# Finite Ultrametric Spaces and Computer Science

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ABSTRACT The purpose of the paper is to describe a few properties of ultrametric spaces (in particular, of finite ones) and to demonstrate some applications of these properties to computer science.

A metric space $(X, d)$ is called *ultrametric* [6] (or *non-Archimedean* [4], or *isosceles* [9]) if its metric satisfies the strong triangle axiom:

$$d(x,z) \leq max[d(x,y), d(y,z)]. \qquad (\Delta)$$

This is usually called the Ultrametric Axiom. Ultrametric spaces were described up to homeomorphism in [3, 21], up to uniform equivalence in [10], and up to isometry in [9, 20]. A survey of their metric [9, 20], geometric [14, 20], uniform [10], and categorical [11–17] properties can be found in the literature. The theory of ultrametric spaces is closely connected with various branches of mathematics. These are number theory (rings $\mathbf{Z}_p$ and fields $\mathbf{Q}_p$ of $p$-adic numbers), algebra (non-Archimedean normed fields), real analysis (the Baire space $B_{\aleph_o}$), general topology (generalized Baire spaces $B_\tau$), $p$-adic analysis (field $\Omega$), $p$-adic functional analysis (algebras of $\Omega$-valued functions), lattice theory [17], Lebesgue measure theory [18], Euclidean geometry [14], category theory and topoi [13, 15, 16], and so on. These relations deal with infinite ultrametric spaces (mainly separable). For applications in computer science, finite spaces are of interest as well.

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## 1 Rationalization of Ultrametrics

Suppose we are going to study a metric space $(X, d)$ (or any other mathematical object supplied with a real valued function) using a computer. The key problem that should be solved first of all is the following. Is it possible to approximate the metric $d(x, y)$ by a rational valued (binary rational valued) metric $r(x, y)$ close to the initial metric $d(x, y)$ in a certain sense? Theorem 1 gives an affirmative answer to this question for a wide class of ultrametric spaces. Theorem 3 does it for the others.

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**Theorem 1.** [19] *Let* $(X,d)$ *be an ultrametric space and* $|V| = |\{d(x,y)|x,y \in X\}| \leq \aleph_0$. *Then for any* $\epsilon > 0$ *and any* $K > 1$ *there exists an ultrametric* $r(x,y)$ *over* $X$ *such that*

*a)* $(X,d)$ *and* $(X,r)$ *are homeomorphic,*

*b)* $(X,d)$ *and* $(X,r)$ *are uniformly equivalent,*

*c) an identity map* $i : (X,d) \to (X,r)$ *is non-expanding,*

*d) an inverse map* $i : (X,r) \to (X,d)$ *is K-Lipschitz, i.e.,* $r(x,y) \leq d(x,y) \leq K \cdot r(x,y)$,

*e) a difference between d and r is at most* $\epsilon$, $d(x,y) \geq r(x,y) \geq d(x,y) - \epsilon$,

*f) all values of new ultrametric* $r(x,y)$ *are binary rational,*

*g) the identity map* $i : (X,d) \to (X,r)$ *induces a map* $i^* : \{d(x,y)|x,y \in X\} \to \{r(x,y)|x,y \in X\}$ *and the latter is one-to-one.*

The requirement $|V| \leq \aleph_0$ is connected with the following property of ultrametric spaces, which does not hold for general metric spaces.

**Theorem 2.** [19, 20] *For any ultrametric space* $(X,d)$ *the set of values of its metric* $V = \{d(x,y)|x,y \in X\}$ *has cardinality no greater than its weight,* $|V| \leq w(X)$.

This shows that the requirement $|V| \leq \aleph_0$ holds, in particular, for all separable ultrametric spaces. On the other hand, if the set $V$ is uncountable then obviously no one-to-one rationalization can exist. However, omitting the requirement g) we can prove the following theorem.

**Theorem 3.** [19] *For any ultrametric space* $(X,d)$ *there exists a uniformly equivalent binary rational ultrametric* $r(x,y)$ *satisfying the statements a)–f) of Theorem 1.*

So we see that for any ultrametric space $(X,d)$ its metric $d(x,y)$ can be rationalized arbitrarily closely in several reasonable senses simultaneously (up to topological equivalence, uniform equivalence, an arbitrary small $\epsilon$-variation, etc.). Note that the assertion e) means that the ultrametrics $d$ and $r$ are close to each other in the metric of uniform convergence on $X \otimes X$, while the assertion d) means the same for the metric of logarithm uniform convergence: $0 \leq \log[d/r] = |\log d - \log r| \leq \log K$.

Unfortunately, the situation with general metric spaces (not ultrametric ones) is worse and more complicated. Moreover, in general, their metrics cannot be rationalized even up to homeomorphism. Actually, if a metric $d(x,y)$ is rational valued then, for any irrational $s$, a closed ball $B(x,s) = \{y|d(x,y) \leq s\}$ is open. Hence $(X,d)$ is small-inductive zero-dimensional, $\operatorname{ind} X = 0$. If the stronger equality holds, $\operatorname{Ind} X = \dim X = 0$ then, in view of a Morita–de Groot theorem [3, 21], $(X,d)$ admits a topologically equivalent ultrametric $\Delta(x,y)$. Theorem 3 then enables us to rationalize it. So small inductive zero-dimensionality $\operatorname{ind} X = 0$ is necessary and (large inductive) zero-dimensionality $\operatorname{Ind} X = 0$ is sufficient for a topological rationalizability of a space. Moreover, the last requirement is necessary and sufficient for a topological rationalizability of general metric space by ultrametrics. This follows from zero-dimensionality Ind $X = 0$ of any ultrametric space and the Morita–de Groot theorem. A criterion for uniform rationalizability of general metric spaces by ultrametrics can be found in [19]. This is large proximate zero-dimensionality $\mathrm{In}\delta X = 0$ of a space (see [10] for the definition). The last property coincides with proximate zero-dimensionality $\delta \dim X = 0$ in the sense of Smirnov [10, 27]. Recall that $\delta \dim X = n$ means that Smirnov's compactification $\sigma X$ of a proximity space $X$ is $n$-dimensional, $\dim(\sigma X) = n$ (see [26, 27]). As mentioned above small inductive zero-dimensionality ind $X = 0$ is necessary and (large inductive) zero-dimensionality Ind $X = \dim X = 0$ is sufficient for a topological rationalizability of a space by general metrics. Thus the following problem naturally arises.

**Problem 1.** There exist metric spaces with non-equal dimensions, ind $X = 0$ and Ind $X > 0$ (e.g., Roy's space [25] or Mrowka's space [22]). *Is it possible to introduce in such a space a rational-valued metric $r(x, y)$ up to topological equivalence at least?*

## 2 Distance Function and Lattice of Balls in Ultrametric Spaces

If an ultrametric space $(X, d)$ is finite, then all the requirements mentioned above are obviously satisfied. Moreover, in this case the set $V = \{d(x, y) | x \neq y \in X\}$ of non-zero values of the metric $d$ has additional computational properties.

**Theorem 4.** *For any finite ultrametric space $(X, d)$ consisting of $n + 1$ points, the set $V$ of values of its metric contains at most $n$ elements.*

**Proof.** For $n = 1$ and $|X| = n + 1 = 2$, $|V| = 1$. Let $n = 2$ and $|X| = n + 1 = 3$. It is well known [6] that the strong triangle inequality implies that any three points form an isosceles triangle with base no greater than the sides. Thus $|V| \leq 2$. Suppose the theorem is true for any $k$-point space, where $k \leq n$, and let $|X| = n + 1$. By Lemma 3 [14], $n + 1$ points of an ultrametric space can be enumerated in such a way that

$$\begin{aligned} \min\{d(a_i, a_j) | i \neq j\} &= d(a_0, a_1) \leq d(a_0, a_2) \leq \cdots \leq d(a_0, a_n) \\ &= \max\{d(a_i, a_j)\}. \end{aligned} \tag{$*$}$$

Let us examine the chain $(*)$ from right to left looking for the first place where there is a strict inequality. Two different cases should be considered here.

Case 1. The inequalities $(*)$ have the form

$$\begin{aligned}\min\{d(a_i,a_j)|i\neq j\} &= d(a_0,a_1)\le d(a_0,a_2)\\ &\le\cdots\le d(a_0,a_{n-1})<d(a_0,a_n)=\max\{d(a_i,a_j)\}.\end{aligned}$$

Then it follows from the axiom $(\Delta)$ that the point $a_n$ is at the same distance from any point $a_k$ for $k<n$, $d(a_0,a_n)=d(a_k,a_n)$. By the inductive assumption, the set $V_{n-1}=\{d(a_k,a_j)|k,j<n\}$ has cardinality no greater than $n-1$. Thus the set $V=V_{n-1}\cup\{d(a_0,a_n)\}$ has cardinality $\le n$.

Case 2. The inequalities $(*)$ have the form

$$\begin{aligned}\min\{d(a_i,a_j)|i\neq j\} &= d(a_0,a_1)\le\cdots\le d(a_0,a_{m-1})<d(a_0,a_m)\\ &=\cdots=d(a_0,a_n)=\max\{d(a_i,a_j)\}.\end{aligned}$$

Then axiom $(\Delta)$ implies that all points $a_k$ for $k=m,\ m+1,\dots,n$, are at the same distance $d(a_0,a_n)$ from any point $a_j$ for $j=0,1,\dots,k-1$. By the inductive assumption, the sets $V_{m-1}=\{d(a_k,a_j)|k,j<m\}$ and $V_m=\{d(a_k,a_j)|k,j\ge m\}$ have cardinalities $|V_{m-1}|\le m-1$ and $|V_m|\le (n-m+1)-1=n-m$. Since $V\subseteq V_{m-1}\cup V_m\cup\{d(a_0,a_n)\}$, $|V|\le (m-1)+(n-m)+1=n$. ∎

For general metric spaces a potency of the set $V$ satisfies the inequality $|V|\le n(n+1)/2$ and increases quadratically as $n\to\infty$. For ultrametric spaces it does linearly. This improves estimation of computer memory capacity and rate of computation.

A few other properties of ultrametric spaces follow from the structure of the set of balls of a space. This leads us to two results of different kinds connecting the theory of ultrametric spaces with lattice theory and computational modeling. To do it we, first of all, should refine the notion of radius of a ball. Usually a set $B(a,s)=\{x|d(x,a)\le s\}$ is called a *closed ball of radius* $s$ with a center located at $a$. However, such a notion gives us too many balls in a space (at least continuously many balls in each non-empty set). However, it seems more natural to say that there are only two balls in a one-point space $X=\{a\}$, namely, the empty set $\oslash$ and $\{a\}$, and only four balls in a two-point space $X=\{a,b\}$, namely, the empty set $\oslash$, two balls of radius zero (= points $a$ and $b$), and the whole space $X$ (= the ball of radius $d(a,b)$ with a center located at $a$ or $b$). That is why we call a number s, written above, to be *a nominal radius* of a ball and introduce the following definition.

*An actual radius of ball* $B(a,s)$ is a number $r=\sup\{d(a,x)|x\in B(a,s)\}$.

Obviously, $r\le s$. In an ultrametric space, balls have a lot of surprising properties, *exempli gratia*,

- *Any point of a ball is its center, i.e.,* $B(a,r)=B(x,r)$ *for any point* $x\in B(a,r)$.

- *An actual radius of a ball is equal to its diameter, i.e.,* $r = \sup\{d(x, y)|x, y \in B(a, s)\}$.
- *Any two balls are either disjoint or one of them is a subset of the other.*
- *If the balls* $B(a, s)$ *and* $B(b, t)$ *are disjoint, then* $d(a, b) = d(x, y)$ *for any* $x \in B(a, s)$ *and any* $y \in B(b, t)$.

Moreover, it turns out that the set of balls of an ultrametric space is a lattice $L(X)$ and there is a duality between ultrametric spaces and a certain class of lattices.

**Theorem** [17]. *For any ultrametric space* $(X, d)$ *there is a complete, atomic, tree-like, and real graduated lattice* $(\mathbf{L}(X),\ \sup, \cap, r(B(\alpha)))$ *and for any complete, atomic, tree-like, and real graduated lattice* $(L, \vee, \wedge, r(\alpha))$ *there is an ultrametric space* $(\mathbf{A}(L),\ \Delta)$ *such that*

- *the space* $(X, d)$ *is isometric to the space* $(\mathbf{A}(\mathbf{L}(X)),\ \Delta)$;
- *the lattice* $(L, \wedge, \vee, r(\alpha))$ *is isomorphic to the lattice* $(\mathbf{L}(\mathbf{A}(L)),\ \cap, \sup, r(B(\alpha)))$.

A similar theorem holds for morphisms. This means that there is an isomorphism functor between the category **ULTRAMETR** of ultrametric spaces and non-expanding maps and the category **LAT**$*$ of complete, atomic, tree-like, and real graduated lattices and isotonic, semi-continuous, non-extensive maps (see [17] for proofs, definitions and details).

**Theorem 5**. *For any finite ultrametric space* $(X, d)$ *consisting of* $n$ *points, the set* $\mathbf{L}(X)$ *of its closed balls contains at most* $2n$ *elements,* $|\mathbf{L}(X)| \leq 2|X|$.

**Proof**. For $n = 1, X = \{a\}$ is a singleton and $\mathbf{L}(X) = \{\emptyset, \{a\}\}$, thus $|\mathbf{L}(X)| = 2 = 2|X|$. Suppose the theorem holds for any $k$-point space with $k \leq n$ and let $|X| = n + 1$. Following similar arguments as above we enumerate the points of space in the same manner and consider two cases.

Case 1. $\min\{d(a_i, a_j)|i \neq j\} = d(a_0, a_1) \leq d(a_0, a_2) \leq \cdots \leq d(a_0, a_{n-1}) < d(a_0, a_n) = \max\{d(a_i, a_j)\}$. Denote by $X'$ the set $X' = \{a_0, a_1, a_2, \ldots, a_{n-1}\}$. By the inductive assumption, $|\mathbf{L}(X')| \leq 2n$. Since $d(a_0, a_n) = d(a_k, a_n) > d(a_k, a_m) \forall k, m < n$, we have $\mathbf{L}(X) = \mathbf{L}(X') \cup X$, thus $|\mathbf{L}(X)| \leq 2n + 1 < 2(n + 1) = 2|X|$.

Case 2. $\min\{d(a_i, a_j)|i \neq j\} = d(a_0, a_1) \leq \cdots \leq d(a_0, a_{m-1}) < d(a_0, a_m) = \cdots = d(a_0, a_{n-1}) = d(a_0, a_n) = \max\{d(a_i, a_j)\}$. Denote by $X_{m-1}$ and $X_m$ the sets $X_{m-1} = \{a_0, a_1, a_2, \ldots, a_{m-1}\}$ and $X_m = \{a_m, a_{m+1}, \ldots, a_n\}$. By the inductive assumption, $|\mathbf{L}(X_{m-1})| \leq 2|X_{m-1}| = 2m$ and $|L(Xm)| \leq 2|X_m| = 2(n - m + 1)$. As mentioned above all points $a_k$ for $k \geq m$, are at the same distance $d(a_0, a_n)$ from all points $a_j$ for $j < m$. Thus $\mathbf{L}(X) \subseteq \mathbf{L}(X_{m-1}) \cup \mathbf{L}(X_m) \cup \{X\}$. The empty set $\oslash$ belongs to both of the lattices $\mathbf{L}(X_{m-1})$ and $\mathbf{L}(X_m)$ and is counted in both of them. Therefore $|\mathbf{L}(X)| \leq 2m + 2(n - m + 1) - 1 + 1 = 2(n + 1) = 2|X|$. ∎

**Example.** Let $X = \{0, 1, 2, \ldots, n - 1\}$ be a subset of natural numbers equipped with the following ultrametric $d(k, m) = \max(k, m)$. The set $\mathbf{L}(X)$ consists of $n$ singletons $\{k\}$ (= balls of radius zero), the empty set $\oslash$, and $n - 1$ balls $B_m = \{0, 1, 2, \ldots, m\} = B(0, m)$ of radius $m$ for $m = 1, 2, \ldots, n - 1$. Thus $|\mathbf{L}(X)| = 2n$ and the estimation $|\mathbf{L}(X)| \leq 2|X|$ cannot be improved.

**Note.** The inequality $|\mathbf{L}(X)| = 2|X|$ does not hold for infinite spaces. Let $X = \mathbf{Q}_+$ be a set of non-negative rationals with the same metric $d(x, y) = \max(x, y)$ (we call it *the max-metric*). The set $\mathbf{L}(X) = \{B(0, r) | r \in \mathbf{R}_+\}$ has cardinality of the continuum.

For general metric spaces a potency $|\mathbf{L}(X)|$ of the set of balls increases quadratically as $n \to \infty$. For ultrametric spaces it increases linearly as the cardinality $|V|$ does. Since all closed balls $B(a, s) = \{x | d(x, a) \leq s\}$ are open they form a clopen base for the topology of $X$. This enables us to construct a computational model for a space $X$ in the sense of Lawson [8]. Following this way Flagg and Kopperman studied a problem of existence of an **algebraic** computational model and proved the following.

**Theorem** [2] *A topological space $X$ has an algebraic computational model if and only if it is separable, complete and ultrametric* (see [8, 2] and references there).

# 3 Categorical Operations and Embedding Theorems

Let us consider a category **METR** ($\mathbf{METR_c}$) of all metric spaces (of diameter not greater than $c$) and non-expanding maps. These are the maps $f : (X, d) \to (Y, r)$, which do not enlarge distances, i.e., $r(f(x),\ f(y)) \leq d(x, y)$ for all $x$ and $y$ in $X$. It is known that sums and products, pull-backs and push-outs, equalizers and co-equalizers, and limits of direct and inverse spectra exist in $\mathbf{METR_c}$ and that the subcategory of ultrametric spaces and the same maps $\mathbf{ULTRAMETR_c}$ is closed in $\mathbf{METR_c}$ with respect to these operations (see [11, 15, 17]). As for the category **METR** we have the following.

**Proposition 1.** *A sum of objects does not exist in* **METR** *even for two singletons.*

**Proposition 2.** *For any finite family $\{(X_k, d_k)|k = 1, 2, \dots, n\}$ of metric spaces, there exists a product $(X_\Pi, d_\Pi)$ of these spaces in* **METR** *(called* a metric product*)* $m\Pi\{(X_k, d_k)\} = (X_\Pi, d_\Pi)$, *where $X_\Pi = \Pi X_k$ is a product of the sets $X_k$ in the category* **SET**, *and $d_\Pi(\{x_k\}, \{y_k\}) = \max\{d_k(x_k, y_k)|k = 1, 2, \dots, n\}$.*

**Proposition 3.** *A metric product of an infinite family $\{(X_\alpha, d_\alpha)|\alpha \in I\}$ of metric spaces exists iff there is $c > 0$ such that for almost all of $(X_\alpha, d_\alpha)$, diam $(X_\alpha, di_\alpha) < c$.*

**Proof.** The part "if" of the statement is obvious, $\{d_\Pi(\{x_\alpha\}, \{y_\alpha\}) = \sup d_\alpha(x_\alpha, y_\alpha)|\alpha \in I\}$ is a finite number for any pair of points $\{x_\alpha\}, \{y_\alpha\}$ in $m\Pi\{(X_\alpha, d_\alpha)\}$. Consider a family $\{(X_\alpha, d_\alpha)|\alpha \in I\}$, which does not satisfy the assertion of the proposition. Then there exists a subfamily $\{(X_n, d_n)|n \in \mathbb{N}\}$ and points $x_n, y_n \in X_n$, such that $d_n(x_n, y_n) \geq n$. Suppose that the product $m\Pi\{(X_\alpha, d_\alpha)\}$ does exist. Then there are non-expanding projections $p_\alpha : m\Pi\{(X_\alpha, d_\alpha)\} \to (X_\alpha, d_\alpha)$ satisfying the standard requirements. Let $Y = \{a\}$ be a singleton and let $f_\alpha, g_\alpha : Y \to (X_\alpha, d_\alpha)$ be two families of maps such that $f_n(a) = x_n, g_n(a) = y_n \forall n \in \mathbf{N}, f_a(a) = g_\alpha(a) = x_\alpha$ for $\alpha \in \mathbf{I} \backslash \mathbf{N}$. Since $m\Pi\{(X_\alpha, d_\alpha)\} = (X_\Pi, d_\Pi)$ is a product in the category **METR** there are the maps $f : Y \to X_\Pi$ and $g : Y \to X_\Pi$ such that $p_\alpha \cdot f = f_\alpha$ and $p_\alpha \cdot g = g_\alpha \forall \alpha \in \mathbf{I}$. Since projections $p_\alpha$ are non-expanding the distance $d_\Pi(f(a), g(a))$ should satisfy the inequality $d_\Pi(f(a), g(a)) \geq d_\alpha(p_\alpha f(a), p_\alpha g(a)) = d_\alpha(f_\alpha(a), g_\alpha(a)) \forall \alpha \in \mathbf{I}$, in particular, $d_\Pi(f(a), g(a)) \geq d_n(f_n(a), g_n(a)) = d_n(x_n, y_n) \geq n, \forall n \in \mathbf{N}$. This implies that $d_\Pi(f(a), g(a)) = \infty$. ■

**Note.** If $\sup\{\text{diam}(X_\alpha, d_\alpha)\} = c < \infty$ then the product $m\Pi\{(X_\alpha, d_\alpha)\}$ in **METR** coincides with the product of the same spaces in $\mathbf{METR_c}$.

**Lemma 1.** *Subcategories* **ULTRAMETR** *and* $\mathbf{ULTRAMETR_c}$ *are closed in* **METR** *and* $\mathbf{METR_c}$ *with respect to product.*

**Proof.** The proof for $\mathbf{METR_c}$ can be found in [15] and that for **METR** is similar. ■

Denote by $(D_\alpha, d_\alpha)$ a space consisting of two real numbers $\{0, c_\alpha\}$ with a metric $d_\alpha(0, c_\alpha) = c_\alpha$. Since two-point spaces are obviously ultrametric and any subset of an ultrametric space is ultrametric again, we obtain the following.

**Corollary 1.** *Any subset of a metric product $m\Pi\{(D_\alpha, d_\alpha)|\alpha \in I\}$ of an arbitrary family of two-point spaces, is ultrametric.*

**Theorem 6.** *For any ultrametric space $(X, d)$ there exists a family $\{(D_\alpha, d_\alpha)|\alpha \in I\}$ of two-point spaces such that $(X, d)$ is isometric to a subspace of their metric product and a number of factors in the product is not greater than the weight of $X, |I| \leq w(X)$.*

**Proof**. For the sake of simplicity, suppose first that $X$ is of finite diameter. Let $Z \subset X$ be a dense subset of cardinality $\tau$, $\tau = |Z| = w(X)$. For any $z \in Z$, consider a set of all open balls $B^\circ(z,r) = \{x|d(z,x) < r\}$ of radius $r$ such that there is $y \in X$ with $d(z,y) = r$ (= the balls with non-empty spheres). By Theorem 2, the set of all such balls is of cardinality at most $\tau$. Denote by $\mathbf{L}^\circ(X)$ a set of all these balls for all $z \in Z$, i.e., $\mathbf{L}^\circ(X) = \{B^\circ(z,r)|z \in Z\} = \{B(\alpha)|\alpha \in I\}$. Clearly $|\mathbf{L}^\circ(X)| = |\mathbf{I}| = \tau \cdot \tau = \tau$ (here we consider spaces of infinite weight, for finite spaces see Theorem 9 below). For any such a ball $B^\circ(z,r) = B(\alpha)$, let us define a function $f_\alpha : X \to D_\alpha = \{0, c_\alpha\}$ as follows: $f_\alpha|_{B(\alpha)} = 0, f_\alpha|_{X\setminus B(\alpha)} = c_\alpha$, where $c_\alpha = r$ is a radius of the ball $B(\alpha) = B^\circ(z,r)$. Properties of balls mentioned in Section 2 imply that $f_\alpha$'s are non-expanding. A categorical product $f = \Pi f_\alpha$ is non-expanding as well and it maps $X$ in the metric product $m\Pi\{D_\alpha|\alpha \in I\}$. For any two points $x$ and $y$ in $X$ there is a point $z \in Z$ such that $x \in B^\circ(z,r)$, $y \notin B^\circ(z,r)$ and $r = d(x,y) = d(z,y)$. For any $\alpha, d_\alpha(f_\alpha(x), f_\alpha(y)) = c_\alpha$ if $x \in B(\alpha)$ and $y \notin B(\alpha)$ or $x \notin B(\alpha)$ and $y \in B(\alpha)$, and $d_\alpha(f_\alpha(x),\ f_\alpha(y)) = 0$ otherwise. Consequently, $d_\Pi(f(x), f(y)) = d_\Pi(\{f_\alpha(x)\},\ \{f_\alpha(y)\}) = \sup\{d_\alpha(f_\alpha(x),\ f\alpha(y))|\alpha \in I\} = \sup\{c_\alpha|c_\alpha \le d(x,y)\} = d(x,y)$. Thus $f$ is an isometric embedding. ■

To embed a space of infinite diameter we are to study a category **ULTRAMETR*** (**METR***) of all (ultra-) metric spaces $(X, d(x,y), x^*)$ with a base point $x^*$ (pointed metric spaces) and non-expanding maps that take a base point to a base point. In contrast to Proposition 1 we have

**Proposition 4**. *For any family* $\{(X_\alpha,\ d_\alpha,\ x_\alpha*)|\alpha \in I\}$ *of pointed metric spaces, there exists a sum* $(X_\sum,\ d_\sum,\ x^*)$ *of these spaces in the category* **METR*** $m\sum\{(X_\alpha,\ d_\alpha,\ x_\alpha*)|\alpha \in I\} = (X_\sum,\ d_\sum,\ x^*)$, *where* $(X_\sum,\ x^*)$ *is a sum of the sets* $(X_\alpha, x_\alpha*)$ *in the category* **SET***, *and* $d_\sum(x_\alpha,\ x^*) = d_\sum(x_\alpha, x_\alpha^*), d_\sum(x_\alpha, y_\alpha) = d_\alpha(x_\alpha, y_\alpha), d_\sum(x_\alpha, y_\beta) = d_\alpha(x_\alpha, x_\alpha^*) + d_\beta(y_\beta, x_\beta^*)$ *for* $\alpha \neq \beta$.

**Proposition 5**. *For any family* $\{(X_\alpha, d_\alpha, x_\alpha^*)|\alpha \in I\}$ *of pointed ultrametric spaces, there exists a sum* $(X_\sum, d_\sum, x^*)$ *of these spaces in the category* **ULTRAMETR*** *(called* a pointed sum*)* $m\sum\{(X_\alpha, d_\alpha, x_\alpha^*)|\alpha \in I\} = (X_\sum, d_\sum, x^*)$, *where* $(X_\sum, x^*)$ *is a sum of the sets* $(X_\alpha, x_\alpha^*)$ *in the category* **SET***, *and* $d_\sum(x_\alpha, x^*) = d_\sum(x_\alpha, x_\alpha^*), d_\sum(x_\alpha, y_\alpha) = d_\alpha(x_\alpha, y_\alpha), d_\sum(x_\alpha, y_\alpha) = \max[d_\alpha(x_\alpha, x_\alpha^*), d_\beta(y_\beta, x_\beta^*)]$ *for* $\alpha \neq \beta$.

**Proposition 6**. *For any family* $\{(X_\alpha, d_\alpha, x_\alpha^*)|\alpha \in I\}$ *of pointed metric spaces, there exists a product* $(X_\Pi, d_\Pi, x^*)$ *of these spaces in the category* **METR*** *(called a* pointed product*)* $m\Pi^*\{(X_\alpha, d_\alpha, x_\alpha^*)|\alpha \in I\} = (X_\Pi, d_\Pi, x^*)$. *Here* $x^* = \{x_\alpha^*\}, d_\Pi(\{x_\alpha\}, \{y_\alpha\}) = \sup\{d_\alpha(x_\alpha, y_\alpha)|\alpha \in I\}$, *and* $X_\Pi$ *is a subset of the Cartesian product* $\Pi X_\alpha$ *of the sets* $X_\alpha$ *consisting of those points* $\{x_\alpha\} \in \Pi X_\alpha$ *that are at a finite distance from the point* $x^*, \sup\{d_\alpha(x_\alpha, x_\alpha^*)|\alpha \in I\} < \infty$.

**Lemma 2.** *The subcategory* **ULTRAMETR*** *is closed in* **METR*** *with respect to multiplication* (see [15]).

Coming back to the proof of Theorem 6 we choose an arbitrary point $x^* \in X$ as a base point in $X$, choose 0 as a base point in every $D_\alpha$, and modify slightly the definition of the functions $f_\alpha(x)$. Namely let $f_\alpha(x)$ be the same as above if $x^* \in B(\alpha)$ and let it be $c_\alpha - f_\alpha(x)$ otherwise. Then new functions $f_\alpha(x)$ take $x^*$ to $0_\alpha \in D_\alpha$ for all $\alpha$, the product $f = \Pi f_\alpha$ takes $x^*$ to $\mathbf{0} = \{0_\alpha\} \in m\Pi^*\{D_\alpha\}$, and it maps $(X, d, x^*)$ in $m\Pi^*\{(D_\alpha, d_\alpha, 0_\alpha)\}$ in view of the definition of the pointed product. The rest of the proof is obvious. ■

Compare the last theorem with the classic theorems on embedding of topological spaces in a product of two-point spaces.

**Theorem** (P.Alexandroff, 1936 [1]). *Any $T_0$-space of weight $\leq \tau$ can be embedded homeomorphically in the Alexandroff cube $F^\tau$ of weight $\tau$.*

Recall that the Alexandroff cube $F^\tau = \Pi\{F_\alpha | \alpha \in \mathbf{I}\}$ is a topological product (a product in the category **TOP**) of $\tau$ connected two-point spaces $F_\alpha = \{0,1\}_\alpha$, i.e., the spaces with open sets $\oslash, \{0\}$, and $\{0,1\}$.

**Theorem** (P.Alexandroff, 1936 [1]). *Any small inductive zero-dimensional space of weight $\leq \tau$ (*ind $X = 0$*) can be embedded homeomorphically in the Cantor cube $D^\tau$ of weight $\tau$ ( $=$ a topological product of $\tau$ discrete two-point spaces $D_\alpha = \{0,1\}_\alpha$ ) .*

A close analogy between these theorems is obvious. The way that they differ is probably more interesting. First, the weight of spaces $F^\tau$ and $D^\tau$ is exactly $\tau$ whereas the weight of a metric product $m\Pi\{(D_\alpha, d_\alpha) | \alpha \in \mathbf{I}\}$ depends on the "sizes" of factors; it is generally greater, $\tau \leq w(m\Pi\{D_\alpha | \alpha \in I\}) \leq 2^\tau$. E.g., a product $m\Pi\{\{0, 1/n\} | n \in \mathbf{N}\}$ of a countable set of two-point spaces $\{0, 1/n\}$ is separable. It is easy to show a countable dense subset there. And the product $X_\Pi = m\Pi\{\{0,1\}_n | n \in \mathbf{N}\}$ of a countable set of two-point spaces $\{0,1\}$ is a metrically discrete set of cardinality of the continuum, $d(x,y) = 1 \forall x, y \in X_\Pi$. Thus its weight is also $c$. Furthermore, in Alexandroff's theorems, all factors are topologically equivalent to each other, their product is a universal space in the category **TOP**, whereas the factors $D_\alpha = \{0_\alpha, c_\alpha\}$ in Theorem 6 are not metrically equivalent to each other. Theorem 6 assigns, for each space $(X, d)$, its own particular family of two-point spaces, and their product is not metrically universal.

The problem of the existence of a universal ultrametric space for all spaces of a given weight was studied in [9, 20]. For any cardinal $\tau$, there is constructed an ultrametric space $LW_\tau$, which contains isometrically all ultrametric space of weight $\leq \tau$ (a *$\tau$-universal space*). However, sometimes this space turns out to be too large. In the rest of the section, we describe a smaller universal space $LV_\tau \subset LW_\tau$, relate the spaces $LV_\tau$ and $LW_\tau$ to the notions of metric sum and product, and deduce a theorem on embedding of ultrametric spaces in Banach spaces.

**Definition** [20]. Let $\tau$ be an arbitrary cardinal, $\omega(\tau)$ be the first ordinal of potency $\tau, W_\tau = \{\alpha|\alpha < \omega(\tau)\}$ be the set of all ordinals smaller than $\omega(\tau)$ (= the set of all ordinals of potency $< \tau$). A function $f : \mathbf{Q}_+ \to W_\tau$ is called *eventually* 0*-valued* (or *eventually vanished*) if there is a number $X(f)$ such that $f(x) = 0 \ \forall\ x > X$. For any two such maps $f$ and $g$ let us define a distance $\Delta(f,g)$ by the equality $\Delta(f,g) = \sup\{x|f(x) \neq g(x)\}$. Denote by $(LW_\tau, \Delta)$ the space of all these functions equipped with the metric $\Delta$.

It is easy to prove that $\Delta$ is a metric satisfying the strong triangle axiom and that $(LW_\tau, \Delta)$ is a complete ultrametric space [9, 20].

**Theorem 7.** *Every ultrametric space of weight $\leq \tau$ can be embedded isometrically in $LW_\tau$. The weight of $LW_\tau$ equals its potency and equals $\tau^{\aleph_o}, |LW_\tau| = w(LW_\tau) = \tau^{\aleph_o}$.*

**Proof.** The key idea of the proof is used in the proof of Theorem 8 below for $\tau = \aleph_0$. For arbitrary $\tau$ as well as for details see [20]. ■

For any cardinal $\tau \leq c, \tau^{\aleph_o} = c$. The weight of the universal space $w(LW_\tau) = \tau^{\aleph_o} = c$ seems to be too great, especially for finite $\tau$. However, the following proposition shows that a weight of a $\tau$-universal space cannot be smaller even for $\tau = 2$.

**Proposition 7.** *If an ultrametric space $(U,d)$ contains isometrically all two-point spaces, then its weight is not less than that of the continuum, $w(U,d) \geq c$.*

**Proof.** It follows directly from Theorem 2. ■

So for all cardinals $\tau \leq c$, the weight of the universal space $LW_\tau$ is the smallest of possible ones. Cardinals $\tau > c$ can be divided in two classes: namely, satisfying $\tau^{\aleph_o} = \tau$ and $\tau^{\aleph_o} > \tau$, respectively. For all cardinals in the first class the weight of $LW_\tau$ is $\tau^{\aleph_o} = \tau$, thus it is the smallest weight of a $\tau$-universal space. For cardinals from the second class we have the following problem [20].

**Problem 2** [20]. Does there exist a $\tau$-universal ultrametric space $(U,d)$ with weight smaller than $\tau^{\aleph_o}$ for cardinals $\tau$ such that $\tau^{\aleph_o} > \tau > c$? In particular, does there exist one of weight $\tau$?

A partial answer to this problem was given by J. Vaughan [31]. Analyzing the proof of Theorem 7 [20], he has mentioned that actually every ultrametric spaces of weight $\leq \tau$ is embedded there in a somewhat smaller space (denoted by $LW_\tau^\backprime$), that is a closure of the set of all "bounded" eventually 0-valued functions [31]. A function $f$ is called *bounded* if it maps $\mathbf{Q}_+$ into a set $[0,\beta]$ for some $\beta < \omega(\tau)$. Vaughan proved that the weight of $LW_\tau^\backprime$ is $\tau \cdot \sum\{\alpha^{\aleph_o}|\alpha < \tau\}$, [31]. Thus for cardinals from the second class that satisfy the inequality $\sum\{\alpha^{\aleph_o}|\alpha < \tau\} \leq \tau$ (we call them *Vaughan's cardinals*), there exists a $\tau$-universal space of weight $\tau$, namely $LW_\tau^\backprime$ [31]. Vaughan also shows that, under the Singular Cardinal Hypothesis (a set-theoretic assumption, whose negation implies the existence of measurable cardinals), every cardinal $\tau > c$ is a Vaughan cardinal [31]. So the problem is solved in the affirmative under ZFC + SCH. However, the question is still open in ZFC without any additional set-theoretic axioms (see [20, 31 and 32] for details and discussion on the relation of Problem 2 to the problem of large cardinals).

In fact, as mentioned in [20], even the subset $LV_\tau \subset LW_\tau$ of all monotone left semi-continuous functions $f : \mathbf{Q}_+ \to W_\tau$ is enough to contain all spaces of weight $\leq \tau$. It is easily proved that $LV_\tau \subset LW^{\backslash}_\tau$. The difference between $LW_\tau$ and $LW^{\backslash}_\tau$ is inessential for finite $\tau$ (see Theorem 8 below), clearly $LW^{\backslash}_\tau = LW_{\tau-1}$, whereas that between $LW_\tau$ and $LV_\tau$ is great whenever $\tau \geq 2$. For cardinals of uncountable confinality $LW^{\backslash}_\tau = LW_\tau$ [31].

Consider the simplest case, $\tau = 2$. The space $LV_2$ consists of all monotone decreasing left semi-continuous functions $f : \mathbf{Q}_+ \to \{0, 1\}$. Every such a function is of the form $f(q) = f_t(q) = 1$ for $q \leq t, f_t(q) = 0$ for $q > t$. Thus it can be identified with the real number $t \in \mathbf{R}_+$ and the space $LV_2$ is none other than a set of positive numbers equipped with the max-metric, $(LV_2, \Delta) = \mathbf{R}_+, \max)$. The latter is an ultrametric space of weight $c$ and it is obviously universal for all two-point spaces; any two-point space $\{0, d\}$ can be embedded in $(\mathbf{R}_+, \max)$ by the identity $i(0) = 0,\ i(d) = d$. Moreover, this is just the same as the pointed sum of all two-point spaces of positive "lengths".

**Proposition 8.** *The space* $(LV_2, \Delta) = (\mathbf{R}_+, \max)$ *is naturally isometric to the pointed sum of all two-point spaces of positive "lengths"* $(LV_2, \Delta) = m\sum^*\{\{0, r\} | r \in \mathbf{R}_+\}$.

To get a universal space of a type of metric product it is enough to take a pointed product of the same two-point spaces $m\Pi^*\{\{0, r\} | r \in \mathbf{R}_+\}$. There are continuously many factors in that product. Better, it is enough to multiply a countable number of factors to reach universality, namely, a countable set of two-point spaces whose lengths are dense in $\mathbf{R}_+$, e.g., spaces of rational lengths, $m\Pi^*\{\{0, q\} | q \in \mathbf{Q}_+\}$. We call this space a *rational cube*, and denote it by $q^Q$.

**Proposition 9.** *The rational cube* $q^Q$ *is a* 2*-universal ultrametric space.*

**Proof.** An isometric embedding $i : \{0, d\} \to m\Pi^*\{\{0, q\} | q \in \mathbf{Q}_+\}$ defined as $i(0) = \mathbf{0} = \{0_q\}, i(d) = \{x_q\}$, where $x_q = q$ for $q \leq d, x_q = 0$ for $q > d$, is a desired one. ■

The space $LW_2$ is also 2-universal, by Theorem 7 (and also by the inclusion $LW_2 \supset LV_2$). The amazing fact is that these two spaces coincide with one another.

**Proposition 10.** *The universal ultrametric space* $LW_2$ *is naturally isometric to the rational cube* $q^Q = m\Pi^*\{\{0, q\} | q \in \mathbf{Q}_+\}$.

**Proof**. The desired isometry takes any function $f \in LW_2$ to a point $\{x_q\} \in \Pi\{\{0,q\}|q \in \mathbf{Q}_+\}$ defined by the equalities $x_q = q$ if $f(q) = 1, x_q = 0$ if $f(q) = 0$. Since $f$ is eventually 0-valued there is $p > 0$ such that $f(q) = 0$ for all $q > p$. Thus $d_\Pi(\mathbf{0}, \{x_q\}) = \sup\{d_q(0,x_q)|q \in \mathbf{Q}_+\} = \sup\{q|x_q \neq 0\} = \sup\{q|f(q) = 1\} \leq \sup\{q|q \leq p\} = p$. Therefore the point $\{x_q\}$ belongs to the *pointed* product $m\Pi^*\{\{0,q\}|q \in \mathbf{Q}_+\} = q^Q$. On the contrary, an inverse map $i^{-1}$ takes any point $\{x_q\} \in q^Q$ to a function $f : \mathbf{Q}_+ \to \{0,1\}$ defined as follows: $f(q) = 1$ if $x_q = q, f(q) = 0$ if $x_q = 0$. Since $\{x_q\}$ belongs to the *pointed* product, $d_\Pi(\mathbf{0}, \{x_q\}) = \sup\{d_q(0,x_q)|q \in \mathbf{Q}_+\} = \sup\{x_q|x_q \neq 0\} = \sup\{q|x_q \neq 0\}$ is finite. Thus there is $p > 0$ such that $x_q = 0\ \forall q > p$, hence the corresponding function $f(q)$ eventually vanishes, $f(q) \in LW_2$. Isometry is obvious. ∎

As mentioned above the weight of $LW_\tau$ as well as that of $LV_\tau$ is $c$ for all $\tau$ from 2 to $c$. $LV_2$ is 2-universal but it is not $n$-universal for any $n > 2$, for $n = 3$ either; this follows, for instance, from the fact that there is no equilateral triangle in $LV_2 = (\mathbf{R}_+, \max)$. Similarly, $LV_n$ is $n$-universal but not $(n+1)$-universal. Fortunately, the space $LW_2$ turns out to be $\tau$-universal not only for every finite $\tau = n$, but also for all cardinals no greater than $c$. We prove this theorem below for separable spaces.

**Theorem 8**. *The space $(LW_2, \Delta) = q^Q$ is metrically universal for all separable ultrametric spaces.*

**Proof**. To prove the theorem, we follow [20] with a minor modification. Recall the method of embedding of a given separable ultrametric space $(X,d)$ into $LW_{\aleph_o}$ (in fact, in $LV_{\aleph_o}$), [20]. First, we choose a countable dense subset $Y = \{a_0, a_1, \dots, a_n, \dots\}$ in $X$ and define an isometric embedding $i : Y \to LV_{\aleph_o}$ inductively. Denote by $f_n(q)$ the image $i(a_n)$ and put $i(a_0) = f_0(q) \equiv 0$, and $f_1(q) = f_0(q)$ for $q > d(a_0,a_1)$ and $f_1(q) = 1$ for $q \leq d(a_0,a_1)$. Obviously, $\Delta(f_0,f_1) = d(a_0,a_1)$. This provides us with the base of induction. Suppose the points $a_0, a_1, \dots, a_{n-1}$ are already embedded isometrically in $LV_{\aleph_o}$ in such a way that $f_k(\mathbf{Q}_+) \subseteq \{0,1,\dots,k\}$ for $k < n$. To embed the point $a_n$, we compute $\min\{d(a_n,a_k)|k < n\} = d_n$, take a point $a_m$ such that $d(a_n,a_m) = d_n$, and put $f_n(q) = f_m(q)$ for $q > d_n$, and $f_n(q) = n$ for $q \leq d_n$. If there are a few such points $a_m$ we can take any of them. It is proved in [20] that $f_n(q)$ is well defined and $i : Y \to LV_{\aleph_o}$ is an isometry. Now, to embed $Y$ in $LW_2$, we choose a countable family $\{Q_n|n \in \mathbf{N}\}$ of pair-wise disjoint dense subsets in $\mathbf{Q}_+$, take characteristic functions $\chi_n$ of the sets $Q_n$ , and replace every function $f$, which equals $k$ on a segment $(s,t]$, by a function which equals $\chi_k$ there. More precisely, if the functions $f_0(q), f_1(q), \dots, f_{n-1}(q)$ are already defined, we put $f_n(q) = f_m(q)$ for $q > d_n$, and $f_n(q) = \chi_n(q)$ for $q \leq d_n$. The specific form of the sets $Q_n$ is inessential. For example, $Q_1$ may be a set of binary rational numbers $Q_1 = \{m/2^n|m,n \in \mathbf{N}\}$, $Q_2$ may be a set of ternary rational numbers, $Q_2 = \{m/3^n|m \neq 3^n, m,n \in \mathbf{N}\}, Q_3$ may be a set of 5-nary rational numbers, and so on. Since the $Q_n$ are dense and pair-wise disjoint, it can be proved as in [20] that $\Delta(f_k, f_n) = d(a_k, a_n)$ for any $k < n$, hence $i : Y \to LW_2$ is an isometry. Since $LW_2$ is complete the closure $[i(Y)]$ in $LW_2$, contains an isometric image of $(X, d)$. ■

Returning to the case of finite spaces we prove the following analogue to Theorem 6.

**Theorem 9.** *For any finite ultrametric space $(X, d)$ consisting of $n + 1$ points there is a family of at most $n$ two-point spaces $\{(D_k, d_k)|k \leq m\}$, where $m \leq n$, such that $(X, d)$ is isometric to a subspace of their metric product $m\Pi\{(D_k, d_k)|k \leq m\}$.*

**Proof.** For $n = 0, 1$, and $2$ the theorem is obvious. Moreover, here we have an equality $m = n$ instead of the inequality $m \leq n$. Suppose the theorem is proved for all at most $n$-point spaces. Following the way that we proved Theorems 4 and 5, we take a space $X = \{a_0, a_1, \dots, a_n\}$ enumerated in the same manner and consider the following two cases.

Case 1. $\min\{d(a_i, a_j)|i \neq j\} = d(a_0, a_1) \leq d(a_0, a_2) \leq \cdots \leq d(a_0, a_{n-1}) < d(a_0, a_n) = \max\{d(a_i, a_j)\}$. By the inductive assumption, the set $X_{n-1} = \{a_0, a_1, a_2, \dots, a_{n-1}\}$ can be embedded isometrically in a product $m\Pi\{(D_k, d_k)|k \leq m\}$, where $m \leq n - 1$. The point $a_n$ is at the same distance $d(a_0, a_n)$ from any point $a_j (j < n)$, in view of the axiom $(\Delta)$. Thus the space $X$ is isometric to a subset of the metric product $m\Pi\{(D_k, d_k)|k \leq m\} \otimes \{0, d(a_0, a_n)\} = m\Pi\{(D_k, d_k)|k \leq m + 1\}$, where $m + 1 \leq n$.

Case 2. $\min\{d(a_i, a_j)|i \neq j\} = d(a_0, a_1) \leq \cdots \leq d(a_0, a_{p-1}) < d(a_0, a_p) = \cdots = d(a_0, a_{n-1}) = d(a_0, a_n) = \max\{d(a_i, a_j)\}$. Denote by $X_p$ and $X_{p+1}$ the sets $X_p = \{a_0, a_1, a_2, \dots, a_{p-1}\}$ and $X_{p+1} = \{a_p, a_{p+1}, \dots, a_n\}$. By the inductive assumption, the space $X_p$ can be embedded isometrically in a product $m\Pi\{(D_k, d_k)|k \leq m\}$, where $m \leq p - 1$ whereas the space $X_{p+1}$ can be embedded isometrically in a product $m\Pi\{(Di, d_i)|i \leq h\}$, where $h \leq (n - p + 1) - 1 = n - p$. As mentioned above all points $a_k$ for $k \geq p$, are at the same distance $d(a_0, a_n)$ from all points $a_j$ for $j < p$ and $d(a_0, a_n) = \max\{d(a_i, a_j)\}$. Therefore $(X, d)$ can be embedded in the product $m\Pi\{(D_k, d_k)|k \leq m\} \otimes m\Pi\{(D_i, d_i)|i \leq h\} \otimes \{0, d(a_0, a_n)\} = m\Pi\{(D_k, d_k)|k \leq m + h + 1\}$, where $m + h + 1 \leq n$. ■

**Note.** The inequality $m \leq n$ cannot be strengthened in general. For any $n \geq 3$, there exist $(n+1)$-point ultrametric spaces that cannot be embedded in $m\Pi\{(D_k, d_k)|k \leq m\}$ for any $m < n$. A few such 4-point spaces are drawn in Figures 2–5 below.

The following particular case of a metric product seems to be the most important for applications in functional analysis. Suppose all factors $(X_\alpha, d_\alpha)$ in a product $m\Pi\{(X_\alpha, d_\alpha)|\alpha \in \mathbf{I}\} = (X_\Pi, d_\Pi)$, are the same space

$(X,d)$ of finite diameter. Then the product $m\Pi\{(X_\alpha,d_\alpha)|\alpha\in\mathbf{I}\}=(X,d)^{\mathbf{I}}$ is none other than a space of all maps from $\mathbf{I}$ to $X$ equipped with a metric of uniform convergence. For an unbounded space $(X,d)$, the pointed product $m\Pi^*\{(X,d,x^*)_\alpha|\alpha\in\mathbf{I}\}$ is a set of all "bounded" maps from $I$ to $(X,d)$ equipped with the same metric. In particular, if $(X,d,x^*)$ is the real line $\mathbf{R}$ with a base point 0, then the pointed product $m\Pi^*\{\mathbf{R}_t|t\in\mathbf{I}\}$ coincides with the set of all bounded real-valued functions $f:\mathbf{I}\to\mathbf{R}$ with the usual norm of uniform convergence $\|f(t)\|=\sup\{|f_{(t)}|,\ t\in\mathbf{I}\}$. Thus an embedding in a product of two-point spaces is at the same time an embedding in Banach space $B^\tau$ with the standard sup-norm, $\|x\|_{\sup}=\sup\{|x_t|,t\in\mathbf{I}\}$. This provides us with an interesting characteristic of ultrametric spaces as subsets of Banach spaces.

**Criterion 1.** Ultrametric spaces are none other than subsets of Banach spaces $B^\tau=(\mathbf{R}^\tau,\|x\|_{\sup})$, whose projections onto any coordinate axis are at most two-point sets.

Recall that by Kuratowski's Theorem [7], any metric space can be embedded in $B^\tau$. Combining the last criterion with Theorem 9 we get the following

**Corollary 2.** *Any finite ultrametric space $(X,d)$ consisting of $n+1$ points can be embedded isometrically in the $m$-dimensional Banach space $B^m=(\mathbf{R}^m,\|x\|_{\sup})$ of dimension $m\le n$ in such a way that projection of the image of $X$ onto any coordinate axis is two-point space.*

It is natural to compare the last corollary and criterion with the well-known theorems on embedding of ultrametric spaces into Euclidean spaces.

**Theorem.** [14] *Every ultrametric space of weight $\tau$ can be isometrically embedded in the generalized Hilbert space $H^\tau$ of weight $\tau$.*

**Theorem.** [14] *Every ultrametric space of cardinality $\Psi$ can be isometrically embedded as a closed subset in the algebraically $\Psi$-dimensional Euclidean space $E^\Psi$, but not in $E^\sigma$ for $\sigma<\Psi$.*

**Theorem** [14]. *Every finite ultrametric space consisting of $n+1$ points can be isometrically embedded in the $n$-dimensional Euclidean space $E^n$ as points in general position. No ultrametric space consisting of $n+1$ points can be isometrically embedded in $E^k$ for $k<n$.*

In other words $n+1$ points of an ultrametric space can be considered as vertices of an $n$-dimensional simplex lying in $E^n$. The last theorems provide us with another characteristic of ultrametric spaces.

**Criterion 2.** Ultrametric spaces are none other than the set of vertexes of simplexes lying in Euclidean (Hilbert) spaces, whose two-dimensional faces are isosceles triangles with bases not greater than sides.

There is a close analogy as well as an essential difference between these kinds of embedding. First, a weight of $\tau$-dimensional Hilbert space $H^\tau$ is equal to $\tau$ whereas a weight of $B^\tau = (\mathbf{R}^\tau, \|x\|_{\sup})$ is $\aleph_0^\tau$ ($= 2^\tau$ for infinite $\tau$). Further the smallest dimension of the Euclidean space, which contains a given $(n+1)$-point ultrametric space $X$ is n for all $(n+1)$-point ultrametric spaces. And the smallest dimension of the similar Banach space $B^m = (\mathbf{R}^m, \|x\|_{\sup})$ containing a given $(n+1)$-point ultrametric space $(X, d)$, depends on the metric $d$ of a space $X$. It can be smaller than $n$, more precisely, $[\log_2(n+1)] \le m \le n$. The difference arises whenever $n \ge 3$. Let us illustrate it for the smallest nontrivial value, $n = 3$. In view of the last theorem, any four-point ultrametric space can be viewed as vertexes of tetrahedron $\subset E^3$ of one of the following six types.

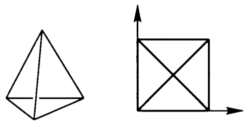

Figure 1

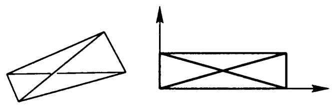

Figure 6

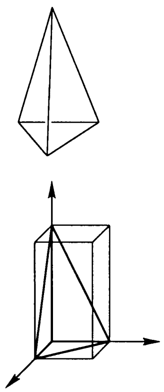

Figure 2

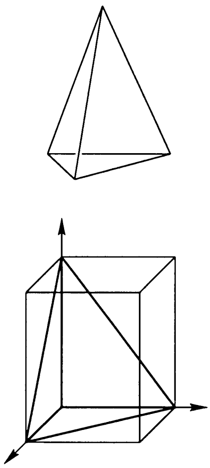

Figure 3

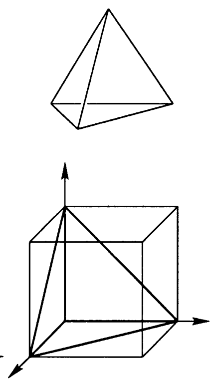

Figure 4

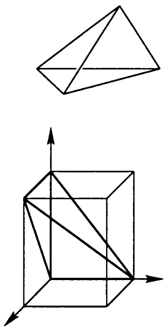

Figure 5

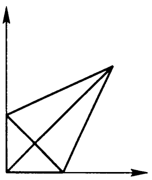

Figure 2a

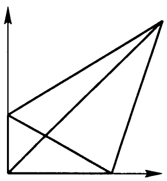

Figure 3a

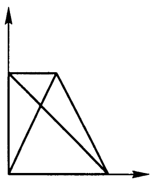

Figure 4a

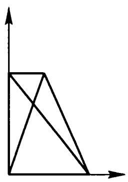

Figure 5a

The set $V = \{d(x,y) | x \neq y \in X\}$ of non-zero values of the metric $d$ has cardinality $|V| = 1, 2, 3, 2, 3, 2$ in each of these cases respectively. In case 1 (a regular tetrahedron), there is an obvious possibility to embed $X$ in **two**-dimensional Banach space $B^2$, as vertexes of a square, see Figure 1. Here all segments, both sides and diagonals, have the same Banach length, $|V| = 1$. The same possibility exists in case 6, although $|V| = 2$ there (see

Figure 6). In cases 3 and 5, $|V| = 3$ therefore it is impossible to embed $X$ in a product of two two-point spaces. It is impossible to find such an embedding in cases 2 and 4 either. However, there exist embeddings of all these types of simplexes in two-dimensional Banach space with three-point projections on coordinate axes (see Figures 2a–5a).

Embedding of ultrametric spaces into Euclidean spaces enables us to apply the theory of ultrametric spaces to linear and convex programming (to the simplex method, in particular). We may hope that the theorems on embedding in Banach spaces will also find appropriate applications. We complete this section by the following problems.

**Problem 3.** Is it possible to embed an arbitrary ultrametric space into an arbitrary Banach space (a Banach space equipped with an arbitrary norm, not necessarily the sup-norm) of appropriate weight, in particular, of the same weight?

**Problem 4.** Does there exist, for any natural $n$, a number $N(n)$ such that any $n$-point ultrametric space can be embedded isometrically in every $N(n)$-dimensional Banach space?

As mentioned above, $N(n) = n - 1$ for Euclidean spaces [14]. Among general Banach spaces, the spaces $L_p$ of Lebesgue integrable functions on $\mathbf{R}$ are the most important. For these spaces, the embedding problem was stated by Prof. Sergey Nikolski in the beginning of the 1970s [30].

**Problem 5** [S. Nikolski, 30]. Is it possible to embed any separable ultrametric space in the space $L_p$ for any $p \geq 1$.

Prof. Israel Gelfand has recently mentioned that for $p = 1$ the affirmative answer to Nikolskiy's problem follows from the next theorem.

**Theorem** [18]. *Every separable ultrametric space can be embedded isometrically in the Lebesgue space $L(\mathbf{R})$.*

Here $L(\mathbf{R})$ denotes a space of Lebesgue measurable subsets of $\mathbf{R}$ with the metric being equal to the measure of symmetrical difference $d(A, B) = \mu(A\Delta B)$.

**Corollary** (I.Gelfand). *Every separable ultrametric space can be embedded isometrically in the space $L_1(\mathbf{R})$ of Lebesgue integrable functions on $\mathbf{R}$ with the norm $\|f(x)\|_1 = \int |f(x)|dx$.*

To prove this, it is enough to assign a characteristic function $\chi_A(x)$ to any measurable subset $A \subset \mathbf{R}$ and note that $\|\chi_A(x) - \chi_B(x)\|_1 = \mu(A\Delta B)$. For $p = 2, L_2(\mathbf{R})$ is none other than Hilbert space $H$, so the problem is also solved in the affirmative by Theorem [14] adduced above. In 1975 A. Timan gave a very partial answer to Nikolski's problem for a certain type of countable spaces [29]. However, in general, the problem is still open.

# 4 Reflectivity and Scanning Programs

Lemmas 1 and 2 as well as others that concern limits of direct and inverse spectra, pull-backs and push-outs, equalizers and co-equalizers, and other categorical operations show that the standard categorical procedures being applied to ultrametric spaces give us again ultrametric spaces [11, 15]. Moreover, we always obtain ultrametric spaces as the result of the action on these spaces of basic metric functors. These are: a trimming functor $\mathbf{METR} \to \mathbf{METR_c}$; a completion functor $\mathbf{METR} \to$ Complete-$\mathbf{METR}$; an orbital functor (a functor of passing to invariant metric $(X, d) \to (X, d_G)$ and to a space of orbits $X/_G$) for spaces acted on by a compact group $G$; the Hausdorff exponential functor $(X, d) \to (\mathrm{Hexp}X, d_H)$, where Hexp $X$ is the space of all (bounded) closed subsets of $X$ with the Hausdorff metric $d_H(A, B) = \inf\{\epsilon | A \subset O_\epsilon(B), B \subset O_\epsilon(A)\}$; functors of passing to various functional spaces with metric of uniform convergence (see Section 3 above), and so on [11, 12, 17]. These can be resumed as a non-formal principle: **'Ultrametrics generate Ultrametrics'**. A reader would probably think that ultrametric spaces form an absolutely closed class with no relations to other spaces. Fortunately, this is not so. Non-expanding (uniform, continuous) maps connect ultrametric spaces with all other metric (uniform, topological) spaces. The latter are at the same time images and pre-images of ultrametric spaces under "nice" maps.

**Theorem 10.** *For every cardinal $\tau$, there is a complete ultrametric space $L^*(\tau)$ such that any metric space of weight $\leq \tau$ is an image of $L^*(\tau)$ under a non-expanding open map.*

In other words $L^*(\tau)$ is a universal pre-image of all metric spaces of weight $\leq \tau$ (*the initial object*). This theorem generalizes a theorem due to Holsztynski [5], who studied initial objects in certain subcategories of $\mathbf{METR}$ (without the requirement for a map to be *open*, and under fairly strong additional assumptions, which imply, in particular, common boundedness of weights, cardinalities, and diameters of the considered spaces). Note that for all spaces of diameter at most $c$ and cardinality at most $\sigma$, the theorem is trivial without the requirement 'f is open'; a discrete space of cardinality $\sigma$ and pair-wise distances $= c$, is obviously a universal inverse image. Theorem 10 can be easily deduced from the next one.

**Theorem** [13]. *For every cardinal $\tau$, there is a complete ultrametric space $L_\tau$ of weight $\tau$ such that any metric space $(X, d)$ of weight $\leq \tau$ is an image of a subspace $L(X) \subset L_\tau$ under a non-expanding open map $f : L(X) \to X$ with compact pre-images of points and totally bonded pre-images of compact subsets $K \subset X$.*

**Proof.** The space $L_\tau$ is naturally defined. It is a set of infinite (in both directions) sequences $\mathbf{a} = (\dots, 0, \alpha_{-m}, \alpha_{-m+1}, \dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ of ordinals $\alpha_n < \omega(\tau)$ containing only finitely many non-zero terms with negative subscript. A metric $d(\mathbf{a},\mathbf{b})$ on $L_\tau$ is defined in a Baire manner $d(\mathbf{a},\mathbf{b}) = 2^{n-1}$, where $n = \min\{k|\alpha_k \neq \beta_k\}$. For the rest of a proof see [13]. ∎

The space $L_\tau$ generalizes the Baire space $B_\tau$ in the same way as a Laurent series generalizes a Taylor series in complex analysis. That is why we denote it by $L_\tau$. We can even replace $L_\tau$ by $B_\tau$ if we reduce the requirement '$f$ is non-expanding' to '$f$ is continuous'. This theorem contains a few well known theorems. E.g., requirement '$f$ is continuous, open, and compact' gives us Nagami's Theorem [23]. Omitting the third assertion we get Morita's Theorem [21] (proved by Hausdorff [4] for separable spaces with $B_\tau$ replaced by $B_{\aleph_o}$). Morita's Theorem was generalized to all $T_0$-spaces of countable character by $V$. Ponomarev [24]. Note that generally the map $f$ cannot be made perfect (pre-images of compact subsets need not be compact) because perfect maps do not augment dimension, whereas all ultrametric spaces are zero-dimensional.

To prove Theorem 10 we choose a sequence $\mathbf{0} = (\dots, 0_{-m}, \dots 0_{-1}, 0_0, 0_1, \dots, 0_n, \dots)$ a base point in $L_\tau$, choose any point $x^*$ in $X$ and verify that, by definition [13], $L(X)$ contains $\mathbf{0}$, and the map $f : L(X) \to X$ takes $\mathbf{0}$ to $x^*$. Then the space $L^*(\tau)$ can be defined as a pointed product of all pair-wise non-isometric subsets $L(X) \subset L_\tau$ containing $\mathbf{0}$. The following proposition completes the proof.

**Proposition 11.** *Canonic projections $p_\beta : m\Pi\{(X_\alpha, d_\alpha, x^*_\alpha)|\alpha \in I\} \to (X_\beta, d_\beta, x^*_\beta)$ of a pointed product onto any factor $(X_\beta, d_\beta, x^*_\beta)$ are open.*

**Proof.** Let $G$ be an open subset of $m\Pi^*\{(X_\alpha, d_\alpha, x^*_\alpha)|\alpha \in \mathbf{I}\}$. For any $\{x_\alpha\}$ in $G$ there is a positive $\epsilon$ such that $O_\epsilon(\{x_\alpha\}) \subset G$. Let $O_\epsilon(x_\beta)$ be an $\epsilon$-neighborhood $O_\epsilon(x_\beta)$ of the point $x_\beta = p_\beta(\{x_\alpha\})$ in $X_\beta$ and $y_\beta \in O_\epsilon(x_\beta)$. Then the set of all $\{z_\alpha\}$ such that $z_\alpha = x_\alpha \forall\alpha \neq \beta, z_\beta = y_\beta \in O_\epsilon(x_\beta)$, is contained in $O_\epsilon(\{x_\alpha\})$. Hence $p_\beta(O_\epsilon(\{x_\alpha\})) = O_\epsilon(x_\beta)$. Therefore, $p_\beta(G)$ is open. ∎

**Problem 6.** Does there exist a universal ultrametric pre-image of weight $\tau$ for all metric spaces of weight at most $\tau$ under non-expanding open mappings?

On the other hand ultrametric spaces are not only pre-images but also images of general metric spaces under non-expanding maps. Moreover, for any metric space $(X,d)$, there exists a greatest element $(uX, d_u)$ in the set of all ultrametric images of $X$ under non-expanding maps. This is called *an ultrametrization* of $X$. The word "greatest" is explained in the following theorem.

**Theorem 11.** *For every metric space $(X,d)$, there are an ultrametric space $(uX, d_u)$ and a non-expanding surjection $u : (X,d) \to (uX, d_u)$ such that for any non-expanding map $f : (X,d) \to (Y,r)$ from $X$ to ar-*

*bitrary ultrametric spaces* $(Y, r)$ *there exists a unique non-expanding map* $uf : (uX, d_u) \to (Y, r)$ *such that* $uf \cdot u = f$.

**Proof**. To prove the theorem we recall the notions of $\epsilon$-chain, $\epsilon$-connectedness, and Cantor connectedness (due to G. Cantor and F. Hausdorff) and describe their behavior under non-expanding maps. Let $(X, d)$ be a metric space and $a, b \in X$. A sequence $a = x_0, x_1, \dots, x_{n-1}, x_n = b$ is called *an $\epsilon$-chain* between $a$ and $b$ provided $d(x_{k-1}, x_k) \leq \epsilon$ for any $k \leq n$. Two points $a$ and $b$ are called *$\epsilon$-linked* if there is an $\epsilon$-chain between them. They are called *linked* if they are $\epsilon$-linked for any $\epsilon > 0$. A space $(X, d)$ is *Cantor connected (linked)* if any two points in $X$ are linked [10]. It is easily verified that the binary relation '$x \sim y$ iff $x$ and $y$ are linked', is an equivalence relation. Denote by $[x]$, $uX$, and $u : X \to uX$ the equivalence class of a point $x$, the quotient space $X/\sim$, and the natural projection $u : X \to X/\sim$, respectively. To introduce a metric $d_u$ on $uX$, we put $d_u(x, y) = \inf\{\epsilon | x \text{ and } y \text{ are } \epsilon\text{-linked}\}$. It is obvious that $d_u(x, y) \geq 0, d_u(x, x) = 0, d_u(x, y) = d_u(y, x)$. Next, if $x$ and $y$ are $\epsilon$-linked, and $y$ and $z$ are $\delta$-linked, then $x$ and $z$ are $\max[\epsilon, \delta]$-linked. This implies that $d_u(x, y)$ satisfies the ultrametric Axiom $(\Delta)$. Thus $d_u(x, y)$ is a pseudo-ultrametric on $X$. Clearly $d_u(x, y) = 0$ iff $x$ and $y$ are linked, hence $d_u(x, y)$ is well defined on the quotient space and it is an ultrametric there, i.e., it satisfies the axiom '$d([x], [y]) = 0$ implies $[x] = [y]$'. Since any pair of points $x$ and $y$ is a $d(x, y)$-chain between them, $d_u(x, y) \leq d(x, y)$. Therefore the natural projection $u : (X, d) \to (uX, d_u)$ is non-expanding. ■

**Lemma 3**. *No two points $a$ and $b$ in an ultrametric space* $(Y, r)$ *are $\epsilon$-linked for any* $\epsilon < r(a, b)$.

**Proof**. For the chain $a = x_0, x_1, x_2 = b$ consisting of three points, this follows from the Axiom $\Delta$ (= the isosceles property). The general case can be proved by induction over a length of chain. ■

This property completely characterizes ultrametric spaces (among general metric spaces) because of the following

**Lemma 4**. *A metric space* $(X, d)$ *is ultrametric if and only if no two points $a$ and $b$ in $X$ are $\epsilon$-linked for any* $\epsilon < d(a, b)$.

**Proposition 12**. *Any non-expanding map takes an $\epsilon$-chain to an $\epsilon$-chain.*

**Corollary 3**. *Non-expanding maps preserve Cantor connectedness.*

Let $(Y, r)$ be an ultrametric space and $f : X \to Y$ be a non-expanding map. By Lemma 3, for any two points $x$ and $y$ in $X$, the images $f(x)$ and $f(y)$ are not $\epsilon$-linked for any $\epsilon < r(f(x), f(y))$. In view of Proposition 12, $x$ and $y$ are not $\epsilon$-linked for any $\epsilon < r(f(x), f(y))$. Hence $d_u(x, y) \geq r(f(x), f(y))$. This implies, first, that for $x \sim y, f(x) = f(y)$, i.e., the map $uf : uX \to Y$, defined as $uf([x]) = f(x)$, is well-defined and completes the diagram below. Second, it is non-expanding in view of the inequality $d_u([x],[y]) = d_u(x,y) \geq r(f(x),f(y)) = r(uf([x]),uf([y]))$.

$$\begin{tikzcd}
(uX, d_u) \arrow[r, "uf"] & (Y, r) \\
(X, d) \arrow[u, "u"] \arrow[ur, "f"'] &
\end{tikzcd}$$

**Corollary 4**. *The subcategory* **ULTRAMETR** *is a reflective subcategory in* **METR**.

**Proof**. For any non-expanding map $f : (X,d) \to (Z,d')$ of general metric spaces, a composition $u \cdot f : X \to Z \to uZ$ is a non-expanding map from $X$ to an ultrametric space $uZ$. Hence it can be lifted to a map $uf : uX \to uZ$ such that the following diagram commutes

$$\begin{tikzcd}
(uX, d_u) \arrow[r, "uf"] & (uZ, d'_u) \\
(X, d) \arrow[u, "u"] \arrow[r, "f"] & (Z, d') \arrow[u, "u"']
\end{tikzcd}$$

The properties $u(fg) = uf \cdot ug$ and $u1_X = 1_{uX}$ are obvious. Thus $u$ is a covariant reflective functor from **METR** to **ULTRAMETR**. ■

**Example**. Let $X = X_1 \cup X_2 = [-1,0) \cup \{1/2^n | n \geq 0\}$ be a subset of the real line $\mathbf{R}$ with the usual metric. Then the space $uX = \{0\} \cup \{1/2^n | n \geq 1\}$ is a subset of $(\mathbf{R}_+, \max)$ with the max-metric. $X_1 = [-1,0)$ is open in $X$ whereas $u(X_1) = \{0\}$ is not open in $uX$, $X_2$ is closed in $X$ whereas $u(X_2)$ is not closed in $uX$. Thus the reflective map $u$ is neither open, nor closed in general.

**Definition**. Two metric spaces $X$ and $Z$ are said to be *ultrametric equivalent* (or *u-equivalent*) if their ultrametrizations are isometric, $uX = uZ$.

It follows from Corollary 3 that all Cantor connected spaces (and only they) are $u$-equivalent to a singleton.

**Theorem 12**. Any metric space consisting of $n$ points is $u$-equivalent to an $n$-point subset of the real line.

**Proof**. Since any metric space is $u$-equivalent to its own ultrametrization, it is enough to prove the theorem for ultrametric spaces only. An ultrametrization of an $n$-point metric space is $n$-point. For $n = 1$ and 2 the assertion of the theorem is trivial. A three-point ultrametric space $X = \{a_0, a_1, a_2\}$ with $d(a_0, a_1) = d_1 \leq d(a_0, a_2) = d(a_0, a_2) = d_2$, is $u$-equivalent to three points $0, d_1$ and $d_1 + d_2$ in $\mathbf{R}$. Suppose, for any $k$-point ultrametric space $X_k = \{a_0, a_1, \dots, a_{k-1}\}$, for $k \leq n$, there is a space $Z_k = \{b_0, b_1, \dots, b_{k-1}\} \subset \mathbf{R}$ with $b_0 < b_1 < \dots < b_{k-1}$, which is $u$-equivalent to $X_k$. Using Lemma 3 [14] for the last time throughout the paper, we see that, in case 1, it is enough to add a point $b_n = b_{n-1} + d(a_0, a_n)$ to the space $Z_n$ to get a $u$-equivalent space $Z_{n+1} = Z_n \cup \{b_n\}, uZ_{n+1} = uX (= X)$. In case 2, we take the spaces $Z_p = \{b_0, b_1, b_2, \dots b_{p-1}\}$ and $Z_{p+1} = \{b_p, b_{p+1}, \dots b_n\}$, which are $u$-equivalent to $X_p$ and $X_{p+1}$ respectively, and move $Z_{p+1}$ as a rigid set along the real line in such a way that $b_p - b_{p-1} = d(a_0, a_p) = \dots = d(a_0, a_n)$. Location of the set $Z = Z_p \cup Z_{p+1}$ on the real line implies that $uZ = X$. ■

Using this theorem E. V. Schepin created an effective scanning algorithm and realized it on an IBM-compatible computer [28]. Other applications of the last theorem to the problem of pattern recognition will be published in future.

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