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Mealy Morphisms of Enriched Categories

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Abstract We define and study the properties of a notion of morphism of enriched categories, intermediate between strong functor and profunctor. Suggested by bicategorical considerations, it turns out to be a generalization of Mealy machine, well-known since the 1950's in the theory of computation. When the base category is closed we construct a classifying category for Mealy morphisms, as we call them. This is also seen to give the free tensor completion of an enriched category.

Keywords Enriched category · Strong functor · Profunctor · Mealy machine

Mathematics Subject Classifications (2010) 18D10 · 18D15 · 18D20 · 18D25 · 18B20

1 Introduction

As pointed out in [1], a monoidal category **V** can be considered as a one object bicategory, and a category enriched in **V** as a lax morphism of bicategories from a chaotic bicategory to **V**. Strong functors do not correspond exactly to either lax or oplax transformations but to a restricted class of them which are identities on the objects, recently studied under the name *icons* [3].

Although this is a natural concept in the present context, one might expect the bicategorical perspective to yield interesting concepts. It is this idea that we pursue here.

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When considering oplax (or lax) transformations what emerges is a notion intermediate between strong functor and profunctor, which we call Mealy morphism because of its similarity to Mealy machines from the theory of computation [8].

In Section 1 we give the basic definitions and properties leading to the bicategory $\mathbf{V}$-$\mathcal{M}ealy$. We relate our basic concepts to $\mathbf{V}$-functors and $\mathbf{V}$-natural transformations. In Section 2 we consider examples, in particular Mealy machines and how they relate to the present work. Section 3 studies the relationship with $\mathbf{V}$-profunctors. We obtain a locally full and faithful embedding of bicategories $\mathbf{V}$-$\mathcal{M}ealy \hookrightarrow \mathbf{V}$-$\mathcal{P}rof$, under the usual cocompleteness conditions on $\mathbf{V}$. In Section 4 we assume $\mathbf{V}$ is closed. We get a Mealy morphism classifier much like the partial morphism classifier of topos theory. It plays the same role for Mealy morphisms as the "$\mathbf{V}$-presheaf" category does for profunctors. It also turns out to give the free tensor completion of a $\mathbf{V}$-category.

The notion of Mealy morphism has been in the air for some time. In particular, we had significant discussions with Michel Thiébaud in the 1980's, occasioned by the work of Walters [10, 11], Street [9] and Betti et al. [2] on categories enriched in a bicategory. We believe that this is an interesting concept and take this opportunity to record the definitions, and in the process develop some of its basic properties.

2 Mealy Morphisms

A monoidal category $\mathbf{V}$ may be considered as a one-object bicategory $\mathcal{V}$. For notation purposes we note that composition is given by $\otimes$ thus:

$$\begin{tikzcd} & * \arrow[dr, "V'"] & \\ * \arrow[ur, "V"] \arrow[rr, "V'\otimes V"'] & & * \end{tikzcd}$$

If $\mathbf{A}$ is a $\mathbf{V}$-category we can construct a bicategory $\mathcal{O}b(\mathbf{A})$ whose objects are those of $\mathbf{A}$ but with a single morphism $(A, A') : A \to A'$ between any two objects and only identity 2-cells. We can also construct a lax morphism

$$\mathbf{A} : \mathcal{O}b(\mathbf{A}) \to \mathcal{V}$$

$$A \mapsto \star$$

$$(A, A') \mapsto \mathbf{A}(A, A') : \star \to \star$$

$$\begin{tikzcd} & A' \arrow[dr, "(A',A'')"] & \\ A \arrow[ur, "(A,A')"] \arrow[rr, "(A,A'')"'] & & A'' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & * \arrow[dr, "\mathbf{A}(A',A'')"] \arrow[d, phantom, "\Downarrow\circ"] & \\ * \arrow[ur, "\mathbf{A}(A,A')"] \arrow[rr, "\mathbf{A}(A,A'')"'] & {} & * \end{tikzcd}$$

$$
A \xrightarrow{(A,A)} A \quad \mapsto \quad
\begin{tikzcd}
* \arrow[r, bend left, "I"{name=U}] \arrow[r, bend right, "\mathbf{A}(A,A)"'{name=D}] & * \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{id}"]
\end{tikzcd}
$$

In fact it is clear that the data and conditions for a $\mathbf{V}$-category are exactly the same as for a lax morphism. This was already in [1].

In this context, a $\mathbf{V}$-functor $F : \mathbf{A} \to \mathbf{B}$ can be viewed as a diagram

$$
\begin{tikzcd}
\mathcal{O}b(\mathbf{A}) \arrow[rr, "F"] \arrow[dr, "\mathbf{A}"'] & \arrow[d, phantom, "\overset{\phi}{\Longleftarrow}"] & \mathcal{O}b(\mathbf{B}) \arrow[dl, "\mathbf{B}"] \\
& \mathbf{V} &
\end{tikzcd}
$$

where F is nothing more than a function on the objects and ϕ is a lax transformation $\mathbf{B} \circ F \to \mathbf{A}$ which is the identity on the objects. Recall that a lax transformation assigns to each object A of $\mathcal{O}b(\mathbf{A})$ a morphism $\phi(A) : (\mathbf{B} \circ F)(A) \to \mathbf{A}(A)$, i.e. $\phi(A) : \star \to \star$, or an object of $\mathbf{V}$, and to each morphism $(A, A') : A \to A'$ in $\mathcal{O}b(\mathbf{A})$ a cell

$$
\begin{tikzcd}
\mathbf{B} \circ F(A) \arrow[r, "\phi A"] \arrow[d, "\mathbf{B}\circ F(A,A')"'] & \mathbf{A}(A) \arrow[d, "\mathbf{A}(A,A')"] \arrow[dl, phantom, "\overset{\phi(A,A')}{\Longleftarrow}"] \\
\mathbf{B} \circ F(A') \arrow[r, "\phi A'"'] & \mathbf{A}(A')
\end{tikzcd}
\quad \text{i.e.} \quad
\begin{tikzcd}
* \arrow[r, "\phi A"] \arrow[d, "\mathbf{B}(FA,FA')"'] & * \arrow[d, "\mathbf{A}(A,A')"] \arrow[dl, phantom, "\overset{\phi(A,A')}{\Longleftarrow}"] \\
* \arrow[r, "\phi A'"'] & *
\end{tikzcd}
$$

or a morphism $\phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{B}(FA, FA')$, the $\phi(A, A')$ respecting composition and identities in the obvious sense (spelled out below). The condition that ϕ is the identity on objects is just that $\phi A = I$ for all A. Then $\phi(A, A')$ can be replaced by a morphism $\mathbf{A}(A, A') \to \mathbf{B}(FA, FA')$ which is the strength of F.

We could equally well have viewed a $\mathbf{V}$-functor as a diagram

$$
\begin{tikzcd}
\mathcal{O}b(\mathbf{A}) \arrow[rr, "F"] \arrow[dr, "\mathbf{A}"'] & \arrow[d, phantom, "\overset{\psi}{\Longrightarrow}"] & \mathcal{O}b(\mathbf{B}) \arrow[dl, "\mathbf{B}"] \\
& \mathbf{V} &
\end{tikzcd}
$$

with ψ an oplax transformation which is the identity on the objects. When the ϕA are not identities, the two definitions are not equivalent but dual (reverse $\otimes$). We choose the first version so as to make our embedding of Mealy morphisms into profunctors covariant.

A Mealy morphism is a pair (F, ϕ) as above without the "identity on objects" restriction. We spell it out in detail.

Definition 1 Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathbf{V}$-categories. A *Mealy morphism* $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$ consists of

(MM1) An *object function* $F : \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{B}$,
(MM2) A *state function* $\phi : \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{V}$,
(MM3) For each pair A, A' in $\mathbf{A}$, a *transition function*

$$\phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{B}(FA, FA'),$$

satisfying

(MM4)

$$\begin{tikzcd} I \otimes \phi A \arrow[r, "id \otimes \phi A"] \arrow[d, "\cong"'] & \mathbf{A}(A, A) \otimes \phi A \arrow[d, "\phi(A,A)"] \\ \phi A \otimes I \arrow[r, "\phi A \otimes id"'] & \phi A \otimes \mathbf{B}(FA, FA) \end{tikzcd}$$

(MM5)

$$\begin{tikzcd} \mathbf{A}(A', A'') \otimes \mathbf{A}(A, A') \otimes \phi A \arrow[rr, "\circ \otimes \phi A"] \arrow[d, "\mathbf{A}(A',A'') \otimes \phi(A,A')"'] & & \mathbf{A}(A, A'') \otimes \phi A \arrow[dd, "\phi(A,A'')"] \\ \mathbf{A}(A', A'') \otimes \phi A' \otimes \mathbf{B}(FA, FA') \arrow[d, "\phi(A',A'') \otimes \mathbf{B}(FA,FA')"'] & & \\ \phi A'' \otimes \mathbf{B}(FA', FA'') \otimes \mathbf{B}(FA, FA') \arrow[rr, "\phi A'' \otimes \circ"'] & & \phi A'' \otimes \mathbf{B}(FA, FA'') \end{tikzcd}$$

Mealy morphisms are easily composed as they are lax transformations. Given $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$ and $(G, \gamma) : \mathbf{B} \rightharpoonup \mathbf{C}$ the composite is $(GF, \phi \otimes \gamma F)$ where $(\phi \otimes \gamma F)A = \phi A \otimes \gamma FA$ and

$$\begin{tikzcd} \mathbf{A}(A, A') \otimes \phi A \otimes \gamma FA \arrow[rr, "(\phi \otimes \gamma F)(A,A')"] \arrow[dr, "\phi(A,A') \otimes \gamma FA"'] & & \phi A' \otimes \gamma FA' \otimes \mathbf{C}(GFA, GFA') \\ & \phi A' \otimes \mathbf{B}(FA, FA') \otimes \gamma FA \arrow[ur, "\phi A' \otimes \gamma(FA,FA')"'] & \end{tikzcd}$$

Of course, as composition is defined in terms of $\otimes$, we should not expect it to be strictly associative. However Mealy morphisms will be the arrows of a bicategory. The definition of a 2-cell is not quite straightforward though. Modifications don't work because the lax transformations in question have different codomains. The definition we take is a generalization of **V**-natural transformation which takes the state functions into account in a way that makes Theorems 2 and 4 valid.

Definition 2 Let $(F, \phi), (K, \kappa) : \mathbf{A} \rightharpoonup \mathbf{B}$ be Mealy morphisms. A *Mealy cell* $t : (F, \phi) \to (K, \kappa)$ is given by:

(MC1) morphisms $tA : \phi A \to \kappa A \otimes \mathbf{B}(FA, KA)$ satisfying
(MC2)

$$\begin{tikzcd}
\mathbf{A}(A, A') \otimes \phi A \arrow[r, "\phi(A,A')"] \arrow[d, "\mathbf{A}(A,A')\otimes tA"'] & \phi A' \otimes \mathbf{B}(FA, FA') \arrow[d, "tA'\otimes\mathbf{B}(FA,FA')"] \\
\mathbf{A}(A, A') \otimes \kappa A \otimes \mathbf{B}(FA, KA) \arrow[d, "\kappa(A,A')\otimes\mathbf{B}(FA,KA)"'] & \kappa A' \otimes \mathbf{B}(FA', KA') \otimes \mathbf{B}(FA, FA') \arrow[d, "\kappa A'\otimes\circ"] \\
\kappa A' \otimes \mathbf{B}(KA, KA') \otimes \mathbf{B}(FA, KA) \arrow[r, "\kappa A'\otimes\circ"'] & \kappa A' \otimes \mathbf{B}(FA, KA')
\end{tikzcd}$$

If a Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$ has its state function constantly I, i.e. $\phi A = I$ for every A, then in (MM3) we can drop the $\phi A, \phi A'$ (by pre and post composing with canonical isomorphisms) to get morphisms $\phi_{A,A'} : \mathbf{A}(A, A') \to \mathbf{B}(FA, FA')$, and (MM4) and (MM5) are the conditions for this to be the strength of a **V**-functor $F : \mathbf{A} \to \mathbf{B}$. So a **V**-functor F gives in this way a Mealy morphism $F_\bullet$. Furthermore, if $t : F_\bullet \to K_\bullet$ is a Mealy cell, dropping the κA in (MC1) gives the data for a **V**-natural transformation, $t_A : I \to \mathbf{B}(FA, KA)$, and (MC2) reduces to **V**-naturality. If $t : F \to K$ is a **V**-natural transformation, we denote the corresponding Mealy cell $F_\bullet \to K_\bullet$ by $t_\bullet$.

Mealy cells can be composed both vertically and horizontally.

If $(F, \phi), (K, \kappa), (M, \mu) : \mathbf{A} \rightharpoonup \mathbf{B}$ are Mealy morphisms and $t : (F, \phi) \to (K, \kappa)$ and $s : (K, \kappa) \to (M, \mu)$ are Mealy cells, the vertical composite $s \cdot t$ is defined by

$$\begin{tikzcd}
\phi A \arrow[r, "(s\cdot t)A"] \arrow[d, "tA"'] & \mu A \otimes \mathbf{B}(FA, MA) \\
\kappa A \otimes \mathbf{B}(FA, KA) \arrow[r, "sA\otimes\mathbf{B}(FA,KA)"'] & \mu A \otimes \mathbf{B}(KA, MA) \otimes \mathbf{B}(FA, KA) \arrow[u, "\mu A\otimes\circ"']
\end{tikzcd}$$

Horizontal composition is best defined in terms of whiskering. If t is as above and $(G, \gamma) : \mathbf{B} \rightharpoonup \mathbf{C}$, then $((G, \gamma)t)(A)$ is the composite

$$\phi A \otimes \gamma FA \xrightarrow{tA\otimes\gamma FA} \kappa A \otimes \mathbf{B}(FA, KA) \otimes \gamma FA$$
$$\xrightarrow{KA\otimes\gamma(FA,KA)} \kappa A \otimes \gamma KA \otimes \mathbf{C}(GFA, GKA).$$

If $(H, \eta) : \mathbf{D} \to \mathbf{A}$, then

$$(t(H, \eta))(D) = \eta D \otimes tHD : \eta D \otimes \phi HD \to \eta D \otimes \kappa HD \otimes \mathbf{B}(FHD, KHD).$$

There is an equivalent definition of Mealy cell better suited to horizontal composition. The above definition generalizes the usual notion of $\mathbf{V}$-natural transformation, when the $\phi A = \kappa A = I$, which is convenient for vertical composition and whiskering. $\mathbf{V}$-natural transformations can also have an "arrows only" definition in terms of morphisms

$$\mathbf{A}(A, A') \to \mathbf{B}(FA, GA').$$

which works well for horizontal composition. The situation is similar for Mealy cells.

Consider the following data:

(MC′1) morphisms $\tau(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \kappa A' \otimes \mathbf{B}(FA, KA')$ satisfying
(MC′2)

$$\begin{tikzcd}
\mathbf{A}(A', A'') \otimes \mathbf{A}(A, A') \otimes \phi A \arrow[rr, "\mathbf{A}(A', A'') \otimes \tau(A, A')"] \arrow[d, "\mathbf{A}(A', A'') \otimes \phi(A, A')"'] \arrow[dr, "\circ \otimes \phi A"] & & \mathbf{A}(A', A'') \otimes \kappa A' \otimes \mathbf{B}(FA, KA') \arrow[d, "\kappa(A', A'') \otimes \mathbf{B}(FA, KA')"] \\
\mathbf{A}(A', A'') \otimes \phi A' \otimes \mathbf{B}(FA, FA') \arrow[d, "\tau(A', A'') \otimes \mathbf{B}(FA, FA')"'] & \mathbf{A}(A, A'') \otimes \phi A \arrow[dr, "\tau(A, A'')"] & \kappa A'' \otimes \mathbf{B}(KA', KA'') \otimes \mathbf{B}(FA, KA') \arrow[d, "\kappa A'' \otimes \circ"] \\
\kappa A'' \otimes \mathbf{B}(FA', KA'') \otimes \mathbf{B}(FA, FA') \arrow[rr, "\kappa A'' \otimes \circ"'] & & \kappa A'' \otimes \mathbf{B}(FA, KA'')
\end{tikzcd}$$

Theorem 1 *There is bijection between Mealy cells* $(F, \phi) \to (K, \kappa)$*, i.e. data* (MC1) *satisfying conditions* (MC2)*, and the data* (MC′1) *satisfying* (MC′2).

Proof Given a Mealy cell t, define $\tau(A, A')$ to be the common value of the commutative square (MC2). Conversely given τ satisfying (MC′2), define tA to be the composite

$$\phi A \xrightarrow{\cong} I \otimes \phi A \xrightarrow{id \otimes \phi A} \mathbf{A}(A, A) \otimes \phi A \xrightarrow{\tau(A, A)} \kappa A \otimes \mathbf{B}(FA, KA).$$

The details of the bijection are straightforward and omitted.

Now given Mealy cells $\tau : (F, \phi) \to (K, \kappa) : \mathbf{A} \rightharpoonup \mathbf{B}$ and $\upsilon : (G, \gamma) \to (L, \lambda) : \mathbf{B} \to \mathbf{C}$, their horizontal composition $\upsilon\tau : (G, \gamma)(F, \phi) \to (L, \lambda)(K, \kappa)$ is specified by

$$\begin{tikzcd}
\mathbf{A}(A, A') \otimes \phi A \otimes \gamma FA \arrow[rr, "\upsilon\tau(A, A')"] \arrow[dr, "\tau(A, A') \otimes \gamma FA"'] & & \kappa A' \otimes \lambda KA' \otimes \mathbf{C}(GFA, LKA') \\
& \kappa A' \otimes \mathbf{B}(FA, \kappa A') \otimes \gamma FA \arrow[ur, "\kappa A' \otimes \upsilon(FA, KA')"'] &
\end{tikzcd}$$

The identity $\iota : (F, \phi) \to (F, \phi)$ is just

$$\iota(A, A') = \phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{B}(FA, FA').$$

□

The properties of these compositions are summarized in the following statements.

Theorem 2

(1) $\mathbf{V}$-*categories, Mealy morphisms and Mealy cells form a bicategory* $\mathbf{V}$-$\mathcal{M}ealy$.
(2) $(\ )_\bullet : \mathbf{V}\text{-}\mathcal{C}at \to \mathbf{V}\text{-}\mathcal{M}ealy$ *is a locally full and faithful strong morphism of bicategories.*

Definition 3 A $\mathbf{V}$-category $\mathbf{B}$ is *tensored* if for every V in $\mathbf{V}$ and B in $\mathbf{B}$ there is an object $V \otimes B$ and a "coevaluation" morphism $\lambda : V \to \mathbf{B}(B, V \otimes B)$ such that for every X in $\mathbf{V}$, B' in $\mathbf{B}$, and morphism $x : X \otimes V \to \mathbf{B}(B, B')$ there is a unique morphism $\widehat{x} : X \to \mathbf{B}(V \otimes B, B')$ such that

$$\begin{tikzcd} X \otimes V \arrow[rr, "\widehat{x}\otimes\lambda"] \arrow[dr, "x"'] & & \mathbf{B}(V \otimes B, B') \otimes \mathbf{B}(B, V \otimes B) \arrow[dl, "\circ"] \\ & \mathbf{B}(B, B') & \end{tikzcd}$$

Less formally, we have bijections

$$\frac{x : X \to \mathbf{B}(V \otimes B, B')}{x^\vee : X \otimes V \to \mathbf{B}(B, B')}$$

natural in X and $\mathbf{V}$-natural in B'. This last condition needs some explanation, as it doesn't make sense as it stands. By $\mathbf{V}$-natural in B' we mean that if x corresponds to $x^\vee$ and $y : Y \to \mathbf{B}(B', B'')$ then

$$Y \otimes X \xrightarrow{y\otimes x} \mathbf{B}(B', B'') \otimes \mathbf{B}(V \otimes B, B') \xrightarrow{\circ} \mathbf{B}(V \otimes B, B'')$$

corresponds to

$$Y \otimes V \otimes X \xrightarrow{y\otimes x^\vee} \mathbf{B}(B', B'') \otimes \mathbf{B}(B, B') \xrightarrow{\circ} \mathbf{B}(B, B'').$$

If we introduce the notation $y \circ x$ for the first composite and a similar one for the second. then the condition is $(y \circ x)^\vee = y \circ x^\vee$. For $Y = I$, this condition reduces to naturality in B', so we can think of it as naturality with parameters, hence $\mathbf{V}$-naturality.

One can think of $V \otimes B$ as the coproduct of V copies of B. Then $\lambda : V \to \mathbf{B}(B, V \otimes B)$ is the V-family of coproduct injections $B \to V \otimes B$. If we let $X = I$, the universal property says that for any V-family x of morphisms $B \to B'$ there exists a unique morphism $\widehat{x} : V \otimes B \to B'$ which when composed with the family of injections gives the family x.

Theorem 3 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathbf{V}$*-categories with* $\mathbf{B}$ *tensored. Then the embedding*

$$\mathbf{V}\text{-}\mathcal{C}at(\mathbf{A},\mathbf{B}) \hookrightarrow \mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A},\mathbf{B})$$

has a left adjoint.

Proof Let $(F,\phi) : \mathbf{A} \rightharpoonup \mathbf{B}$ be a Mealy morphism. Define $\widehat{F} : \mathbf{A} \to \mathbf{B}$ by $\widehat{F}(A) = \phi A \otimes FA$ with strength

$$st_{\widehat{F}} : \mathbf{A}(A,A') \to \mathbf{B}(\phi A \otimes FA, \phi A' \otimes FA')$$

the unique morphism such that

$$\begin{tikzcd}
\mathbf{A}(A,A') \otimes \phi A \arrow[rr, "st_{\widehat{F}} \otimes \lambda"] \arrow[d, "\phi(A,A')"'] & & \mathbf{B}(\phi A \otimes FA, \phi A' \otimes FA') \otimes \mathbf{B}(FA, \phi A \otimes FA) \arrow[d, "\circ"] \\
\phi A' \otimes \mathbf{B}(FA, FA') \arrow[dr, "\lambda \otimes \mathbf{B}(FA,FA')"'] & & \mathbf{B}(FA, \phi A' \otimes FA') \\
& \mathbf{B}(FA', \phi A' \otimes FA') \otimes \mathbf{B}(FA, FA') \arrow[ur, "\circ"'] &
\end{tikzcd}$$

Let $G : \mathbf{A} \to \mathbf{B}$ be a $\mathbf{V}$-functor and $t : F \to G_\bullet$ a Mealy cell whose components are

$$tA : \phi A \to I \otimes \mathbf{B}(FA, GA).$$

Then we get a $\mathbf{V}$-natural transformation, $\widetilde{t}$, whose components $\widetilde{t}A : I \to \mathbf{B}(\phi A \otimes FA, GA)$ are the unique morphisms such that

$$\begin{tikzcd}
I \otimes \phi A \arrow[r, "\widetilde{t}A \otimes \lambda"] \arrow[d, "I \otimes tA"'] & \mathbf{B}(\phi A \otimes FA, GA) \otimes \mathbf{B}(FA, \phi A \otimes FA) \arrow[d, "\circ"] \\
I \otimes I \otimes \mathbf{B}(FA, GA) \arrow[r, "\cong"'] & \mathbf{B}(FA, GA)
\end{tikzcd}$$

That $\widehat{F}$ is a $\mathbf{V}$-functor, $\widetilde{t}$ a $\mathbf{V}$-natural transformation and that $t \mapsto \widetilde{t}$ is a bijection are tedious calculations which we omit. □

Of course the universal property of $(\widehat{\ \ })$ makes it into a functor but we give its value on Mealy cells for future reference. For a Mealy cell $t : (F,\phi) \to (G,\gamma)$ with components $tA : \phi A \to \gamma A \otimes \mathbf{B}(FA, GA)$, $\widehat{t}$ is the $\mathbf{V}$-natural transformation whose component at A is the unique morphism

$$\widehat{t}A : I \to \mathbf{B}(\phi A \otimes FA, \gamma A \otimes GA)$$

such that

$$\begin{tikzcd}
I \otimes \phi A \arrow[r, "\widehat{t}A\otimes\lambda"] \arrow[d, "\cong"'] & \mathbf{B}(\phi A \otimes FA, \gamma A \otimes GA) \otimes \mathbf{B}(FA, \phi A \otimes FA) \arrow[d, "\circ"] \\
\phi A \arrow[d, "tA"'] & \mathbf{B}(FA, \gamma A \otimes GA) \\
\gamma A \otimes \mathbf{B}(FA, GA) \arrow[r, "\lambda\otimes\mathbf{B}(FA,GA)"'] & \mathbf{B}(GA, \gamma A \otimes GA) \otimes \mathbf{B}(FA, GA) \arrow[u, "\circ"']
\end{tikzcd}$$

3 Examples

Example 1 Let $\mathbf{I}$ be the $\mathbf{V}$-category with one object $*$ and $\mathbf{I}(*, *) = I$. A Mealy morphism $(F, \phi) : \mathbf{I} \rightharpoonup \mathbf{B}$ consists of:

(1) an object $B = F*$ of $\mathbf{B}$;
(2) an object $V = \phi*$ of $\mathbf{V}$;
(3) a morphism $\phi(*, *) : \mathbf{I}(*, *) \otimes \phi* \to \phi * \otimes \mathbf{B}(F*, F*)$, i.e. a morphism $I \otimes V \to V \otimes \mathbf{B}(B, B)$.

(MM4) alone determines $\phi(*, *)$ to be

$$I \otimes V \xrightarrow{\cong} V \otimes I \xrightarrow{V\otimes\, id} V \otimes \mathbf{B}(B, B).$$

So we can identify a Mealy morphism $\mathbf{I} \to \mathbf{B}$ with a pair (V, B).

Given another pair (V', B'), a Mealy cell between the corresponding Mealy morphisms $\mathbf{I} \to \mathbf{B}$ is given by a morphism

$$t(*) : \phi* \to \phi' * \otimes \mathbf{B}(F*, F'*)$$

i.e. a morphism of $\mathbf{V}$

$$b : V \to V' \otimes \mathbf{B}(B, B')$$

satisfying (MC2), which is automatic in this case. We call it b because we are thinking of it as a kind of morphism in $\mathbf{B}$.

This describes the category $\widetilde{\mathbf{B}}_0 = \mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{I}, \mathbf{B})$. The category $\mathbf{V}\text{-}\mathcal{C}at(\mathbf{I}, \mathbf{V})$ is equivalent to $\mathbf{B}_0$ the underlying category of $\mathbf{B}$. The full inclusion $H_0 : \mathbf{B}_0 \to \widetilde{\mathbf{B}}_0$ expressing part (2) of Theorem 2 is given by

$$H_0(B) = (I, B),$$

$$H_0(b) = I \xrightarrow{b} \mathbf{B}(B, B') \xrightarrow{\cong} I \otimes \mathbf{B}(B, B').$$

When $\mathbf{B}$ is tensored, the reflection of Theorem 3 gives $L : \widetilde{\mathbf{B}}_0 \to \mathbf{B}_0$, $L(V, B) = V \otimes B$. The adjointness bijection

$$\frac{L(V, B) \to B'}{(V, B) \to (I, B')}$$

reduces to

$$\frac{I \to \mathbf{B}(V \otimes B, B')}{V \to \mathbf{B}(B, B')}$$

which is the tensor bijection of Definition 3 with $X = I$. It will turn out that, given some closedness conditions on $\mathbf{V}$, $\widetilde{\mathbf{B}}_0$ will be the underlying category of a $\mathbf{V}$-category, H_0 the underlying functor of a $\mathbf{V}$-functor H, and $\mathbf{B}$ being tensored will be equivalent to H having a strong left adjoint.

Example 2 A Mealy morphism $(F, \phi) : \mathbf{A} \to \mathbf{I}$ consists of a function $F : \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{I} \cong 1$, of which there is exactly one, and a function $\phi : \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{V}$ together with morphisms

$$\phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{I}(FA, FA') \cong \phi A'$$

satisfying (MM4) and (MM5). This is exactly a $\mathbf{V}$-functor $\phi : \mathbf{A} \to \mathbf{V}$ formulated in a way that does not use closedness of $\mathbf{V}$ (see [13]).

A Mealy cell $t : (F, \phi) \to (K, \kappa)$ is a family of morphisms

$$tA : \phi A \to \kappa A \otimes \mathbf{I}(FA, KA) \cong \kappa A$$

satisfying (MC2), which is exactly a $\mathbf{V}$-natural transformation, again formulated so as not to use closedness of $\mathbf{V}$ [13]. Thus

$$\mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A}, \mathbf{I}) \simeq \mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \mathbf{V}).$$

Example 3 For a general Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$, $\phi : \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{V}$ is not the object part of a functor $\mathbf{A} \to \mathbf{V}$ as might be supposed, nor for that matter is F the object part of a functor $\mathbf{A} \to \mathbf{B}$, although one wouldn't expect that.

When $\mathbf{V}$ is the category of sets, $\mathbf{Set}$, with cartesian product, the situation is nicer. The morphism

$$\phi(A, A') : \mathbf{A}(A, A') \times \phi A \to \phi A' \times \mathbf{B}(FA, FA')$$

has two components:

$$\phi_1(A, A') : \mathbf{A}(A, A') \times \phi A \to \phi A'$$

which makes ϕ into a functor $\mathbf{A} \to \mathbf{Set}$, and

$$\phi_2(A, A') : \mathbf{A}(A, A') \times \phi A \to \mathbf{B}(FA, FA')$$

which makes F into a functor $\bar{F} : \mathbf{El}(\phi) \to \mathbf{B}$. Here $\mathbf{El}(\phi)$ is the category of elements of ϕ whose objects are pairs (A, x), $x \in \phi A$, and morphisms $(A, x) \to (A', x')$ given by morphisms of $\mathbf{A}$, $a : A \to A'$, such that $\phi(a)(x) = x'$. Then $\bar{F}(A, x) = FA$ and $\bar{F}(a) = \phi_2(a, x)$.

In this way, we may think of a Mealy morphism $\mathbf{A} \rightharpoonup \mathbf{B}$ as a span

$$\begin{tikzcd} & \mathbf{E} \arrow[dl, "P"'] \arrow[dr, "\bar{F}"] & \\ \mathbf{A} & & \mathbf{B} \end{tikzcd}$$

where P is a discrete opfibration and $\bar{F}$ is a functor which is *objectwise constant on the fibres*.

We don't completely understand this last condition but it could hardly be otherwise for a general $\mathbf{V}$. This description of Mealy morphisms suggest that they might be thought of as partial functors in the same way as profunctors are thought of as categorical relations.

A Mealy cell $t : (F, \phi) \to (K, \kappa)$ is given by morphisms $tA : \phi A \to \kappa A \times \mathbf{B}(FA, KA)$ which also have two components:

$$t_1 A : \phi A \to \kappa A$$

which defines a natural transformation $t_1 : \phi \to \kappa$, or equivalently a functor T over $\mathbf{A}$

$$\begin{tikzcd} \mathbf{El}(\phi) \arrow[rr, "T"] \arrow[dr] & & \mathbf{El}(\kappa) \arrow[dl] \\ & \mathbf{A} & \end{tikzcd}$$

and

$$t_2 A : \phi A \to \mathbf{B}(FA, KA)$$

which is equivalent to a natural transformation $\bar{t}$

$$\begin{tikzcd} \mathbf{El}(\phi) \arrow[rr, "T"] \arrow[dr, "\bar{F}"'] & \arrow[d, phantom, "\overset{\bar{t}}{\Rightarrow}"] & \mathbf{El}(\kappa) \arrow[dl, "\bar{G}"] \\ & \mathbf{B} & \end{tikzcd}$$

Example 4 The name "Mealy morphism" was chosen because the concept is a natural generalization of Mealy machine introduced by George Mealy in 1955 [8]. According to Wikipedia [12], a Mealy machine consists of:

(1) A (finite) set S of *states*;
(2) A (finite) set Σ, the *input alphabet*;

(3) A (finite) set Λ, the *output alphabet*;
(4) An *output function* $g : \Sigma \times S \to \Lambda$;
(5) A *transition function* $t : \Sigma \times S \to S$;
(6) A *start state* $s_0 \in S$.

The way it is run is as follows. We take a word, $a_n a_{n-1} \ldots a_2 a_1$, in the input alphabet and start in state s_0. We get an output $b_1 = g(a_1, s_0)$ and a new state $s_1 = t(a_1, s_0)$, then we get another output $b_2 = g(a_2, s_1)$ and a new state $s_2 = t(a_2, s_1)$, and so on, $b_k = g(a_k, s_{k-1})$ and $s_k = t(a_k, s_{k-1})$. Thus we input a word $a_n \ldots a_2 a_1$ and get an output word $b_n \ldots b_2 b_1$ and the machine is left in state s_n, ready for another input. It is clear that we are acting on words in Σ and Λ, so if Σ^* and Λ^* are the free monoids on Σ and Λ respectively, the g and t extend to functions $\bar{g} : \Sigma^* \times S \to \Lambda^*$ and $\bar{t} : \Sigma^* \times S \to S$ which can be combined into one

$$\phi : \Sigma^* \times S \to S \times \Lambda^*.$$

The $\bar{t}$ is supposed to be an action of Σ^* on S and $\bar{g}$ is a kind of "twisted homomorphism", $\bar{g}(e, s) = e$ and $\bar{g}(ww', s) = \bar{g}(w, \bar{t}(w', s))\bar{g}(w', s)$. These conditions are equivalent to ϕ satisfying (MM4) and (MM5).

It is a small step to arbitrary sets of states (not just finite) and arbitrary monoids where the elements are equivalence classes of words. Then we can replace the monoids by categories where words can be formed only if they meet certain typing conditions, and then the states should also be typed.

That the formalism also works for categories enriched in an arbitrary monoidal category is significant. It is tempting to speculate about applications to non-deterministic machines or to quantum computing.

It is not clear what Mealy cells correspond to in terms of machines. They go between machines with the same inputs and outputs, but different states and operating instructions of course. A straightforward translation gives the following. A morphism of Mealy machines

$$(S, \Sigma, \Lambda, g, t) \to (S', \Sigma, \Lambda, g', t')$$

(we ignore the start states) is a pair of functions $u : S \to S$ and $v : S \to \Lambda^*$ such that for every $a \in \Sigma$ and $s \in S$, $u(t(a, s)) = t(a, \mu(s))$ (equivariant) and $v(s)g(a, s) = g(a, u(s))v(s)$ (intertwines).

4 Profunctors

Mealy morphisms can be viewed as certain profunctors. Recall (see [4, 5]) that a profunctor $P : \mathbf{A} \to\!\!\bullet\!\!> \mathbf{B}$ is given by an object function $P : \mathrm{Ob}\mathbf{B} \times \mathrm{Ob}\mathbf{A} \to \mathrm{Ob}\mathbf{V}$ and left and right actions

$$\lambda : \mathbf{A}(A, A') \otimes P(B, A) \to P(B, A')$$

$$\rho : P(B, A) \otimes \mathbf{B}(B', B) \to P(B', A)$$

satisfying unit laws (two of them) and associativity laws (three of them). One may think of a profunctor as a $\mathbf{V}$-functor $P : \mathbf{B}^{op} \otimes \mathbf{A} \to \mathbf{V}$ although unless $\mathbf{V}$ is symmetric (or at least braided), $\mathbf{B}^{op}$ does not make sense nor does the $\otimes$ of $\mathbf{V}$-categories.

In any case, this suggests that we might equivalently define a profunctor as having a "biaction"

$$\beta : \mathbf{A}(A, A') \otimes P(B, A) \otimes \mathbf{B}(B', B) \to P(A', B')$$

satisfying a single unit law and a single associativity law, although there seems to be no advantage to doing so.

A morphism of profunctors $t : P \to Q$ is an equivariant family of morphims

$$t(B, A) : P(B, A) \to Q(B, A).$$

Composition of profunctors is expressed in terms of colimits, represented schematically by

$$\begin{tikzcd}
 & P(B, A) \otimes Q(C, B) \arrow[dr, "\gamma_B"] & \\
P(B, A) \otimes \mathbf{B}(B', B) \otimes Q(C, B') \arrow[ur, "P(B,A)\otimes\rho_Q"] \arrow[dr, "\lambda_P\otimes Q(C,B')"'] & & Q \otimes P(C, A) \\
 & P(B', A) \otimes Q(C, B') \arrow[ur, "\gamma_{B'}"'] &
\end{tikzcd}$$

where there is one node of the form $P(B, A) \otimes Q(C, B)$ for each B (two illustrated) and one of the form $P(B, A) \otimes \mathbf{B}(B', B') \otimes Q(C, B')$ for each pair B, B'. For the composite $Q \otimes P$ all these colimits must exist in $\mathbf{V}$ and be preserved by the functors $V \otimes -$ and $- \otimes V$ for each V.

Given a Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$, let $(F, \phi)_*(B, A) = \phi A \otimes \mathbf{B}(B, FA)$, $\lambda : \mathbf{A}(A, A') \otimes (F, \phi)_*(B, A) \to (F, \phi)_*(B, A')$ be as in

$$\begin{tikzcd}
\mathbf{A}(A, A') \otimes \phi A \otimes \mathbf{B}(B, FA) \arrow[rr, "\lambda"] \arrow[dr, "\phi(A,A')\otimes\mathbf{B}(B,FA)"'] & & \phi A' \otimes \mathbf{B}(B, FA') \\
 & \phi A' \otimes \mathbf{B}(FA, FA') \otimes \mathbf{B}(B, FA) \arrow[ur, "\phi A'\otimes\circ"'] &
\end{tikzcd}$$

and let

$$\rho : (F, \phi)_*(B, A) \otimes \mathbf{B}(B', B) \to (F, \phi)_*(B', A)$$

be

$$\phi A \otimes \mathbf{B}(B, FA) \otimes \mathbf{B}(B', B) \xrightarrow{\phi A \circ} \phi A \otimes \mathbf{B}(B', FA).$$

Given a Mealy cell $t : (F, \phi) \to (K, \kappa)$ let $t_*(B, A) : (F, \phi)_*(B, A) \to (K, \kappa)_*(B, A)$ be the composite

$$\phi A \otimes \mathbf{B}(B, FA) \xrightarrow{tA \otimes \mathbf{B}(B,FA)} \kappa A \otimes \mathbf{B}(FA, KA) \otimes \mathbf{B}(B, FA)$$

$$\xrightarrow{\kappa A \otimes \circ} \kappa A \otimes \mathbf{B}(B, KA).$$

Theorem 4

(1) $(F, \phi)_*$ *is a profunctor* $\mathbf{A} \longrightarrow\!\!\!\!\!\bullet\,\, \mathbf{B}$.
(2) t_* *is a morphism of profunctors* $(F, \phi)_* \to (K, \kappa)_*$.
(3) $(\ )_* : \mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A}, \mathbf{B}) \to \mathbf{V}\text{-}\mathcal{P}rof(\mathbf{A}, \mathbf{B})$ *is a full and faithful functor.*
(4) *The composite* $Q \otimes (F, \phi)_*$ *exists for all profunctors* $Q : \mathbf{B} \longrightarrow\!\!\!\!\!\bullet\,\, \mathbf{C}$.
(5) $(G, \gamma)_* \otimes (F, \phi)_* \cong ((G, \gamma)(F, \phi))_*$ *canonically.*

Proof (1), (2) and (3) are long but straightforward calculations which we omit.
(4) We have a formula for $Q \otimes (F, \phi)_*$:

$$Q \otimes (F, \phi)_*(C, A) = \phi A \otimes Q(C, FA)$$

with actions

$$\lambda : \mathbf{A}(A, A') \otimes (Q \otimes (F, \phi)_*(C, A)) \to Q \otimes (F, \phi)_*(C, A')$$

given by

$$\begin{tikzcd}
\mathbf{A}(A, A') \otimes \phi A \otimes Q(C, FA) \arrow[rr, "\lambda"] \arrow[dr, "\phi(A,A') \otimes Q(C,FA)"'] & & \phi A' \otimes Q(C, FA') \\
& \phi A' \otimes \mathbf{B}(FA, FA') \otimes Q(C, FA) \arrow[ur, "\phi A' \otimes \lambda_Q"'] &
\end{tikzcd}$$

and

$$\rho : (Q \otimes (F, \phi)_*(C, A)) \otimes \mathbf{C}(C', C) \to Q \otimes (F, \phi)_*(C', A)$$

by

$$\phi A \otimes Q(C, FA) \otimes \mathbf{C}(C', C) \xrightarrow{\phi A \otimes \rho_Q} \phi A \otimes Q(C', FA).$$

That $Q \otimes (F, \phi)_*$ so defined is actually profunctor composition follows from the lemma below. Indeed, the diagram represented schematically by

$$\begin{tikzcd}[column sep=small]
 & \phi A \otimes \mathbf{B}(B, FA) \otimes Q(C, B) \arrow[dr, "\phi A \otimes \circ"] & \\
\phi A \otimes \mathbf{B}(B, FA) \otimes \mathbf{B}(B', B) \otimes Q(C, B') \arrow[ur, "\phi A \otimes \mathbf{B}(B, FA) \otimes \lambda_Q"] \arrow[dr, "\phi A \otimes \circ \otimes Q(C, B')"'] & & \phi A \otimes Q(C, FA) \\
 & \phi A \otimes \mathbf{B}(B', FA) \otimes Q(C, B') \arrow[ur, "\phi A \otimes \circ"'] &
\end{tikzcd}$$

is $\phi A \otimes -$ applied to an instance of the absolute colimit given in the lemma, i.e. with $R = Q(C, -)$ and $B_0 = FA$. Note that A and C are kept constant.

(5) is an easy calculation using the formulas from (4). □

Remark 1 We take statement (3) of the above theorem as justifying our definition of Mealy cell.

Lemma 1 Let $R : \mathbf{B} \to \mathbf{V}$ *be a* $\mathbf{V}$*-functor and* B_0 *an object of* $\mathbf{B}$*, then the diagram represented schematically by*

$$\begin{tikzcd}
 & \mathbf{B}(B, B_0) \otimes RB \arrow[dr, "st_R B"] & \\
\mathbf{B}(B, B_0) \otimes \mathbf{B}(B', B) \otimes RB' \arrow[ur, "\mathbf{B}(B, B_0) \otimes st_R B'"] \arrow[dr, "\circ \otimes RB'"'] & & RB_0 \\
 & \mathbf{B}(B', B_0) \otimes RB' \arrow[ur, "st_R B'"'] &
\end{tikzcd}$$

is an absolute colimit diagram.

Proof See [6], Theorem 1. □

Corollary 1 *If* $\mathbf{V}$ *has small colimits preserved by* $V \otimes -$ *and* $- \otimes V$*, then* $(\ )_*$ *is a locally full and faithful embedding of bicategories*

$$\mathbf{V}\text{-}\mathcal{M}ealy \to \mathbf{V}\text{-}\mathcal{P}rof.$$

Proof Cocompleteness of $\mathbf{V}$ is required to make $\mathbf{V}$-$\mathcal{P}rof$ into a bicategory. Most of the statement is contained in the previous theorem. All that remains is to check that $(\ )_*$ preserves whiskering which is a straightforward calculation. □

Thus we see that Mealy morphisms can be identified with certain profunctors. They emerge as a kind of morphism of $\mathbf{V}$-categories intermediate between

V-functors and **V**-profunctors, closer to **V**-functors. We think of them as partially defined functors, although this is not a perfect analogy.

Example 5 Suppose **V** has an initial object preserved by $V \otimes -$ and $- \otimes V$. Then we have discrete **V**-categories. Let **A** and **B** be two of them on sets A and B of objects. A **V**-functor $\mathbf{A} \to \mathbf{B}$ is determined by a function $A \to B$. A **V**-profunctor is a $B \times A$ matrix of **V**-objects. Profunctor composition is given by matrix multiplication whenever the required coproducts exist and are well-behaved.

A Mealy morphism $(F, \phi) : \mathbf{A} \to \mathbf{B}$ consists of functions $F : A \to B$ and $\phi : A \to \mathrm{Ob}\mathbf{V}$ and nothing more, the structure morphism being uniquely determined. The profunctor $(F, \phi)_*$ is the $B \times A$ matrix whose ba^{th} entry is 0 unless $Fa = b$ in which case it's $\phi(a)$. So $(F, \phi)_*$ is a matrix in which each column has at most one non-zero entry, e.g.

$$\begin{bmatrix} 0 & 0 & 0 \\ V_1 & 0 & V_3 \\ 0 & 0 & 0 \\ 0 & V_2 & 0 \end{bmatrix}$$

for $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $F(i) = 2i \bmod 4$, $\phi(i) = V_i$.

Although matrix multiplication in general requires well-behaved coproducts we see very clearly here that such a matrix can be multiplied by an arbitrary one on the left, e.g.

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ V_1 & 0 & V_3 \\ 0 & 0 & 0 \\ 0 & V_2 & 0 \end{bmatrix} = \begin{bmatrix} W_{12} \otimes V_1 & W_{14} \otimes V_2 & W_{12} \otimes V_3 \\ W_{21} \otimes V_1 & W_{24} \otimes V_2 & W_{22} \otimes V_3 \end{bmatrix}$$

because all but one of each summand is 0.

The matrix $(F, \phi)_*$ can be written as a product of a function (matrix) times a diagonal matrix, e.g.

$$\begin{bmatrix} 0 & 0 & 0 \\ V_1 & 0 & V_3 \\ 0 & 0 & 0 \\ 0 & V_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & I \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{bmatrix},$$

uniquely if none of the V_i is 0. Thus we may think of (F, ϕ) as a weighted function.

5 The Mealy Morphism Classifier

The action of a Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$,

$$\phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{B}(FA, FA')$$

could equivalently be written as

$$\bar{\phi}(A, A') : \mathbf{A}(A, A') \to \mathbf{V}(\phi A, \phi A' \otimes \mathbf{B}(FA, FA'))$$

were **V** closed, and this looks like the strength morphisms for a **V**-functor from **A** into something, $\widetilde{\mathbf{B}}$ say. This is indeed the case. We will construct a **V**-category $\widetilde{\mathbf{B}}$

which classifies Mealy morphisms, analogous to the partial morphism classifier $\widetilde{\mathbf{B}}$ of topos theory. $\widetilde{\mathbf{B}}$ will be the enrichment in $\mathbf{V}$ of the category $\widetilde{\mathbf{B}}_0$ introduced in Section 2.

For the rest of the paper we assume that $\mathbf{V}$ is closed, i.e. the functors $(\ ) \otimes V : \mathbf{V} \to \mathbf{V}$ have right adjoints $\mathbf{V}(V, -)$ for all V in $\mathbf{V}$. This makes $\mathbf{V}$ itself into a $\mathbf{V}$-category. (Note, in passing, that requiring right adjoints for $V \otimes (\ )$ instead, does not.) The counit for the adjunction $\mathrm{ev}_V V' = \mathrm{ev} : \mathbf{V}(V, V') \otimes V \to V'$ is called *evaluation*, and the unit $h_V V' = h : V' \to \mathbf{V}(V, V' \otimes V)$ is sometimes called *coevaluation*, for lack of a better name.

Given a $\mathbf{V}$-category $\mathbf{B}$ we define a new one, $\widetilde{\mathbf{B}}$ as follows. An object is a pair (V, B) with V in $\mathbf{V}$ and B in $\mathbf{B}$. The $\mathbf{V}$-hom is given by $\widetilde{\mathbf{B}}((V, B), (V', B')) = \mathbf{V}(V, V' \otimes \mathbf{B}(B, B'))$. Units $I \to \widetilde{\mathbf{B}}((V, B), (V, B))$ are given by adjoint transpose

$$\frac{I \otimes V \xrightarrow{\cong} V \otimes I \xrightarrow{V \otimes\, id} V \otimes \mathbf{B}(B, B)}{id_{(V,B)} : I \to \mathbf{V}(V, V \otimes \mathbf{B}(B, B))}$$

and composition

$$\mathbf{V}(V', V'' \otimes \mathbf{B}(B', B'')) \otimes \mathbf{V}(V, V' \otimes \mathbf{B}(B, B') \to \mathbf{V}(V, V'' \otimes \mathbf{B}(B, B''))$$

as the adjoint transpose of the composite of two evaluations and composition in $\mathbf{B}$:

$$\mathbf{V}(V', V'' \otimes \mathbf{B}(B', B'')) \otimes \mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \otimes V$$

$$\xrightarrow{1 \otimes\, ev} \mathbf{V}(V', V'' \otimes \mathbf{B}(B', B'')) \otimes V' \otimes \mathbf{B}(B, B')$$

$$\xrightarrow{ev\, \otimes 1} V'' \otimes \mathbf{B}(B', B'') \otimes \mathbf{B}(B, B') \xrightarrow{V'' \otimes \circ} V'' \otimes \mathbf{B}(B, B'').$$

Proposition 1 $\widetilde{\mathbf{B}}$ *is a* $\mathbf{V}$*-category.*

Proof This is a large but straightforward calculation. □

Example 6 For $\mathbf{I}$ the one object discrete $\mathbf{V}$-category, $\widetilde{\mathbf{I}} \cong \mathbf{V}$. Indeed, an object of $\widetilde{\mathbf{I}}$ is $(V, *)$ and $\widetilde{\mathbf{I}}((V, *), (V', *)) = \mathbf{V}(V, V' \otimes \mathbf{I}(*, *)) \cong \mathbf{V}(V, V')$.

Example 7 For $\mathbf{V} = \mathbf{Set}$, $\widetilde{\mathbf{B}}$ is equivalent to the full subcategory of Fam$\mathbf{B}$, the category of families in $\mathbf{B}$, determined by constant families of objects. Fam$\mathbf{B}$ is the coproduct completion of $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ the copower completion.

There is a canonical Mealy morphism

$$(E, \epsilon) : \widetilde{\mathbf{B}} \rightharpoonup \mathbf{B},$$

$$E(V, B) = B, \quad \epsilon(V, B) = V,$$

$$\epsilon((V, B), (V', B')) = \mathrm{ev} : \mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \otimes V \to V' \otimes \mathbf{B}(B, B').$$

Theorem 5

(1) *(E, ϵ) is a Mealy morphism.*
(2) *(E, ϵ) is the universal Mealy morphism into $\mathbf{B}$ in the sense that composing with it gives an equivalence of categories*

$$\mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \widetilde{\mathbf{B}}) \xrightarrow{\sim} \mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A}, \mathbf{B})$$

for every $\mathbf{A}$.
(3) *$\widetilde{(\ )}$ is a right biadjoint to the "inclusion"*

$$(\ )_{\bullet} : \mathbf{V}\text{-}\mathcal{C}at \to \mathbf{V}\text{-}\mathcal{M}ealy.$$

Proof

(1) Straightforward calculation.
(2) We construct a pseudo-inverse to the functor "composing with (E, ϵ)". Given a Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$, define a $\mathbf{V}$-functor $\overline{(F, \phi)} : \mathbf{A} \to \widetilde{\mathbf{B}}$ by $\overline{(F, \phi)}(A) = (\phi A, FA)$ with strength given by

$$\cfrac{\cfrac{st_{\widetilde{(F,\phi)}} : \mathbf{A}(A, A') \to \widetilde{\mathbf{B}}((\phi A, FA), (\phi A', FA'))}{\mathbf{A}(A, A') \to \mathbf{V}(\phi A, \phi A' \otimes \mathbf{B}(FA, FA'))}}{\phi(A, A') : \mathbf{A}(A, A') \otimes \phi A \to \phi A' \otimes \mathbf{B}(FA, FA').}$$

Given another Mealy morphism $(G, \gamma) : \mathbf{A} \rightharpoonup \mathbf{B}$ and a mealy cell $t : (F, \phi) \to (G, \gamma)$ we construct a $\mathbf{V}$-natural transformation $\bar{t} : \overline{(F, \phi)} \to \overline{(G, \gamma)}$ by

$$\cfrac{\cfrac{\cfrac{\bar{t}A : I \to \widetilde{\mathbf{B}}(\overline{(F, \phi)}A, \overline{(G, \gamma)}A)}{\bar{t}A : I \to \widetilde{\mathbf{B}}((\phi A, FA), (\gamma A, GA))}}{\bar{t}A : I \to \mathbf{V}(\phi A, \gamma A \otimes \mathbf{B}(FA, GA)}}{tA : \phi A \to \gamma A \otimes \mathbf{B}(FA, GA).}$$

The details that $\overline{(F, \phi)}$ is a $\mathbf{V}$-functor, $\bar{t}$ is a $\mathbf{V}$-natural transformation, $\overline{(\ \ )}$ is a functor pseudo-inverse to composing with (K, κ) are straightforward and are omitted.
(3) is a more global statement of (2).

□

The universal property of $\widetilde{\mathbf{B}}$ makes $\widetilde{(\ )}$ into a strong morphism of bicategories $\mathbf{V}\text{-}\mathcal{M}ealy \to \mathbf{V}\text{-}\mathcal{C}at$. Given a Mealy morphism $(F, \phi) : \mathbf{A} \rightharpoonup \mathbf{B}$, there is a unique (up to isomorphism) $\mathbf{V}$-functor $\widetilde{(F, \phi)} : \widetilde{\mathbf{A}} \to \widetilde{\mathbf{B}}$ such that

$$\begin{tikzcd} \widetilde{\mathbf{A}} \arrow[r, harpoon, "\widetilde{(F,\phi)}"] \arrow[d, harpoon, "{(E,\epsilon)}"'] & \widetilde{\mathbf{B}} \arrow[d, harpoon, "{(E,\epsilon)}"] \\ \mathbf{A} \arrow[r, harpoon, "{(F,\phi)}"'] & \mathbf{B} \end{tikzcd}$$

We record for use below the formula for $(\widetilde{F,\phi})$. Of course, $(\widetilde{F,\phi})$ looks a lot like $(\overline{F,\phi})$. It is after all just $\overline{(F,\phi)(E,\epsilon)}$.

$$(\widetilde{F,\phi})(V,A) = (V\otimes\phi A, FA)$$

$$\begin{array}{c}
st_{(\widetilde{F,\phi})} : \widetilde{\mathbf{A}}((V,A),(V',A')) \to \widetilde{\mathbf{B}}((V\otimes\phi A, FA),(V'\otimes\phi A', FA')) \\
\hline
st_{(\widetilde{F,\phi})} : \mathbf{V}(V, V'\otimes\mathbf{A}(A,A')) \to \mathbf{V}(V\otimes\phi A, V'\otimes\phi A'\otimes\mathbf{B}(FA,FA')) \\
\hline
\end{array}$$

$$\begin{tikzcd}
\mathbf{V}(V, V'\otimes\mathbf{A}(A,A'))\otimes V\otimes\phi A \arrow[rr] \arrow[dr, "ev\otimes\phi A"'] & & V'\otimes\phi A'\otimes\mathbf{B}(FA,FA') \\
& V'\otimes\mathbf{A}(A,A')\otimes\phi A \arrow[ur, "V'\otimes\phi(A,A')"'] &
\end{tikzcd}$$

The unit for the adjunction $(\ )_\bullet \dashv \widetilde{(\ )}$ is given by

$$H : \mathbf{B} \to \widetilde{\mathbf{B}}$$

$HB = (I, B)$ and

$$st_H : \mathbf{B}(B,B') \to \widetilde{\mathbf{B}}((I,B),(I,B')) \cong \mathbf{V}(I, I\otimes\mathbf{B}(B,B'))$$

is the canonical isomorphism. Thus H is $\mathbf{V}$-full and faithful.

The counit is $(E,\epsilon) : \widetilde{\mathbf{B}} \rightharpoonup \mathbf{B}$.

Proposition 2 $H_\bullet = (H,\eta)$ *is left adjoint to* (E,ϵ) *in* $\mathbf{V}$-$\mathcal{M}ealy$.

Proof $(E,\epsilon)H_\bullet \cong 1_{\mathbf{B}}$ and the counit $e : H_\bullet(E,\epsilon) \to (1_{\widetilde{\mathbf{B}}}, \iota)$ is given by

$$\begin{array}{c}
e(V,B) : \epsilon(V,B)\otimes\eta E(V,B) \to \iota(V,B)\otimes\widetilde{\mathbf{B}}(H_\bullet(E,\epsilon)(V,B), 1_{\widetilde{B}}(V,B)) \\
\hline
V\otimes I \to I\otimes\widetilde{B}((I,B),(V,B)) \\
\hline
V\otimes I \to I\otimes\mathbf{V}(I, V\otimes\mathbf{B}(B,B)) \\
\hline
V\otimes \mathrm{id} : V\otimes I \to V\otimes\mathbf{B}(B,B)
\end{array}$$

□

This proposition becomes transparent if we embed $\mathbf{V}$-$\mathcal{M}ealy$ into $\mathbf{V}$-$\mathcal{P}rof$. The profunctor $(E,\epsilon)_*$ is given by

$$(E,\epsilon)_*(B,(V',B')) = V'\otimes\mathbf{B}(B,B') \cong \widetilde{\mathbf{B}}((I,B),(V',B')) = \widetilde{\mathbf{B}}(HB,(V',B'))$$

so $(E,\epsilon)_* \cong H^*$ which is right adjoint to H_*. Of course, this is just a sketch as a complete proof would require showing that the actions agree as well.

Theorem 6

(1) *The pseudo-monad* $\widetilde{(\ )}$ *on* $\mathbf{V}$-$\mathcal{C}at$ *is of Kock-Zöberlien type.*
(2) $\mathbf{V}$-$\mathcal{M}ealy$ *is biequivalent to the Kleisly category for* $\widetilde{(\ )}$.

Proof

(1) According to [7, Section 2.4], a pseudo-adjunction $(F, U, \eta : 1 \to UF, \epsilon : FU \to 1)$ produces a monad of Kock-Zöberlien type if it satisfies either of the two equivalent conditions: $U\epsilon \dashv \eta U\mathbf{A}$ with counit the "triangle isomorphism"; $F\eta\mathbf{B} \dashv \epsilon F\mathbf{B}$ with unit the "triangle isomorphism". Adjunctions satisfying these equivalent conditions were called KZ-adjunctions.
We use the second condition for the adjunction $(\ )_\bullet \dashv \widetilde{(\ )}$. The unit is $H : \mathbf{B} \to \widetilde{\mathbf{B}}$ and the counit $(E, \epsilon) : \widetilde{\mathbf{B}} \rightharpoonup \mathbf{B}$. So we only need to see that $H_\bullet \dashv (E, \epsilon)$ which is Proposition 2.

(2) $\mathbf{V}$-$\mathcal{M}ealy$ is biequivalent to the Kleisly category for $\widetilde{(\ )}$ because $(\ )_\bullet$ is bijective on objects.

□

$\widetilde{\mathbf{B}}$ has another universal property. It is the free $\mathbf{V}$-category with tensors generated by $\mathbf{B}$.

Proposition 3 $\widetilde{\mathbf{B}}$ *is tensored.*

Proof The following natural bijections

$$\begin{array}{c} X \to \widetilde{\mathbf{B}}(W \otimes V, B), (V', B')) \\ \hline X \to \mathbf{V}(W \otimes V, V' \otimes \mathbf{B}(B, B')) \\ \hline X \otimes W \otimes V \to V' \otimes \mathbf{B}(B, B') \\ \hline X \otimes W \to \mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \end{array}$$

show that $(W \otimes V, B)$ is the tensor $W \otimes (V, B)$. □

Proposition 4 $\mathbf{B}$ *is tensored if and only if* $H : \mathbf{B} \to \widetilde{\mathbf{B}}$ *has a left* $\mathbf{V}$*-adjoint.*

Proof If $\mathbf{B}$ is tensored, then $L : \widetilde{\mathbf{B}} \to \mathbf{B}$ given by

$$L(V, B) = V \otimes B$$

and

$$st_L : \widetilde{\mathbf{B}}((V, B), (V', B')) \to \mathbf{B}(V \otimes B, V' \otimes B')$$

$$\begin{tikzcd}
\mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \otimes V \arrow[r] \arrow[d, "ev"'] & \mathbf{B}(B, V' \otimes B') \\
V' \otimes \mathbf{B}(B, B') \arrow[r, "\lambda \otimes \mathbf{B}(B, B')"'] & \mathbf{B}(B', V' \otimes B') \otimes \mathbf{B}(B, B') \arrow[u, "\circ"']
\end{tikzcd}$$

gives a left $\mathbf{V}$-adjoint to H.

Conversely, if H has a left $\mathbf{V}$-adjoint L, the isomorphisms

$$\mathbf{B}(L(V, B), B') \xrightarrow{\cong} \widetilde{\mathbf{B}}((V, B), (I, B')) \cong \mathbf{V}(V, \mathbf{B}(B, B'))$$

give natural bijections

$$\frac{X \to \mathbf{B}(L(V, B), B')}{X \otimes V \to \mathbf{B}(B, B')}$$

so $L(V, B)$ is $V \otimes B$. □

We are now in a position to give a converse to Theorem 3 in the case where $\mathbf{V}$ is closed.

Proposition 5 $\mathbf{B}$ *is tensored if and only if for every* $\mathbf{V}$*-category* $\mathbf{A}$

$$(\ )_\bullet : \mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \mathbf{B}) \to \mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A}, \mathbf{B})$$

has a left adjoint L_A*, pseudo-natural in* $\mathbf{A}$.

Proof By Theorem 5, $\mathbf{V}\text{-}\mathcal{M}ealy(\mathbf{A}, \mathbf{B})$ is equivalent to $\mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \widetilde{\mathbf{B}})$ and $(\ )_\bullet$ is composition with H

$$\mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \mathbf{B}) \to \mathbf{V}\text{-}\mathcal{C}at(\mathbf{A}, \widetilde{\mathbf{B}}),$$

which has a left adjoint, pseudo-natural in $\mathbf{A}$, if and only if H has a left adjoint in $\mathbf{V}\text{-}\mathcal{C}at$.

□

Suppose $\mathbf{A}$ and $\mathbf{B}$ are tensored and $F : \mathbf{A} \to \mathbf{B}$ is a $\mathbf{V}$-functor. Then there is a canonical comparison

$$\tau_F : V \otimes FA \to F(V \otimes A)$$

corresponding to

$$V \xrightarrow{\lambda} \mathbf{A}(A, V \otimes A) \xrightarrow{st_F} \mathbf{B}(FA, F(V \otimes A)).$$

In fact, giving natural morphisms τ_F satisfying unit and associativity laws is equivalent to giving the strength morphisms st_F (see [13]).

Definition 4 *F preserves tensors* if the τ_F are isomorphisms.

Tensored $\mathbf{V}$-categories with tensor preserving $\mathbf{V}$-functors and $\mathbf{V}$-natural transformations form a 2-category $\mathbf{V}\text{-}\mathcal{T}en$, a locally full sub 2-category of $\mathbf{V}\text{-}\mathcal{C}at$.

Theorem 7

(1) *Composing with* $H : \mathbf{B} \to \widetilde{\mathbf{B}}$ *gives an equivalence of categories*

$$\mathbf{V}\text{-}\mathcal{T}en(\widetilde{\mathbf{B}}, \mathbf{C}) \xrightarrow{\sim} \mathbf{V}\text{-}\mathcal{C}at(\mathbf{B}, \mathbf{C})$$

for all tensored categories $\mathbf{C}$.

(2) $\widetilde{(\ )}$ *is left 2-adjoint to the inclusion* $\mathbf{V}\text{-}\mathcal{T}en \hookrightarrow \mathbf{V}\text{-}\mathcal{C}at$.

(3) $\mathbf{V}\text{-}\mathcal{T}en$ *is biequivalent to the 2-category of Eilenberg–Moore algebras for* $\widetilde{(\ )}$.

Proof

(1) Note from the proof of Proposition 1 that $(V, B) \cong V \otimes (I, B) = V \otimes HB$ so this tells us how to extend a $\mathbf{V}$-functor $F : \mathbf{B} \to \mathbf{C}$ to a tensor preserving one $G : \widetilde{\mathbf{B}} \to \mathbf{C}$, at least on the objects: $G(V, B) = V \otimes FB$. The strength of G

$$st_G : \widetilde{\mathbf{B}}((V, B), (V', B')) \to \mathbf{C}(G(V, B), G(V', B'))$$

is given by the universal property of tensor

$$\frac{\mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \xrightarrow{st_G} \mathbf{C}(V \otimes FB, V' \otimes FB')}{\begin{tikzcd} \mathbf{V}(V, V' \otimes \mathbf{B}(B, B')) \otimes V \arrow[r] \arrow[d, "ev"'] & \mathbf{C}(FB, V' \otimes FB') \\ V' \otimes \mathbf{B}(B, B') \arrow[r, "\lambda \otimes st_F"'] & \mathbf{C}(FB', V' \otimes FB') \otimes \mathbf{C}(FB, FB') \arrow[u, "\circ"'] \end{tikzcd}}$$

We omit the straightforward calculation showing that G is a $\mathbf{V}$-functor.
For a $\mathbf{V}$-natural transformation $t : F \to F'$ we get a $\mathbf{V}$-natural transformation $u : G \to G'$ between the extensions of F and F' in the obvious way:

$$u(V, B) : G(V, B) \to G'(V, B) = V \otimes tB : V \otimes FB \to V \otimes F'B.$$

We omit the tedious but straightforward calculations showing that u is $\mathbf{V}$-natural and that this extension construction gives a pseudo-inverse to composing with H.

(2) is merely a global reformulation of (1).

(3) As $\widetilde{(\ )}$ is a Kock-Zöberlien monad, it is sufficient to show that $\mathbf{B}$ is tensored if and only if the unit $H : \mathbf{B} \to \widetilde{\mathbf{B}}$ has a left $\mathbf{V}$-adjoint, which is precisely Proposition 4.

□

Corollary 2 *$\mathbf{V}$-$\mathcal{M}ealy$ is biequivalent to the full sub 2-category of $\mathbf{V}$-$\mathcal{T}en$ determined by objects of the form $\widetilde{\mathbf{B}}$.*

Proof $\mathbf{V}$-$\mathcal{M}ealy$ is biequivalent to the Kleisly 2-category of $\widetilde{(\ )}$ which in turn is biequivalent to full sub 2-category of Eilenberg–Moore algebras determined by the free objects. □

Thus we may consider a Mealy morphism $\mathbf{A} \rightharpoonup \mathbf{B}$ as a tensor preserving $\mathbf{V}$-functor $\widetilde{\mathbf{A}} \to \widetilde{\mathbf{B}}$ and this respects composition and 2-cells. This is analogous to considering a ($\mathbf{Set}$)-profunctor $\mathbf{A} \longrightarrow\!\!\!\!\!\bullet\;\; \mathbf{B}$ as a cocontinuous functor $\mathbf{Set}^{\mathbf{A}^{op}} \to \mathbf{Set}^{\mathbf{B}^{op}}$. In fact, in the $\mathbf{Set}$ case, $\widetilde{\mathbf{B}}$ can be identified with the full subcategory of $\mathbf{Set}^{\mathbf{B}^{op}}$ determined by copowers of representables, i.e. functors of the form $I \times \mathbf{B}(-, B)$. Thus we see $\widetilde{(\ )}$ as a submonad of $\mathbf{Set}^{(\ )^{op}}$, to the extent that $\mathbf{Set}^{(\ )^{op}}$ is a monad at all. This also holds more generally for $\mathbf{V}$ symmetric and cocomplete.

For more general $\mathbf{V}$, we can still define contravariant $\mathbf{V}$-functors $F : \mathbf{B}^{op} \to \mathbf{V}$ as object functions $F : \mathrm{Ob}\mathbf{B} \to \mathrm{Ob}\mathbf{V}$ with associative and unitary actions

$$FB \otimes \mathbf{B}(B', B) \to FB',$$

and $\mathbf{V}$-natural transformations between them as equivariant families $tB : FB \to GB$. If $\mathbf{B}$ is small and $\mathbf{V}$ is complete and closed then we can define a structure of $\mathbf{V}$-category on $\mathbf{V}$-presheaves, $\widehat{\mathbf{B}}$, by the equalizer

$$\widehat{\mathbf{B}}(F, G) \rightarrowtail \prod_{B} \mathbf{V}(FB, GB) \rightrightarrows \prod_{B,B'} \mathbf{V}(FB \otimes \mathbf{B}(B', B), GB').$$

Then a profunctor $\mathbf{A} \twoheadrightarrow \mathbf{B}$ can be viewed as a $\mathbf{V}$-functor $\mathbf{A} \to \widehat{\mathbf{B}}$. The contravariant representables $\mathbf{B}(-, B) : \mathbf{B}^{op} \to \mathbf{V}$ are $\mathbf{V}$-presheaves, and if F is a $\mathbf{V}$-presheaf so is $V \otimes F$ for any V in $\mathbf{V}$. So $\widetilde{\mathbf{B}}$ can be embedded in $\widehat{\mathbf{B}}$ via

$$(V, B) \mapsto V \otimes \mathbf{B}(-, B)$$

an embedding which is $\mathbf{V}$-full and faithful, as hinted at by the following isomorphisms

$$\begin{aligned}\widehat{\mathbf{B}}(V \otimes \mathbf{B}(-, B), V' \otimes \mathbf{B}(-, B')) &\cong \mathbf{V}(V, \widehat{\mathbf{B}}(\mathbf{B}(-, B), V' \otimes \mathbf{B}(-, B')))\\ &\cong \mathbf{V}(V, V' \otimes \mathbf{B}(B, B')).\end{aligned}$$

Note that, although we can define covariant $\mathbf{V}$-functors $\mathbf{B} \to \mathbf{V}$, indeed if $\mathbf{V}$ is right closed they are exactly $\mathbf{V}$-functors, the functor category $\mathbf{V}\text{-}\mathcal{C}at(\mathbf{B}, \mathbf{V})$ is not enriched in $\mathbf{V}$ in general, no matter how we order the tensor products.

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