The Algebraic Approach II: Dioids, Quantales and Monads

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Abstract. The algebraic approach to formal language and automata theory is a continuation of the earliest traditions in these fields which had sought to represent languages, translations and other computations as expressions (e.g. regular expressions) in suitably-defined algebras; and grammars, automata and transitions as relational and equational systems over these algebras that have such expressions as their solutions.

As part of a larger programme to algebraize the classical results of formal language and automata theory, we have recast and generalized the Chomsky hierarchy as a complete lattice of dioid algebras. Here, we will formulate a general construction by ideals that yields a family of adjunctions between the members of this hierarchy.

In addition, we will briefly discuss the extension of the dioid hierarchy to semirings and power series algebras.

Keywords: Monad, Ideal, Adjunction, Category, Dioid, Semiring, Quantale, Kleene.

1 Preliminaries

1.1 The Algebraic Point of View

In the standard formulation of formal languages and automata, which we will refer to henceforth as the classical theory, a language is usually regarded as a subset of a free monoid $M = X^*$. In contrast, in the *Algebraic Approach*, a formal language is viewed as an algebraic entity residing in a partially ordered monoid. Through the conventional identification $x \leftrightarrow \{x\}$, the point of view grounded in set theory is algebraized, with each set actually being viewed as a *sum* of its elements, e.g.,

$$\{a^m b^m : m \geq 0\} = \bigcup_{m \geq 0} \{a\}^m \{b\}^m \leftrightarrow \sum_{m \geq 0} a^m b^m \,.$$

In the classical theory, the process of algebraization ended abruptly at the type 3 level in the Chomsky hierarchy: the regular languages and their corresponding algebra of regular expressions. Attempts were made to extend this process to the type 2 level (i.e., context-free expressions) [2,3,4], but did not find particularly

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fruitful applications; e.g., no algebraic reformulation of parsing theory. A significant step, however, in this direction had already been taken early on [12], the result being the Chomsky-Schützenberger Theorem for context-free languages. However, no theory of context-free expressions arose from this result. In recent times, we've begun to see renewed progress in this direction [5].

Much of what stood in the way may have been the difficulty in clarifying the algebraic foundation underlying the theory of regular expressions. In what algebra(s) do these objects live? A diversity of answers emerged, as had been noted in [8], in which adjunctions were constructed to embody the hierarchical relation $\mathcal{R} \leq \omega \leq \mathcal{P}$.

Though the large number of inequivalent formulations may seem to be a setback, in fact, as we have seen in [1], a complete lattice of monadic dioids can be defined which presents no less than an embodiment and significant generalization of the Chomsky Hierarchy, itself. For, in addition to the operators $\mathcal{F}M$, $\mathcal{R}M$, ωM and $\mathcal{P}M$ defining, respectively, the finite, rational, countable and general subsets of a monoid M, we also have operators $\mathcal{C}M$, $\mathcal{S}M$ and $\mathcal{T}M$ defining, respectively, the context-free, context-sensitive and Turing-computable subsets of M. Correspondingly, one may then seek to define adjunctions between the members of the larger hierarchy

$$\mathcal{F} \leq \mathcal{R} \leq \mathcal{C} \leq \mathcal{S} \leq \mathcal{T} \leq \omega \leq \mathcal{P}$$

and, indeed, between all the members of the monadic dioid lattice, itself.

A precursor to the results formulated here may be found in [8], where adjunctions are defined connecting the operators $\mathcal{R} \leq \omega \leq \mathcal{P}$ and their corresponding categories of dioids, which we shall term $\mathbf{D}\mathcal{R}$, $\mathbf{D}\omega$ and $\mathbf{D}\mathcal{P}$. The functors

$$\mathbf{D}\mathcal{R} \to \mathbf{D}\omega \to \mathbf{D}\mathcal{P}$$

are constructed by defining appropriate families of ideals for the respective algebras, while the opposite members of the respective adjoint pairs

$$\mathbf{D}\mathcal{R} \leftarrow \mathbf{D}\omega \leftarrow \mathbf{D}\mathcal{P}$$

give us the structure-reducing forgetful functors. Conway [9] had earlier provided a construction for the adjunction formed of the pair $\mathbf{Q}_{\mathcal{R}}^{\mathcal{P}} : \mathbf{D}\mathcal{R} \to \mathbf{D}\mathcal{P}$, and $\mathbf{Q}_{\mathcal{P}}^{\mathcal{R}} : \mathbf{D}\mathcal{P} \to \mathbf{D}\mathcal{R}$.

These constructions may also be viewed as results in Kleene algebra, whereby a given *-continuous Kleene algebra is extended to a form that has closure and distributivity under a larger family of subsets. Expanding on this point of view, the adjunctions relating the pairs $\mathcal{R} \leq \mathcal{C}$, $\mathcal{R} \leq \mathcal{S}$ and $\mathcal{R} \leq \mathcal{T}$ may be viewed as operations that give us a fixed-point closure of a given *-continuous Kleene algebra for $\mathcal{C}$, or a relational closure for $\mathcal{S}$ and $\mathcal{T}$. Concrete realizations of these constructions, in particular for $\mathcal{C}$, would then provide us with an algebraization of the classical result known as the Chomsky-Schützenberger theorem (thus, also resolving a question raised in the closing section of [5]).

More generally, denoting by $\mathbf{D}\mathcal{A}$ the category of $\mathcal{A}$-diods and $\mathcal{A}$-morphisms,[1] a desired outcome would be to reflect the hierarchy of monadic dioids by a hierarchy of adjunctions $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{B}$ where $\mathcal{A} \leq \mathcal{B}$, such that $\mathbf{Q}^{\mathcal{C}}_{\mathcal{B}} \circ \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} = \mathbf{Q}^{\mathcal{C}}_{\mathcal{A}}$, whenever $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$.

1.2 Monadic Operators

Different families of it languages over an alphabet X are defined through their corresponding families of subsets of a monoid X^*. When expressed the algebraic setting formulated in [1], each family is identified as a it monadic operator, $\mathcal{A} : M \mapsto \mathcal{A}M$ that freely extends a monoid M to an $\mathcal{A}$-dioid. Reviewing the basic results of [1], such an operator, $\mathcal{A}M$, $(\mathbf{A}_1)$ is defined as a family of subsets of the monoid M; $(\mathbf{A}_2)$ contains all the finite subsets of M; $(\mathbf{A}_3)$ is closed under products (thus making $\mathcal{A}M$ a monoid); $(\mathbf{A}_4)$ is closed under unions from $\mathcal{A}\mathcal{A}M$; and $(\mathbf{A}_5)$ respects homomorphisms in the sense that if $f : M \to N$ is a monoid homomorphism, then[2] $\widetilde{f}(U) \in \mathcal{A}N$ for all $U \in \mathcal{A}M$. Though property $\mathbf{A}_3$ is not a part of the classical theory, we are able to prove the equivalence of the combination of $\mathbf{A}_3$ and $\mathbf{A}_4$ with a property that is classical: that $(\mathbf{A}_5)$ $\mathcal{A}$ respects $\mathcal{A}$-substitutions[3] – if $\sigma : M \to \mathcal{P}N$ is an $\mathcal{A}$-substitution, then $\hat{\sigma}(U) \in \mathcal{A}N$ for all $U \in \mathcal{A}M$. Finally, we are able to show, given the surjectivity of the monoid homomorphism $f : M \to N$, the surjectivity of its lift $\widetilde{f} : \mathcal{A}M \to \mathcal{A}N$ (property $\mathbf{A}_6$, [1]).

Using the notation $x > A$ to denote when x is an upper bound of a set A, one may then define M to be $(\mathbf{D}_0)$ $\mathcal{A}$-*additive* if every $U \in \mathcal{A}M$ has a least upper bound $\sum U \in M$; $(\mathbf{D}_1)$ $\mathcal{A}$-*separable* if for all $x > aUb$ there exists $x > U$ such that $x \geq aub$, where $a, b \in M$ and $U \in \mathcal{A}M$; and $(\mathbf{D}_2)$ strongly $\mathcal{A}$-separable if for all $x > UV$ there exist $u > U$, $v > V$ such that $x \geq uv$, where $U, V \in \mathcal{A}M$. Finally, a monoid homomorphism $f : M \to N$ is said to be $(\mathbf{D}_3)$ $\mathcal{A}$-*continuous* if for all $y > \widetilde{f}(U)$ there exists $x > U$ such that $y \geq f(x)$, where $U \in \mathcal{A}M$.

When a monoid is $\mathcal{A}$-additive both forms of separability equivalently reduce to more familiar form as the following distributivity identities: $(\mathbf{D}_{1'})$ $a, b \in M, U \in \mathcal{A}M \to \sum(aUd)$; and $(\mathbf{D}_{2'})$ $U, V \in \mathcal{A}M \to \sum(UV) = \sum U \cdot \sum V$. Also, in such monoids, for order-preserving monoid homomorphisms, $f : M \to M'$, $\mathcal{A}$-additivity reduces equivalently to the condition that $(\mathbf{D}_{3'})$: $U \in \mathcal{A}M \to f(\sum U) = \sum \widetilde{f}(U)$. Finally, an $\mathcal{A}$-*dioid* is a partially ordered monoid M satisfying $\mathbf{D}_0$ and $\mathbf{D}_1$, and an $\mathcal{A}$-*morphism* is an order-preserving monoid homomorphism that satisfies $\mathbf{D}_3$.

The following results may then be proven:

Theorem 1 (*The Universal Property*, **[1]**)**.** *The free $\mathcal{A}$ dioid extension of a monoid M is $\mathcal{A}M$. Equivalently, this may be stated as follows: that $\eta_M : M \to$*

[1] Defined in [1], these will be reviewed here in the following section.

[2] In here, and in the following, we will denote the image of a function f on a set U by $\widetilde{f}(U) \equiv \{f(u) : u \in U\}$.

[3] Recalling [1], an $\mathcal{A}$-*substitution* is a monoid homomorphism $\sigma : M \to \mathcal{A}N$ and is uniquely determined by its extension $\hat{\sigma} : \mathcal{P}M \to \mathcal{P}N$, given by $\hat{\sigma} : U \subseteq M \mapsto \bigcup_{u \in U} \sigma(u)$ to a (unit-preserving) quantale homomorphism [1].

$\mathcal{A}M$, $m \mapsto \{m\}$ is a monoid homomorphism and that a monoid homorphism $f : M \to D$ to an $\mathcal{A}$-dioid D extends uniquely to an $\mathcal{A}$-morphism $f^* : \mathcal{A}M \to D$; i.e., such that $f = f^* \circ \eta_M$.

Theorem 2 (*Hierarchical Completeness*, **[1]**). *Monadic operators form a complete lattice with top $\mathcal{A}M = \mathcal{P}M$ and bottom $\mathcal{A}M = \mathcal{F}M$, with lattice meet defined for a family Z be a family of monadic operators by $(\wedge \mathsf{Z})\, M = \bigcap_{\mathcal{A} \in \mathsf{Z}} \mathcal{A}M$.*

We will use $\geq$ and $\leq$ to denote the lattice ordering relation, $\mathcal{A} \leq \mathcal{B} \leftrightarrow \mathcal{A} \cap \mathcal{B} = \mathcal{A}$ and $\mathcal{B} \geq \mathcal{A} \leftrightarrow \mathcal{A} \leq \mathcal{B}$.

2 Ideals and Quantales

In this section, we will define the quantale completion for each variety of monadic dioid. The construction is accomplished through a suitably defined family of ideals and is similar to that used to define the completion of a lattice.

2.1 Ideals, Basic Properties

Corresponding to each operator $\mathcal{A}$ is a variety of ideals that will be termed $\mathcal{A}$-ideals. The definition makes use of the following closure, which is generic to partial orderings.

Definition 1. *For a partially ordered set D, let $U' = \{x \in D : \forall y > U : y \geq x\}$.*

If a set U has a least upper bound $\sum U$, then the relation $y > U$ is equivalent to $y \geq \sum U$. Therefore, defining the *interval* $\langle a \rangle \equiv \{x \in D : x \leq a\}$ we have the following properties.

Theorem 3. *For a partially ordered set D:*

- *$\{a\}' = \langle a \rangle$;*
- *if 0 is the minimal element of D, then $\emptyset' = \{0\}$; and*
- *if $U \subseteq D$ has a least upper bound $\sum U$ then $U' = \langle \sum U \rangle$.*

We may define the family $\mathcal{A}[D]$ of $\mathcal{A}$-ideals in the general setting of partially ordered monoids, D. The sole requirement we impose on such ideals $I \subseteq D$ is that ($\mathbf{I}_1$): for all $U \in \mathcal{A}D$ and $a, b \in D$, if $aUb \subseteq I$, then $aU'b \subseteq I$.

Since $\{a\}' = \langle a \rangle$, property $\mathbf{I}_1$ implies that an $\mathcal{A}$-ideal I must also be closed downward with respect to the partial ordering $\leq$, ($\mathbf{I}_2$): $x \leq d \in I \to x \in I$.

Though the definition is generic to partially ordered monoids, its primary application will be to $\mathcal{A}$-dioids, D. In such a case, an $\mathcal{A}$-ideal of D may be equivalently defined by property ($\mathbf{I}_3$): $U \in \mathcal{A}D$, $U \subseteq I \to \sum U \in I$. We prove this in the following.

Corollary 1. *For $\mathcal{A}$-dioids, D, $\mathbf{I}_1$ is equivalent to $\mathbf{I}_2$ and $\mathbf{I}_3$.*

Proof. Taking $a = b = 1$ in $\mathbf{I}_1$, leads to the result $\mathbf{I}_3$. For the converse, we note that the $\mathcal{A}$-separability property $\mathbf{D}_1$ of D implies for $U \in \mathcal{A}D$ and $a, b \in D$ that

$$a \left\langle \sum U \right\rangle b = aU'b \subseteq (aUb)' = \left\langle \sum (aUb) \right\rangle .$$

Combined with $\mathbf{I}_2$ and $\mathbf{I}_3$, this leads to $\mathbf{I}_1$.

For $\mathcal{A} = \mathcal{F}, \mathcal{R}$, equivalent definitions of $\mathcal{A}$-ideals may be formulated in the general setting of dioids. In particular, since $\emptyset' = \{0\}$, property $\mathbf{I}_1$ requires that $0 \in I$.

Corollary 2. *Let D be a dioid. Then for an $\mathcal{A}$-ideal $I \subseteq D$,*

$\mathbf{IF}_0$ $I \neq \emptyset$;
$\mathbf{IF}_1$ $0 \in I$;
$\mathbf{IF}_2$ $d, e \in I \to d + e \in I$.

Moreover, an $\mathcal{F}$-ideal $I \subseteq D$ is equivalently defined by $\mathbf{I}_2$, $\mathbf{IF}_1$ *(or, equivalently,* $\mathbf{IF}_0$*) and* $\mathbf{IF}_2$.

Proof. All three properties $\mathbf{IF}_0$, $\mathbf{IF}_1$ and $\mathbf{IF}_2$ follow from $\mathbf{I}_3$, for the case $\mathcal{A} = \mathcal{F}$. Taking $a = b = 1$ with $U = \emptyset$ yields $\mathbf{IF}_1$ from which $\mathbf{IF}_0$ follows, while taking $a = b = 1$ with $U = \{d, e\}$ yields $\mathbf{IF}_2$. Similarly, for $\mathcal{A}$-dioids, the result follows in virtue of the inclusion $\mathcal{F}D \subseteq \mathcal{A}D$.

Conversely, suppose $\mathbf{I}_2$, $\mathbf{IF}_1$ and $\mathbf{IF}_2$ hold, and that $U = \{u_1, \ldots, u_n\} \subseteq D$ with $n \geq 0$. Then we have $n = 0 \to \sum U = \sum \emptyset = 0 \in I$ by $\mathbf{IF}_1$ and $\mathbf{I}_2$, and $n > 0 \to \sum U = u_1 + \ldots + u_n \in I$ by $\mathbf{IF}_2$.

For the operator $\mathcal{R}$, we have the following characterization:

Corollary 3. *An $\mathcal{R}$-ideal $I \subseteq D$ of an $\mathcal{R}$-dioid D is an $\mathcal{F}$-ideal of D for which:*

$\mathbf{IR}_1$ *if $ab^n c \in I$ for all $n \geq 0$, then $ab^* c \in I$.*

Proof. If $I \subseteq D$ is an $\mathcal{R}$-ideal, from $a \{b\}^* c = \{ab^n c : n \geq 0\} \subseteq I$, we conclude that $ab^*c = \sum a \{b\}^* c \in I$, by $\mathbf{I}_3$. To prove the converse, for an $\mathcal{F}$-ideal $I \subseteq D$ satisfying $\mathbf{IR}_1$, we need to inductively establish, for $U \in \mathcal{R}D$, that $aUd \subseteq I \to a (\sum U) d \in I$. The argument is quite analogous to that used to establish the equivalence of $\mathcal{R}$-dioids and *-continuous Kleene algebras. We already have the property for finite subsets, by assumption. Showing that the property is preserved by sums, products, stars is easy, noting the following

$$a \left(\sum U \cup V\right) d = a \left(\sum U\right) d + a \left(\sum V\right) d ,$$

$$a \left(\sum UV\right) d = a \left(\sum U\right) \left(\sum V\right) d = \sum_{v \in V} a \left(\sum U\right) vd ,$$

$$a \left(\sum U^*\right) d = \sum_{n \geq 0} a \left(\sum U\right)^n d$$

and using $\mathbf{IR}_1$ in conjunction with the last equality.

In general $\mathcal{A}$-ideals will form a hierarchy closed under intersection. This is a consequence of the following:

Theorem 4. *For a partially ordered monoid D, $\mathsf{Y} \subseteq \mathcal{A}[D] \to \cap\mathsf{Y} \in \mathcal{A}[D]$.*

Proof. Let $\mathsf{Y} \subseteq \mathcal{A}[D]$. Then suppose $U \in \mathcal{A}D$ with $aUb \subseteq \cap\mathsf{Y}$. Then for any $\mathcal{A}$-ideal $I \in \mathsf{Y}$, we have $aUb \subseteq \cap\mathsf{Y} \subseteq I \to aU'b \subseteq I$, by $\mathbf{I}_1$. Hence, $aU'b \subseteq \cap\mathsf{Y}$, thus making Y an $\mathcal{A}$-ideal too. For the special case where $\mathsf{Y} = \emptyset$, we set $\cap\emptyset = D$ and note that D is an ideal of itself.

As a result it follows that $\mathcal{A}[D]$ forms a complete lattice under the subset ordering $\subseteq$ with D as the maximal element. One may therefore define the ideal-closure of arbitrary sets:

Definition 2. *Let D be a partially ordered monoid and $U \subseteq D$. Then $\langle U \rangle_{\mathcal{A}} = \cap\{I \in \mathcal{A}[D] : U \subseteq I\}$.*

Basic properties, generic to partially ordered monoids, include the following:

Theorem 5. *In any partially ordered monoid D, if $U, V \subseteq D$ then*

$$U \subseteq \langle U \rangle_{\mathcal{A}} ,$$

$$U \subseteq V \to \langle U \rangle_{\mathcal{A}} \subseteq \langle V \rangle_{\mathcal{A}} ,$$

$$U \in \mathcal{A}[D] \leftrightarrow U = \langle U \rangle_{\mathcal{A}} .$$

For brevity, in the following we will usually omit the index and just write $\langle U \rangle$ for $\langle U \rangle_{\mathcal{A}}$, where the context permits. In the special case of $\mathcal{A}$-dioids, the following results also hold:

Corollary 4. *Let D be an $\mathcal{A}$-dioid. Then $\langle \emptyset \rangle = \langle 0 \rangle$ is the minimal $\mathcal{A}$-ideal in D; and each interval $\langle a \rangle = \langle \{a\} \rangle$, for $a \in D$, is a principal $\mathcal{A}$-ideal in D.*

More generally, if D is already an $\mathcal{A}$-dioid, then $U' = \langle \sum U \rangle$ for any $U \in \mathcal{A}D$, so that these subsets generate principal ideals.

Lemma 1. *Let D be an $\mathcal{A}$-dioid. Then for any $U \in \mathcal{A}D$, then $\langle U \rangle_{\mathcal{D}} = \langle \sum U \rangle$.*

This then shows that the ideals generated by the subsets from $\mathcal{A}D$ will be in a one-to-one correspondence with D itself, when D has the structure of an $\mathcal{A}$-dioid. Taking the ideals generated from a larger family $\mathcal{B}D$ provides the natural candidate for the extension of D to a $\mathcal{B}$-dioid. If we could define the product and sum operations on ideals, then this would provide a basis for extending the $\mathcal{A}$-dioid D to a $\mathcal{B}$-dioid for an operator $\mathcal{B} > \mathcal{A}$. We would simply take those ideals generated from $\mathcal{B}D$.

In the most general case, where $\mathcal{B} = \mathcal{P}$, the family of ideals generated is just $\mathcal{A}[D]$, itself. The entire collection of ideals should then yield a full-fledged quantale structure. In fact, this is what we will examine next.

2.2 Defining a Quantale Structure on Ideals

The family $\mathcal{A}[D]$, when provided with a suitable algebraic structure, will define the extension of D to a dioid with the structure characteristic of a $\mathcal{P}$-dioid or quantale with identity 1: a complete upper semilattice in which distributivity applies to all subsets. As a result, we will be able to define the map $\mathbf{Q}_{\mathcal{A}} : D \to \mathcal{A}[D]$ that yields a functor $\mathbf{Q}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{P}$ from the category $\mathbf{D}\mathcal{A}$ of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms to the category $\mathbf{D}\mathcal{P}$ of quantales (with units) and quantale (unit-preserving) morphisms.

Products. The product of two ideals should preserve the correspondence $\langle U \rangle = \langle \sum U \rangle$ that holds in $\mathcal{A}$-dioids D with respect to $\mathcal{A}$-ideals generated by subsets from $\mathcal{A}D$. But this would require that

$$\left\langle \sum U \sum V \right\rangle \leftrightarrow \langle UV \rangle \leftrightarrow \left\langle \sum UV \right\rangle .$$

Therefore, the product should satisfy the property $\langle U_1 V_1 \rangle_{\mathcal{A}} = \langle U_2 V_2 \rangle_{\mathcal{A}}$ whenever $\langle U_1 \rangle_{\mathcal{A}} = \langle U_2 \rangle_{\mathcal{A}}$ and $\langle V_1 \rangle_{\mathcal{A}} = \langle V_2 \rangle_{\mathcal{A}}$. We will prove this is so by showing, in particular, the following result. For brevity, we will again omit the subscript.

Lemma 2 (*The Product Lemma*)**.** *Suppose D is a dioid and that $U, V \subseteq D$. Then $\langle UV \rangle = \langle \langle U \rangle \langle V \rangle \rangle$.*

Proof. One direction is already immediate: from $U \subseteq \langle U \rangle$ and $V \subseteq \langle V \rangle$, we get $UV \subseteq \langle U \rangle \langle V \rangle$. Consequently, $\langle UV \rangle \subseteq \langle \langle U \rangle \langle V \rangle \rangle$. In the other direction, if we can show that $\langle U \rangle \langle V \rangle \subseteq \langle UV \rangle$ then it will follow that

$$\langle \langle U \rangle \langle V \rangle \rangle \subseteq \langle \langle UV \rangle \rangle = \langle UV \rangle .$$

To this end, let $Y = \{y \in D : yV \subseteq \langle UV \rangle\}$ and $Z = \{z \in D : \langle U \rangle z \subseteq \langle UV \rangle\}$. Then clearly $YV \subseteq \langle UV \rangle$ and $U \subseteq Y$. So, if we can show that Y is an ideal, it will then follow that $\langle U \rangle \subseteq \langle Y \rangle = Y$, from which we could conclude $\langle U \rangle V \subseteq \langle UV \rangle$. From this, in turn, it will follow that $V \subseteq Z$, while $\langle U \rangle Z \subseteq \langle UV \rangle$. So, if we can also show that Z is an ideal, then we will be able to conclude that $\langle V \rangle \subseteq \langle Z \rangle = Z$ and, from this, that $\langle U \rangle \langle V \rangle \subseteq \langle U \rangle Z \subseteq \langle UV \rangle$.

Suppose, then, that $aWb \subseteq Y$, where $a, b \in D$ and $W \in \mathcal{A}D$. Then, for each $v \in V$, by definition of Y, we have $aWbv \subseteq \langle UV \rangle$. Applying property $\mathbf{I}_1$ to the ideal $\langle UV \rangle$, we conclude that $aW'bv \subseteq \langle UV \rangle$. Therefore, it follows that $aW'bV \subseteq \langle UV \rangle$ and, from this, that $aW'b \subseteq Y$. Thus, Y is an ideal.

The argument showing that Z is an ideal is similar. Suppose $aWb \subseteq Z$, again, with $a, b \in D$ and $W \in \mathcal{A}D$. Then, for each $u \in \langle U \rangle$, by definition of Z, we have $uaWb \subseteq \langle UV \rangle$. Again applying property $\mathbf{I}_1$ to the ideal $\langle UV \rangle$, we conclude that $uaW'b \subseteq \langle UV \rangle$, from this it follows that $\langle U \rangle aW'b \subseteq \langle UV \rangle$ and $aW'b \subseteq Z$.

This clears the way for us to define products over subsets of D.

Definition 3. *Let D be a dioid, and $U, V \subseteq D$. Then define $U \cdot V \equiv \langle UV \rangle$.*

Lemma 3. *Let D be a dioid. Then $\mathcal{A}[D]$ is a partially ordered monoid with product $U, V \mapsto U \cdot V$, identity $\langle\{1\}\rangle$ and ordering $\subseteq$.*

Proof. Let $U, V, W \subseteq D$ Then

$$\langle\{1\}\rangle \cdot \langle V\rangle = \langle\{1\} V\rangle = \langle V\rangle = \langle V\{1\}\rangle = \langle V\rangle \cdot \langle\{1\}\rangle ,$$

$$\langle U\rangle \cdot (\langle V\rangle \cdot \langle W\rangle) = \langle U\rangle \cdot \langle VW\rangle = \langle UVW\rangle = \langle UV\rangle \cdot \langle W\rangle = (\langle U\rangle \cdot \langle V\rangle) \cdot \langle W\rangle .$$

We can treat this algebra as an inclusion of the monoid structure of D, itself, through the correspondence $x \leftrightarrow \langle x\rangle$. But in general, it will not be an embedding, unless D also possesses the structure of an $\mathcal{A}$-dioid. This result is captured by the following property:

Theorem 6. *If D is a $\mathcal{A}$-dioid, then for $a, b \in D$, $\langle a\rangle \cdot \langle b\rangle = \langle ab\rangle$. Thus, $\langle\rangle_{\mathcal{A}} : D \to \mathcal{D}[D]$, is a monoid embedding with the unit $\langle 1\rangle$.*

Proof. This follows from the relation between principal ideals and intervals, which generally holds in dioids:

$$\langle a\rangle \cdot \langle b\rangle = \langle\{a\}\rangle_{\mathcal{A}} \cdot \langle\{b\}\rangle_{\mathcal{A}} = \langle\{a\}\{b\}\rangle_{\mathcal{A}} = \langle\{ab\}\rangle_{\mathcal{A}} = \langle ab\rangle .$$

The one-to-one ness of $a \mapsto \langle a\rangle$ is a consequence of the anti-symmetry property of partial orders.

Sums. In a similar way, we would like to preserve the correspondence $U \leftrightarrow \langle\sum U\rangle$ with respect to the sum operator. So, if $U \in \mathcal{A}D$, then we should be able to express $\langle U\rangle_{\mathcal{A}}$ as a sum over its component principal ideals, $\langle U\rangle_{\mathcal{A}} = \sum_{u\in U}\langle u\rangle = \cup_{u\in U}\langle u\rangle$. In order for this to work, we need to know that if $\langle U_\alpha\rangle_{\mathcal{A}} = \langle V_\alpha\rangle_{\mathcal{A}}$ for all $\alpha \in A$, then $\left\langle\bigcup_{\alpha\in A} U_\alpha\right\rangle_{\mathcal{A}} = \left\langle\bigcup_{\alpha\in A} V_\alpha\right\rangle_{\mathcal{A}}$. In particular, we will prove the following result (omitting the subscript again, for brevity):

Lemma 4 (*The Sum Lemma***).** *Let D be a dioid and $\mathsf{Y} \subseteq \mathcal{P}D$. Then $\langle\bigcup \mathsf{Y}\rangle = \left\langle\bigcup_{V\in\mathsf{Y}}\langle V\rangle\right\rangle$.*

Proof. Unlike the *Product Lemma* (lemma 2), this result may be established directly without an inductive proof. Suppose $\mathsf{Y} \subseteq \mathcal{P}D$. For $V \in \mathsf{Y}$, we then have the following line of argumentation

$$V \in \mathsf{Y} \to V \subseteq \bigcup \mathsf{Y} \to \langle V\rangle \subseteq \left\langle\bigcup \mathsf{Y}\right\rangle .$$

here, we can continue and argue as follows

$$\bigcup_{V\in\mathsf{Y}}\langle V\rangle \subseteq \left\langle\bigcup \mathsf{Y}\right\rangle \to \left\langle\bigcup_{V\in\mathsf{Y}}\langle V\rangle\right\rangle \subseteq \left\langle\left\langle\bigcup \mathsf{Y}\right\rangle\right\rangle = \left\langle\bigcup \mathsf{Y}\right\rangle .$$

Going in the opposite direction, we have the inclusions $V \subseteq \langle V\rangle$, for each $V \in \mathsf{Y}$. Therefore,

$$\bigcup \mathsf{Y} \subseteq \bigcup_{V\in\mathsf{Y}}\langle V\rangle \to \left\langle\bigcup \mathsf{Y}\right\rangle \subseteq \left\langle\bigcup_{V\in\mathsf{Y}}\langle V\rangle\right\rangle .$$

This clears the way for us to define a summation operator over $\mathcal{P}D$.

Definition 4. *Let D be a dioid and $\mathsf{Y} \subseteq \mathcal{P}D$. Then, define $\sum \mathsf{Y} \equiv \langle \bigcup \mathsf{Y} \rangle$.*

Theorem 7. *Let D be a dioid. Then $\mathsf{Y} \mapsto \sum \mathsf{Y}$ is the least upper bound operator over $\mathcal{A}[D]$.*

Proof. Suppose $\mathsf{Y} \subseteq \mathcal{P}D$ and $I \in \mathcal{A}[D]$ and upper bound. That is, assume that $V \subseteq I$ for all $V \in \mathsf{Y}$. Then it follows that

$$\bigcup \mathsf{Y} \subseteq I \to \sum \mathsf{Y} = \left\langle \bigcup \mathsf{Y} \right\rangle \subseteq \langle I \rangle = I\,.$$

But clearly $\sum \mathsf{Y}$ is, itself, an upper bound of Y. Indeed, for all $V \in \mathsf{Y}$, we have

$$V \subseteq \bigcup \mathsf{Y} \subseteq \left\langle \bigcup \mathsf{Y} \right\rangle = \sum \mathsf{Y}\,.$$

Therefore, $\sum \mathsf{Y}$ is the least upper bound of Y.

We can also prove that the Σ operator is distributive.

Lemma 5. *Let D be a dioid, $U, V \subseteq D$ and $\mathsf{Y} \subseteq \mathcal{P}D$. Then $U \cdot \sum \mathsf{Y} \cdot V = \sum_{W \in \mathsf{Y}} U \cdot W \cdot V$.*

Proof. This is a direct consequence of definition 4 and theorem 7 with

$$U \cdot \sum \mathsf{Y} \cdot V = U \cdot \left\langle \bigcup \mathsf{Y} \right\rangle \cdot V = \left\langle U \left(\bigcup \mathsf{Y} \right) V \right\rangle = \left\langle \bigcup_{W \in \mathsf{Y}} UWV \right\rangle$$

while

$$\sum_{w \in \mathsf{Y}} U \cdot W \cdot V = \sum_{W \in \mathsf{Y}} \langle UWV \rangle = \left\langle \bigcup_{W \in \mathsf{Y}} UWV \right\rangle\,.$$

Quantale Structure. Finally, this leads to the result

Theorem 8. *For any dioid D and monadic operator $\mathcal{A}$, $\mathcal{A}[D]$ is a quantale with a unit $\langle\{1\}\rangle$. Moreover, if D is an $\mathcal{A}$-dioid, then the map $\mathbf{Q}_{\mathcal{A}} : D \to \mathcal{A}[D]$ is an $\mathcal{A}$-morphism.*

Proof. In general, the restriction of the map $\langle\rangle_{\mathcal{A}} : \mathcal{A}D \to \mathcal{A}[D]$ is an order-preserving monoid homomorphism since $\langle\{1\}\rangle_{\mathcal{A}} = \langle 1 \rangle$ and $\langle U \rangle_{\mathcal{A}} \cdot \langle V \rangle_{\mathcal{A}} = \langle UV \rangle_{\mathcal{A}}$. When the dioid D also happens to have the structure of an $\mathcal{A}$-dioid, then the correspondence reduces to an embedding $\mathbf{Q}_{\mathcal{A}} : D \to \mathcal{A}[D]$ into the principal ideals of D, for in that case, we have $\langle U \rangle_{\mathcal{A}} = \langle \sum U \rangle$, for all $U \in \mathcal{A}D$. The result is then an extension of the $\mathcal{A}$-dioid D to a quantale $\mathcal{A}[D]$.

Morphisms. Finally, we should have consistency with respect to $\mathcal{A}$-morphisms $f : D \to D'$. In particular, we'd like to have the property that $\left\langle \widetilde{f}(U) \right\rangle_{\mathcal{A}} = \left\langle \widetilde{f}(V) \right\rangle_{\mathcal{A}}$ whenever $\langle U \rangle_{\mathcal{A}} = \langle V \rangle_{\mathcal{A}}$. This result, too, will be true. We will prove it in the following form (once again, omitting the subscript for brevity).

Lemma 6 (*The Morphism Lemma*)**.** *Let D, D' be dioids and $f : D \to D'$ an $\mathcal{A}$-morphism. Then for all $U \subseteq D$, $\left\langle \widetilde{f}(U) \right\rangle = \left\langle \widetilde{f}(\langle U \rangle) \right\rangle$.*

Proof. The forward inclusion is easy since

$$U \subseteq \langle U \rangle \to \widetilde{f}(U) \subseteq \widetilde{f}(\langle U \rangle) \to \left\langle \widetilde{f}(U) \right\rangle \subseteq \left\langle \widetilde{f}(\langle U \rangle) \right\rangle .$$

To prove the converse inclusion, define $X = \left\{ x \in D : f(x) \in \left\langle \widetilde{f}(U) \right\rangle \right\}$. Then X is an $\mathcal{A}$-ideal. For if $V \in \mathcal{A}D$ and $a, b \in D$ with $aVb \subseteq X$, then $\widetilde{f}(aVb) \subseteq \left\langle \widetilde{f}(U) \right\rangle$. Since f is a monoid homomorphism, then $\widetilde{f}(aVb) = f(a)\widetilde{f}(V) f(b)$. Moreover, by property $\mathbf{A}_4$, since $V \in \mathcal{A}D$, then $\widetilde{f}(V) \in \mathcal{A}D'$. Therefore, applying $\mathbf{I}_1$ to the ideal $\left\langle \widetilde{f}(U) \right\rangle$, we get $f(a)\widetilde{f}(V)' f(b) \subseteq \left\langle \widetilde{f}(U) \right\rangle$. If we can then show that $f(V') \subseteq f(V)'$, then it will follow that

$$\widetilde{f}(aV'b) = f(a)\widetilde{f}(V') f(b) \subseteq f(a)\widetilde{f}(V)' f(b) \subseteq \left\langle \widetilde{f}(U) \right\rangle ,$$

so that $aV'b \subseteq X$, thus proving that X is an ideal. With that given, then noting $U \subseteq X$, we would have $\langle U \rangle \subseteq \langle X \rangle = X$, and finally $\widetilde{f}(\langle U \rangle) \subseteq \widetilde{f}(X) \subseteq \left\langle \widetilde{f}(U) \right\rangle$.

It is at this point that the $\mathcal{A}$-additivity of f comes into play. Let $x \in V'$. Pick any upper bound $y > \widetilde{f}(V)$. Then by the $\mathcal{A}$-additivity of f, we have $y \geq f(v)$ for some upper bound $v > V$. By definition of V', it then follows that $x \leq v$. In turn, by the order-preserving property of f (which is a part of the definition of an $\mathcal{A}$-morphism), it follows that $f(x) \leq f(v) \leq y$. Thus, $f(x) \in \widetilde{f}(V)'$.

This result clears the way to unambiguously defining the lifting of f to a mapping $f_{\mathcal{A}} : \mathcal{A}[D] \to \mathcal{A}[D']$ over the respective quantales.

Definition 5. *Let D, D' be dioids, and $f : D \to D'$ an $\mathcal{A}$-morphism. Then define $f_{\mathcal{A}}(U) \equiv \left\langle \widetilde{f}(U) \right\rangle_{\mathcal{A}}$, for $U \subseteq D$.*

Theorem 9. *Let D, D' be dioids, $f : D \to D'$ an $\mathcal{A}$-morphism. Then $f_{\mathcal{A}} : \mathcal{A}[D] \to \mathcal{A}[D']$ is an identity-preserving quantale homomorphism; or, equivalently, a $\mathcal{P}$-morphism.*

Proof. The identity $\langle 1 \rangle = \langle \{1\} \rangle$ is clearly preserved, since $f_{\mathcal{A}}(\langle 1 \rangle) = \langle \{f(1)\} \rangle = \langle 1 \rangle$. Products are preserved, since

$$f_{\mathcal{A}}(U \cdot V) = f_{\mathcal{A}}(\langle UV \rangle) = \left\langle \widetilde{f}(UV) \right\rangle = \left\langle \widetilde{f}(U)\widetilde{f}(V) \right\rangle$$

while

$$f_{\mathcal{A}}(U) \cdot f_{\mathcal{A}}(V) = \left\langle \widetilde{f}(U) \right\rangle \cdot \left\langle \widetilde{f}(V) \right\rangle = \left\langle \widetilde{f}(U)\widetilde{f}(V) \right\rangle ,$$

for $U, V \subseteq D$. Finally, suppose $\mathsf{Y} \subseteq \mathcal{P}D$. Then

$$f_{\mathcal{A}}\left(\sum \mathsf{Y}\right) = f_{\mathcal{A}}\left(\left\langle \bigcup \mathsf{Y} \right\rangle\right) = \left\langle \widetilde{f}\left(\bigcup \mathsf{Y}\right) \right\rangle = \left\langle \bigcup_{U \in \mathsf{Y}} \widetilde{f}(U) \right\rangle ,$$

while

$$\sum_{U \in \mathsf{Y}} f_{\mathcal{A}}(U) = \sum_{U \in \mathsf{Y}} \left\langle \widetilde{f}(U) \right\rangle = \left\langle \bigcup_{U \in \mathsf{Y}} \widetilde{f}(U) \right\rangle ,$$

which establishes our result. In particular, the Morphism Lemma (lemma 6) is made use of in the second equality of each reduction to remove the inner bracket.

Free Quantale Extensions. This is the final ingredient needed to show that $\mathbf{Q}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{P}$ is a functor. Moreover, we may also show that the extension provided by the function is a free extension, in the sense of satisfying an appropriate universal property.

A functor must preserve identity morphisms. This is almost immediate. In fact, letting D be an $\mathcal{A}$-dioid, then for the identity morphism $1_D : D \to D$, we have for $U \subseteq D$, $(1_D)_{\mathcal{A}}(U) = \left\langle \widetilde{1_D}(U) \right\rangle_{\mathcal{A}} = \langle U \rangle_{\mathcal{A}}$. Restricted to $U \in \mathcal{A}[D]$, this produces the result $(1_D)_{\mathcal{A}}(U) = \langle U \rangle_{\mathcal{A}} = U$. The preservation of the functor under composition is given by the following result.

Theorem 10. *Let D, D', D'' be dioids with $f : D' \to D''$ and $g : D \to D'$ being $\mathcal{A}$-morphisms. Then $(f \circ g)_{\mathcal{A}} = f_{\mathcal{A}} \circ g_{\mathcal{A}}$.*

Proof. Let $U \subseteq D$. Then

$$f_{\mathcal{A}} \circ g_{\mathcal{A}}(U) = f_{\mathcal{A}}(\langle \widetilde{g}(U) \rangle) = \left\langle \widetilde{f}(\langle \widetilde{g}(U) \rangle) \right\rangle = \left\langle \widetilde{f}(\widetilde{g}(U)) \right\rangle .$$

Reducing the left-hand side, we get

$$(f \circ g)_{\mathcal{A}}(U) = \left\langle (\widetilde{f \circ g})(U) \right\rangle = \left\langle \widetilde{f}(\widetilde{g}(U)) \right\rangle .$$

Thus, we finally arrive at the result

Corollary 5. *Let $\mathbf{Q}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{P}$ be given by $\mathbf{Q}_{\mathcal{A}} D \equiv \mathcal{A}[D]$, for $\mathcal{A}$-dioids D, and $\mathbf{Q}_{\mathcal{A}} f \equiv f_{\mathcal{A}}$, for $\mathcal{A}$-morphisms $f : D \to D'$ between $\mathcal{A}$-dioids D and D'. Then $\mathbf{Q}_{\mathcal{A}}$ is a functor.*

The universal property is stated as follows. Letting Q denote a quantale with identity, we may define $\mathbf{Q}^{\mathcal{A}} Q$ as the algebra Q, itself, with only the $\mathcal{A}$-dioid structure. This map is actually a functor $\mathbf{Q}^{\mathcal{A}} : \mathbf{D}\mathcal{P} \to \mathbf{D}\mathcal{A}$ which is termed a *forgetful functor*. It is nothing more than the identity map, where the extra structure associated with a $\mathcal{P}$-dioid, not already present as part of the $\mathcal{A}$-dioid structure, is forgotten.

The universal property states that any $\mathcal{A}$-morphism $f : D \to \mathbf{Q}^{\mathcal{A}} Q$ from an $\mathcal{A}$-dioid D should extend uniquely to a unit-preserving quantale morphism (or $\mathcal{P}$-morphism) $f^* : \mathcal{A}[D] \to Q$. The sense in which this is an extension is that it works in conjunction with the unit $\mathcal{A}$-morphism $\eta_D : D \to \mathcal{A}[D]$ defined by $\eta_D(d) = \langle d \rangle$, with $f(d) = f^*(\langle d \rangle)$. The functor pair $\left(\mathbf{Q}_{\mathcal{A}}, \mathbf{Q}^{\mathcal{A}}\right)$ comprises an adjunction between $\mathbf{D}\mathcal{A}$ and $\mathbf{D}\mathcal{P}$ with a unit $D \mapsto \eta_D$. We will not directly prove this result here, since it will be superseded by the more general result in the following section.

3 A Hierarchy of Adjunctions

If we restrict the family of $\mathcal{A}$-ideals to those generated by $\mathcal{B}$-subsets, then we may obtain a representation for a $\mathcal{B}$-algebra. Therefore, let us define the following:

Definition 6. *Let D be a dioid, and $\mathcal{A}, \mathcal{B}$ be monadic operators. Then define $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D = \{\langle U \rangle_{\mathcal{A}} : U \in \mathcal{B}D\}$.*

This is a generalization of our previous construction, with $\mathcal{A}[D] = \mathbf{Q}^{\mathcal{P}}_{\mathcal{A}} D$; or, $\mathbf{Q}_{\mathcal{A}} = \mathbf{Q}^{\mathcal{P}}_{\mathcal{A}}$. The algebra $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ is closed under products. For, if $U, V \in \mathcal{B}D$, then $\langle U \rangle \cdot \langle V \rangle = \langle UV \rangle \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$, since $UV \in \mathcal{B}D$, by $\mathbf{A}_2$. Similarly, $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ is also closed under sums from $\mathcal{B}\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$. Let $\mathsf{Z} \in \mathcal{B}\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$. Since $U \in \mathcal{B}D \mapsto \langle U \rangle_{\mathcal{A}} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ is a monoid homomorphism, then by $\mathbf{A}_6$ it follows that $\mathsf{Z} = \{\langle U \rangle_{\mathcal{A}} : U \in \mathsf{Y}\}$ for some $\mathsf{Y} \in \mathcal{B}\mathcal{B}D$. But, then we can write

$$\sum \mathsf{Z} = \sum_{U \in \mathsf{Y}} \langle U \rangle_{\mathcal{A}} = \left\langle \bigcup \mathsf{Y} \right\rangle_{\mathcal{A}} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D \,,$$

since, by $\mathbf{A}_3$, $\bigcup Y \in \mathcal{B}D$. Together, this proves the following result:

Theorem 11. *Let D be a dioid and $\mathcal{A}$, $\mathcal{B}$ be monadic operators. Then $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ is a $\mathcal{B}$-dioid.*

We also have closure under the lifting of $\mathcal{A}$-morphisms:

Theorem 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be monadic operators. If D and D' are dioids and $f : D \to D'$ is an $\mathcal{A}$-morphism, then $I \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D \to f_{\mathcal{A}}(I) \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D'$.*

Proof. Let $I = \langle U \rangle$, with $U \in \mathcal{B}D$. Then $\widetilde{f}(U) \in \mathcal{B}D'$, by $\mathbf{A}_4$. Therefore $f_{\mathcal{A}}(I) = \left\langle \widetilde{f}(U) \right\rangle_{\mathcal{A}} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D'$.

This allows us to generalize our previous result to the following:

Theorem 13. *Let $\mathcal{A}$ and $\mathcal{B}$ be monadic operators. Define $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{B}$ by: $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D = \{\langle U \rangle_{\mathcal{A}} : U \in \mathcal{B}D\}$ for $\mathcal{A}$-dioids D, as before; $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} f = f_{\mathcal{A}}$ for $\mathcal{A}$-morphisms $f : D \to D'$. Then $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}$ is a functor.*

Theorem 14. *Let $\mathcal{A}$ and $\mathcal{B}$ be monadic operators with $\mathcal{A} \geq \mathcal{B}$. Then $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{D}\mathcal{B}$ is the forgetful functor. In particular, for $\mathcal{A} = \mathcal{B}$, $\mathbf{Q}^{\mathcal{A}}_{\mathcal{A}}$ is the identity functor on $\mathbf{D}\mathcal{A}$.*

Proof. Under the stated condition, every ideal reduces to a principal ideal

$$U \in \mathcal{B}D \subseteq \mathcal{A}D \to \langle U \rangle_{\mathcal{A}} = \left\langle \sum U \right\rangle \,.$$

This establishes a one-to-one correspondence between $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ and D. Previously, we pointed out that the product is preserved with $\langle x \rangle \cdot \langle y \rangle = \langle xy \rangle$ for $x, y \in D$, and we already know that $\langle 1 \rangle = \langle \{1\} \rangle_{\mathcal{A}}$ is the identity. This shows that $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} D$ and D are isomorphic as monoids.

Here, we can show that sums over $\mathcal{B}\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D$ exist in $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D$ without using property $\mathbf{A}_6$ for $\mathcal{B}$. Suppose $\mathsf{Z} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D$. Since the map $\left(\mathsf{Q}^{\mathcal{B}}_{\mathcal{A}}\right)^{-1} : \langle x\rangle \mapsto x$ is a monoid isomorphism then

$$V = \widetilde{\left(\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}\right)}^{-1}(\mathsf{Z}) = \{x \in D : \langle x\rangle \in \mathsf{Z}\} \in \mathcal{B}D\,,$$

by $\mathbf{A}_4$. Therefore,

$$\sum \mathsf{Z} = \left\langle \sum_{v\in V} \langle v\rangle \right\rangle_{\mathcal{A}} = \left\langle \sum_{v\in V} \{v\} \right\rangle_{\mathcal{A}} = \langle V\rangle_{\mathcal{A}} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D\,.$$

Therefore, $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D$ is a $\mathcal{B}$-dioid.

Thus, we only need to show that $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}} : x \in D \mapsto \langle x\rangle$ is $\mathcal{B}$-additive. To that end, let $U \in \mathcal{B}D$. Then, we have

$$\sum \widetilde{\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}}(U) = \sum_{u\in U} \langle u\rangle = \left\langle \bigcup_{u\in U} \{u\} \right\rangle_{\mathcal{A}} = \langle U\rangle_{\mathcal{A}} = \left\langle \sum U \right\rangle\,.$$

This shows that, as a $\mathcal{B}$-dioid, $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}D$ is isomorphic to D.

Finally, we already know that $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}f = f_{\mathcal{A}}$ preserves arbitrary sums, for $\mathcal{A}$-morphisms $f : D \to D'$. Therefore, $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}$ is a $\mathcal{B}$-morphism. This establishes our result.

Finally, the following theorem shows the sense in which the hierarchy of monadic dioids may be considered as a chain of free extensions.

Theorem 15. *Let $\mathcal{A}$ and $\mathcal{B}$ be monadic operators with $\mathcal{A} \leq \mathcal{B}$. Then $\mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}$ is a left adjoint of $\mathbf{Q}^{\mathcal{A}}_{\mathcal{B}}$.*

Before proceeding with the proof, it will first be necessary to describe in more detail the result being sought out here. We are seeking to show that the functors $\mathbf{E} = \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}$ and $\mathbf{U} = \mathbf{Q}^{\mathcal{A}}_{\mathcal{B}}$ forms an adjunction between the categories $\mathbf{D}\mathcal{A}$ and $\mathbf{D}\mathcal{B}$. This requires showing that there is a one-to-one correspondence between $\mathcal{A}$-morphisms $f : A \to \mathbf{U}B$ and $\mathcal{B}$-morphisms $g : \mathbf{E}A \to B$, for any $\mathcal{A}$-dioid A and $\mathcal{B}$-dioid B; that is natural, in the sense that it respects compositions on both sides. Let the correspondence be denoted by the following rules

$$\frac{f : A \to \mathbf{U}B}{f^* : \mathbf{E}A \to B}\,, \frac{g : \mathbf{E}A \to B}{g_* : A \to \mathbf{U}B}\,.$$

To implement the one-to-one nature of the correspondence, we require

$$\frac{f : A \to \mathbf{U}B}{\left(f^*\right)_* = f}\,, \frac{g : \mathbf{E}A \to B}{\left(g_*\right)^* = g}\,.$$

To implement the *naturalness* condition, we require

$$\frac{g : A' \to A, f : A \to \mathbf{U}B, h : B \to B'}{\left(\mathbf{U}h \circ f \circ g\right)^* = h \circ f^* \circ \mathbf{E}g}\,.$$

The candidate chosen for this correspondence is $f^*\left(\langle U\rangle_{\mathcal{A}}\right) = \sum\sum \widetilde{f}\,(U)$. But we must first show that this is well-defined. This is done through the following lemma, which is an elaboration of an argument presented originally in [8].

Lemma 7. *Let A be an $\mathcal{A}$-dioid and B a $\mathcal{B}$-dioid with $f : A \to \mathbf{U}B$ an $\mathcal{A}$-morphism. For each $U \in \mathcal{B}A$, $\sum \widetilde{f}\,(U) = \sum \widetilde{f}\left(\langle U\rangle_{\mathcal{A}}\right)$.*

Proof. It is important to note that this is also an existence result. Though $\widetilde{f}\,(U) \in \mathcal{B}B$, by $\mathbf{A}_4$, it need not be the case that $\widetilde{f}\left(\langle U\rangle_{\mathcal{A}}\right) \in \mathcal{B}B$. Therefore, there is no guarantee at the outset that the latter be summable in B.

However, we do have the following result. Making use of the Morphism Lemma (lemma 6), we know that

$$\left\langle \widetilde{f}\,(U)\right\rangle_{\mathcal{A}} = \left\langle \widetilde{f}\left(\langle U\rangle_{\mathcal{A}}\right)\right\rangle_{\mathcal{A}}$$

for any $U \in \mathcal{B}A$. Moreover, since $\widetilde{f}\,(U) \in \mathcal{B}B$, by $\mathbf{A}_4$, then the sum $\sum \widetilde{f}\,(U) \in \mathcal{B}B$ is defined, and we can write

$$\left\langle \widetilde{f}\left(\langle U\rangle_{\mathcal{A}}\right)\right\rangle_{\mathcal{A}} = \left\langle \widetilde{f}\,(U)\right\rangle_{\mathcal{A}} = \left\langle \sum \widetilde{f}\,(U)\right\rangle .$$

This shows that $\sum \widetilde{f}\,(U)$ is an upper bound of $\widetilde{f}\left(\langle U\rangle_{\mathcal{A}}\right)$. But it is already the least upper bound of the smaller set $\widetilde{f}\,(U)$. Therefore, it must be the least upper bound of the larger set, as well.

On the basis of this result, the map $f^* : \mathbf{E}A \to B$ is well defined. With this matter resolved, we can then proceed to the proof of Theorem 15.

Proof (of Theorem 15). That fact that $f \mapsto f^*$ is one-to-one comes from showing that f is recovered from the principal ideals by $f\,(x) = \langle f^*\left(\langle x\rangle\right)\rangle$. In particular, since $\langle x\rangle$ is an interval, then $\sum \widetilde{f}\left(\langle x\rangle\right) = \langle \sum f\,(x)\rangle = f\,(x)$. Therefore,

$$\langle f^*\left(\langle x\rangle\right)\rangle = \left\langle \sum\sum \widetilde{f}\left(\langle x\rangle\right)\right\rangle = \left\langle \sum f\,(x)\right\rangle = f\,(x) \ .$$

To show that $f^* : \mathbf{E}A \to B$ is actually a $\mathcal{B}$-morphism, we must first show that the monoid structure is preserved. For the identity, noting that $f\,(1) = \langle 1\rangle \in \mathbf{U}B$, we have:

$$f^*\left(\langle 1\rangle\right) = \sum\sum \widetilde{f}\left(\{1\}\right) = \sum\sum \{f\,(1)\} = \sum f\,(1) = \sum \langle 1\rangle = 1\,.$$

For products, we can write

$$\sum\sum \widetilde{f}\,(UV) = \sum\sum \widetilde{f}\,(U)\,\widetilde{f}\,(V) = \sum\left(\sum \widetilde{f}\,(U) \sum \widetilde{f}\,(V)\right) .$$

Noting that the sum on the right distributes and applying the definition of f^*, we obtain the result

$$f^*\left(\langle U\rangle_{\mathcal{A}} \cdot \langle V\rangle_{\mathcal{A}}\right) = f^*\left(\langle U\rangle_{\mathcal{A}}\right) f^*\left(\langle V\rangle_{\mathcal{A}}\right) .$$

Next, we must show that the summation operator is preserved over $\mathcal{B}\mathbf{E}A$. Let $\mathsf{Z} \in \mathcal{B}\mathsf{E}A = \mathcal{B}\mathsf{Q}^{\mathcal{B}}_{\mathcal{A}}A$. It's at this point that we use property $\mathbf{A}_6$. Since $U \in \mathcal{B}A \mapsto \langle U \rangle_{\mathcal{A}} \in \mathbf{Q}^{\mathcal{B}}_{\mathcal{A}}A$ is a monoid homomorphism, then we may assume that there is a set $\mathsf{Y} \in \mathcal{B}\mathcal{B}A$ such that $\mathsf{Z} = \{\langle U \rangle_{\mathcal{A}} : U \in \mathsf{Y}\}$. Then the summation $f^* \left(\sum \mathsf{Z}\right) = \sum \sum \widetilde{f} \left(\langle \bigcup Z \rangle\right)_{\mathcal{A}}$ can be rewritten, using the Sum Lemma (lemma 4), with

$$\left\langle \bigcup \mathsf{Z} \right\rangle_{\mathcal{A}} = \left\langle \bigcup_{U \in \mathsf{Y}} \langle U \rangle_{\mathcal{A}} \right\rangle_{\mathcal{A}} = \left\langle \bigcup_{U \in \mathsf{Y}} U \right\rangle_{\mathcal{A}} = \left\langle \bigcup \mathsf{Y} \right\rangle_{\mathcal{A}} .$$

Using the Morphism Lemma (lemma 6), we then have

$$\sum \widetilde{f} \left(\left\langle \bigcup \mathsf{Z} \right\rangle_{\mathcal{A}} \right) = \sum \widetilde{f} \left(\left\langle \bigcup \mathsf{Y} \right\rangle_{\mathcal{A}} \right) = \sum \widetilde{f} \left(\bigcup \mathsf{Y} \right) .$$

The application to the union can be broken down to that on the component sets,

$$\sum \widetilde{f} \left(\bigcup \mathsf{Y} \right) = \sum \bigcup_{U \in \mathsf{Y}} \widetilde{f}\,(U) .$$

Since each set $\widetilde{f}\,(U) \in \mathcal{B}B$ (by property $\mathbf{A}_4$), the least upper bound $\sum \widetilde{f}\,(U) \in \mathcal{B}B$ is defined. The associativity of least upper bounds, which is a general property of partially ordered sets, can then be used to write – making use, again, of the Sum Lemma –

$$\sum \bigcup_{U \in \mathsf{Y}} \widetilde{f}\,(U) = \sum_{U \in \mathsf{Y}} \sum \widetilde{f}\,(U) = \sum_{U \in \mathsf{Y}} \sum \widetilde{f}\,(\langle U \rangle_{\mathsf{A}}) .$$

Similarly, applying associativity again, we can write

$$f^* \left(\sum \mathsf{Z} \right) = \sum \sum_{U \in \mathsf{Y}} \sum \widetilde{f}\,(\langle U \rangle_{\mathcal{A}}) = \sum_{U \in \mathsf{Y}} \sum \sum \widetilde{f}\,(\langle U \rangle_{\mathcal{A}}) .$$

From the other direction, we may write,

$$\sum \widetilde{f^*}\,(\mathsf{Z}) = \sum_{U \in \mathsf{Y}} f^*\,(\langle U \rangle_{\mathcal{A}}) = \sum_{U \in \mathsf{Y}} \sum \sum \widetilde{f}\,(\langle U \rangle_{\mathcal{A}}) ,$$

which establishes preservation of sums over $\mathcal{B}\mathbf{E}A$.

The additional property of naturalness requires showing that this correspondence be well-behaved with respect to composition with morphisms from the respective categories. In particular, for an $\mathcal{A}$-morphism $g : A' \to A$ and a $\mathcal{B}$-morphism $h : B \to B'$, we need to show that $(\mathbf{U}h \circ f \circ g)^* = h \circ f^* \circ \mathbf{E}g$.

To this end, let $U \in \mathcal{B}A'$ and let I denote the interval $\sum \widetilde{f}\,(\widetilde{g}\,(\langle U \rangle_{\mathcal{A}})) \in \mathbf{U}B$. Noting, by the Morphism Lemma that $I = \sum \widetilde{f}\,(\langle \widetilde{g}\,(U) \rangle_{\mathcal{A}})$, we can write

$$(h \circ f^* \circ \mathbf{E}g)\,(\langle U \rangle_{\mathcal{A}}) = h\,(f^*\,(\mathbf{E}g\,(\langle U \rangle_{\mathcal{A}}))) = h \left(\sum I \right)$$

while

$$(\mathbf{U}h \circ f \circ g)^*\,(\langle U \rangle_{\mathcal{A}}) = \sum \sum \widetilde{\mathbf{U}h} \left(\widetilde{f}\,(\widetilde{g}\,(\langle U \rangle_{\mathcal{A}})) \right) = \sum \mathbf{U}h\,(I) .$$

Since I is an interval in B, then $\sum \mathbf{U}h(I) = h(\sum I)$ follows, which establishes the result.

It is worth pointing out that $\mathbf{EU}B \cong B$. The ideal $\langle U\rangle_{\mathcal{A}} = \langle \sum U\rangle$ is principal, noting that $\sum U \in B$ is defined for all $U \in \mathcal{B}B$, since B is a $\mathcal{B}$-dioid. The map $g_{\mathcal{A}}$ applied to this ideal results in

$$g_{\mathcal{A}}(\langle U\rangle_{\mathcal{A}}) = \langle \widetilde{g}(\langle U\rangle_{\mathcal{A}})\rangle_{\mathcal{A}} = \langle \widetilde{g}(U)\rangle_{\mathcal{A}} = \left\langle \sum \widetilde{g}(U)\right\rangle = \left\langle g\left(\sum U\right)\right\rangle$$

for a $\mathcal{B}$-morphism $g : B \to B'$. Therefore, the composition $\mathbf{E} \circ \mathbf{U}$ is just the identity functor on $\mathbf{D}\mathcal{B}$.

Corollary 6. *Let $\mathcal{A}, \mathcal{B}$ be monadic operators with $\mathcal{A} \leq \mathcal{B}$. Then $\mathbf{Q}_{\mathcal{A}}^{\mathcal{B}} \circ \mathbf{Q}_{\mathcal{B}}^{\mathcal{A}}$ is the identity functor on $\mathbf{D}\mathcal{B}$.*

In addition, we may show that the adjunctions behave consistently under compositions.

Corollary 7. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be monadic operators with $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$. Then $\mathbf{Q}_{\mathcal{V}}^{\mathcal{U}} \circ \mathbf{Q}_{\mathcal{W}}^{\mathcal{V}} = \mathbf{Q}_{\mathcal{W}}^{\mathcal{U}}$, for any permutation $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of $\mathcal{A}, \mathcal{B}, \mathcal{C}$.*

Proof. It is actually only necessary to take $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ or $(\mathcal{C}, \mathcal{B}, \mathcal{A})$ as the permitations of $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ since the other cases can be derived by composition using corollary 6.

These two cases result from showing that adjunctions are closed under composition which is a general category-theoretic result. The adjunctions here involve left-adjoints of forgetful functors. However, since the forgetful functors close under composition, and the composition of adjunctions is also an adjunction, then the result follows directly from the uniqueness of left adjoints [10] (Corollary 1, p. 83).

Theorem 16. *The functor $\mathcal{A} : \mathbf{Monoid} \to \mathbf{D}\mathcal{A}$ and the forgetful functor $\hat{\mathcal{A}} : \mathbf{D}\mathcal{A} \to \mathbf{Monoid}$ form an adjunction pair.*

Proof. This is the essence of the properties $\mathbf{A}_1$-$\mathbf{A}_4$. Here, the unit $\eta_M : M \to \mathcal{A}M$ is the inclusion $\eta_M(m) = \{m\}$. The extension of the monoid homomorphism $f : M \to \hat{\mathcal{A}}A$ to an $\mathcal{A}$-morphism $f^* : \mathcal{A}M \to A$ is related to the least upper bound operator by $f^*(U) = \sum \widetilde{f}(U)$, for $U \in \mathcal{A}M$. The naturalness of this correspondence is, in fact, the essential point of Theorem 1. In fact, the construction of $\mathcal{A}$-dioids is a special case of a general construction, through adjunctions, of what are known in category theory as T-algebras [10]. To complete the proof will actually require establishing properties

$\mathbf{D}_4$ $\sum\{m\} = m$ for $m \in D$,
$\mathbf{D}_5$ $\sum_{U \in \mathsf{Y}} \sum U = \sum(\bigcup \mathsf{Y})$, for $\mathsf{Y} \in \mathcal{A}\mathcal{A}D$,
$\mathbf{D}_6$ $f(A) = \sum_{a \in A} f(\{a\})$, for $A \in \mathcal{A}M$, where $f : \mathcal{A}M \to D$ is an $\mathcal{A}$-morphism

which are all elementary consequences for partially ordered sets.

It follows, also, from these considerations that $\mathbf{Q}_{\mathcal{A}}^{\mathcal{B}} \circ \mathcal{A} = \mathcal{B}$ for $\mathcal{A} \leq \mathcal{B}$ and that, under the same condition, $\hat{\mathcal{B}} \circ \mathbf{Q}_{\mathcal{A}}^{\mathcal{B}} = \hat{\mathcal{A}}$.

4 Further Developments

What we have done is construct a hierarchy of monads. For each operator $\mathcal{A}$ there is an adjunction pair $\left(\mathcal{A}, \hat{\mathcal{A}}\right)$ that extends the category of monoids to the category of $\mathcal{A}$-dioids. The unit of the adjunction is the polymorphic function (i.e., natural transformation) $\eta : I_{\mathbf{Monoid}} \to \hat{\mathcal{A}} \circ \mathcal{A}$, given by $\eta_M : M \to \mathcal{A}M$, where $\eta_M(m) = \{m\}$. The monad product $\Sigma : \mathcal{A} \circ \hat{\mathcal{A}} \to I_{\mathbf{D}\mathcal{A}}$ is given by $\Sigma_D : \mathcal{A}D \to D$, where $\Sigma_D(U) = \sum U$.

The incorporation of the idempotency property, $A = A + A$, is the critical feature behind the occurrence of the partially ordered monoid structure. In contrast, in the formal power series approach [6,7,11], addition no longer need be idempotent. Therefore, a natural route of generalization is of the monad hierarchy from dioids to semirings. Unlike the case for dioids, where a Σ operator is already given to us satisfying all of $\mathbf{D}_1, \ldots, \mathbf{D}_6$, for a semiring-based formulation of the foregoing the additional properties $\mathbf{D}_4, \mathbf{D}_5, \mathbf{D}_6$ will also need to be explicitly stipulated.

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