square faces, 3-5 the dihedral angle between triangular and pentagonal faces, and so on.

3. Further Convex Polyhedra with Regular Faces

Each of the facially regular figures described so far has a similar arrangement of faces about each of its vertices. If dissimilar arrangements of faces are allowed about the vertices of a figure, a further range of convex polyhedra with regular faces can be produced. Perhaps the simplest examples would be the square and pentagonal pyramids shown in Diagram 3.1. In each, a set of triangles meets at an apex with a different arrangement of faces at its other vertices.

3.1

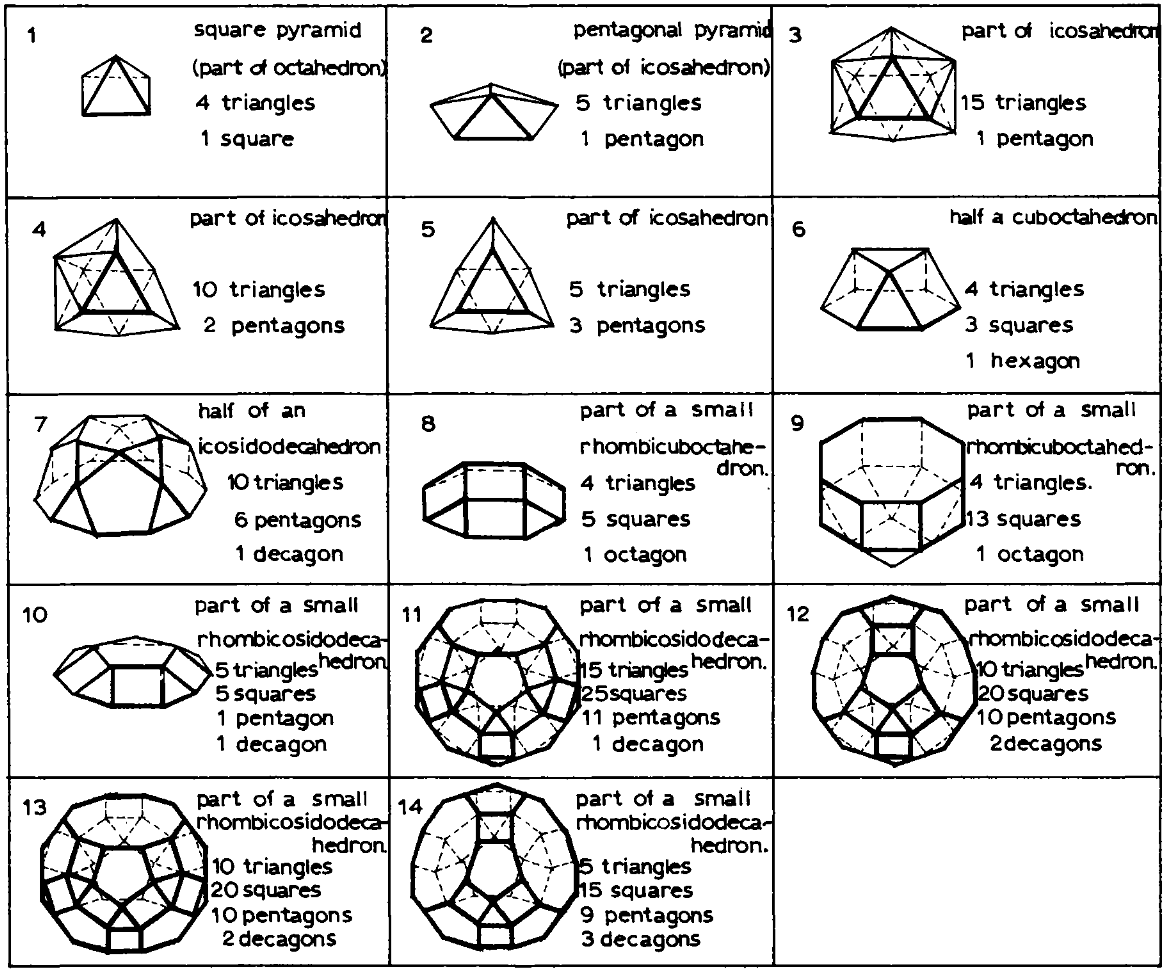
Such figures have received scant attention outside mathematical journals, and it was only in the early 1960s, through the work of such persons as V. A. Zalgaller in Russia and N. W. Johnson at Wheaton College in Norton, Massachusetts, that it was proven that there were ninety-two polyhedra of this type. Note that all of these figures are convex; otherwise, an infinite number of figures would be possible. (Further information on these figures, including their names, can be found in "Convex Polyhedra with Regular Faces," by N. W. Johnson, and in *Convex Polyhedra with Regular Faces*, by V. A. Zalgaller. Though the names of these figures are descriptive, they are not used here, as such names as *hebesphenomegacorona* and *gyroelongated pentagonal bicupola* are so long they might deter the reader.

Description of the Figures

The polyhedra are divided into six groups, to enable readers to familiarise themselves with them without having to build all ninety-two. The number assigned to each figure has no significance other than identification. The first fourteen figures can be formed by cutting parts from Platonic and Archimedean polyhedra. Figures 12 and 13 have the same numbers of faces, edges, and vertices, but their symmetries are different.

The next fifteen figures (Diagram 3.3, Figures 15-29) can be formed by joining Platonic polyhedra, and by joining Platonic and Archimedean polyhedra to the figures described previously. Figures 20 and 21 are different versions of similar figures, formed by adding the pentagonal pyramids to different faces. A similar situation exists with Figures 27 and 28.

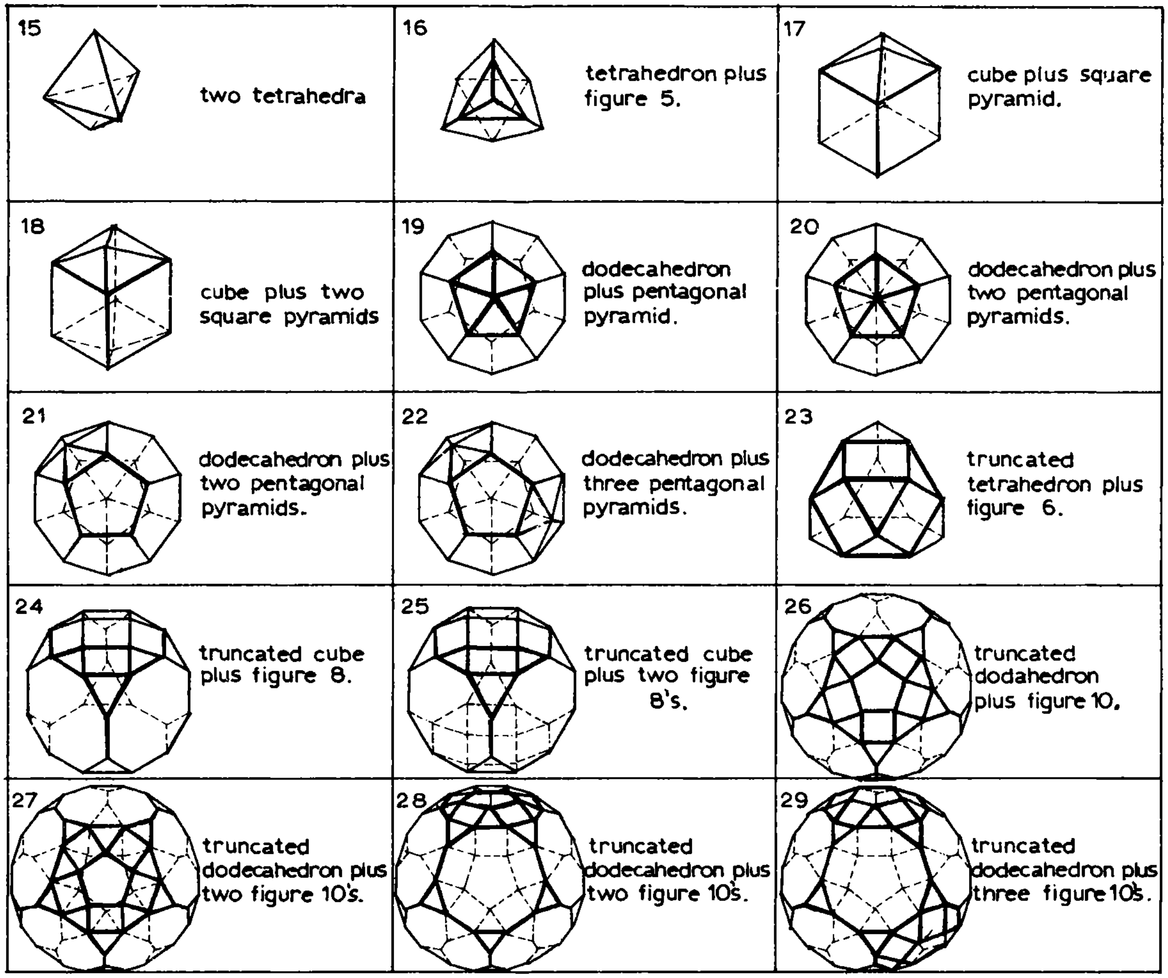
3.2



The next twenty-six figures (Diagram 3.4, Figures 30-55) can be produced by adding already established polyhedra to facially regular prisms. In several cases different figures can be made by rotating one section of a figure relative to the rest of it, for example, Figures 49 and 50, as shown. (Figure 47 is the pseudo rhombicuboctahedron mentioned in Chapter 2.)

The next eleven figures (Diagram 3.5, Figures 56-66) can be produced by adding the first fourteen polyhedra (Figures 1-14) to facially regular antiprisms.

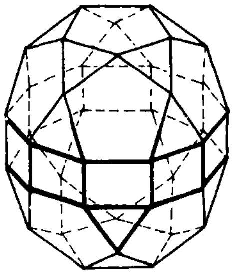
3.3

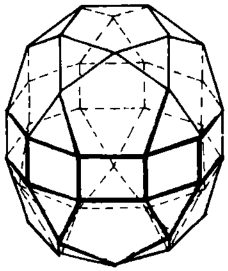


3.4

30	triangular prism plus tetrahedron
31	triangular prism plus two tetrahedra
32	two triangular prisms
33	triangular prism plus square pyramid
34	triangular prism plus two square pyramids
35	triangular prism plus three square pyramids
36	pentagonal prism plus square pyramid
37	pentagonal prism plus two square pyramids
38	pentagonal prism plus pentagonal pyramid
39	pentagonal prism plus two pentagonal pyramids
40	hexagonal prism plus square pyramid
41 42	hexagonal prism plus two square pyramids(2 versions)
43	hexagonal prism plus three square pyramids
44	hexagonal prism plus figure 6
45 46	hexagonal prism plus two figure 6's (2 versions)
47	octagonal prism plus two figure 8's
48	decagonal prism plus figure 7
49 50	decagonal prism plus two figure 7's (2 versions)
51	decagonal prism plus figure 10
52 53	decagonal prism plus two figure 10's(2 versions)
54 55	decagonal prism plus figure 7 and figure 10(2 versions)

examples





figures 49 and 50, two different figures built from similar sets of components.

3.5

56	square antiprism plus square pyramid
57	square antiprism plus two square pyramids
58	hexagonal antiprism plus figure 6
59	hexagonal antiprism plus two figure 6's
60	octagonal antiprism plus figure 8
61	octagonal antiprism plus two figure 8's
62	decagonal antiprism plus figure 7
63	decagonal antiprism plus two figure 7's
64	decagonal antiprism plus figure 10
65	decagonal antiprism plus figure 7 and figure 10
66	decagonal antiprism plus two figure 10's

example

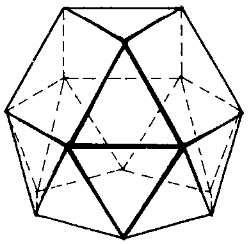
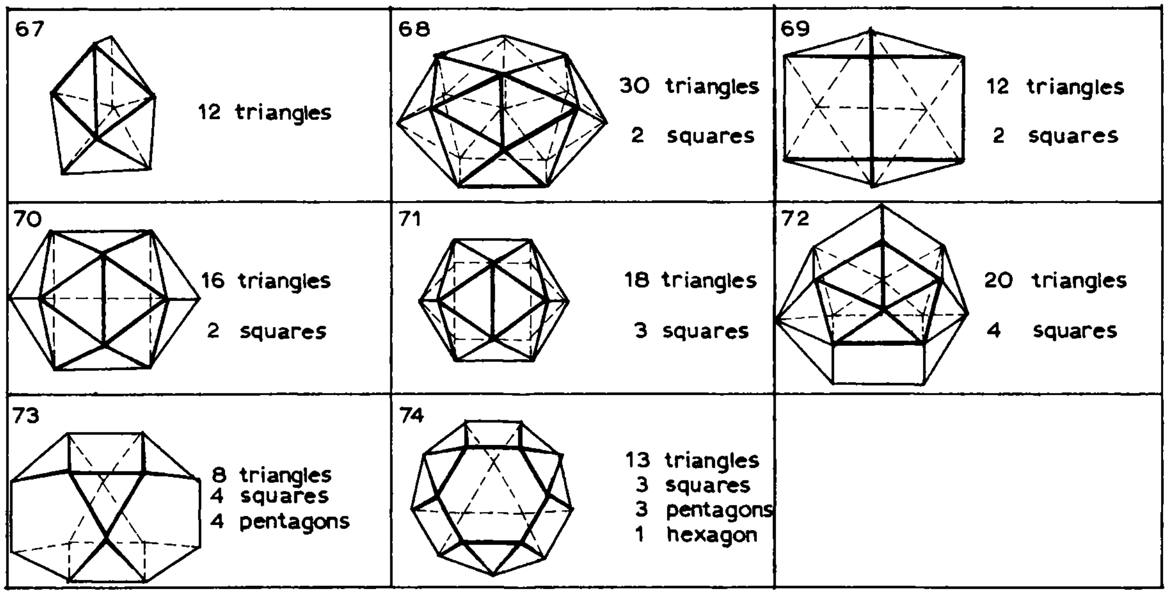


figure 58

3.6



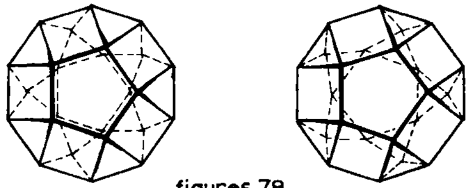
Figures 67 to 74 (Diagram 3.6) cannot be created by cutting away parts of a Platonic or an Archimedean polyhedron, so they exhibit some unusual symmetries.

Seventeen of the final eighteen figures (Figures 75-91) can be formed by joining together Figures 1-14, and the eighteenth (Figure 92) by joining Figure 1 to Figure 69 (Diagram 3.7). There are two versions of some combinations, the different versions obtained by rotating the parts relative to each other.

Some of the figures could be included in more than one of the groups, but for the sake of simplicity they are placed under one heading only. An inspection of a random selection of these figures reveals that, beyond the fact that they are all convex and have regular convex faces, they are very irregular and have few common characteristics. For example, whereas some figures have circumspheres which touch all of their vertices and interspheres which touch all of their edges, many of them do not. Though the sum of the face angles about each vertex of a figure may not be constant,

75	figure 2 plus figure 2
76	figure 6 plus figure 6
77	figure 7 plus figure 7
78	figure 7 plus figure 10 (2 versions)
79	
80	figure 8 plus figure 8 (2 versions)
81	
82	figure 10 plus figure 10 (2 versions)
83	
84	figure 14 plus figure 10
85	figure 14 plus two figure 10's
86	figure 14 plus three figure 10's
87	figure 12 plus figure 10
88	figure 12 plus two figure 10 s
89	figure 13 plus figure 10
90	figure 13 plus two figure 10 s
91	figure 11 plus figure 10
92	figure 69 plus figure 1

examples



figures 78 and 79

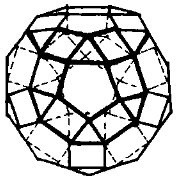


figure 91

the sum of all the face angles of the figure is still

$$360° \times V - 720°,$$

where V is the number of its vertices.

Construction of Models

The easiest way of building these models is to cut out the requisite number of faces and join them, edge to edge, till the model is complete. For those models which can be broken down into smaller parts (Figures 15-66 and 75-92) it is a good idea to build the parts separately, so that they can be joined together in various ways, thus allowing several figures to be explored by using the same parts.

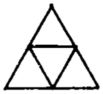
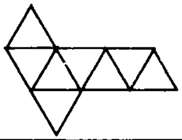
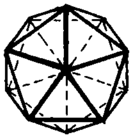
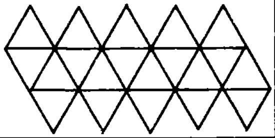
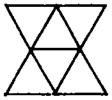
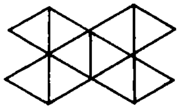
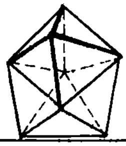
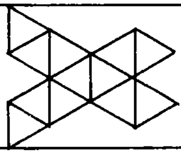
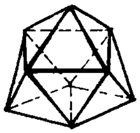
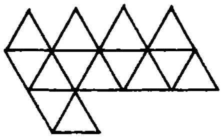
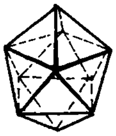
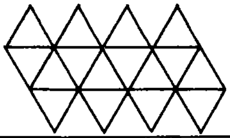
The Convex Deltahedra

A deltahedron is a polyhedron with faces which are all equilateral triangles. There is an infinite number of concave deltahedra, including such figures as a pentagonal dodecahedron whose pentagonal faces have been replaced by concave arrangements of five equilateral triangles. However, there are only eight convex deltahedra: the tetrahedron, the octahedron, and the icosahedron, which are Platonic polyhedra, and Figures 15, 35, 57, 67, and 75 (Diagram 3.8).

Models of these eight polyhedra can be made with toothpicks or other suitable strut materials, each toothpick representing an edge of the figure. Since all the faces are triangles, flexible connectors can be used between the struts and the resulting frameworks will still be rigid. The quickest way of building cardboard models is to draw a sheet of equilateral triangles and then cut out the nets of faces illustrated in right-hand column of the diagram. The names of three of the figures are long and cumbersome, and it is easier to think of them as the twelve-faced, fourteen-faced, and sixteen-faced deltahedra, respectively.

Each Platonic polyhedron has the same number of congruent faces meeting at each of its vertices, and the five nonregular convex deltahedra have congruent faces but not the same number at each vertex. Each Archimedean polyhedron has the same arrangement of different regular polygons about each of its vertices, and the remaining eighty-seven figures described in this chapter have different arrangements of different regular polygons about their vertices. Hence, it could be claimed that these eighty-seven figures have a relationship to the Archimedean polyhedra similar to the relationship between the nonregular deltahedra and the Platonic polyhedra.

3.8

THE EIGHT CONVEX DELTAHEDRA			
name	sketch	data	net diagram
tetrahedron		4 faces 4 vertices 6 edges	
octahedron		8 faces 6 vertices 12 edges	
icosahedron		20 faces 12 vertices 30 edges	
triangular dipyramid		6 faces 5 vertices 9 edges	
pentagonal dipyramid		10 faces 7 vertices 15 edges	
dodecadeltahedron (twelve-faced deltahedron)		12 faces 8 vertices 18 edges	
tetracaidecadeltahedron (fourteen-faced deltahedron)		14 faces 9 vertices 21 edges	
heccaidecadeltahedron (sixteen-faced deltahedron)		16 faces 10 vertices 24 edges	

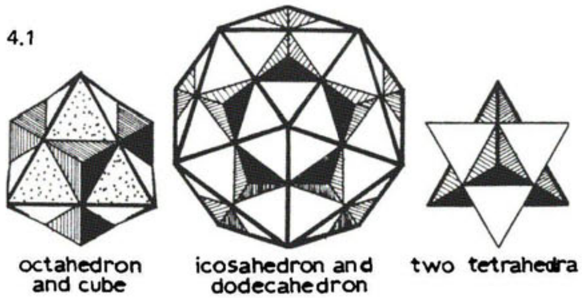
4. The Duals of Archimedean Polyhedra, Prisms, and Antiprisms

One of the clearest explanations of duality can be found, not in a mathematical dictionary, but in the *Pocket Oxford Dictionary*, where the adjective *dual* is defined as meaning "of double nature, forming a pair." When dealing with polyhedra, the word *dual* is used as an abbreviation for the words *dual solid* or *dual polyhedron*.

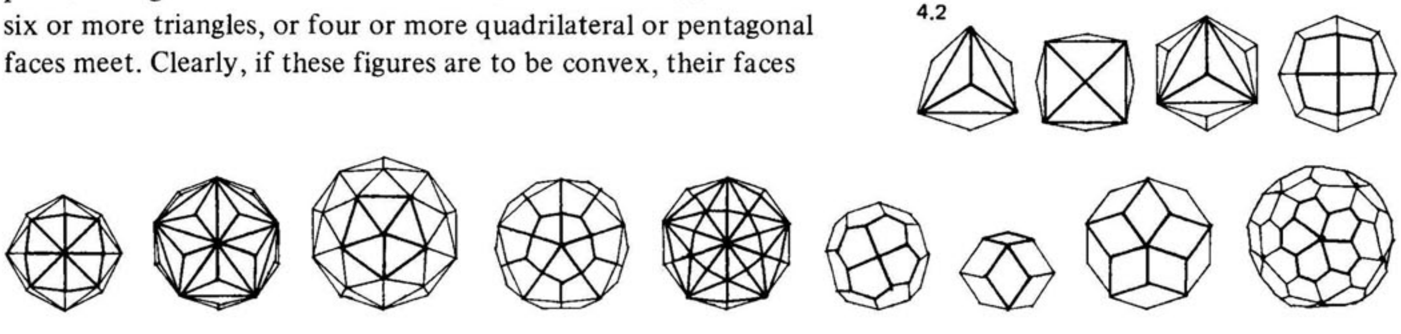
In Chapter 1 it was shown how the octahedron and the cube could be placed about a common intersphere so that their edges touched that sphere at twelve common points, and so that each vertex lay outside a face of the other figure, as in Diagram 4.1. An icosahedron and a dodecahedron can be arranged in a similar way, as can a pair of tetrahedra. Each figure in the pair is the dual of the other figure.

In Chapter 2 it was found that the Archimedean polyhedra, facially regular prisms, and facially regular antiprisms could not be paired in that way and that the duals of those figures were an entirely different family of polyhedra. The sketches in Diagram 4.2 show the duals of the Archimedean polyhedra. It can be seen that the same number of edges do not meet at each vertex of a particular figure. It can also be seen that there are vertices where six or more triangles, or four or more quadrilateral or pentagonal faces meet. Clearly, if these figures are to be convex, their faces

4.1



4.2



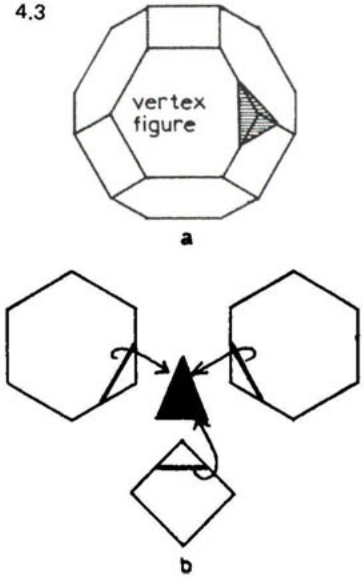
cannot be regular polygons, as the sum of the face angles about some of their vertices would be equal to or greater than 360°. Though the faces are not regular, all of the faces of a particular figure are congruent.

The Shapes of the Faces

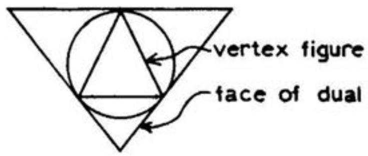
The shapes of the faces of a dual of an Archimedean polyhedron, a facially regular prism, or a facially regular antiprism can be found by drawing the vertex figure of the appropriate facially regular polyhedron, by joining the midpoints of the edges which join at a vertex, as shown in Diagram 4.3a for the truncated octahedron. It can be seen that the vertex figure of this polyhedron is an isosceles triangle, whose edge lengths can be found by drawing each face of the Archimedean polyhedron and measuring the lengths of the diagonals, as in Diagram 4.3b.

The next step is to inscribe the vertex figure with a circle (Diagram 4.4). Since the vertices of the vertex figure represent the midpoints of the edges of the Archimedean polyhedron, this circle will be a small circle on the intersphere of the Archimedean polyhedron. Both the Archimedean polyhedron and its dual can be arranged about this common intersphere to touch it at common points on the same small circle. Hence, the face of the dual of the Archimedean polyhedron can be drawn by constructing the tangents of the circle at those points, as shown. Since all the vertex figures of an Archimedean polyhedron are congruent, all the small circles are the same size, and hence all the faces of its dual are congruent. (This construction is mentioned in *Mathematical Models* by H. M. Cundy and A. P. Rollett, who ascribed it to Dorman Luke.)

4.3



4.4



Drawing the Faces

Though the faces of these figures can be constructed by the method described in the preceding paragraphs, it is quicker to use the data given in Diagrams 4.12 and 4.13. Some readers may be worried about accuracy, as it is impossible to construct an exact angle of, say, 112°53′ with the average protractor, but there will be no problems if it is within about half a degree. The drawings of the individual faces in the table give an accurate idea of their shapes, but because of their small size, they should not be scaled to establish relative edge lengths. Where edge lengths on a particular face are equal, they are crossed with a similar number of lines; where they are unequal, each different length of edge is crossed with a different number of lines.

Triangular faces, such as those of the triakis tetrahedron, can be constructed by measuring off a suitable length for the longest edge and drawing two lines at a suitable inclination from its ends (Diagram 4.5). The lines intersect to define the third vertex of the triangle. Check to see that the angle at this vertex is the right size; if it is, the edge lengths are in the right proportions.

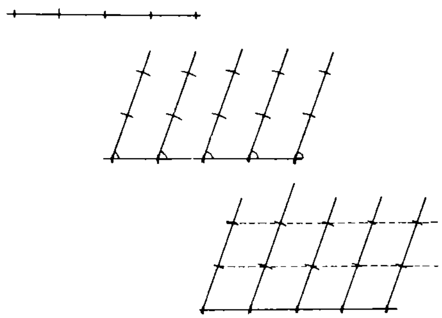
Since all four of their edges are equal in length, rhombic faces can be produced very quickly by drawing a line and dividing it into equal edge lengths. The next step is to construct a series of parallel lines at a suitable inclination to that line and mark off equal edge lengths along those parallel lines (Diagram 4.6). Joining those points produces a series of suitably-shaped rhombic faces.

The kite-shaped faces have two different edge lengths and can be drawn by constructing two edges of equal length at the correct inclination to one another, as shown in Diagram 4.7. The next step is to construct the other two edges at the correct inclination to the previous edges. Those edges will intersect to define the final vertex. Check all angles, in case an error has been made.

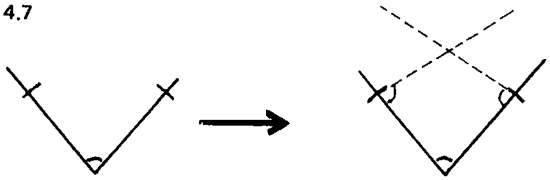
Both types of pentagonal face have three edges which are equal in length. Construct the three edges at the correct angle to one another, as shown in Diagram 4.8; then draw the final two edges at the correct angles to the end lines. These two lines of equal length intersect to define the final vertex, after which all the angles should be checked for accuracy.

4.5

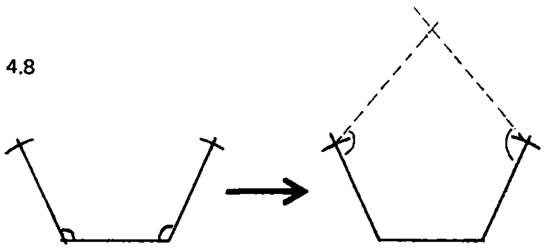
4.6



4.7



4.8

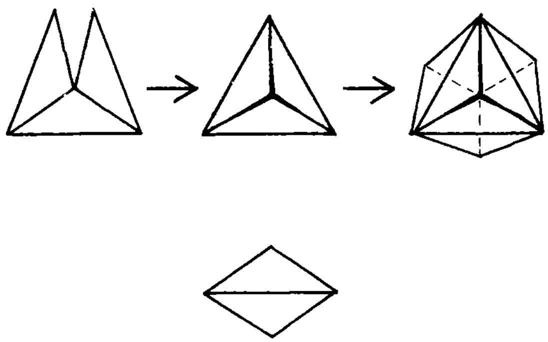


As with the Archimedean polyhedra, there is no need to construct models of all the figures, as a selection of three or four will be sufficient for many people. It is a good idea to build models of the duals of the facially regular figures which have already been built, in order to make comparisons between them.

Net diagrams for these figures can be found in such books as *Natural Structure* by Robert Williams, but they take a long time to draw out and it is quicker to cut out the faces individually or in small groups. Whatever method is used, a lot of time can be saved by using templates and patterns, as described in Appendix 3. When joining the faces, remember that all the face angles at a particular vertex should be the same.

The right-hand columns in Diagrams 4.12 and 4.13 attempt to show how groups of faces can be formed first and then the figure built by joining those groups. Asterisks indicate parts with similar vertices in the two right-hand columns of both tables. Nine of the figures look like Platonic polyhedra with shallow pyramids added to each face. Those figures can be made by making the requisite number of pyramids and joining them as if they were the faces of the appropriate Platonic polyhedron. For example, the first figure in Diagram 4.12, the triakis tetrahedron, can be made by joining the twelve triangles to form four separate shallow pyramids, as in Diagram 4.9. Then those pyramids should be joined, as if each were a triangular face of a tetrahedron, as shown, to complete the figure. An alternative technique is to cut out the triangles in pairs, as in the sketch at bottom. This method is harder to visualise than the other but it tends to produce neater models.

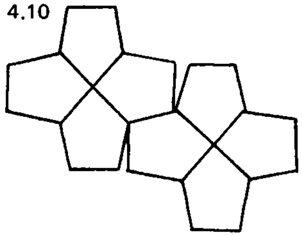
4.9



The quickest way of making the two figures with rhombic faces is to cut out the faces and then join them, edge to edge, until the figure is complete. The wider angles on the faces are joined together in groups of three, and the narrower angles in groups of four or five, depending on the figure being built.

The two figures with pentagonal faces are slightly harder to join. One method is to join the requisite number of faces in clusters of four or five, depending on the figure, as shown in the table. These clusters of faces can then be joined as shown in Diagram 4.10.

4.10



Nomenclature

The names of these dual figures are long. Some people prefer to remember a figure as "the dual of such and such a figure" rather than by its individual name, but the names are descriptive and fairly easy to remember. Triakis, tetrakis, pentakis, and hexakis literally mean three times, four times, five times, and six times, respectively, so a triakis tetrahedron is a tetrahedron whose faces have been divided into three parts. A rhombic dodecahedron is a figure with twelve rhombic faces, and a rhombic triacontrahedron is a figure with thirty rhombic faces. A trapezoidal icositetrahedron is a figure with twenty-four kite-shaped faces, and a pentagonal icositetrahedron is a figure with twenty-four pentagonal faces.

Some General Characteristics of the Duals of the Archimedean Polyhedra, the Facially Regular Prisms, and the Facially Regular Antiprisms

The most important characteristic of these polyhedra is that they are just as regular as the facially regular figures described in Chapter 2, though in different ways. All the faces of the facially regular figures are regular, but each figure has more than one type of face; the faces of their duals are not regular, but each figure has only one type of face.

The vertex figures of each polyhedron described in Chapter 2 are congruent but not regular; the vertex figures of each of their duals are regular but not congruent. Since their vertex figures are regular, these figures are sometimes called vertically regular polyhedra. Because the face angles of the vertically regular figures are not neat, whole numbers of degrees, some readers may wonder why they cannot be rounded off. If that were done, their vertex figures would no longer be regular polygons.

Each figure described in Chapter 2 had a similar number of edges meeting at each vertex, but most had more than one dihedral angle. Each of their duals has one dihedral angle but does not have the same number of edges meeting at each vertex. The sums of the face angles about each vertex of a vertically regular polyhedron are not constant, but the sum of all the face angles is

$$360° V - 720°,$$

where V is the number of its vertices.

Each of the facially regular figures described in Chapter 2 had a circumsphere which touched all of its vertices and an intersphere to touch all of its edges, but no insphere to touch all of its faces. Each of their vertically regular duals has an insphere which touches every face and an intersphere to touch every edge, but no circumsphere that touches every vertex. It was noted in Chapter 1 that an important characteristic of a regular polyhedron was that it had a circumsphere which could touch all of its vertices, an intersphere to touch all of its edges, and an insphere to touch all of its faces.

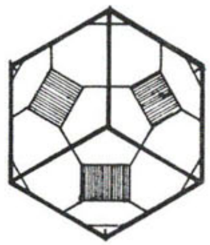
Further Relationships to the Archimedean Polyhedra

The names of four of the Archimedean polyhedra, the small and great rhombicuboctahedra and rhombicosidodecahedra, have the prefix *rhomb*. Since the faces of those figures are clearly squares and not rhombuses, some readers may question the terminology. However, it is highly appropriate, as each of the figures can be circumscribed by either a rhombic dodecahedron or a rhombic triacontrahedron, as shown in Diagram 4.11, so that its square faces lie on the rhombic faces of the circumscribing polyhedron.

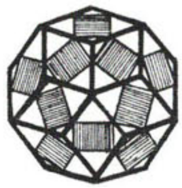
The rhombic dodecahedron and the rhombic triacontrahedron are the duals of the cuboctahedron and the icosidodecahedron, respectively. In Chapter 2 the two polyhedra are classified with the octahedron as two-frequency figures. It is interesting to find that the dual of the octahedron, the cube, like the rhombic dodecahedron and the rhombic triacontrahedron, has quadrilateral faces.

4.11

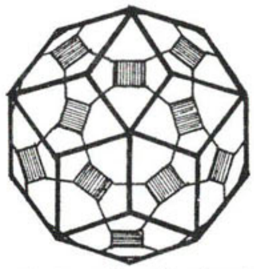
small rhombicuboctahedron in rhombic dodecahedron



great rhombicuboctahedron in rhombic dodecahedron



small rhombicosidodecahedron in rhombic triacontrahedron

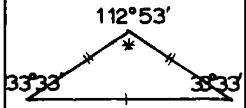
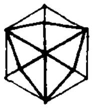
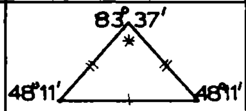
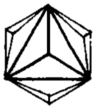
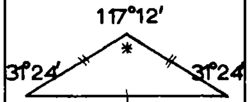
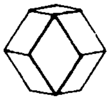
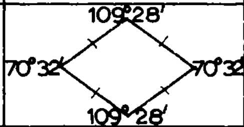
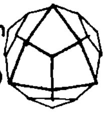
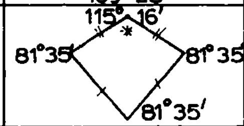
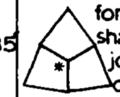
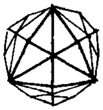
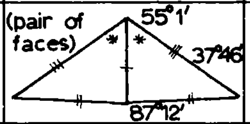
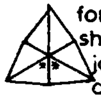
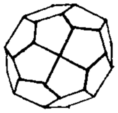
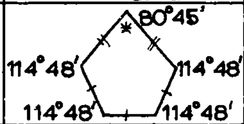
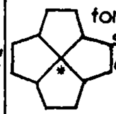
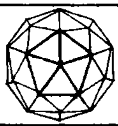
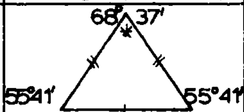
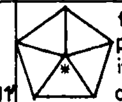
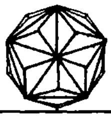
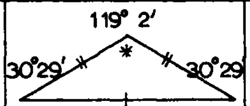


great rhombicosidodecahedron in rhombic triacontrahedron

The duals of the snub cube and the snub dodecahedron, the pentagonal icositetrahedron and the pentagonal hexecontrahedron, have pentagonal faces. In Chapter 2 it was mentioned that the icosahedron could be regarded as a snub tetrahedron. It is interesting to note that the dual of this third snub figure is the pentagonal dodecahedron, which also has pentagonal faces.

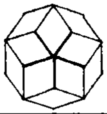
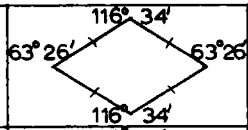
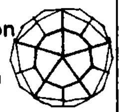
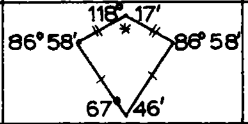
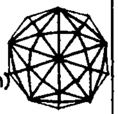
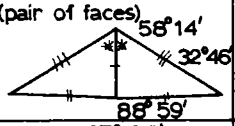
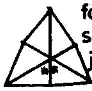
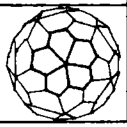
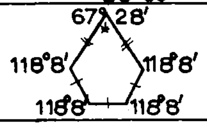
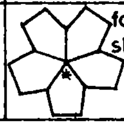
Conclusion

Since they do not seem to be as regular as the facially regular polyhedra, relatively little work has been done on the duals of the Archimedean Polyhedra and of the facially regular prisms and antiprisms. These figures, however, have now been demonstrated to be just as regular as the facially regular polyhedra, and the author feels that many important discoveries are yet to be made about them, particularly in the realm of joining them, along the lines described in Chapter 5.

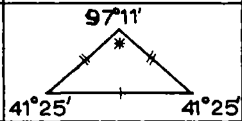
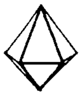
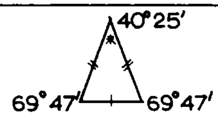
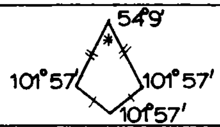
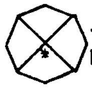
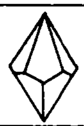
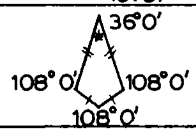
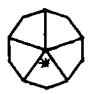
DATA FOR THE DUALS OF THE ARCHIMEDEAN POLYHEDRA				
FIGURE	SKETCH	FACES EDGES VERTICES DIHEDRAL ANGLE	FACE DETAILS	ASSEMBLY
triakis tetrahedron (dual of truncated tetrahedron)		12 faces 18 edges 8 vertices 129° 31'		 form 4 triangular pyramids and join as if building a tetrahedron.
tetrakis hexahedron (dual of truncated octahedron)		24 faces 36 edges 14 vertices 143° 7'		 form 6 square pyramids and join as if building a cube.
triakis octahedron (dual of truncated cube)		24 faces 36 edges 14 vertices 147° 21'		 form 8 triangular pyramids and join as if building an octahedron.
rhombic dodecahedron (dual of cuboctahedron)		12 faces 24 edges 14 vertices 120° 0'		wider angles meet in groups of three, smaller angles in groups of four.
trapezoidal icositetrahedron (dual of small rhombicuboctahedron)		24 faces 48 edges 26 vertices 138° 7'		 form 8 triangular shapes as shown and join as if building an octahedron.
hexakis octahedron (dual of great rhombicuboctahedron)		48 faces 72 edges 26 vertices 155° 5'		 form 8 triangular shapes as shown and join as if building an octahedron.
pentagonal icositetrahedron (dual of snub cube)		24 faces 60 edges 38 vertices 136° 19'		 form 6 shapes as shown and join as described in text.
pentakis dodecahedron (dual of truncated icosahedron)		60 faces 90 edges 32 vertices 156° 43'		 form 12 pentagonal pyramids and join as if building a dodecahedron.
triakis icosahedron (dual of truncated dodecahedron)		60 faces 90 edges 32 vertices 160° 37'		 form 20 triangular pyramids and join as if building an icosahedron.

(continued)

4.12 (continued)

rhombic triacontrahedron (dual of icosidodecahedron)		30 faces 60 edges 32 vertices 144° 0′		wider angles meet in groups of three, smaller angles in groups of five.
trapezoidal hexecontrahedron (dual of small rhombicosidodecahedron)		60 faces 120 edges 62 vertices 154° 7′		 form 20 triangular shapes as shown and join as if building an icosahedron.
hexakis icosahedron (dual of great rhombicosidodecahedron)		120 faces 180 edges 62 vertices 164° 53′	(pair of faces) 	 form 20 triangular shapes as shown and join as if building an icosahedron.
pentagonal hexecontrahedron (dual of snub dodecahedron)		60 faces 150 edges 92 vertices 153° 11′		 form 12 shapes as shown and join as described in text.

4.13

DATA FOR THE DUALS OF SOME FACIALLY REGULAR PRISMS AND ANTIPRISMS				
FIGURE	SKETCH	FACES EDGES VERTICES DIHEDRAL ANGLE	FACE DETAILS	ASSEMBLY
triangular dipyramid (dual of triangular prism)		6 faces 9 edges 5 vertices 98° 13′		 form two triangular pyramids, then join them.
pentagonal dipyramid (dual of pentagonal prism)		10 faces 15 edges 7 vertices 119° 6′		 form two pentagonal pyramids, then join them.
trapezoidal octahedron (dual of square antiprism)		8 faces 16 edges 10 vertices 105° 8′		 form two pyramids and join.
trapezoidal decahedron (dual of pentagonal antiprism)		10 faces 20 edges 12 vertices 116° 34′		 form two pyramids and join.

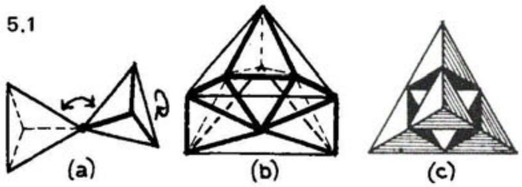
5. Joining Polyhedra

Though polyhedra can be joined in a haphazard way, careful planning usually results in more exciting configurations. An attempt is made here to describe several different approaches. But, either intentionally or by oversight, certain methods of joining figures have been omitted, so this chapter should not be regarded as an exhaustive treatment of the subject.

Six Types of Contact

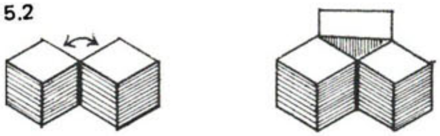
There are six basic ways in which one polyhedron can be joined to another. The first method is to join the vertex of one polyhedron to the vertex of another polyhedron, as shown by the two tetrahedra in Diagram 5.1a. If they are joined with a flexible connector, the one figure is free to move in several directions relative to the other. Such rotations can be prevented by using a nonflexible connector or by adding cables or struts as in 5.1b. A more interesting way of preventing rotations is to add extra polyhedra to the configuration until it is stabilised. Diagram 5.1c shows four tetrahedra joined vertex to vertex so that they form a stable configuration and cannot move relative to each other.

5.1



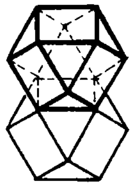
The second method of joining polyhedra is to join the edge of one figure to the edge of another, as shown by the two cubes on the left of Diagram 5.2. If the joint between the two figures is flexible, it acts like a hinge, allowing the figures to rotate about it independently. This can be prevented by using a nonflexible connector or by adding cables or struts, as before. The polyhedra can also be stabilised by adding further polyhedra, as shown by the three cubes which have been joined together, edge to edge in a stable configuration, at the right of Diagram 5.2. (Some other configurations of figures joined edge to edge are described later in this chapter.)

5.2



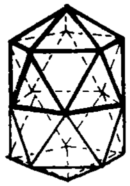
The most obvious way of joining polyhedra together is to join the face of one figure to a congruent face of another figure. The one figure cannot move relative to the other without destroying the face-to-face contact. Diagram 5.3 shows two cuboctahedra with a square face of the one joined to a square face of the other.

5.3



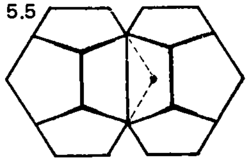
The fourth method is to allow the vertex of the one figure to penetrate the other figure until a common series of edges along which the figures can be joined is found. Diagram 5.4 shows the vertex of one icosahedron penetrating another icosahedron until the two figures can be fused along a common pentagonal circuit of edges.

5.4



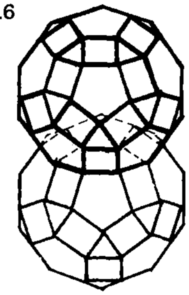
The fifth method is to allow the edge of one figure to penetrate the other figure until a suitable common plane at which the two figures may be joined is established. Diagram 5.5 shows the edge of one pentagonal dodecahedron penetrating another pentagonal dodecahedron until four of the vertices of each figure coincide and define a common plane at which the two figures can be joined.

5.5



The sixth method of joining polyhedra is to allow the face of one polyhedron to penetrate another polyhedron till a suitable common plane at which the two figures may be joined is found. In Diagram 5.6 the two small rhombicuboctahedra share a common plane defined by a decagonal circuit of edges on each figure.

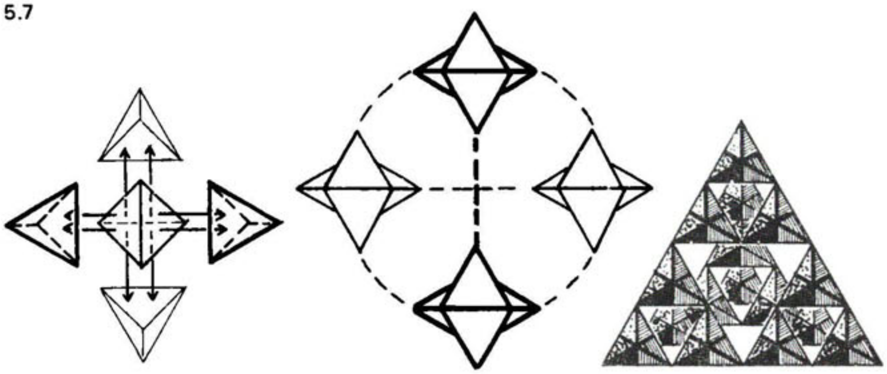
5.6



The last three methods of joining may appear to be less important than the first three, but they can be very useful in some situations. When figures are joined by the last three methods, the internal faces can be omitted. Polyhedra can also be joined in other ways, such as by joining the vertex of one figure to an edge of another, but those methods are not as important as the six basic methods, so they have been omitted.

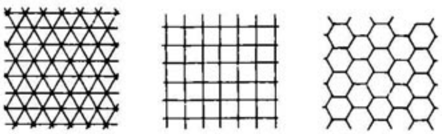
Several of the six methods can be used in the same configuration. Diagram 5.7a shows four tetrahedra joined to the faces of a fifth tetrahedron to produce a figure whose four outer vertices define the vertices of a regular tetrahedron. Four such figures can then be joined vertex to vertex to form a figure whose four outer vertices define the vertices of an even larger tetrahedron, as shown in Diagram 5.7b. Four of those figures can be joined to produce an even larger figure like the one in Diagram 5.7c, and so on.

5.7

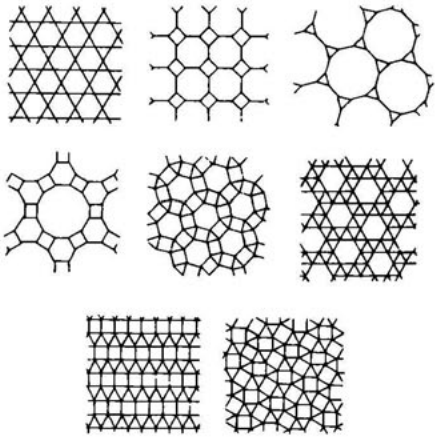


(a) (b) (c)

5.8



5.9



Plane Tessellations

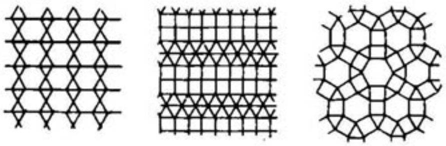
A plane tessellation, or tiling pattern, is a two-dimensional pattern of polygons. There are three regular tessellations, where the same number of congruent regular polygons meet at each "vertex" (Diagram 5.8). They can be regarded as two-dimensional equivalents of the Platonic polyhedra.

There are eight semiregular tessellations, where the same number of different regular convex polygons meet, in the same order, about each "vertex" (Diagram 5.9). They can be regarded as two-dimensional equivalents of the Archimedean polyhedra, facially regular prisms, and facially regular antiprisms.

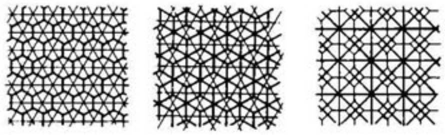
There are tessellations, like those in Diagram 5.10, where different types of regular convex polygons meet in different arrangements about the vertices. Those tessellations can be regarded as the two-dimensional equivalents of the ninety-two figures with regular faces described in Chapter 3.

Tessellations which are the duals of existing tessellations can be created by joining the midpoints of the faces of the existing tessellation (Diagram 5.11). Note that the dual tessellation of a regular tessellation is another regular tessellation, as the dual of a Platonic polyhedron is another Platonic polyhedron.

5.10



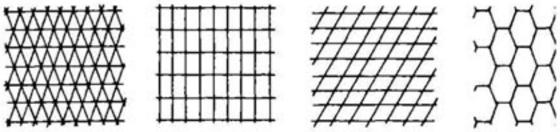
5.11



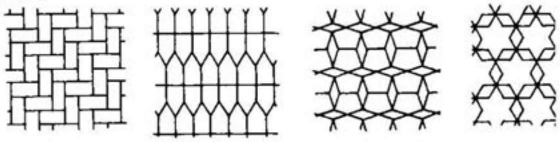
An existing tessellation can be elongated and distorted (Diagram 5.12). And tessellations can be formed from nonregular polygons of one or more types and from star-shaped polygons (Diagram 5.13).

Besides being interesting in their own right, tessellations can be very useful when systems of joined polyhedra are investigated, as a section cut through that system always produces a tessellation of some kind. If a section is cut through a stack of cubes, cutting parallel to their faces, a tessellation of squares is produced. If there are no gaps between the polyhedra, each polygon of the tessellation represents a section through a polyhedron; if there are gaps between the polyhedra, some of the polygons represent these voids. Note that the regular pentagon and the regular decagon do not appear in the more symmetrical tessellations, which explains why figures with five-way symmetries, such as the icosahedron and the dodecahedron, are harder to join in an ordered way than are such figures as the cube. (Further information and further tessellations appear in Critchlow, *Order in Space*, and in Fejes Tóth, *Regular Figures*.)

5.12



5.13

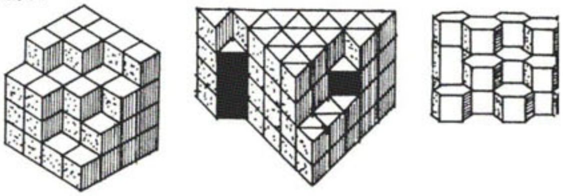


Close-Packing Polyhedra

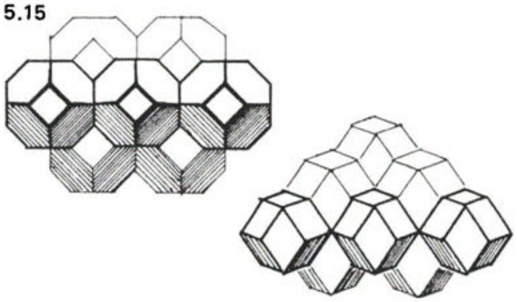
Certain polyhedra can be fitted together so that no gaps occur between them. A familiar example of this would be a child's wooden building blocks, which can be stacked together without leaving any gaps between the individual cubes. Such arrangements are known as close-packing systems of polyhedra. They have received a lot of attention because of their potential applications in packaging and architecture. The close-packing system of cubes is an example of a close-packing system where all the polyhedra are identical. Identical triangular prisms can also be joined together without spaces between them, as can identical hexagonal prisms (Diagram 5.14).

Of the Platonic polyhedra, only cubes will close-pack by themselves, and of the Archimedean polyhedra, only truncated octahedra will (Diagram 5.15). Of the duals of the Archimedean polyhedra, only rhombic dodecahedra will close-pack by them-

5.14



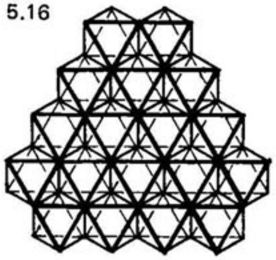
5.15



selves (Diagram 5.15). Despite the fact that so few facially regular and vertically regular polyhedra can pack space by themselves, an infinite number of irregular figures will do so. Many of them can be considered distortions and variations of the figures already cited.

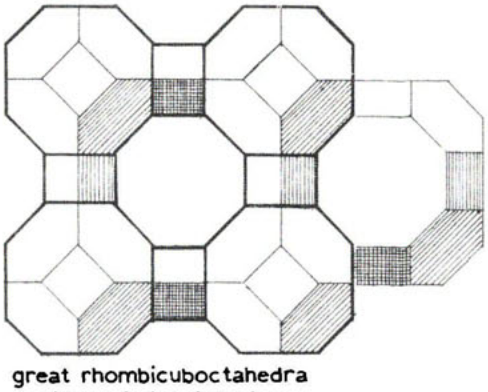
Polyhedra in a close-packing arrangement need not be identical. There are several combinations of different facially regular polyhedra which can be close-packed, such as the combination of tetrahedra and octahedra shown in Diagram 5.16. Since the edges of both of those polyhedra define stable triangulated networks, models of that arrangement of polyhedra can be made from toothpicks or other such materials. A very successful roof truss, called a space frame, is based on that framework. A space frame is very light in weight and can span large distances without intermediate supports. Since all the edges in such a framework are the same length, standard-length components can be used to make it. Besides its value in engineering, this close-packing arrangement of polyhedra can be used as a basis for creating more complex combinations of figures, as is shown later in this chapter.

5.16

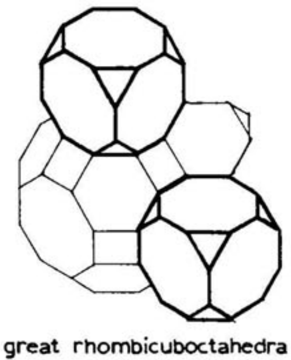


Many other close-packing combinations can be made from mixtures of facially regular polyhedra, such as the ones shown in Diagram 5.17. Obviously, there are also many close-packing combinations of prisms, which could be likened to tessellations made from a thick material. (Other close-packing systems of

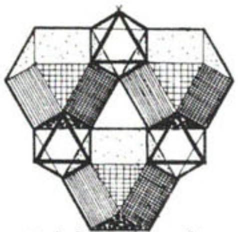
5.17



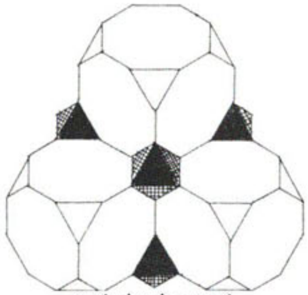
great rhombicuboctahedra
truncated octahedra
and cubes



great rhombicuboctahedra
truncated tetrahedra
and truncated cubes



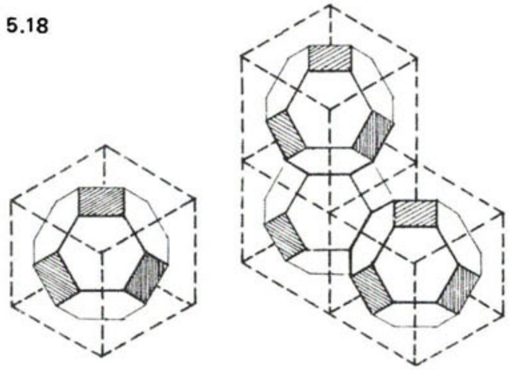
octahedra and
cuboctahedra



octahedra and
truncated cubes

polyhedra appear in such books as Williams, *Natural Structure*, and Critchlow, *Order in Space*.) It will be found that polyhedra whose faces are pentagons or decagons and polyhedra which are snub figures do not appear in close-packing arrangements of facially regular convex polyhedra. On the other hand, there is also an infinite number of sets of close-packing irregular polyhedra, many of which can be regarded as variations on existing arrangements of facially regular figures. Though much attention has been given to close-packing arrangements of polyhedra, other ways of joining polyhedra should not be ignored.

5.18



A Three-dimensional Approach to Joining Polyhedra

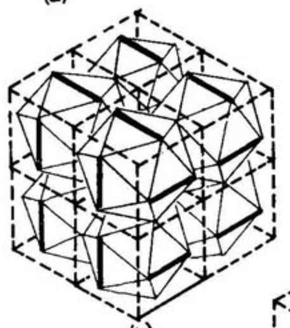
Polyhedra can be joined together in such a way that there are open spaces in the arrangement. Such arrangements can be made by omitting certain cells of a close-packing arrangement of polyhedra, but there are many other arrangements which cannot be evolved from a close-packing arrangement in this way.

Many polyhedra can be arranged symmetrically inside an imaginary cube. If these imaginary cubes are close-packed, the enclosed polyhedra can be joined. For example, a truncated octahedron can be arranged inside an imaginary cube so that its six square faces touch the faces of that cube (Diagram 5.18). Several such imaginary cubes, each containing a truncated octahedron can be joined together face-to-face in a close-packing arrangement. In joining the imaginary cubes, the square faces of the truncated octahedra coincide and can be joined to produce an orderly three-dimensional arrangement of polyhedra. This is not a close-packing arrangement of truncated octahedra, as voids are formed about the vertices of the imaginary cubes.

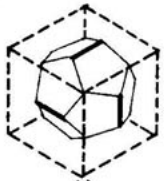
By this approach, polyhedra which were not featured in the close-packing arrangements shown earlier can be joined together. For example, an icosahedron can be placed inside a cube so that an edge of the icosahedron touches each face of the cube, as shown in Diagram 5.19a. Then a series of imaginary cubes, each containing a similar icosahedron, can be joined face to face. When

5.19

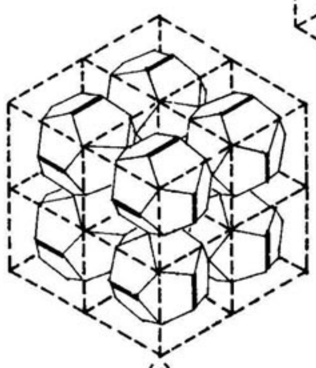
(a)



(b)



(c)



(d)

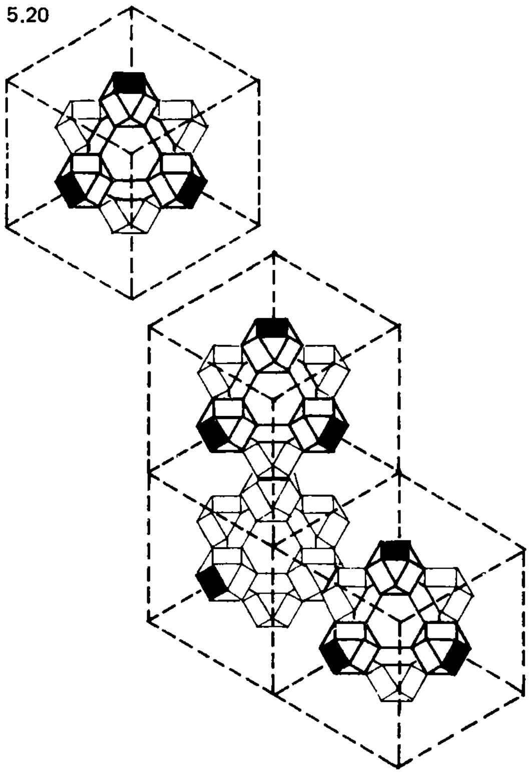
the faces of those imaginary cubes are joined, the edges of the individual icosahedra coincide, and they can be joined together to produce an orderly arrangement of icosahedra, as in Diagram 5.19b. Pentagonal dodecahedra can also be arranged inside cubes with their edges touching the faces of the cubes (Diagram 5.19c). Then those imaginary cubes can be joined, allowing the dodecahedra to be joined in an orderly arrangement as in Diagram 5.19d. Note that in both of these examples the figures are joined edge to edge, so several icosahedra and dodecahedra must be joined before a stable configuration is obtained.

Instead of single figures, groups of polyhedra can also be arranged inside imaginary cubes so that they touch the faces of those cubes symmetrically. Diagram 5.20 shows a truncated octahedron with a cuboctahedron joined to each of its six square faces. That group of figures has been fitted inside an imaginary cube so that an outer square face of each cuboctahedron touches a face of the circumscribing cube. Similar imaginary cubes, each containing a similar group of figures, could be joined together face-to-face, thereby joining the outer square faces of the cuboctahedra, as they are in the right-hand sketch of Diagram 5.20.

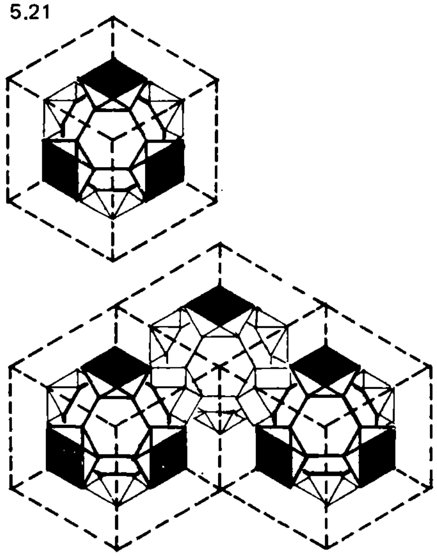
Half-polyhedra can be added to the central polyhedron instead of complete ones. Diagram 5.21 shows a half-cuboctahedron added to each square face of a truncated octahedron, inside the circumscribing cube. When those cubes are joined, there is only one cuboctahedron between the truncated octahedra, as shown.

A minor drawback of using cubes as the imaginary circumscribing polyhedra is that the individual polyhedra are clearly arranged along lines which intersect one another at right angles. Since people are so familiar with this sort of symmetry, the resulting arrangements may appear obvious and simple. However, it is also possible to circumscribe polyhedra, or groups of polyhedra, by tetrahedra and to join the circumscribed polyhedra, using the interrelationships between the tetrahedra in the close-packing system of octahedra and tetrahedra. The symmetries of such arrangements are not so familiar as those obtained from the cube, so they tend to look more interesting than the others.

5.20



5.21

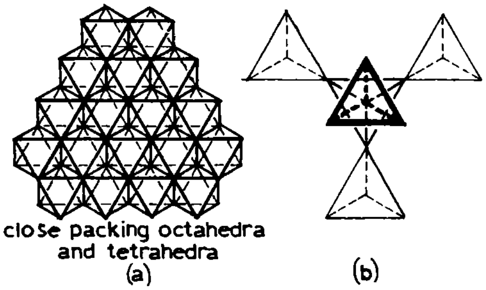


The tetrahedra in a close-packing arrangement of octahedra and tetrahedra touch each other vertex to vertex. Diagram 5.22b shows part of the close-packing system where it can be seen that there are five tetrahedra – a central one and one joined to each of its vertices. This network of figures can be extended by adding tetrahedra to the peripheral tetrahedra in the same way as the peripheral tetrahedra were added to the original central one. Diagram 5.22c shows how each tetrahedron is rotated relative to each of its adjoining tetrahedra.

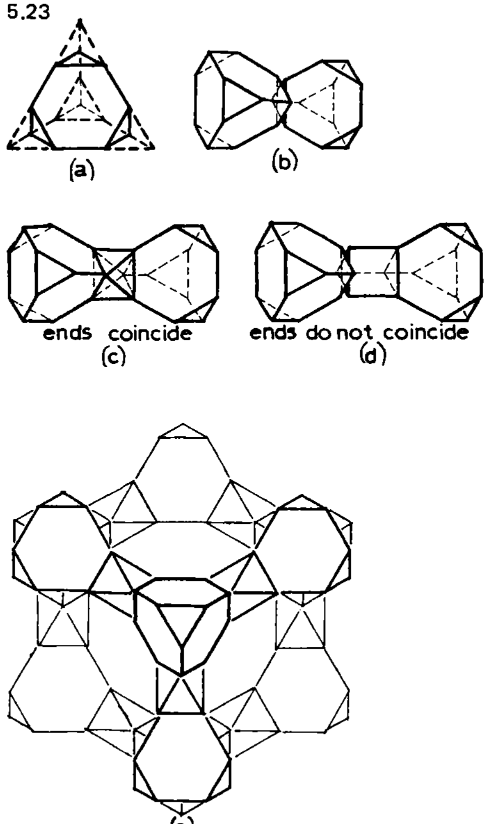
To join polyhedra, or groups of polyhedra, along these lines, the figures, or groups of figures, must be circumscribed by an imaginary tetrahedron. For example, a truncated tetrahedron can be fitted inside an imaginary tetrahedron as in Diagram 5.23a. If two of the imaginary tetrahedra are brought together in the vertex-to-vertex relationship already mentioned, it is apparent that the triangular faces of the truncated tetrahedra can be joined, but that the one face will be twisted relative to the other face as in Diagram 5.23b. Another polyhedron whose opposite triangular faces are twisted relative to one another can be inserted between the truncated tetrahedra so that all of the faces can be joined exactly together. (For example, Sketches c and d of Diagram 5.23 show how an octahedron can be placed between the two truncated tetrahedra as its opposite triangles are twisted relative to each other but a triangular prism would be no good as its end triangles lie directly over each other.) Diagram 5.23e shows octahedra and truncated tetrahedra joined in that way. That arrangement of figures can be extended by adding more octahedra and truncated tetrahedra. The spaces between the polyhedra in that type of arrangement can be as interesting and worth studying as the arrangement of polyhedra itself.

Instead of an octahedron, any other polyhedron, or group of polyhedra, can be put between the triangular faces of the truncated tetrahedra, provided their opposite faces are equilateral triangles and are twisted relative to one another. For example, two octahedra cannot be placed between them, as the one end triangular face lies directly over the other end triangular face (Diagram 5.24), but three octahedra can, as their end faces are twisted relative to each other, as shown (right). The bottom

5.22



5.23

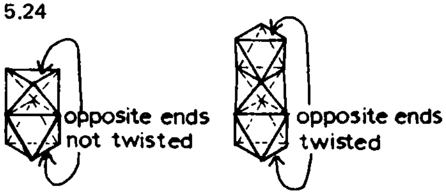


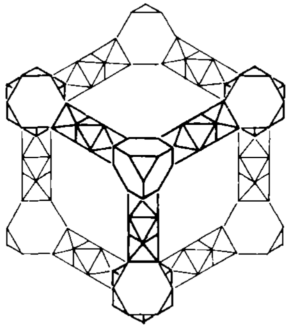
sketch shows the truncated tetrahedra joined by sets of three octahedra.

Other figures, such as the truncated octahedron, can be circumscribed by an imaginary tetrahedron so that a hexagonal face is adjacent to each vertex. When two such imaginary tetrahedra are joined, the hexagonal faces of the truncated octahedron lie directly over one another and can be joined directly (Diagram 5.25). If a more open arrangement of figures is wanted, polyhedra can be inserted between the individual truncated octahedra, provided their opposite end hexagons are not twisted. A hexagonal prism or a pair of hexagonal antiprisms can be placed between the figures with the desired effect, but a single hexagonal antiprism would give an unwanted twist to the arrangement. The lower sketch in Diagram 5.25 shows an arrangement of truncated octahedra and hexagonal prisms.

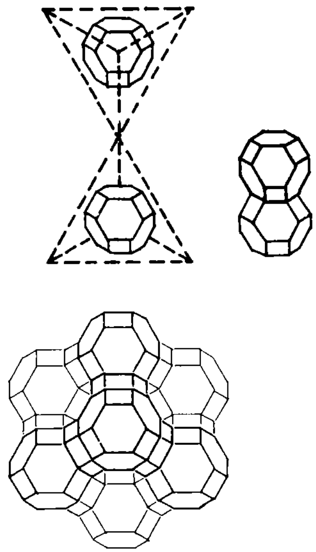
Though only a few examples of this approach to joining polyhedra have been mentioned, the possibilities are tremendous. Many other arrangements of close-packing polyhedra, such as truncated octahedra, rhombic dodecahedra, and triangular and hexagonal prisms can be used as bases for this method of joining polyhedra.

5.24





5.25

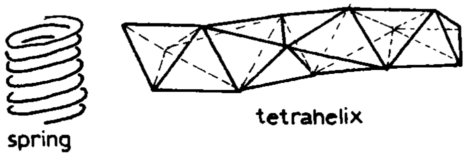


Helical Combinations of Polyhedra

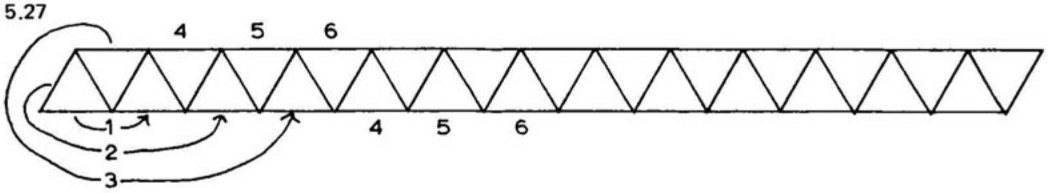
A helix is a curve which runs around an imaginary cone or cylinder, a common example being a spring. Tetrahedra can be joined together face to face to create a form which can be likened to a twisted column with triangular faces (Diagram 5.26). The edges of that arrangement of tetrahedra follow helical lines, so the figure is often referred to as a tetrahelix. The figure can be long or short, depending on the number of tetrahedra incorporated in it.

There are several ways of building a tetrahelix, the most obvious being to make a series of tetrahedra and join them face to face. However, it is much quicker to make it from a long strip of equilateral triangles joined edge to edge, as indicated in Diagram 5.27. Note that some of the faces meet in a concave way. A figure built in the second way may not be as strong as one made by the

5.26



5.27

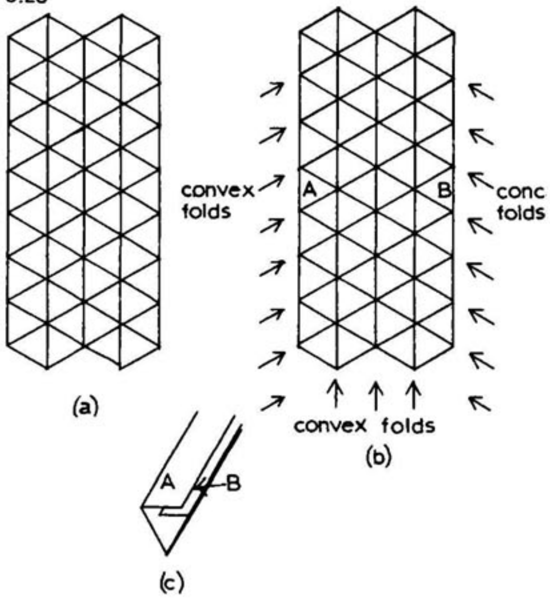


first method, as it lacks the bracing of the internal faces, but it is much quicker to build and looks a lot neater.

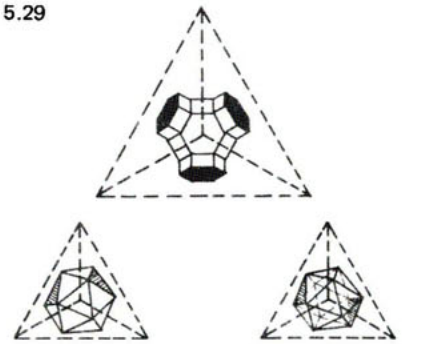
The quickest and neatest way of building a model of the tetrahelix was shown the author by R. Buckminster Fuller. In that method a long strip of paper is divided into four columns of equilateral triangles (Diagram 5.28a). This strip of paper should then be creased so that two sets of parallel lines are creased in one direction and the third set in the other direction (Diagram 5.28b). The paper should then be smoothed out and the column of triangles marked *A* placed over the column of triangles marked *B* to create a triangular strut, as shown in Diagram 5.28c. This strut should then be given a twist so that the triangles in Column A slide till they each rest over different triangles in Column B. The figure can then be glued, the ends closed, and the surplus triangles removed to create a model of a tetrahelix. Since the edges of a tetrahelix define a stable framework, good models can be made from toothpicks.

The tetrahelix is the simplest helix that can be built from polyhedra, and it can be used as the basis for many other helices. Many polyhedra can be circumscribed by a tetrahedron so that four of their faces touch the faces of that tetrahedron, and such figures can be joined into helices by circumscribing them with imaginary tetrahedra and then joining those tetrahedra. Very complex helices can be generated by forming a cluster of polyhedra inside each imaginary tetrahedron, such as the truncated octahedron and the four hexagonal prisms shown in the sketch of Diagram 5.29. When building a helix of octahedra or icosahedra, it is important to join those faces which line up with the faces of an inscribing tetrahedron, as indicated in Diagram 5.29. Very little work has been done on joining polyhedra in this way, and it is certainly worthy of further investigation.

5.28



5.29



Rotating Rings of Polyhedra

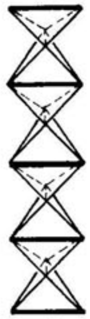
The opposite edges of a regular tetrahedron are at right angles to one another. A ring of tetrahedra can be formed by joining the opposite edges of eight tetrahedra (Diagram 5.30). If the joints between the figures are flexible enough to allow each tetrahedron to rotate about its neighbours, the whole ring of tetrahedra can rotate as a smoke ring rotates in the air. In fact, a ring of only six tetrahedra will also rotate, but only to a limited extent, as the individual figures obstruct each other's rotation.

One of the joints between two tetrahedra can be disconnected to form a string of tetrahedra, joined edge to edge (Diagram 5.31). Further tetrahedra can be added to the string and the ends closed to produce rotating rings of larger diameters. If a string of twenty-two or more tetrahedra is formed in this way, a knot can be tied in the string before joining the ends and the figure will still rotate. (Strong joints are needed between the individual polyhedra in that model.)

Any other figure or combination of figures which has a pair of opposite edges at right angles to one another can be joined to produce a rotating ring of polyhedra. Such figures include the truncated tetrahedron, the triakis tetrahedron, the twelve-faced deltahedron, and several of the figures with regular faces described in Chapter 3, as shown in Diagram 5.32, the twelve-faced deltahedron producing a particularly beautiful rotating ring.

Rotating rings of polyhedra can also be produced from figures which do not have pairs of opposite edges at right angles to one another. For example, sixteen octahedra can be joined edge to edge to form a strip (Diagram 5.33). The ends of the strip can be joined after it has been twisted, like the strip of paper shown, to produce a rotating ring of sixteen octahedra. Rotating rings of polyhedra can be made from other figures, as well, though this family of joined polyhedra has received little attention.

5.31



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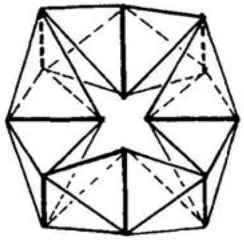
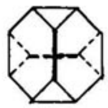
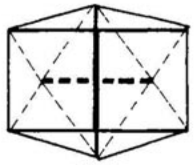


5.32



truncated tetrahedron

triakis tetrahedron



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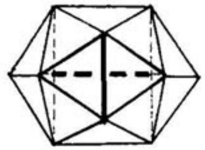

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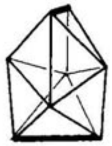
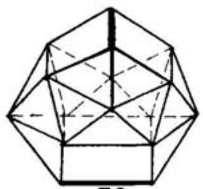
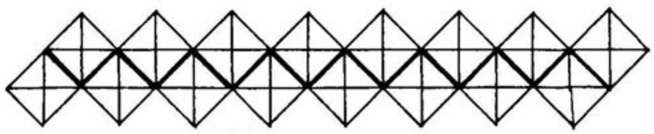


twelve faced deltahedron

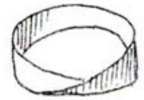


72

5.33



sixteen octahedra joined edge to edge

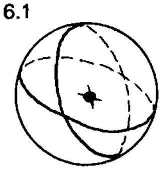


twisted strip of paper

6. The Geodesic Polyhedra of R. Buckminster Fuller and Related Polyhedra

A geodesic line is the shortest distance between two points across a surface. If that surface is curved, a geodesic line across it will usually be curved, too. A geodesic line on the surface of a sphere will be part of a great circle, a great circle being a circle whose center is the center of the sphere and whose plane divides that sphere into two equal parts. In other words, great circles are equators, though they are usually at different inclinations than the equator of a globe, like those in Diagram 6.1. Dr. R. Buckminster Fuller realized that the surface of a sphere could be divided into triangles by a network of intersecting geodesic lines and that structures could be designed so that their main elements either followed those lines or were joined along them. This is the basis of his very successful "geodesic domes." Since those domes are usually made from flat or straight materials, the curved geodesic lines are replaced by chords, resulting in a figure with plane faces and straight edges – in other words, a polyhedron. Strictly speaking, the expression *geodesic polyhedron* is a misnomer, as the edges of the polyhedron are chords to geodesic lines, but, since the word *geodesic* is used so extensively in this way, it is used in this volume. This chapter describes some of these polyhedra, explaining some of their advantages as geometric bases for domes.

6.1



A Simplified Approach to the Generation of Geodesic Polyhedra

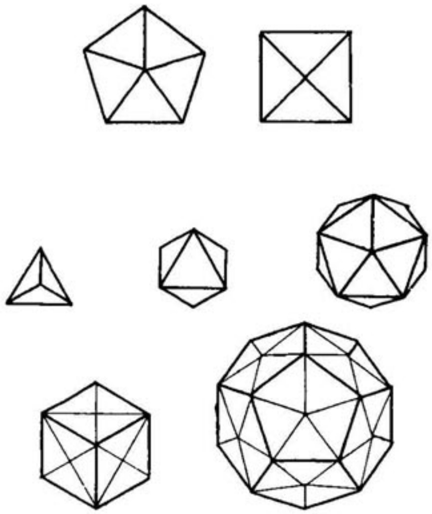
Geodesic polyhedra can be evolved in a variety of ways, so a simplified approach, involving many assumptions, will be given first, followed by some of the alternatives.

Each polyhedron is defined by a network of geodesic lines which intersect to define its vertices, while its edges are defined by chords between those vertices. The first problem to consider is the arrangement of the geodesic lines over the surface of the sphere. Clearly, the lines can be arranged in a random manner, but the calculation of such a figure could be lengthy and the resulting figure unsymmetrical, making the standardisation of its components very difficult. It is therefore usual to subdivide an existing polyhedron in an orderly manner with a series of lines and then to project those lines and the edges of the polyhedron onto the circumscribing sphere as geodesic lines. The polyhedron used as a basis for the geodesic polyhedron is called the principal polyhedron. To start, the Platonic polyhedra will be used as principal polyhedra.

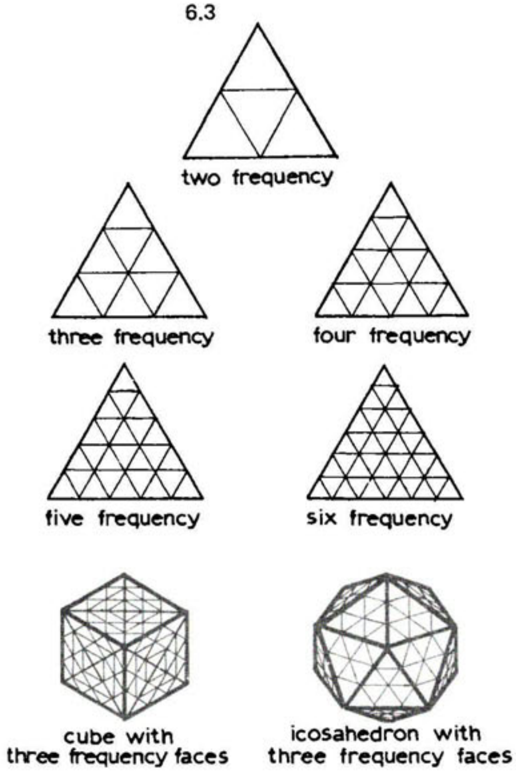
There are many ways in which the faces of a principal polyhedron can be subdivided, but just one method – the Alternate Method – will be described at this stage. The Alternate Method can only be used on faces which are triangles or which have been triangulated, so if a cube or a dodecahedron is to be used as a principal polyhedron, its faces must first be triangulated. This can be done by drawing diagonals across their faces, but it is preferable to draw lines from each vertex to the face center, as in Diagram 6.2, for reasons which will be apparent later. When that is done, the triangulated Platonic polyhedra are as shown in the lower set of sketches.

Each triangular face can then be subdivided into a series of smaller triangles by lines running parallel to the original edges of the triangles, as shown in the upper part of Diagram 6.3. Each face has been named according to the number of parts into which its original edges have been divided (that is, the original edges of a three-frequency face have been divided into three parts, the edges of a four-frequency face into four parts, and so on). Only low-frequency subdivisions are shown here, but the idea can be extended to much higher frequencies. At this stage, the figures are Platonic polyhedra whose plane faces are subdivided into a series of small triangles, as shown by the two examples on the bottom line of the diagram.

6.2



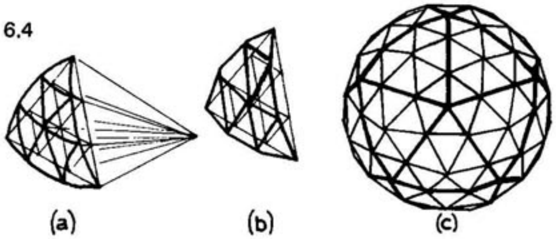
6.3



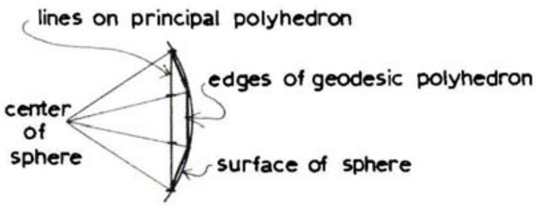
The next step is to imagine that the subdivided polyhedron is surrounded by its circumscribing sphere, with its vertices touching that sphere. Each edge of the circumscribed polyhedron, together with each of the lines which subdivide its faces, should be projected from the center of the polyhedron onto the surface of the circumscribing sphere, as in Diagram 6.4a. Each of those projected lines defines a geodesic line which is an arc of a great circle. The geodesic lines intersect to define the vertices of the geodesic polyhedron. Those vertices can be joined by chords to the great circle arcs, as in Diagram 6.4b to define the edges of the polyhedron, such as the three-frequency icosahedron of Diagram 6.4c.

When the subdivided principal polyhedron is projected onto the circumscribing sphere, each of its edges and each line drawn across a face becomes longer. Some of those lines are projected further outwards and become longer than the others, so the edges of the polyhedron are not all the same length and many triangular faces are neither equilateral nor isosceles. Before the lines and edges are projected onto the circumscribing sphere, the sums of the face angles about many of the vertices are equal to 360°, which produces plane surfaces. Once the edges and lines are projected outwards to define the new polyhedron, the sum of the face angles about each vertex becomes less than 360°. It can be interesting to compare models of a figure before and after its edges are projected outwards.

6.4



6.5



Comparisons of the Five Platonic Polyhedra as Principal Polyhedra

It is usually an advantage if a dome has as few different types of face or edge as possible. Diagram 6.6 lists the numbers of faces and edges for some of the lower-frequency figures, the numbers in brackets being the numbers of different types of faces or edges each figure has. Notice that the figures derived from the icosahedron and the dodecahedron tend to have fewer types of components for a given number of faces or edges. This is because the icosahedron and the triangulated dodecahedron start off with

6.6

	tetrahedron		octahedron		icosahedron		cube		dodecahedron	
	faces	edges	faces	edges	faces	edges	faces	edges	faces	edges
1 frequency	4 (1)	6 (1)	8 (1)	12 (1)	20 (1)	30 (1)	24 (1)	36 (2)	60 (1)	90 (2)
2 frequency	16 (2)	24 (2)	32 (2)	48 (2)	80 (2)	120 (2)	96 (3)	144 (5)	240 (3)	380 (5)
3 frequency	36 (2)	54 (3)	72 (2)	108 (3)	180 (2)	270 (3)	216 (6)	324 (10)	540 (6)	810 (10)
4 frequency	64 (5)	96 (6)	128 (5)	192 (6)	320 (5)	480 (6)	384 (10)	576 (16)	960 (10)	1440 (16)

more faces than the other Platonic polyhedra, so they generate more facets when they are subdivided. This is an important reason for using those two figures as principal polyhedra, though there are other factors to consider, too.

Six triangles meet at most of the vertices of these polyhedra. But it is impossible to design a convex polyhedron which has six triangles meeting at every vertex, as such a figure would have $6V$ face angles (V being the number of vertices). A triangulated figure with $6V$ face angles has $2V$ triangular faces, as each triangle has three face angles. Hence, the sum of all the face angles of such a figure is

$$2V \times 180° \; (= 360° \times V).$$

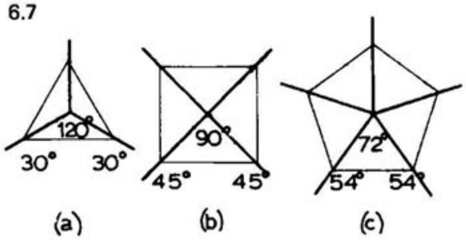
It has been established earlier that the total sum of all the face angles of a convex polyhedron is always equal to

$$360° \times V - 720°,$$

so such a figure cannot be a convex polyhedron. The vertices where fewer than six triangles meet can be regarded as the places where that factor of 720° is removed, to make the figure a convex polyhedron. For the geodesic polyhedra described so far, the vertices where fewer than six triangles meet are those defined by the original vertices of the tetrahedron, the octahedron, and the icosahedron or by the face centers of the cube and the dodecahedron.

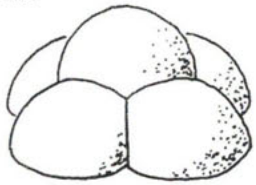
The faces surrounding each vertex form a shallow pyramid whose apex is that vertex. The more faces a figure has, the flatter the pyramids, until with a very large figure they are almost flat. When only three edges meet at a vertex, as with a figure derived from the tetrahedron, the triangles about that vertex are almost 120° 30° 30° isosceles triangles, as in Diagram 6.7a. Though the triangles are almost that shape, the larger angles must always be less than 120°. It is apparent that there is a great difference in length between the longer and the shorter edges of such triangles. Where four edges meet at a vertex, as in figures derived from the cube and the octahedron, the triangles about such a vertex are almost right-angled isosceles triangles like the one in Diagram 6.7b, though the angles must always be less than 90°. The difference between the longer and shorter edges is much less in that type of triangle. Where five triangles meet at a vertex, as in figures derived from the icosahedron and dodecahedron, the faces about such a vertex are almost 72° 54° 54° triangles, as in Diagram 6.7c, and the difference in length between the longer and shorter edges is less than in the previous examples. This is one reason why the differences between the longest and the shortest edge lengths always tends to be greater for polyhedra derived from the tetrahedron than for those derived from the cube and the octahedron. For polyhedra derived from the cube and the octahedron, those differences are, in turn, greater than those between the edge lengths of figures derived from the icosahedron and the dodecahedron. For practical reasons, it is often preferable that the edges of a dome be approximately the same length and hence its faces approximately the same size, another reason for basing a dome on a geodesic polyhedron derived from the icosahedron or the dodecahedron.

6.7



At times other considerations may be more important. Chapter 5 showed that cubes could be close-packed and that icosahedra and dodecahedra could not. When domes are to be joined (as in Diagram 6.8), it is often easier to use figures derived from the cube or the octahedron than to use figures derived from the icosahedron or the dodecahedron. On the whole, figures derived from the icosahedron and the dodecahedron are often the best to use, though figures derived from other principal polyhedra should not be ignored.

6.8

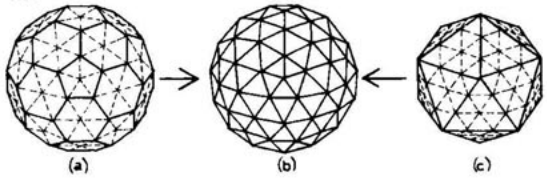


Other Principal Polyhedra

In most cases a suitable geodesic polyhedron can be derived from one of the five Platonic polyhedra, but there are situations where a figure derived from a different principal polyhedron may be required. Such figures can be evolved from the chosen principal polyhedron exactly as before, by dividing its faces into triangles and then subdividing the triangles to a suitable frequency.

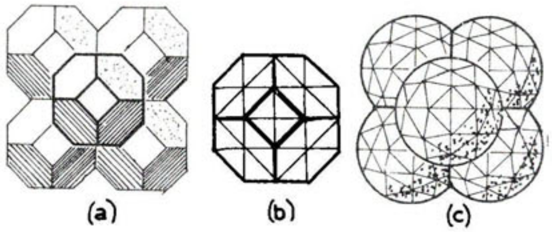
Any of the Archimedean polyhedra can be used as a principal polyhedron, though it will often be found later that almost identical figures could have been derived from one of the Platonic polyhedra. For example, the figure derived by triangulating the faces of a truncated icosahedron (Diagram 6.9a) is almost the same as the figure derived by subdividing the faces of an icosahedron to three frequencies (Diagram 6.9c). This kind of relationship is worth remembering when building models or doing calculations.

6.9



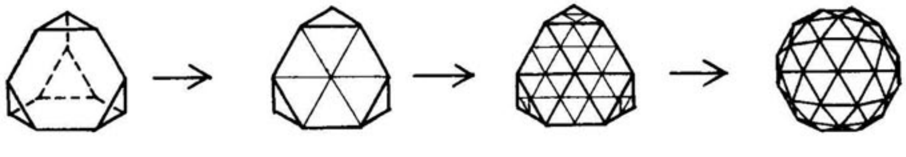
There are two Archimedean polyhedra which are particularly important as principal polyhedra. As was demonstrated in Chapter 5, the truncated octahedron is the only Archimedean polyhedron which close-packs by itself (Diagram 6.10a). Each face of the truncated octahedron can be triangulated and the triangles subdivided to whatever frequency is required (Diagram 6.10b). Clusters of domes, packing together like soap bubbles as in Diagram 6.10c, can be derived from such figures. (The geometric data for such an arrangement is given in Appendix 2.)

6.10



The second Archimedean polyhedron which is important as a principal polyhedron is the truncated tetrahedron. Geodesic polyhedra are derived from this figure in the same way as before, by triangulating the faces and subdividing them to a suitable frequency before projecting the lines onto the circumscribing sphere, as shown in Diagram 6.11. These figures cannot be derived from one of the Platonic polyhedra in the normal way, and their

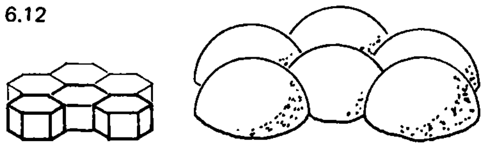
6.11



importance is that they have many of the symmetries of the tetrahedron without having all of its disadvantages. Geodesic polyhedra derived from the tetrahedron always have four vertices where only three edges meet, and, as was discussed earlier, there is a great difference between such a figure's longest and shortest edges. The truncated tetrahedron, however, has twelve vertices, each of which defines a vertex on the geodesic polyhedron where five edges meet. This means that such figures have less variation in edge length than similarly-sized figures derived from the tetrahedron. And, although the higher-frequency figures have an unfortunately large number of different-sized edges, the lower-frequency figures may be worth considering in certain situations. Geometric data for some of those figures appear in Appendix 2.

Facially regular prisms and antiprisms can be used as principal polyhedra. The most important of them is the hexagonal prism, as hexagonal prisms can be close-packed (Diagram 6.12a). The figures should be triangulated and subdivided as before and then projected onto their circumscribing spheres to generate the desired figures. Such geodesic polyhedra can be used to cover a given area without having spaces between the individual figures (Diagram 6.12b).

6.12

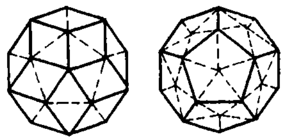


(a) (b)

The irregular polyhedra with regular faces described in Chapter 3 are not very symmetrical or spherical, so it is difficult to use them as bases for approximately spherical polyhedra. However, they can be used as bases for nonspherical shapes. Since their faces are all equilateral triangles, the five nonregular deltahedra can be particularly useful.

The duals of the Archimedean polyhedra, facially regular prisms, and facially regular antiprisms are often suggested as principal polyhedra, particularly the rhombic triacontrahedron. But when the faces of that figure are triangulated (Diagram 6.13a), the polyhedra derived from it are the same as those derived from a triangulated dodecahedron (Diagram 6.13b). Many other figures derived from the vertically regular figures can also be derived from facially regular figures, instead.

6.13



(a) (b)

There are three drawbacks to using a vertically regular figure as a principal polyhedron. The first is that they do not have circumspheres which touch all of their vertices, which tends to complicate calculations. The second is that many of the figures have great differences in the lengths of their edges, which produces polyhedra with similar characteristics. The third is that some of the figures have many edges meeting at some of their vertices and comparatively few at others. If such figures were used as geometric bases of domes, it would make the design of connections and junctions very difficult.

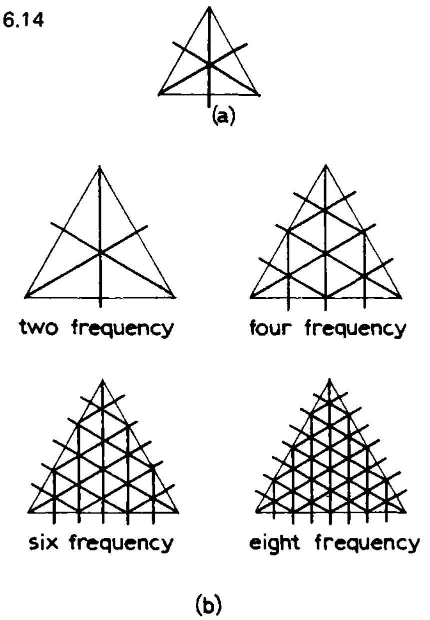
Other Ways of Subdividing the Faces of a Principal Polyhedron

The Alternate Method of subdividing the faces of the principal polyhedron is one of the most widely used methods, but there are others, such as the Triacon Method. The Triacon Method was formerly the more widely used, but there is a tendency nowadays to prefer the Alternate Method, as it is easier to visualise and use. In the Triacon Method, as with the Alternate Method, the first step is to divide nontriangular faces into triangles by joining their centers to their vertices. The next step is to join the vertices of each triangle to the midpoints of their opposite edges, dividing the triangle into six parts (Diagram 6.14a). Each of the lines defines an axis, and higher-frequency subdivisions can be achieved by drawing additional lines parallel to these lines, as shown in Diagram 6.14b. As with the Alternate Method, each subdivision is named according to the number of parts into which the edges of the original triangles are subdivided. Notice that in the Triacon Method the faces can only be divided into frequencies which are even numbers.

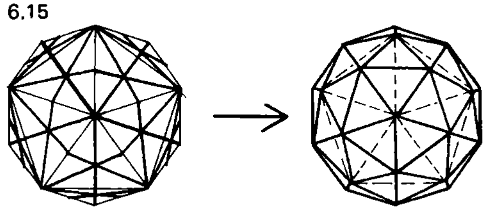
In Diagram 6.14 it can be seen that triangles which are half the size of the others occur around the edges of each large triangle. When the edges and lines are projected onto the circumscribing sphere to define the new polyhedron, the edges defined by the edges of the original triangles of the principal polyhedron are omitted so that pairs of "half-triangles" become whole triangles, as shown in the pair of sketches in Diagram 6.15, which show the icosahedron subdivided to two frequencies by the Triacon Method.

The figure in Diagram 6.15 could have been generated from a

6.14



6.15



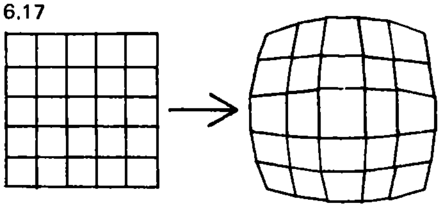
6.16

	tetrahedron		octahedron		icosahedron		cube		dodecahedron	
	faces	alternate method.	faces	alternate method.	faces	alternate method.	faces	alternate method.	faces	alternate method.
2 frequency	12	—	24	1F cube	60	1F dodecah.	72	3F octah.	180	3F icosah.
4 frequency	48	—	96	2F cube	240	2Fdodecah.	288	6F octah.	720	6F icosah.
6 frequency	108	—	216	3F cube	540	3Fdodecah.	648	9F octah.	1620	9F icosah.
8 frequency	192	—	384	4F cube	960	4Fdodecah.	1152	12Foctah.	2880	12Ficosah.

triangulated dodecahedron as a one-frequency Alternate Method figure. With the exception of figures derived from a tetrahedron, all figures generated from a Platonic polyhedron by the Triacon Method can also be derived from a different Platonic polyhedron by the Alternate Method. Diagram 6.16 shows the numbers of faces of various figures, together with the name of the figure from which each can be derived by the Alternate Method. It is useful to be aware of the relationship between the two methods, as the one may be easier to visualise in some cases than the other. An awareness of this kind of relationship can be particularly useful when attempting calculations.

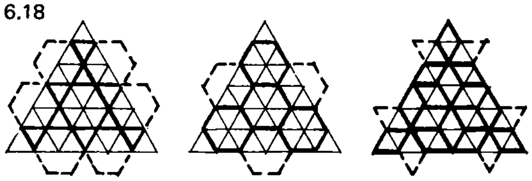
Though all the geodesic polyhedra described so far have had triangular faces, they can have other kinds of faces. For example, each face of a regular cube can be divided into a set of squares and the lines and edges projected onto the circumscribing sphere (Diagram 6.17). Such a figure has quadrilateral faces which are not necessarily flat.

6.17



Three important facial patterns can be derived from geodesic polyhedra with triangular faces. Each of the figures in Diagram 6.18 represents a face of a principal polyhedron, subdivided to six frequencies by the Alternate Method, over which one of the three patterns has been sketched. Parts of some of the faces are located on adjacent faces of the principal polyhedron, as indicated by dotted lines.

6.18

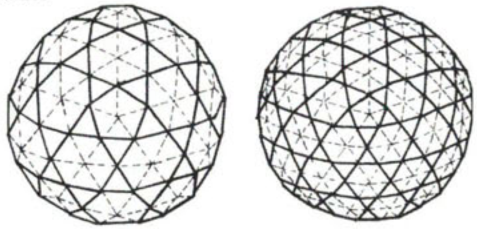


The first pattern consists of hexagons and triangles and can be superimposed on any figure whose faces have been triangulated to a multiple of two frequencies. The polygons formed about the vertices or face centers of the principal polyhedron depend upon the figure being used; the examples in Diagram 6.19 show the four- and six-frequency icosahedra, each of which has pentagons about the vertices of the original icosahedron.

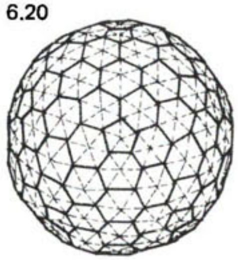
The second pattern consists of hexagons and can be superimposed on any figure whose faces have been triangulated to a multiple of three frequencies. Pentagons, squares, or triangles are formed at some of the vertices, depending on the principal polyhedron. Diagram 6.20 shows the six-frequency icosahedron, which has twelve pentagonal faces and one hundred ten hexagonal faces.

The third pattern consists of diamond shapes and can be superimposed over any figure whose faces have been subdivided to a multiple of three frequencies. All the faces of that type of figure are diamond-shaped, like the six-frequency icosahedron in Diagram 6.21.

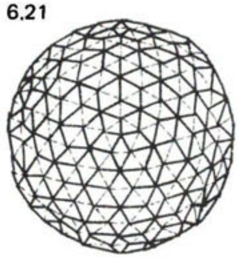
6.19



6.20



6.21



General Note on Figures with Nontriangular Faces

A plane can always be constructed to pass through three points, but that plane will not necessarily pass through a fourth given point. So, unless a face of a polyhedron is triangular, it may not define a single plane. If it is critical that those faces be flat, the geometry of the figure will often have to be recalculated. A plane cutting through a sphere will always define a circle on the surface of that sphere, so, if a face of that figure is to be flat and all of its vertices touch the circumscribing sphere, all of its vertices must touch a common small circle on that sphere. Another problem with a polyhedron which has nontriangular faces is that its edges do not define a stable framework. Such a figure cannot be built with a strut representing each edge, if the connectors are flexible and allow the struts to move.

Circumspheres, Interspheres, and Inspheres

Though these polyhedra are usually generated by subdividing a regular polyhedron in an orderly way, they do not have the degree of regularity exhibited by many of the polyhedra described earlier.

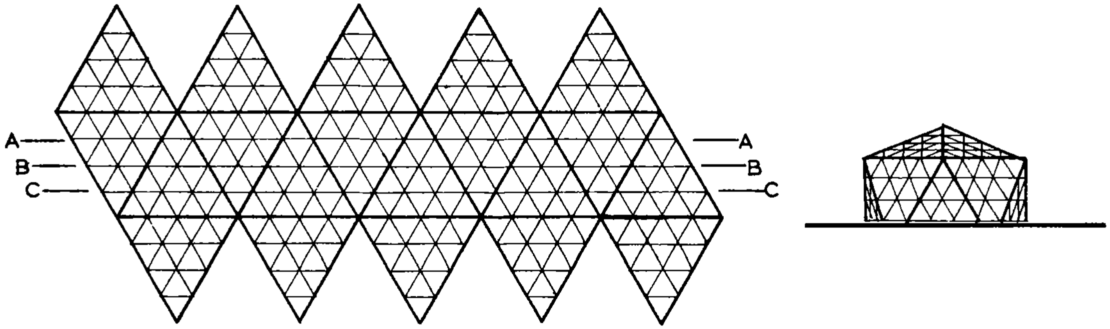
Each of the Platonic polyhedra has a circumsphere, an intersphere, and an insphere; each Archimedean polyhedron a circumsphere and an intersphere; and each of their vertically regular duals, an intersphere and an insphere. A geodesic polyhedron can only have one of those spheres, at the most, and so far it has been assumed that all of the vertices of such a figure touch a common circumsphere. However, the polyhedron could be designed so that all of its edges touched a common intersphere or all of its faces, a common insphere. But little work has been done on such figures to date.

Small-Circle Variations on Great-Circle Geometry

Once a Principal polyhedron is subdivided into a series of triangles and before its lines and edges are projected onto its circumscribing sphere, circuits of lines and edges can be traced around the figure. Three such circuits (A-A, B-B, and C-C) are marked on the net diagram (Diagram 6.22). If that figure were to be truncated along one of the circuits at this stage, the portion remaining would sit on level ground without gaps occurring between it and the ground, as shown in the right-hand sketch of Diagram 6.22.

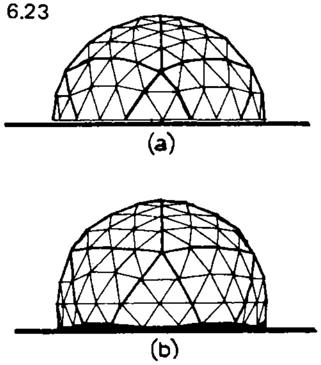
Unfortunately, when the lines and edges on the principal polyhedron are projected outwards, they generate separate great-circle arcs on the circumscribing sphere. Each of the circuits (identified by the letters *A-A*, *B-B*, and *C-C* in Diagram 6.22) is formed from ten different lines on ten different faces of the

6.22



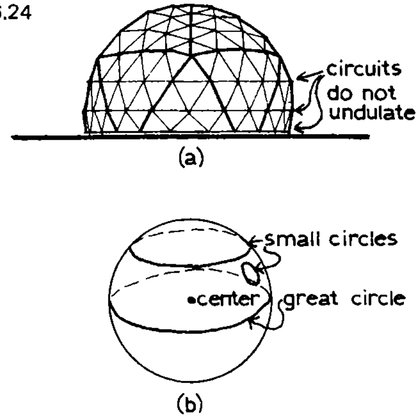
icosahedron. Each of the lines generates a separate great-circle arc. Since a complete great circle is an equator which divides the sphere into two equal parts and since the components of the circuit *B-B* divide the figure exactly in two, the arcs generated by each of these lines form part of a common great circle. If the figure is truncated along this circuit, as in Diagram 6.23a, it sits evenly on the ground, as all of its edges lie on the same plane. On the other hand, the lines in circuits *A-A* and *C-C* do not divide the figure into two equal parts, so the center of the circuit lies above or below the center of the polyhedron. When the lines in circuits *A-A* or *C-C* are projected outwards from the center of the polyhedron, they form ten different great-circle arcs. Each arc will arc upwards or downwards in relation to the circuit, depending on whether that circuit is above or below the equator. So if the figure were truncated along circuit *A-A* or along circuit *C-C* as in Diagram 6.23b, it would not sit evenly on the ground. This could be critical in many situations. Generally, it should never be assumed that a set of edges formed by a particular line or edge will be continued by another set of edges unless it is clear that they are parts of a common equator.

6.23



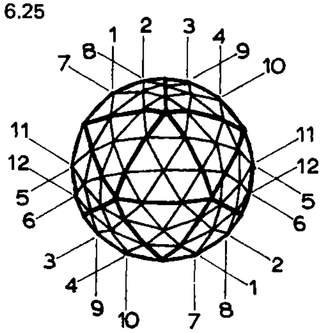
The circuits of edges indicated by the letters *A-A* and *C-C* in the net diagram of Diagram 6.22 can be made to flow smoothly round the figure as shown in Diagram 6.24a. In such a case all the edges of one of those circuits lie on a single common plane which does not pass through the center of the circumscribing sphere. In other words, the edges do not follow equators or great circles round the figure but trace circles like the tropics of Cancer and Capricorn on a globe. Such circles are called small circles, a small circle being *any* circle on a sphere which is not a great circle. Another definition of a small circle is that it is any circle on a sphere whose center does not coincide with the center of that sphere. Since the expression *small circle* can be used for very small circles as well as very large ones, the expression *lesser circle* is sometimes used to describe large small circles like the tropics of Cancer and Capricorn. (*Lesser circle* is really an archaic synonym for *small circle*.)

6.24



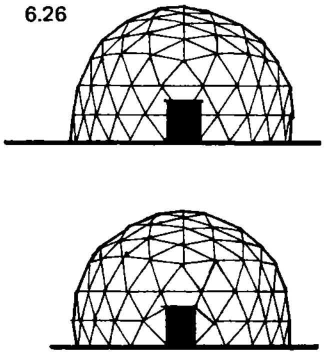
Though it may be difficult to tell the difference between a model of a figure using great-circle geometry and a model of a figure using small-circle geometry, the latter figure is quite different and must be calculated from a different set of equations. Though a figure can be made with only one or two of its circuits following small circles, it is usually better to make as many circuits as possible to follow small circles around the figure, to make it as symmetrical as possible. On the sketch of the three-frequency icosahedron in Diagram 6.25 the edges define no fewer than twelve small circles. Also in that drawing it can be seen that, for reasons of symmetry, all the edges of the original icosahedron define great-circle arcs on the circumscribing sphere. Since the majority of the edges of such a figure no longer follow geodesic arcs, a purist might not regard such figures as true geodesic polyhedra (R. Buckminster Fuller refers to them as "Truncatable Geodesics"). Geometric data for some of them appears in Appendix 2.

6.25



The main advantage of the small-circle figure is that a dome based on one of them can sit on level ground without any gaps between its base and the ground, even if that dome is not an exact hemisphere. Another advantage of small-circle figures is that they often have convenient horizontal circuits of edges, which facilitate fixing things on the insides of the domes. A third advantage of the small-circle figure is that, whereas the circuits of edges on a true geodesic polyhedron have to change directions as they move from one great-circle arc to another, disrupting the flow of lines inside the dome, the circuits of a small-circle figure flow smoothly round the figure. The disadvantages of this type of figure are that it has a greater number of different types of component and a bigger difference between the sizes of its largest and smallest components.

6.26

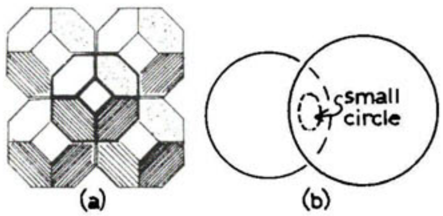


Further Geometric Modifications

One of the most common reasons for modifying the geometry of a polyhedron occurs when it is being used for such a purpose as a dome, and a door or similar fixture will not fit into one of the faces. In such a situation, certain of the figure's vertices can be relocated, as shown in the bottom sketch of Diagram 6.26.

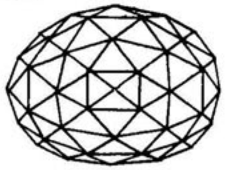
Another situation in which a certain amount of modification is required occurs when a cluster of geodesic polyhedra is based on a close-packing arrangement of polyhedra such as the truncated octahedra in Diagram 6.27a. Though the principal polyhedra close-pack, their circumscribing spheres are larger and penetrate one another, with common boundaries which are small circles, as depicted in Diagram 6.27b. In effect, there no longer are complete circumscribing spheres, and some vertices and edges are truncated by the adjoining figure. In such a situation the appropriate vertices and edges must be relocated on the boundary between the two spheres. The geometric data for a close-packing arrangement of two-frequency truncated octahedra, showing these modifications, is given in Appendix 2.

6.27



Finally, it should be realised that geodesic polyhedra need not be spherical (see Diagram 6.28). Several ellipsoidal polyhedra devised by Peter Calthorpe are described in Pacific Domes, *Domebook Two*.

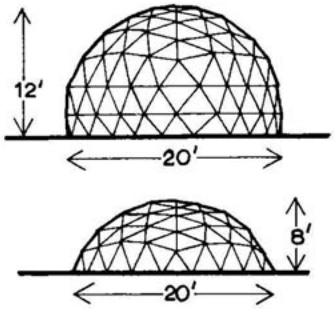
6.28



The Truncation of Geodesic Polyhedra

A complete polyhedron is not usually used as the basis for a dome, so the figure is truncated at a suitable place. Since a polyhedron can be truncated at many positions to give domes of several heights, it may be necessary to describe how much of a figure is being used. One way of doing this is to state the height and diameter of the figure, as in Diagram 6.29.

6.29



Another good way of describing how much of a triangulated polyhedron is being used is to express the number of triangles on the truncated figure as a fraction of the total number of triangles in the complete figure. Net Diagram 6.30 shows the icosahedron, with its faces subdivided to three-frequencies, before its lines and edges are projected onto the circumscribing sphere. It is assumed that the top vertex in the diagram is the top of the dome and that the figure is to be truncated horizontally at various positions to create domes of various heights. These domes will range in size from a very shallow one with only five triangular faces, up to the complete figure with 180 faces. The fractions on the left of the

6.30

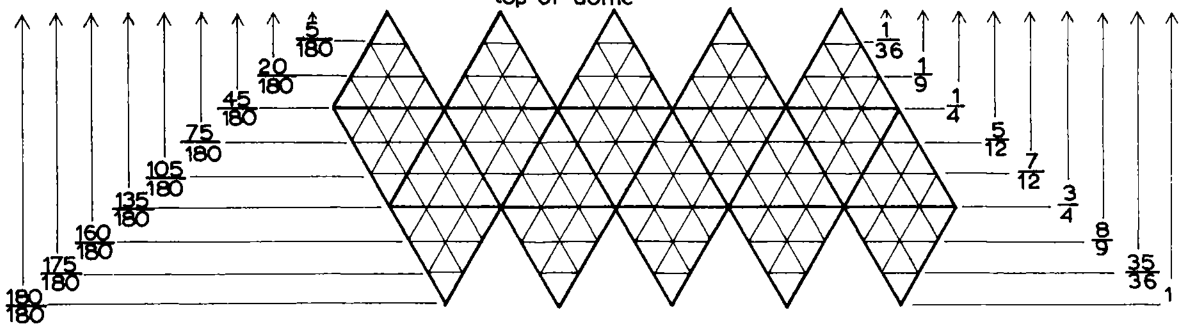
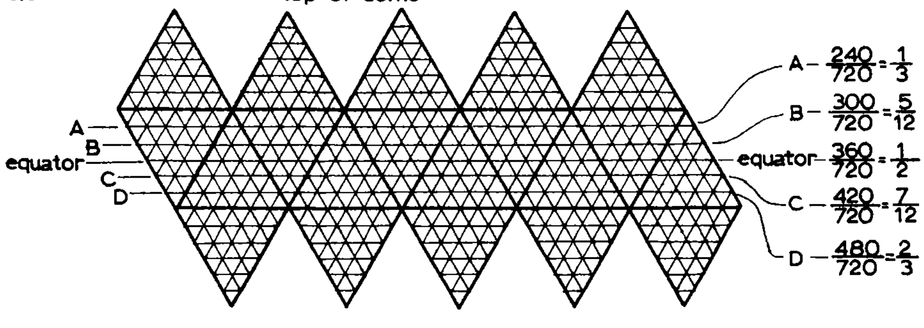


diagram express the numbers of triangles in each dome as fractions of the total number of triangles in the complete figure. These fractions can be simplified as shown on the right of the diagram.

This method may appear to be more complex than the one used in *Domebook Two*, where a figure truncated above the equator is a 3/8 dome and a figure truncated below the equator is a 5/8 dome, but it is more accurate. The problem with the *Domebook* notation is that ambiguities occur with the higher-frequency figures like the six-frequency icosahedron in Diagram 6.31. The questions are whether a 3/8 truncation should be along line *A* or line *B* and whether a 5/8 truncation should be along line *C* or line *D*. A six-frequency icosahedron has 720 triangular faces, so the diagram shows the numbers of triangles above each line, expressed as a fraction of 720. All of the fractions can be simplified as shown to give a simple, unambiguous indication of where the figure is truncated.

6.31



Before a model of a polyhedron can be constructed, the shapes of its faces must be known. Until recently, it would have been necessary to have made a calculation such as the one shown in Example 3 in Appendix 1. Fortunately, data is now available, making calculations unnecessary in most cases. Several tables of data for geodesic polyhedra appear in Appendix 2, and further data can be found in such publications as *Domebook Two*.

A model can be constructed if its face angles are known, but it is a lot easier to build it if its edge lengths are known. The edge lengths are usually expressed as proportions of the radius of the circumscribing sphere. Since each edge is a chord to that circumscribing sphere, those proportions are called chord factors. The first step is to choose the radius of the figure to be constructed and to multiply each chord factor by that radius to give the actual lengths of the edges of that figure.

To construct a three-frequency icosahedron with a radius of 10 centimeters (about 4 inches), first find the chord factors for the figure. Chord factors are often given to many decimal places, but for this example they can be rounded off to four decimal places as follows:

$$a = 0.3486$$
$$b = 0.4035$$
$$c = 0.4124$$

Since the chord factors are expressed in decimals, it is a lot simpler to work in metric units, as inches are usually divided into inconvenient fractions. Each chord factor should be multiplied by the chosen radius of 10 centimeters as follows:

$$a\ (= 0.3486 \times 10\ \text{cm.}) = 3.486\ \text{cm.}$$
$$b\ (= 0.4035 \times 10\ \text{cm.}) = 4.035\ \text{cm.}$$
$$c\ (= 0.4124 \times 10\ \text{cm.}) = 4.124\ \text{cm.}$$

Those dimensions can now be used to construct the faces of the figure, which can be joined to make a polyhedron about 20 centimeters, or 8 inches, in diameter. It is important not to confuse radius for diameter when multiplying the chord factors, otherwise the figure will be much larger than originally intended.

Construction of Models: Examples

Models can be built as frameworks of struts, each strut representing an edge of the figure, but most people find it quicker and easier to build models from card, especially if the following points are observed:

1. The card should not be too thick, or it will be hard to bend and join accurately, thus forming inaccurate models.
2. The faces should be cut out in groups whenever possible. Besides reducing the time spent in cutting and joining, it results in neater, more accurate models. Studying the symmetries of the figure will help the reader discover suitable groups of faces.
3. Patterns, such as those described in Appendix 3 save a considerable amount of time, as well as ensuring that components are the same size and fit together properly.

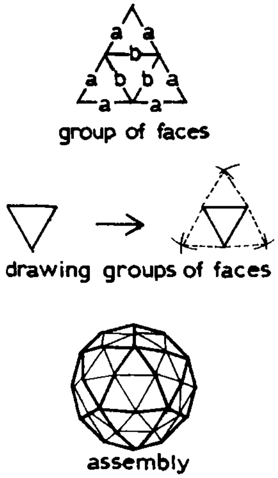
The models should be made as accurately as possible, especially the higher-frequency ones, as an inaccurate model may not fit together. It is a good idea to try a small lower-frequency figure before attempting a larger one. (Though a half-degree error in a three-frequency icosahedron may not be too serious, that same inaccuracy can make it impossible to join the faces of a six-frequency icosahedron.)

Sometimes groups of faces on a model are pushed inwards accidentally, creating a concave surface on the figure. If all the faces of the figure have not been fixed into position, those faces can be pushed out again from the inside. If the figure has been completed, the apparent tragedy can be overcome, either by sucking them out with one's mouth or by slipping a needle beneath them and levering them out. The following examples show how, with a little planning, much time can be saved when building models. In each example the letters *a, b, c,* and so on are the same as those used to identify the chord factors in the tables in Appendix 2.

The Two-Frequency Icosahedron

Each face of the icosahedron generates a congruent set of four faces, which can be drawn out as a group. Having converted the chord factors into suitable edge lengths, the first step is to construct the central triangle of the group, an equilateral triangle of edge length b. Next, a pair of compasses should be set to length a and, by constructing intersecting arcs from the vertices of the central triangle, the three isosceles triangles with edges a, a, b, can be defined, as shown in Diagram 6.32. Twenty such groups of faces should be cut out and scored along their internal edges to allow the figure to bend into shape when it is assembled. The groups of faces are joined together as if each group were a face of an icosahedron.

6.32

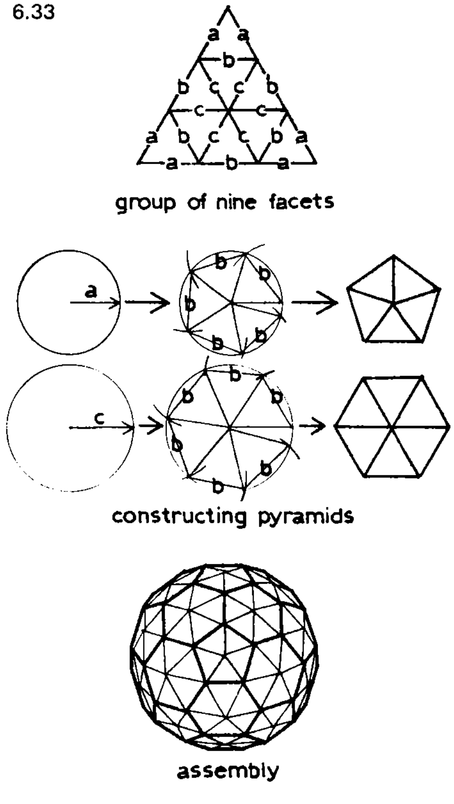


The Three-Frequency Icosahedron

Each face of the icosahedron generates a congruent set of nine facets. There are at least two quick ways of building this figure, but in either method the first step is to convert the chord factors into suitable edge lengths. In the first method, a circle of radius a should be drawn and its perimeter divided with five chords of length b. The chords will not divide up the whole of the perimeter; there will be a small part left over. The ends of the chords should be joined to the center of the circle to create a pentagonal shape with a small gap about 6° wide, caused by the odd part of the perimeter, as shown in Diagram 6.33. The shape should be cut out and the edges creased or scored and then the gap or "sinus" closed to create a shallow pentagonal pyramid. Twelve such pyramids will be needed. Next, draw a circle of radius c and divide its perimeter with six chords of length b. Again, there will be a sinus about 6° wide, which can be closed when the unit has been cut out, to create a shallow hexagonal pyramid. Twenty of these hexagonal pyramids will be needed. The final step is to join the shallow pentagonal and hexagonal pyramids as if they were the pentagonal and hexagonal faces of a truncated icosahedron.

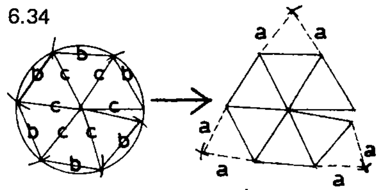
An alternative way of building the same figure is to construct the complete cluster of nine facets generated by each face of the

6.33

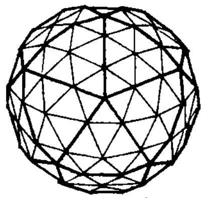


principal polyhedron. First, draw a circle of radius c and divide its perimeter with six chords of length b. Join the ends of the chords to the center of the circle to create a hexagonal shape with a sinus, as shown in Diagram 6.34. From the vertices of this hexagonal shape draw intersecting arcs with a pair of compasses set to the length a to define the final three vertices of the other three triangles. Twenty groups of faces like this one will be needed, and once their internal edges have been scored they can be joined together as if each group were a triangular face of an icosahedron.

6.34



drawing group of nine faces



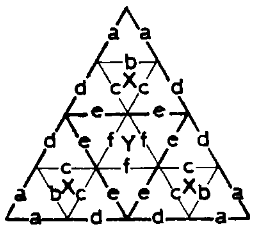
assembly

The Four-Frequency Icosahedron

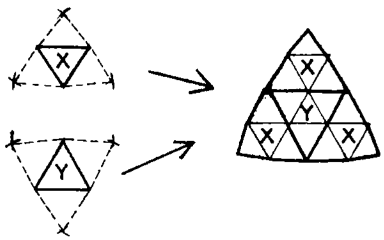
The sixteen faces generated by each face of the icosahedron can be divided into four groups, each group consisting of four triangles as in Diagram 6.35. There are two types of group, designated by the letters X and Y in the diagram. Each group can be drawn by constructing the central triangle and then constructing the other three triangles from its vertices. Twenty groups of type Y and sixty groups of type X will be needed. Once they have been cut out and their edges scored and creased, they can be joined into larger clusters of sixteen triangles by joining three groups X to each group Y, as shown. The large clusters of sixteen triangles should then be joined as if they were curved triangular faces that were being used to make an icosahedron with curved faces.

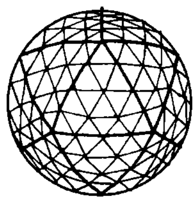
Though the construction of only a few low-frequency polyhedra has been described, similar techniques can be used to speed up the construction of the higher-frequency figures, too. It is worth repeating that the faces of the higher-frequency figures need to be drawn out, cut, and joined very accurately, so it is a good idea to try simpler models first.

6.35



group of sixteen faces





assembly

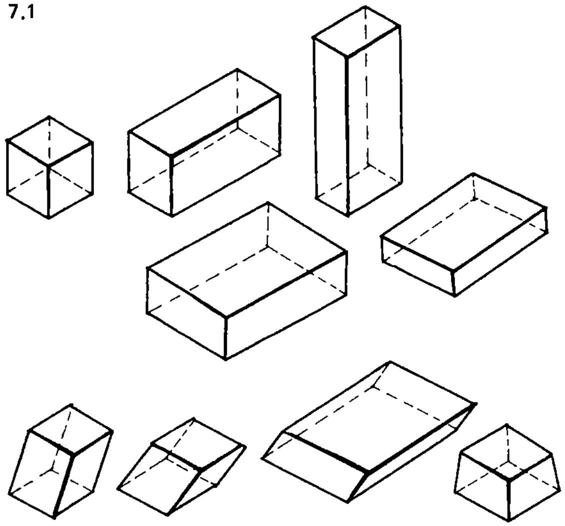
7. Some Irregular Polyhedra

Most of the polyhedra described so far have possessed a high degree of order and symmetry, but any polyhedron can be distorted and modified to give many variations. For example, a large variety of figures can be created by stretching the faces of a regular cube or by altering its face angles or by adjusting both its faces and its angles, as illustrated in Diagram 7.1.

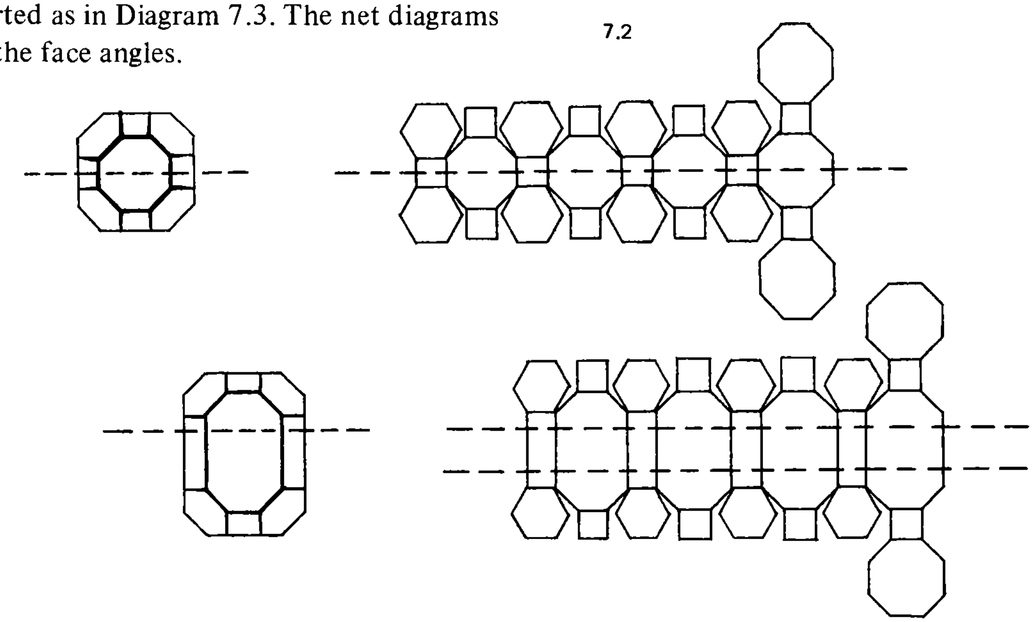
The great rhombicuboctahedron serves as an example of how faces can be stretched in Diagram 7.2. The top two sketches show how a band of squares and octagons can be traced around that polyhedron and its net diagram. The band of squares and octagons can be stretched, as in the lower pair of diagrams, to create an elongated figure. The figure could also be stretched in other directions.

Changing the face angles of a polyhedron is a lot trickier, but a simple example would be the way in which an elongated hexagonal prism can be distorted as in Diagram 7.3. The net diagrams illustrate the changes in the face angles.

7.1



7.2



The main point to remember is that a polyhedron need not remain as one of the ordered shapes described earlier, but it can be distorted and changed as the situation may demand.

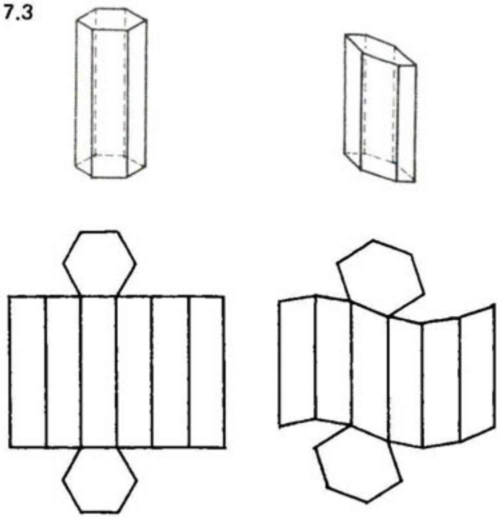
Truncations of Existing Polyhedra

A truncated cube can be formed from a cube by cutting away the portions of that cube which surround its vertices. A second truncation can be made which removes larger portions of the cube, to produce a cuboctahedron. Even larger portions of the cube can be removed as a third truncation, to reveal a truncated octahedron. If more of the cube is cut away, the dual of the cube, the octahedron, is produced. Further truncation of the cube simply diminishes the size of the octahedron (Diagram 7.4).

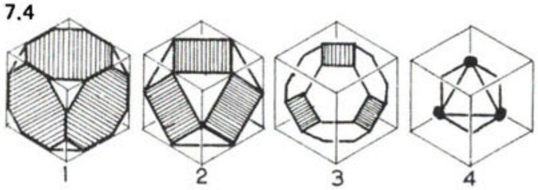
If similar truncations are made of the other Platonic polyhedra, figures which are Platonic or Archimedean polyhedra are produced, but if other polyhedra are truncated, some interesting new polyhedra can be created. The first example (Diagram 7.5) shows the truncated tetrahedron, whose edges are indicated by the thinner lines in the diagrams. The first truncation of this figure produces a polyhedron in which a triangular face replaces each vertex of the original figure, a hexagon replaces each original triangular face, and a dodecagonal face replaces each original hexagon. Some of the faces of the new figure are not regular. In the second truncation, the new triangular faces become larger, and the dodecagons and hexagons of the first truncation become hexagons and triangles, respectively. In the third truncation the triangles and hexagons which lie on the faces of the original truncated tetrahedron become smaller, and the triangles formed about each vertex of the original figure become so large that they truncate one another and become hexagons. In the fourth truncation the hexagons and triangles which lay on the faces of the original truncated tetrahedron disappear, and the resulting figure is the dual of the truncated tetrahedron, the triakis tetrahedron.

If truncations are made to the triakis tetrahedron instead of the truncated tetrahedron, the same figures can be produced, though in the reverse order, as in Diagram 7.6.

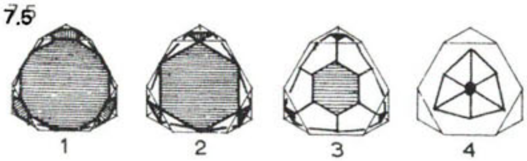
7.3



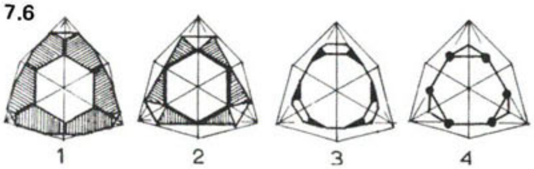
7.4



7.5



7.6



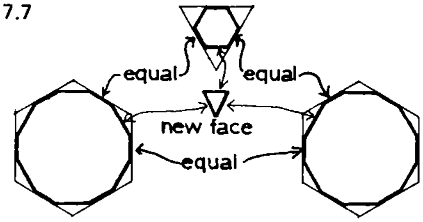
The geometry of a figure produced as the second truncation is fixed by the geometry of the original figure, but a certain amount of variation is possible with the figures produced as the first and third truncations. For example, in a first truncation very little of the original figure may be removed, or so much may be removed that it is almost a second truncation. The shapes of the faces of a first truncation can be established by drawing out the faces of the original figure; in Diagram 7.7 they are the triangular and hexagonal faces of a truncated tetrahedron. On these faces mark the new faces, taking care that the edge of one face is the same length as the corresponding edge of the adjacent face. The edge lengths of the new figure can be measured or calculated from such a diagram.

The shapes of the faces of a second truncation of a figure, such as the second truncation of the truncated tetrahedron, can be worked out by sketching the new faces on a similar set of diagrams, as shown on top in Diagram 7.8. Alternately, they can be worked out from the faces of the dual of the truncated tetrahedron, the triakis tetrahedron. If that figure is used, the vertices of the new figure must be established by measuring equal distances from all similar vertices, as in the bottom sketch of Diagram 7.8. Since the vertices of the new figure touch the midpoints of the edges of the truncated tetrahedron and since the latter figure has an intersphere which touches each edge at its midpoint, that intersphere must be the circumsphere to the new figure. This will be true for all the second truncations of the Archimedean polyhedra.

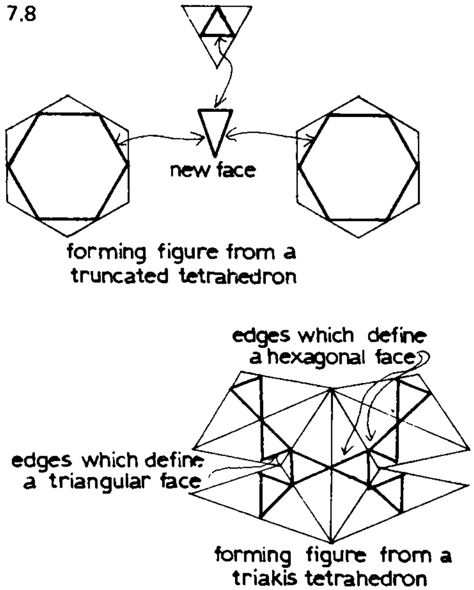
It is usually easier to do a third truncation of an Archimedean polyhedron by making a first truncation of its dual. The faces of the dual of the figure under question should be drawn out and similar distances measured from similar vertices. Joining these points as shown in Diagram 7.9 will establish the size and shape of the nonregular faces and the sizes of the regular hexagons and triangles.

Diagram 7.10 offers further examples of figures produced by truncating Archimedean polyhedra. Since the fourth truncation produces the dual of the figure being truncated, it is not included,

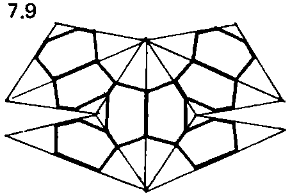
7.7



7.8



7.9



though it is worth remembering that it may be easier to work from the dual than from the original figure. In each sketch the edges of the original polyhedron are shown by the thinner lines, and the edges of the new polyhedra are shown by thicker lines.

In the first example – the first truncation of the truncated octahedron – a triangular face appears for each vertex of the original figure, an octagon replaces each square face, and a twelve-sided face replaces each hexagon. The second truncation enlarges the triangular faces to make the twelve-sided faces into hexagons and the octagonal faces into squares. The third truncation results in smaller hexagonal and square faces, and the triangular faces become so large that they truncate one another to become hexagons.

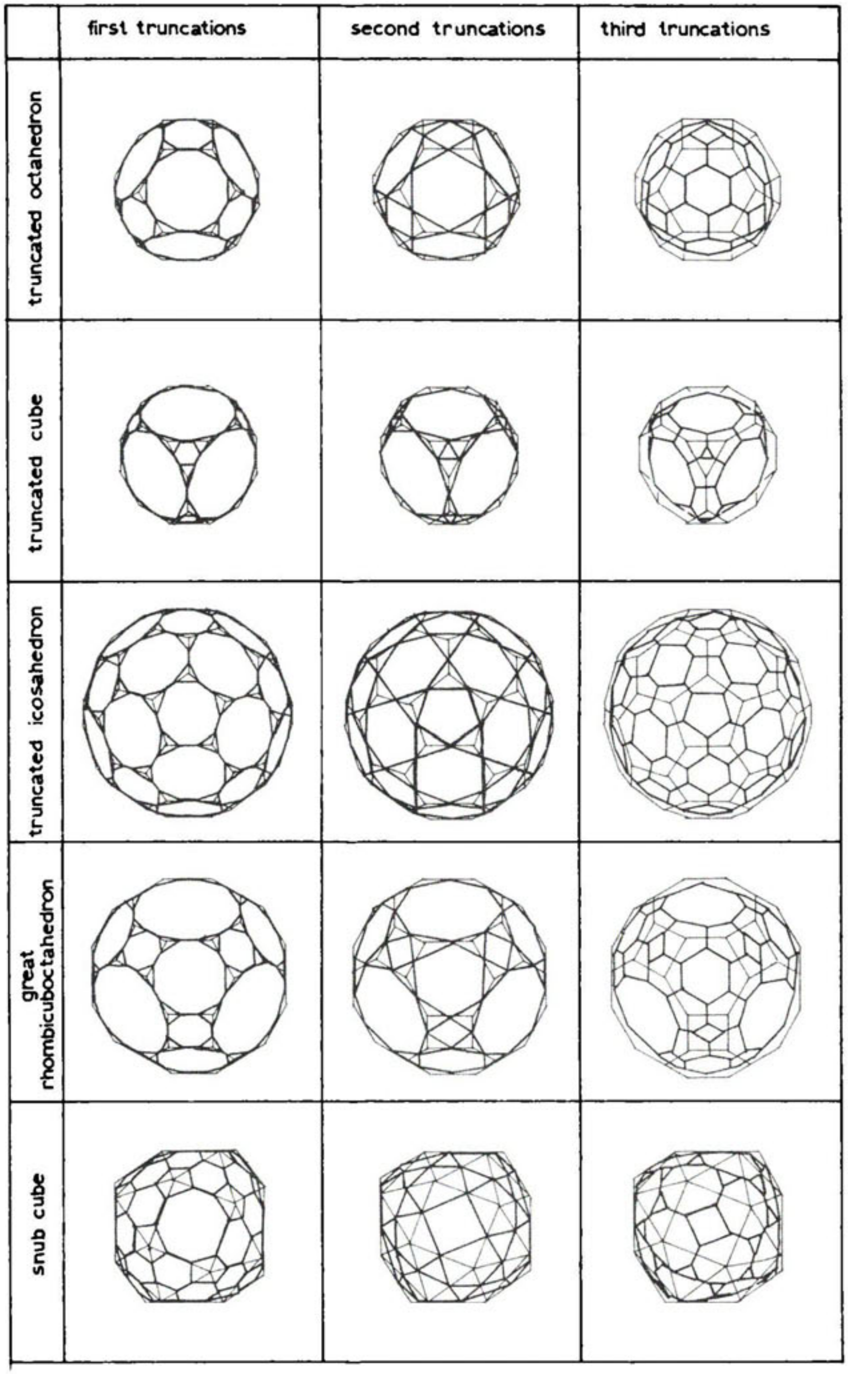
Similarly, with the truncated cube an isosceles triangle appears for each vertex on the first truncation and those triangles are bigger on the second truncation, and at the third truncation they are so large that they truncate one another and become hexagons. It is interesting to note that a sketch of the net diagram for the figure produced by the second truncation of the truncated cube appears in Albrecht Dürer's *Unterweisung der Messung Mit dem Zirkel und Richtscheit*, published in 1525.

The next two sets of sketches in Diagram 7.10 show the figures produced from the truncated icosahedron and the great rhombicuboctahedron. In both cases, triangles appear at the vertices of the original figure at the first truncation, are larger on the second truncation, and form hexagonal faces on the third truncation.

Some Archimedean polyhedra, such as the snub cube, have more than three faces meeting at each vertex. At the first truncation of that figure irregular pentagons appear which are larger on the second truncation and become decagons at the third truncation.

The facially regular prisms and antiprisms can be truncated in the same way as the Archimedean polyhedra. In the pentagonal prism shown as an example in Diagram 7.11, triangular faces are produced in the first truncation, become larger in the second truncation, then become hexagons in the third truncation. Upon

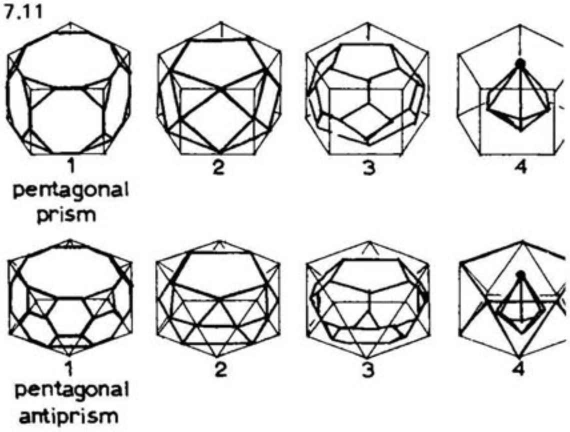
7.10


first truncations
second truncations
third truncations
truncated octahedron
truncated cube
truncated icosahedron
great rhombicuboctahedron
snub cube

the fourth truncation the hexagonal faces become the ten triangular faces of the pentagonal dipyramid, the dual of the pentagonal prism. In the pentagonal antiprism, quadrilateral faces are produced at the first truncation and become larger in the second truncation. At the third truncation the quadrilaterals become octagons, and at the fourth truncation they become the trapezoidal faces of the trapezoidal decahedron, the dual of the pentagonal prism. Truncations of the other facially regular prisms and antiprisms are similar to these two examples.

Figures which are neither facially nor vertically regular can also be truncated, though care must be taken if plane faces are required. For example, any of the figures produced by one of the truncations previously described could be truncated again to create even more complex polyhedra.

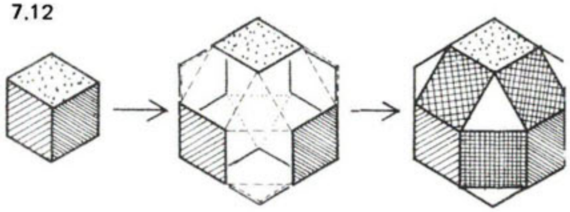
7.11



Insertion of Rectangular Faces Between the Edges of an Existing Polyhedron

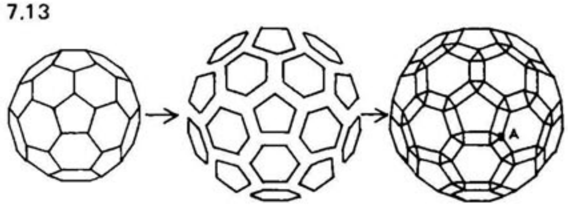
If the six square faces of a cube are forced apart, twelve square faces can be inserted between them as in Diagram 7.12. Eight triangular "gaps" occur between the newly added faces, and the resulting figure is the small rhombicuboctahedron.

7.12



The cuboctahedron, the small rhombicuboctahedron, and the small rhombicosidodecahedron result if squares are added in this way to the other Platonic polyhedra, but a completely new set of figures can be produced by doing this to the Archimedean polyhedra. For example, the faces of a truncated icosahedron can be forced apart and a series of rectangles inserted between its edges, as shown in Diagram 7.13. Triangular "gaps" appear between the newly added rectangular faces of the polyhedron. At first one might think that squares could be inserted between the hexagonal and pentagonal faces of the truncated icosahedron, but if this were so, the "gaps" between them would be equilateral triangles. At certain vertices, such as the one indicated by the letter A, a hexagon, two rectangles, and a triangle meet. If the triangles were equilateral, the sum of the face angles about that vertex would be 360° (120° + 90° + 90° + 60°), so the triangles

7.13



cannot be equilateral and the question of their actual shape arises. It can be answered by returning to the original example of the expansion of the cube.

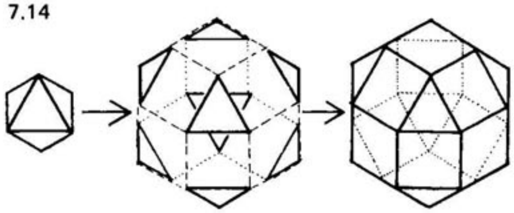
When the squares were inserted between the edges of the cube, eight triangular "gaps" appeared between the newly added square faces. The triangles were the same shape and defined the same planes as the faces of a regular octahedron, the dual of the cube. The small rhombicuboctahedron can also be formed by forcing apart the faces of an octahedron and inserting square faces between its edges, as in Diagram 7.14. In this case six square "gaps" appear between the newly added faces, corresponding to the six square faces of the regular cube.

Returning to the figure derived from the truncated icosahedron, the sixty triangular "gaps" between the rectangular faces are the same shape and define the same planes as the faces of the dual of the truncated icosahedron, the pentakis dodecahedron. Hence, the face angles of the triangular gaps are as they appear in Diagram 7.15, and the sum of the face angles about any vertex is less than 360° (355° 41′, or 356° 57′). Since the triangles are isosceles and the edge lengths of the pentagons and hexagons equal, there are two sizes of rectangle. If desired, one of the sets of rectangles can be regular squares.

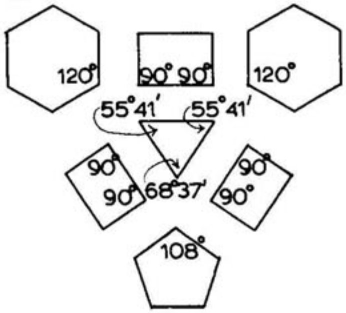
Just as the small rhombicuboctahedron can be created by inserting square faces between the faces of either a cube or its dual (the octahedron), the previously described figure can also be formed by inserting rectangular faces between the faces of the dual of the truncated icosahedron, the pentakis dodecahedron. If the figure is formed from the pentakis dodecahedron, an important variation is possible, as regular squares can be inserted between the edges of the figure. Since the triangles are not regular, there are two sizes of square, as shown in Diagram 7.16, and whereas the pentagonal faces are regular, the hexagonal faces are equiangular but not equilateral.

Diagram 7.17a shows the figure formed when squares are inserted between the faces of a cuboctahedron (shaded) or between the faces of its dual, the rhombic dodecahedron (solid black). Diagram 7.17b shows the figure formed when squares are

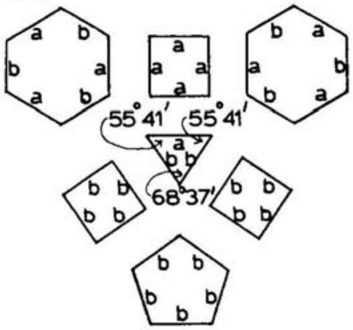
7.14



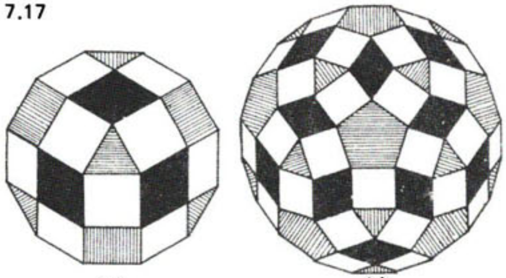
7.15



7.16



7.17

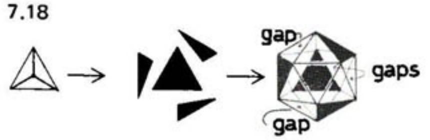

(a) (b)

inserted between the faces of an icosidodecahedron (shaded) or between the faces of its dual, the rhombic triacontrahedron (solid black). The interesting point about these two figures is that all of the edges of a rhombic dodecahedron are the same length, as are all the edges of the rhombic triacontrahedron, which means that equal squares can be inserted between the faces of any of these four polyhedra to produce figures whose edges are equal in length.

Insertion of Triangular Faces Between the Edges of an Existing Polyhedron

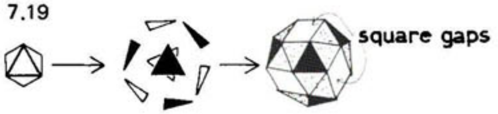
If the faces of a regular tetrahedron are pulled apart, pairs of equilateral triangles can be fitted between them (Diagram 7.18). Besides the four original faces of the tetrahedron and the six pairs of newly added triangles between them, there are also four triangular "gaps" between the pairs of triangles which lie on the same planes as the dual of the tetrahedron (another tetrahedron). The figure produced by this method is the snub version of the tetrahedron, the icosahedron.

7.18



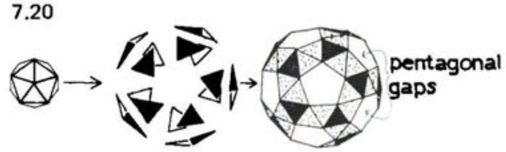
Similarly, the faces of an octahedron can be forced apart and pairs of equilateral triangles inserted between them (Diagram 7.19). Six square "gaps" appear between the equilateral triangles, corresponding to the six square faces of the dual of the octahedron (the regular cube), to produce the snub cube. An alternative way of creating this figure is to insert pairs of equilateral triangles between the faces of a cube, in which case eight triangular "gaps" appear, corresponding to the faces of the octahedron.

7.19



If the faces of an icosahedron are forced apart and pairs of equilateral triangles are inserted between them (Diagram 7.20), twelve pentagonal gaps appear, corresponding to the twelve faces of the dual of the icosahedron, the pentagonal dodecahedron. The resulting figure is the snub dodecahedron. An alternative way of generating this figure is to insert pairs of equilateral triangles between the faces of the dodecahedron, in which case twenty triangular "gaps" appear, corresponding to the twenty faces of the icosahedron.

7.20

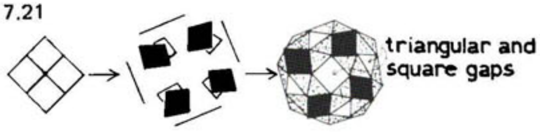


The figures generated from the Platonic polyhedra have all been described in earlier chapters, but if pairs of triangles are fitted between the faces of other polyhedra, another set of polyhedra can be produced. For example, pairs of equilateral triangles can be placed between the faces of a rhombic dodecahedron (Diagram 7.21). Six square and eight triangular gaps appear between the triangles, corresponding to the faces of the dual of the rhombic dodecahedron, the cuboctahedron. The same figure can be produced by putting pairs of triangles between the faces of a cuboctahedron, though in that case uncertainties may arise about the exact shape of the rhombic "gaps" between the triangles.

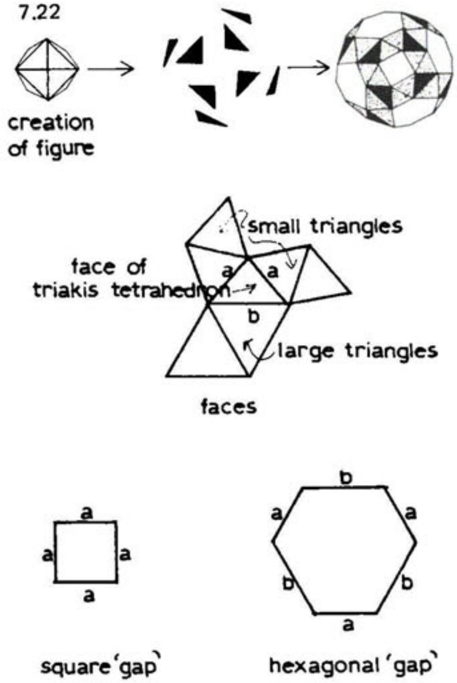
Since the edges of a rhombic dodecahedron are the same length, the triangles inserted between the faces of the rhombic dodecahedron are equilateral and the same size. With the exception of the rhombic triacontrahedron, the other duals of the facially regular polyhedra have more than one edge length, so there are different sizes of equilateral triangles, as shown in Diagram 7.22 for the tetrakis hexahedron. In this figure there will be six square and eight hexagonal gaps between the triangles, corresponding to the faces of the truncated octahedron, the dual of the tetrakis hexahedron. The squares are regular, but the hexagons, though equiangular, have two different edge lengths.

Though only a few examples have been given, it should be apparent that a large range of figures can be created by this method. The word *gap* has been used for the sake of convenience, but it should be understood that each "gap" is a face of the polyhedron and not necessarily a void between the faces.

7.21



7.22



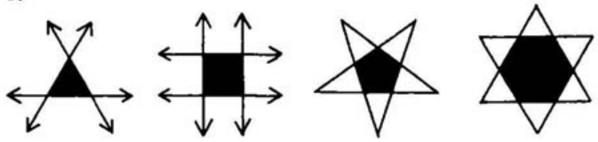
8. The Kepler Poinsot Polyhedra and Related Figures

If the edges of a regular polygon with five or more edges are extended, they intersect to define the extreme vertices of a star-shaped polygon (Diagram 8.1). If the regular polygon has fewer than five edges, its edges never intersect, however far they are extended. Since star-shaped polygons are formed by this process, it is called stellation.

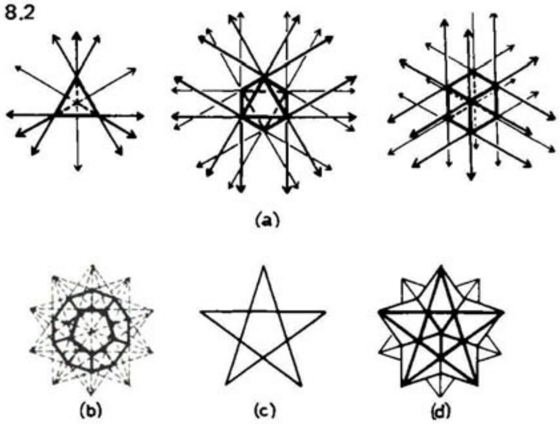
The edges of polyhedra can also be extended in this way, and sometimes they intersect to define new polyhedral forms. As with the triangle and the square, however, no matter how far the edges of a tetrahedron, an octahedron, or a cube are extended (Diagram 8.2a), they never intersect to define the vertices of a new polyhedron. However, if the edges of a regular pentagonal dodecahedron are extended, they meet in groups of five at twelve new vertices (Diagram 8.2b). This figure has twelve faces which are pentagrams or five-pointed star-polygons (Diagram 8.2c) which intersect one another to define a figure called the SMALL STELLATED DODECAHEDRON (Diagram 8.2d).

If the edges of a regular icosahedron are extended, they meet in groups of three at twenty new vertices (Diagram 8.3a). The figure thus produced has twelve intersecting faces which are pentagrams, though it is very different from the small stellated dodecahedron. Since this figure was derived from the icosahedron, it would be tempting to call it a stellated icosahedron, but that would not be an accurate name for a figure with twelve faces. The figure is therefore called the GREAT STELLATED DODECAHEDRON (Diagram 8.3b), *great* being used to distinguish it from the small stellated dodecahedron described earlier. Both figures were described by Johannes Kepler in his book *Harmonices Mundi* in 1619.

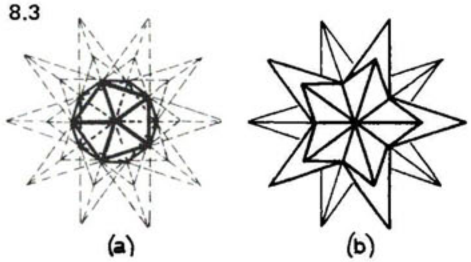
8.1



8.2



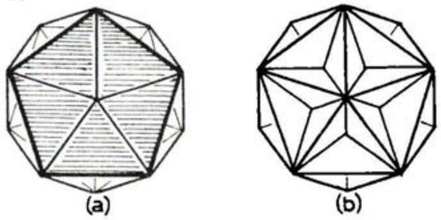
8.3



The duals of the two preceding figures were discovered by Louis Poinsot. They can be evolved by a process called faceting. The basic idea is to join the vertices of an existing polyhedron with a different set of edges and faces.

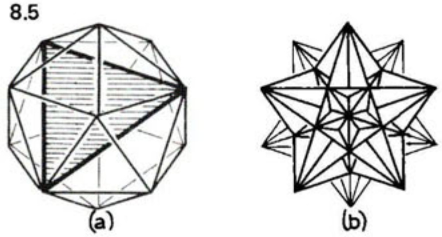
For instance, five of the vertices of a regular icosahedron can be joined to define a regular pentagon (Diagram 8.4a). And twelve intersecting pentagons can be constructed inside an icosahedron to define the GREAT DODECAHEDRON (Diagram 8.4b).

8.4



A large triangle can be constructed inside a regular icosahedron (Diagram 8.5a). If twenty such triangles are constructed, they intersect, and define a figure called the GREAT ICOSAHEDRON (Diagram 8.5b).

8.5

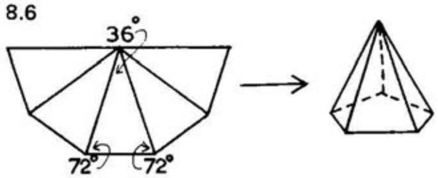


Construction of Models

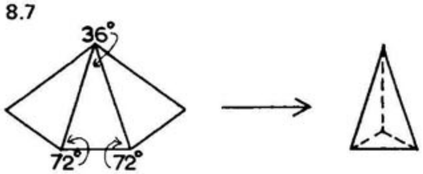
It may take a little longer to build these polyhedra than many of the figures described in earlier chapters, but their sheer beauty makes the effort well worth while. When building them, it may help to refer to the sketches and descriptions given in this chapter.

A SMALL STELLATED DODECAHEDRON can be made by adding a slender pentagonal pyramid, formed from five icosceles triangles (Diagram 8.6), to each face of a pentagonal dodecahedron. It is possible to omit the central dodecahedron and simply join the components edge to edge, but such a figure is not as strong as one built around a dodecahedron.

8.6



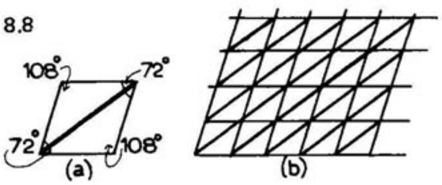
8.7



A GREAT STELLATED DODECAHEDRON can be built by making twenty slender triangular pyramids from isosceles triangles (Diagram 8.7) and joining their bases edge to edge, as if they were the faces of a regular icosahedron. Though a stronger figure can be made by adding the pyramids to a central icosahedron, it is not so critical with this figure, as the pyramids have triangular bases.

A GREAT DODECAHEDRON can be made from thirty rhombic components (Diagram 8.8a) which can be drawn out very quickly, as in Diagram 8.8b. Each rhombic component should be creased along its longer diagonal, and then they should be joined so that those folds define the edges of a regular icosahedron. The edges of the components should be joined to form the edges of the

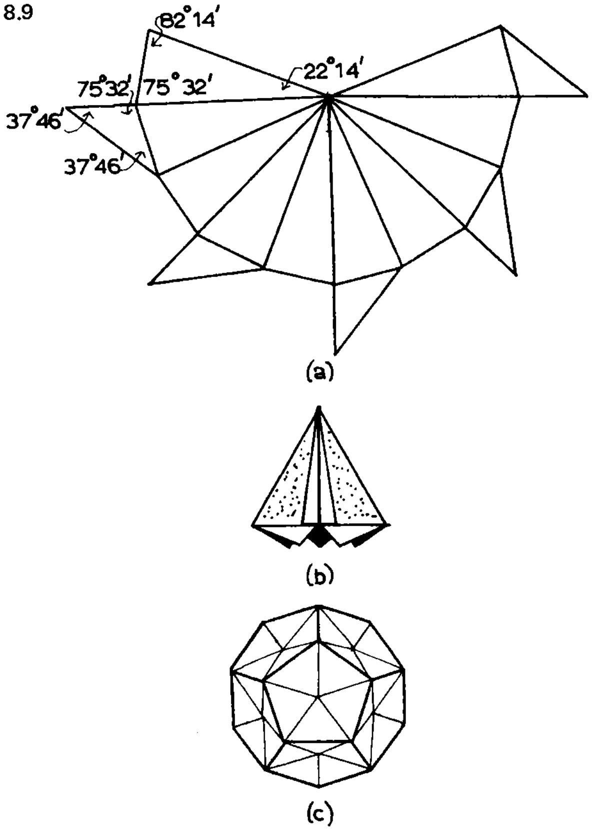
8.8



concave triangular depressions between the edges of the icosahedron.

It takes much longer to build the GREAT ICOSAHEDRON, but it is so beautiful that it is well worth the effort. The first step is to draw out twelve groups of faces, similar to those shown in Diagram 8.9a. The chore can be speeded up by using a pattern, as described in Appendix 3. The components should be cut out, scored, and bent and then joined to produce star-shaped pyramids with protruding bases similar to the one in Diagram 8.9b. The edges of the bases of these pyramids can be joined, edge to edge, as if the pyramids were the pentagonal faces of a regular dodecahedron. A model built this way may not be as strong as some people wish. Those people should first make a concave deltahedron, which is a pentagonal dodecahedron with its pentagonal faces replaced by concave pyramids of five equilateral triangles (Diagram 8.9c). The bases of the star-shaped pyramids can be stuck to the faces of this deltahedron to produce a very sturdy model.

8.9



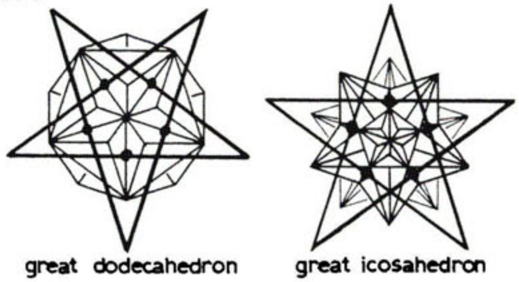
Characteristics of the Kepler Poinsot Polyhedra and Comparisons with the Platonic Polyhedra

Like the Platonic polyhedra, each of the Kepler Poinsot polyhedra has regular faces of one type, though this may be harder to appreciate, since the faces intersect and two of the figures have star-shaped faces. The faces of Kepler Poinsot polyhedra meet in a similar way about each vertex of a particular figure, resulting in congruent regular vertex figures just like Platonic Polyhedra.

A circumsphere can be constructed about each figure to touch all of its vertices, the vertices being the extremities of the faces and not the intermediate intersections of edges or faces. An intersphere will touch all of the midpoints of the edges of a Kepler Poinsot Polyhedron, as the midpoints are measured between the extreme vertices and not between intermediate intersections of edges and faces. Though the face centers of all of these figures are obscured by other faces, an insphere can be constructed inside each figure to touch all of its faces. Of the figures described in the earlier chapters, only the Platonic polyhedra have all three spheres.

One of the characteristics of a regular polyhedron or a regular tessellation is that its dual is another regular polyhedron or tessellation. Though it is harder to visualise, the dual of one Kepler Poinsot polyhedron is another Kepler Poinsot polyhedron. The small stellated dodecahedron (12 faces, 12 vertices, 30 edges) is the dual of the great dodecahedron (12 faces, 12 vertices, 30 edges) and the great stellated dodecahedron (12 faces, 20 vertices, 30 edges) is the dual of the great icosahedron (20 faces, 12 vertices, 30 edges). Diagram 8.10 shows the great dodecahedron and the great icosahedron on which are marked the midpoints of five of their edges. They are the points where each figure touches its intersphere and where the edges of its dual touch that same sphere. The heavier lines show the edges of one of the faces of the duals of these figures, which touch the common interspheres at the same points.

8.10



Finally, the faces of each polyhedron meet at a constant dihedral angle, like the faces of a Platonic polyhedron, hence the regular vertex figures mentioned earlier.

The main difference between these figures and the Platonic polyhedra is that their faces intersect and divide themselves into similar sets of facets. Because of the pentagons and pentagrams found in these figures, many parts of a figure have Golden Proportion relationships to its other parts. The Kepler Poinsot polyhedra are just as regular as the Platonic polyhedra, so they are often called the NONCONVEX REGULAR POLYHEDRA, though – since their discovery was no mean intellectual accomplishment – it is perhaps more fitting to refer to them by the names of their discoverers.

The Five Regular Compounds

In Chapter 1 it was shown that a tetrahedron could be defined by drawing a diagonal across each face of a cube (Diagram 8.11a). In fact, two such tetrahedra can be defined in the same cube to create a compound of two tetrahedra (Diagram 8.11b). Since this

8.11

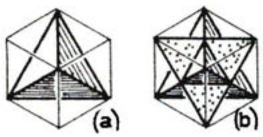
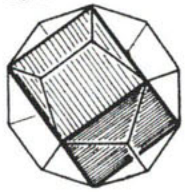


figure has two dual tetrahedra arranged about a common inter-sphere, this compound of two tetrahedra must be a self-dual.

A regular cube can be defined by drawing a diagonal across each face of a pentagonal dodecahedron (Diagram 8.12). A set of five cubes whose faces intersect each other can be fitted inside the same dodecahedron.

8.12



An octahedron can be defined by joining six of the vertices of an icosidodecahedron (Diagram 8.13). A set of five octahedra whose faces intersect each other can be fitted inside an icosido-decahedron in this way. This compound of five octahedra is the dual of the previous figure, the compound of five cubes.

8.13

Since a tetrahedron can be defined by drawing a diagonal across each face of a regular cube and since a compound of five cubes can be formed by joining the vertices of a dodecahedron, a compound of five tetrahedra can be formed inside a dodecahedron. Since two tetrahedra, the one the dual of the other, can be fitted inside the same cube, there will be enantiomorphic versions of this figure, the one version being the dual of the other. An enantiomorphic pair of these figures can be combined to create a compound of ten tetrahedra which will be its own dual.

Further sketches, details, and methods of construction for these five figures appear in Cundy and Rollett, *Mathematical Models*. They are very beautiful, but they take a long time to build.

Further Figures Produced by Stellating and Faceting Polyhedra

Many other figures with intersecting faces can be made by stellating and faceting Platonic polyhedra, Archimedean polyhedra, prisms, and antiprisms. Since this volume is concerned primarily with polyhedra with nonintersecting faces, it is inappro-priate to go into them in more detail here. However, those who wish to pursue the matter further can find them described in Coxeter and others, "Uniform Polyhedra," or in Wenninger, *Polyhedron Models*.

Calculations

Initially it may seem difficult to work out unknown dimensions and angles on polyhedra, but fortunately most problems can be solved through simple trigonometry or by constructing and measuring scale drawings. The trickiest part of most problems is to visualise the steps by which the unknown quantities can be determined from those which are already known. The solution is to cut suitable sections through the figures, thereby translating each three-dimensional problem into a series of two-dimensional problems which are relatively easy to solve. The best way of deciding which sections to cut is by studying a model. This may also reveal a few shortcuts which can reduce the length of the problem.

This appendix starts with a list of useful data and the formulae that are needed for most problems. This is followed by five worked examples. Whenever a formula is used in these examples, a reference is given to the list of formulae so readers can identify its original format. All examples are worked out in algebraic terms, for which numerical values can be substituted when appropriate.

The final example shows how a problem can be solved by constructing and measuring scale drawings – some consolation to those who find formulae difficult to use.

Useful Data and Formulae

$\pi = 3.1416.$

$\tau = \text{Golden Proportion} = \frac{\sqrt{5}+1}{2} : 1 = 1.618 : 1\ .$

There are sixty seconds (60″) in one minute and sixty minutes (60′) in one degree (1°).

Circumference of a circle $= 2\pi r$ (r is the radius).

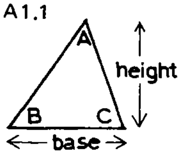
Area of a circle $= \pi r^2$.

Surface area of a sphere $= 4\pi r^2$.

Volume of a sphere $= \frac{4}{3}\pi r^3$.

Area of a triangle $= \dfrac{\text{base} \times \text{height}}{2}$.

A1.1



Sum of the internal angles of a triangle $(A + B + C) = 180°$.

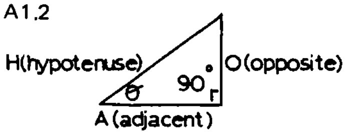
Sum of the internal angles of a convex polygon with N sides $= 180N° - 360°$.

Volume of a tetrahedron $= \dfrac{\text{base area} \times \text{height}}{3}$ (true for all pyramids).

Euler's formula: In a polyhedron, the number of faces plus the number of vertices is equal to the number of edges plus two ($F + V = E + 2$).

Tangents, sines, and cosines can be represented as ratios between the edges of a right-angled triangle (Diagram A1.2).

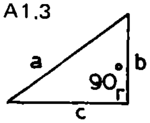
A1.2



$$\tan\theta = \frac{O}{A};\ \sin\theta = \frac{O}{H};\ \cos\theta = \frac{A}{H}.$$

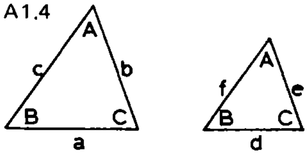
I. *Pythagoras's Theorem:* If a, b, and c are the edge lengths of a right-angled triangle (Diagram A1.3), $a^2 = b^2 + c^2$.

A1.3



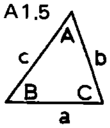
II. *Similar triangles:* If the angles of two triangles are identical (Diagram A1.4), their sides will be proportional $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$.

A1.4



III. *Sine formula* (Diagram A1.5): $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

A1.5



IV. *Cosine formula:*

(IVA) $$a^2 = b^2 + c^2 - 2bc\cos A.$$

This formula can be rearranged as follows:

(IVB) $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

But if $c = b$,

(IVC) $$a^2 = 2b^2(1 - \cos A),$$

(IVD) $$\operatorname{Cos} A = \frac{2b^2 - a^2}{2b^2}.$$

If $c = a$,

(IVE) $$a = \frac{b}{2\cos A},$$

(IVF) $$\cos A = \frac{b}{2a}.$$

Example 1. The volume of a regular tetrahedron.

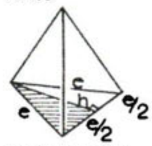
This example does not set out to prove that the volume of a tetrahedron is equal to $\frac{\text{its base-area} \times \text{its height}}{3}$, but it uses that information to find its volume in terms of its edge length (e).

The first problem is to find the area of the base.

The base of a tetrahedron is an equilateral triangle whose area is equal to $\frac{\text{its base} \times \text{its height}}{2}$. The edge length of this triangle is equal to e, and its height (h) is the perpendicular which divides the base into two equal parts, each equal to $e/2$.

Using Pythagoras's theorem on the shaded triangle (Diagram A1.6):

A1.6



tetrahedron

$$e^2 = h^2 + (e/2)^2.$$

And by rearranging this formula:

$$e^2 - (e/2)^2 = h^2.$$

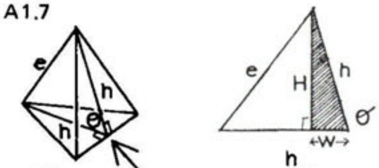
Therefore, $$h^2 = e^2 - e^2/4 = \frac{3e^2}{4}.$$

So $$h = \frac{\sqrt{3}}{2} e.$$

Hence, the area of the base triangle is $\dfrac{e \times \dfrac{\sqrt{3}e}{2}}{2} = \underline{\underline{\dfrac{\sqrt{3}e^2}{4}}}$.

The second step is to find the height of the tetrahedron by cutting a section through it as indicated by the arrow in Diagram A1.7. The perpendiculars across each face are equal to h, which has already been found $\left(\frac{\sqrt{3}e}{2}\right)$.

A1.7



cutting section   section

The section through the tetrahedron will be a triangle, as shown, H being the height of the tetrahedron.

θ can be found by using the cosine formula (version IVD) on the whole triangle as follows:

$$\cos\theta = \frac{2\left(\frac{\sqrt{3}}{2}e\right)^2 - e^2}{2\left(\frac{\sqrt{3}}{2}e\right)^2} = \frac{2\cdot\frac{3}{4}e^2 - e^2}{2\cdot\frac{3}{4}e^2} = \frac{\frac{3}{2}e^2 - e^2}{\frac{3}{2}e^2} = \frac{\frac{1}{2}e^2}{\frac{3}{2}e^2} = \frac{1}{3}.$$

But $\cos\theta = \dfrac{\text{adjacent edge}}{\text{hypotenuse}}$, so considering the shaded triangle:

$\cos\theta = \dfrac{1}{3} = \dfrac{w}{\frac{\sqrt{3}e}{2}}$. Hence $w = \dfrac{\sqrt{3}e}{2}\cdot\dfrac{1}{3} = \dfrac{e}{2\sqrt{3}}$.

Next, using Pythagoras's Theorem on the shaded triangle:

$$h^2 = w^2 + H^2 \text{ or } H^2 = h^2 - w^2.$$

Hence $H^2 = \left(\dfrac{\sqrt{3}}{2}e\right)^2 - \left(\dfrac{e}{2\sqrt{3}}\right)^2 = \dfrac{3}{4}e^2 - \dfrac{e^2}{12} = \dfrac{(9-1)e^2}{12} = \dfrac{2}{3}e^2$.

Hence $\underline{\underline{H = \sqrt{\frac{2}{3}}\,e}}$.

Now that both the base-area and the height of the tetrahedron are known, the volume $\dfrac{(\text{base area} \times \text{height})}{3}$ can be found.

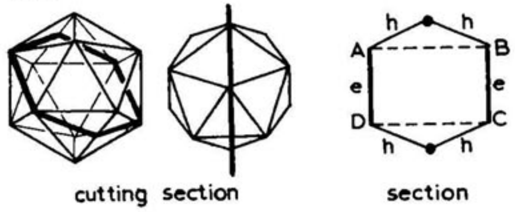
$$\text{Volume}\ \frac{\frac{\sqrt{3}e^2}{4}\cdot\sqrt{\frac{2}{3}}\,e}{3} = \frac{e^3\frac{\sqrt{2}}{4}}{3} = \frac{e^3\frac{1}{2\sqrt{2}}}{3} = \frac{e^3}{6\sqrt{2}}.$$

Finally a value for e can be substituted to give the volume of a regular tetrahedron of a given edge length.

Note that the key to this calculation lay in cutting the right section through the figure.

Example 2. Expressing the radius of the circumsphere, the radius of the intersphere, and the radius of the insphere of a regular icosahedron in terms of its edge length (e).

A1.8



First cut the section through the icosahedron as shown in Diagram A1.8 (two sketches are given for the sake of clarity).

The section is a hexagonal polygon, two edges of which are edges of the icosahedron and the other four perpendiculars (h) of its triangular faces. In Example 1 h was found to be $\frac{\sqrt{3}e}{2}$ for an equilateral triangle of edge length e.

Since the hexagon is neither equilateral nor equiangular, the distances AB and CD must be found.

If a model of the figure is inspected, it can be seen that AB and CD are the diagonals of regular pentagons of edge length e, so the rectangle $ABCD$ is a Golden Rectangle with AD and BC equal to e and AB and CD equal to $1.618e$.

A1.9

Alternately, the length of AB or CD can be found by using the cosine formula (version IVC) on a regular pentagon of edge length e (Diagram A1.9) as follows:

$$g^2 = 2e^2 \, (1 - \cos 108°).$$

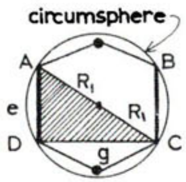
A value for cos 108° does not appear in most logarithm tables, as $-\cos 108°$ is equal to $+\cos 72°$.

So, substituting this in the equation:

$$g^2 = 2e^2 \, (1 + \cos 72°).$$

Hence, g can be found and the section through the icosahedron constructed.

A1.10



The radius of the circumsphere The circumsphere touches each vertex of the icosahedron, and as A and C are opposite vertices, the distance between them is the diameter of the circumsphere (Diagram A1.10). If the radius of the circumsphere is R_1, the length of the diameter AC is $2R_1$.

Using Pythagoras's Theorem on the shaded triangle:

$$(2R_1)^2 = g^2 + e^2 \text{ and by substituting for } g:$$

$$4R_1{}^2 = 2e^2 \, (1 + \cos 72°) + e^2.$$

So $\quad R_1^2 = \dfrac{e^2\ (2 + 2\cos 72^\circ + 1)}{4} = \dfrac{e^2\ (3 + 2\cos 72^\circ)}{4}.$

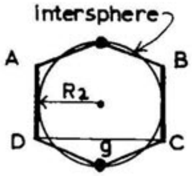
Hence, the radius of the circumsphere $R_1 = \frac{e}{2}\sqrt{3 + 2\cos 72^\circ}$.

A suitable value for e can now be substituted.

The radius of the intersphere Let the intersphere have a radius of R_2.

The intersphere touches the midpoint of each edge, so, using the same section through the figure as before, the perpendicular from the center of the figure to the edge AD is a radius of the intersphere (Diagram A1.11).

A1.11

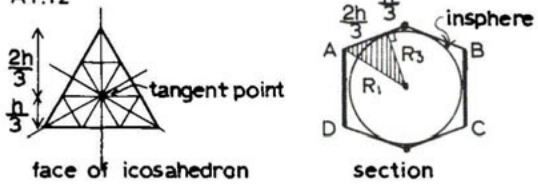


By inspection $R_2 = \dfrac{g}{2} = \dfrac{\sqrt{2e^2\ (1 + \cos 72^\circ)}}{2} = e\sqrt{\dfrac{1 + \cos 72^\circ}{2}}.$

A value for e can now be substituted, as before.

The Radius of the Insphere Let the insphere, which touches the center of each face of the icosahedron, have a radius R_3 (Diagram A1.12). The center of each face lies a third the distance up its height h, as shown.

A1.12



Hence, the radius of the insphere can be shown on the same section through the figure, as before. Since the insphere is a tangent to the face, it touches the face at right angles.

Using Pythagoras's Theorem on the triangle shown:

$$(R_1)^2 = (R_3)^2 + \left(\frac{2h}{3}\right)^2.$$

Hence, $(R_3)^2 = (R_1)^2 - \left(\dfrac{2h}{3}\right)^2$, and by substituting for R_1 and h:

$$(R_3)^2 = \frac{e^2\ (3 + 2\cos 72^\circ)}{4} - \frac{4}{9}\cdot\frac{3e^2}{4}$$

$$= \frac{e^2}{12}(3\,(3 + 2\cos 72^\circ) - 4) = \frac{e^2}{12}(5 + 6\cos 72^\circ).$$

Hence, $R_3 = \dfrac{e}{2}\sqrt{\dfrac{5 + 6\cos 72^\circ}{3}}$.

Now a value for e can be substituted.

It can be seen that the key to solving these problems is to cut a suitable section through the figure.

Example 3. The Edge Lengths of a Three-Frequency Icosahedron

Each face of the original icosahedron is divided into nine equilateral triangles (Diagram A1.13). Then each intersection of edges or subdividing lines is projected onto a common circumscribing sphere to define a vertex of the three-frequency icosahedron. By symmetry, there are three different edge lengths, as indicated by the letters x, y, and z. The problem is to express x, y, and z in terms of the radius of the circumsphere of the figure.

First cut a section through the icosahedron, as shown in Diagram A1.14. This is exactly the same section as was used to calculate the radius of the circumsphere (R_1) in Example 2. where R_1 was found to be equal to $\frac{e}{2}\sqrt{3 + 2 \cos 72°}$.

Since everything is to be expressed in terms of R_1, the formula should be rearranged as follows:

$$e = \frac{2}{\sqrt{3 + 2 \cos 72°}} R_1 .$$

Each edge, such as AD, is divided into three equal lengths of $e/3$ and then projected onto the circumscribing sphere to define the edges x and y (Diagram A1.15c).

The next step is to drop a perpendicular from the center of the figure to the edge of the icosahedron as in Diagram A1.15d. This perpendicular is a radius of the intersphere (R_2), which was found in Example 2 to be equal to $e\sqrt{\frac{1 + \cos 72°}{2}}$.

R_4 is the distance from the center of the figure to a point one-third the way along the edge of the original icosahedron. When this line is extended to the circumscribing sphere, it defines the junction between edge x and edge y.

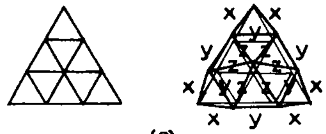
Using Pythagoras's Theorem on the triangle in Diagram A1.15d, whose edges are R_2, R_4, and $e/6$:

$$(R_4)^2 = (R_2)^2 + (e/6)^2 .$$

Hence, R_4 can be found.

Considering the triangles between the edge y and the center of the figure shown in Diagram A1.15d, there are two similar triangles, whose edges are R_1, R_1, y, and $R_4, R_4, e/3$, respectively.

A1.13

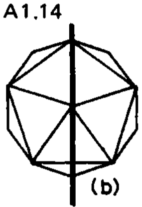


(a)

division of face

projection onto sphere

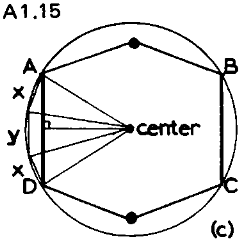
A1.14



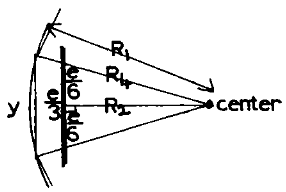
(b)

cutting section

A1.15



(c)

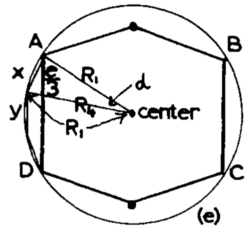


(d) part of sketch c

So $\dfrac{y}{e/3} = \dfrac{R_1}{R_4}$.

Hence $y = \dfrac{e \cdot R_1}{3R_4}$.

A1.16



The value of x can be found by considering the part of the section shown in Diagram A1.15e.

Using the cosine formula (version IVB) on the triangle whose edge lengths are R_1, R_4, and $e/3$:

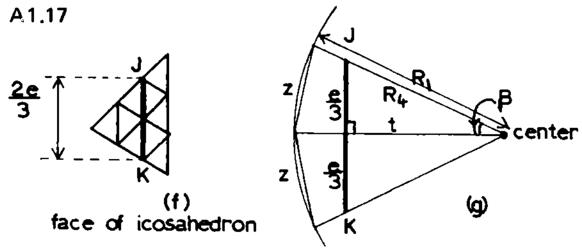
$$\cos \alpha = \frac{(R_1)^2 + (R_4)^2 - (e/3)^2}{2R_1 R_4} .$$

Now x can be found by using the cosine formula (version IVC) on the triangle whose edge lengths are R_1, R_1, and x, as follows:

$$x^2 = 2(R_1)^2 \ (1 - \cos \alpha) .$$

Hence, $x = R_1 \sqrt{2(1 - \cos \alpha)}$.

A1.17



Two edges of length z are produced by projecting the line JK on Diagram A1.17f to the circumscribing sphere. A section should be cut through this line to pass through the center of the figure, as in Diagram A1.17g.

The ends of the line JK are a distance of R_4 from the center of the figure, R_4 having been calculated earlier in the problem.

The distance of the perpendicular from the center of the figure to the line JK is t.

Using Pythagoras's Theorem on the triangle whose edges are t, R_4, and $e/3$, $t^2 = (R_4)^2 - (e/3)^2$.

$$\cos \beta = \frac{\text{adjacent edge}}{\text{hypotenuse}} = \frac{t}{R_2} .$$

Now z can be found by using the cosine formula (version IVC) on the triangle whose edges are R_1, R_1, and z, as follows:

$$z^2 = 2(R_1)^2 \ (1 - \cos \beta) .$$

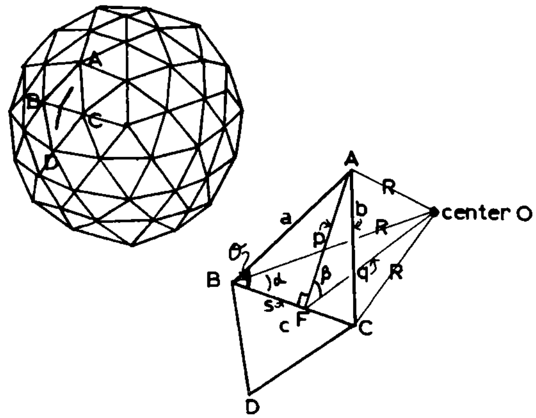
Hence, $z = R_1 \sqrt{2(1 - \cos \beta)}$.

All the edge lengths have now been found, and the next step is to work through the equations with logarithm tables to establish the values of e, R_2, R_4, and t and the sizes of x, y, and z in terms of the radius of the circumscribing sphere (R_1).

Example 4. The Calculation of a Dihedral Angle

The dihedral angle is the angle between two faces where they meet at a common edge. In this example it is assumed that all of the edge lengths are known but are not necessarily equal, all of the faces are triangles, and all of the vertices touch the same circumscribing sphere.

A1.18



The problem is to find the angle between triangle ABC and triangle BCD at their common edge BC in Diagram A1.18.

From the vertices of the triangle ABC, construct lines to the center (O) of the polyhedron. All of these lines will be of length R, the radius of the circumscribing sphere.

Drop a perpendicular (p) from A to meet BC at F.

Using the cosine formula (version IVB) on triangle ABC, whose edge lengths are a, b, and c:

$$\cos\theta = \frac{a^2 + c^2 - b^2}{2ac}.$$

$p = a \sin\theta$.

s (the distance between B and F) $= a \cos\theta$.

Next consider the triangle BCO.

Using the cosine formula (version IVF), $\cos\alpha = \dfrac{c}{2R}$.

q is the distance of F from the center of the figure O.

Using the cosine formula (version IVA) on the triangle BFO,

$$q^2 = s^2 + R^2 - 2Rs \cos\alpha.$$

Now that p, q, and R are known, the cosine formula (version IVB) can be used on triangle AFO to find β.

$$\cos\beta = \frac{p^2 + q^2 - R^2}{2pq}.$$

The next step is to repeat these equations for the triangle BCD. In this series of equations the values for θ, p, s, α, q, and β may be different from the ones already calculated.

The final step is to add together the two values for β to obtain the dihedral angle between the faces.

This example shows how the edge lengths of a three-frequency icosahedron can be found by drawing accurate sections through the figure.

Each edge of the icosahedron in Diagram A1.19 is divided into three equal parts and each face into nine triangular facets. Each intersection of lines is projected onto a common circumsphere to define one of the vertices of a three-frequency icosahedron. By symmetry there will be three different edge lengths, as indicated by the letters x, y, and z in the right-hand diagram. The problem is to express x, y, and z in terms of some fixed dimension.

The first step is to draw the section which cuts along two opposite edges of the icosahedron and passes through its center (Diagram A1.20). The central rectangle in that section is a Golden Rectangle.

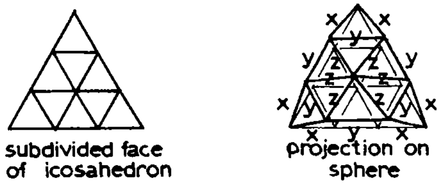
To draw the section, first construct a suitably-sized Golden Rectangle by drawing a square, bisecting one edge, and then drawing an arc to cut an extended edge of the square, as in Diagram A1.21. The lengths of the other four edges of the hexagonal section can be measured on a drawing of an equilateral triangle, which is one of the faces of the icosahedron, as shown.

The next step is to draw diagonals across the Golden Rectangle to find the center of the polyhedron and its circumscribing sphere. Compasses can then be used to draw the outline of the circumscribing sphere, as in Diagram A1.22.

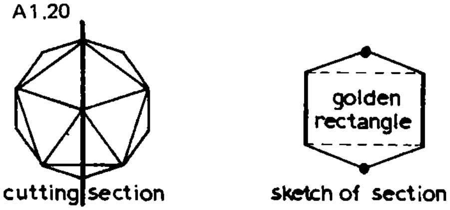
Then divide one of the edges into three equal parts with lines from the center and extend those lines outwards to the circumscribing sphere, as shown. The lines cut the circumscribing sphere to establish vertices of the three-frequency figure, which can be joined to establish edge lengths x and y, as indicated.

The edge length z can be found by constructing the section shown in Diagram A1.23. The distance between the edge of the icosahedron and its center (w), measured along the line which trisects that edge, can be measured from the previous section. The width of the face at that point will be two-thirds the length of the edge of the original icosahedron. Once the triangle whose edges

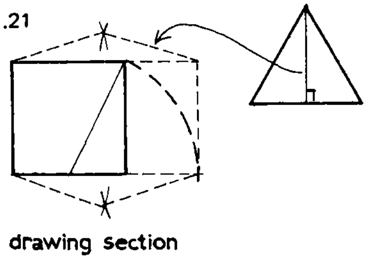
A1.19



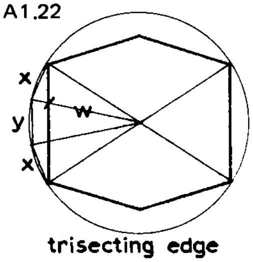
A1.20



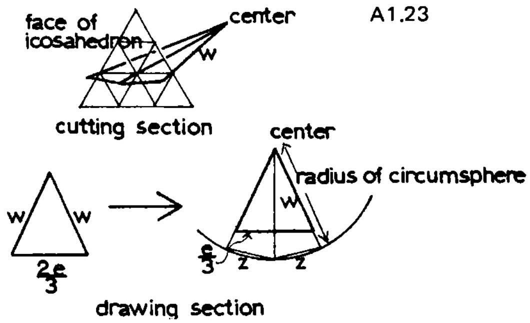
A1.21



A1.22



A1.23



are w, w, and $\frac{2e}{3}$ is constructed, its edges can be extended from the center till they cut the circumsphere, the radius of which can be measured on a previous drawing. Two chords z can be drawn to this circumsphere, as shown in Diagram A1.23, right. The lengths of the three chords x, y, and z can now be measured on the diagrams and then multiplied by an appropriate factor if the ultimate figure is a different size from the diagrams. Thus, all of the edges of the three-frequency icosahedron can be found from these drawings. Many other problems can be solved in this way, and drawings can be a useful way of checking a calculation.

Chord Factors for Geodesic Polyhedra

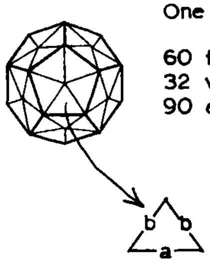
A2.1

CHORD FACTORS — Table one

Figures derived from the DODECAHEDRON by the Alternate Method.

One frequency

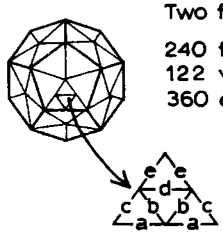
60 faces
32 vertices
90 edges



a = ·713640
b = ·640850

Two frequencies

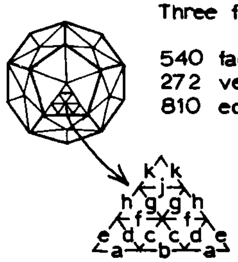
240 faces
122 vertices
360 edges



a = ·362841
b = ·340339
c = ·324734
d = ·376681
e = ·324734 (= c)

Three frequencies

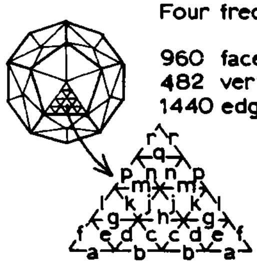
540 faces
272 vertices
810 edges



a = ·237657
b = ·252597
c = ·228242
d = ·225128
e = ·213490
f = ·252116
g = ·225438
h = ·224087
j = ·249539
k = ·213490 (= e)

Four frequencies

960 faces
482 vertices
1440 edges



a = ·175924
b = ·188429
c = ·171879
d = ·169507
e = ·166955
f = ·158465
g = ·185219
h = ·191986
j = ·172304
k = ·169639
l = ·167350
m = ·189191
n = ·165468
p = ·167350 (= l)
q = ·185279
r = ·158465 (= f)

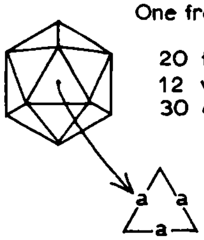
All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by the author.

CHORD FACTORS

Table two

Figures derived from the ICOSAHEDRON by the Alternate Method.

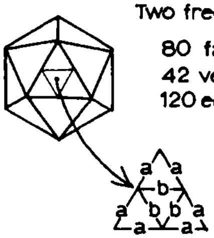
One frequency



20 faces
12 vertices
30 edges

a = 1·051462

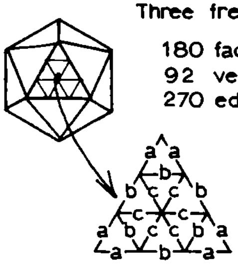
Two frequencies



80 faces
42 vertices
120 edges

a = ·546533
b = ·618034

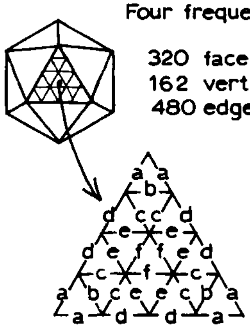
Three frequencies



180 faces
92 vertices
270 edges

a = ·348615
b = ·403548
c = ·412411

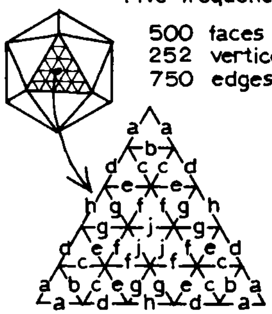
Four frequencies



320 faces
162 vertices
480 edges

a = ·253185
b = ·295242
c = ·294531
d = ·298588
e = ·312869
f = ·324920

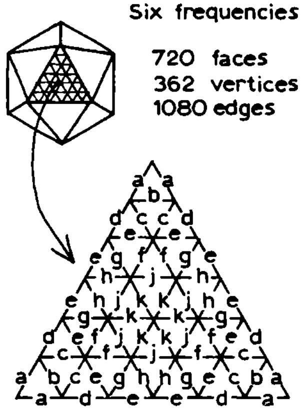
Five frequencies



500 faces
252 vertices
750 edges

a = ·198147
b = ·231790
c = ·225686
d = ·231598
e = ·247243
f = ·255167
g = ·245086
h = ·245346
j = ·261598

Six frequencies



720 faces
362 vertices
1080 edges

a = ·162567
b = ·190477
c = ·181908
d = ·187383
e = ·202820
f = ·205908
g = ·198013
h = · 205908 (=f)
j = ·215354
k = ·216628

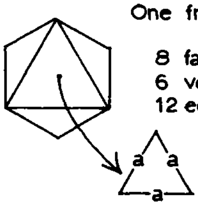
All vertices touch the circumscribing sphere
Each chord factor is expressed in terms of the radius of the circumscribing sphere
These chord factors were developed by Joseph D.Clinton under a N.A.S.A Research Grant (see bibliography).

A2.3

CHORD FACTORS — Table three

Figures derived from the OCTAHEDRON by the Alternate Method.

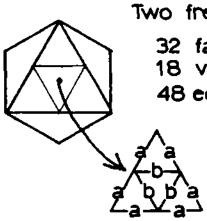
One frequency



8 faces
6 vertices
12 edges

a = 1·414213

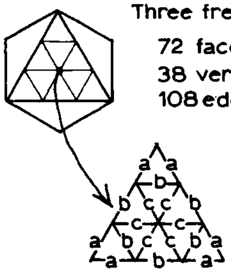
Two frequencies



32 faces
18 vertices
48 edges

a = ·765367
b = 1·000000

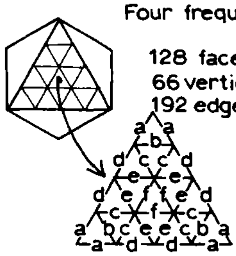
Three frequencies



72 faces
38 vertices
108 edges

a = ·459507
b = ·632456
c = ·671421

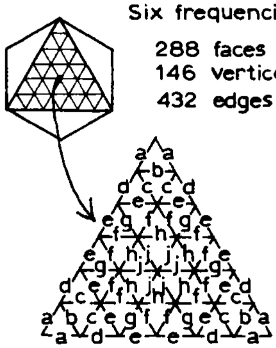
Four frequencies



128 faces
66 vertices
192 edges

a = ·320365
b = ·447214
c = ·438871
d = ·459506
e = ·517638
f = ·577350

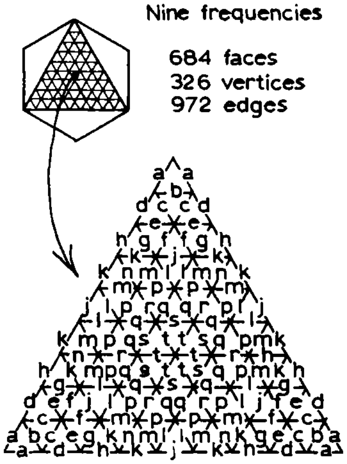
Six frequencies



288 faces
146 vertices
432 edges

a = ·197077
b = ·277350
c = ·241971
d = ·265467
e = ·320364
f = ·331931
g = ·296032
h = ·377964
j = ·385176

Nine frequencies



684 faces
326 vertices
972 edges

a = ·124277
b = ·175412
c = ·141457
d = ·153795
e = ·195190
f = ·178374
g = ·162497
h = ·185086
j = ·220863
k = ·210702
l = ·206355
m = ·215719
n = ·185970
p = ·240797
q = ·245917
r = ·232229
s = ·262613
t = ·264949

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by D.Andrews and A.Steinbrecher.

A2.4

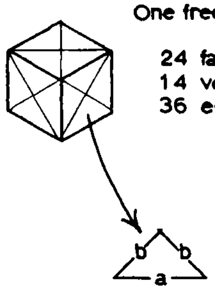
CHORD FACTORS

Table four

Figures derived from the CUBE by the Alternate Method.

One frequency

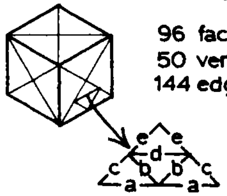
24 faces
14 vertices
36 edges



a = 1·157362
b = ·919402

Two frequencies

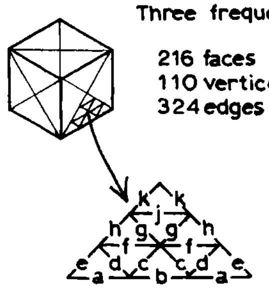
96 faces
50 vertices
144 edges



a = ·605811
b = ·533266
c = ·473130
d = ·650115
e = ·473130 (= c)

Three frequencies

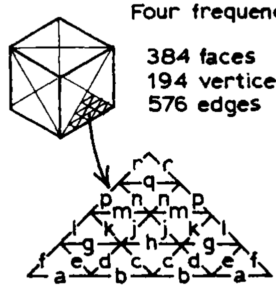
216 faces
110 vertices
324 edges



a = ·381642
b = ·458831
c = ·369777
d = ·345108
e = ·305593
f = ·437710
g = ·342890
h = ·340067
j = ·427099
k = ·305593 (= e)

Four frequencies

384 faces
194 vertices
576 edges



a = ·274774
b = ·338201
c = ·279081
d = ·268070
e = ·250832
f = ·223925
g = ·313678
h = ·351406
j = ·275377
k = ·259338
l = ·252585
m = ·329564
n = ·245520
p = ·252585 (= l)
q = ·314678
r = ·223925 (= f)

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by the author.

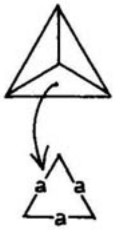
A2.5

CHORD FACTORS — Table five

Figures derived from the TETRAHEDRON by the Alternate Method.

One frequency

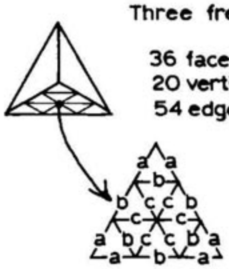
4 faces
4 vertices
6 edges



a = 1·632992

Three frequencies

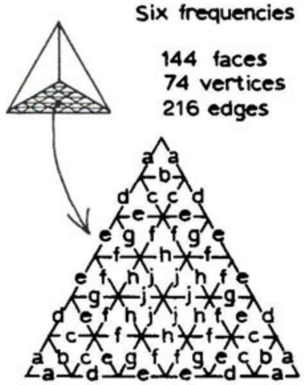
36 faces
20 vertices
54 edges



a = ·509138
b = ·853001
c = ·977847

Six frequencies

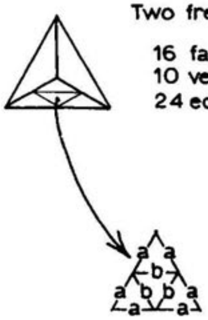
144 faces
74 vertices
216 edges



a = ·194934
b = ·342998
c = ·248791
d = ·314152
e = ·436954
f = ·459500
g = ·335397
h = ·632457
j = ·671424

Two frequencies

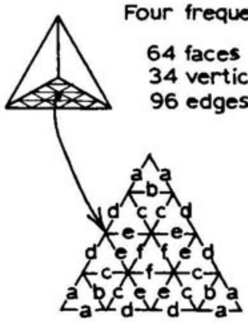
16 faces
10 vertices
24 edges



a = ·919401
b = 1·414211

Four frequencies

64 faces
34 vertices
96 edges



a = ·337424
b = ·577350
c = ·517045
d = ·605812
e = ·765367
f = ·999998

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by the author.

A2.6

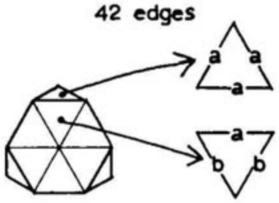
CHORD FACTORS

Table six

Figures derived from the TRUNCATED TETRAHEDRON by the Alternate method

One frequency

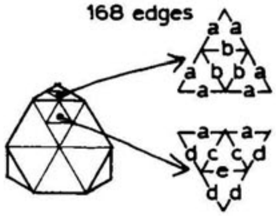
28 faces
16 vertices
42 edges



a = ·852802
b = ·977512

Two frequencies

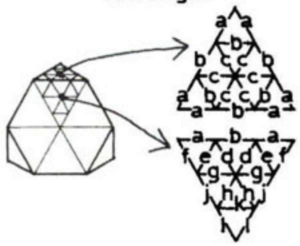
112 faces
58 vertices
168 edges



a = ·436958
b = ·471404
c = ·549006
d = ·505133
e = ·488757

Three frequencies

252 faces
128 vertices
378 edges



a = ·283694
b = ·310460
c = ·314368
d = ·361432
e = ·360108
f = ·324604
g = ·324604
h = ·369699
j = ·367140
k = ·320300
l = ·324604 (=f)

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by the author

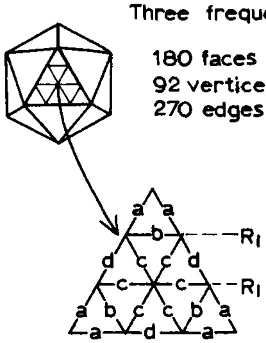
A2.7

CHORD FACTORS — Table seven

SMALL CIRCLE figures derived from the ICOSAHEDRON.

Three frequencies

180 faces
92 vertices
270 edges

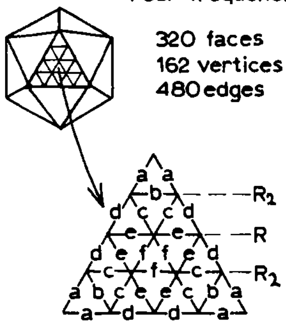


Radius of small circle R_1 ·982246

a = ·330205
b = ·382854
c = ·421209
d = ·440002

Four frequencies

320 faces
162 vertices
480 edges

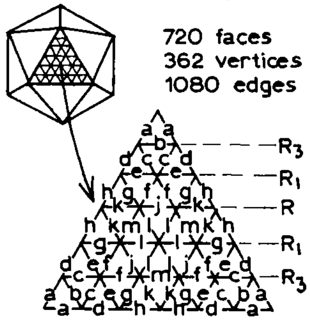


Radius of small circle R_2 ·961045
Radius of great circle R=1·000000

a = ·225149
b = ·262998
c = ·307360
d = ·326479
e = ·312869
f = ·324920

Six frequencies

720 faces
362 vertices
1080 edges



Radius of small circle R_3 ·933454
Radius of small circle R_1 ·982246
Radius of great circle R=1·000000

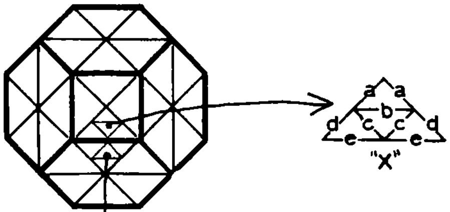
a = ·119351
b = ·140055
c = ·190801
d = 212014
e = ·192355
f = ·212138
g = ·211383
h = ·221396
j = ·196028
k = ·215570
l = ·212138 (= f)
m = ·224828

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by the author and Dr. E.R. Ashworth.

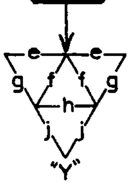
A2.8

CHORD FACTORS	Table eight

Clusters of geodesic polyhedra based on the two frequency truncated octahedron.

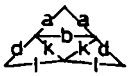


four clusters of faces like this are generated by each square face.



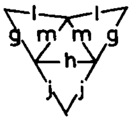
six clusters of faces like this are generated by each hexagonal face.

a =·1971
b =·2774
c =·2420
d =·2655
e =·3204
f =·3319
g =·2960
h =·3780
j =·3852



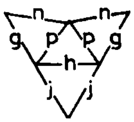
this cluster replaces the group "X" when the group adjoins a hexagonal face of another truncated octahedron.

k =·2000
l =·3274



this cluster of faces replaces the group "Y" when the group adjoins a square face of another truncated octahedron.

m =·2786



this cluster of faces replaces the group "Y" when the group adjoins a hexagonal face of another truncated octahedron.

n = ·3423
p = ·2429

All vertices touch the circumscribing sphere.
Each chord factor is expressed in terms of the radius of the circumscribing sphere.
Calculations by D. Andrews and the author.

Building Models of Polyhedra

Though the concepts and relationships described in this volume are comparatively simple, they are often difficult to visualize. The reader can overcome this and can become familiar with the shapes and symmetries of the figures by building models of them. It takes time and patience, but it saves a lot of time and frustration in the long run. The first models may be inelegant, but a crude model is better than none at all, and practice will soon raise the standards of workmanship. Model-building is not an expensive operation, since the tools and materials can be found in most homes and offices, or can be bought inexpensively from neighborhood stores. Very attractive models can be made from scrap materials like old cartons; the only constraint is the builder's imagination. This appendix is designed to supplement that imagination with some time-saving hints on tools, materials, assembly, and geometry.

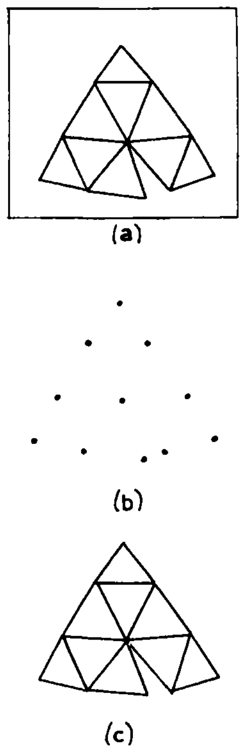
Sheet Materials

Models of polyhedra can be made from almost any stiff material such as construction paper, manila folders, old package wrappings, computer punch cards, or postcards (even picture postcards), to mention just a few examples. The thicker cards are harder to cut and join than the thinner ones, and the author prefers old manila folders and postcards. Though relatively thin, construction paper can also be used to produce sturdy, brightly coloured models. The main problem with construction paper is that it tends to fade when exposed to bright sunlight. Very strong models can be made from thin sheets of plastic or metal, though cutting and joining these materials can be tricky. Strong models can also be made from a very thin plywood, only 1 millimeter thick, which is used for building model aircraft.

Marking Out the Faces

The model builder must be careful to mark out the polyhedral faces as accurately as possible, making frequent checks for errors. But with a little forethought, the chore of drawing out each face individually can be reduced considerably. For example, an accurate template can be made from stout material and the faces marked out by tracing it. The main problem with a template is that it can become worn and rounded at its corners, so the resulting faces are not as accurate as may be required. A much better method, especially if one is attempting to produce a complex configuration of faces, is to draw the assembly carefully on a sheet of paper, as shown in Diagram A3.1a. This drawing can be used as a pattern by placing it on top of the construction material and then using a sharp point such as a pin or the end of a pair of compasses to indicate the positions of significant vertices by pricking the heavier material (Diagram A3.1b). The pinpricks can be joined, as shown in Diagram A3.1c, to reproduce the original pattern. Several pieces of construction material can be marked out this way simultaneously if they are stapled together and the pin is driven through them all, though care must be taken to drive the pin straight through and not at an angle.

A3.1



Cutting and Scoring

The faces should always be cut out carefully, to avoid inaccuracies. Scissors can be used for the thinner materials and model-making knives (for example, Stanley knives) for the thicker, harder ones. (The knives should always have sharp, new blades.) A metal straightedge can be used to guide the blade and to ensure a clean, straight cut; the wooden or plastic edges of a protractor, T square, or ruler should not be used, since they are easily damaged.

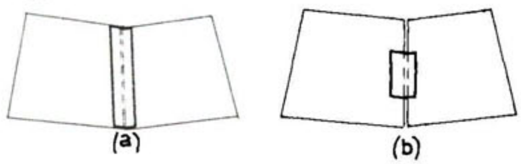
Many sheet materials can be bent cleanly and accurately if they are scored or cut partially through their thickness. Some materials can be scored by pressing down hard with a pencil as one draws the edges, but the thicker materials must be cut partway through with a model-making knife. The cuts should be deep enough to allow the material to bend cleanly but not so deep that they cause the material to split. This requires practice.

Joining the Faces

It is usually quickest to join faces with a suitable tape. The adhesive on ordinary Scotch Transparent Tape tends to dry out, especially on absorbant material, thus allowing the model to disintegrate. Scotch Magic Transparent Tape, however, has a much more durable adhesive and has the additional advantage of being almost invisible if applied carefully. Generally, a model tends to look neater if the tape is hidden inside it, but tape can also form an attractive external feature. Drafting or masking tape does not split or tear as easily as the transparent tapes, and it joins faces firmly if applied carefully. It is not as elegant as the other tapes, but that does not matter if the taping is done on the inside of the model. Tapes with a backing of cloth or plastic make very strong joints, but they are relatively expensive.

In making a good, strong joint, the method of applying the tape can be as critical as the tape itself. First, the tape should run the whole length of the joint, as shown in Diagram A3.2a, rather than stopping short, as in Diagram A3.2b. It is not worth trying to economise on tape that way, since it considerably weakens the model. Any excess tape can be trimmed off afterwards with a razor blade.

A3.2



Second, the tape must be made to adhere as firmly as possible. Finger pressure is not sufficient. It is a good idea to press the tape down firmly with the cap of a ball-point pen or the handle of a knife, to squeeze out any air bubbles and to ensure as much contact between the tape and the card as possible.

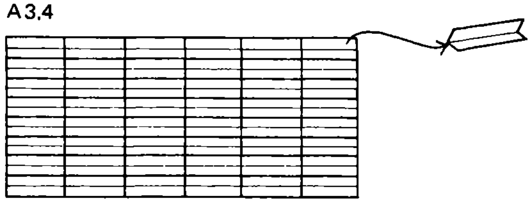
As an alternative to tape, one can join the faces with tabs. One method is to leave margins around each face, as at left in Diagram A3.3. The tabs can then be glued to the undersides of adjacent faces. Since only one tab is needed for each edge of the polyhedron, the model maker must determine where tabs are needed and where they are not. An alternative is to form a tab against every edge and then glue the tabs together, as shown at

A3.3

right in Diagram A3.3. Besides being easier to work out, such a joint forms an internal ridge which strengthens the model.

Another way of joining faces is to cut them out without tabs and then make a set of hinged tabs from a separate sheet of material, as in Diagram A3.4. By applying contact cement to all surfaces to be joined, one can assemble such a model very rapidly. Further, the hinged tabs reinforce the edges, thereby strengthening the model.

A3.4



Glues

Of all the glues and adhesives the author has tried, he finds contact cement the most satisfactory. This adhesive is applied to both surfaces and let dry, after which the surfaces are pressed together, to bond on contact. The model builder must exercise care to align the two pieces correctly, as the glue is very tenacious. The glue does not cause the paper to wrinkle. Since the parts bond instantaneously, clamps and clips are not needed to hold them together while the glue sets.

The Last Piece

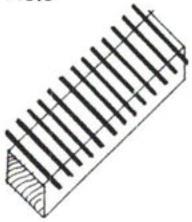
If a model is constructed with tabs or tape on the inside, a problem will arise in affixing the last face. This problem will be less serious if the last face is a small one. For example, in making a truncated cube, which has eight triangular and six octagonal faces, the problem of affixing the last face is less pronounced if the last face is a triangle than if it is an octagon. Sometimes the natural springiness of the tab or tape can create enough pressure to bond the last piece into position, and sometimes a thin blade such as a palette knife can be slipped underneath it to apply the necessary pressure. Another possibility is to put contact cement round all the edges and glue the last piece into position. If all else fails, the last face can be taped into position on the outside with Scotch Magic Transparent Tape.

Stick Models

Very good models of certain polyhedra can be made with frameworks of struts or rods representing their edges. One of the attractions of such models is that one can see the "front" and "back" simultaneously. Many suitable materials, such as dowel (small-diameter lumber rods) swab sticks, toothpicks, drinking straws, knitting needles, and so on, can be bought inexpensively at a neighborhood store. It is possible to buy special kits (consisting of thin wooden rods and plastic connectors) for building models of polyhedra, but, at the time of writing, many of them are badly designed or use a very inferior grade of plastic. Fortunately, it is easy to make one's own connectors from short lengths of small-diameter rubber or plastic tubing, bolted together with small nuts and bolts as shown in Diagram A3.5. Dowel rods of suitable lengths can be slipped into these outlets to serve as the struts of the framework and to define the edges of the polyhedron. Toothpicks cost next to nothing, are easy to cut to length, and can be glued together to form excellent models. The quickest technique is to give the ends of each strut a thin coat of contact cement and then lay them out on a piece of lumber to allow the glue to dry as in Diagram A3.6. Once the glue is dry, the ends can be pressed together to form the model. When a large model is attempted, it may be advisable to build it in stages, allowing each stage to harden overnight.

A3.5

A3.6



Good models can also be made from plastic drinking straws, joined with pins or pieces of wire driven through their ends. An alternative method is to run continuous threads through the straws, tieing the threads together at their ends to connect them.

Though stick models are excellent for a figure whose faces are all triangles, such as an icosahedron or a triangulated geodesic figure, they do not work well for figures such as the cube, whose faces are not triangles. Such a model will distort unless diagonals are added to triangulate the faces and stabilise the figure.

General Notes

To avoid the embarrassment of producing a model of unmanageable proportions, estimate the final size of the model before starting work. For example, a tetrahedron with its four triangular faces will not be very big if its edges are two inches long, but a great rhombicosidodecahedron, with its sixty-two faces, would be very large indeed, if its edges were that same length.

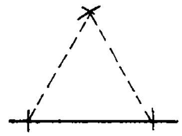
The best ways of improving one's models are practice and experimentation. It is always worth trying out a new material or technique.

Since the faces of many polyhedra are regular polygons, this appendix closes with descriptions of some quick ways of drawing them. (Information on drawing the nonregular faces, such as those of the duals of facially regular figures and those of geodesic figures have been discussed in the appropriate chapters.)

Equilateral Triangles

Mark off the required edge length with compasses (Diagram A3.7). Then, without changing the distance between the compass points, draw an arc from each end of that first edge. The point of intersection of the two arcs defines the third point of the equilateral triangle.

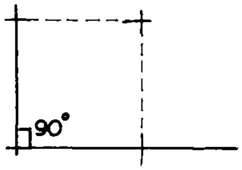
A3.7



Squares

Construct two lines which intersect at right angles, with a pair of compasses or a protractor (Diagram A3.8). Let the point of intersection be one of the vertices, and from it mark off two equal lengths with compasses to establish two more vertices. Then, with the compasses set to that same edge length, draw an arc from each of the last two vertices. The arcs intersect to define the fourth vertex of the square. Finally, as a check on accuracy, make sure that the two diagonals are equal in length.

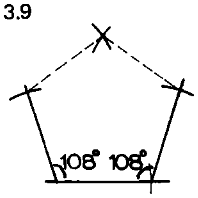
A3.8



Regular Pentagons (Five-Sided Polygons)

Draw one of the edges, and, using a protractor, draw a further line from each end to meet the original edge at 108°, as shown in Diagram A3.9. Use a pair of compasses to mark off two more edges along those lines, and then establish two more vertices. With the compasses set to the same edge length, draw an arc from each of these vertices, the point of intersection being the fifth vertex. Finally, check that all edges are the same length and that each internal angle is 108°.

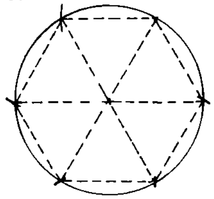
A3.9



Regular Hexagons (Six-Sided Polygons)

Draw a circle whose radius is equal to the edge length of the desired hexagon; then, without changing the radius, use the compasses to mark off a series of arcs, each centered from another arc, to divide the circumference into six equal parts (Diagram A3.10). Join the points formed by the intersections between the arcs and the circumference to form a regular hexagon. It can be seen that the regular hexagon can be subdivided into six equilateral triangles.

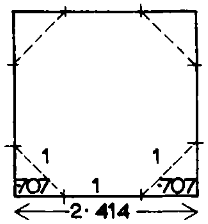
A3.10



Regular Octagons (Eight-Sided Polygons)

Draw a regular square whose edges are 2.414 times the edge length of the desired octagon. Then divide each edge in the proportions .707*e* : *e* : .707*e*, where *e* is the edge length of the octagon (Diagram A3.11). Join these points, as shown, to create the octagon. Then check that all edge lengths are equal.

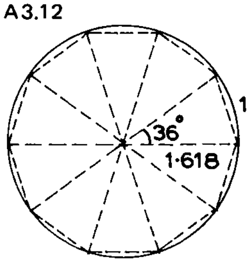
A3.11



Regular Decagons (Ten-Sided Polygons)

Draw a circle whose radius is 1.618 times the edge length of the desired decagon, and use a protractor to divide the circle into ten equal 36° sectors, as shown in Diagram A3.12. The decagon can be constructed by joining the points of intersection as shown. Finally, check that all edges of the decagon are the correct length.

A3.12



Checks on Accuracy

Once a face has been drawn, check it for accuracy, as an inaccurate face will create an inaccurate model and will thus waste a lot of effort. If a face has not come out accurately, find the mistake and draw a fresh one. Inaccurate faces should be discarded or be clearly marked, to prevent their being used accidentally.

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