# Chapter 8

# LOCALES AND TOPOSES AS SPACES

Steven Vickers
*University of Birmingham*

**Second Reader**

Guram Bezhanishvili
*New Mexico State University*

## 1. Introduction

Mac Lane and Moerdijk, 1992, in their thorough introduction to topos theory, start their Prologue by saying:

> A startling aspect of topos theory is that it unifies two seemingly wholly distinct mathematical subjects: on the one hand, topology and algebraic geometry, and on the other hand, logic and set theory. Indeed, a topos can be considered both as a "generalized space" and as a "generalized universe of sets".

This dual nature of topos theory is of great importance, and one can quite reasonably understand Grothendieck's name "topos" as meaning "that of which topology is the study". Mac Lane and Moerdijk are unquestionably masters of the spatial nature of toposes, yet one could easily read through their book without grasping it. The mathematical technology is so firmly expressed in the set theory and the logic that the spatiality is obscured.

The aim in this chapter is to provide a reader's guide to the spatial content of the major texts. Those texts can also provide a more detailed account of original sources and other applications than has been possible here.

We have on the one hand, the logic and set theory, and, on the other, the topology. In a nutshell, the topos connection between them is that the topos acts like a "Lindenbaum algebra" (of formulae modulo equivalence) for a logical theory whose models are the points of a space.

The prototype is Stone's Representation Theorem for Boolean algebras, which relates propositional logic to Hausdorff, totally disconnected topology.



*M. Aiello, I. Pratt-Hartmann and J. van Benthem (eds.), Handbook of Spatial Logics,* 429–496.

However, it takes some work to develop the idea to its full generality. First, the logic is not at all ordinary classical logic. It is an infinitary positive logic known as *geometric* logic. Second, we are in general talking about predicate theories, and for these the appropriate notion of Lindenbaum algebra is not straightforward. It is really the "category of sets generated by the theory". Lastly, "space" of points is not an ordinary topological space—it is a real generalization.

However, the propositional fragment of the predicate logic does correspond more or less to ordinary topological spaces. As a rough picture of the correspondence, in the propositional case we find:

- space $\sim$ logical theory
- point $\sim$ model of the theory
- open set $\sim$ propositional formula
- sheaf $\sim$ predicate formula
- continuous map $\sim$ transformation of models that is definable within geometric logic

These "propositional toposes" are called *localic*, or (with slight abuse of language) *locales*. They are equivalent to the locales introduced in—say—Johnstone, 1982 or Vickers, 1989.

Now the topos theorists discovered some deep facts about the interaction between continuous maps and the logic and set theory of toposes. A map $f : X \to Y$ gives a geometric morphism between the corresponding toposes of sheaves. A topos is sufficiently like the category of sets that a kind of set theory can be modelled in it. Roughly speaking, in sheaves over $X$ it is set theory "continuously parametrized by a variable point of $X$". The map $f$ then comes to be seen as a "generalized" point of $Y$, parametrized by a point of $X$, and this is a point of $Y$ in the non-standard set theory of sheaves over $X$. So by allowing topological reasoning to take place in toposes instead of in the category of ordinary sets, one gains a simple way to reason about the generalized points of $Y$—in other words the maps into $Y$.

However, to make this trick work one has to reason constructively because the internal logic of toposes is not in general classical. And constructive topology does not work well unless one replaces topological spaces with locales. For instance, the Tychonoff theorem and the Heine-Borel theorem hold constructively for locales but not for topological spaces.

Now there is a well known drawback to locales. They do not in general have enough points and for this reason are normally treated with an opaque "point-free" style of argument. However, they do have enough *generalized* points. Since constructive reasoning gives easy access to these, it also allows locales to be discussed in a spatial way in terms of their points. We in fact get a cohesive package of mathematical deals.

1 Constructive reasoning allows maps to be treated as generalized points.

2 Locales give a better constructive topology (better results hold) than ordinary spaces.

3 The constructive reasoning makes it possible to deal with locales as though they were spaces of points.

What's more, the more stringent *geometric* constructivism has an intrinsic continuity—one might almost say it is the logical essence of continuity. The effect of this is that constructions described in conformity with its disciplines are automatically continuous.

The prime aim of this chapter is to explain how this deep connection between logic and topology works out. However, as a spinoff we find that "generalized spaces" corresponding to toposes become more accessible. They are spaces in which the opens are insufficient to define the topological structure, and sheaves have to be used instead.

These ideas are not essentially new. They have been a hidden part of topos theory from the start. Some writers, such as Wraith, 1979, have made quite explicit use of the virtues of geometric logic. Our aim here is to make them less hidden. At the same time we shall also stress a peculiarity of geometric logic, namely that it embodies a geometric *type theory*. This provides a more naturally mathematical mode of working in geometric logic.

For further reading as a standard text on topos theory, we particularly recommend Mac Lane and Moerdijk, 1992. The standard reference text (Johnstone, 2002a, Johnstone, 2002b) is much more complete and ultimately indispensable. In particular, it treats in some depth the notion of "geometric type construct" that is very important for us. However, it can be impenetrable for beginners.

Though the chapter is so closely linked to toposes, many of its techniques can also be used in other (and distinct) constructive foundations such as formal topology in predicative type theory (Sambin, 1987). We refer to Vickers, 2006 and Vickers, 2005 for some of the connections.

## 2. Opens as propositions

Since Tarski, 1938 it has been known that topologies—by which we mean specifically the lattices of open sets for topological spaces—can provide models for intuitionistic propositional logic.

For discrete topologies, i.e. powersets, this is no surprise. Classical propositional logic can be embedded in classical predicate logic by translating each proposition into a predicate with a single variable $x$, and then the standard semantics interprets each proposition as a subset of the carrier for $x$. Logical connectives translate directly into the corresponding set-theoretic operations in the powerset, and classicality of the logic corresponds to the fact (in classical mathematics) that powersets are Boolean algebras.

What is interesting is that when we replace the powerset by a topology (on $X$, say), there is still enough lattice structure to model intuitionistic logic. The connectives $\wedge$ and $\vee$ can still be translated to $\cap$ and $\cup$, which both preserve openness. However the direct set-theoretic correspondent with negation is complementation, and that does not preserve openness. If proposition $P$ is interpreted as open set $U$, then $\neg P$ is interpreted as the *interior* of the complement $X - U$. Similarly, if also $Q$ is interpreted as $V$, then $P \to Q$ is interpreted as the interior of $(X - U) \cup V$.

These latter operations can both be defined directly in terms of the complete lattice structure of the topology. The interior of $X - U$ is the join of all those opens $W$ such that $W \cap U = \emptyset$, while the interior of $(X - U) \cup V$ is the join of those $W$ such that $W \cap U \subseteq V$. Every topology is a Heyting algebra; and since intuitionistic logic freely expresses Heyting algebra structure, any interpretation of the propositional symbols can be extended to all formulae in a way that respects intuitionistic equivalence.

DEFINITION 8.1 *A* frame *is a complete lattice in which the following* frame distributivity law *holds ($a \in A$, $S \subseteq A$):*

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}.$$

*A* frame homomorphism *is a function between frames that preserves joins and finite meets.*

Every topology is a frame, because the axioms for opens sets tell us that $\wedge$ and $\bigvee$ are set theoretic $\cap$ and $\bigcup$.

PROPOSITION 8.2 *Every frame $A$ is a Heyting algebra.*

*Proof* We must show that for every $a, b \in A$ there is an element $a \to b \in A$ such that for all $c \in A$,

$$(*) \qquad c \leq a \to b \Leftrightarrow c \wedge a \leq b.$$

(If this element exists, then it is unique.) We can define

$$a \to b = \bigvee \{x \mid x \wedge a \leq b\}.$$

Now the $(\Leftarrow)$ direction in (*) is immediate, while the $(\Rightarrow)$ follows from frame distributivity, which implies that $(a \to b) \wedge a \leq b$. QED

In fact, frames and complete Heyting algebras are the same things. However, we distinguish between the notions because of the homomorphisms. A frame homomorphism does not necessarily preserve the Heyting arrow, and so is not necessarily a Heyting algebra homomorphism.

Tarski, 1938 went further, and showed a stronger property if the space $X$ is a dense-in-itself separable metric space (such as the real line). If $P_i$ $(i \in \mathbb{N})$ is a countable family of propositional symbols, then an intuitionistic formula $\phi$ in the $P_i$s is an intuitionistic theorem iff, for every interpretation of the $P_i$s in $\Omega X$, the corresponding interpretation of $\phi$ is the whole of $X$. This was explained further in McKinsey and Tarski, 1944 using an embedding of intuitionistic propositional logic in the classical modal logic **S4**, which could then be interpreted in the powerset of $X$ with the $\Box$ modality corresponding to the interior operator—for further details, see 5.

## 2.1 Lindenbaum algebras for classical logic

A Lindenbaum algebra is a lattice of formulae modulo provable equivalence, and they and their generalizations will be key to the whole of this chapter. Let us review how it works for classical logic, where the connection with topology is essentially Stone's Representation Theorem for Boolean algebras (see Johnstone, 1982). If $\Sigma$ is a propositional signature (i.e. a set of propositional symbols) then we write $\mathsf{Sen}_\Sigma$ for the set of sentences constructed over $\Sigma$ using a classically adequate set of connectives. If $T$ is a theory over $\Sigma$ (a set of sentences), then an equivalence relation $\equiv_T$ can be defined by

$$\phi \equiv_T \psi \text{ iff } T \vdash (\phi \leftrightarrow \psi) \text{ in classical logic}$$

and the Lindenbaum algebra for $T$ is defined as $\mathcal{LA}(\Sigma, T) = \mathsf{Sen}_\Sigma / \equiv_T$. It is a Boolean algebra.

The central idea is to use the dual nature of homomorphisms $g : \mathcal{LA}(\Sigma, T) \to \mathcal{LA}(\Sigma', T')$.

*Logically*, $g$ is a logical translation of $(\Sigma, T)$ into $(\Sigma', T')$. It translates propositional symbols into sentences (modulo equivalence) and preserves theoremhood. Isomorphism gives a natural presentation-independent notion of equivalence of theories, by mutual translatability.

*Spatially*, $g$ provides a transformation of models, from $(\Sigma', T')$ to $(\Sigma, T)$ (note the reversal of direction). It is this spatial view that provides the link with topology, for a model transformation arises this way iff it is continuous with respect to certain topologies on the model spaces. This is shown by Stone's Theorem.

Because classical logic is complete, the transformation of ordinary models suffices to determine the Lindenbaum algebra homomorphism. However, we also get an alternative view by considering generalized models. Then there is a *generic* model whose transformation determines that of all the other (specific) models, and this idea becomes important for incomplete logics.

Let $A$ be a Boolean algebra. An *interpretation* of $\Sigma$ in $A$ is a function $M : \Sigma \to A$, and this extends uniquely to a function $\overline{M} : \mathsf{Sen}_\Sigma \to A$ that evaluates the connectives by the corresponding Boolean operations on $A$. Then $M$ is a *model* of $T$ iff $\overline{M}(\phi) = 1$ for every $\phi \in T$. Models are preserved by Boolean algebra homomorphisms $f : A \to B$ – the composite $f \circ M$ is also a model in $B$. We write $\mathrm{Mod}_A(T)$ for the set of models of $T$ in $A$.

Those were the generalized models referred to above. The standard models, interpreting propositions as truth values, are found by taking $A = \mathbf{2} = \{0, 1\}$.

The *generic* model $M_T$ of $T$ is a particular model in the Lindenbaum algebra. It interprets each proposition symbol $P \in \Sigma$ as the equivalence class of $P$ as sentence. Clearly, by definition of the Lindenbaum algebra we have $\phi \equiv_T \top$ for every $\phi \in T$, and so $\overline{M_T}(\phi) = 1$, so $M_T$ is a model. It has a universal property: any model can be got, uniquely, by applying a Boolean algebra homomorphism to the generic model.

PROPOSITION 8.3 *Let* $(\Sigma, T)$ *be a classical propositional theory, and* $A$ *a Boolean algebra. Then the function* $f \mapsto f \circ M_T$*, taking Boolean algebra homomorphisms* $\mathcal{LA}(\Sigma, T) \to A$ *into* $\mathrm{Mod}_A(T)$*, is a bijection.*

*Proof* Let $M : \Sigma \to A$ be a model of $T$. Suppose $\phi \equiv_T \psi$. The classical proof of $T \vdash (\phi \leftrightarrow \psi)$ will involve only finitely many elements of $T$, say $t_1, \ldots, t_n$. Because of the nature of classical proofs (this requires some checking) it will imply that $\overline{M}(t_1) \wedge \cdots \wedge \overline{M}(t_n) \le \overline{M}(\phi) \leftrightarrow \overline{M}(\psi)$ in $A$. But each $\overline{M}(t_i)$ is 1, because $M$ is a model, and hence $\overline{M}(\phi) \leftrightarrow \overline{M}(\psi) = 1$ and $\overline{M}(\phi) = \overline{M}(\psi)$. It follows that $\overline{M}$ factors (uniquely) through $\mathcal{LA}(\Sigma, T)$ as $f \circ M_T$ where $f$ is a Boolean algebra homomorphism. QED

The generic model $M_T$ corresponds to the identity homomorphism on $\mathcal{LA}(\Sigma, T)$.

As a consequence of the proposition, we see there is a function

$$\vDash : \mathrm{Mod}_A(T) \times \mathcal{LA}(\Sigma, T) \to A$$

with $\vDash (M, \phi)$ the image of $\phi$ under the homomorphism corresponding to $M$. In the particular case of $A = \mathbf{2}$, $\vDash (M, \phi) = 1$ iff $M \vDash \phi$ in the usual sense.

Now consider a homomorphism $g : \mathcal{LA}(\Sigma, T) \to \mathcal{LA}(\Sigma', T')$. By the proposition, $g$ corresponds to a model of $T$ in $\mathcal{LA}(\Sigma', T')$. This gives a logical translation of $(\Sigma, T)$ in $(\Sigma', T')$ – the propositional symbols in $\Sigma$ are interpreted as formulae over $\Sigma'$, in such a way that the axioms in $T$ all become provable from $T'$.

But one can also view this from the model side. A model of $T'$ in $A$ is a homomorphism from $\mathcal{LA}(\Sigma', T')$ to $A$, and precomposing this with $g$ gives a homomorphism from $\mathcal{LA}(\Sigma, T)$ to $A$, in other words a model of $T$. (Note the reversal of direction!) Thus $g$ gives a uniform way of transforming models of $T'$ into models of $T$.

Trivially, $g = g \circ \mathrm{Id}_{\mathcal{LA}(\Sigma',T')}$ is completely determined by its transformation of the generic model. The generic model is non-standard, but we also find $g$ is determined by its transformation of the standard models.

PROPOSITION 8.4 *Let* $(\Sigma, T)$ *and* $(\Sigma', T')$ *be two propositional theories, and let* $f, g : \mathcal{LA}(\Sigma, T) \to \mathcal{LA}(\Sigma', T')$ *be two homomorphisms inducing model transformations* $F, G : \mathrm{Mod}_{\mathbf{2}}(T') \to \mathrm{Mod}_{\mathbf{2}}(T)$. *If* $F = G$ *then* $f = g$.

*Proof* It suffices to show that $f(P) \equiv_{T'} g(P)$ for every $P \in \Sigma$. By completeness, it suffices to show that for every standard model $M'$ of $T'$, $f(P)$ and $g(P)$ have the same truth value at $M'$. But those truth values are the same as those for $P$ at $F(M')$ and $G(M')$ respectively, and they are equal. QED

Not every transformation of standard models is induced by a homomorphism. Stone's Representation Theorem represents $\mathcal{LA}(\Sigma, T)$ using a topological space $\mathrm{Mod}_{\mathbf{2}}(T)$, with sets of the form $\{M \mid M \vDash \phi\}$ ($\phi \in \mathcal{LA}(\Sigma, T)$) providing a base of opens. (In fact, they are the clopens.) This is a *Stone space* – Hausdorff and totally disconnected (see Johnstone, 1982). The Theorem shows that the homomorphisms correspond to the *continuous* maps between the model spaces.

DEFINITION 8.5 *Let* $(\Sigma, T)$ *and* $(\Sigma', T')$ *be two propositional theories. We define a* map *from* $(\Sigma, T)$ *to* $(\Sigma', T')$ *to be a homomorphism from* $\mathcal{LA}(\Sigma', T') \to \mathcal{LA}(\Sigma, T)$ *(or, equivalently, a model of* $(\Sigma', T')$ *in* $\mathcal{LA}(\Sigma, T)$*).*

*Note* – In this chapter, the word "map" will always carry connotations of continuity. A map between topological spaces is understood to be continuous.

By emphasizing the model transformations, and defining "maps" to go in the same direction (opposite to that of the homomorphisms), we try to foster a view that the theory represents its "space of models". Propositional theories and maps between them form a category. Here are some simple but important examples.

EXAMPLE 8.6 *(1) The theory* $(\emptyset, \emptyset)$ *(no symbols or axioms) corresponds to the one-element space* 1. *It has a unique, vacuous model in any Boolean algebra and is final in the category of theories.* $\mathcal{LA}(\emptyset, \emptyset) = \mathbf{2}$, *and for any theory* $(\Sigma, T)$ *the maps from* $(\emptyset, \emptyset)$ *to* $(\Sigma, T)$ *are equivalent to the standard models of* $(\Sigma, T)$.

*(2) The inconsistent theory* $(\emptyset, \{\bot\})$ *corresponds to the empty space* $\emptyset$. *It has no model except in the one-element Boolean algebra* $\mathbf{1}$, *which is* $\mathcal{LA}(\emptyset, \{\bot\})$. *It is initial in the category of theories.*

*(3) The theory* $(\{P\}, \emptyset)$ *corresponds to the discrete 2-element space* 2, *having two standard models* $P \mapsto 0$ *and* $P \mapsto 1$. *Models in* $A$ *are elements of* $A$. *In Stone's Theorem, this corresponds to the fact that for any space* $X$, *clopens*

*are equivalent to maps $X \to 2$. Its Lindenbaum algebra $\mathcal{LA}(\{P\}, \emptyset)$ is the Boolean algebra $\mathbf{4} = \{0, P, \neg P, 1\}$, which is freely generated by $P$.*

## 2.2 Frames as Lindenbaum algebras

The Stone topologies, with the Boolean algebra of clopens forming a base and corresponding to classical propositional logic, are very special. We generalize this by changing to propositional *geometric* logic, for which the Lindenbaum algebras are *frames*, playing the role of topologies (with all opens, not just clopens).

From Tarski's results on interpreting intuitionistic logic in topologies, one might expect the logic here to be intuitionistic. However, full intuitionistic logic is too strong. If $f$ is a continuous map, then its inverse image function, restricted to open sets, will act as the corresponding homomorphism between Lindenbaum algebras. In general this does not preserve the Heyting arrow, though it is a frame homomorphism. Hence we need a logic that corresponds to the structure of frame rather than of Heyting algebra.

DEFINITION 8.7 *Let $\Sigma$ be a set (of propositional symbols).* Geometric formulae *over $\Sigma$ are constructed from the symbols in $\Sigma$ using $\top$ (true), $\wedge$ and arbitrary – possibly infinitary – disjunctions $\bigvee$. A* geometric theory *over $\Sigma$ is a set of axioms of the form $\phi \to \psi$, where $\phi$ and $\psi$ are geometric formulae.*

Coherent *formulae and theories are defined in the same way, but without any infinitary disjunctions. This is sometimes known as* positive *logic.*

Note that because of the limitations of the logic, a theory is not simply a set of formulae. The logical rules are best described in sequent form. (A theory is in effect a set of axiomatic sequents, and we shall often write its axioms as sequents, using $\vdash$. We shall also write $\dashv\vdash$ for bidirectional entailment.) The rules are *identity*

$$\phi \vdash \phi,$$

*cut*

$$\frac{\phi \vdash \psi \qquad \psi \vdash \chi}{\phi \vdash \chi},$$

the *conjunction* rules

$$\phi \vdash \top, \quad \phi \wedge \psi \vdash \phi, \quad \phi \wedge \psi \vdash \psi, \quad \frac{\phi \vdash \psi \quad \phi \vdash \chi}{\phi \vdash \psi \wedge \chi},$$

the *disjunction* rules

$$\phi \vdash \bigvee S \quad (\phi \in S), \quad \frac{\phi \vdash \psi \quad (\text{all } \phi \in S)}{\bigvee S \vdash \psi}$$

and *frame distributivity*

$$\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}.$$

Note that $\bigvee \emptyset$ plays the role of $\perp$ (false). Note also that frame distributivity allows us to reduce every formula to a disjunction of finite conjunctions of symbols from $\Sigma$. Hence although the formulae as defined syntactically form a proper class, modulo equivalence they form a set (classically, and according to at least some constructive foundations).

We write $\Omega[T]$ for the Lindenbaum algebra of $T$, i.e. the set of geometric formulae modulo equivalence provable from $T$. The logical rules imply that it is a frame. There is an obvious notion of model of $T$ in any frame, and $\Omega[T]$ has a particular generic model $M_T$ given by interpreting each propositional symbol as its equivalence class of formulae.

PROPOSITION 8.8 *(cf. Proposition 8.3.) Let* $(\Sigma, T)$ *be a geometric theory, and let* $A$ *be a frame. Then the function* $f \mapsto f \circ M_T$*, taking frame homomorphisms* $\Omega[T] \to A$ *into* $\mathrm{Mod}_A(T)$*, is a bijection.*

*Proof* If $M : \Sigma \to A$ is a model then it extends to a function $\overline{M}$ on the class of formulae. All the logical rules will be valid in $A$ under this interpretation, and the axioms in $T$ will all hold because $M$ is a model, so it follows that $\overline{M}$ factors (uniquely) via $\Omega[T]$. QED

Standard models are given, as usual, by interpreting the propositional symbols as truth values. However, we write $\Omega$ for the frame of truth values, by contrast with $\mathbf{2}$ for the Boolean algebra. (This follows the topos-theoretic notation.) Constructively they are different, allowing for the fact that geometric logic is a positive logic. A geometric truth value is equivalent to a subset of a singleton, while a Boolean truth value is a *decidable* subset of a singleton.

DEFINITION 8.9 *Let* $A$ *be a frame. Then the geometric theory* $\mathrm{Th}_A$ *is presented as follows. For the signature* $\Sigma$*, introduce a propositional symbol* $P_a$ *for each* $a \in A$*, and then take axioms*

$$\begin{aligned} P_a &\to P_b \qquad (a \leq b \textit{ in } A) \\ P_a \wedge P_b &\to P_{a\wedge b} \qquad (a, b \in A) \\ \top &\to P_1 \\ P_{\bigvee S} &\to \textstyle\bigvee_{a\in S} P_a \qquad (S \subseteq A) \end{aligned}$$

All the theory is saying is that the finite meets and arbitrary joins in $A$ should be treated logically as finite conjunctions and arbitrary disjunctions. Hence the connectives of propositional geometric logic correspond directly to the frame structure. From this it follows that the models of $\mathrm{Th}_A$ in a frame $B$ are the frame homomorphisms from $A$ to $B$, and so we see that $\Omega[\mathrm{Th}_A] \cong A$.

A standard model of $\mathrm{Th}_A$ can also be described by saying which propositional symbols $P_a$ are assigned the truth value $\mathbf{true}$, and hence by a subset $F \subseteq A$ satisfying certain conditions corresponding to the axioms. The first axiom says that $F$ is an *upper* subset of $A$, the next two that $F$ is a *filter*, and then the fourth that it is a *completely prime* filter. (Note: the standard texts contain various other descriptions of the standard models, but they are constructively inequivalent.)

## 2.3 Locales

Let us define, conceptually, a *locale* to be a "propositional geometric theory pretending to be a space", using the ideas of Sec. 2.1, which took the logical theory as the starting point. That is to say, the locale *is* the theory, but repackaged in a spatial language of points and maps instead of models and Lindenbaum algebra homomorphisms. What makes this repackaging significant is the fact that geometric logic is incomplete—in general, there are not enough standard models to account for all the frame homomorphisms (cf. Proposition 8.4). Thus the spatial side (in terms of standard models) and the logical side (in terms of Lindenbaum algebras) become mathematically inequivalent. However, the logical side still contains good topological results; indeed, in constructive mathematics they are often better than the spatial ones. The localic repackaging makes it much easier to see this topological content.

The usual definition is that a locale *is* a frame. We prefer to say it is the propositional geometric theory, and that it *has* a frame. This makes it easier to see locales as a special case of toposes, which arise from predicate geometric theories. In addition, in certain foundational schools such as predicative type theory, the frames are problematic. They are constructed using the powerset, and that is impredicative. The main account in this school is the formulation as "formal topology" (Sambin, 1987).

A formal topology gives a base $S$ (so every frame element is to be a join of base elements) and the *cover* relation $\lhd$, which describes when one basic open is to be covered by a set of others. This then corresponds to a propositional geometric theory in which $S$ provides the propositional symbols, and the cover relation provides axioms to say one symbol entails a disjunction of others. (For these purposes, the notion of map can be defined in a more primitive way that does not rely on constructing the frame.) The variant notion of *inductively generated formal topology* (Coquand et al., 2003) is even closer to the propositional geometric theory in that it does not require the complete cover relation but just a part from which the rest can be deduced.

Let us review the ideas of Sec. 2.1 in the light of Sec. 2.2.

DEFINITION 8.10 *1 A* locale *is (presented by) a propositional geometric theory. If the theory is $T$, we write $[T]$ for the locale. The locale $[T]$ should be conceptualized as "the space of models of $T$".*

*2 If $X$ is a locale, then $\Omega X$ denotes its Lindenbaum algebra, a frame.*

*3 The* opens *of $X$ are the elements of $\Omega X$.*

*4 If $X$ and $Y$ are locales, then a* map $f : X \to Y$ *is a frame homomorphism $f^* : \Omega Y \to \Omega X$. We write* $\mathrm{Map}(X, Y)$ *for the set of maps from $X$ to $Y$. Locales and maps form a category* **Loc**, *dual to the category* **Fr** *of frames.*

*5 If $X$ and $Y$ are locales then* $\mathrm{Map}(X, Y)$ *is partially ordered by the* specialization order, *$f \sqsubseteq f'$ if $f^*(U) \leq f'^*(U)$ for every $U \in \Omega Y$.*

*6 If $X$ and $W$ are locales then the* (generalized) points *of $X$* at stage (of definition) *$W$ are the maps $W \to X$. Think of these as points of $X$ "continuously parametrized by a variable point of $W$". The points of $[T]$ are just the models of $T$ in $\Omega W$.*

*7 If $X$ is a locale then the identity map $X \to X$, a point of $X$ at stage $X$, is the* generic *point of $X$.*

*8 If $f : X \to Y$ is a map, then postcomposition $f \circ -$ transforms points of $X$ (at any stage) to points of $Y$ (at the same stage). We shall often write $f(x)$ for $f \circ x$.*

The specialization order is already present (as a preorder) in ordinary topology: $x \sqsubseteq x'$ if every neighbourhood of $x$ also contains $x'$ (i.e. $x$ is in the closure of $\{x'\}$). It is often neglected there, because for Hausdorff spaces (more precisely for $T_1$ spaces) it is discrete: $x \sqsubseteq x'$ iff $x = x'$.

An important fact about the specialization order is that it has directed joins. In any poset $(P, \leq)$, a family of elements $(x_i)_{i \in I}$ is *directed* if $I$ is inhabited and, for any $i, j \in I$, there is some $k \in I$ such that $x_i \leq x_k$ and $x_j \leq x_k$. We also say $P$ is a *directed complete* poset (or *dcpo*) if it has a join for every directed family. We use an arrow, as in $\bigsqcup^{\uparrow}_i x_i$, to indicate that a join is of a directed family.

PROPOSITION 8.11 *Let $X$ and $Y$ be locales. Then* $\mathrm{Map}(X, Y)$ *is directed complete with respect to $\sqsubseteq$. The directed joins are preserved by composition on either side.*

*Proof* Let $(f_i)_{i \in I}$ be directed in $\mathrm{Map}(X, Y)$. The join $\bigsqcup^{\uparrow}_i f_i$ is given as a frame homomorphism by

$$(\bigsqcup\nolimits^{\uparrow}_i f_i)^*(U) = \bigvee\nolimits_{i \in I} f_i^*(U).$$

QED

The directed joins are less familiar from ordinary topology. This is partly because so many familiar spaces have discrete specialization order, but also because in the absence of sobriety (Sec. 2.4) the directed joins may be missing. However, they are fundamental in computer science and provide a means for providing the semantics of recursive algorithms. (See, e.g., Plotkin, 1981, Gierz et al., 1980, Vickers, 1989.) From the proposition we see one essential feature of maps, namely that they preserve directed joins of points. (This is known as Scott continuity.)

DEFINITION 8.12 *(1) The* one-point *locale* $1$ *is presented by the empty theory over the empty signature.* $\Omega 1$ *is* $\Omega$. *(Note—we shall also write* $1$ *for a singleton* set. *In practice this ambiguity should not cause problems.) The* global *points of a locale* $X$ *are its points at stage* $1$*, i.e. the maps* $1 \to X$. *Thus the global points of* $[T]$ *are the standard models of* $T$.

*(2) The* empty *locale* $\emptyset$ *is presented by the inconsistent theory* $\{\top \to \bot\}$ *over the empty signature.* $\Omega\emptyset$ *is a one-element frame. It has no points except at stage* $\emptyset$.

*(3) The* Sierpiński *locale* $\mathbb{S}$ *is presented by the empty theory over a one-element signature* $\{P\}$. *Its points are equivalent to subsets of the set* $1$. *We usually write* $\top$ *for the subset* $1$ *itself (an open point), and* $\bot$ *for the empty subset (a closed point). Note that* $\bot \sqsubseteq \top$.

REMARK 8.13 *The opens* $U'$ *of any locale* $X$ *are equivalent to the maps* $U : X \to \mathbb{S}$*, with* $U' = U^*(P)$. *If* $f : X \to Y$ *is a map, then*

$$f^*(U') = f^*(U^*(P)) = (U \circ f)^*(P),$$

*and hence corresponds to* $U \circ f$. *Hence we can talk about opens and inverse image functions purely in the language of maps.*

## 2.4 Locales compared with spaces

Now we have this language of points, opens and maps, all deriving from the single notion of geometric theory, we shall compare it with ordinary topology.

Given a locale $X$, let us write $\mathrm{pt}(X)$ for its set of global points, maps $x : 1 \to X$. If $U : X \to \mathbb{S}$ is an open, then the composite $U \circ x$ is a global point of $\mathbb{S}$, and hence a subset of 1. We write $x \models U$ iff $U \circ x = \top$ (i.e. $x^*(U) = 1$), and $\mathrm{ext}(U)$ (the *extent* of $U$) for $\{x \in \mathrm{pt}(X) \mid x \models U\}$. Because $x^* : \Omega X \to \Omega$ is a frame homomorphism, we find that the sets $\mathrm{ext}(U)$ form a topology on $\mathrm{pt}(X)$.

Now let $f : X \to Y$ be a map of locales, giving a point transformer $\mathrm{pt}(f) : \mathrm{pt}(X) \to \mathrm{pt}(Y)$. If $V$ is an open of $Y$, then

$$\begin{aligned} x \in \mathrm{pt}(f)^{-1}(\mathrm{ext}(V)) &\Leftrightarrow f \circ x \vDash V \\ &\Leftrightarrow V \circ f \circ x = \top \\ &\Leftrightarrow x \vDash \mathrm{ext}(f^*(V)). \end{aligned}$$

It follows that $\mathrm{pt}(f)^{-1}(\mathrm{ext}(V)) = \mathrm{ext}(f^*(V))$, and so $\mathrm{pt}(f)$ is continuous.

- For each locale $X$, its global points form a topological space $\mathrm{pt}(X)$.
- For each map of locales, the corresponding transformation $\mathrm{pt}(f)$ of global points is continuous.

This looks promising, but there is not an exact match between locales and topological spaces, and we need to understand that. The central connection, a categorical adjunction, is summarized in the following result. For an element $x$ of a topological space $X$, we write $\mathfrak{N}_x = \{U \in \Omega X \mid x \in U\}$ for the set of open neighbourhoods of $x$. This is a completely prime filter in $\Omega X$ .

PROPOSITION 8.14 *Let $X$ be a topological space and $Y$ a locale. Then there is a bijection between*

1. *maps (continuous, as always) $f : X \to \mathrm{pt}(Y)$, and*
2. *maps $g : [\mathrm{Th}_{\Omega X}] \to Y$ (homomorphisms $g^* : \Omega Y \to \Omega X$).*

*Proof* Consider the following condition on pairs $(f, \phi)$ where $f : X \to \mathrm{pt}(Y)$ and $\phi : \Omega Y \to \Omega X$ are arbitrary functions:

$$(\forall x \in X,\ \forall V \in \Omega Y)\ (f(x) \in \mathrm{ext}(V) \leftrightarrow x \in \phi(V)).$$

This is equivalent to

$$(\forall V \in \Omega Y)\ \phi(V) = f^{-1}(\mathrm{ext}(V))$$

and, considering the points of $\mathrm{pt}(Y)$ as completely prime filters of $\Omega X$ and remembering that $f(x) \in \mathrm{ext}(V)$ iff $V \in f(x)$, to

$$(\forall x \in X)\ f(x) = \phi^{-1}(\mathfrak{N}_x).$$

It follows that $\phi$ is determined by $f$, and $f$ is determined by $\phi$.

Under these conditions, it follows that $f$ is continuous and $\phi$ is a frame homomorphism. Conversely, if $f$ is continuous then inverse image $f^{-1}$ gives a corresponding $\phi$; and given a frame homomorphism $\phi$, we find that each $\phi^{-1}(\mathfrak{N}_x)$ is a completely prime filter. QED

The first mismatch between spaces and locales is that *not every space comes from a locale.*

Let $(X, \Omega X)$ be a topological space, with $\Omega X$ the topology—the family of open sets. Consider the locale $[\mathrm{Th}_{\Omega X}]$, whose global points are the completely prime filters of $\Omega X$. For every point $x \in X$, its open neighbourhood filter $\mathfrak{N}_x$ is a completely prime filter. However, two points $x$ and $y$ might have the same open neighbourhood filter—every open containing $x$ also contains $y$, and vice versa. A space is called $T_0$ if this never happens, i.e. if $\mathfrak{N}_x = \mathfrak{N}_y$ then $x = y$.

In addition, there may be a completely prime filter that is not $\mathfrak{N}_x$ for any point $x$.

EXAMPLE 8.15 *Consider the set $\mathbb{N}$ of natural numbers with a topology in which the opens are the upper sets. The completely prime filter of non-empty upper sets is not the open neighbourhood filter of any point.*

A space is *sober* if the correspondence $x \mapsto \mathfrak{N}_x$ is a bijection between points and completely prime filters. For sober spaces, we might just as well consider them as locales—we lose nothing by using the geometric theories to study sober topological spaces.

PROPOSITION 8.16 *Let $X$ and $Y$ be sober spaces. Then there is a bijection between maps $f : X \to Y$, and maps $g : [\mathrm{Th}_{\Omega X}] \to [\mathrm{Th}_{\Omega Y}]$ (homomorphisms $g^* : \Omega Y \to \Omega X$).*

*Proof* Apply Proposition 8.14 with $[\mathrm{Th}_{\Omega Y}]$ substituted for the locale $Y$. Sobriety assures us that the sober space $Y$ is homeomorphic to $\mathrm{pt}([\mathrm{Th}_{\Omega Y}])$. QED

Any "point-free" approach, constructing the points out of the logic, is inevitably sober. Many well-behaved spaces, for instance all Hausdorff spaces, are sober, and any space can be "soberified" by replacing it by the space of completely prime filters.

The second mismatch between spaces and locales is that *not every locale comes from a space*. This arises out of an important logical fact, that *geometric logic is not complete.*

Stone's Theorem showed how each Boolean algebra is isomorphic to a sub-Boolean-algebra of a powerset. In logical terms, there are always enough standard models to distinguish between inequivalent sentences. This is a consequence of completeness.

By contrast, locales do not always have enough global points to discriminate between the opens. For each locale $X$, the extent homomorphism $\mathrm{ext} : \Omega X \to \mathcal{P}\,\mathrm{pt}(X)$ defines a topology on $\mathrm{pt}(X)$. The locale $X$ is *spatial* if ext is 1-1, but not all locales are spatial.

For example, let $\mathbb{R}$ be the real line with its usual topology. Let $T$ be $\mathrm{Th}_{\Omega\mathbb{R}}$ extended by extra axioms $\neg\neg U \to U$ for every $U \in \Omega\mathbb{R}$. ($\neg\neg$ is the Heyting double negation in $\Omega\mathbb{R}$. Concretely, $\neg\neg U$ is the interior of the closure of $U$.) $\mathrm{pt}[T]$ is a subspace of $\mathbb{R}$, comprising those reals $x$ such that for every $U$, if $x \in \neg\neg U$ then $x \in U$. There are no such $x$, for consider $U = (-\infty, x) \cup (x, \infty)$, which has $\neg\neg U = \mathbb{R}$. But $\neg\neg$ is a *nucleus* (see e.g. Johnstone, 1982), and an immediate consequence of the general theory is that the opens of $[T]$ are equivalent to the regular opens of $\mathbb{R}$, i.e. those $U$ for which $\neg\neg U = U$. Hence $[T]$ is a non-trivial locale with no global points, hence non-spatial.

Logically, spatiality is the same as completeness, but there is a difference of emphasis. Completeness refers to the ability of the logical reasoning (from rules and axioms) to generate all the equivalences that are valid for the models: if not, then it is the logic that is considered incomplete. Spatiality refers to the existence of enough models to discriminate between logically inequivalent formulae: if not, then the class of models is incomplete.

In classical mathematics, most important locales are spatial; but this can rely on the axiom of choice to find sufficient points. In constructive mathematics many important locales (such as the real line) behave better in non-spatial form, and if we spatialize by topologizing the global points, then important theorems (such as the Heine-Borel Theorem) become false. This has led to a common misconception that constructive topology is deficient in theorems. This is actually not true, and the purpose of this chapter is to show how topology and constructive reasoning are intimately related. However, an important step is to forego any dependence on spatiality, on relying on a space being carried by an untopologized *set* of points.

Happily, constructive reasoning itself contains the key to dealing with non-spatiality. Spatiality is an issue when we try to deal with a locale in terms of its global points, of which there might not be enough. But there are enough generalized points. For example, a map of locales is defined by its action on the generic point. The generalized points live in non-Boolean lattices (or, as we shall shortly see, in the non-classical mathematics of sheaves), and it is convenient to deal with them using constructive mathematics as a tool.

## 2.5 Example: the localic reals

As an adaptation of the localic reals in Johnstone, 1982, we present a propositional geometric theory $T_{\mathbb{R}}$ with propositional symbols $P_{q,r}$ ($q, r \in \mathbb{Q}$, the rationals) and axioms:

$$P_{q,r} \wedge P_{q',r'} \leftrightarrow \bigvee \{P_{s,t} \mid \max(q,q') < s < t < \min(r,r')\}$$
$$\top \to \bigvee \{P_{q-\varepsilon,q+\varepsilon} \mid q \in \mathbb{Q}\} \text{ if } 0 < \varepsilon \in \mathbb{Q}$$

PROPOSITION 8.17 *In the theory $T_{\mathbb{R}}$, we can derive the following.*

$$
\begin{aligned}
P_{q,r} &\dashv\vdash \bigvee\{P_{s,t} \mid q < s < t < r\} \\
P_{q,r} \wedge P_{q',r'} &\dashv\vdash P_{\max(q,q'),\min(r,r')} \\
P_{q,r} &\vdash \bot \qquad \textit{if } r \leq q \\
P_{q',r'} &\vdash P_{q,r} \qquad \textit{if } q \leq q' \textit{ and } r' \leq r \\
P_{q,t} &\vdash P_{q,r} \vee P_{s,t} \qquad \textit{if } q < s < r < t
\end{aligned}
$$

*Proof* These are all straightforward except the last. If $q < s < r < t$ then let $\varepsilon = (r - s)/2$. We have

$$
\begin{aligned}
P_{q,t} &\vdash \bigvee\{P_{u-\varepsilon,u+\varepsilon} \mid u \in \mathbb{Q}\} \wedge P_{q,t} \\
&\dashv\vdash \bigvee\{P_{\max(u-\varepsilon,q),\min(u+\varepsilon,t)} \mid u \in \mathbb{Q}\}.
\end{aligned}
$$

Now for any $u \in \mathbb{Q}$, we cannot have both $u + \varepsilon > r$ and $u - \varepsilon < s$, for then $r - \varepsilon < u < s + \varepsilon$, so $r - s < 2\varepsilon$, contradiction. Hence either $u + \varepsilon \leq r$, in which case $P_{\max(u-\varepsilon,q),\min(u+\varepsilon,t)} \vdash P_{q,r}$, or $u - \varepsilon \geq s$, in which case $P_{\max(u-\varepsilon,q),\min(u+\varepsilon,t)} \vdash P_{s,t}$. QED

In Sec. 4.7 we shall see how the models of this are equivalent to Dedekind sections of the rationals.

We have a model of the theory in $\Omega\mathbb{R}$, interpreting $P_{q,r}$ as the open interval $(q, r) = \{x \in \mathbb{R} \mid q < x < r\}$, and hence a frame homomorphism $\alpha : \Omega[T_{\mathbb{R}}] \to \Omega\mathbb{R}$. Clearly it is onto, since the open intervals $(q, r)$ form a base of opens. It is also 1-1, for suppose $\phi$ and $\psi$ are elements of $\Omega[T_{\mathbb{R}}]$ (geometric formulae) such that $\alpha(\phi) \subseteq \alpha(\psi)$. We show that $\phi \vdash \psi$. Any finite meet of symbols $P_{q,r}$ is a join of such symbols, and it follows that any formula $\phi$ is equivalent to a join of such symbols. Hence it suffices to show that if $\alpha(P_{q,r}) \subseteq \alpha(\psi)$ then $P_{q,r} \vdash \psi$. From Proposition 8.17, it suffices to show that if $\alpha(P_{q,r}) \subseteq \alpha(\psi)$ and $q < q' < r' < r$ then $P_{q',r'} \vdash \psi$. Let $S = \{s \in \mathbb{Q} \mid q' \leq s \leq r' \text{ and } P_{q',s} \vdash \psi\}$. $S$ is non-empty (because $q' \in S$) and bounded above (by $r'$), and so it has a supremum, a real number $x$. Since $q' \leq x \leq r'$ we have $x \in \alpha(P_{q,r})$ and so $x \in \alpha(\psi)$. Since $\psi$ too is a join of symbols $P_{t,u}$, we can find one such that $P_{t,u} \vdash \psi$ and $x \in \alpha(P_{t,u})$, i.e. $t < x < u$. If $t \leq q'$, then $P_{q',u} \vdash \psi$. On the other hand, suppose $q' < t$. Choose a rational $t'$ with $t < t' < x$. Then by definition of $x$ we have $P_{q',t'} \vdash \psi$. Again, but this time using Proposition 8.17, we get $P_{q',u} \vdash \psi$. It follows that $\min(r', u) \in S$, so $x \leq \min(r', u) \leq x$. Since $x < u$, it follows that $x = r' < u$, so $P_{q',r'} \vdash \psi$.

It follows that the locale $[T_{\mathbb{R}}]$ is spatial, with $\Omega[T_{\mathbb{R}}] \cong \Omega\mathbb{R}$. However, the proof just given is classical, in particular in its assumption that a non-empty set of rationals, bounded above, has a real-valued supremum. There are

(Fourman and Hyland, 1979) non-classical examples where $[T_{\mathbb{R}}]$ is not spatial. This might be seen as a defect of the locale $[T_{\mathbb{R}}]$, but in fact this non-spatial locale has better constructive behaviour than the space $\mathbb{R}$. For example, the Heine-Borel Theorem holds for the locale but not, in general, for the space (Fourman and Grayson, 1982).

## 3. Predicate geometric logic

In a sense, the propositional geometric logic is all that is needed for treating "topologies as Lindenbaum algebras". However, there are good reasons for extending these ideas to the case of predicate logic.

The first is Grothendieck's discovery that there are certain situations that involve topology and continuity, but where topological spaces are inadequate for expressing them. He invented toposes to cover these situations, and said, "toposes are generalized topological spaces". The generalization is essentially that from propositional to predicate geometric logic, and the toposes are a categorical version of Lindenbaum algebra appropriate to this predicate case. Thus it would be more accurate to say that toposes are generalized locales. Again we can view the topos as a space of models, but in general there are not enough opens to define the generalized topological structure, and sheaves must be used instead. Another point of generalization is that the classes $\mathrm{Map}(X, Y)$ are no longer posets but have category structure—the specialization order is replaced by specialization morphisms. (Specialization morphisms between points correspond to homomorphisms between models.)

A second reason for studying the predicate logic is that quite often it is natural to replace a propositional geometric theory by an equivalent predicate theory. The reason this is possible in any but the most trivial cases is a remarkable consequence of having infinitary disjunctions. These allow sorts to be characterized uniquely up to isomorphism as, for example, the natural numbers. This means that there is an intrinsic type theory in predicate geometric logic, and one can work not so much in geometric *logic* as in a geometric *mathematics*, which turns out to have an intrinsic continuity.

To fix our logical terminology, we say that a *many-sorted, first-order signature* has a set of sorts, a set of predicate symbols, and a set of function symbols. Each predicate or function symbol has an *arity* stipulating the number and sorts of its arguments, and (for a function) the sort of its result. A predicate symbol with no arguments is *propositional*, while a function with no arguments is a *constant*. We shall express the arities of predicates and functions thus:

$$\begin{array}{ll} P \subseteq A_1, ..., A_n & \text{(for a predicate)} \\ P \subseteq 1 & \text{(for a proposition)} \\ f : A_1, ..., A_n \to B & \text{(for a function)} \\ c : B & \text{(for a constant)} \end{array}$$

We shall also freely use vector notation, writing e.g. $\vec{A}$ instead of $A_1, \ldots, A_n$. In many situations, as here, this is to be understood as representing a product.

DEFINITION 8.18 *Let $\Sigma$ be a many-sorted, first-order signature. If $\vec{x}$ is a (finite) vector of distinct variables, each with a given sort, then a* geometric formula *over $\Sigma$ in* context *$\vec{x}$ is a formula built up using term formation from the variables $\vec{x}$ and the function symbols of $\Sigma$, and formula formation from the terms and the predicate symbols from $\Sigma$ using $=$, $\wedge$, $\top$, $\bigvee$ (possibly infinitary) and $\exists$. Note that, even with infinitary disjunctions, a formula is allowed only finitely many free variables, since they all have to be taken from the finite context $\vec{x}$. Not all the variables in the context have to be used in the formula.*

*A* geometric theory *over $\Sigma$ is a set of axioms of the form*

$$(\forall \vec{x})\ (\phi \rightarrow \psi)$$

*where $\phi$ and $\psi$ are geometric formulae over $\Sigma$ in context $\vec{x}$. (We shall also commonly write such axioms in sequent form $\phi \vdash_{\vec{x}} \psi$.)*

*A geometric theory is* coherent *if all disjunctions used in it are finitary. (Note that Mac Lane and Moerdijk, 1992, X.3 uses the word "geometric" to mean "coherent".)*

When we need to make explicit reference to the context of a term or formula, we shall use notation such as $(\vec{x}.t)$ or $(\vec{x}.\phi)$.

DEFINITION 8.19 *Let $T_1$ and $T_2$ be two geometric theories. A* theory morphism *$F$ from $T_1$ to $T_2$ comprises the following data.*

1. *To each sort $A$ of $T_1$, there is assigned a sort $F(A)$ of $T_2$. After this, each arity $\alpha$ for $T_1$ can be translated to an arity $F(\alpha)$ for $T_2$.*

2. *To each function symbol $f$ of $T_1$, with arity $\alpha$, there is assigned a function symbol $F(f)$ of $T_2$, with arity $F(\alpha)$. After this, each term in context $(\vec{x}.t)$ for $T_1$ can be translated to a term in context $(\vec{x}.F(t))$ for $T_2$.*

3. *Similarly, to each predicate symbol $P$ of $T_1$, with arity $\alpha$, there is assigned a predicate symbol $F(P)$ of $T_2$, with arity $F(\alpha)$. After this, each formula in context $(\vec{x}.\phi)$ for $T_1$ can be translated to a formula in context $(\vec{x}.F(\phi))$ for $T_2$.*

4. *To each axiom $(\forall \vec{x})\ (\phi \rightarrow \psi)$ of $T_1$, there is an axiom $(\forall \vec{x})\ (F(\phi) \rightarrow F(\psi))$ of $T_2$.*

Note—theory morphisms are presentation-dependent, and do not provide the general notion of map.

## 3.1 Logical rules

For the logical rules of predicate geometric logic we again follow the account in Johnstone, 2002b, D1.3.1. They are expressed using sequents of the form $\phi \vdash_{\vec{x}} \psi$ where $\phi$ and $\psi$ are formulae in context $\vec{x}$.

Labelling the turnstile with the context allows us to give a clean treatment of empty carriers, following Mostowski (see Lambek and Scott, 1986). This is exemplified by the following non-geometric deduction:

$$\begin{array}{l} (\forall x)\ \phi(x) \\ \phi(a) \\ (\exists x)\ \phi(x). \end{array}$$

This purports to prove a sequent $(\forall x)\ \phi(x) \vdash (\exists x)\ \phi(x)$, but that is invalid with an empty carrier. The true conclusion is that we can make the inference *in the context* $a$, which we write $(\forall x)\ \phi(x) \vdash_a (\exists x)\ \phi(x)$. This is valid provided $a$ is interpreted. But from that we can not infer $(\forall x)\ \phi(x) \vdash (\exists x)\ \phi(x)$. (Some such device is necessary in constructive logic, where excluding empty carriers would be a serious problem. But it ought also to be better known in classical logic. See also Example 8.27.)

The rules of predicate geometric logic are those of the propositional logic (with the context labels added) together with the following: *substitution* is

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi(\vec{s}/\vec{x}) \vdash_{\vec{y}} \psi(\vec{s}/\vec{x})}$$

where $\vec{s}$ is a vector of terms in context $\vec{y}$, with sorts matching those of $\vec{x}$; the *equality* rules

$$\top \vdash_x x = x, \qquad (\vec{x} = \vec{y}) \wedge \phi \vdash_{\vec{z}} \phi(\vec{y}/\vec{x})$$

($\vec{z}$ has to include all the variables in $\vec{x}$ and $\vec{y}$, as well as those free in $\phi$); the *existential* rules

$$\frac{\phi \vdash_{\vec{x},y} \psi}{(\exists y)\phi \vdash_{\vec{x}} \psi}, \qquad \frac{(\exists y)\phi \vdash_{\vec{x}} \psi}{\phi \vdash_{\vec{x},y} \psi};$$

and the *Frobenius* rule

$$\phi \wedge (\exists y)\psi \vdash_{\vec{x}} (\exists y)(\phi \wedge \psi).$$

Note that the substitution rule justifies *context weakening*

$$\frac{\phi \vdash_{\vec{x}} \psi}{\phi \vdash_{\vec{x},y} \psi}.$$

In other words, a deduction in one context will still be valid if we add extra variables, though not if we remove unused variables (which is what was done in the example of $(\forall x)\ \phi(x) \vdash (\exists x)\ \phi(x)$).

| Syntax | Interpretation |
|---|---|
| sort $A$ | *carrier* set $\{M\|A\}$ |
| sort tuple $\vec{A} = (A_1, \ldots A_n)$ | $\{M\|\vec{A}\} = \prod_{i=1}^{n}\{M\|A_i\}$ |
| predicate $P \subseteq \vec{A}$ | subset $\{M\|P\} \subseteq \{M\|\vec{A}\}$ |
| proposition $P \subseteq 1$ | subset $\{M\|P\} \subseteq 1$ |
| function $f : \vec{A} \to B$ | function $\{M\|f\} : \{M\|\vec{A}\} \to \{M\|B\}$ |
| constant $c : B$ | element $\{M\|c\} \in \{M\|B\}$ |
| formula in context $(\vec{x}.\phi)$ | subset $\{M\|\vec{x}.\phi\} \subseteq \{M\|\sigma(\vec{x})\}$ |
| term in context $(\vec{x}.t)$ | function $\{M\|\vec{x}.t\} : \{M\|\sigma(\vec{x})\} \to \{M\|\sigma(t)\}$ |

*Figure 8.1.* Interpretations of syntactic elements.

## 3.2 Models

The notion of standard model (in sets) is as expected, except that we *allow empty carriers*. The logical rules, with the context attached to the turnstile, are designed to be sound for empty carriers. We shall also introduce a novel notation that is useful when dealing with more than one interpretation at the same time.

The interpretation of different syntactic elements is defined in Table 8.1. The notation $\sigma(t)$ denotes the sort of a term $t$, and similarly for a tuple of terms. Once the signature ingredients are interpreted (arbitrarily), the interpretation of terms and formulae in context follows structurally in an evident way, so that $\{M|\vec{x}.\phi\}$ is the set of value tuples (in $\{M|\vec{A}\}$) for which $\phi$ holds, and $\{M|\vec{x}.t\}$ yields a result in $\{M|\sigma(t)\}$ for any value tuple substituted for $\vec{x}$.

If $T$ is a geometric theory over $\Sigma$, then we say that an interpretation $M$ of $\Sigma$ is a *model* of $T$ if, for every axiom $\phi \vdash_{\vec{x}} \psi$ in $T$, we have $\{M|\vec{x}.\phi\} \subseteq \{M|\vec{x}.\psi\}$.

We also define a notion of homomorphism.

DEFINITION 8.20 *Let $\Sigma$ be a signature and let $M, N$ be two interpretations of $\Sigma$. Then a* homomorphism $h : M \to N$ *comprises a function $\{h|A\} : \{M|A\} \to \{N|A\}$ for each sort $A$, subject to the following conditions. We shall write $\{h|\vec{A}\}$ for the product function $\prod_i\{h|A_i\} : \{M|\vec{A}\} \to \{N|\vec{A}\}$. For each predicate $P \subseteq \vec{A}$ in $\Sigma$ and for each function $f : \vec{A} \to B$ we require*

$$\{M|P\} \subseteq \{h|\vec{A}\}^{-1}(\{N|P\})$$
$$\{h|B\} \circ \{M|f\} = \{N|f\} \circ \{h|\vec{A}\}.$$

*Informally, we may say for any suitable value tuples $\vec{a}$ in $M$, that if $P(\vec{a})$ holds in $M$, then $P(h(\vec{a}))$ holds in $N$; and that $f(h(\vec{a})) = h(f(\vec{a}))$.*

The two conditions, for predicates and functions, are not independent. If the function $f$ is instead described by its graph, then the predicate condition for the graph is equivalent to the function condition for $f$.

Obviously homomorphisms can be composed, and there are identity homomorphisms, and so for any theory $T$ we have a category $\mathrm{Mod}(T)$ of models of $T$.

PROPOSITION 8.21 *Let $\Sigma$ be a signature, let $M, N$ be two interpretations of $\Sigma$, and let $h : M \to N$ be a homomorphism.*

*1 Let $(\vec{x}.t)$ be a term in context. Then*

$$\{N|\vec{x}.t\} \circ \{h|\sigma(\vec{x})\} = \{h|\sigma(t)\} \circ \{M|\vec{x}.t\}.$$

*2 If $(\vec{x}.\phi)$ is any geometric formula in context, then $\{h|\sigma(\vec{x})\}$ restricts to a function*

$$\{h|\vec{x}.\phi\} : \{M|\vec{x}.\phi\} \to \{N|\vec{x}.\phi\}.$$

*Proof* Induction on the formation of terms and formulae. QED

The result relies fundamentally on the positivity of the logic. For a logic with negation, the homomorphism condition that we gave for predicates is not liftable through negation. For this reason in classical logic one may see a different notion of homomorphism.

REMARK 8.22 *Categorically, Definition 8.20 amounts to saying that for sorts $B$, sort tuples $\vec{A}$ and predicates $P \subseteq \vec{A}$, we have functors $|B\}$, $|\vec{A}\}$ and $|P\} : \mathrm{Mod}(T) \to \mathbf{Set}$ with natural transformations $|P\} \to |\vec{A}\}$; and for functions $f : \vec{A} \to B$ there are natural transformations $|f\} : |\vec{A}\} \to |B\}$. The Proposition says this extends to formulae and terms, with natural transformations $|\vec{x}.\phi\} \to |\sigma(\vec{x})\}$ and $|\vec{x}.t\} : |\sigma(\vec{x})\} \to |\sigma(t)\}$.*

If $F : T_1 \to T_2$ is a theory morphism, then for every model $M$ of $T_2$ there is a corresponding model $F^*M$ of $T_1$, defined by

$$\{F^*M|-\} = \{M|F(-)\}.$$

This is called the *$F$-reduct* of $M$. $F^*$ gives a functor from $\mathrm{Mod}(T_2)$ to $\mathrm{Mod}(T_1)$ (note the reversed direction!).

## 3.3 Cartesian theories

These provide some important examples of geometric theories. They also provide the setting for some key constructions of universal algebra (including initial and free algebras) that turn out to be "geometric" in nature (Sec. 3.4).

The best known and simplest amongst the Cartesian theories are the finitary algebraic theories, where "finitary" refers to the requirement that all operators should have finite arity. Note that, unlike Johnstone, 2002b, Definition D1.1.7(a), we allow them to be many-sorted.

DEFINITION 8.23 *A* finitary algebraic theory *is a geometric theory presented with no predicate symbols, and with axioms all of the form*

$$(\forall \vec{x})\ (\top \to s = t)$$

*where $s$ and $t$ are two terms in context $\vec{x}$. (In other words, the axioms are* equational laws.*)*

Cartesian theories generalize these, essentially by allowing operators to be partial. In that generality they are slightly difficult to formalize and have appeared in various guises. The definition here (following Johnstone, 2002b, D1.3.4) is due to Coste. Equivalent are the essentially algebraic theories of Freyd, the left exact theories and sketches (see Barr and Wells, 1984) and the quasi-equational theories of Palmgren and Vickers, 2007.

As the references make clear, Cartesian theories are intimately associated with *Cartesian categories*, i.e. categories with all finite limits. A functor between them that preserves all finite limits is a *Cartesian functor*.

Most of the examples of Cartesian theories in this chapter are in fact finitary algebraic. Readers may safely omit the following definition if they wish, and consider Theorems 8.25 and 8.26 in the finitary algebraic case.

DEFINITION 8.24 (JOHNSTONE, 2002B, D1.3.4) *Let $\Sigma$ be a many-sorted, first-order signature, and let $T$ be a coherent theory over it.*

*The formulae in context that are* Cartesian *relative to $T$ are as follows: atomic formulae, $\top$, equations and conjunctions of Cartesian formulae; and $(\vec{x}.(\exists y)\phi)$ provided $(\vec{x}, y.\phi)$ is Cartesian, and the following sequent is derivable from $T$:*

$$\phi \wedge \phi(y'/y) \vdash_{\vec{x}yy'} y = y'$$

*The theory $T$ is Cartesian if there is a well-founded partial order on its axioms, such that for every axiom $(\forall \vec{x})(\phi \to \psi)$, the formulae in context $(\vec{x}.\phi)$ and $(\vec{x}.\psi)$ are Cartesian relative to the previous axioms in $T$.*

The essential point of this definition is that existential quantification can be used only when it is provably unique. This allows a mechanism for dealing with partial operations, by replacing them by their graphs. For example, the theory of categories is Cartesian but not algebraic. Composition is partial, with $f \circ g$ defined iff the domain of $f$ is equal to the codomain of $g$.

As is well-known, every algebraic theory $T$ has an initial model—that is to say, the category $\mathrm{Mod}(T)$ has an initial object. A consequence of this is that reduct functors have left adjoints. This generalizes to Cartesian theories.

THEOREM 8.25 (INITIAL MODEL THEOREM) *Let $T$ be a Cartesian theory. Then $T$ has an initial model, in other words a model $M_0$ such that for every other model $M$ there is a unique homomorphism $M_0 \to M$.*

*Proof* In the case where $T$ is an algebraic theory, the initial model is got by taking all terms, and then factoring out a congruence generated from the equational laws. (The construction of Lindenbaum algebras, as propositions modulo provable equality, is a particular instance of this.) Palmgren and Vickers, 2007 show how a similar proof can cover Cartesian theories, by working in a logic of partial terms. The construction first takes all partial terms, and then factors out a partial congruence (not necessarily reflexive) of provable equality, in which self-equality of a term is equivalent to its being defined.

More traditional proofs rely on first forming a *syntactic category*, a Cartesian category $\mathcal{C}_T$ such that models of $T$ are equivalent to Cartesian functors from $\mathcal{C}_T$ to $\mathbf{Set}$ (Johnstone, 2002b, Theorem D1.4.7), and then appealing to Kennison's Theorem (Barr and Wells, 1984, Theorem 4.2.1). QED

THEOREM 8.26 (FREE MODEL THEOREM) *Let $T_1$ and $T_2$ be Cartesian theories, and let $F : T_1 \to T_2$ be a theory morphism. Then the reduct functor $F^* : \mathrm{Mod}(T_2) \to \mathrm{Mod}(T_1)$ has a left adjoint* $\mathrm{Free}_F$.

*Proof* Just to sketch the proof, let $M$ be a model of $T_1$. We can define a new Cartesian theory $T$ whose models are pairs $(N, f)$, where $N$ is a model of $T_2$ and $f : M \to F^*(N)$ is a homomorphism. $T$ is got by augmenting $T_2$ with constant symbols for the elements of $M$, and equations to say that the interpretation of those constants respects the structure of $T_1$. Then an initial model of $T$ can be taken for $\mathrm{Free}_F(M)$. QED

The best known examples are where $T_1$ and $T_2$ are both single-sorted algebraic, with $T_1$ the theory with (one sort and) no operators or laws. Its models are sets. There is a unique theory morphism $F : T_1 \to T_2$, and $F^*$ picks out the carrier but forgets all the algebraic structure. Then $\mathrm{Free}_F$ constructs the free $T_2$-model on a set.

Note that this theorem, and the initial model theorem on which it depends, in general rely critically on the fact that we allow empty carriers.

EXAMPLE 8.27 *Consider the algebraic theory with two sorts $A$ and $B$, two constants $s$ and $t$ of sort $B$, a unary operator $f : A \to B$ and axioms*

$$(\forall x : A)\ (\top \to s = f(x))$$
$$(\forall x : A)\ (\top \to t = f(x)).$$

*Its initial model $M_0$ has $\{M_0|A\} = \emptyset$, $\{M_0|B\} = \{s, t\}$.*

Examples like this are sometimes used in equational reasoning to suggest that "equality is not transitive if empty carriers are allowed". This is because the equational laws can be presented as $s = f(x)$ and $t = f(x)$, but in $M_0$ we cannot deduce $s = t$. In our treatment we see that the equalities are in context, and in effect, "equality in context $x$" *is* transitive. We can deduce $\top \vdash_x s = t$, which implies $s = t$ *provided* we can interpret the variable $x$ in $A$.

## 3.4 Geometric types

We have presented geometric theories using simple sorts that are declared in the signature and thereafter cannot be manipulated in any way. However, it would be an obvious convenience if we could perform mathematical constructions on those sorts to derive new ones. Following Johnstone, 2002b, D4.1 we shall use the word *type* for these generalized sorts, and reserve the word *sort* for what was declared in the signature.

An important feature of geometric logic is that its infinitary disjunctions allow us to characterize some type constructors uniquely up to isomorphism by using geometric structure and axioms.

A fundamental example is the natural numbers. Consider the geometric theory with a single sort $N$, constant $0$, unary operator $s$ and axioms

$$\begin{aligned}(\forall x : N)\ (s(x) = 0 \longrightarrow \bot)\\ (\forall x, y : N)\ (s(x) = s(y) \longrightarrow x = y)\\ (\forall x : N)\ \bigvee_{n \in \mathbb{N}} x = s^n(0).\end{aligned}$$

Here, $s^n(0)$ stands for the term $s(...(s(0))...)$ with $n$ occurrences of $s$. The notation $s^n$ is not a formal part of the logic, but a metasyntax used to describe the set of formulae over which the disjunction is taken. In any model, $N$ can and must be interpreted as a set that is isomorphic to the natural numbers, by a unique isomorphism under which the constant $0$ corresponds to the natural number $0$, and the function $s$ corresponds to the successor operation $n \longmapsto n+1$. Hence, modulo isomorphism, this theory is just a variant of the trivial theory with empty signature and no axioms.

In fact, just within the logic we can prove that $N$ has a universal property that characterizes the natural numbers: it is an initial model for the single-sorted algebraic theory of *induction algebras*, which has constant $\varepsilon$, unary operator $t$ and no axioms.

THEOREM 8.28 *Let a geometric theory have $N, 0, s$ as axiomatized above. Then, in any model, $N$ is interpreted as an initial induction algebra.*

*Proof* Let $(A, \varepsilon, t)$ be an induction algebra $A$. Of course $N$ is itself an induction algebra under $0$ and $s$; we must show there is a unique induction algebra homomorphism from $N$ to $A$. For uniqueness, suppose $f : N \to A$ is a homomorphism. We show that (in sequent form)

$$y = f(x) \vdash\dashv_{xy} \bigvee_{n \in \mathbb{N}} (x = s^n(0) \wedge y = t^n(\varepsilon)).$$

For $\dashv$, we have $\top \vdash f(s^n(0)) = t^n(\varepsilon)$ by induction on $n$. For $\vdash$, combine this with

$$\top \vdash_x \bigvee_{n \in \mathbb{N}} (x = s^n(0)).$$

For existence of $f$, we show that the formula in context

$$(xy.\,\Gamma) \equiv \bigvee_{n \in \mathbb{N}} (x = s^n(0) \wedge y = t^n(\varepsilon))$$

is the graph of an induction algebra homomorphism, that is to say

$$\begin{aligned}
\Gamma \wedge \Gamma(y'/y) &\vdash_{xyy'} y = y' \\
\top &\vdash_x (\exists y)\Gamma \\
\top &\vdash \Gamma(0, \varepsilon/x, y) \\
\Gamma &\vdash_{xy} \Gamma(s(x), t(y)/x, y).
\end{aligned}$$

The first two of these state that the relation $\Gamma$ is single-valued and total, and hence the graph of a function, and the remaining two state that the function preserves the induction algebra operations.

These are all easy except perhaps for the first. For that, we want

$$x = s^n(0) \wedge y = t^n(\varepsilon) \wedge x = s^{n'}(0) \wedge y' = t^{n'}(\varepsilon) \vdash_{xyy'} y = y'.$$

If $n = n'$ that is obvious, while for $n \neq n'$ we can prove $x = s^n(0) \wedge x = s^{n'}(0) \vdash \bot$. QED

Let us stress the fact that the proof was within the formality of geometric logic. Thus it will be valid not only for the standard semantics in sets, but also for other semantics such as (as we shall see) in sheaves.

As is well known, there can be no such characterization of the natural numbers in finitary logic. Our ability to do it in geometric logic derives from the power of the infinitary disjunctions, and this extends to other "geometric" type constructs.

One might respond to this power by regarding explicit geometric type constructions as unnecessary, since they can be reduced to first order logic. By contrast we shall instead feel justified in using an explicit geometric type theory, since it does not transcend the scope of geometric logic.

In the present state of our knowledge we do not have a formal type theory along these lines. Instead, we shall use types informally, using the geometric type constructs wherever sorts can occur, and also introducing any functions and predicates that are associated with those types. In Remark 8.34 we shall see a semantic characterization of which type constructs are geometric.

By *geometric type theory* we shall understand a geometric theory in which geometric type constructs are used in this way. They have the same expressive power as geometric theories. An interpretation must interpret those types and the associated predicates and functions in the intended way.

A *coherent type theory* is a geometric type theory in which all disjunctions are finite. These are stronger in expressive power than coherent theories, since the type constructs may implicitly use infinitary disjunctions for their justification.

The first example of geometric type constructor, and one of the simplest, is the Cartesian product. (In fact all finite categorical limits are geometric.) Other examples arise out of the following general principle. Suppose $F : T_1 \to T_2$ is a theory morphism between Cartesian theories. Then the free model construction $\mathrm{Free}_F$ is geometric. Note that this will usually construct not only types, but also functions. In the following example, the principle can be applied with both theories algebraic (possibly many sorted) – exercise! But in each case it is also possible to specify the type geometrically and give a geometric proof of its universal property.

EXAMPLE 8.29 *1 The* natural numbers $\mathbb{N}$.

*2 The* list type $A^*$ *over a type* $A$*. In* $\mathbf{Set}$ *its elements are finite lists of elements of* $A$*.* $A^*$ *is the free monoid (having associative binary operation with a 2-sided identity element) over* $A$*; see Johnstone, 2002a, A2.5.15 for another treatment.*

*3* Coproducts *(in* $\mathbf{Set}$*, disjoint unions).*

*4* Coequalizers *(or, more particularly, quotients of equivalence relations).*

One very important type constructor is the *finite powerset* $\mathcal{F}$. This is discussed extensively in Johnstone, 2002b, D5.4 under the notation of $K$. (Constructively, the particular notion of finiteness being used is *Kuratowski finiteness*. The Kuratowski finite subsets of a set $S$ are the elements of the $\cup$-subsemilattice of $\mathcal{P}S$ generated by the singletons. $S$ is Kuratowski finite if it is a Kuratowski finite subset of itself.) It is a geometric construction because $\mathcal{F}A$ is the free semilattice over $A$. (A *semilattice* is a monoid $(A, 0, \vee)$ in which $\vee$ is *commutative* ($x \vee y = y \vee x$) and *idempotent* ($x \vee x = x$).) Johnstone, 2002b, D5.4 gives an alternative description analogous to that of list objects; see also Vickers, 1999.

A particularly important feature of the finite power type is that it enables us to internalize universal quantification, *provided* it is finitely bounded. Suppose $(x : A, y : B.\ \phi)$ is a formula in context. Then so is $(S : \mathcal{F}A, y : B.\ (\forall x \in S)\ \phi)$. It is interpreted as follows. Consider $\{M|xy.\phi\}$ as a function from $\{M|A\}$ to $\mathcal{P}\{M|B\}$. The codomain of this is a semilattice under $\cap$, and so we get a semilattice homomorphism from $\mathcal{F}\{M|A\}$ to $\mathcal{P}\{M|B\}$. This transposes to a subset of $\{M|\mathcal{F}A \times B\}$, the interpretation of $(\forall x \in S)\ \phi$.

### 3.5 Dedekind sections

Each real number $x$ is characterized by its *Dedekind section*, two sets of rationals:

$$L = \{q \in \mathbb{Q} \mid q < x\},$$
$$R = \{r \in \mathbb{Q} \mid x < r\}.$$

(Variants of this are possible, with $\leq$ instead of $<$. But they do not yield a geometric theory.) The idea then is to *define* the real number $x$ to be the pair $(L, R)$ of subsets of $\mathbb{Q}$, and (for $q, r \in \mathbb{Q}$) *define* $q < x$ if $q \in L$ and $x < r$ if $r \in R$.

The pairs $x = (L, R)$ that arise in this way are characterized by the following properties.

1. There is some rational $q$ with $q < x$.
2. If $q < q' < x$ then $q < x$.
3. If $q < x$ then there is some rational $q'$ with $q < q' < x$.
4. There is some rational $r$ with $x < r$.
5. If $x < r' < r$ then $x < r$.
6. If $x < r$ then there is some rational $r'$ with $x < r' < r$.
7. It is impossible to have $q < x < q$.
8. If $q < r$ then either $q < x$ or $x < r$.

((8) looks more obvious contrapositively: if $r \leq x \leq q$ then $r \leq q$. But we are axiomatizing $<$, not $\leq$.)

We can rewrite this, though at some cost in clarity, to a coherent type theory Ded with two predicates $L, R \subseteq \mathbb{Q}$ and axioms

$$\begin{aligned}
&(\exists q : \mathbb{Q})\ L(q)\\
&(\forall q, q' : \mathbb{Q})\ (q < q' \wedge L(q') \longrightarrow L(q))\\
&(\forall q : \mathbb{Q})\ (L(q) \longrightarrow (\exists q' : \mathbb{Q})\ (q < q' \wedge L(q')))\\
&(\exists r : \mathbb{Q})\ R(r)\\
&(\forall r, r' : \mathbb{Q})\ (r' < r \wedge R(r') \longrightarrow R(r))\\
&(\forall r : \mathbb{Q})\ (R(r) \longrightarrow (\exists r' : \mathbb{Q})\ (r' < r \wedge R(r')))\\
&(\forall q : \mathbb{Q})\ (L(q) \wedge R(q) \longrightarrow \bot)\\
&(\forall q, r : \mathbb{Q})\ (q < r \longrightarrow L(q) \vee R(r))
\end{aligned}$$

To see that this is indeed a coherent type theory, we must show that $\mathbb{Q}$ is a geometric type construction, and in addition that $<$ (on rationals) is geometric. The standard construction of $\mathbb{Q}$ is in stages. First, the natural numbers $\mathbb{N}$ have already been mentioned as an example of a geometric type. The arithmetic operations of addition and multiplication can then be defined in the following way. Addition is the unique operation $+ : \mathbb{N} \times \mathbb{N}$ that satisfies

$$\begin{gathered}(\forall n : \mathbb{N})\ 0 + n = n \\ (\forall m, n : \mathbb{N})\ s(m) + n = s(m + n).\end{gathered}$$

Hence if the symbol $+$ is declared and those axioms are added to the theory, there is no change to the models – the operation $+$ is forced to be interpreted in the intended way. So its use as a standard mathematical symbol is shorthand for that declaration with axioms. Similarly, we can define all the relations $=, \neq, <, >, \leq$ and $\geq$ as subsets of $\mathbb{N} \times \mathbb{N}$. (In this case, we can even define them as operations $\mathbb{N} \times \mathbb{N} \rightarrow 2 = 1 + 1$. This is because the relations are decidable.) For instance, $<$ is the unique relation satisfying

$$\begin{gathered}(\forall n : \mathbb{N})\ 0 < s(n) \\ (\forall m : \mathbb{N})\ (m < 0 \rightarrow \bot) \\ (\forall m, n : \mathbb{N})\ (s(m) < s(n) \rightarrow m < n) \\ (\forall m, n : \mathbb{N})\ (m < n \rightarrow s(m) < s(n)).\end{gathered}$$

Next, the integers $\mathbb{Z}$ are got as a quotient of $\mathbb{N} \times \mathbb{N}$ by an equivalence relation $\sim_1$, defined by $(m, n) \sim_1 (m', n')$ iff $m + n' = m' + n$. The pair $(m, n)$ represents the integer $m - n$. Again, it is possible to define arithmetic and inequalities.

Finally, the rationals $\mathbb{Q}$ are got as a quotient of $\mathbb{Z} \times \{n \in \mathbb{N} \mid n \neq 0\}$ by an equivalence relation $\sim_2$, defined by $(p, q) \sim_2 (p', q')$ iff $pq' = p'q$. (The pair $(p, q)$ represents the rational $p/q$.) The inequality $<$ is defined by $(p, q) < (p', q')$ if $pq' < p'q$.

So we see that $\mathbb{Q}$ and much of its accompanying structure are all geometric and can be used as needed in coherent type theories. This is emphatically *not* the case with the reals $\mathbb{R}$. As a set, $\mathbb{R}$ is described as a subset of $\mathcal{P}\mathbb{Q} \times \mathcal{P}\mathbb{Q}$, and the powerset constructor $\mathcal{P}$ is *not* amongst the geometric type constructors. (See Remark 8.34.) That is why we have to access the reals in a different way, by defining a theory whose models they are.

In Sec. 2.5 we saw the localic reals, given by a propositional geometric theory. In fact (Sec. 4.7), that geometric theory is equivalent to this one, despite the fact that one is propositional and the other is predicate with type constructs. In Sec. 4.6 we shall see how Ded retains a propositional character from the fact that it has no sorts declared. Its types are all constructed out of nothing.

# 4. Categorical logic

For propositional logic, the standard semantics interprets propositions as truth values. For a more general semantics, we interpreted propositions as elements of more general lattices—Boolean algebras for classical logic, frames for geometric logic. Then the Lindenbaum algebra was the lattice (of the appropriate kind) freely generated by a generic model.

For predicate logic, the standard semantics is in sets. Categorical logic generalizes this by interpreting the symbols in a category, of a kind appropriate to the logic, and then the analogue of the Lindenbaum algebra is a category. For geometric logic, the appropriate categories are Grothendieck toposes and the Lindenbaum algebra for a predicate geometric theory is the classifying topos. Our aim now is to explore how the technology of Lindenbaum algebras (as used for locales) extends to this setting.

We start with an introduction to categorical logic, following Johnstone, 2002b. A more elementary introduction can be found in Goldblatt, 1979.

## 4.1 Interpreting logic in a category

We assume that the reader has an elementary knowledge of category theory, including the basic definition, limits and colimits, and adjunctions.

We shall normally write composition of morphisms in applicative order, using "$\circ$". However, on occasion it will be convenient to use diagrammatic order instead, with ";". Thus the composition of morphisms

$$\xrightarrow{f}\xrightarrow{g}$$

will be written usually as $g \circ f$, but occasionally as $f; g$.

Suppose $\Sigma$ is a many-sorted, first-order signature. The usual notion of interpretation of $\Sigma$ in the category $\mathbf{Set}$ of sets, as given in Table 8.1, can be extended to *any* category $\mathcal{C}$ with finite products. Sets, functions, elements and subobjects become objects, morphisms, morphisms with domain $1$ (a terminal object, i.e. nullary product) and subobjects in $\mathcal{C}$.

If $A$ is an object in $\mathcal{C}$, we shall write $\mathrm{Sub}_{\mathcal{C}}(A)$ (or often just $\mathrm{Sub}(A)$) for the class of subobjects of $A$. We shall generally assume also that $\mathcal{C}$ is *well-powered*, i.e. that each $\mathrm{Sub}_{\mathcal{C}}(A)$ is a set. In our situations $\mathrm{Sub}_{\mathcal{C}}(A)$ is a meet semilattice, with greatest lower bounds given by pullbacks of subobjects. If $f : A \to B$ is a morphism, then pullback gives a meet semilattice homomorphism $f^* : \mathrm{Sub}_{\mathcal{C}}(B) \to \mathrm{Sub}_{\mathcal{C}}(A)$, called *inverse image*. (Exercise: in $\mathbf{Set}$ that is exactly what it is.)

As before, any interpretation of the ingredients of $\Sigma$ in $\mathcal{C}$ can (given suitable categorical structure in $\mathcal{C}$) be extended recursively to terms and formulae in context. We shall assume initially that $\mathcal{C}$ has at least all finite limits, in other words that it is *Cartesian*. First, we deal with terms.

*Variables:* $(x.x)$ is a term in context, interpreted by the identity morphism on $\{M|\sigma(x)\}$.

*Substitution:* Suppose $(\vec{x}.t)$ is a term in context, and $(\vec{w}.\vec{s})$ is a vector of terms in context that is type compatible with $\vec{x}$ (i.e. $\sigma(\vec{s}) = \sigma(\vec{x})$). (Note: we have the same context $\vec{w}$ for every component of $\vec{s}$.) Then $(\vec{w}.t(\vec{s}/\vec{x}))$ is a term in context. Its interpretation is given by

$$\begin{aligned}&\langle\{M|\vec{w}.\vec{s}\}\rangle\,;\{M|\vec{x}.t\}:\\ &\qquad\{M|\sigma(\vec{w})\}\to\{M|\sigma(\vec{x})\}\to\{M|\sigma(t)\}\end{aligned}$$

where $\langle\{M|\vec{w}.\vec{s}\}\rangle$ denotes the product tupling $\langle\{M|\vec{w}.s_1\},...,\{M|\vec{w}.s_n\}\rangle$ of morphisms.

Substitution also covers *context weakening*. Suppose $(\vec{x}.t)$ is a term in context, and $w$ is a variable not in $\vec{x}$. Then $(\vec{x},w.\ \vec{x})$ is a vector of terms in context, and the substitution $(\vec{x},w.\ t(\vec{x}/\vec{x}))$ gives a term in context $(\vec{x},w.\ t)$. For its semantics, note that $\langle\{M|\vec{x},w.\ \vec{x}\}\rangle$ is the product projection $\{M|\sigma(\vec{x})\}\times\{M|\sigma(w)\}\to\{M|\sigma(\vec{x})\}$.

Now we deal with formulae. Exercise! Check that these all make sense in $\mathbf{Set}$.

*Substitution:* Suppose $(\vec{x}.\phi)$ is a formula in context, and $(\vec{w}.\vec{t})$ is a vector of terms in context such that $\sigma(\vec{t}) = \sigma(\vec{x})$. Then $(\vec{w}.\phi(\vec{t}/\vec{x}))$ is a formula in context given by an inverse image (a pullback),

$$\{M|\vec{w}.\phi(\vec{t}/\vec{x})\} = \left\langle\{M|\vec{w}.\vec{t}\}\right\rangle^*(\{M|\vec{x}.\phi\}).$$

Again, this also covers *context weakening*. If $(\vec{x}.\phi)$ is a formula in context, and $w$ is a variable not in $\vec{x}$, then $(\vec{x},w.\ \phi)$ is also a formula in context given by substituting $\vec{x}$ for $\vec{x}$.

*Equality:* Let $(\vec{x}.t)$ and $(\vec{x}.t')$ be two terms in context, with $\sigma(t) = \sigma(t')$. Then $(\vec{x}.\ t = t')$ is a formula in context interpreted by an equalizer

$$\{M|\vec{x}.\ t=t'\}\hookrightarrow\{M|\sigma(\vec{x})\}\ \underset{\{M|\vec{x}.t'\}}{\overset{\{M|\vec{x}.t\}}{\rightrightarrows}}\ \{M|\sigma(t)\}.$$

*Conjunction:* Let $(\vec{x}.\phi)$ and $(\vec{x}.\psi)$ be two formulae in context. Then the conjunction rules imply that $(\vec{x}.\phi\wedge\psi)$ must be interpreted by the greatest lower bound of subobjects $\{M|\vec{x}.\phi\}$ and $\{M|\vec{x}.\psi\}$ in $\{M|\sigma(\vec{x})\}$. For $\top$, the nullary conjunction, we have $\{M|\vec{x}.\top\} = \{M|\sigma(\vec{x})\}$.

For the remaining geometric connectives, $\bigvee,\bot,\exists$, we need extra structure on the category $\mathcal{C}$.

For *existential quantification*, $\mathcal{C}$ must have *images*: the image of $f:A\to B$, if it exists, is the smallest subobject of $B$ through which $f$ factors. A consequence of $\mathcal{C}$ having images (for all morphisms) is that there are "direct image" functions $\exists_f : \mathrm{Sub}(A) \to \mathrm{Sub}(B)$, left adjoint to $f^*$. (In **Set** the adjunction appears as $f(S) \subseteq T$ iff $S \subseteq f^{-1}(T)$.) As was first understood by Lawvere, this is exactly the content of the existential rules. Hence if $(\vec{x}, y.\ \phi)$ is a formula in context, so is $(\vec{x}.\ (\exists y)\ \phi)$ and it is interpreted by

$$\{M|\vec{x}.\ (\exists y)\ \phi\} = \exists_\pi(\{M|\vec{x}, y.\ \phi\})$$

where $\pi : \{M|\sigma(\vec{x})\} \times \{M|\sigma(y)\} \to \{M|\sigma(\vec{x})\}$ is the product projection.

For *disjunction,* the disjunction rules imply that if $(\vec{x}.\phi_i)$ are formulae in context, then $\{M|\vec{x}.\bigvee_i \phi_i\}$ has to be the least upper bound of the subobjects $\{M|\vec{x}.\phi_i\}$ in $\{M|\sigma(\vec{x})\}$. Hence our categorical structure must include those least upper bounds. *False* is just a nullary disjunction. $\{M|\vec{x}.\bot\}$ must be the least subobject of $\{M|\sigma(\vec{x})\}$.

*Negation, implication, universal quantification:* These connectives enter geometric logic only at the level of axioms, and we interpret them in a different way. Suppose we have formulae in context $(\vec{x}.\phi)$ and $(\vec{x}.\psi)$. Then an interpretation $M$ *satisfies the axiom* $(\forall \vec{x})\ (\phi \to \psi)$, symbolically

$$M \vDash (\forall \vec{x})\ (\phi \to \psi),$$

if $\{M|\vec{x}.\phi\} \leq \{M|\vec{x}.\psi\}$. Negation can also be treated this way, by taking $\neg\phi$ as $\phi \to \bot$.

As usual, if $T$ is a theory over signature $\Sigma$, then $M$ is a model of $T$ if it satisfies every axiom in $T$.

At this point it is very useful to consider the notion of *generalized element*, analogous to generalized points.

DEFINITION 8.30 *Let $\mathcal{C}$ be a category and $A$ an object in it. A* generalized element *of $A$ is any morphism whose codomain is $A$. The domain of the morphism is called the* stage of definition *of the generalized element.*

*A generalized element at stage 1 is a* global element.

Consider for instance the above assertion $M \vDash (\forall \vec{x})\ (\phi \to \psi)$. We might like this to mean that every element of $\{M|\sigma(\vec{x})\}$ that satisfies $\phi$ (i.e. it factors via $\{M|\vec{x}.\phi\}$) also satisfies $\psi$. But this is only a weak assertion if there is a shortage of morphisms from 1 to $\{M|\sigma(\vec{x})\}$.

Our interpretation of the assertion $M \vDash (\forall \vec{x})\ (\phi \to \psi)$ can now be explained naturally in terms of generalized elements, for it says that every *generalized* element of $\phi$ is also in $\psi$. To see this one way round, consider the inclusion $\{M|\vec{x}.\phi\} \to \{M|\sigma(\vec{x})\}$. This is a generalized element (with stage of definition $\{M|\vec{x}.\phi\}$) that satisfies $\phi$. In fact it is the *generic* element of $\phi$ in $M$. If it is also to satisfy $\psi$, then we get $\{M|\vec{x}.\phi\} \leq \{M|\vec{x}.\psi\}$. Conversely, every generalized element of $\phi$ factors through the generic element, so if this is in $\psi$ so too is every generalized element of $\phi$.

Normally, when we intend our language to be interpreted in categories like this then by "element" we shall mean generalized element.

Homomorphisms between interpretations are defined just as in Definition 8.20, modulo obvious changes—the carrier functions $\{f|A\}$ become morphisms, $\{h|\vec{A}\}^{-1}$ becomes $\{h|\vec{A}\}^{*}$, subset inclusion $\subseteq$ becomes subobject inclusion $\leq$, etc. Proposition 8.21 still holds, by induction on the formation of terms and formulae.

Again we have a category $\mathrm{Mod}_{\mathcal{C}}(T)$ of models of $T$ in $\mathcal{C}$, and for any theory morphism $F : T_1 \to T_2$, we have an *F-reduct* functor $F^* : \mathrm{Mod}_{\mathcal{C}}(T_2) \to \mathrm{Mod}_{\mathcal{C}}(T_1)$.

## 4.2 Grothendieck toposes

To interpret geometric logic categorically, we shall use *Grothendieck toposes*. These are usually defined as "categories of sheaves over Grothendieck topologies", but that is in effect referring to a representation theorem, Giraud's Theorem (8.67). This says that a category is equivalent to such a category of sheaves iff it has certain structure and properties. Since the structure and properties can be related directly to the geometric logic, we shall use it as our definition. It can be described in various ways; our presentation here is the *$\infty$-pretopos with separating set of objects* of Johnstone, 2002b, Theorem C2.2.8 (vii) and Johnstone, 2002a, A1.4.

DEFINITION 8.31 *A category $\mathcal{E}$ is a* Grothendieck topos *if it has the following properties.*

1 *$\mathcal{E}$ has all finite limits.*

2 $\mathcal{E}$ has images *(p. 458) and image factorization is preserved by pullback.*

3 *$\mathcal{E}$ is* well-powered *(i.e. for every object $A$, the class $\mathrm{Sub}_{\mathcal{E}}(A)$ is a set).*

4 *For each object $A$, the poset $\mathrm{Sub}_{\mathcal{E}}(A)$ has arbitrary joins (least upper bounds), and they are preserved by pullbacks $f^*$.*

5 *Every set-indexed family of objects of $\mathcal{E}$ has a disjoint coproduct. (A coproduct $A = \sum_i A_i$ is* disjoint *if all the coproduct injections $A_i \to A$ are monic, and the meet of $A_i$ and $A_j$ in $\mathrm{Sub}_{\mathcal{E}}(A)$ is less than $\bigvee\{A_k \mid k = i \text{ and } k = j\}$. Hence if $i \neq j$ then $A_i$ and $A_j$ are disjoint subobjects of* $\mathrm{Sub}_{\mathcal{E}}(A)$.*)*

6 *Every equivalence relation $R \rightrightarrows A$ in $\mathcal{E}$ is a kernel pair. (Fuller details can be found in Mac Lane and Moerdijk, 1992, Appendix, Theorem 1 or Johnstone, 2002a, A1.3.6. If a pair of morphisms $a, b : R \to A$ is such that $\langle a, b\rangle : R \to A \times A$ is monic, then they can be thought of as a relation*

*on $A$. Then the usual notions of reflexive, symmetric and transitive can be translated into categorical terms by the usual logical interpretation. Our condition then says there is a* quotient *morphism $q : A \to B$ such that $(a, b)$ is the kernel pair of $q$, i.e. they complete the pullback square of $q$ pulled back against itself.)*

7 *$\mathcal{E}$ has a* separating *set $S$ (not a proper class) of objects (i.e. if $f, g : A \to B$ are such that for every $u : C \to A$ with $C \in S$ we have $f \circ u = g \circ u$, then $f = g$).*

Condition (1) says that $\mathcal{E}$ is *Cartesian*, and enables us to interpret the logic of conjunction and equality, as well as substitution.

Adding condition (2) makes $\mathcal{E}$ *regular* (Johnstone, 2002a, A1.3.3) and enables us to interpret the logic of $\exists$; preservation under pullback gives the Frobenius rule. It also enables the technique (e.g. Theorem 8.28) of defining a morphism by its graph.

Adding conditions (3) and (4) makes $\mathcal{E}$ *geometric* (Johnstone, 2002a, A1.4.18) and enables us to interpret the logic of arbitrary disjunction; preservation under pullback gives the frame distributivity rule. This structure is already sufficient for interpreting all the first-order part of geometric logic.

Adding condition (5) makes $\mathcal{E}$ $\infty$*-positive*, and then adding (6) makes it an $\infty$*-pretopos* (Johnstone, 2002a, A1.4.19). These enable us to interpret the geometric type theory. In particular, the Initial and Free Model Theorems 8.25 and 8.26 still work for models of Cartesian theories in $\mathcal{E}$. The proofs can be checked to go through; more explicitly, the proof in Palmgren and Vickers, 2007 is valid in Heyting pretoposes, and that includes Grothendieck toposes.

Condition (7) is a "smallness" condition. It allows us to deduce—in Giraud's Theorem—that although $\mathcal{E}$ is large, it can still be generated from a small structure.

We wish to find the appropriate notion of Lindenbaum algebra, in the form of a Grothendieck topos, for a geometric theory. This will be called the *classifying topos* for the theory. The central result for an ordinary Lindenbaum algebra was Proposition 8.3. We shall replace "Boolean algebra" by "Grothendieck topos", but we also need to know the appropriate notion of "homomorphism of Grothendieck topos".

All the structure needed for geometric logic and type theory (including the image factorization and the joins of subobjects) can be constructed using finite limits and arbitrary colimits, and conversely those are geometric type constructs (characterizable uniquely up to isomorphism by geometric structure and axioms). We shall therefore be interested in functors between toposes that preserve colimits and finite limits. A functor preserves colimits if it has a right adjoint, and for Grothendieck toposes the converse can also be shown

(Johnstone, 2002b, Remark C2.2.10). A *geometric morphism* is an adjoint pair of functors between toposes for which the left adjoint (which preserves all colimits) preserves finite limits.

At this point we are going to introduce some non-standard notation arising out of the fundamental split personality of toposes—spatial (generalized spaces) or logical (generalized universes of sets). The real interest of geometric morphisms is that they are the topos analogue of continuous map: they are a notion from the spatial side. In fact, we shall often refer to them as maps. On the other hand the Grothendieck toposes as we know them up till now, the categories with structure that was used to interpret logic, are very much the generalized universes of sets. We shall introduce notation that distinguishes between the two sides in the same way as we distinguished between locales and frames. Thus although technically we are dealing with those categories, we shall use notation that allows them to pretend to be spaces.

If we declare a symbol (e.g. $X$) to denote a topos, we shall nonetheless reserve its use for the topos in its spatial aspect. When we want to refer to it in its logical aspect, in other words the actual category, we shall write $\mathcal{S}X$. We shall call the objects of $\mathcal{S}X$ the *sheaves* over the topos rather than the objects of the topos. The symbol $\mathcal{S}$ can be read as standing for "sheaves".

We therefore define:

DEFINITION 8.32 *Let $X$ and $Y$ be two toposes. A* geometric morphism *(or* map*) $f : X \to Y$ is a pair of functors*

$$f^* : \mathcal{S}Y \to \mathcal{S}X$$
$$f_* : \mathcal{S}X \to \mathcal{S}Y$$

*such that $f^*$ is left adjoint to $f_*$ and $f^*$ preserves finite limits.*

*$f^*$ is called the* inverse image part*, and $f_*$ the* direct image part.

*We write* $\mathrm{Map}(X, Y)$ *for the class of geometric morphisms from $X$ to $Y$.*

Note the directions! The structure preserving functor is $f^*$, and this goes in the *reverse* direction to the geometric morphism $f$. As we shall see later, $f^*$ is analogous to the inverse image function for a continuous map between topological spaces. (However, the "direct image part" is not analogous to the direct image function.)

PROPOSITION 8.33 *Let $f : X \to Y$ be a geometric morphism. Then $f^*$ preserves free model constructions for Cartesian theories.*

*Proof* (See Johnstone, 2002b, D5.3.7 for the case of free algebras over sets for a single-sorted theory.) Let $\alpha : T_1 \to T_2$ be a theory morphism between two Cartesian theories. Let $A$ be a $T_1$-algebra in $\mathcal{S}Y$, and let $h : A \to \alpha^*(T_2\langle A\rangle)$

be a free $T_2$-algebra over it. ($h$ is a $T_1$-homomorphism.) It is required to show that $f^*(h) : f^*(A) \to f^*(\alpha^*(T_2\langle A\rangle))$ is a free $T_2$-algebra over $f^*(A)$.

For models of Cartesian theories, any functor that preserves finite limits will transform models to models. This applies to both $f^*$ and $f_*$. They also both preserve model reduction $\alpha^*$. Moreover, the adjunction of $f^*$ and $f_*$ extends to models: there is a bijection between homomorphisms $f^*(A) \to B$ in $\mathcal{S}X$ and homomorphisms $A \to f_*(B)$ in $\mathcal{S}Y$. If $B$ is a reduct $\alpha^*(B')$, then we see that $T_1$-homomorphisms $f^*(A) \to B$ are equivalent to $T_2$-homomorphisms $T_2\langle A\rangle \to f_*(B')$ and hence to $T_2$-homomorphisms $f^*(T_2\langle A\rangle) \to B'$. Hence $f^*(T_2\langle A\rangle)$ is the free $T_2$-model over $f^*(A)$, as required. QED

REMARK 8.34 *We can now state a general semantic characterization of "geometric type construct". They are those constructs that can be carried out in any Grothendieck topos, and are preserved by inverse image functors of geometric morphisms. (Remember that those inverse image functors are the analogues of homomorphisms between Lindenbaum algebras.) Those we have seen include* finite limits *(in set-theoretic terms: products, singletons (as terminal object), fibred products (pullbacks) and equalizers);* arbitrary colimits *(disjoint unions, quotients);* images*;* free model constructions *for theory morphisms between Cartesian theories (including the natural numbers, finite powersets and list objects); and* integers $\mathbb{Z}$ *and* rationals $\mathbb{Q}$*, and associated structure including arithmetic and inequalities.*

*These are type constructs that can be permitted, informally, in a geometric or coherent type theory.*

Our next result is the analogue of the specialization order on maps between locales (Definition 8.10).

THEOREM 8.35 *Let $X$ and $Y$ be toposes. Then:*

1. $\mathrm{Map}(X, Y)$ *is a category. The morphisms are called* specialization morphisms, *or* natural transformations. *If $\alpha$ is a specialization morphism from $f$ to $g$, then we write $\alpha : f \Rightarrow g$.*

2. *Composition with maps on either side is functorial.*

3. *Composition with maps satisfies the "interchange law". Suppose $\alpha : f \Rightarrow g$ in $\mathrm{Map}(X, Y)$, and $\beta : h \Rightarrow k$ in $\mathrm{Map}(Y, Z)$. Then the following diagram commutes.*

$$\begin{array}{ccc} h \circ f & \xrightarrow{\beta \circ f} & k \circ f \\ h \circ \alpha \downarrow & & \downarrow k \circ \alpha \\ h \circ g & \xrightarrow[\beta \circ g]{} & k \circ g \end{array}$$

*This allows us to define a* horizontal composition $\beta \circ \alpha : h \circ f \Rightarrow k \circ g$.

*Proof* 1. Mac Lane and Moerdijk, 1992, Sec. VII.1. A morphism $\alpha$ from $f$ to $g$, is defined as a natural transformation from $f^*$ to $g^*$. These are equivalent to natural transformations from $g_*$ to $f_*$. (Note the reversal of direction.)

2, 3. These are obvious and come from horizontal composition of natural transformations. (Mac Lane, 1971) QED

An important feature of maps is that we can take filtered colimits. These are a categorical generalization of the directed joins of locale maps (Proposition 8.11).

DEFINITION 8.36 *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is* filtered *if it satisfies the following conditions.*

1 *$\mathcal{C}$ has an object.*

2 *If $A$ and $B$ are objects of $\mathcal{C}$, then there is an object $C$ with morphisms $f : A \to C$ and $g : B \to C$.*

3 *If $A$ and $B$ are objects of $\mathcal{C}$, and $f, g : A \to B$, then there is an object $C$ and morphism $h : B \to C$ such that $h \circ f = h \circ g$.*

To put this more concisely, $\mathcal{C}$ is filtered iff every finite diagram in $\mathcal{C}$ has a cocone. A poset is filtered iff it is directed.

Composition with maps preserves filtered colimits, and this preservation of filtered colimits is an important property of maps, analogous to *Scott continuity*.

THEOREM 8.37 *Let $X$ and $Y$ be Grothendieck toposes. Then:*

1 $\mathrm{Map}(X, Y)$ *has all filtered colimits.*

2 *Composition with maps on either side preserves filtered colimits.*

*Proof* 1. Suppose we have a filtered diagram of maps $f_i$. Then $\mathrm{colim}_i\, f_i$ is calculated by

$$(\mathop{\mathrm{colim}}_i f_i)^*(B) = \mathop{\mathrm{colim}}_i(f_i^*(B))$$

Regardless of filteredness, this will preserve colimits. The filteredness ensures that it preserves finite limits, because filtered colimits commute with finite limits.

2. For precomposition by $g : W \to X$, this follows from the fact that $g^*$ preserves colimits. For postcomposition by $h : Y \to Z$, it is trivial. QED

## 4.3 Elementary toposes

Grothendieck toposes have the structure needed to interpret geometric logic. Surprisingly, they also turn out to have structure for interpreting full first-order logic and even higher-order logic, though that structure is not geometric—it is not preserved by inverse image functors. (In this it is like the Heyting arrow, which exists in frames but is not preserved by frame homomorphisms.) This led to a generalized notion of topos, the *elementary topos*, which embodies the finitary part of that fuller structure.

This structure allows $\neg$, $\rightarrow$ and $\forall$ as connectives for constructing formulae, and so allows the coherent axioms to be formulae. It is only with this step that the differences between classical and constructive logic become visible. Characteristically classical axioms such as

$$\begin{array}{ll} \neg\neg\phi \rightarrow \phi & \text{(double negation rule)} \\ \phi \vee \neg\phi & \text{(excluded middle)} \end{array}$$

cannot be stated in coherent form, since they require negation to be used as a connective.

A minimal definition uses the notion of *powerobject* $\mathcal{P}(A)$, an object whose elements at stage $B$ are the subobjects of $B \times A$ (so the global elements are the subobjects of $A$, like a powerset).

DEFINITION 8.38 *An* elementary topos *is a Cartesian category with a power object $\mathcal{P}(A)$ for every object $A$.*

The standard texts (Mac Lane and Moerdijk, 1992, IV.1, Johnstone, 2002a) show how much more structure can be deduced from this. In particular, an elementary topos is Cartesian closed: if $A$ and $B$ are objects, then there is a further object $B^A$, the *exponential*, whose elements at stage $C$ are in bijection with the morphisms from $C \times A$ to $B$. It also has finite colimits, and the subobject pullback functions $f^*$ have both left adjoints $\exists_f$—as required in Sec. 4.1 to interpret $\exists$—and right adjoints $\forall_f$—which are needed for $\forall$.

The powerobject $\mathcal{P}(1)$ is known as the *subobject classifier*, $\Omega$. Its elements at stage $A$ are the subobjects of $A$. In particular, its global elements are the subobjects of 1 and can be thought of as truth values. In $\mathbf{Set}$, $\Omega \cong 2$ where 2 is defined as the coproduct $1 + 1$. But, logically, this implies excluded middle, and does not hold in general. 2 is the object of *decidable* truth values.

Note that we obtain type constructors in elementary toposes that are *non-geometric*—not preserved by inverse image functors. Important examples include $\Omega$, powersets, function sets (exponentials) and the set of reals.

The existence of the subobject classifier has a big effect on the way the logic is interpreted. Subobjects of $A$ are now equivalent to their characteristic morphisms $A \rightarrow \Omega$, and so logical formulae can be interpreted as *terms of type*

$\Omega$. This is formalized in the *Mitchell-Bénabou language* (see Mac Lane and Moerdijk, 1992, VI.5). Moreover, the logical connectives are interpreted as operations on $\Omega$. This account of interpreting logic in toposes is rather different in appearance from the one we have described, though for coherent logic they are equivalent.

An elementary topos need not have a *natural numbers object* (characterized as initial induction algebra). However, it is of vital importance when it is present since then analogues of Theorems 8.25 and 8.26 will hold. Johnstone, 2002b, D5.3.5 covers the case of free algebras over sets for finitarily presented single-sorted algebraic theories.

## 4.4 Classifying toposes

We can now give the definition of classifying topos, as Lindenbaum algebra for predicate geometric theory. This is the analogue of Proposition 8.3, though note that we have replaced homomorphisms by "maps", going in the opposite direction.

Suppose $T$ is a geometric type theory, $X$ a Grothendieck topos and $M$ a model of $T$ in $\mathcal{S}X$. Then for every Grothendieck topos $W$ we have a functor

$$(-)^*(M) : \mathrm{Map}(W, X) \to \mathrm{Mod}_{\mathcal{S}W}(T)$$

that takes a map $f$ to the model $f^*(M)$. This is indeed a model, because $f^*$ preserves all the geometric structure used to define modelhood.

Note also that it is a *functor*—natural transformations between maps are taken to homomorphisms between models (exercise!).

DEFINITION 8.39 *Let $T$ be a geometric type theory. A* classifying topos *for $T$ is a Grothendieck topos $[T]$, equipped with a* generic *model $G$ of $T$ in $\mathcal{S}[T]$, such that for every Grothendieck topos $W$ the functor $(-)^*(G)$ is an equivalence of categories.*

We adapt the definition from Johnstone, 2002a, B4.2.1(b) with changes of notation.

Note the effect of having only an equivalence. The correspondence between models and maps is only up to isomorphism—if $M$ is a model in $\mathcal{S}W$, then there is some map $f : W \to [T]$ such that $M \cong f^*(G)$. However, given maps $f$ and $g$, there is a bijection between homomorphisms $f^*(G) \to g^*(G)$ and natural transformations $f \Rightarrow g$. A consequence of this is that the classifying topos itself is defined only up to categorical equivalence.

We now have an alternative reading to the symbol $\mathcal{S}$. If $\mathcal{S}$ stands for "sets", then $\mathcal{S}[T]$ can be read as "the category of sets with a model of $T$ freely adjoined". This is in line with some existing notation (Johnstone, 2002a, B4.2.1) and is analogous to notation such as $\mathbb{R}[X]$ for a polynomial ring.

There are some crucial results that cannot be proved without a closer examination of the structure of classifying toposes. We defer that to Sec. 5, and meanwhile look at the use of classifying toposes. The crucial results are:

- Every geometric theory has a classifying topos (Theorem 8.65).
- Every geometric type theory has a classifying topos (Theorem 8.66, with some restrictions on the generality).
- Every Grothendieck topos classifies some geometric theory (Theorem 8.67).
- For propositional geometric theories, the maps between the locales are equivalent to the maps between their classifying toposes (Theorem 8.71). Hence for these the locale and topos treatments are equivalent, and locales can be considered a special case of toposes.

Just as for locales, we define a *point* of a topos $X$ at *stage* $W$ to be a map from $W$ to $X$. Then maps act as point transformers by postcomposition.

The empty theory $(\emptyset, \emptyset)$ with no symbols and no axioms has a unique model in any category, given by the vacuous interpretation, so $\mathrm{Mod}_{\mathcal{S}W}(\emptyset, \emptyset)$ is the category with one object and one (identity) morphism. We write $1$ for its classifying topos.

PROPOSITION 8.40 $\mathcal{S}1 \simeq \mathbf{Set}$.

*Proof* Mac Lane and Moerdijk, 1992, Sec. VII.1. Every set $A$ is a coproduct of copies of the terminal object (singleton set), which we shall also write $1$. (There should be no confusion between the different 1s.) Hence for any Grothendieck topos $X$ an inverse image functor $!^* : \mathbf{Set} \to \mathcal{S}X$ has to take each set $A$ to a coproduct of an $A$-indexed family of copies of $1$. Moreover, any such functor preserves finite limits. (This is non-trivial, and relies on the properties of Grothendieck toposes.) The category of such functors is equivalent to the category with one object and one morphism. In other words, there does exist such a functor, and for any two such functors (with different choices of coproducts) there is a unique natural isomorphism between them. It is easy to show from the adjunction that if $B$ is an object of $\mathcal{S}X$ then $!_*(B)$ is isomorphic to the set of global elements of $B$, morphisms $1 \to B$. QED

As before, points at stage $1$ are called *global*. The global points of $[T]$ are equivalent to the models of $T$ in $\mathbf{Set}$.

Let $\mathbb{O}$ be the theory with one sort and no functions, predicates or axioms. Categorically, it is the theory of "objects", since a model in $\mathcal{S}X$ is just an object of $\mathcal{S}X$, and its classifying topos is called the *object classifier* (not to be confused with the subobject classifier $\Omega$ that exists in any elementary topos). In generalization of the terminology for spaces, we call the objects of $\mathcal{S}X$ the *sheaves* over $X$, and they are equivalent to maps from $X$ to $[\mathbb{O}]$. Since the global points of $[\mathbb{O}]$ are sets, our intuition is that $[\mathbb{O}]$ is "the space of sets", and in Sec. 5 we shall see why it is a reasonable intuition to think of a sheaf over $X$ as a continuous map from $X$ to a space of sets.

## 4.5 Maps between classifying toposes

Now that our "Lindenbaum algebras" are Grothendieck toposes, we can—as we have seen—interpret large amounts of ordinary mathematics internally in them. This makes them very different from the lattices we used for propositional logics, and this has a profound effect on the way we can use these logical techniques. It makes it possible to treat classifying toposes $[T]$ in a very spatial way.

Suppose $T_1$ and $T_2$ are two geometric theories. By definition of classifying toposes, a geometric morphism $f : [T_1] \to [T_2]$ is equivalent to a model $M$ of $T_2$ in $\mathcal{S}[T_1]$. Now all the objects and morphisms in $\mathcal{S}[T_1]$ are constructed out of the generic model $G$ of $T_1$, and indeed can be constructed using finite limits and arbitrary colimits. It follows that $M$ too has to be constructed out of the generic $T_1$-model. Let us portray this naively as a model transformation.

1 We declare "Let $G$ be a model of $T_1$."

2 We construct a model $M$ of $T_2$.

Within the scope of the declaration (1), our logic and mathematics are to be interpreted in $\mathcal{S}[T_1]$ with $G$ the generic $T_1$-model. This means it must be constructively valid. We thus have a temporary change of mathematics. Back outside the scope of the declaration, returning to our ambient mathematics, we find our model construction gives a geometric morphism $f : [T_1] \to [T_2]$.

The same technique also works for natural transformations. If we define *two* models $M$ and $M'$, and a homomorphism $\theta_G : M \to M'$, then that gives us two maps $f, f' : [T_1] \to [T_2]$ and a natural transformation $\theta : f \Rightarrow f'$.

On the face of it, in step (2) we could use any mathematics validly interpretable in $\mathcal{S}[T_1]$. For instance, we might use $\Omega$ or function types, since in fact $\mathcal{S}[T_1]$ is an elementary topos. However, there are good reasons for restricting to geometric constructions.

If we have a point $x$ of $[T_1]$ at stage $W$—that is to say, a model $x^*(G)$ of $T_1$ in $\mathcal{S}W$, $G$ being the generic model of $T_1$, then we can apply $f$ to it by composition and get a model $x^*(M)$ of $T_2$ in $\mathcal{S}W$, corresponding to $f \circ x$. If the construction of $M$ from $G$ (in $\mathcal{S}[T_1]$) is geometric, then it is preserved by $x^*$, and so the same construction constructs $x^*(M)$ out of $x^*(G)$. Hence the geometric construction works uniformly, not only for the generic point but for all points.

We therefore see that geometric morphisms between classifying toposes can be viewed as *geometric* model transformations.

It is the geometric working that enables us to view a topos spatially as comprehending all generalized points, because it allows us to transport our mathematics from one stage of definition to another along inverse image functors. Since, as we shall see later, geometric morphisms generalize continuous maps in topology, another way to view the role of the geometric constructions is that they have an intrinsic continuity.

This same view of map also provides a good way to think about generalized points. A point of $X$ at stage $Y$, in other words a map $Y \to X$ is conveniently thought of as a point of $X$ "parametrized by" a variable point of $Y$.

EXAMPLE 8.41 Reduct maps. *Let $F : U \to T$ be a theory morphism between geometric theories (Definition 8.19). Then every model of $T$ is trivially a model of $U$ by model reduction. This defines a* reduct map $\mathrm{Red}_F : [T] \to [U]$.

In fact *any* geometric morphism can be expressed as a reduct map.

THEOREM 8.42 *Let $f : [T] \to [U]$ be a geometric morphism. Then there is a geometric theory $T'$ equivalent to $T$ and with a theory morphism $F : U \to T'$ such that $f$ factors as $[T] \simeq [T'] \xrightarrow{\mathrm{Red}_F} [U]$.*

*Proof* In $\mathcal{S}[T]$ we have the generic model $G$ of $T$ and a model $M$ of $U$ given by $f$. The result can be proved using the conventional techniques of Sections 5.3 and 5.4. It appears in detail in Viglas, 2004. However, here is a more conceptual reason. Each ingredient of $M$ can be constructed from the ingredients of $G$ using colimits and finite limits. $T$ can be extended with sorts for such colimits or finite limits, together with structure and axioms to force then to be those colimits or limits. The extended theory is equivalent to $T$, since its models are determined up to unique isomorphism by their $T$-reducts. But there is also an obvious theory morphism from $U$. QED

From this one can easily deduce results such as the following.

PROPOSITION 8.43 *Let $f_i : Y_i \to X$ be a map between Grothendieck toposes ($i = 1, 2$). Then there is a pseudo-pullback square*

$$\begin{array}{ccc} Z & \xrightarrow{p_1} & Y_1 \\ p_2 \downarrow & \cong & \downarrow f_1 \\ Y_2 & \xrightarrow[f_2]{} & X \end{array}$$

*(A* pseudo-*pullback is like a pullback except that the square is required to commute only up to isomorphism.)*

*Proof* Suppose $Y_i$ and $X$ classify theories $T_i$ and $U$. We can factor each $f_i$ as an equivalence followed by a reduct map for a theory morphism $F_i : U \to T'_i$. Hence, we may assume that each $f_i$ is already a reduct map. Now define $P$ as follows. It has $T_1$ and $T_2$ put together disjointly, giving a theory morphism from each $T_i$. This now has two copies of $U$, the images of the two theory morphisms. Add function symbols and axioms to make mutually inverse homomorphisms between those two copies of $U$. Then a model of $P$ comprises a model $M_i$ of each $T_i$, and an isomorphism between their $U$-reducts, and this is exactly what is needed for the pseudo-pullback property for $Z = [P]$. QED

Hence, the points of $Z$ are equivalent to triples $(y_1, y_2, \theta)$ where each $y_i$ is a point of $Y_i$ and $\theta : f_1(y_1) \cong f_2(y_2)$ is an isomorphism. When $X = 1$, we get a *product* $Y_1 \times Y_2$ whose points are pairs of points from $Y_1$ and $Y_2$.

By similar means we can construct a topos $Z'$ whose points $(y_1, y_2, \theta)$ have $\theta : f_1(y_1) \to f_2(y_2)$ a homomorphism. The resulting square

$$\begin{array}{ccc} Z' & \xrightarrow{p'_1} & Y_1 \\ p'_2 \downarrow & \overset{\theta}{\Leftarrow} & \downarrow f_1 \\ Y_2 & \xrightarrow[f_2]{} & X \end{array}$$

is called a *comma square*. It does not commute, but has a natural transformation from $f_1 \circ p'_1$ to $f_2 \circ p'_2$.

Our next result shows vividly how geometric morphisms between classifying toposes can appear like functors between model categories—indeed, by taking points one extracts the functors. But as a geometric morphism, it carries extra information that it has continuity properties—for example, that it preserves filtered colimits. (This fact about geometric morphisms was exploited in Viglas, 2004 for proving that certain functors preserved filtered colimits.)

Its notion of adjunction between toposes is technically possible because the natural transformations make the category of toposes and maps into a 2-category. An adjunction between $X$ and $Y$ comprises maps $F : X \to Y$ and $G : Y \to X$ (the left and right adjoints), and natural transformations $\eta : \mathrm{Id}_X \Rightarrow G \circ F$ and $\varepsilon : F \circ G \Rightarrow \mathrm{Id}_Y$ such that the two composites

$$(F \circ \eta); (\varepsilon \circ F) : F \Rightarrow F \circ G \circ F \Rightarrow F$$
$$(\eta \circ G); (G \circ \varepsilon) : G \Rightarrow G \circ F \circ G \Rightarrow G$$

are both identities (cf. Mac Lane, 1971, IV.1 Theorem 2(v)).

This can all be worked through in terms of geometric transformations. However, Viglas, 2004 simplifies it greatly. Once the maps $F$ and $G$ have been defined, it suffices to use a geometric argument of the following form (where $x$ and $y$ are points of $X$ and $Y$ respectively). It is analogous to a more familiar characterization of adjunction, but the geometricity guarantees all the functoriality and naturality required.

- For each specialization (homomorphism) $\phi : x \Rightarrow G(y)$, define $\alpha(\phi) : F(x) \Rightarrow y$.
- For each $\psi : F(x) \Rightarrow y$, define $\beta(\psi) : x \Rightarrow G(y)$.
- Show $\beta(\alpha(\phi)) = \phi$ and $\alpha(\beta(\psi)) = \psi$.

THEOREM 8.44 *Let $F : T_1 \to T_2$ be a morphism between two Cartesian theories. Then the reduct map* $\mathrm{Red}_F : [T_2] \to [T_1]$ *has a left adjoint* $\mathrm{Free}_F : [T_1] \to [T_2]$.

*Proof* Constructing a free $T_2$-model over a $T_1$-model is geometric, and so the map $\mathrm{Free}_F$ is defined by saying for any $T_1$-model $M$, $\mathrm{Free}_F(M)$ is the free $T_2$-model over it.

The adjunction arises here because the corresponding adjunction within any Grothendieck topos is geometric. (This can be proved from Proposition 8.33.)

QED

## 4.6 Localic toposes

We now have two ways to deal with propositional geometric theories: as locales or as toposes (which in this case are called *localic*). Theorem 8.71 will show that locales and localic toposes are equivalent, but for the moment we look at some of the topos behaviour in its own right.

DEFINITION 8.45 *A topos is* localic *if it classifies a propositional geometric theory.*

*A geometric type theory is* essentially propositional *if it has no sorts.*

The theory Ded of Dedekind sections (Sec. 3.5) is essentially propositional.

We conjecture that the next result holds more generally, for instance when type constructs are applied to propositions (as subsingletons). However, our restricted proof is at least enough to cover our applications.

THEOREM 8.46 *Let $T$ be an essentially propositional geometric type theory whose types can all be constructed in the empty theory. Then $[T]$ is localic.*

*Proof* If $\tau$ is a type in $T$, it has an interpretation $[|\tau|]_X$ in any Grothendieck topos $\mathcal{S}X$. We write $[|\tau|]$ for $[|\tau|]_1$ (in $\mathbf{Set}$).

We may assume without loss of generality that $T$ is presented without any function symbols, but only predicates. This is because any function symbol can be replaced by a predicate for its graph, with axioms for single-valuedness and totality. We now show how $T$ may be converted into an equivalent propositional geometric theory $T'$.

The propositions of $T'$ are as follows. If $S \subseteq \tau_1 \times \ldots \times \tau_n$ is a predicate symbol in $T$, then for each $\vec{a} \in \prod_{i=1}^{n}[|\tau_i|]$ we introduce a proposition $\bar{S}_{\vec{a}}$. Now for each formula in context $(\vec{x}.\phi)$ in $T$, and for each $\vec{a} \in \prod_i [|\sigma(x_i)|]$, we define a formula $\bar{\phi}_{\vec{a}}$ in $T'$ by induction as follows.

1 $(\overline{\bigvee_j \phi_j})_{\vec{a}} = \bigvee_j (\overline{\phi_j})_{\vec{a}}$, and similarly for conjunctions.

2 $\overline{(x_i = x_j)}_{\vec{a}} = \bigvee\{\top \mid a_i = a_j\}$.

3 $(\overline{(\exists y)\ \phi})_{\vec{a}} = \bigvee\{\bar{\phi}_{\vec{a},b} \mid b \in [|\sigma(y)|]\}$.

Finally, for each axiom $(\forall \vec{x})\ (\phi \longrightarrow \psi)$ of $T$, we give $T'$ axioms $\bar{\phi}_{\vec{a}} \longrightarrow \bar{\psi}_{\vec{a}}$ $(\vec{a} \in \prod_i [|\sigma(x_i)|])$.

Note that, even if $T$ is a *coherent* type theory (no infinitary disjunctions) and *finitely presented* (only finitely many symbols and axioms), $T'$ is likely to have infinitely many symbols and axioms, and infinitary disjunctions.

We now show that $T$ and $T'$ are equivalent. In **Set**, $[|\tau|] \cong \sum_{a \in [|\tau|]} 1$, and since this is geometric it also holds in any $\mathcal{S}X$. It follows that subobjects of $[|\tau|]_X$ correspond to $[|\tau|]$-indexed families of subobjects of 1 in $\mathcal{S}X$. Hence structures for $T$ are equivalent to structures for $T'$. Now suppose that $M$ is a structure for $T$, and $M'$ the corresponding structure for $T'$. By structural induction on the formula $\phi$, one can then show that for any formula in context $(\vec{x}.\phi)$, the subobject $\{M|\phi\}$ of $\prod_i \{M|\sigma(x_i)\}$ corresponds to the family of subobjects $\{M'|\bar{\phi}_{\vec{a}}\}$ for $\vec{a} \in \prod_i [|\sigma(x_i)|]$. From this one deduces that $M$ is a model for $T$ iff $M'$ is a model for $T'$. QED

PROPOSITION 8.47 *Let $X$ be a localic topos and let $x$ and $x'$ be points of it. Then there is at most one homomorphism from $x$ to $x'$.*

*Proof* We can take $X = [T]$ where $T$ is propositional. But then with no sorts, a homomorphism needs to supply no carrier functions. The sole requirement is that if $P$ is a propositional symbol and $\{x|P\}$ holds (topologically, $x$ is in the open $P$), then so does $\{x'|P\}$. QED

Hence if $X$ is localic then $\mathrm{Map}(Y, X)$ is a preorder. We write $x \sqsubseteq x'$ if there is a homomorphism from $x$ to $x'$; this is called the *specialization order* on points. Later we shall prove that this agrees with the specialization order we have already defined for locales.

Sec. 4.5 showed that maps between toposes can be defined as geometric model transformations, and this still applies to localic toposes. But Theorem 8.71 will show that those also define maps between the locales. If the locales are spatial, Proposition 8.16 shows that we then get continuous maps between the spaces. Thus geometricity of the model transformation is enough to guarantee continuity. As a logical approach to continuity, geometric logic works by starting with ordinary logic and then *removing* the structure (e.g. negation) that makes it possible to define non-continuous functions. Compare this with other approaches, such as topology itself, or the modal logic of interior, that work by *adding* structure to support a bureaucracy of continuity proofs.

## 4.7 Example: the reals

We have now seen two geometric theories that purport to represent the real line. In Sec. 2.5, $T_{\mathbb{R}}$ was a propositional geometric theory described as the localic reals, while in Sec. 3.5 $\mathrm{Ded}$ was a predicate geometric theory whose models are the Dedekind sections. We now show that they are equivalent.

By the proof of Theorem 8.46 we see that $T_{\mathbb{R}}$ is equivalent to a theory $T'_{\mathbb{R}}$ with a single predicate symbol $P \subseteq \mathbb{Q}^2$, and axioms

$$P(q,r) \wedge P(q',r') \vdash\dashv_{qrq'r'} (\exists st)(P(s,t) \wedge \max(q,q') < s < t < \min(r,r'))$$
$$0 < \varepsilon \vdash_{\varepsilon} (\exists q) P(q-\varepsilon, q+\varepsilon)$$

Given a model of $T'_{\mathbb{R}}$, we define a Dedekind section $(L,R)$ geometrically by

$$L = \{q \in \mathbb{Q} \mid (\exists r)P(q,r)\}$$
$$R = \{r \in \mathbb{Q} \mid (\exists q)P(q,r)\}.$$

It is easy to see that this is a Dedekind section. If $q < r$, let $\varepsilon = (r-q)/2$ and find $s$ such that $P(s-\varepsilon, s+\varepsilon)$. If $q > s-\varepsilon$ and $r < s+\varepsilon$ then $q+\varepsilon > s > r-\varepsilon$ and $r-q < 2\varepsilon$, a contradiction. Hence either $q \leq s-\varepsilon \in L$ or $r \geq s+\varepsilon \in R$. (Note that the order on $\mathbb{Q}$ is decidable, so we can use this proof by contradiction.) Also, $P(q,r)$ holds iff $q \in L$ and $r \in R$.

Conversely, suppose $(L,R)$ is a Dedekind section, and define $P(q,r)$ if $q \in L$ and $r \in R$. The first axiom for $P$ is clearly satisfied. For the second, take $\varepsilon > 0$ and find $q_0 \in L$ and $r_0 \in R$ so that $P(q_0, r_0)$. Find $n \in \mathbb{N}$ such that $r_0 - q_0 < 2^{n+1}\varepsilon$. By induction on $n$, we show that there is some $u$ with $P(u-\varepsilon, u+\varepsilon)$. If $n = 0$, we can take $u = (q_0+r_0)/2$, for $u-\varepsilon < q_0 < r_0 < u+\varepsilon$. Now suppose $n \geq 1$. Let $s_i = q_0 + i(r_0-q_0)/4$ ($0 \leq i \leq 4$). Since $s_1 < s_2 < s_3$, we have (1) either $s_1 \in L$ or $s_2 \in R$, and (2) either $s_2 \in L$ or $s_3 \in R$. Examining the possibilities, we can find $q_1$ and $r_1$ from amongst the $s_i$s with $P(q_1, r_1)$ and $r_1 - q_1 = (r_0-q_0)/2 < 2^n\varepsilon$.

The above geometric constructions give us maps $f : [T'_{\mathbb{R}}] \to [\mathrm{Ded}]$ and $g : [\mathrm{Ded}] \to [T'_{\mathbb{R}}]$. Composing them, we see that $g \circ f \cong \mathrm{Id}_{[T'_{\mathbb{R}}]}$ and $f \circ g \cong \mathrm{Id}_{[\mathrm{Ded}]}$. Hence the two theories are equivalent.

We have proved this solely on the hypothesis that the classifying toposes exist. We have not had to analyse the structure of the classifying toposes at all, beyond the knowledge that they are Grothendieck toposes and have generic models.

Having shown these theories are equivalent, it is possible now to *define* the real line $\mathbb{R}$ to be the classifying topos $[\,\mathrm{Ded}]$, the "space of Dedekind sections". This may seem heavy-handed. However, it tells us what the real numbers are (the points of $\mathbb{R}$, i.e. the models of Ded). It also defines the topology. The opens of $[\,\mathrm{Ded}]$ (i.e. the subobjects of 1 in $\mathcal{S}[\,\mathrm{Ded}]$—see Sec. 4.4) are equivalent to those of $[T_{\mathbb{R}}]$, and Theorem 8.71 will show that they are equivalent to elements of the frame $\Omega[T_{\mathbb{R}}]$ as defined in Sec. 2.5.

Let us use this definition of $\mathbb{R}$ to define a map.

EXAMPLE 8.48 *Addition* $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is defined as follows. If* $x$ *and* $y$ *are points of* $\mathbb{R}$*, then* $q < x + y$ *if* $q = q_1 + q_2$ *for some* $q_1 < x$ *and* $q_2 < y$*, and* $x + y < r$ *if* $r = r_1 + r_2$ *for some* $x < r_1$ *and* $y < r_2$*.*

See how we have defined a map (a geometric morphism) just by defining a geometric construction on points. Remarkably, this is enough to guarantee continuity of the corresponding function between spaces.

## 5. Sheaves as predicates

We have already defined sheaves as the objects of classifying toposes. This broadens the normal usage, which defines sheaves in a more technical way. In this section we analyse more closely the structure of classifying toposes and come to the technical definition. This will enable us to prove the crucial results announced in Sec. 4.4.

Sheaves were defined first over topological spaces. Grothendieck subsequently generalized the definition to sheaves over a *site*, but as he stressed, the category of sheaves (the Grothendieck topos) is more important than the site (which is in effect a particular form of geometric theory). The reason for this is essentially our "Lindenbaum algebra" methodology—it is in terms of the topos that we get a good definition of map.

In propositional logic each proposition $\phi$ corresponds to a map $|\phi\}$ from models to truth values. In the geometric context this corresponds to a map from the locale to $\mathbb{S}$, and in ordinary topology this is an open.

In predicate logic, each formula $(\vec{x}.\phi)$ corresponds to a function $|\vec{x}.\phi\}$ from models to sets. In a geometric context this is a map from the classifying topos to the object classifier $[\mathbb{O}]$, and in ordinary topology it is *sheaves* that provide the corresponding "continuous set-valued map".

### 5.1 Sheaves over a topological space

Suppose $X$ is a space and $S$ is a "continuous set-valued function" on it. For this to make sense, we certainly need a set $S(x)$ for each point $x$; this is called the *stalk* of $S$ at $x$. If we let $Y$ be the disjoint union of the stalks we shall have a projection $\pi : Y \to X$, and the stalks $S(x)$ are recoverable as fibres $\pi^{-1}(\{x\})$.

We shall introduce a class of continuous maps $\pi$, the *local homeomorphisms*, for which it turns out that sheaves can be derived by using the fibres as the stalks.

Each stalk is to be a *set*, and we take from that that the fibre $\pi^{-1}(\{x\})$, as a subspace of $Y$, should have the *discrete* topology. Thus within fibres, the topology of $Y$ should be discrete. On the other hand, across the fibres we might argue that the topology of $Y$ should be no finer than is got from $X$ in order that the dependence of $\pi^{-1}(\{x\})$ on $x$ should be "continuous".

Those are vague, but let us suggest that the second "across fibre" condition requires $\pi$ to be an open map—it takes open sets to open sets.

For the first "within fibre" condition, consider that a space $Z$ is discrete iff the diagonal inclusion $\Delta : Z \hookrightarrow Z \times Z$, $\Delta(z) = (z, z)$, is open. To see this, take $z \in Z$. By definition of the product topology, there are open neighbourhoods $U$ and $V$ of $z$ such that $U \times V \subseteq \Delta(Z)$. By replacing $U$ and $V$ by their intersection, we might as well assume $U = V$. Now if $z' \in U$ then $(z, z') \in U \times U \subseteq \Delta(Z)$, so $z' = z$. Hence $U = \{z\}$ and so $\{z\}$ is open, so the topology is discrete.

Generalizing to $\pi : Y \to X$, we use a "fibrewise discrete" property that the inclusion $\Delta : Y \hookrightarrow Y \times_X Y$ should be open. ($Y \times_X Y$ is the *fibred product*, or *pullback*, $\{(y_1, y_2) \mid \pi(y_1) = \pi(y_2)\}$.) This is more than enough to imply that every fibre is discrete in its subspace topology.

PROPOSITION 8.49 *Let $\pi : Y \to X$ be a map of spaces. Then the following conditions are equivalent.*

1. *$\pi$ and the diagonal inclusion $\Delta : Y \hookrightarrow Y \times_X Y$ are both open.*
2. *Each $y \in Y$ has an open neighbourhood $V$ such that $\pi$ restricted to $V$ is a homeomorphism onto an open neighbourhood of $\pi(y)$.*

*Proof* First, note that $\Delta(Y)$ is open in $Y \times_X Y$ (as subspace of $Y \times Y$) iff every $(y, y) \in \Delta(Y)$ has a basic open neighbourhood in $Y \times_X Y$ that is contained in $\Delta(Y)$, in other words we can find neighbourhoods $V_1$ and $V_2$ of $y$ such that $V_1 \times V_2 \cap Y \times_X Y \subseteq \Delta(Y)$. By restricting to $V_1 \cap V_2$ we might as well assume $V_1 = V_2$. The condition $V \times V \cap Y \times_X Y \subseteq \Delta(Y)$ says that $\pi$ is 1-1 on $V$. To summarize, $\Delta$ is open iff every $y \in Y$ has an open neighbourhood $V$ on which $\pi$ is 1-1.

(1)⇒(2): If $y \in Y$, choose $V$ as above. $\pi$ is a continuous bijection from $V$ onto $\pi(V)$, and since $\pi$ is open, we deduce that this bijection is a homeomorphism.

(2)⇒(1): Let $W$ be open in $Y$. If $y \in W$, then we can find $V_y$ in condition (2) and $\pi(W \cap V_y)$ is open. $\pi(W)$ is the union of these open sets $\pi(W \cap V_y)$ and hence is open. Hence $\pi$ is an open map. Openness of $\Delta$ follows from what we have already said. QED

DEFINITION 8.50 *Let* $\pi : Y \to X$ *be a continuous map between two topological spaces.* $\pi$ *is a* local homeomorphism *(* over $X$*) if it satisfies the equivalent conditions of the proposition.*

If $\pi_i : Y_i \to X$ are two local homeomorphisms over $X$, then a morphism from $\pi_1$ to $\pi_2$ is a map $f : Y_1 \to Y_2$ such that $\pi_2 \circ f = \pi_1$. We obtain a category $\mathbf{LocHom}_X$ of local homeomorphisms over $X$.

It will turn out from a long train of argument that $\mathbf{LocHom}_X$ is a Grothendieck topos. Local homeomorphisms are equivalent to sheaves as presheaves, and then from the more general topos theory they are classifying toposes. However, it is an illuminating exercise to prove it directly. The geometric constructions needed in $\mathbf{LocHom}_X$ can all be constructed stalkwise by elementary means.

For any map $\pi : Y \to X$, a *local section* of $\pi$ is a map $\sigma : U \to Y$, with $U$ open in $X$, such that $\pi \circ \sigma = \mathrm{Id}_U$. An open $V$ as in Proposition 8.49 (2) is equivalent to a local section of $\pi$ whose image is open. The other main definition of sheaf uses sections, through the notion of *presheaf*: a presheaf on any category $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to the category $\mathbf{Set}$ of sets. For a topological space, a presheaf on $X$ is defined to be a presheaf on $\Omega X$. As with any poset, the objects of $\Omega X$ are its elements, and the morphisms are the pairs $(U, V)$ with $U \subseteq V$ – in other words, there is a single morphism from $U$ to $V$ provided $U \subseteq V$. A presheaf $F$ on $\Omega X$ has a set $F(U)$ for each $U \in \Omega X$, and if $U \subseteq V$ there is a *restriction* from $F(V)$ to $F(U)$, which we shall normally write $\sigma \mapsto \sigma|U$.

A morphism of presheaves is just a natural transformation. For presheaves over a space $X$, this means that a morphism from $F$ to $G$ has a family of functions $f_U : F(U) \to G(U)$ $(U \in \Omega X)$ that commute with the restriction maps.

DEFINITION 8.51 *Let $X$ be a topological space. A presheaf $F$ on $X$ is a* sheaf *if it satisfies the following* pasting *condition.*

*Let $U_i \in \Omega X$ ($i \in I$), and suppose for each $i$ we have $\sigma_i \in F(U_i)$ such that for all $i, j$ we have $\sigma_i|(U_i \cap U_j) = \sigma_j|(U_i \cap U_j)$. Then there is a unique $\sigma \in F(\bigcup_i U_i)$ such that for all $i$ we have $\sigma|U_i = \sigma_i$.*

*A morphism of sheaves is just a presheaf morphism. We get a category $\mathcal{S}X$ of sheaves over $X$.*

Note the uniqueness. As an immediate consequence, by taking $I = \emptyset$ we see that if $F$ is a sheaf then $F(\emptyset)$ is a singleton. Note also that the same definition of presheaf and sheaf work over a locale.

EXAMPLE 8.52 *Let* $\pi : Y \to X$ *be a local homeomorphism, and let the presheaf* $\mathrm{Sect}(\pi)$ *be defined by*

$$\mathrm{Sect}(\pi)(U) = \{\sigma : U \to Y \mid \sigma \textit{ is a local section of } \pi\}.$$

*The restriction maps are ordinary domain restriction of functions. Then* $\mathrm{Sect}(\pi)$ *is a sheaf.*

*The process is functorial. If* $\pi_i : Y_i \to X$ $(i = 1, 2)$ *are two local homeomorphisms over* $X$*, and* $f : Y_1 \to Y_2$ *is a morphism between them, then composition with* $f$ *gives a sheaf morphism* $\mathrm{Sect}(\pi_1) \to \mathrm{Sect}(\pi_2)$*. We get a functor* $\mathrm{Sect} : \mathbf{LocHom}_X \to \mathcal{S}X$*.*

THEOREM 8.53 $\mathrm{Sect} : \mathbf{LocHom}_X \to \mathcal{S}X$ *is an equivalence of categories.*

*Proof* (Sketch) It is necessary to show that the functor $\mathrm{Sect}$ is full and faithful, and essentially surjective. From $\mathrm{Sect}(\pi)$ we can recover the stalks, since the stalk at $x$ is the colimit of the sets $\mathrm{Sect}(\pi)(U)$ as $U$ ranges over the open neighbourhoods of $x$. Furthermore, we can recover the topology since the images of the local sections form a base. Starting from an arbitrary sheaf $F$, the same construction yields a local homeomorphism whose sheaf of sections is isomorphic to $F$—this proves essential surjectivity.

Faithfulness is easy, but for fullness one must show that if $\pi_i : Y_i \to X$ $(i = 1, 2)$ then every sheaf morphism $\alpha : \mathrm{Sect}(\pi_1) \to \mathrm{Sect}(\pi_2)$ comes from a morphism $f$ from $\pi_1$ to $\pi_2$. If $y \in Y_1$, find a section $\sigma : U \to Y_1$ whose image contains $y$. Then $f(y)$ is defined as $\alpha_U(\sigma)(\pi_1(y))$. One must prove that this definition is independent of choice of $\sigma$, that $f$ is continuous, that $\pi_2 \circ f = \pi_1$ and that $\mathrm{Sect}(f) = \alpha$. QED

## 5.2 Sheaves and local homeomorphisms for toposes

For any topos $X$, the sheaves over $X$ (the objects of $\mathcal{S}X$) are equivalent to the maps $X \to [\mathbb{O}]$. Hence, by the methods of Sec. 4.5, to define a sheaf $S$ we declare "let $x$ be a point of $X$" and then, geometrically, define a set $S(x)$. We therefore think of $S$ as a continuous set-valued map on $X$. (However, except on global points, these are not sets in the sense of set theory, with the structure all defined through the $\in$ relation. Geometric type theory is not done that way.) We call $S(x)$ the *stalk* of $S$ at $x$, and this notation also suggests we might view the sheaf as a set parametrized by a variable point of $X$.

Both the point transformation and the parametrization involve a radically new notion of continuity, since $[\mathbb{O}]$ has far too few opens to be a useful topological space in anything like the conventional sense. An open of $[\mathbb{O}]$, a map $[\mathbb{O}] \to \mathbb{S}$, is a geometric definition of a truth value for each set $S$. There are three obvious ways to do this: constant $\top$, constant $\bot$ and by the formula $(\exists a \in S)\ \top$. In effect, we have three open subspaces of "the space of sets": the whole space, the empty space, and the space of inhabited sets. We shall later (Example 8.74) be able to prove that—at least classically—these are the only three, and from the localic point of view $[\mathbb{O}]$ cannot be distinguished from $\mathbb{S}$. (Technically, $\mathbb{S}$ is the "localic reflection" of $[\mathbb{O}]$—Definition 8.72.)

EXAMPLE 8.54 *Let $T$ be a geometric theory. The functors and natural transformations $|B\}$, $|\vec{x}.\phi\}$, $|\vec{x}.t\}$, etc. of Remark 8.22 define sheaves and sheaf morphisms.*

Stalks can be gathered together to make a new topos, analogous to a local homeomorphism. Let $\mathbb{O}, elt$ be the theory with one sort and one constant symbol, and let $p : [\mathbb{O}, elt] \to [\mathbb{O}]$ be the obvious reduct map (which forgets the constant). The points of $[\mathbb{O}, elt]$ are pairs $(A, a)$ where $A$ is a set (or, in general, a sheaf over a topos) and $a$ a global element of $A$.

PROPOSITION 8.55 *Let $X$ be a topos and $A : X \to [\mathbb{O}]$ a sheaf. Let $X/A$ be the topos given by the pseudo-pullback*

$$\begin{array}{rcl} X/A & \longrightarrow & [\mathbb{O}, elt] \\ A^*p\downarrow & \cong & \downarrow p \\ X & \underset{A}{\longrightarrow} & [\mathbb{O}] \end{array}$$

*Its points are pairs $(x, a)$ with $x$ a point of $X$ and $a \in A(x)$. Then $\mathcal{S}(X/A)$ is equivalent to the* slice *category $(\mathcal{S}X)/A$, whose objects are morphisms in $\mathcal{S}X$ with codomain $A$.*

*Proof* $\mathcal{S}(X/A)$ is got from $\mathcal{S}X$ by freely adjoining a global element $e : 1 \to A$. From this one can construct, for any $\nu : C \to A$ in $\mathcal{S}X$, the pullback along $e$ giving an object $e^*C$, and the result says in effect that every object of $\mathcal{S}(X/A)$ comes from some $\nu$ in this way.

It is straightforward to check that $(\mathcal{S}X)/A$ is a Grothendieck topos. (The corresponding fact for elementary toposes is the "Fundamental Theorem of Topos Theory", Johnstone, 2002a, A2.3.) We have a functor $q^* : \mathcal{S}X \to \mathcal{S}X/A$, with $q^*(B)$ the projection $B \times A \to A$, and it preserves colimits and finite limits. In $\mathcal{S}X/A$ the final object is $\mathrm{Id}_A : A \to A$, and $q^*A$ has a global element $e$ given by the diagonal $\Delta : A \to A \times A$. Every object $\nu : C \to A$ is the pullback of $q^*(\nu)$ against $e$.

Now suppose we have a map $f : Y \to X$ with a global element $e' : 1 \to f^*(A)$. The result amounts to showing that $f^*$ factors via $q^*$ and a functor $r^* : \mathcal{S}X/A \to \mathcal{S}Y$ that preserves colimits and finite limits, and takes $e$ to $e'$. Clearly $r^*(\nu)$ has to be (up to isomorphism) defined as the pullback of $f^*(\nu)$ against $e'$, after which it remains only to check that it has the required properties. QED

Note a corollary to this. A map $X/B \to X/A$ over $X$ is equivalent to a global element of $(A \times B \to B)$ in $\mathcal{S}X/B$, and this is just a morphism $B \to A$ in $\mathcal{S}X$. Hence sheaf morphisms are equivalent to maps between the corresponding fibred spaces.

In the pseudo-pullback square in the proof of Proposition 8.55, the map $A^*p$ on the left maps $(x, a)$ to $x$. The maps that arise in this way from sheaves over $X$ are called *local homeomorphisms* or *étale* maps (see Johnstone, 2002b, C3.3.4). For each point $x$, the pseudo-pullback of $A^*p$ against $x$ is in effect the stalk at $x$. In fact, we have three equivalent categories to represent sheaves over $X$: $\mathcal{S}X$, $\mathrm{Map}(X, [\mathbb{O}])$ and the category of local homeomorphisms with codomain $X$. In Joyal and Tierney, 1984 the local homeomorphisms are characterized in a way analogous to the first condition of Proposition 8.49, using a topos notion of open map.

## 5.3 Sites

A canonical form of geometric theory is that deriving from a *site*. We give the definition from Johnstone, 2002a, A2.1.9.

DEFINITION 8.56 *Let $\mathcal{C}$ be a small category. A* coverage *$J$ on $\mathcal{C}$ assigns to each object $A$ of $\mathcal{C}$ a collection $J(A)$ of families $(f_i : A_i \to A \mid i \in I)$ of morphisms targeted at $A$, subject to the condition that for each such family in $J(A)$, and for each morphism $g : B \to A$, there is a family $(h_{i'} : B_{i'} \to B \mid i' \in I')$ in $J(B)$ such that each $g \circ h_{i'}$ factors via some $f_i$.*

*A category equipped with a coverage is called a* site.

The definition in Mac Lane and Moerdijk, 1992, III.2 Definition 1 is slightly different, as a category equipped with a *Grothendieck topology*. In this, the covering families are all required to be *sieves*, i.e. closed under precomposition. The difference is explained in Johnstone, 2002b, C2.1.8 (where a Grothendieck topology is called a *Grothendieck coverage*). Any coverage generates a Grothendieck topology that is equivalent to it for its intended purposes.

DEFINITION 8.57 *Let $(\mathcal{C}, J)$ be a site. Then the geometric theory* $\mathrm{CtsFlat}$ $(\mathcal{C}, J)$ *of* continuous flat functors over $(\mathcal{C}, J)$ *has sorts $X_A$ and functions $u_f : X_A \to X_B$ for the objects $A$ and morphisms $f : A \to B$ of $\mathcal{C}$, and axioms*

$$(\forall x : X_A)\ u_{\mathrm{Id}_A}(x) = x \ (A \in \mathrm{Ob}(\mathcal{C}))$$

$$(\forall x : X_A)\ u_g(u_f(x)) = u_{g\circ f}(x) \ (f : A \to B,\ g : B \to C)$$

$$\bigvee_{A\in\mathrm{Ob}(\mathcal{C})}(\exists x : X_A)\ \top$$

$$(\forall x : X_A, y : X_B) \bigvee_{C\in\mathrm{Ob}(\mathcal{C})}\bigvee_{f:C\to A}\bigvee_{g:C\to B}(\exists z : X_C)\ (x = u_f(z) \wedge y = u_g(z))$$

$$\begin{aligned}&(\forall x : X_A)\ (u_f(x) = u_g(x)\\ &\quad\longrightarrow \bigvee_{C\in\mathrm{Ob}(\mathcal{C})}\bigvee\{(\exists z : X_C)\ x = u_h(z) \mid h : C \to A, f \circ h = g \circ h\}\\ &\qquad\quad (f, g : A \to B)\end{aligned}$$

$$(\forall x : X_A) \bigvee_{i\in I}(\exists y : X_{A_i})\ x = u_{f_i}(y) \ ((f_i : A_i \to A \mid i \in I) \text{ in } J(A))$$

Its models in a Grothendieck topos $\mathcal{E}$ are the *continuous filtering functors* (or *continuous flat functors*) from $\mathcal{C}$ to $\mathcal{E}$ (Mac Lane and Moerdijk, 1992, VII Sec. 7-9). The first two axiom schemas stipulate functoriality, the next three are the flatness (or filtering property) and the final one is the continuity. Note that if $\mathcal{C}$ has all finite limits, then (Mac Lane and Moerdijk, 1992, VII.9 Corollary 3) the flat functors from $\mathcal{C}$ to a Grothendieck topos are exactly the finite limit preserving functors.

Without the final axiom in Definition 8.57 we have the theory $\mathrm{Flat}(\mathcal{C})$ of *flat functors* over $\mathcal{C}$.

EXAMPLE 8.58 *Any Cartesian theory $T$ is equivalent to* $\mathrm{Flat}(\mathcal{C})$ *where $\mathcal{C}$ is the opposite of the category of finitely presented $T$-models. For a discussion of some non-Cartesian theories of the form* $\mathrm{Flat}(\mathcal{C})$*, as well as the constructive notion of "finite" used in "finitely presented" (stronger than Kuratowski finiteness) see Vickers, 2001.*

## 5.4 Sheaves as presheaves

In this section we return to the main question left over from Sec. 4.4: what are the sheaves over a classifying topos? It is only when this has been answered that we can be sure classifying toposes exist. We outline the proof in stages: first, theories $\mathrm{Flat}(\mathcal{C})$; then $\mathrm{CtsFlat}(\mathcal{C}, J)$; then geometric theories in general; and then geometric type theories.

LEMMA 8.59 *If $\mathcal{C}$ is a small category then the presheaf topos $\mathbf{Set}^{\mathcal{C}^{op}}$ classifies* $\mathrm{Flat}(\mathcal{C})$.

*Proof* (Sketch. cf. Proposition 8.40.) For any Grothendieck topos $X$ we want a correspondence between flat functors $F : \mathcal{C} \to \mathcal{S}X$ and functors $\mathbf{Set}^{\mathcal{C}^{op}} \to \mathcal{S}X$ preserving colimits and finite limits.

The Yoneda embedding $\mathcal{Y} : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$ acts as a *free cocompletion* of $\mathcal{C}$ (Mac Lane and Moerdijk, 1992, I.5 Corollary 4): any functor from $\mathcal{C}$ to a cocomplete category factors uniquely (up to isomorphism) via $\mathcal{Y}$ and a colimit preserving functor. This gives an equivalence between functors $\mathcal{C} \to \mathcal{S}X$ and colimit preserving functors $\mathbf{Set}^{\mathcal{C}^{op}} \to \mathcal{S}X$.

However, $\mathbf{Set}^{\mathcal{C}^{op}}$ is also a Grothendieck topos (Mac Lane and Moerdijk, 1992, I). The content of Mac Lane and Moerdijk, 1992, VII.7 Theorem 2 is then that flatness of the functor $\mathcal{C} \to \mathcal{S}X$ is equivalent to finite limit preservation by the colimit preserving functors $\mathbf{Set}^{\mathcal{C}^{op}} \to \mathcal{S}X$. (Note also that $\mathcal{Y}$ is flat.) QED

EXAMPLE 8.60 *(See Example 8.58.) Let $T$ be a Cartesian theory, and let $\mathcal{C}$ be its category of finitely presented models so that $T$ is equivalent to* $\mathrm{Flat}(\mathcal{C}^{op})$. *Then $\mathcal{S}[T] \simeq \mathbf{Set}^{\mathcal{C}}$. Thus, as set-valued map on points (models of $T$), a sheaf* *is determined by its action on the finitely presented models. This also follows from the fact that every model is a filtered colimit of finitely presented models, and maps preserve filtered colimits.*

We now turn to theories $\mathrm{CtsFlat}(\mathcal{C}, J)$. Let $F : \mathcal{C} \to \mathcal{F}$ be a flat functor to a Grothendieck topos. The continuity axiom says that for each covering family $(f_i : A_i \to A)_{i \in I}$ in $J(A)$,

$$(\forall x : F(A)) \bigvee_{i \in I} (\exists y : F(A_i))\; x = F(f_i)(y).$$

Categorically, this says that the cotupled morphism $f = [F(f_i)]_{i \in I} : \sum_{i \in I} F(A_i) \to F(A)$ is epi, in other words that its image is the whole of $F(A)$.

To analyse this, we calculate what the image is in general. The result content is roughly that of Mac Lane and Moerdijk, 1992, VII.7 Lemma 2. However, we sketch a "geometric" proof that does not rely on the subobject classifier.

LEMMA 8.61 *Let $(\mathcal{C}, J)$ be a site, and let $F : \mathcal{C} \to \mathcal{S}X$ be a flat functor to a Grothendieck topos. Let $(f_i : A_i \to A)_{i \in I}$ be in $J(A)$. Then the image of*

$$f = [F(f_i)]_{i \in I} : \sum_{i \in I} F(A_i) \to F(A)$$

*is the colimit of a diagram $F \circ \Delta$ as follows. Let $\mathcal{D}$ be the full subcategory of the slice category $\mathcal{C}/A$ whose objects are morphisms $g : C \to A$ that factor through some $f_i$. $\Delta$ is the obvious functor from $\mathcal{D}$ to $\mathcal{C}$, taking $(g : C \to A)$ to $C$.*

*Proof* Let $\mu_i : F(A_i) \to \sum_i F(A_i)$ be the coproduct injection, and let $f = m \circ q$ be the image factorization of $f$.

We define a cocone from $F \circ \Delta$ to $\mathrm{Im}\, f$ as follows. If $g : C \to A$ factors via some $f_i$, then $F(g)$ factors via $f$ and hence uniquely via $m$, as $m \circ \nu_g$ (say).

Now suppose we have a cocone from $F \circ \Delta$ to some $K$, given by morphisms $\nu'_g : F(C) \to K$ for each $g : C \to A$ in $\mathrm{Ob}(\mathcal{D})$. If we are to have a colimit morphism $\alpha : \mathrm{Im}\, f \to K$, then it is determined uniquely by $\alpha \circ q$ (because $q$ is epi) and hence by the morphisms $\alpha \circ q \circ \mu_i = \alpha \circ \nu_{f_i} = \nu'_{f_i}$. This proves uniqueness for $\alpha$.

The morphisms $\nu'_{f_i}$ give us a morphism $\alpha' = [\nu'_{f_i}]_i : \sum_i F(A_i) \to K$, and we should like $\alpha'$ to factor as $\alpha \circ q$ for some $\alpha : \mathrm{Im}\, f \to K$. This will be as required, since if $g = f_i \circ g' : C \to A$ then

$$\nu'_g = \nu'_{f_i} \circ F(g') = \alpha' \circ \mu_i \circ F(g') = \alpha \circ q \circ \mu_i \circ F(g') = \alpha \circ \nu_{f_i} \circ F(g') = \alpha \circ \nu_g.$$

To prove existence of $\alpha$, we interpret logic in $\mathcal{S}X$. Viewing $\mathrm{Im}\, f$ as a quotient of $\sum_i F(A_i)$, it suffices to show that if $f(a) = f(b)$ $(a, b \in \sum_i F(A_i))$ then

$\alpha'(a) = \alpha'(b)$. Suppose $a = \mu_i(a')$, with $a' \in F(A_i)$, and similarly $b = \mu_j(b')$, so $F(f_i)(a') = F(f_j)(b')$. By flatness of $F$ we can find an object $C$ in $\mathcal{C}$, morphisms $g : C \to A_i$ and $h : C \to A_j$ with $f_i \circ g = f_j \circ h$, and $c \in F(C)$ with $a' = F(g)(c)$ and $b' = F(h)(c)$. Then

$$\begin{aligned}\alpha'(a) &= \alpha' \circ \mu_i \circ F(g)(c) = \nu'_{f_i} \circ F(g)(c) = \nu'_{f_i \circ g}(c) \\ &= \nu'_{f_j \circ h}(c) = \nu'_{f_j} \circ F(h)(c) = \alpha' \circ \mu_j \circ F(h)(c) = \alpha'(b).\end{aligned}$$

QED

At this point we can introduce the notion of *sheaf* over a site. (In Mac Lane and Moerdijk, 1992, III.4 this is for a slightly different definition of site, but the difference is not of great significance here.)

DEFINITION 8.62 *Let $(\mathcal{C}, J)$ be a site, and let $S$ be a presheaf over $\mathcal{C}$. $S$ is a* sheaf *if it has the following* pasting *property.*

*Suppose $(f_i : A_i \to A)_{i \in I}$ is in $J(A)$. Suppose for each $i \in I$ we have $x_i \in S(A_i)$, with the family $(x_i)_{i \in I}$ "matching" in the sense that if $C$ is an object of $\mathcal{C}$ and $g_i : C \to A_i$, $g_j : C \to A_j$ are morphisms with $f_i \circ g_i = f_j \circ g_j$ then $S(g_i)(x_i) = S(g_j)(x_j)$. Then there is a unique $x \in S(A)$ such that $x_i = S(f_i)(x)$ for all $i$.*

LEMMA 8.63 *Suppose $F : \mathcal{C} \to \mathcal{S}X$ is flat, and let $f : X \to [\mathrm{Flat}(\mathcal{C})]$ be the corresponding map. Then $F$ is continuous iff for every object $U$ of $\mathcal{S}X$, the presheaf $f_*(U)$ is a sheaf.*

*Proof* Suppose $(f_i : A_i \to A)_{i \in I}$ is in $J(A)$. By Yoneda's Lemma, if $S$ is a presheaf then a family $(x_i)_{i \in I}$ of elements $x_i \in S(A_i)$ corresponds to a family $(\xi_i)_{i \in I}$ of morphisms $\xi_i : \mathcal{Y}(A_i) \to S$. Let $\Delta : \mathcal{D} \to \mathcal{C}$ be the diagram described in Lemma 8.61. Then we find that the family $(x_i)_i$ is matching iff the family $(\xi_i)_i$ extends (uniquely) to a cocone from $\mathcal{Y} \circ \Delta$ to $S$. The existence of $x$ is equivalent to the factorization of this cocone through $\mathcal{Y}(A)$. If $S$ is of the form $f_*(U)$, then the cocone of presheaves corresponds to a cocone in $\mathcal{S}X$ from $f^* \circ \mathcal{Y} \circ \Delta = F \circ \Delta$ to $U$. By Lemma 8.61 we know $F$ is continuous iff for every covering $(f_i)$ we have that the colimit of $F \circ \Delta$ is $F(A)$, i.e. for every $U$, every cocone $F \circ \Delta \to U$ factors via $F(A) = f^* \circ \mathcal{Y}(A) \to U$ and this gives us our $\mathcal{Y}(A) \to f_*(U)$ as required for finding $x$. Hence $F$ is continuous iff for every covering and for every $U$ we can perform the pasting with $f_*(U)$. But this just says that every $f_*(U)$ is a sheaf. QED

If we define $\mathrm{Sh}(\mathcal{C}, J)$ to be the full subcategory of $\mathbf{Set}^{\mathcal{C}^{op}}$ comprising the sheaves, then we see that the maps $X \to [\mathrm{Flat}(\mathcal{C})]$ corresponding to continuous flat functors are the ones whose direct image part factors via $\mathrm{Sh}(\mathcal{C}, J)$.

THEOREM 8.64 *If* $(\mathcal{C}, J)$ *is a site then* $\mathrm{Sh}(\mathcal{C}, J)$ *is a classifying topos for* $\mathrm{CtsFlat}(\mathcal{C}, J)$.

*Proof* This is the content of Mac Lane and Moerdijk, 1992, VII.9 Corollary 2, which states that—in the conventional notation—there is an equivalence of categories between $\mathrm{Map}(\mathcal{E}, \mathrm{Sh}(\mathcal{C},J))$ and the category of continuous filtering functors $\mathcal{C} \to \mathcal{E}$ (i.e. models of the site theory). In our notation we can thus take $\mathrm{Sh}(\mathcal{C},J)$ as $\mathcal{S}[\mathrm{CtsFlat}(\mathcal{C}, J)]$. In outline, the rest of the proof is as follows.

First, $\mathrm{Sh}(\mathcal{C}, J)$ is indeed a topos. (This includes the fact that it is an elementary topos. This is perhaps unexpected, since the argument from classifying toposes worked with the geometric structure.)

Next, the inclusion $\mathrm{Sh}(\mathcal{C}, J) \to \mathbf{Set}^{\mathcal{C}^{op}}$ is the direct image part of a geometric morphism. Proving the existence of the inverse image part, the "associated sheaf functor" or *sheavification*, is of fundamental importance. If $S$ is already a sheaf, then it is its own sheavification.

After all that, proving that $\mathrm{Sh}(\mathcal{C}, J)$ classifies flat continuous functors is more or less Lemma 8.63. QED

We can also calculate the stalks explicitly. Let $x$ be a global point, a continuous flat functor from $\mathcal{C}$ to $\mathbf{Set}$, and $S$ a sheaf. The stalk $S \circ x$ can be calculated in two stages (Mac Lane and Moerdijk, 1992, Sec. VII.5). First, let $U_0$ be the disjoint union over all objects $A$ of $\mathcal{C}$ of the products $x(A) \times S(A)$. Next, if $f : A \to B$ is a morphism in $\mathcal{C}$, and $a \in x(A)$ and $b \in S(B)$, we identify $(a, S(f)(b))$ and $(x(f)(a), b)$ in $U_0$ and generate an equivalence relation $\sim$ thereby. Then the stalk is $U_0/\sim$. This construction is geometric, and can be reproduced for non-global points.

THEOREM 8.65 *Every geometric theory is equivalent to a site theory* $\mathrm{CtsFlat}$ $(\mathcal{C}, J)$*, and hence has a classifying topos.*

*Proof* Let $T$ be a geometric theory over signature $\Sigma$. By Johnstone, 2002b, Lemma D1.3.8, every geometric formula in context over $\Sigma$ is logically equivalent to one of the form $\bigvee_i (\exists \vec{y}_i)\phi_i$ where each $\phi_i$ is a Horn formula (a conjunction of equations and predicate symbols applied to terms). It follows that each axiom in $T$ is equivalent to a set of axioms of the form $\psi \vdash_{\vec{x}} \bigvee_i (\exists \vec{y}_i)\phi_i$. Moreover, by replacing $\phi_i$ by $\psi \wedge \phi_i$ and using the distributivity and Frobenius rules, we may assume that $\phi_i \vdash_{\vec{x}\vec{y}_i} \psi$. From $\Sigma$ can be constructed (Johnstone, 2002b, D1.4) a *syntactic category* $\mathcal{C}$, Cartesian (i.e. with all finite limits), such that in any Cartesian category $\mathcal{D}$ we have that interpretations of $\Sigma$ in $\mathcal{D}$ are equivalent to Cartesian (finite limit preserving) functors from $\mathcal{C}$ to $\mathcal{D}$; and recall that because $\mathcal{C}$ is Cartesian, flat functors from $\mathcal{C}$ to a topos are the same as Cartesian functors. The objects of $\mathcal{C}$ are the Horn formulae in context, modulo renaming of variables, and the morphisms are the formulae that are "provably the graphs of functions", modulo logical equivalence. Now suppose $\psi \vdash_{\vec{x}} \bigvee_i (\exists \vec{y}_i)\phi_i$ is one of the axioms in $T$. In $\mathcal{C}$ we have diagrams

$$\begin{array}{lcl} (\vec{x}, \vec{y}_i.\phi_i) & \hookrightarrow & (\vec{x}, \vec{y}_i.\top) \\ \downarrow & & \downarrow \\ (\vec{x}.\psi) & \hookrightarrow & (\vec{x}.\top) \end{array}$$

where the right-hand arrow is the product projection, and the left-hand arrow follows from our assumption that $\phi_i \vdash_{\vec{x}\vec{y}_i} \psi$. We take those left-hand arrows, as $i$ varies, as covering $(\vec{x}.\psi)$, and use these covers to generate a coverage $J$ of $\mathcal{C}$. Models of $T$ are equivalent to models of $\mathrm{CtsFlat}(\mathcal{C}, J)$. QED

We should now like a result of the form "every geometric type theory has a classifying topos". This is difficult, since our notion of geometric type theory is only informal. The following argument from Johnstone, 2002a, B4.2 uses a particular restricted formalization that nonetheless seems ample to cover examples that arise in practice.

THEOREM 8.66 *Normally, geometric type theories have classifying toposes. (The proof is not completely general.)*

*Proof* Johnstone, 2002a, Definition B4.2.7(c) gives a definition of geometric theory that includes features of geometric type theory. According to that definition, a geometric theory $T$ is built up in a finite sequence $T_0, \ldots, T_n = T$. $T_0$ declares finitely many sorts, and each subsequent step is of one of two forms. A *simple functional extension* $T_{i+1}$ of $T_i$ declares a function symbol $f : F_1 \to F_2$, where $F_1$ and $F_2$ are geometric types. A *simple geometric quotient* $T_{i+1}$ of $T_i$ is based on a morphism $u : F_1 \to F_2$ of geometric types. $T_{i+1}$ adds axioms

$$u(x) = u(x') \vdash_{x,x':F_1} x = x'$$
$$\top \vdash_{y:F_2} (\exists x : F_1)\, y = u(x)$$

and thus forces $u$ to be an isomorphism.

In each case, if $T_i$ has a classifying topos, then we can identify the geometric types $(F_1, F_2)$ and morphisms $(u)$ with objects and morphisms of $\mathcal{S}[T_i]$, and one can construct a classifying topos for $T_{i+1}$. Hence every geometric theory by that definition has a classifying topos.

These two steps provide a completely general way of introducing function symbols, and also axioms $\phi \vdash_{\vec{x}} \psi$, for satisfaction of the axiom is equivalent to saying that the inclusion morphism $\psi \wedge \phi \to \phi$ is an isomorphism. As for predicate symbols $P \subseteq \vec{A}$, these can be introduced with a sort $P'$ and function $i_P : P' \to \vec{A}$ which must then be constrained to be monic (to give a subobject corresponding to $P$). This is done by an axiom

$$i_P(x) = i_P(x') \vdash_{x,x':P'} x = x'.$$

Hence all the ingredients of geometric type theory can be introduced by these steps. QED

Since only finitely many steps are allowed, it would seem that the geometric type theory according to that definition should be finitely presented—only finitely many symbols and axioms. However, in practice one can get round that by internalizing the indexing set of an infinite family of symbols or axioms. For example, consider modules over a ring $R$. The algebraic theory of these would normally be presented with a (possibly infinite) $R$-indexed family of unary operators $\sigma_r$ for scalar multiplication. But the set $R$ is a constant geometric type (a coproduct of an $R$-indexed family of copies of 1) over any theory, and modules $M$ can equivalently be presented using an operator $\sigma : R \times M \to M$. (Exercise: formulate this using simple functional extensions and simple geometric quotients.)

THEOREM 8.67 *Let $\mathcal{E}$ be a category. Then the following are equivalent.*

1 *$\mathcal{E}$ is a Grothendieck topos (as defined in Definition 8.31).*

2 *$\mathcal{E}$ is equivalent to* $\mathrm{Sh}(\mathcal{C}, J)$ *for some site* $(\mathcal{C}, J)$.

3 *$\mathcal{E}$ is classifying topos for some geometric theory.*

*Proof* (1)⇔(2) is known as *Giraud's Theorem.* See Johnstone, 2002b, C2.2.8, where condition (vii) is our condition (1). For an alternative version, see Mac Lane and Moerdijk, 1992, Appendix, Theorem 1. (2) is usually taken as the definition of Grothendieck topos.

(2)⇔(3): Theorems 8.64 and 8.65. QED

## 5.5 Sheaves for locales

We now turn to the question of how continuous maps between spaces and locales relate to geometric morphisms between toposes.

PROPOSITION 8.68 *Let $X$ be a Grothendieck topos. Then* $\mathrm{Sub}_{\mathcal{S}X}(1)$ *is a frame.*

*Proof* Johnstone, 2002b, C1.4.7. In fact $\mathrm{Sub}_{\mathcal{S}X}(S)$ is a frame for any sheaf $S$. QED

Now for any propositional geometric theory $T$, topos models in $\mathcal{S}X$ are equivalent to frame models in $\mathrm{Sub}_{\mathcal{S}X}(1)$. It follows that $T$ and $\mathrm{Th}_{\Omega[T]}$ (Definition 8.9) are equivalent with respect to topos models.

Let $A$ be a frame. As a poset it can also be considered a category, and we can define a coverage $J$ on it as follows. Let $a \in A$, and let $\{b_i \mid i \in I\} \subseteq \{b \mid$

$b \leq a\}$. Then $\{b_i \mid i \in I\} \in J(a)$ if $a \leq \bigvee_{i \in I} b_i$. (Exercise: this is indeed a coverage in the sense of Definition 8.56.)

PROPOSITION 8.69 *The theories* $\mathrm{CtsFlat}(A, J)$ *and* $\mathrm{Th}_A$ *are equivalent.*

*Proof* As a category, $A$ is Cartesian (products are meets, and equalizers are trivial). Hence, flatness of a functor is equivalent to preservation of finite limits. The top element of $A$ must map to the terminal object 1, and all the other elements of $A$ to subobjects of 1 (because if a functor preserves finite limits then it preserves monics, and all the morphisms in $A$ are monic). Hence a flat functor over $A$ is equivalent to a function $A \to \mathrm{Sub}(1)$ that preserves finite meets. Continuity then says that the function preserves arbitrary joins too. QED

If $A$ is a frame, then for a presheaf $S : A^{op} \to \mathbf{Set}$, if $a \leq b$ in $A$ and $x \in S(b)$, then we write $x|_a$ for $S(a \leq b)(x)$, the *restriction* of $x$ to $a$.

THEOREM 8.70 *Let $A$ be a frame, and let $X$ be the topos* $[\mathrm{Th}_A]$.

1 *A sheaf over $X$ is equivalent to a sheaf over the locale for $A$ (Definition 8.51, replacing $\Omega X$ by $A$, and $\cap$ and $\bigcup$ by $\wedge$ and $\bigvee$).*

2 *There is an order isomorphism between* $\mathrm{Sub}_{\mathcal{S}X}(1)$*, the set of subsheaves of* $1$ *over $X$, and $A$.*

*Proof* (1) is calculated directly from Definition 8.62 using Theorem 8.64 and Proposition 8.69. For (2), the terminal sheaf 1 is defined by $1(a) = 1$ (i.e. some singleton) for every $a \in A$. This can be calculated directly, but it also follows from the fact that the embedding $\mathcal{S}X \to \mathbf{Set}^{A^{op}}$ is a right adjoint and hence preserves all limits, and finite limits in $\mathbf{Set}^{A^{op}}$ are calculated argumentwise. Now the subsheaves of 1 are the sheaves $S$ for which every $S(a)$ is a subsingleton.

For every $b \in A$ we have a subsheaf $S_b$ of 1 defined by $S_b(a) = 1$ iff $a \leq b$. (In fact these make up the generic point of $X$ in $\mathcal{S}X$.) Clearly if $S_b = S_{b'}$ then $b \leq b' \leq b$, so $b = b'$. On the other hand, suppose $S$ is a subsheaf of 1 and let $b$ be the join of those $a \in A$ for which $S_a$ is inhabited. By pasting we find that $S(b)$ is inhabited, and it follows that $S = S_b$. QED

It follows that for any propositional geometric theory $T$ we have $\mathrm{Sub}_{\mathcal{S}[T]}(1) \cong \Omega[T]$.

THEOREM 8.71 *Let $T$ and $T'$ be propositional theories. Then there is an equivalence between*

1 *locale maps* $[T] \to [T']$, *and*

2 *topos maps* $[T] \to [T']$.

*Proof* A topos map $[T] \to [T']$ is equivalent to a model of $\mathrm{Th}_{\Omega[T']}$ in $\mathcal{S}[T]$, i.e. a frame homomorphism $\Omega[T'] \to \mathrm{Sub}_{\mathcal{S}[T]}(1) \cong \Omega[T]$. QED

Referring back to Proposition 8.16, we see that for sober spaces, continuous maps are equivalent to geometric morphisms between the corresponding toposes. We have now justified the key fact that underlies this chapter: toposes generalize topological spaces (at least in the sober case), and geometric morphisms are the continuous maps at topos generality.

We now know that locales and localic toposes are equivalent. We write $X$ without any bias either way, and refer concretely to the frame as $\Omega X$ and to the category of sheaves as $\mathcal{S}X$. More generally, for any Grothendieck topos $X$ we can write $\Omega X$ for the frame $\mathrm{Sub}_{\mathcal{S}X}(1)$ without creating any ambiguity in the localic case. We call its elements *opens* of $X$, equivalent to maps $X \to \mathbb{S}$.

DEFINITION 8.72 *Let $X$ be a Grothendieck topos. Then the* localic reflection *of $X$ is the locale* $\mathrm{Loc}(X)$ *whose frame is* $\Omega X$.

PROPOSITION 8.73 *Let $X$ be a Grothendieck topos. Then there is a map* $\alpha : X \to \mathrm{Loc}(X)$ *such that any map $f : X \to Y$ with $Y$ a locale factors uniquely (up to isomorphism) via $\alpha$.*

*Proof* This is immediate from the fact that if $Y$ is a locale, then geometric morphisms from $X$ to $Y$ are equivalent to frame homomorphisms from $\Omega Y$ to $\mathrm{Sub}_{\mathcal{S}X}\, 1$. QED

If $\phi : x \Rightarrow y$ is a specialization morphism between points of $X$, then $\alpha(x) \sqsubseteq \alpha(y)$. Hence $x$ and $y$ are identified by $\alpha$ if there are specialization morphisms going in both directions between them. Thus the localic reflection can lose a lot of structure.

EXAMPLE 8.74 *Consider the object classifier* $[\mathbb{O}]$. *Classically, if $A$ and $B$ are two sets then there is a function from $A$ to $B$ unless $A$ is inhabited and $B$ is empty. Hence we might expect* $\mathrm{Loc}([\mathbb{O}])$ *to have two points for two classes of sets: inhabited, and empty. We can calculate that in fact* $\mathrm{Loc}([\mathbb{O}]) \simeq \mathbb{S}$. *The theory $\mathbb{O}$ is algebraic, and its category of finitely presented algebras is the category* Fin *of finite sets. (Constructively, this is "finite" in a strong sense, meaning isomorphic to $\{1, \ldots, n\}$ for some natural number $n$.) Hence* $\mathcal{S}[\mathbb{O}] \simeq \mathbf{Set}^{\mathrm{Fin}}$. *A sheaf* $S : \mathrm{Fin} \to \mathbf{Set}$ *is a subsheaf of* 1*—an open—iff every $S(A)$ is a subsingleton, and we find it is determined up to isomorphism by* $S(0) \subseteq S(1) \subseteq 1$. *It can be calculated that the frame of these is isomorphic to* $\Omega\mathbb{S}$.

*Thinking of* $[\mathbb{O}]$ *as a generalized space, we now see how far it is from being an* un*generalized space. Its opens are simply too few to characterize the generalized topological structure and we have to use sheaves instead.*

## 6. Summary of toposes

Let us summarize the key points of this story.

1 The usual semantics of first-order logic provides meaning in sets: sorts are sets, function symbols (and terms generally) are functions, and predicates (and formulae) are subsets of products. This tells us, for each theory, what are the *models* of that theory.

2 Categorical logic uses the same idea to provide meaning in more general categories: sorts are objects, function symbols and terms are morphisms, and predicates and formulae are subobjects of products. It tells us what the models of a theory are in more general categories.

3 The logic has to be matched to the categorical structure. The ability to interpret logical connectives, and the validity of logical axioms in an interpretation, both depend on the structure and properties of the category.

4 It is natural to form axioms in two stages as $(\forall \vec{x})(\phi \to \psi)$. Then $\phi$ and $\psi$ are formulae, using connectives appropriate to the categorical structure, and the form of the axioms compares two subobjects (for $\phi$ and $\psi$) and uses minimal categorical structure.

5 The logic we are particularly interested in, *geometric logic*, is interpreted in Grothendieck toposes. However, it is only a fragment of what can be interpreted there. Its formulae use $\wedge$, $\bigvee$, $=$ and $\exists$.

6 It is related to *geometric morphisms* between toposes, in that the geometric logic is preserved by the inverse image functors of geometric morphisms.

7 To emphasize the difference between spatial and logical aspects of toposes, we use a non-standard notation with simple symbols to denote a topos "as generalized topological space", and we apply an $\mathcal{S}$ to denote the same topos "as generalized universe of sets" (in other words, the category discussed above where the logic is interpreted). Thus a geometric morphism $f : X \to Y$ comprises two functors $f^* : \mathcal{S}Y \to \mathcal{S}X$ and $f_* : \mathcal{S}X \to \mathcal{S}Y$.

8 There are type constructors that can be considered to be within the scope of geometric logic. These include free algebra constructions. Although we have not defined the precise range of these type constructors, we have introduced the phrase *geometric type theory* for theories that use those finitary constructors we know to be of this kind. They are equivalent in expressive power to geometric theories.

9 *Coherent theories* and *coherent type theories* are similar to the geometric versions but do not use infinitary disjunctions. Coherent type theories are intermediate in expressive power between coherent theories and geometric theories. It is found in practice that once the finitary type constructors are brought in, the infinitary disjunctions of geometric logic are often not needed.

10 We define a *(generalized) point* of a topos $X$ to be a geometric morphism whose codomain is $X$. It is a *global* point if its domain (its *stage of definition)* is the topos 1 where $\mathcal{S}1 = \mathbf{Set}$.

11 Each geometric type theory $T$ has a *classifying topos* $[T]$ whose points at stage $Y$ are the models of $T$ in $\mathcal{S}Y$. $\mathcal{S}[T]$ is generated by a "generic" model of the theory and is an analogue of Lindenbaum algebra for a predicate geometric theory.

12 The Grothendieck toposes are the classifying toposes for geometric type theories. They can be constructed as toposes of sheaves over sites.

13 A geometric morphism from $X$ to $Y$ transforms, by composition, points of $X$ (at any stage of definition) to points of $Y$.

14 By the definition of classifying topos, we define a geometric morphism from $[T_1]$ to $[T_2]$ by constructing a model of $T_2$ in $\mathcal{S}[T_1]$. Since $\mathcal{S}[T_1]$ is generated by a generic model of $T_1$, this appears formally as declaring, "Let $M$ be a model of $T_1$," and then constructing a model of $T_2$ out of it. To be valid in $\mathcal{S}[T_1]$, the construction must be intuitionistically valid; and to be uniform over all stages of definition it must be geometric.

15 Thus we think of Grothendieck toposes as generalized spaces of models, and geometric morphisms as maps between those spaces.

For some examples of the techniques in use, see Vickers, 1999, Vickers, 2001 and Vickers, 2004. In particular, Vickers, 2001 discusses toposes $X$ for which $\mathcal{S}X$ is a presheaf category, with reference to examples such as the simplicial sets Mac Lane and Moerdijk, 1992, Sec. VIII.8.

## 7. Other directions

We have focused on the relationship between geometric logic and the categorical structure of Grothendieck toposes, to give an introduction to how toposes can be understood as generalized topological spaces. However, the connections between logic and toposes go far beyond this and most of the standard texts describe a range of broader applications. We now briefly mention just a few other aspects of topos theory that are relevant to the logic of space.

## 7.1 Fibred locales

We have already seen how a map $f : X \to Y$ can be understood as a generalized point of $Y$, continuously parametrized by a variable point of $X$. In terms of the non-classical mathematics of sheaves, this is a model in $\mathcal{S}X$ of whatever theory $Y$ classifies.

However, we can also look at the parametrization the other way round. For each point $y$ of $Y$, we get a fibre $X_y = f^{-1}(\{x\})$—indeed, this still makes sense for toposes, by taking the pseudo-pullback of $f$ along $y$. Hence this is a space "parametrized by a variable point of $Y$". We have seen one example of this already, in sheaves and local homeomorphisms. There is a particular "localic" kind of map $f$ between toposes, essentially meaning that $X$ is presented by no new sorts relative to $Y$ (and in particular any map between locales is localic). This gives a notion of "fibred locale" over $Y$, and it turns out that this is equivalent to doing locale theory constructively in $\mathcal{S}Y$.

Joyal and Tierney, 1984 give a straightforward approach to this using frames and we shall sketch that. (Vickers, 2004 gives a more geometric account.) The notion of frame (and frame homomorphism) can be defined in any elementary topos. However, the theory is not finitary algebraic and makes essential use of the elementary topos structure: to define arbitrary joins on $A$ requires a morphism from the powerobject $\mathcal{P}(A)$ to $A$.

Frame structure is preserved by direct image functors $f_*$ (though not by $f^*$), and the subobject classifier is $\Omega$ is always a frame. Hence for any map $f : X \to Y$, $f_*(\Omega_X)$ is a frame in $\mathcal{S}Y$. On the other hand, given a frame $A$ in $\mathcal{S}Y$, we can replicate the construction of the category of sheaves to get a localic map $p : Z \to Y$ such that $p_*(\Omega_Z) \cong A$. In fact we find a duality between frames in $\mathcal{S}Y$ and fibred locales over $Y$.

EXAMPLE 8.75 *Let $S$ be a sheaf over a topos $X$, and let $f : X/S \to X$ be the map of Proposition 8.55. One can calculate that the subobject classifier in $\mathcal{S}X/S$ is $S \times \Omega_X \to S$ and its image under $f_*$ is $\mathcal{P}(S)$. Relative to $X$, it is therefore the discrete locale (i.e. all subsets open) corresponding to $S$.*

## 7.2 Powerlocales

*Powerlocales* are the localic analogue of hyperspaces, spaces whose points are subspaces of other spaces. If $X$ is a locale, then there are various kinds of powerlocales whose points are different kinds of sublocales (the localic analogue of subspace) of $X$.

In some ways the starting point is the Vietoris powerlocale $\mathrm{V}X$, which bears a direct relationship to the Vietoris hyperspace and was first studied in Johnstone, 1985. In computer science an analogous "Plotkin powerdomain" has been used to give semantics for non-deterministic programs—that is, programs for which the result is in some sense a range of points. It was noticed (Smyth,

1978) that its topology is generated by two coarser topologies that give two powerdomains that are interesting in their own right, and these were transferred (Robinson, 1986) to locales to give the upper and lower powerlocales $\mathrm{P}_U\, X$ and $\mathrm{P}_L\, X$. Computer science applications in localic form have appeared in Abramsky, 1991a and Abramsky, 1991b. The three principal powerlocales (Vietoris, upper, lower) are summarized in Vickers, 1997. Their relationship with the predicative mathematics of formal topology is discussed in Vickers, 2006 and Vickers, 2005. More recently (Johnstone and Vickers, 1991, Vickers, 2004, Vickers and Townsend, 2004) it has been noticed that both the upper and lower powerlocales embed in a larger *double powerlocale*, which can be got as either $\mathrm{P}_U\, \mathrm{P}_L\, X$ or $\mathrm{P}_L\, \mathrm{P}_U\, X$ (they are homeomorphic).

Each powerlocale has a good logical content, long understood in computer science. Given a locale $X$, each powerlocale embodies a logical theory whose models are certain kinds of *sublocales* of $X$. A sublocale is in effect a theory got by adding extra axioms to that for $X$, thus specifying a part of the class of models of $X$. Some topological properties of $X$, compactness being a good example, can be discussed in terms of points of the powerlocales (Vickers, 1995, Vickers, 2006).

The logical approach relies on the idea that, given a logic of points, we get a "logic of parts", reminiscent of modal logic. For each property $U$ of points, an open of the original locale, we get two properties of parts: $\Box U$ says that the part is wholly inside $U$, while $\Diamond U$ says that the part has at least one point in $U$ (i.e. it *meets* $U$). Suitable axioms for the properties $\Box U$ are that $\Box$ preserves finite meets and also directed joins – this latter turns out to be necessary for good results, and imposes a compactness condition on the parts. From these we get the upper powerlocale. A suitable property for $\Diamond$ is that it preserves all joins, and from that we get the lower powerlocale. Taking the properties $\Box U$ and $\Diamond U$ together, we need extra axioms to show their interaction:

$$\Box U \wedge \Diamond V \to \Diamond(U \wedge V),$$
$$\Box(U \vee V) \to \Box U \vee \Diamond V.$$

From these we get the Vietoris powerlocale.

At that first stage, the powerlocales are defined directly in terms of the frames. However, one can also investigate them as theory constructions. That is to say, if the original space (the "logic of points") is given as a theory rather than as a frame, we show how to gain theories of the powerlocales. The proofs uses "coverage theorems", results that transform a presentation of the frame by generators and relations into a presentation of the same structure but by generators and relations with respect to different algebraic operators.

### 7.3 Modal logic

One direction that might particularly interest readers of this book is the connection with modal logic. The pointers that follow here were supplied by the Second Reader of this chapter. Classical as well as non-classical modalities have been studied along topos-theoretic lines by Reyes with others: see Lavendhomme et al., 1989; Reyes, 1991; Makkai and Reyes, 1995; Reyes and Zolfaghari, 1991; Reyes and Zolfaghari, 1996. Categorical semantics for super-intuitionistic and modal predicate logics were developed by Ghilardi (Ghilardi, 1989; Ghilardi, 1991; Ghilardi, 1992) and Shehtman and Skvortsov (Shehtman and Skvortsov, 1990; Skvortsov and Shehtman, 1993; Skvortsov, 1996; Skvortsov, 2003); see also Suzuki, 1990; Suzuki, 1993; Isoda, 1997; Nagaoka and Isoda, 1997; Shirasu, 1998. A modal intuitionistic calculus of nuclei was developed by Goldblatt (Goldblatt, 1981; Goldblatt, 1979).

## 8. Conclusions

It seems obvious, even trite, that a logic of finite conjunction and arbitrary disjunction might be related to the finite intersections and arbitrary unions of open sets in topology. Locale theory shows how propositional geometric theories can be studied topologically. Nonetheless, geometric logic is very peculiar from the perspective of traditional logic. Its incompleteness seems a grave disadvantage, while its type-theoretic content in a first-order logic comes as a surprise.

Our basic message is that in a constructive geometric mathematics, topology appears as an emergent feature: the logical theories describe classes of models with an intrinsic topology (in Grothendieck's generalized sense, using sheaves when there are not enough opens), and mathematical constructions have an intrinsic continuity.

Paradoxically, the constructivity provides the way around the incompleteness. Normally one thinks of constructivity as the enemy of completeness, because so many completeness proofs are classical. But by allowing for constructive mathematics one gains access to a more complete range of models of each geometric theory. Amongst the Grothendieck toposes each theory has its classifying topos, equipped with the generic model. It serves as "generalized Lindenbaum algebra", but can also be thought of as "the space of models". Geometric morphisms are logic-preserving functors between the toposes, but can also be used (in the reverse direction) as continuous maps of models, at any stage. This is without reference to the concrete class of standard models, of which there might anyway be insufficient because of the incompleteness. The propositional fragment can alternatively be treated using locales (and frames as Lindenbaum algebras), but the two treatments are equivalent. In the spatial case the geometric morphisms recover the known notion of continuous map between spaces (modulo issues of sobriety).

That broad story underlies much of topos theory, though there are also many deep non-geometric uses of Grothendieck toposes.

The type theoretic content is an unfamiliar development in first order logic. In a sense it is superfluous, since it does not essentially extend the scope of geometric logic. Nonetheless it makes the logic more convenient and in particular it can be used to eliminate infinitary disjunctions in favour of finitary constructions.

Combining *coherent* logic with some of the geometric type constructors, we get a coherent type theory. This must be less expressive than geometric logic, yet it is already enough to capture important topological examples such as the real line. An exciting thought is that this may provide an example of topology emerging from a *finitary* type theory, with finite coproducts and the inductive construction of free models. Such a coherent type theory would be better described in its own terms, with a corresponding class of categories to interpret it. A promising candidate class for these categories is the arithmetic universes of Joyal. By contrast with Grothendieck toposes, these categories do not automatically have function spaces or subobject classifiers. This is going to require a much more careful syntactic formulation of the coherent type theory, probably including aspects of dependent type theory. Some preliminary results have been found in Maietti, 2003.

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