Categorical Semantics of Linear Logic

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Proof Theory is the result of a tumultuous history, developed on the periphery of mainstream mathematics. Hence, its language is often idiosyncratic: sequent calculus, cut-elimination, subformula property, etc. This survey is designed to guide the novice reader and the itinerant mathematician on a smooth and engaging path through the subject – focused on the symbolic mechanisms of cut-elimination, and their transcription as coherence diagrams in categories with structure. This spiritual journey at the meeting point of linguistic and algebra is demanding at times, but unusually rewarding: to date, no language (formal or informal) has been studied as thoroughly in mathematics as the language of proofs.

We start the survey by a short introduction to Proof Theory (Chapter 1) followed by an informal explanation of the principles of Denotational Semantics (Chapter 2) analogous to a Representation Theory for proofs, generating invariants modulo cut-elimination. After describing in full detail the cut-elimination procedure of linear logic (Chapter 3), we explain how to transcribe it into the language of categories with structure. We review two alternative constructions of a $*$-autonomous category, or monoidal category with duality (Chapter 4). After giving a 2-categorical account of lax and colax monoidal adjunctions (Chapter 5) and recalling the notions of monoids and monads (Chapter 6) we relate four different categorical axiomatizations of propositional linear logic appearing in the litterature (Chapter 7).

Keywords: Proof Theory, Linear Logic, Cut-Elimination Theorem, Categorical Semantics, Monoidal Categories, Linearly Distributive Categories, $*$-autonomous Categories, Monoidal Adjunctions.

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1 Proof theory: a short introduction

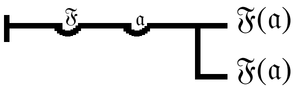
From vernacular proofs to formal proofs: Gottlob Frege

By nature and taste, the mathematician studies properties of specific mathematical objects, like rings, topological spaces, C^*-algebras, etc. This practice involves a high familiarity with proofs, and with their elaboration. Hence, building a proof is frequently seen as an art, or at least as a craft, among mathematicians. Any chair is fine to sit on, but some chairs are more elegant than others. Similarly, the same theorem may be established by beautiful or by ugly means ; but the experienced mathematician will always look for an elegant proof.

In his daily work, the mathematician thinks of a proof as a rational argument exchanged on a blackboard, or written in a book — without further inquiry: the proof is a vehicle of thought, not an object of formal investigation. In that respect, the logician interested in Proof Theory is a peculiar kind of mathematician, one who investigates *inside mathematics* the linguistic event of convincing someone else, or oneself, by a mathematical argument.

Proof Theory has a short and turbulent history, which started in 1879 with a booklet of eighty-eight pages, published by Gottlob Frege at the age of 31. In this short monograph, Gottlob Frege described the first formal notation ever imagined for proofs — which he calls *Begrifftschrift* in German, a neologism translated as *ideography*. Gottlob Frege compares his ideography to a microscope which translates *vernacular proofs* exchanged between mathematicians into *formal proofs* which may be studied like any other mathematical object.

In this formal language, proofs are written in two stages. First, a formula is represented as 2-dimensional graphical structures: for instance, the syntactic tree



is a graphical notation for the first-order and second-order formula written

$$\forall \mathfrak{F}.\ \forall \mathfrak{a}.\ \mathfrak{F}(\mathfrak{a}) \Rightarrow \mathfrak{F}(\mathfrak{a})$$

in our contemporary notation. Then, a proof is represented as a sequence of such formulas, constructed incrementally according to a series of *derivation rules*, or logical principles.

Looking for Foundations: David Hilbert

Gottlob Frege had terrible difficulties to convince the mathematical community of his time. Most of his articles were rejected by mainstream mathematical journals, and he often ended up publishing them in condensed and non technical form in slightly obscure philosophical journals. The ideography was saved from oblivion two decades later by Bertrand Russell – whose curiosity in this extraordinary work was aroused by Giuseppe Peano. At about the same time, David Hilbert got interested in logic, and more specifically, in Gottlob Frege's ideography. It is significant that David Hilbert raised as early as 1900 a purely proof-theoretic problem in his famous communication of twenty-three open problems at the International Congress of Mathematicians in Paris. The second open problem of the list consists indeed of showing that arithmetic is consistent, that is, without contradiction.

David Hilbert further develops the idea in his monograph of 1925 on the Infinite [25]. He explains there that he hopes to establish, by purely *finite* combinatorial arguments on formal proofs, that there exists no contradiction in mathematics — in particular no contradiction in arguments involving *infinite* objects in arithmetic and analysis. This finitary program was certainly influenced by his successful work in Algebraic Geometry, which is also based on the finitary principle of reducing the infinite to the finite. Kurt Gödel established three decades later with his Incompleteness Theorem (1931), that Hilbert's program was a hopeless dream: consistency of arithmetics cannot be established by purely arithmetical arguments.

Consistency of Arithmetics: Gerhard Gentzen

Hilbert's dream was fruitful nonetheless. Gerhard Gentzen, who was a student of Hermann Weyl, established the consistency of arithmetics in 1936, by a purely combinatorial argument. Of course, we have just mentioned Gödel's incompleteness theorem, which says that this proof of consistency cannot be performed inside arithmetic. Accordingly, Gentzen used in his argument a transfinite induction up to Cantor's ordinal ϵ_0 – and this part of the reasoning lies outside arithmetics. We should recall here that the ordinal ε_0 is the first ordinal in Cantor's epsilon hierarchy: it is defined as the smallest ordinal which cannot be described starting from zero, and using addition, multiplication and exponentiation of ordinals to the base ω.

Like many mathematicians and philosophers of his time, Gerhard Gentzen was fascinated by the idea of providing safe *Grundlagen* ("foundations" in German) for science and knowledge. By proving consistency of arithmetic, Gentzen hoped to secure this part of mathematics from the kind of antinomies or paradoxes discovered around 1900 in Set Theory by Cesare Burali-Forti, Georg Cantor, and David Russell. Today, this foundational motivation does not seem as relevant as it was in the early 1930s. Most mathematicians believe that reasoning by finite induction on natural numbers is fine, and does not lead to contradictions in arithmetics. And it seems difficult to convince the skeptical ones that finite induction is safe, by exhibiting Gentzen's argument based on transfinite induction!

The sequent calculus

However, Gentzen's work should not be reduced to a useless and historical bibelot, hanging in a Cabinet of Curiosity. On the contrary, it is regarded by our contemporaries as one of the most important and influential works ever produced in Logic and Proof Theory. But the traditional perspective is reversed: what matters is not the consistency result in itself, but rather the method invented by Gerhard Gentzen in order to establish the result.

This methodology is based on a formal innovation: the *sequent calculus*; and a discovery: the *cut-elimination theorem.* Together, they offer an elegant and flexible framework to formalize proofs — either in classical or in intuitionistic logic. This framework improves in many ways the formal proof systems designed previously by Gottlob Frege, Bertrand Russell, and David Hilbert. We find it useful to explain here the fundamental principles underlying this calculus and procedure, since this survey on categorical semantics is based on them.

Formulas

For simplicity, we restrict ourselves to propositional logic without quantifiers, either on first-order entities (elements) or second-order entities (propositions or sets). We also do not consider first-order variables. In this very elementary logic, a formula A is simply defined as a binary rooted tree

- with nodes labelled by a conjunction (noted $\wedge$), a disjunction (noted $\vee$), or an implication (noted $\Rightarrow$),
- with leaves labelled by the constant true (noted T), the constant false (noted F) or a propositional variable (ranging over A, B or C).

A typical formula is the so-called Peirce's law:

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

which cannot be proved in intuitionistic logic, but can be proved in classical logic, as we shall see later in this introductory chapter.

Sequents

A *sequent* is defined as a pair of sequences of formulas $A_1, ..., A_m$ and $B_1, ..., B_n$ separated by a symbol $\vdash$ in the following way:

$$A_1, \ldots, A_m \vdash B_1, \ldots, B_n. \tag{1}$$

The sequent (1) should be understood as the stating that the conjunction of all the formulas $A_1, \ldots, A_m$ implies the disjunction of all the formulas $B_1, \ldots, B_n$, which may be written as follows:

$$A_1 \wedge \ldots \wedge A_m \quad \Rightarrow \quad B_1 \vee \ldots \vee B_n.$$

Three easy sequents

The simplest example of sequent is the following:

$$A \vdash A \tag{2}$$

which states that the formula A implies the formula A. Another very simple sequent is

$$A, B \vdash A \tag{3}$$

which states that the conjunction of the formulas A and B implies the formula A. Yet another typical sequent is

$$A \vdash A, B \tag{4}$$

which states that the formula A implies the disjunction of the formulas A and B.

Philosophical interlude: truth values and tautologies

The specialists in Proof Theory are generally reluctant to justify the definition of their sequent calculus by the external notion of "truth value" of a formula in a model. However, the notion of "truth value" has been so emphasized by Alfred Tarski after Frege, and is so widespread today, that it may serve as a guideline for the novice reader who discovers Gerhard Gentzen's sequent calculus for the first time. It will always be possible to explain the conceptual deficiencies of the notion later, and the necessity to reconstruct it from inside Proof-Theory.

In this perspective, the sequent (1) states that in any model $\mathcal{M}$ in which the formulas $A_1, ..., A_m$ are all true, then at least one of the formulas $B_1, \ldots, B_n$ is also true. One remarkable point of course is that nobody knows which formula is satisfied among $B_1, \ldots, B_n$. This makes all the spice of sequent calculus! One may carry on in this line, and observe that the three sequents (2), (3) and (4) are tautologies in the model-theoretic sense that they happen to be true in any model $\mathcal{M}$. For instance, the tautology (2) states that a formula A is true in $\mathcal{M}$ whenever the formula A is true; and the tautology (4) states that the formula A or the formula B is true in $\mathcal{M}$ when the formula A is true.

Proofs: deriving tautologies from tautologies

What is more interesting from the proof-theoretic point of view is that tautologies may be deduced mechanically from tautologies, by applying well-chosen rules of logic. For instance, the two tautologies (3) and (4) may be deduced from the tautology (2) in the following way. Suppose that one has established that a given sequent

$$\Gamma_1, \Gamma_2 \vdash \Delta$$

describes a tautology — where Γ_1 and Γ_2 and Δ denote sequences of formulas. It is not difficult to establish then that the sequent

$$\Gamma_1, B, \Gamma_2 \vdash \Delta$$

is a tautology. The sequent $\Gamma_1, B, \Gamma_2 \vdash \Delta$ states indeed that at least one of the formulas in Δ is true when all the formulas in Γ_1 and Γ_2 and moreover the formula B are true. But this statement follows immediately from the fact that the sequent $\Gamma_1, \Gamma_2 \vdash \Delta$ is a tautology. Similarly, we leave the reader establish that whenever a sequent

$$\Gamma \vdash \Delta_1, \Delta_2$$

is a tautology, then the sequent

$$\Gamma \vdash \Delta_1, B, \Delta_2$$

is also a tautology, for every formula B and every pair of sequences of formulas Δ_1 and Δ_2.

The rules of logic: weakening and axiom

We have just identified two simple recipes to deduce a tautology from another tautology. The two rules of logic are called *Left Weakening* and *Right Weakening*. They reflect a fundamental principle of classical and intuitionistic logic, that a formula $A \Rightarrow B$

may be established just by proving the formula B, without using the hypothesis A. Like the other rules of logic, they are written down vertically in the sequent calculus, with the starting sequent on top, and the resulting sequent at bottom, separated by a line, in the following way:

$$\frac{\Gamma_1, \Gamma_2 \vdash \Delta}{\Gamma_1, B, \Gamma_2 \vdash \Delta} \text{ Left Weakening} \tag{5}$$

and

$$\frac{\Gamma \vdash \Delta_1, \Delta_2}{\Gamma \vdash \Delta_1, B, \Delta_2} \text{ Right Weakening} \tag{6}$$

Gerhard Gentzen's sequent calculus is based on the principle that *a proof* describes a series of rules of logic like (5) and (6) applied to an elementary tautology like (2). For homogeneity, the sequent (2) itself is identified as the result of a specific logical rule, called the *Axiom*, which deduces the sequent (2) from no sequent at all. The rule is thus written as follows:

$$\frac{}{A \vdash A} \text{ Axiom}$$

Now, the sequent calculus takes advantage of the horizontal notation for sequents, and of the vertical notation for rules, to write down proofs as 2-dimensional entities. For instance, the informal proof of sequent (3) is written as follows in the sequent calculus:

$$\frac{\dfrac{}{A \vdash A} \text{ Axiom}}{A, B \vdash A} \text{ Left Weakening} \tag{7}$$

The rules of logic: contraction and exchange

Another fundamental principle of classical and intuitionistic logic is that the formula $A \Rightarrow B$ is proved when the formula B is deduced from the hypothesis formula A used several times. This principle is reflected in the sequent calculus by two additional rules of logic, called *Left Contraction* and *Right Contraction*, formulated as follows:

$$\frac{\Gamma_1, A, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, \Gamma_2 \vdash \Delta} \text{ Left Contraction} \tag{8}$$

and

$$\frac{\Gamma \vdash \Delta_1, A, A, \Delta_2}{\Gamma \vdash \Delta_1, A, \Delta_2} \text{ Right Contraction} \tag{9}$$

Another important principle of classical and intuitionistic logic is that the order of hypothesis and conclusions does not really matter in a proof. This is reflected in the sequent calculus by the *Left Exchange* and *Right Exchange* rules:

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta} \text{ Left Exchange}$$

and

$$\frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2} \text{ Right Exchange}$$

The rules of logic: structural rules vs. logical rules

According to Gentzen, the rules of logic should be separated into three classes:

- the axiom rule,
- the structural rules: weakening, contraction, exchange, and cut,
- the logical rules.

We have already encountered all the structural rules, except for the cut rule. This rule deserves a special discussion, and will be introduced later for that reason. The structural rules manipulate the formulas of the sequent, but do not alter them. In contrast, the task of each logical rule is to introduce a new logical connective in a formula, either on the lefthand side or righthand side of the sequent. Consequently, there exist two logical rules for each connective of the logic. The left and right introduction rules associated with conjunction are:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \text{Left} \wedge$$

and

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ \text{Right} \wedge$$

The left and right introduction rules associated with disjunction are:

$$\frac{\Gamma_1, A \vdash \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ \text{Left} \vee$$

and

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2}\ \text{Right} \vee$$

The left and right introduction rules associated with implication are:

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \Rightarrow B \vdash \Delta_1, \Delta_2}\ \text{Left} \Rightarrow$$

and

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ \text{Right} \Rightarrow$$

In each of these rules, the two sequences of formulas Γ and Δ are arbitrary.

Formal proofs as derivation trees

Since we have already constructed a few formal proofs in our sequent calculus for classical logic, it may be the proper time to give a general definition. From now on, a formal *proof* is defined as a derivation tree constructed according to the rules of the sequent calculus. By *derivation tree*, we mean a rooted tree in which:

- every leaf is labelled by an axiom rule,
- every node is labelled by a rule of the sequent calculus,
- every edge is labelled by a sequent.

A derivation tree should satisfy the expected consistency property relating the sequents on the edges to the rules on the nodes. In particular, the arity of a node in the derivation tree follows from the number of sequents on top of the rule: for instance, a node labelled with the *Left* $\wedge$ rule has arity one, whereas a node labelled with the *Right* $\wedge$ rule has arity two. Note that every derivation tree has a root, which is a node labelled by a rule of the sequent calculus. The *conclusion* of the proof is defined in the expected way as the sequent $\Gamma \vdash \Delta$ obtained by that rule.

Philosophical interlude: the anti-realist tradition in Proof Theory

Once the sequent calculus is understood and accepted by the novice reader, the specialist in Proof Theory will generally advise forgetting any guideline related to Model Theory, like truth-values or tautologies. This is a pervasive dogma of Proof Theory, which could simply follow from a naive application of Ockham's razor: now that proofs can be produced mechanically by a symbolic device (the sequent calculus) independently of any notion of truth... why should we remember any of the "ancient" model-theoretic explanations?

In fact, the philosophical position generally adopted in Proof Theory since Gentzen is far more radical – even if this remains generally implicit in daily mathematical work. This position is generally called *anti-realist* to stress the antagonism with the *realist* position. We will only sketch the debate in a few words here. For the realist, the world is constituted of a fixed set of objects, independent of the mind and of its symbolic representations. Thus, "truth" amounts to a proper correspondence between the words and symbols emanating from the mind, and the objects and external things of the world. For the anti-realist, on the other hand, the very question "what objects is the world made of?" requires already a theory or a description. In that case, "truth" amounts rather to some kind of ideal coherence between our various beliefs and experiences.

The anti-realist position in Proof Theory may be summarized in four technical tropisms:

- The sequent calculus generates formal proofs, and these formal proofs should be studied as autonomous entities, just like any other mathematical object.
- The notion of "logical truth" in model-theory is based on the realist idea of the existence of an external world: the model. This is too redundant to be useful: what information does the statement that "the formula $A \wedge B$ is true if and only if the formula A is true and the formula B is true" provide ?
- So, the "meaning" of the connectives of logic arises from their introduction rules in the sequent calculus, and not from an external and realist concept of truth-value. These introduction rules are inherently justified by the structural properties of proofs, like cut-elimination, or the subformula property.

- Gödel's completeness theorem may be reunderstood in this way: every model $\mathcal{M}$ plays the role of a potential non recursive refutator which may be simulated by some kind of infinite non recursive proof — this leading to a purely proof-theoretic exposition of the completeness theorem.

This position is advocated today by Jean-Yves Girard in a series of sharp commentaries [22] later developed in his work on ludics [23].

Two exemplary proofs in classical logic

There is a famous principle in classical logic that the disjunction of a formula A and of its negation $\neg A$ is necessarily true. This principle, called the Tertium Non Datur in Latin ("the third is not given") is nicely formulated by the formula

$$(A \Rightarrow B) \ \vee \ A$$

which states that for every formula B, either the formula A holds, or the formula A implies the formula B. This very formula is established by the following in our sequent calculus for classical logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Axiom}}{A \vdash B, A}\ \text{Right Weakening}}{\vdash A \Rightarrow B, A}\ \text{Right} \Rightarrow}{\vdash (A \Rightarrow B) \ \vee \ A}\ \text{Right} \vee \tag{10}$$

The proof works for every formula B, and may be specialized to the falsity formula $\bot$. From this follows a proof of the formula:

$$\neg A \ \vee \ A$$

where we identify the negation $\neg A$ of the formula A to the formula $A \Rightarrow \bot$ which states that the formula A implies falsity.

We have mentioned above that Peirce's formula:

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

may be established in classical logic. Indeed, we write below the proof of the formula in the sequent calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Axiom}}{A \vdash B, A}\ \text{Right Weakening}}{\vdash A \Rightarrow B, A}\ \text{Right} \Rightarrow \qquad \dfrac{}{A \vdash A}\ \text{Axiom}}{(A \Rightarrow B) \Rightarrow A \vdash A, A}\ \text{Left} \Rightarrow}{(A \Rightarrow B) \Rightarrow A \vdash A}\ \text{Right Contraction}}{\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}\ \text{Right} \Rightarrow$$

Note that the main part of the proof of the Tertium Non Datur appears at the very top left of that proof. In fact, it is possible to prove that the two formulas are equivalent in intuitionistic logic: in fact, each of them may be taken as an additional axiom of intuitionistic logic, in order to obtain classical logic.

Cut-elimination

At this point, all the rules of our sequent calculus for classical logic have been introduced... except perhaps the most fundamental one: the *cut-rule*, formulated as follows:

$$\frac{\Gamma_1 \vdash A, \Delta_1 \qquad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\text{Cut}$$

The cut-rule reflects the famous deduction principle of logic: Modus Ponens ("affirmative mode" in Latin) which states that the formula B may be deduced from the two formulas A and $A \Rightarrow B$ taken together. Suppose given two proofs π_1 and π_2 of the sequents $\vdash A$ and $A \vdash B$:

$$\begin{array}{c}\pi_1\\ \vdots\\ \hline \vdash A\end{array} \qquad\qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline A \vdash B\end{array}$$

The cut-rule may be applied to the two derivation trees so as to obtain a proof

$$\begin{array}{c}\pi_3\\ \vdots\\ \hline \vdash B\end{array} \quad = \quad \cfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline A \vdash B\end{array}}{\vdash B}\,\text{Cut} \tag{11}$$

of the sequent $\vdash B$. This is Modus Ponens translated in the sequent calculus.

Despite the fact that it captures Modus Ponens, the most fundamental principle of logic, Gentzen made the extraordinary observation that the cut-rule may be forgotten from the point of view of provability, or what can be proved in logic! In technical terms, this says that the cut-rule is *admissible* in classical logic, as well as in intuitionistic logic: every sequent $\Gamma \vdash \Delta$ which may be proved by a proof π may be also proved by a proof π' in which the cut-rule does not appear at any stage of the proof. Such a proof is called *cut-free*.

Gerhard Gentzen called this property the *cut-elimination theorem*, or *Hauptsatz* in German. Applied to our previous example (11) the property states that there exists an alternative cut-free proof

$$\begin{array}{c}\pi_4\\ \vdots\\ \hline \vdash B\end{array} \tag{12}$$

of the sequent $\vdash B$.

The subformula property and the consistency of logic

The cut-elimination theorem is the backbone of modern Proof Theory. It is remarkable for instance that three fundamental properties of formal logic follow quite directly from this single theorem:

- the subformula property,
- the consistency of the logic,

- the completeness theorem.

Let us discuss the subformula property first. A formula D is called a subformula of a formula AcB in three cases only:

- when the formula D is equal to the formula AcB,
- when the formula D is subformula of the formula A,
- when the formula D is subformula of the formula B,

where AcB means either $A \Rightarrow B$, or $A \wedge B$ or $A \vee B$. And the constant formula F (resp. T) is the only subformula of the formula F (resp. T).

The subformula property states that every provable formula A may be established by a proof π in which only subformulas of the formula A appear. This remarkable property follows immediately from the cut-elimination theorem. Suppose indeed that a formula A is provable in the logic. This simply means that there exists a proof of the sequent $\vdash A$. By cut-elimination, there exists a cut-free proof π of the sequent $\vdash A$. A simple inspection of the rules of our sequent calculus shows that this cut-free proof π contains only subformulas of the original formula A.

Then, the consistency of the logic follows easily from the subformula property. Suppose indeed that the constant formula F is provable in the logic. Then, by the subformula property, there exists a proof π of the sequent $\vdash F$ which contains only subformulas of the formula F. Since the formula F is the only subformula of itself, every sequent appearing in the proof π should be a sequence of F:

$$F, \ldots, F \ \vdash \ F, \ldots, F.$$

Any logical rule in the proof π would introduce a connective of logic $\Rightarrow$ or $\wedge$ or $\vee$, which is not possible. From this follows that, besides some axiom rules, the proof π is made of structural rules only. From this follows easily that every sequent in the proof π is empty on the lefthand side:

$$\vdash \ F, \ldots, F.$$

Since no such sequent can be obtained as result of an Axiom rule, one concludes that there exists no proof π of the formula F in our logic. This is precisely the statement of consistency.

The completeness theorem is slightly more difficult to deduce from the cut-elimination theorem. The interested reader will find a detailed proof of the theorem in the first chapter of the Handbook of Proof Theory, exposed by Samuel Buss [14].

The cut-elimination procedure

In order to establish the cut-elimination theorem, Gentzen introduced a series of symbolic transformations on proofs. Each of these rules tranforms a proof π containing a cut-rule into a proof π' with the same conclusion. In practice, the resulting proof π' will involve several cut-rules; but the complexity of these cut-rules will be strictly less than the complexity of the cut-rules in the original proof π. Consequently, the rewriting rules may be iterated until one reaches a cut-free proof. Termination of the procedure is far from obvious: it is precisely to prove termination that Gentzen uses a transfinite induction up to Cantor's ordinal ε_0. This provides an effective *cut-elimination procedure* which transforms any proof of the sequent $\Gamma \vdash \Delta$ into a cut-free proof of the same sequent. The cut-elimination theorem follows immediately.

This procedural aspect of cut-elimination is the starting point of denotational semantics, whose task is precisely to provide mathematical invariants of proofs under cut-elimination procedure. The exercise is far from obvious. One difficulty comes from the symbolic intricacy of the cut-elimination. We will see in Chapter 3 that describing in full details the cut-elimination procedure of a reasonable logic like linear logic already takes a dozen meticulous pages.

Intuitionistic logic

Intuitionistic logic has been introduced and developed by Luitzen Egbertus Jan Brouwer at the beginning of the 20th century, in order to provide safer foundations for mathematics. Brouwer rejected the idea of formalizing mathematics – but his own student Arend Heyting committed the outrage, and produced in 1930 a formal system for intuitionistic logic, based on the idea that the Tertium Non Datur principle of classical logic should be rejected.

A surprising and remarkable observation of Gerhard Gentzen is that an equivalent formalization of intuitionistic logic is obtained simply by restricting the sequent calculus for classical logic to "intuitionistic" sequents:

$$\Gamma \vdash A$$

with exactly one formula A on the righthand side. The reader will easily check for illustration that the proof (10) of the sequent

$$\vdash (A \Rightarrow B) \vee A$$

cannot be performed in the intuitionistic fragment of classical logic: one needs the ability to contract on the righthand side of the sequent in order to perform the proof.

Linear logic

Gentzen's idea to describe intuitionistic logic by limiting classical logic to particular sequents seems just too simplistic and too arbitrary to work... But it works, and deeper reasons must explain this unexpected success. This reflection is precisely the starting point of linear logic. It appears indeed that the key feature of intuitionistic sequent calculus, compared to classical sequent calculus, is that the Weakening and Contraction rules can be only applied on the lefthand side of the sequents (=the hypothesis), and not on the righthand side (= the conclusion).

Linear logic is based on the idea that the Weakening and Contraction rules do not apply to *any* formula, but only to very particular kinds of modal formulas. Two modalities are involved: the modality ! (pronounced "of course") and the modality ? (pronounced "why not"). Weakening and Contraction apply to formulas $!A$ on the lefthand side, and to formulas $?A$ on the righthand side.

Informally speaking, the intuitionistic sequent

$$A, B \vdash C$$

is translated as

$$!A, !B \vdash C$$

where the of course modality on the formulas $!A$ and $!B$ indicates that the two formulas may be weakened and contracted at will.

First-order logic

In this short introduction to Proof Theory, we have chosen to limit ourselves to the propositional fragment of classical logic: no variables, no quantification. This simplifies matters, and captures the essence of Gerhard Gentzen's ideas. Here, we would like to indicate the logical principles underlying first-order classical logic, and illustrate the logic at work on a remarkable formula, called the drinker formula.

In order to define first-order logic, one needs:

- an infinite set $\mathcal{V}$ of first-order variable symbols, ranging over x, y, z,
- a set $\mathcal{F}$ of symbols with a specified arity, ranging over f, g,
- a set $\mathcal{R}$ of relation symbols with a specified arity, ranging over R, Q.

The *terms* of the logic are constructed from the function symbols and the first-order variables. Hence, any first-order variable x is a term, and $f(t_1, ..., t_k)$ is a term if t_1,...,t_k are terms, and f has arity k. In particular, any function symbol f of arity 0 defines a term. The *atomic formulas* or the logic are defined as a relation symbol substituted by terms. Hence, $R(t_1, ..., t_k)$ is an atomic formula if t_1,...,t_k are terms, and R has arity k.

The formulas of first-order logic are constructed as in the propositional case, except that:

- propositional variables A, B, C are replaced by atomic formulas $R(t_1, ..., t_k)$,
- every node of the formula is either a propositional connective $\wedge$ or $\vee$ or $\Rightarrow$ as in the propositional case, or a universal quantifier $\forall x$, or an existential quantifier $\exists x$.

So, a typical first-order formula looks like:

$$\forall y.R(f(x), y).$$

One should be aware that this formula, in which the quantifier $\forall x$ binds the first-order variable x, is treated as the same formula as:

$$\forall z.R(f(x), z).$$

We will not discuss here the usual distinction between a *free* and a *bound* occurrence of a variable in a first-order formula; nor describe how a free variable x of a first-order formula $A(x)$ is substituted without capture of variable by a term t, in order to define a formula $A(t)$. These definitions may be found in many textbooks. It should be enough to illustrate the definition by mentioning that the formula

$$A(x) = \forall y.R(f(x), y)$$

applied to the term $t = g(y)$ defines the formula

$$A(t) = \forall z.R(f(g(y)), z).$$

Except for those syntactic details, the sequent calculus works just as in the propositional case. The left introduction of the universal quantifier

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x.A(x) \vdash \Delta} \text{ Left } \forall$$

and the right introduction of the existential quantifier

$$\frac{\Gamma \vdash A(t), \Delta}{\Gamma \vdash \exists x.A(x), \Delta} \text{ Right } \exists$$

may be performed for any term t of the language without any restriction. On the other hand, the right introduction of the universal quantifier

$$\frac{\Gamma \vdash A(x), \Delta}{\Gamma \vdash \forall x.A(x), \Delta} \text{ Right } \forall$$

and the left introduction of the existential quantifier

$$\frac{\Gamma, A(x) \vdash \Delta}{\Gamma, \exists x.A(x) \vdash \Delta} \text{ Left } \exists$$

may be applied only if the first-order variable x does not appear in any formula of the contexts Γ and Δ. Note that the formula $A(x)$ may contain other free variables than x.

Let us illustrate these rules with the following first-order formula, called the drinker formula:

$$\exists y.\{A(y) \Rightarrow \forall x.A(x)\} \tag{13}$$

which states that for every formula $A(x)$ with first-order variable x, there exists an element y of the ontology such that if $A(y)$ holds, then $A(x)$ holds for every element x of the ontology. The element y is thus the witness for the universal validity of $A(x)$. The name of "drinker formula" comes from the following case study: suppose that x ranges over the customers of a pub, and that $A(x)$ means that the customer x is not drinking beer; then, there exists a particularly addicted customer y (the drinker) such that, if any customer x in the pub is drinking beer, then the customer y is also drinking beer. The existence of such a customer y in the pub is far from obvious, but it may be established by purely logical means in classical logic!

The drinker formula has been thoroughly analysed by Jean-Louis Krivine [28] who generally replaces it with a formula expressed only with universal quantification, equivalent in classical logic:

$$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \quad \Rightarrow \quad B.$$

Here, B stands for any formula of the logic. The original formulation (13) of the drinker formula is then obtained by replacing the formula B by the falsity formula $\bot$, and by applying the series of equivalences in classical logic:

$$
\begin{array}{rl}
 & \neg\ \forall y.\{\neg(A(y) \Rightarrow \forall x.A(x))\} \\
\equiv & \exists y.\{\neg\neg(A(y) \Rightarrow \forall x.A(x))\} \\
\equiv & \exists y.\{A(y) \Rightarrow \forall x.A(x)\}
\end{array}
$$

where, again, we write $\neg A$ for the formula $(A \Rightarrow \bot)$. The shortest proof of the drinker formula in classical logic is then:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$A(x_0) \vdash A(x_0)$}
\RightLabel{Right Weakening}
\UnaryInfC{$A(x_0) \vdash \forall x.A(x), A(x_0)$}
\RightLabel{Right $\Rightarrow$}
\UnaryInfC{$\vdash A(x_0) \Rightarrow \forall x.A(x), A(x_0)$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$B \vdash B$}
\RightLabel{Left $\Rightarrow$}
\BinaryInfC{$(A(x_0) \Rightarrow \forall x.A(x)) \Rightarrow B \vdash A(x_0), B$}
\RightLabel{Left $\forall$}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash A(x_0), B$}
\RightLabel{Right $\forall$}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash \forall x.A(x), B$}
\RightLabel{Left Weakening}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\}, A(y)_0 \vdash \forall x.A(x), B$}
\RightLabel{Right $\Rightarrow$}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash A(y)_0 \Rightarrow \forall x.A(x), B$}
\AxiomC{}
\RightLabel{Axiom}
\UnaryInfC{$B \vdash B$}
\RightLabel{Left $\Rightarrow$}
\BinaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\}, (A(y)_0 \Rightarrow \forall x.A(x)) \Rightarrow B \vdash B, B$}
\RightLabel{Left $\forall$}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\}, \forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash B, B$}
\RightLabel{Contraction}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash B, B$}
\RightLabel{Contraction}
\UnaryInfC{$\forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \vdash B$}
\RightLabel{Right $\Rightarrow$}
\UnaryInfC{$\vdash \forall y.\{(A(y) \Rightarrow \forall x.A(x)) \Rightarrow B\} \Rightarrow B$}
\end{prooftree}
$$

An historical remark on Gerhard Gentzen's system LK

The reader familiar with Proof Theory will notice that our presentation of classical logic departs in several ways from Gentzen's original presentation. One main difference is that Gentzen's original sequent calculus **LK** contains *two* right introduction rules for disjunction:

$$
\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \text{Right } \vee_1 \qquad\qquad \dfrac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \text{Right } \vee_2
$$

whereas the sequent calculus presented here contains only one introduction rule:

$$
\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2}\ \text{Right } \vee
$$

We know since the discovery of linear logic, and the clarifications it offers, that the two presentations of classical logic are very different in nature. The introduction rules of the sequent calculus **LK** are called *additive* whereas the presentation chosen here are *multiplicative*. However, it is possible to simulate the multiplicative rule inside the original system **LK**, in the following way:

$$
\begin{prooftree}
\AxiomC{$\Gamma \vdash \Delta_1, A, B, \Delta_2$}
\RightLabel{Right $\vee_1$}
\UnaryInfC{$\Gamma \vdash \Delta_1, A \vee B, B, \Delta$}
\RightLabel{Right $\vee_2$}
\UnaryInfC{$\Gamma \vdash \Delta_1, A \vee B, A \vee B, \Delta$}
\RightLabel{Right Contraction}
\UnaryInfC{$\Gamma \vdash \Delta_1, A \vee B, \Delta_2$}
\end{prooftree}
$$

Conversely, the two additive introduction rules of the sequent calculus **LK** can be simulated in our sequent calculus, in the following way:

$$\dfrac{\dfrac{\Gamma \vdash \Delta_1, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\text{ Right Weakening}}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2}\text{ Right }\vee_1 \qquad \dfrac{\dfrac{\Gamma \vdash \Delta_1, B, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2}\text{ Right Weakening}}{\Gamma \vdash \Delta_1, A \vee B, \Delta_2}\text{ Right }\vee_1$$

Note however that the Weakening and the Contraction rules play a key role in the back and forth translations between the additive and the multiplicative sequent calculi. Indeed, the two logical systems (additive and multiplicative) become different, but remarkably complementary, in linear logic — where the Weakening and the Contraction rules are limited to modal formulas.

Notes and references

We advise the interested reader to look directly at the original papers by Gentzen, collected and edited by Manfred Szabo in [18]. More recent material can be found in Jean-Yves Girard's monographs on Proof Theory [19] and [20] as well as in the Handbook of Proof Theory [14] edited by Samuel Buss.

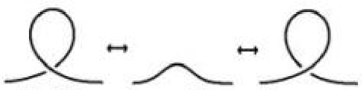
2 Semantics: proof invariants and categories

2.1 Proof invariants organize themselves as categories

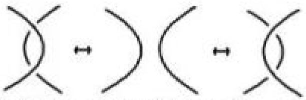
In order to better understand linear logic, we look for *invariants* of proofs under cut-elimination. Any such invariant is a function

$$\pi \mapsto [\pi]$$

which associates to every proof π of linear logic a mathematical entity $[\pi]$ called the *denotation* of the proof. Invariance under cut-elimination means that the denotation $[\pi]$ coincides with the denotation $[\pi']$ of any proof π' obtained by applying the cut-elimination procedure to the proof π. An analogy comes to mind with Knot Theory, and more specifically the induced Representation Theory: by definition, a knot invariant is a function which associates to every knot an entity (typically, a number or a polynomial) which remains unaltered under the action of the three Reidemeister moves:



Reidemeister Type I move



Reidemeister Type II move

Reidemeister Type III moves

We are looking for similar invariants for proofs, this time with respect to the proof transformations occurring in the course of cut-elimination. We will see that, just like in Representation Theory, the construction of such invariants is achieved by constructing the suitable kind of categories and functors.

Note that invariance is not enough: we are looking for *modular* invariants. What does that mean? Suppose given three formulas A, B, C, together with a proof π_1 of the sequent $A \vdash B$ and a proof π_2 of the sequent $B \vdash C$. We have already described the *cut-rule* in classical logic and in intuitionistic logic. The same cut-rule exists in linear logic. When applied to the proofs π_1 and π_2, it leads to the following proof π of the sequent $A \vdash C$:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A \vdash B} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B \vdash C}}{A \vdash C}\ \text{Cut}$$

Now, we declare an invariant *modular* when the denotation of the proof π may be deduced directly from the denotations $[\pi_1]$ and $[\pi_2]$ of the proofs π_1 and π_2. In this case, there exists a binary operation $\circ$ on denotations satisfying

$$[\pi] \quad = \quad [\pi_2] \circ [\pi_1].$$

The very design of linear logic (and of its cut-elimination procedure) ensures that this composition law is associative and has a left and a right identity. What do we mean? This point deserves to be clarified. First, consider associativity. Suppose given a formula D and a proof π_3 of the sequent $C \vdash D$. By modularity, the two proofs

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{A \vdash B} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{B \vdash C}}{A \vdash C}\,\text{Cut} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{C \vdash D}}{A \vdash D}\,\text{Cut}$$

and

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{A \vdash B} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{B \vdash C} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{C \vdash D}}{B \vdash D}\,\text{Cut}}{A \vdash D}\,\text{Cut}$$

have respective denotations

$$[\pi_3] \circ ([\pi_2] \circ [\pi_1]) \qquad \text{and} \qquad ([\pi_3] \circ [\pi_2]) \circ [\pi_1].$$

The two proofs are equivalent from the point of view of cut-elimination. Indeed, depending on the situation, the procedure may transform the first proof into the second proof, or conversely, the second proof into the first proof. This illustrates what logicians call a *commutative conversion*: in that case a conversion permuting the order of the two cut rules. By invariance, the denotations of the two proofs coincide. This establishes associativity of composition:

$$[\pi_3] \circ ([\pi_2] \circ [\pi_1]) \quad = \quad ([\pi_3] \circ [\pi_2]) \circ [\pi_1].$$

What about the left and right identities? There is an obvious candidate for the identity on the formula A, which is the denotation id_A associated to the proof

$$\dfrac{}{A \vdash A}\,\text{Axiom}$$

Given a proof π of the sequent $A \vdash B$, the cut-elimination procedure transforms the two proofs

$$\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \qquad \dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A \vdash B}}{A \vdash B}\,\text{Cut}$$

and

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A \vdash B} \qquad \dfrac{}{B \vdash B}\,\text{Axiom}}{A \vdash B}\,\text{Cut}$$

into the proof

$$\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A \vdash B}$$

Modularity and invariance imply together that

$$[\pi] \circ id_A \quad = \quad id_B \circ [\pi] \quad = \quad [\pi].$$

From this, we deduce that every modular invariant of proofs gives rise to a category. In this category, every formula A defines an object $[A]$, which may rightly be called the *denotation* of the formula; and every proof

$$\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{A \vdash B}$$

denotes a morphism

$$[\pi] : [A] \longrightarrow [B]$$

which, by definition, is invariant under cut-elimination of the proof π.

2.2 A tensor product in linear logic

The usual conjunction $\wedge$ of classical and intuitionistic logic is replaced in linear logic by a conjunction akin to the *tensor product* of linear algebra, and thus noted $\otimes$. We are thus tempted to look for denotations satisfying not just invariance and modularity, but also *tensoriality*. By tensoriality, we mean two related things. First, the denotation $[A \otimes B]$ of the formula $A \otimes B$ should follow directly from the denotations of the formula A and B, by applying a binary operation (also noted $\otimes$) on the denotations of formulas:

$$[A \otimes B] \quad = \quad [A] \otimes [B].$$

Second, given two proofs

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad\qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2}$$

the denotation of the proof π

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}\ \text{Right}\,\otimes}{A_1 \otimes B_1 \vdash A_2 \otimes B_2}\ \text{Left}\,\otimes$$

should follow from the denotations of the proofs π_1 and π_2 by applying a binary operation (noted $\otimes$ again) on the denotations of proofs:

$$[\pi] \quad = \quad [\pi_1] \otimes [\pi_2].$$

These two requirements imply together that the linear conjunction $\otimes$ of linear logic defines a *bifunctor* on the category of denotations. We check this claim as an exercise. Consider four proofs

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{A_2 \vdash A_3} \qquad \dfrac{\begin{matrix}\pi_4\\ \vdots\end{matrix}}{B_2 \vdash B_3}$$

with respective denotations

$$f_1 = [\pi_1], \qquad f_2 = [\pi_2], \qquad f_3 = [\pi_3], \qquad f_4 = [\pi_4].$$

The cut-elimination procedure transforms the proof

$$\dfrac{\text{Left}\,\otimes\ \dfrac{\text{Right}\,\otimes\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}}{A_1 \otimes B_1 \vdash A_2 \otimes B_2} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{A_2 \vdash A_3} \qquad \dfrac{\begin{matrix}\pi_4\\ \vdots\end{matrix}}{B_2 \vdash B_3}}{A_2, B_2 \vdash A_3 \otimes B_3}\ \text{Right}\,\otimes}{A_2 \otimes B_2 \vdash A_3 \otimes B_3}\ \text{Left}\,\otimes}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\ \text{Cut}$$

with denotation

$$(f_3 \otimes f_4) \circ (f_1 \otimes f_2)$$

into the proof

$$\dfrac{\dfrac{\text{Cut}\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{A_2 \vdash A_3}}{A_1 \vdash A_3} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2} \qquad \dfrac{\begin{matrix}\pi_4\\ \vdots\end{matrix}}{B_2 \vdash B_3}}{B_1 \vdash B_3}\ \text{Cut}}{A_1, B_1 \vdash A_3 \otimes B_3}\ \text{Right}\,\otimes}{A_1 \otimes B_1 \vdash A_3 \otimes B_3}\ \text{Left}\,\otimes$$

with denotation

$$(f_3 \circ f_1) \otimes (f_4 \circ f_2).$$

By invariance, the equality

$$(f_3 \otimes f_4) \circ (f_1 \otimes f_2) \quad = \quad (f_3 \circ f_1) \otimes (f_4 \circ f_2)$$

holds in the underlying category of denotations. This ensures that the first equation of bifunctoriality is satisfied. One deduces in a similar way the other equation

$$id_{[A]\otimes[B]} \quad = \quad id_{[A]} \otimes id_{[B]}$$

by noting that the cut-elimination procedure transforms the proof

$$\dfrac{}{A \otimes B \vdash A \otimes B}\ \text{Axiom}$$

into the proof

$$\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B}\ \text{Axiom}}{A, B \vdash A \otimes B}\ \text{Right} \otimes}{A \otimes B \vdash A \otimes B}\ \text{Left} \otimes$$

by the *η-expansion* rule described in Chapter 3, Section 3.5.

2.3 Proof invariants organize themselves as monoidal categories (1)

We have just explained the reasons why the operation ⊗ defines a bifunctor on the category of denotations. We can go further, and show that this bifunctor defines a monoidal category — not exactly in fact, but nearly so. The reader will find the notion of monoidal category recalled in Chapter 4.

A preliminary step in order to define a monoidal category is to choose a unit object e in the category. The choice is almost immediate in the case of linear logic. In classical and intuitionistic logic, the truth value T standing for "true" behaves as a kind of *unit* for conjunction, since the two sequents

$$A \wedge T \vdash A \qquad \text{and} \qquad A \vdash A \wedge T$$

are provable for every formula A of the logic. In linear logic, the truth value T is replaced by a constant 1 which plays exactly the same role for the tensor product. In particular, the two sequents

$$A \otimes 1 \vdash A \qquad \text{and} \qquad A \vdash A \otimes 1$$

are provable for every formula A of linear logic. The *unit* of the category is thus defined as the denotation $e = [1]$ of the formula 1.

Now, we construct three isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),$$

$$\lambda_A : e \otimes A \longrightarrow A, \qquad \rho_A : A \otimes e \longrightarrow A$$

indexed on the objects A, B, C of the category, which satisfy all the coherence and naturality conditions of a monoidal category. The associativity morphism α is defined as the denotation of the proof $\pi_{A,B,C}$ below:

$$\dfrac{\dfrac{\dfrac{\text{Axiom}\ \dfrac{}{A \vdash A} \qquad \dfrac{\dfrac{}{B \vdash B}\ \text{Axiom} \qquad \dfrac{}{C \vdash C}\ \text{Axiom}}{B, C \vdash B \otimes C}\ \text{Right} \otimes}{A, B, C \vdash A \otimes (B \otimes C)}\ \text{Right} \otimes}{A \otimes B, C \vdash A \otimes (B \otimes C)}\ \text{Left} \otimes}{(A \otimes B) \otimes C \vdash A \otimes (B \otimes C)}\ \text{Left} \otimes$$

The two morphisms λ and ρ are defined as the respective denotations of the two proofs below:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{1, A \vdash A}\,\text{Left } 1}{1 \otimes A \vdash A}\,\text{Left } \otimes$$

and

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom}}{A, 1 \vdash A}\,\text{Left } 1}{A \otimes 1 \vdash A}\,\text{Left } \otimes$$

The naturality and coherence conditions on α, λ and ρ are not particularly difficult to establish. For instance, naturality of α means that for every three proofs

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{C_1 \vdash C_2}$$

with respective denotations:

$$f_1 = [\pi_1], \qquad f_2 = [\pi_2], \qquad f_3 = [\pi_3].$$

the following categorical diagram commutes:

$$\begin{tikzcd}
(A_1 \otimes B_1) \otimes C_1 \arrow[r, "\alpha"] \arrow[d, "(f_1 \otimes f_2) \otimes f_3"'] & A_1 \otimes (B_1 \otimes C_1) \arrow[d, "f_1 \otimes (f_2 \otimes f_3)"] \\
(A_2 \otimes B_2) \otimes C_2 \arrow[r, "\alpha"] & A_2 \otimes (B_2 \otimes C_2)
\end{tikzcd} \tag{14}$$

where, for this time, and for clarity's sake only, we do not distinguish between the formula, say $(A_1 \otimes B_1) \otimes C_1$, and its denotation $[(A_1 \otimes B_1) \otimes C_1]$. We would like to prove that this diagram commutes. Consider the two proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}}{A_1 \otimes B_1 \vdash A_2 \otimes B_2} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{C_1 \vdash C_2}}{A_1 \otimes B_1, C_1 \vdash (A_2 \otimes B_2) \otimes C_2}}{(A_1 \otimes B_1) \otimes C_1 \vdash (A_2 \otimes B_2) \otimes C_2} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A_2 \vdash A_2} \quad \dfrac{\dfrac{}{B_2 \vdash B_2} \quad \dfrac{}{C_2 \vdash C_2}}{B_2, C_2 \vdash B_2 \otimes C_2}}{A_2, B_2, C_2 \vdash A_2 \otimes (B_2 \otimes C_2)}}{A_2 \otimes B_2, C_2 \vdash A_2 \otimes (B_2 \otimes C_2)}}{(A_2 \otimes B_2) \otimes C_2 \vdash A_2 \otimes (B_2 \otimes C_2)}}{(A_1 \otimes B_1) \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)}\,\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A_1 \vdash A_1} \quad \dfrac{\dfrac{}{B_1 \vdash B_1} \quad \dfrac{}{C_1 \vdash C_1}}{B_1, C_1 \vdash B_1 \otimes C_1}}{A_1, B_1, C_1 \vdash A_1 \otimes (B_1 \otimes C_1)}}{A_1 \otimes B_1, C_1 \vdash A_1 \otimes (B_1 \otimes C_1)}}{(A_1 \otimes B_1) \otimes C_1 \vdash A_1 \otimes (B_1 \otimes C_1)} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2} \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{C_1 \vdash C_2}}{B_1, C_1 \vdash B_2 \otimes C_2}}{B_1 \otimes C_1 \vdash B_2 \otimes C_2}}{A_1, B_1 \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)}}{A_1 \otimes (B_1 \otimes C_1) \vdash A_2 \otimes (B_2 \otimes C_2)}}{(A_1 \otimes B_1) \otimes C_1 \vdash A_2 \otimes (B_2 \otimes C_2)}\,\text{Cut}$$

By modularity, the two proofs have

$$\alpha \circ ((f_1 \otimes f_2) \otimes f_3) \qquad \text{and} \qquad (f_1 \otimes (f_2 \otimes f_3)) \circ \alpha.$$

as respective denotations. Now, the two proofs reduce by cut-elimination to the same proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2}}{A_1, B_1 \vdash A_2 \otimes B_2}\,\text{Right}\otimes \quad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{C_1 \vdash C_2}}{A_1, B_1, C_1 \vdash (A_2 \otimes B_2) \otimes C_2}\,\text{Right}\otimes}{A_1, B_1 \otimes C_1 \vdash (A_2 \otimes B_2) \otimes C_2}\,\text{Left}\otimes}{A_1 \otimes (B_1 \otimes C_1) \vdash (A_2 \otimes B_2) \otimes C_2}\,\text{Left}\otimes$$

which is simply the original proof of associativity in which every axiom step

$$\dfrac{}{A \vdash A} \qquad \dfrac{}{B \vdash B} \qquad \dfrac{}{C \vdash C}$$

has been replaced by the respective proof

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1 \vdash A_2} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{B_1 \vdash B_2} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{C_1 \vdash C_2}$$

The very fact that the two proofs reduce to the same proof, and that denotation is invariant under cut-elimination, ensures that the equality

$$\alpha \circ ((f_1 \otimes f_2) \otimes f_3) \quad = \quad (f_1 \otimes (f_2 \otimes f_3)) \circ \alpha.$$

holds. We conclude that the categorical diagram (14) commutes, and thus, that the family α of associativity morphisms is natural. The other naturality and coherence conditions required of a monoidal category are established in just the same way.

2.4 Proof invariants organize themselves as monoidal categories (2)

In order to conclude that the tensor product $\otimes$ defines a monoidal category of denotations, there only remains to check that the three morphisms α, λ and ρ are isomorphisms. Interestingly, this is not necessarily the case! The expected inverse of the three morphisms α, λ and ρ are the denotations $\overline{\alpha}$, $\overline{\lambda}$ and $\overline{\rho}$ of the three proofs below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}\,\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}\,\text{Axiom}}{A, B \vdash A \otimes B}\,\text{Right}\otimes \quad \dfrac{}{C \vdash C}\,\text{Axiom}}{A, B, C \vdash (A \otimes B) \otimes C}\,\text{Right}\otimes}{A, B \otimes C \vdash (A \otimes B) \otimes C}\,\text{Left}\otimes}{A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C}\,\text{Left}\otimes$$

and

$$\dfrac{\text{Right 1}\;\dfrac{}{\vdash 1} \qquad \dfrac{}{A \vdash A}\;\text{Axiom}}{A \vdash 1 \otimes A}\;\text{Right}\otimes$$

and

$$\dfrac{\text{Axiom}\;\dfrac{}{A \vdash A} \qquad \dfrac{}{\vdash 1}\;\text{Right 1}}{A \vdash A \otimes 1}\;\text{Right}\otimes$$

It is not difficult to deduce the following two equalities from invariance and modularity:

$$\lambda \circ \overline{\lambda} \;=\; id_A, \qquad\qquad \rho \circ \overline{\rho} \;=\; id_A.$$

On the other hand, and quite surprisingly, none of the four expected equalities

$$\overline{\lambda} \circ \lambda \;=\; id_{e\otimes A}, \qquad\qquad \overline{\rho} \circ \rho \;=\; id_{A\otimes e},$$

$$\overline{\alpha} \circ \alpha \;=\; id_{(A\otimes B)\otimes C}, \qquad\qquad \alpha \circ \overline{\alpha} \;=\; id_{A\otimes(B\otimes C)},$$

is necessarily satisfied by the category of denotations. Typically, modularity ensures that the morphism $\overline{\rho} \circ \rho$ denotes the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\;\text{Axiom}}{A, 1 \vdash A}\;\text{Left 1}}{A \otimes 1 \vdash A}\;\text{Left}\otimes \qquad \dfrac{\dfrac{}{A \vdash A}\;\text{Axiom} \qquad \dfrac{}{\vdash 1}\;\text{Right 1}}{A \vdash A \otimes 1}\;\text{Right}\otimes}{A \otimes 1 \vdash A \otimes 1}\;\text{Cut}$$

which is transformed by cut-elimination into the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\;\text{Axiom}}{A, 1 \vdash A}\;\text{Left 1}}{A \otimes 1 \vdash A}\;\text{Left}\otimes \qquad \dfrac{}{\vdash 1}\;\text{Right 1}}{A \otimes 1 \vdash A \otimes 1}\;\text{Right}\otimes \tag{15}$$

Strictly speaking, invariance, modularity and tensoriality do not force that the proof (15) has the same denotation as the η-expansion of the identity:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\;\text{Axiom} \qquad \dfrac{}{1 \vdash 1}\;\text{Axiom}}{A, 1 \vdash A \otimes 1}\;\text{Right}\otimes}{A \otimes 1 \vdash A \otimes 1}\;\text{Left}\otimes \tag{16}$$

at least if we are careful to define the cut-elimination procedure of linear logic in the slightly unconventional but right way given in Chapter 3.

2.5 Proof invariants organize themselves as monoidal categories (3)

However, we are not very far at this point from obtaining a monoidal category of denotations. To that purpose, it is sufficient indeed to add a series of equalities to invariance, modularity and tensoriality. For every two proofs π_1 and π_2, we require first that the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, C, D \vdash A}}{\Gamma, C \otimes D \vdash A}\,\text{Left}\otimes \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{\Gamma, C \otimes D, \Delta \vdash A \otimes B}\,\text{Right}\otimes \tag{17}$$

has the same denotation as the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma, C, D \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{\Gamma, C, D, \Delta \vdash A \otimes B}\,\text{Right}\otimes}{\Gamma, C \otimes D, \Delta \vdash A \otimes B}\,\text{Left}\otimes \tag{18}$$

obtained by "permuting" the left and right introduction of the tensor product. We require symmetrically that the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta, C, D \vdash B}}{\Delta, C \otimes D \vdash B}\,\text{Left}\otimes}{\Gamma, \Delta, C \otimes D \vdash A \otimes B}\,\text{Right}\otimes \tag{19}$$

has the same denotation as the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta, C, D \vdash B}}{\Delta, C \otimes D \vdash B}\,\text{Right}\otimes}{\Gamma, \Delta, C \otimes D \vdash A \otimes B}\,\text{Left}\otimes \tag{20}$$

obtained by "permuting" the left and right introduction of the tensor product. We also require that the two proofs

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A}}{\Gamma, 1 \vdash A}\,\text{Left }1 \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{\Gamma, 1, \Delta \vdash A \otimes B}\,\text{Right}\otimes \tag{21}$$

and

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{1, \Delta \vdash A}\,\text{Left }1}{\Gamma, 1, \Delta \vdash A \otimes B}\,\text{Right}\otimes \tag{22}$$

have the same denotation as the proof

$$\cfrac{\cfrac{\cfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \cfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B}\ \text{Right} \otimes}{\Gamma, 1, \Delta \vdash A \otimes B}\ \text{Left } 1 \tag{23}$$

obtained by "relocating" the left introduction of the unit 1 from the sequent $\Gamma \vdash A$ or the sequent $\Delta \vdash B$ to the sequent $\Gamma, \Delta \vdash A \otimes B$.

Once these four additional equalities satisfied, the original hypothesis of invariance, modularity and tensoriality of denotations implies the desired equalities:

$$\overline{\lambda} \circ \lambda = id_{e \otimes A}, \qquad \overline{\rho} \circ \rho = id_{A \otimes e},$$

$$\overline{\alpha} \circ \alpha = id_{(A \otimes B) \otimes C}, \qquad \alpha \circ \overline{\alpha} = id_{A \otimes (B \otimes C)}.$$

Hence, the three morphisms α, λ and ρ are isomorphisms in the category of denotations, with respective inverse $\overline{\alpha}$, $\overline{\lambda}$ and $\overline{\rho}$. We conclude in that case that the category of denotations is monoidal.

Remark. The discussion above is mainly intended to the amaze the insider. The cut-elimination procedure described in Chapter 3 is designed extremely carefully in order to avoid unnecessary proof transformations. Once this strict cut-elimination policy is adopted, the equalities mentioned above are not necessarily satisfied: consequently, the category of denotations is not necessarily monoidal. On the other hand, all existing cut-elimination procedures appearing in the literature are strictly more permissive than ours, in the sense that more proof transformations are accepted as valid. We will see in Chapter 3 that when such a permissive policy is adopted, the three principles of invariance, modularity and tensoriality imply the equalities just mentioned: hence, the category of denotations is monoidal in that case.

2.6 Conversely, what is a categorical model of linear logic?

We have recognized that every (invariant, modular, tensorial) denotation defines a monoidal category of denotations, at least when the cut-elimination procedure is sufficiently permissive. There remains to investigate the converse question: what axioms a given monoidal category $\mathbb{C}$ should satisfy in order to define a modular and tensorial invariant of proofs? The general principle of the interpretation is that every sequent

$$A_1, \ldots, A_m \ \vdash \ B$$

of linear logic will be interpreted as a morphism

$$[A_1] \otimes \cdots \otimes [A_m] \ \longrightarrow \ [B]$$

in the category $\mathbb{C}$, where we write $[A]$ for the object which denotes the formula A in the category. This object $[A]$ is computed by induction on the size of the formula A in the expected way. Typically,

$$[A \otimes B] \ = \ [A] \otimes [B]$$

This explains why the category $\mathbb{C}$ should admit, at least, a tensor product. It is useful to write

$$[\Gamma] \; = \; [A_1] \otimes \cdots \otimes [A_m]$$

for the denotation of the context

$$\Gamma \; = \; A_1, \ldots, A_m$$

as an object of the category $\mathbb{C}$. Every proof of the sequent

$$\Gamma \; \vdash \; B$$

is thus interpreted as a morphism

$$[\Gamma] \; \longrightarrow \; [B]$$

in the category $\mathbb{C}$. A proof π is then interpreted by induction on the "depth" of its derivation tree. Typically, the axiom rule

$$\frac{}{A \vdash A}\;\text{Axiom}$$

is interpreted as the identity morphism on the interpretation of the formula A.

$$id_{[A]} \quad : \quad [A] \; \longrightarrow \; [A].$$

Also typically, given two proofs

$$\frac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad\qquad \frac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta \vdash B}$$

interpreted as morphisms

$$f \;:\; [\Gamma] \;\longrightarrow\; [A] \qquad\qquad g \;:\; [\Delta] \;\longrightarrow\; [B]$$

in the category $\mathbb{C}$, the proof

$$\frac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \vdash B}\;\text{Right}\otimes$$

is interpreted as the morphism

$$[\Gamma] \otimes [\Delta] \xrightarrow{\quad f \otimes g \quad} [A] \otimes [B]$$

in the monoidal category $\mathbb{C}$.

Beyond these basic principles, the structures and properties required of a category $\mathbb{C}$ in order to provide an invariant of proofs depends on the fragment (or variant) of linear logic one has in mind: commutative or non-commutative, classical or intuitionistic, additive or non-additive, etc. In each case, we sketch below what kind of axioms a monoidal category $\mathbb{C}$ should satisfy in order to define an invariant of proofs.

Commutative vs. non-commutative logic

Linear logic is generally understood as a commutative logic, because there exists a canonical proof of the sequent $A \otimes B \vdash B \otimes A$ for every formula A and B. The proof is constructed as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{Axiom} \qquad \dfrac{}{A \vdash A}\,\text{Axiom}}{B, A \vdash B \otimes A}\,\text{Right}\,\otimes}{A, B \vdash B \otimes A}\,\text{Exchange}}{A \otimes B \vdash B \otimes A}\,\text{Left}\,\otimes$$

For this reason, usual (commutative) linear logic is interpreted in monoidal categories equipped with a symmetry, thus called *symmetric* monoidal categories; see Section 4.3 in Chapter 4 for a definition.

On the other hand, several non-commutative variants of linear logic have been introduced in the literature, in which the exchange rule:

$$\dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\,\text{Exchange}$$

has been removed, or replaced by a restricted version. These non-commutative variants of linear logic are interpreted in monoidal categories, possibly equipped with a suitable notion of permutation, like a braiding; see Section 4.2 in Chapter 4 for a definition.

Classical linear logic and duality

In his original article, Jean-Yves Girard introduced a *classical* linear logic, in which sequents are monolateral:

$$\vdash \ A_1, \cdots, A_n.$$

The main feature of the logic is a duality principle, based on an involutive negation:

- every formula A has a negation $A^{\perp}$,
- the negation of the negation $A^{\perp\perp}$ of a formula A is the formula A.

From this, a new connective $\parr$ can be defined by duality:

$$(A \parr B) \quad = \quad (B^{\perp} \otimes A^{\perp})^{\perp}.$$

This leads to an alternative presentation of linear logic, based this time on bilateral sequents:

$$A_1, \cdots, A_m \ \vdash \ B_1, \cdots, B_n \tag{24}$$

We have seen in Chapter 1 that in classical logic, this bilateral sequent stands for the formula

$$A_1 \wedge \cdots \wedge A_m \ \Rightarrow \ B_1 \vee \cdots \vee B_n.$$

Similarly, in linear logic, it stands for the formula

$$A_1 \otimes \cdots \otimes A_m \ \multimap \ B_1 \parr \cdots \parr B_n$$

where $\multimap$ is implication in linear logic. The notion of linearly distributive category introduced by Robin Cockett and Robert Seely, and recalled in Chapter 4 of this survey, is a category equipped with *two* monoidal structures $\otimes$ and $\bullet$ precisely to interpret such a bilateral sequent (24) as a morphism

$$[A_1] \otimes \ldots \otimes [A_m] \;\longrightarrow\; [B_1] \bullet \ldots \bullet [B_n].$$

in the category.

Intuitionistic linear logic and linear implication $\multimap$

The *intuitionistic* fragment of linear logic was later extracted from classical linear logic by restricting the bilateral sequents (24) to "intuitionistic" sequents

$$A_1, \cdots, A_m \;\vdash\; B$$

in which several formulas may appear on the lefthand side of the sequent, but only one formula appears on the righthand side. We have seen in the introduction (Chapter 1) that Heyting applied the same trick to classical logic in order to formalize intuitionistic logic. Hence the name "intuitionistic" linear logic.

Duality generally disappears in the usual formalizations of intuitionistic linear logic: the original connectives of linear logic are limited to the tensor product $\otimes$, the unit 1, and the linear implication $\multimap$. The right introduction of linear implication is performed by the rule:

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ \text{Right} \multimap$$

which may be interpreted in a monoidal closed category; see Chapter 4 of this survey for a definition.

The additive conjunction & of linear logic

One important aspect of linear logic is the discovery that there exists *two* different conjunctions in logic:

- a "multiplicative" conjunction called "tensor" and noted $\otimes$ because it behaves like a tensor product in linear algebra,
- another "additive" conjunction called "with" and noted & and which behaves like a cartesian product in linear algebra.

In intuitionistic linear logic, the right introduction of the connective & is performed by the rule:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\& B} \tag{25}$$

The left introduction of the connective & is performed by two different rules:

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A\& B, \Delta \vdash C} \qquad\qquad \frac{\Gamma, B, \Delta \vdash C}{\Gamma, A\& B, \Delta \vdash C} \tag{26}$$

The additive conjunction & is generally interpreted as a cartesian product in a monoidal category $\mathbb{C}$. Suppose indeed that $\Gamma = X_1, ..., X_m$ and that π_A and π_B are two proofs

$$\begin{array}{c}\pi_A\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad\qquad \begin{array}{c}\pi_B\\ \vdots\\ \hline \Gamma \vdash B\end{array}$$

of the sequents on top of the right introduction rule (25), interpreted by the morphisms:

$$f \; : \; [\Gamma] \longrightarrow [A] \qquad\qquad g \; : \; [\Gamma] \longrightarrow [B]$$

in the monoidal category $\mathbb{C}$. In order to interpret the proof

$$\frac{\begin{array}{c}\pi_A\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_B\\ \vdots\\ \hline \Gamma \vdash B\end{array}}{\Gamma \vdash A\&B}\;\text{Right }\& \tag{27}$$

we suppose from now on that every pair of objects A and B in the category $\mathbb{C}$ has a cartesian product noted $A\&B$. Then, the two morphisms f and g give rise to a unique morphism

$$\langle f, g \rangle \; : \; [\Gamma] \longrightarrow [A]\&[B]$$

making the diagram

$$\begin{tikzcd}
& & {[A]} \\
{[\Gamma]} \arrow[r, "\langle f,g\rangle"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & {[A]\&[B]} \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
& & {[B]}
\end{tikzcd}$$

commute in the category $\mathbb{C}$. In the diagram, the two morphisms π_1 and π_2 denote the first and second projection of the cartesian product. Now, we define the interpretation of the formula $A\&B$ as expected:

$$[A\&B] \; = \; [A]\&[B]$$

and interpret the proof (27) as the morphism $\langle f, g \rangle$.

The two left introduction rules (26) are interpreted by precomposing with the first or second projection of the cartesian product $[A]\&[B]$. Consider a proof

$$\begin{array}{c}\pi\\ \vdots\\ \hline \Gamma, B, \Delta \vdash C\end{array}$$

interpreted as the morphism

$$f \; : \; [\Gamma] \otimes [A] \otimes [\Delta] \longrightarrow [C]$$

in the category $\mathbb{C}$. Then, the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{\Gamma, B, \Delta \vdash C}}{\Gamma, A\&B, \Delta \vdash C}\ \text{Left}\ \&_1$$

is interpreted as the morphism:

$$[\Gamma] \otimes\ [A\&B] \otimes [\Delta] \xrightarrow{[\Gamma]\otimes\pi_1\otimes[\Delta]} [\Gamma] \otimes\ [A] \otimes [\Delta] \xrightarrow{\quad f \quad} [C].$$

Exponential modality

The main difficulty of the field is to understand the categorical properties of the exponential modality ! of linear logic. This question has been much debated in the past, sometimes with extreme vigour. It seems however that we have reached a state of agreement, or at least relative equilibrium, in the last few years. People have realized indeed that all the axiomatizations appearing in the literature converge to a unique notion: a well-behaved (that is: symmetric monoidal) adjunction

$$\begin{tikzcd} \mathbb{M} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{L} \arrow[ll, bend left, "M"] \end{tikzcd}$$

between:

- a symmetric monoidal closed category $\mathbb{L}$,
- a cartesian category $\mathbb{M}$.

By cartesian category, we mean a category with finite products: the category has a terminal object, and every pair of objects A and B has a cartesian product.

An adjunction $M \dashv L$ satisfying these properties is called a *linear-non-linear adjunction*; see Definition 19 at the beginning of Chapter 7. It provides a categorical model of *intuitionistic* linear logic, and a categorical model of *classical* linear logic when the category $\mathbb{C}$ is not only symmetric monoidal closed, but also $*$-autonomous. In this model, the exponential modality ! is interpreted as the comonad

$$! \ = \ M \circ L$$

induced on the category $\mathbb{L}$ by the linear-non-linear adjunction. We will come back to this point in Chapter 7 of the survey, where we review four alternative definitions of a categorical model of linear logic, and extract in each case a particular linear-non-linear adjunction.

2.7 Proof invariants as free categories

It is worth explaining a second time that Proof Theory is in many respects similar to Knot Theory, as understood in Representation Theory. In Knot Theory, every object in a monoidal category equipped with a *braiding* and a *left duality* defines an invariant of knots under the Reidemeister moves. See Chapter 4 for a definition of braiding and left duality. The invariant is then computed as follows: one defines a category $\mathcal{T}$ with natural numbers as objects, and knots (or rather *tangles*) as morphisms. One shows that:

- the category $\mathcal{T}$ is monoidal with braiding and left duality,
- there exists a unique structure preserving functor F from this category $\mathcal{T}$ to any monoidal category with braiding and left duality.

The notions of braiding and duality are given in Chapter 4. By structure preserving, we mean that the functor should transport the monoidal structure and the braiding of the category of tangles to the category $\mathbb{C}$.

By analogy, Denotational Semantics may be called the Representation Theory of proofs. For instance, it is possible to construct a free symmetric monoidal closed category over a category $\mathbb{C}$. Then, an invariant of proofs in intuitionistic multiplicative linear logic is the same thing as a structure preserving functor from this category to a symmetric monoidal closed category.

2.8 Notes and references

Several variants of non-commutative linear logic have been introduced in the litterature, starting with the cyclic linear logic formulated by Jean-Yves Girard, and described by David Yetter in [36]. The intuitionistic fragment of this cyclic linear logic happens to coincide with a sequent calculus devised by Jim Lambek [30] as early as 1958 in order to parse sentences in English and other vernacular languages.

One motivation for cyclic linear logic is topological: cyclic linear logic generates exactly the *planar proofs* of linear logic. By planar proof, one means a proof whose proof-net is planar, see [21]. Cyclic linear logic was later extended in several ways: to a non-commutative logic by Paul Ruet [1], to a planar logic by Paul-André Melliès [32], and more recently to a permutative logic by Jean-Marc Andreoli, Gabriele Pulcini and Paul Ruet [2]. Again, these logics are mainly motivated by the topological properties of the proof they generate: planarity, etc. Another motivation is provided by the Curry-Howard isomorphism relating Proof Theory to Programming Language Theory. Frank Pfenning and Jeff Polakow study in [33] a non-commutative extension of intuitionistic linear logic, in which non-commutativity captures the stack discipline involved in the standard continuation passing style translations.

There remains a lot to be understood and clarified on the various non-commutative logics, in particular on the semantic side. In that direction, one should mention the early work by Rick Blute and Phil Scott on Hopf algebras and cyclic linear logic [10, 12]. In Chapter 4, we will investigate two non-commutative variants of well-known categorical models of multiplicative linear logic: the linearly distributive categories introduced by Robin Cockett and Robert Seely in [15], and the non symmetric $*$-autonomous categories formalized by Michael Barr in [4].

3 Linear logic and its cut-elimination procedure

In this chapter, we introduce propositional linear logic, understood now as a formal proof system. First, we describe the sequent calculus of classical linear logic (LL) and explain how to specialize to its intuitionistic fragment (ILL). Then, we describe in full detail the cut-elimination procedure in the intuitionistic fragment. Finally, we return to classical linear logic and describe briefly the cut-elimination procedure in the general system.

3.1 Classical linear logic

The formulas

The formulas of propositional linear logic are constructed by an alphabet of four nullary constructors called *units*:

$$0 \qquad 1 \qquad \bot \qquad \top$$

two unary constructors called *modalities*:

$$!A \qquad ?A$$

and four binary constructors called *connectives*:

$$A \oplus B \qquad A \otimes B \qquad A \parr B \qquad A \& B$$

Each constructor receives a specific name in the folklore of linear logic. Each constructor is also classified: *additive*, *multiplicative*, or *exponential*, depending on its nature and affinities with other constructors. This is recalled in the table below.

$\oplus$	plus	The additives
0	zero: the unit of $\oplus$	
$\&$	with	
$\top$	top: the unit of $\&$	
$\otimes$	tensor product	The multiplicatives
1	one: the unit of $\otimes$	
$\parr$	parallel product	
$\bot$	bottom: the unit of $\parr$	
$!$	bang (or shriek)	The exponential modalities
$?$	why not	

The sequents

The sequents are *monolateral*

$$\vdash A_1, \ldots, A_n$$

understood as *sequences* of formulas, not sets. In particular, the same formula A may appear twice (consecutively) in the sequence: this is precisely what happens when the contraction rule applies.

The sequent calculus

A proof of propositional linear logic is constructed according to a series of rules presented in Figure 1. Note that there is no distinction between "Left" and "Right" introduction rules, since every sequent is monolateral.

Axiom	$\frac{}{\vdash A^\perp, A}$	Cut	$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$
$\otimes$	$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$	$\parr$	$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$
1	$\frac{}{\vdash 1}$	$\perp$	$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$
$\oplus_1$	$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$	&	$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$
$\oplus_2$	$\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$		
0	no rule	$\top$	$\frac{}{\vdash \Gamma, \top}$
Contraction	$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$	Weakening	$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}$
Dereliction	$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$	Promotion	$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$

Figure 1: Sequent calculus of linear logic (LL)

3.2 Intuitionistic linear logic

The formulas

The formulas of propositional intuitionistic linear logic (with additives) are constructed by an alphabet of two *units*:

$$1 \qquad \top$$

one modality:

$$!A$$

and three connectives:

$$A \otimes B \qquad A \multimap B \qquad A \& B$$

The connective $\multimap$ is called linear implication.

The sequents

The sequents are *intuitionistic*, that is, bilateral

$$A_1, \ldots, A_m \ \vdash \ B$$

with a *sequence* of formulas $A_1, ..., A_m$ on the lefthand side, and a *unique* formula B on the righthand side.

The sequent calculus

A proof of propositional intuitionistic linear logic is constructed according to a series of rules presented in Figure 2. We follow the tradition, and call "intuitionistic linear logic" the intuitionistic fragment without the connective & nor unit $\top$. Then, "intuitionistic linear logic *with finite products*" is the logic extended with the four rules of Figure 3.

3.3 Cut-elimination in intuitionistic linear logic

The cut-elimination procedure is described as a series of symbolic transformations on proofs in Sections 3.4 – 3.11.

3.4 Cut-elimination: commuting conversion cut vs. cut

3.4.1 Commuting conversion cut vs. cut (first case)

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_2, A, \Upsilon_3 \vdash B} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_1, B, \Upsilon_4 \vdash C}}{\Upsilon_1, \Upsilon_2, A, \Upsilon_3, \Upsilon_4 \vdash C}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, \Gamma, \Upsilon_3, \Upsilon_4 \vdash C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_2, A, \Upsilon_3 \vdash B}}{\Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\,\text{Cut} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_1, B, \Upsilon_4 \vdash C}}{\Upsilon_1, \Upsilon_2, \Gamma, \Upsilon_3, \Upsilon_4 \vdash C}\,\text{Cut}$$

and conversely. In other words, the two proofs are equivalent from the point of view of the cut-elimination procedure. This point has already been mentioned in Section 2.1 of Chapter 2: this commutative conversion ensures that composition is associative in the category induced by any invariant and modular denotation of proofs.

Axiom	$\dfrac{}{A \vdash A}$
Cut	$\dfrac{\Gamma \vdash A \qquad \Upsilon_1, A, \Upsilon_2 \vdash B}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}$
Left $\otimes$	$\dfrac{\Upsilon_1, A, B, \Upsilon_2 \vdash C}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}$
Right $\otimes$	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
Left $\multimap$	$\dfrac{\Gamma \vdash A \qquad \Upsilon_1, B, \Upsilon_2 \vdash C}{\Upsilon_1, \Gamma, A \multimap B, \Upsilon_2 \vdash C}$
Right $\multimap$	$\dfrac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}$
Left 1	$\dfrac{\Upsilon_1, \Upsilon_2 \vdash A}{\Upsilon_1, 1, \Upsilon_2 \vdash A}$
Right 1	$\dfrac{}{\vdash 1}$
Promotion	$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$
Dereliction	$\dfrac{\Upsilon_1, A, \Upsilon_2 \vdash B}{\Upsilon_1, !A, \Upsilon_2 \vdash B}$
Weakening	$\dfrac{\Upsilon_1, \Upsilon_2 \vdash B}{\Upsilon_1, !A, \Upsilon_2 \vdash B}$
Contraction	$\dfrac{\Upsilon_1, !A, !A, \Upsilon_2 \vdash B}{\Upsilon_1, !A, \Upsilon_2 \vdash B}$
Exchange	$\dfrac{\Upsilon_1, A_1, A_2, \Upsilon_2 \vdash B}{\Upsilon_1, A_2, A_1, \Upsilon_2 \vdash B}$

Figure 2: Sequent calculus of intuitionistic linear logic (ILL)

3.4.2 Commuting conversion cut vs. cut (second case)

Another commuting conversion is this one. The proof

Left $\&_1$	$\dfrac{\Upsilon_1, A, \Upsilon_2 \vdash C}{\Upsilon_1, A\&B, \Upsilon_2 \vdash C}$
Left $\&_2$	$\dfrac{\Upsilon_1, B, \Upsilon_2 \vdash C}{\Upsilon_1, A\&B, \Upsilon_2 \vdash C}$
Right $\&$	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\&B}$
True	$\dfrac{}{\Gamma \vdash \top}$

Figure 3: Addendum to figure 2: ILL with finite products

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, B, \Upsilon_3 \vdash C}}{\Upsilon_1, A, \Upsilon_2, \Delta, \Upsilon_3 \vdash C}\,\text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta, \Upsilon_3 \vdash C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Delta \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, B, \Upsilon_3 \vdash C}}{\Upsilon_1, \Gamma, \Upsilon_2, B, \Upsilon_3 \vdash C}\,\text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta, \Upsilon_3 \vdash C}\,\text{Cut}$$

and conversely.

3.5 Cut-elimination: the η-expansion steps

3.5.1 The tensor product

The proof

$$\dfrac{}{A \otimes B \vdash A \otimes B}\,\text{Axiom}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Axiom} \qquad \dfrac{}{B \vdash B}\,\text{Axiom}}{A, B \vdash A \otimes B}\,\text{Right}\otimes}{A \otimes B \vdash A \otimes B}\,\text{Left}\otimes$$

3.5.2 The linear implication

The proof

$$\frac{}{A \multimap B \vdash A \multimap B}\,\text{Axiom}$$

is transformed into the proof

$$\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\text{Axiom} \qquad \cfrac{}{B \vdash B}\,\text{Axiom}}{A, A \multimap B \vdash B}\,\text{Left} \multimap}{A \multimap B \vdash A \multimap B}\,\text{Right} \multimap$$

3.5.3 The tensor unit

The proof

$$\frac{}{1 \vdash 1}\,\text{Axiom}$$

is transformed into the proof

$$\cfrac{\cfrac{}{\vdash 1}\,\text{Right } 1}{1 \vdash 1}\,\text{Left } 1$$

3.5.4 The exponential modality

The proof

$$\frac{}{!A \vdash !A}\,\text{Axiom}$$

is transformed into the proof

$$\cfrac{\cfrac{\cfrac{}{A \vdash A}\,\text{Axiom}}{!A \vdash A}\,\text{Dereliction}}{!A \vdash !A}\,\text{Promotion}$$

3.6 Cut-elimination: the axiom steps

3.6.1 Axiom steps

The proof

$$\cfrac{\cfrac{}{A \vdash A}\,\text{Axiom} \qquad \cfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Upsilon_1, A, \Upsilon_2 \vdash B}}{\Upsilon_1, A, \Upsilon_2 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\cfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{\Upsilon_1, A, \Upsilon_2 \vdash B}$$

3.6.2 Conclusion vs. axiom

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{}{A \vdash A}\,\text{Axiom}}{\Gamma \vdash A}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Gamma \vdash A}$$

3.7 Cut-elimination: the exchange steps

3.7.1 Conclusion vs. exchange (the first case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, B, A, \Upsilon_2 \vdash C}\,\text{Exchange}}{\Upsilon_1, B, \Gamma, \Upsilon_2 \vdash C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, \Gamma, B, \Upsilon_2 \vdash C}\,\text{Cut}}{\Upsilon_1, B, \Gamma, \Upsilon_2 \vdash C}\,\text{Series of Exchanges}$$

3.7.2 Conclusion vs. exchange (the second case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, B, A, \Upsilon_2 \vdash C}\,\text{Exchange}}{\Upsilon_1, \Gamma, A, \Upsilon_2 \vdash C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash B\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2 \vdash C\end{array}}{\Upsilon_1, A, \Gamma, \Upsilon_2 \vdash C}\,\text{Cut}}{\Upsilon_1, \Gamma, A, \Upsilon_2 \vdash C}\,\text{Series of Exchanges}$$

3.8 Cut-elimination: principal formula vs. principal formula

In this section and the next, we explain how the cut-elimination procedure transforms a proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash B\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\,\text{Cut}$$

in which the conclusion A and the hypothesis A are both *principal* in their respective proofs π_1 and π_2. In this section, we treat the cases in which the last rules of the proofs π_1 and π_2 introduces:

- the tensor product (Section 3.8.1),
- the linear implication (Section 3.8.2),
- the tensor unit (Section 3.8.3).

For clarity's sake, we treat separately in Section 3.9 the three cases where the last rule of the proof π_1 is a promotion rule, and the last rule of the proof π_2 is a "structural rule": a dereliction, a weakening or a contraction.

3.8.1 The tensor product

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \otimes B}\,\text{Right}\otimes \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2 \vdash C\end{array}}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}\,\text{Left}\otimes}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2 \vdash C\end{array}}{\Upsilon_1, A, \Delta, \Upsilon_2 \vdash C}\,\text{Cut}}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\,\text{Cut}$$

A choice has been made here, since the cut-rule on the formula $A \otimes B$ is replaced by a cut-rule on the formula B, followed by a cut-rule on the formula A. Instead, the cut-rule on A may have been applied before the cut-rule on B. However, this choice is innocuous, because the two derivations resulting from this choice are equivalent – modulo the conversion rule given in Section 3.4.2.

3.8.2 The linear implication

The proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A, \Delta \vdash B}}{\Delta \vdash A \multimap B}\ \text{Right} \multimap \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash C}}{\Upsilon_1, \Gamma, A \multimap B, \Upsilon_2 \vdash C}\ \text{Left} \multimap}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A, \Delta \vdash B}}{\Gamma, \Delta \vdash B}\ \text{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash C}}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

3.8.3 The tensor unit

The proof

$$\dfrac{\dfrac{}{\vdash 1}\ \text{Right } 1 \qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, \Upsilon_2 \vdash A}}{\Upsilon_1, 1, \Upsilon_2 \vdash A}\ \text{Left } 1}{\Upsilon_1, \Upsilon_2 \vdash A}\ \text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Upsilon_1, \Upsilon_2 \vdash A}$$

3.9 Cut-elimination: promotion vs. dereliction and structural rules

In this section, we explain how the cut-elimination procedure transforms a proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{!\Gamma \vdash A}}{!\Gamma \vdash !A}\ \text{Promotion} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, !A, \Upsilon_2 \vdash B}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\ \text{Cut}$$

in which the hypothesis $!A$ is principal in the proof π_2. There are exactly three cases to treat, depending on the last rule of the proof π_2:

- a dereliction (Section 3.9.1),
- a weakening (Section 3.9.2),
- a contraction (Section 3.9.3).

The interaction with an exchange step has already been treated in Section 3.7.

3.9.1 Promotion vs. dereliction

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \overline{!\Gamma \vdash A}\end{matrix}}{!\Gamma \vdash !A}\,\text{Promotion} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \overline{\Upsilon_1, A, \Upsilon_2 \vdash B}\end{matrix}}{\Upsilon_1, !A, \Upsilon_2 \vdash B}\,\text{Dereliction}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \overline{!\Gamma \vdash A}\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \overline{\Upsilon_1, A, \Upsilon_2 \vdash B}\end{matrix}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\,\text{Cut}$$

3.9.2 Promotion vs. weakening

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \overline{!\Gamma \vdash A}\end{matrix}}{!\Gamma \vdash !A}\,\text{Promotion} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \overline{\Upsilon_1, \Upsilon_2 \vdash B}\end{matrix}}{\Upsilon_1, !A, \Upsilon_2 \vdash B}\,\text{Weakening}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{matrix}\pi_2\\ \vdots\\ \overline{\Upsilon_1, \Upsilon_2 \vdash B}\end{matrix}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\,\text{Series of Weakenings}$$

3.9.3 Promotion vs. contraction

The proof

$$
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{!\Gamma \vdash A}}{!\Gamma \vdash !A}\text{Promotion}
\qquad
\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, !A, !A, \Upsilon_2 \vdash B}}{\Upsilon_1, !A, \Upsilon_2 \vdash B}\text{Contraction}
}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\text{Cut}
$$

is transformed into the proof

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{!\Gamma \vdash A}}{!\Gamma \vdash !A}\text{Promotion}
\qquad
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{!\Gamma \vdash A}}{!\Gamma \vdash !A}\text{Promotion}
\qquad
\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, !A, !A, \Upsilon_2 \vdash B}
}{\Upsilon_1, !A, !\Gamma, \Upsilon_2 \vdash B}\text{Cut}
}{\Upsilon_1, !\Gamma, !\Gamma, \Upsilon_2 \vdash B}\text{Cut}}{\Upsilon_1, !\Gamma, \Upsilon_2 \vdash B}\text{Series of Contractions and Exchanges}
$$

3.10 Cut-elimination: secondary conclusion

In this section, we explain how the cut-elimination procedure transforms a proof

$$
\dfrac{
\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A}
\qquad
\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, A, \Upsilon_2 \vdash B}
}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\text{Cut}
$$

in which the conclusion A is *secondary* in the proof π_1. This leads us to a case analysis, in which we describe how the proof evolves depending on the last rule of the proof π_1. The six cases are treated in turn:

- a left introduction of the linear implication,
- a dereliction,
- a weakening,
- a contraction,
- an exchange,
- a left introduction of the tensor product (low priority)
- a left introduction of the tensor unit (low priority).

The last two cases are treated at the end of the section because they are given a lower priority in the procedure.

3.10.1 Left introduction of the linear implication

The proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_2, B, \Upsilon_3 \vdash C}}{\Upsilon_2, \Gamma, A \multimap B, \Upsilon_3 \vdash C}\,\text{Left} \multimap \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, \Gamma, A \multimap B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_2, B, \Upsilon_3 \vdash C} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, \Gamma, A \multimap B, \Upsilon_3, \Upsilon_4 \vdash D}\,\text{Left} \multimap$$

3.10.2 A generic description of the structural rules: dereliction, weakening, contraction, exchange

Four cases remain to be treated in order to describe entirely how the cut-elimination procedure transforms a proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_4 \vdash B}}{\Upsilon_1, \Gamma, \Upsilon_4 \vdash B}\,\text{Cut}$$

in which the conclusion A is *secondary* in the proof π_1. Each case depends on the last rule of the proof π_1, which may be:

- a dereliction,
- a weakening,
- a contraction,
- an exchange.

Each of the four rules is of the form

$$\dfrac{\Upsilon_2, \Phi, \Upsilon_3 \vdash A}{\Upsilon_2, \Psi, \Upsilon_3 \vdash A}$$

where the context Φ is transformed into the context Ψ in a way depending on the specific rule:

- dereliction: the context Φ consists of a formula C, and the context Ψ consists of the formula $!C$,
- weakening: the context Φ is empty, and the context Ψ consists of a formula $!C$,

- contraction: the context Φ consists of two formulas $!C, !C$ and the context Ψ consists of the formula $!C$,
- exchange: the context Φ consists of two formulas C, D and the context Ψ consists of the two formulas D, C.

By hypothesis, the proof π_1 decomposes in the following way:

$$\dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_2, \Phi, \Upsilon_3 \vdash A}}{\Upsilon_2, \Psi, \Upsilon_3 \vdash A}\text{ the specific rule}$$

The proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_2, \Phi, \Upsilon_3 \vdash A}}{\Upsilon_2, \Psi, \Upsilon_3 \vdash A}\text{ the specific rule} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, A, \Upsilon_4 \vdash B}}{\Upsilon_1, \Upsilon_2, \Psi, \Upsilon_3, \Upsilon_4 \vdash B}\text{ Cut}$$

is then transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_2, \Phi, \Upsilon_3 \vdash A} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, A, \Upsilon_4 \vdash B}}{\Upsilon_1, \Upsilon_2, \Phi, \Upsilon_3, \Upsilon_4 \vdash B}\text{ Cut}}{\Upsilon_1, \Upsilon_2, \Psi, \Upsilon_3, \Upsilon_4 \vdash B}\text{ the specific rule}$$

3.10.3 Left introduction of the tensor (with low priority)

The proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Upsilon_2, A, B, \Upsilon_3 \vdash C}}{\Upsilon_2, A \otimes B, \Upsilon_3 \vdash C}\text{ Left }\otimes \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, A \otimes B, \Upsilon_3, \Upsilon_4 \vdash D}\text{ Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Upsilon_2, A, B, \Upsilon_3 \vdash C} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, A, B, \Upsilon_3, \Upsilon_4 \vdash D}\text{ Cut}}{\Upsilon_1, \Upsilon_2, A \otimes B, \Upsilon_3, \Upsilon_4 \vdash D}\text{ Left }\otimes$$

3.10.4 Left introduction of the tensor unit (with low priority)

The proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Upsilon_2, \Upsilon_3 \vdash A\end{array}}{\Upsilon_2, 1, \Upsilon_3 \vdash A}\,\text{Left 1} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_4 \vdash B\end{array}}{\Upsilon_1, \Upsilon_2, 1, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Upsilon_2, \Upsilon_3 \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_4 \vdash B\end{array}}{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, 1, \Upsilon_3, \Upsilon_4 \vdash B}\,\text{Left 1}$$

3.11 Cut-elimination: secondary hypothesis

In this section, we explain how the cut-elimination procedure transforms a proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash B\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\,\text{Cut}$$

in which the hypothesis A is *secondary* in the proof π_2. This leads us to a long case analysis, in which we describe how the proof evolves depending on the last rule of the proof π_2. The nine cases are treated in turn in the section:

- the right introduction of the tensor,
- the left introduction of the linear implication,
- the four structural rules: dereliction, weakening, contraction, exchange,
- the left introduction of the tensor (low priority),
- the left introduction of the tensor unit (low priority),
- the right introduction of the linear implication (low priority).

The last three cases are treated at the end of the section, because they are given a low priority in the procedure.

3.11.1 Right introduction of the tensor (first case)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash B\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Delta \vdash C\end{array}}{\Upsilon_1, A, \Upsilon_2, \Delta \vdash B \otimes C}\,\text{Right} \otimes}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta \vdash B \otimes C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash B\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\,\text{Cut} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Delta \vdash C\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2, \Delta \vdash B \otimes C}\,\text{Right} \otimes$$

3.11.2 Right introduction of the tensor (second case)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash C\end{array}}{\Delta, \Upsilon_1, A, \Upsilon_2 \vdash B \otimes C}\,\text{Right} \otimes}{\Delta, \Upsilon_1, \Gamma, \Upsilon_2 \vdash B \otimes C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \vdash B\end{array} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2 \vdash C\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash C}\,\text{Cut}}{\Delta, \Upsilon_1, \Gamma, \Upsilon_2 \vdash B \otimes C}\,\text{Right} \otimes$$

3.11.3 Left introduction of the linear implication (first case)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_2, A, \Upsilon_3 \vdash B\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, C, \Upsilon_4 \vdash D\end{array}}{\Upsilon_1, \Upsilon_2, A, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Left} \multimap}{\Upsilon_1, \Upsilon_2, \Gamma, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_2, A, \Upsilon_3 \vdash B}}{\Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\,\text{Cut} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, \Gamma, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Left} \multimap$$

3.11.4 Left introduction of the linear implication (second case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_3 \vdash B} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, C, \Upsilon_4 \vdash D}}{\Upsilon_1, A, \Upsilon_2, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Left} \multimap}{\Upsilon_1, \Gamma, \Upsilon_2, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_3 \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, C, \Upsilon_4 \vdash D}}{\Upsilon_1, \Gamma, \Upsilon_2, C, \Upsilon_4 \vdash D}\,\text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, \Upsilon_3, B \multimap C, \Upsilon_4 \vdash D}\,\text{Left} \multimap$$

3.11.5 Left introduction of the linear implication (third case)

The proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_2 \vdash B} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_3, A, \Upsilon_4 \vdash D}}{\Upsilon_1, \Upsilon_2, B \multimap C, \Upsilon_3, A, \Upsilon_4 \vdash D}\,\text{Left} \multimap}{\Upsilon_1, \Upsilon_2, B \multimap C, \Upsilon_3, \Gamma, \Upsilon_4 \vdash D}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_2 \vdash B} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, C, \Upsilon_3, A, \Upsilon_4 \vdash D}}{\Upsilon_1, C, \Upsilon_3, \Gamma, \Upsilon_4 \vdash D}\,\text{Cut}}{\Upsilon_1, \Upsilon_2, B \multimap C, \Upsilon_3, \Gamma, \Upsilon_4 \vdash D}\,\text{Left} \multimap$$

3.11.6 A generic description of the structural rules: dereliction, weakening, contraction, exchange

Four cases remain to be treated in order to describe how the cut-elimination procedure transforms a proof

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{\Gamma \vdash A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2 \vdash B}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B}\ \text{Cut}$$

in which the hypothesis A is *secondary* in the proof π_2. Each case depends on the last rule of the proof π_2, which may be:

- a dereliction,
- a weakening,
- a contraction,
- an exchange.

Each of the four rules is of the form

$$\dfrac{\Upsilon_1, \Phi, \Upsilon_2 \vdash B}{\Upsilon_1, \Psi, \Upsilon_2 \vdash B}$$

where the context Φ is transformed into the context Ψ in a way depending on the specific rule:

- dereliction: the context Φ consists of a formula C, and the context Ψ consists of the formula $!C$,
- weakening: the context Φ is empty, and the context Ψ consists of a formula $!C$,
- contraction: the context Φ consists of two formulas $!C, !C$ and the context Ψ consists of the formula $!C$,
- exchange: the context Φ consists of two formulas C, D and the context Ψ consists of the two formulas D, C.

From this follows that the proof π_2 decomposes as a proof of the form

$$\dfrac{\dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, A, \Upsilon_2, \Phi, \Upsilon_3 \vdash C}}{\Upsilon_1, A, \Upsilon_2, \Psi, \Upsilon_3 \vdash C}\ \text{the specific rule}$$

or as a proof of the form

$$\dfrac{\dfrac{\begin{matrix}\pi_3\\ \vdots\end{matrix}}{\Upsilon_1, \Phi, \Upsilon_2, A, \Upsilon_3 \vdash C}}{\Upsilon_1, \Psi, \Upsilon_2, A, \Upsilon_3 \vdash C}\ \text{the specific rule}$$

depending on the relative position of the secondary hypothesis A and of the contexts Φ and Ψ among the hypothesis of the proof π_2. In the first case, the proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2, \Phi, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, A, \Upsilon_2, \Psi, \Upsilon_3 \vdash B}\text{ the specific rule}}{\Upsilon_1, \Gamma, \Upsilon_2, \Psi, \Upsilon_3 \vdash B}\text{ Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2, \Phi, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, \Gamma, \Upsilon_2, \Phi, \Upsilon_3 \vdash B}\text{ Cut}}{\Upsilon_1, \Gamma, \Upsilon_2, \Psi, \Upsilon_3 \vdash B}\text{ the specific rule}$$

In the second case, the proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, \Phi, \Upsilon_2, A, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, \Psi, \Upsilon_2, A, \Upsilon_3 \vdash B}\text{ the specific rule}}{\Upsilon_1, \Psi, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\text{ Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \hline \Upsilon_1, \Phi, \Upsilon_2, A, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, \Phi, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\text{ Cut}}{\Upsilon_1, \Psi, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\text{ the specific rule}$$

3.11.7 Left introduction of the tensor (first case) (with low priority)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2, B, C, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, A, \Upsilon_2, B \otimes C, \Upsilon_3 \vdash B}\text{ Left }\otimes}{\Upsilon_1, \Gamma, \Upsilon_2, B \otimes C, \Upsilon_3 \vdash B}\text{ Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, \Upsilon_2, B, C, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, \Gamma, \Upsilon_3, B, C, \Upsilon_3 \vdash B}\text{ Cut}}{\Upsilon_1, \Gamma, \Upsilon_3, B \otimes C, \Upsilon_3 \vdash B}\text{ Left }\otimes$$

3.11.8 Left introduction of the tensor (second case) (with low priority)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash C\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2, C, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, A \otimes B, \Upsilon_2, C, \Upsilon_3 \vdash B}\,\text{Left } \otimes}{\Upsilon_1, A \otimes B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash C\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Upsilon_1, A, B, \Upsilon_2, C, \Upsilon_3 \vdash B\end{array}}{\Upsilon_1, A, B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\,\text{Cut}}{\Upsilon_1, A \otimes B, \Upsilon_2, \Gamma, \Upsilon_3 \vdash B}\,\text{Left } \otimes$$

3.11.9 Left introduction of the tensor unit (with low priority)

Just as in

Each of the five rules is of the form

$$\dfrac{\Upsilon_1, \Phi, \Upsilon_2 \vdash B}{\Upsilon_1, \Psi, \Upsilon_2 \vdash B}$$

where the context Φ is transformed into the context Ψ in a way depending on the specific rule:

the context Φ is empty, and the context Ψ consists of the formula 1,

3.11.10 Right introduction of the linear implication (with low priority)

The proof

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \hline B, \Upsilon_1, A, \Upsilon_2 \vdash C\end{array}}{\Upsilon_1, A, \Upsilon_2 \vdash B \multimap C}\,\text{Right } \multimap}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B \multimap C}\,\text{Cut}$$

is transformed into the proof

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \hline \Gamma \vdash A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \hline B, \Upsilon_1, A, \Upsilon_2 \vdash C\end{array}}{B, \Upsilon_1, \Gamma, \Upsilon_2 \vdash C}\,\text{Cut}}{\Upsilon_1, \Gamma, \Upsilon_2 \vdash B \multimap C}\,\text{Right } \multimap$$

4 Monoidal categories and duality

After recalling the usual definition of a monoidal category, we describe two alternative ways to duality and the notion of $*$-autonomous category (read star-autonomous). On one hand, a $*$-autonomous category may be seen as a symmetric monoidal closed category equipped with a dualizing object. This is developed in Sections 4.1—4.7 according to the topography below.

$$\begin{tikzcd}
 & \text{Monoidal} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Left closed} \arrow[dr] & \text{Symmetric monoidal} \arrow[d] & \text{Right closed} \arrow[dl] \\
 & \text{Symmetric monoidal closed} \arrow[d] & \\
 & *\text{-autonomous} &
\end{tikzcd}$$

On the other hand, a $*$-autonomous category may also be seen as a symmetric linearly distributive category equipped with a duality. The notion of linearly distributive category and its connection to $*$-autonomous categories is developed in Sections 4.8—4.11 following the topography below.

$$\begin{tikzcd}
 & \text{Monoidal} \arrow[d] & \\
 & \text{Linearly distributive} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Left duality} \arrow[dr] & \text{Symmetric linearly distributive} \arrow[d] & \text{Right duality} \arrow[dl] \\
 & *\text{-autonomous} &
\end{tikzcd}$$

4.1 Monoidal categories

A monoidal category $\mathbb{C}$ is a category with a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ associative up to a natural isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

and with a unit object e for the bifunctor, up to natural isomorphisms

$$\lambda_A : e \otimes A \longrightarrow A, \qquad \rho_A : A \otimes e \longrightarrow A.$$

The structure maps α, λ, ρ must satisfy two axioms. First, the pentagonal diagram

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "A\otimes\alpha"']
\end{tikzcd}$$

should commute for all objects A, B, C, D of the category. Second, the triangular diagram

$$\begin{tikzcd}
(A \otimes e) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes B"'] & & A \otimes (e \otimes B) \arrow[dl, "A\otimes\lambda"] \\
& A \otimes B &
\end{tikzcd}$$

should commute for all objects A and B of the category. Note that for clarity's sake, we generally drop the indices on the structure maps α, λ, ρ in our diagrams, and write A instead of id_A in compound morphisms like $A \otimes \alpha = id_A \otimes \alpha$.

The pentagon and triangle axioms ensure that any diagram made of structure maps commutes in the category $\mathbb{C}$. This property is called the *coherence property* of monoidal categories. It implies among other things that the structure morphisms $\lambda_e : e \otimes e \longrightarrow e$ and $\rho_e : e \otimes e \longrightarrow e$ coincide. This point is worth stressing, since the equality of these two maps is often given as a third axiom of monoidal categories. The equality follows in fact from the pentagon and triangle axioms. We clarify this point in Proposition 2, after the preliminary Proposition 1.

Proposition 1 *The triangles*

$$\begin{tikzcd}
(e \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\lambda\otimes B"'] & & e \otimes (A \otimes B) \arrow[dl, "\lambda"] \\
& A \otimes B &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes e \arrow[rr, "\alpha"] \arrow[dr, "\rho"'] & & A \otimes (B \otimes e) \arrow[dl, "A\otimes\rho"] \\
& A \otimes B &
\end{tikzcd}$$

commute in any monoidal category $\mathbb{C}$.

Proof. The proof is based on the observation that the functor $e \otimes - : \mathbb{C} \longrightarrow \mathbb{C}$ is full and faithful, because λ is a natural isomorphism from this functor to the identity functor. So, two morphisms $f, g : A \longrightarrow B$ coincide iff the morphisms $e \otimes f, e \otimes g : e \otimes A \longrightarrow e \otimes B$

coincide as well. In particular, the first triangle of the proposition commutes iff the triangle

$$\begin{tikzcd}
e \otimes ((e \otimes A) \otimes B) \arrow[rr, "e\otimes\alpha"] \arrow[dr, "e\otimes(\lambda\otimes B)"'] & & e \otimes (e \otimes (A \otimes B)) \arrow[dl, "e\otimes\lambda"] \\
& e \otimes (A \otimes B) &
\end{tikzcd}$$

commutes. Now, this triangle commutes iff the triangle obtained by adjoining a pentagon on top of it

$$\begin{tikzcd}
((e \otimes e) \otimes A) \otimes B \arrow[rr, "\alpha"] \arrow[d, "\alpha\otimes B"'] & & (e \otimes e) \otimes (A \otimes B) \arrow[dd, "\alpha"] \\
(e \otimes (e \otimes A)) \otimes B \arrow[d, "\alpha"'] & & \\
e \otimes ((e \otimes A) \otimes B) \arrow[rr, dotted, "e\otimes\alpha"] \arrow[dr, "e\otimes(\lambda\otimes B)"'] & & e \otimes (e \otimes (A \otimes B)) \arrow[dl, "e\otimes\lambda"] \\
& e \otimes (A \otimes B) &
\end{tikzcd}$$

commutes as well — this comes from the fact that α is an isomorphism. We leave as an exercise to the reader the elementary “diagram-chase” proving that this last triangle commutes, with its two borders equal to:

$$((e \otimes e) \otimes A) \otimes B \xrightarrow{(\rho\otimes A)\otimes B} (e \otimes A) \otimes B \xrightarrow{\alpha} e \otimes (A \otimes B).$$

This establishes that the first triangle of the proposition commutes. The second triangle is shown to commute in a similar way. □

Proposition 2 *The two morphisms λ_e and ρ_e coincide in any monoidal category $\mathbb{C}$.*

Proof. Naturality of λ implies that the diagram

$$\begin{tikzcd}
e \otimes (e \otimes B) \arrow[r, "\lambda"] \arrow[d, "e\otimes\lambda"'] & e \otimes B \arrow[d, "\lambda"] \\
e \otimes B \arrow[r, "\lambda"] & B
\end{tikzcd}$$

commutes. From this follows that the two structure morphisms

$$e \otimes (e \otimes B) \xrightarrow{\lambda} e \otimes B \qquad\qquad e \otimes (e \otimes B) \xrightarrow{e\otimes\lambda} e \otimes B$$

coincide — because the morphism $\lambda : e \otimes B \longrightarrow B$ is an isomorphism. This is the crux of the proof. Then, one instantiates the object A by the unit object e in the first triangle of Proposition 1, and replaces the morphism λ by the morphism $e \otimes \lambda$, to obtain that the triangle

$$\begin{tikzcd}
(e \otimes e) \otimes B \arrow[r, "\alpha"] \arrow[d, "\lambda\otimes B"'] & e \otimes (e \otimes B) \arrow[d, "e\otimes\lambda"] \\
e \otimes B \arrow[r, phantom, "="] & e \otimes B
\end{tikzcd}$$

commutes for every object B of the category $\mathbb{C}$. The triangular axiom of monoidal categories indicates then that the two morphisms:

$$(e \otimes e) \otimes B \xrightarrow{\lambda_e \otimes B} e \otimes B \qquad\qquad (e \otimes e) \otimes B \xrightarrow{\rho_e \otimes B} e \otimes B$$

coincide for every object B, and in particular for the object $B = e$. This shows that the two morphisms $\lambda_e \otimes e$ and $\rho_e \otimes e$ coincide. Just as in the proof of Proposition 1, we conclude from the fact that the functor $- \otimes e : \mathbb{C} \longrightarrow \mathbb{C}$ is full and faithful: the two morphisms λ_e and ρ_e coincide. □

One is generally interested in combining objects $A_1, ..., A_n$ of a monoidal category $\mathbb{C}$ using the "monoidal structure" or "tensor product" of the category, in order to obtain an object like $\bigotimes_i A_i$. Unfortunately, the tensor product is only associative up to natural isomorphism. Thus, there are generally several candidates for $\bigotimes_i A_i$. Typically, $(A_1 \otimes A_2) \otimes A_3$ and $A_1 \otimes (A_2 \otimes A_3)$ are two isomorphic objects of the category, candidates for the tensor product of A_1, A_2, A_3. This is the reason why the coherence property is so useful: it enables to "identify" the various candidates for $\bigotimes_i A_i$ in a *coherent* way. One may thus proceed "as if" the isomorphisms α, λ, ρ were identities.

This aspect of coherence is important. It may be expressed in a quite elegant and conceptual way. A monoidal category is *strict* when its structure maps α, λ and ρ are identities. So, in a strict monoidal category, there is only *one* candidate for $\bigotimes_i A_i$. The coherence theorem states that every monoidal category is *equivalent* to a strict monoidal category. Equivalence of monoidal categories is expressed conveniently in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations. We come back to this point, and provide all definitions, in Chapter 5.

Exercise. Show that in every monoidal category $\mathbb{C}$, the set of endomorphisms of the unit object e defines a *commutative* monoid for the composition, in the sense that $f \circ g = g \circ f$ for every two morphisms $f, g : e \longrightarrow e$. Show moreover that composition coincides with tensor product up to the isomorphism $\rho_e = \lambda_e$, in the sense that $f \otimes g = \rho_e^{-1} \circ (f \circ g) \circ \rho_e$. ■

4.2 Braided monoidal categories

A braided monoidal category $\mathbb{C}$ is a monoidal category equipped with a braiding. A braiding is a natural isomorphism

$$\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$$

making the hexagonal diagrams

$$\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\gamma"] & (B \otimes C) \otimes A \arrow[dr, bend left, "\alpha"] & \\
(A \otimes B) \otimes C \arrow[ur, bend left, "\alpha"] \arrow[dr, bend right, "\gamma\otimes C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[ur, bend right, "B\otimes\gamma"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\gamma"] & C \otimes (A \otimes B) \arrow[dr, bend left, "\alpha^{-1}"] & \\
A \otimes (B \otimes C) \arrow[ur, bend left, "\alpha^{-1}"] \arrow[dr, bend right, "A\otimes\gamma"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"] & (A \otimes C) \otimes B \arrow[ur, bend right, "\gamma\otimes B"'] &
\end{tikzcd}$$

commute. Note that the second hexagon is just the first one in which the morphism γ has been replaced by its inverse γ^{-1}.

The braiding and the unit of the monoidal category are related in the following way.

Proposition 3 *The triangles*

$$\begin{tikzcd}
A \otimes e \arrow[rr, "\gamma"] \arrow[dr, "\rho"'] & & e \otimes A \arrow[dl, "\lambda"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
e \otimes A \arrow[rr, "\gamma"] \arrow[dr, "\lambda"'] & & A \otimes e \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

commute in any braided monoidal category $\mathbb{C}$*.*

Proof. The main idea is to fill the first commutative hexagon with five smaller commutative diagrams:

$$\begin{tikzcd}
& A \otimes (e \otimes C) \arrow[r, "\gamma"] \arrow[d, "A\otimes\lambda"] \arrow[dr, phantom, "(b)"] & (e \otimes C) \otimes A \arrow[d, "\lambda\otimes A"] \arrow[dr, bend left, "\alpha"] \arrow[dr, phantom, "(c)"] & \\
(A \otimes e) \otimes C \arrow[ur, bend left, "\alpha"] \arrow[ur, phantom, "(a)"] \arrow[r, "\rho\otimes C"] \arrow[dr, bend right, "\gamma\otimes C"'] \arrow[dr, phantom, "(\bullet)"] & A \otimes C \arrow[r, "\gamma"] & C \otimes A & e \otimes (C \otimes A) \arrow[l, "\lambda"'] \\
& (e \otimes A) \otimes C \arrow[u, "\lambda\otimes C"] \arrow[r, "\alpha"] \arrow[ur, phantom, "(e)"] & e \otimes (A \otimes C) \arrow[ul, "\lambda"] \arrow[ur, bend right, "e\otimes\gamma"'] \arrow[ur, phantom, "(d)"] &
\end{tikzcd}$$

In clockwise order, these diagrams commute (a) by the triangle axiom of monoidal categories, (b) by naturality of γ, (c) by Proposition 1, (d) by naturality of λ, (e) by

Proposition 1. From this and the fact that γ is an isomorphism, follows that diagram ($\bullet$) commutes.

Now, one instantiates diagram ($\bullet$) with $C = e$. Just as in the proofs of Proposition 1 and 2, one takes advantage of the fact that the functor $-\otimes e : \mathbb{C} \longrightarrow \mathbb{C}$ is full and faithful, to deduce that the first triangle of the proposition commutes. The second triangle of the proposition is shown to commute in a similar way. □

4.3 Symmetric monoidal categories

A symmetric monoidal category $\mathbb{C}$ is a braided monoidal category whose braiding is a symmetry. A symmetry is a braiding satisfying $\gamma_{B,A} = \gamma^{-1}_{A,B}$ for all objects A, B of the category. Note that, in that case, the second hexagonal diagram may be dropped in the definition of braiding, since this diagram commutes for $\gamma_{A,B}$ iff the first hexagonal diagram commutes for $\gamma_{B,A} = \gamma^{-1}_{A,B}$.

4.4 Monoidal closed categories

A *left closed structure* in a monoidal category $(\mathbb{C}, \otimes, e)$ is the data of

- an object $A \multimap B$,
- a morphism $eval_{A,B} : A \otimes (A \multimap B) \longrightarrow B$,

for every two objects A and B of the category $\mathbb{C}$. The morphism $eval_{A,B}$ is called the *left evaluation* morphism. It must satisfy the following universal property. For every morphism

$$f : A \otimes X \longrightarrow B$$

there exists a unique morphism

$$h : X \longrightarrow A \multimap B$$

making the diagram

$$\begin{tikzcd}
A \otimes X \arrow[d, "A\otimes h"'] \arrow[dr, "f"] & \\
A \otimes (A \multimap B) \arrow[r, "eval_{A,B}"'] & B
\end{tikzcd} \tag{28}$$

commute.

A monoidal closed category $\mathbb{C}$ is a monoidal category equipped with a left closed structure. There are several alternative definitions of a closed structure, which we review here.

It follows from the universality property (28) that every object A of the category $\mathbb{C}$ defines an endofunctor

$$B \mapsto (A \multimap B) \tag{29}$$

of the category $\mathbb{C}$. Besides, for every object A, this functor is right adjoint to the functor

$$B \mapsto (A \otimes B). \tag{30}$$

This means that there exists a bijection between the sets of morphisms

$$\mathbb{C}(A \otimes B, C) \;\cong\; \mathbb{C}(B, A \multimap C) \tag{31}$$

natural in B and C. This provides an alternative definition of a left closed structure: a right adjoint to the functor (30), for every object A. The reader interested in the notion of adjunction will find a comprehensive study of the notion in Chapter 5.

The parameter theorem (see Theorem 3 in Chapter IV, Section 7 of MacLane's book [31]) then enables us to structure the family of functors (29) indexed by objects of A, as a bifunctor

$$(A, B) \mapsto A \multimap B \;:\; \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbb{C}. \tag{32}$$

contravariant in its first argument, covariant in its second argument. This bifunctor is defined as the unique bifunctor making the bijection (31) natural in A, B and C. This provides yet another alternative definition of left closed structure: a bifunctor (32) and a bijection (31) natural in A, B and C.

Exercise. Show that in a monoidal closed category $\mathbb{C}$ with monoidal unit e, every object A is isomorphic to the object $e \multimap A$. Show moreover that the isomorphism between A and $e \multimap A$ is natural in A. ■

4.5 Monoidal biclosed categories

A monoidal biclosed category is a monoidal category equipped with a left closed structure as well as a right closed structure. By definition, a *right closed structure* in a monoidal category $(\mathbb{C}, \otimes, e)$ is the data of

- an object $A \multimapinv B$,
- a morphism $evar_{A,B} : (B \multimapinv A) \otimes A \longrightarrow B$,

for every two objects A and B of the category $\mathbb{C}$. The morphism $evar_{A,B}$ is called the *right evaluation* morphism. It must satisfy a similar universal property as the left evaluation morphism in Section 4.4, that for every morphism

$$f : X \otimes A \longrightarrow B$$

there exists a unique morphism

$$h : X \longrightarrow B \multimapinv A$$

making the diagram below commute:

$$\begin{tikzcd}
X \otimes A \arrow[d, "h \otimes A"'] \arrow[dr, "f"] & \\
(B \multimapinv A) \otimes A \arrow[r, "evar_{A,B}"'] & B
\end{tikzcd} \tag{33}$$

As for the left closed structure in Section 4.4, this is equivalent to the property that the endofunctor

$$B \mapsto (B \otimes A)$$

has a right adjoint

$$B \mapsto (B \multimapinv A)$$

for every object A of the category. The parameter theorem ensures then that this family of functors indexed by the object A defines a bifunctor

$$\multimapinv \quad : \quad \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbb{C}$$

and a family of bijections

$$\mathbb{C}(B \otimes A, C) \;\cong\; \mathbb{C}(B, C \multimapinv A) \tag{34}$$

natural in the objects A, B and C.

4.6 Symmetric monoidal closed categories

A symmetric monoidal closed category $\mathbb{C}$ is a monoidal category equipped with a symmetry and a left closed structure. It is not difficult to show that any symmetric monoidal closed category is also equipped with a right closed structure, defined as follows:

- the object $B \multimapinv A$ is defined as the object $A \multimap B$,
- the right evaluation morphism $evar_{A,B}$ is defined as

$$(A \multimap B) \otimes A \xrightarrow{\gamma_{A \multimap B, A}} A \otimes (A \multimap B) \xrightarrow{eval_{A,B}} B$$

Symmetric monoidal closed categories provide the necessary structure to interpret the formulas and proofs of the *multiplicative* and *intuitionistic* fragment of linear logic. The symmetry interprets *exchange*, the operation of permuting formulas in a sequent, while the tensor product and closed structure interpret the multiplicative conjunction and implication of the logic, respectively.

This logical perspective on categories with structure is often enlightening, both for logic and for categories. By way of illustration, there is a famous principle in intuitionistic logic that every formula A implies its double negation $\neg\neg A$. This principle holds also in intuitionistic *linear* logic. In that case, the negation of a formula A is given by the formula $A \multimap \perp$, where $\perp$ stands for the multiplicative formula *False* — or in fact, when there exists no such formula *False* available, for any formula of the logic. So, there is a proof π in intuitionistic linear logic that every formula A implies its double negation $(A \multimap \perp) \multimap \perp$.

Exactly the same phenomenon happens in any symmetric monoidal closed category, and in fact in any monoidal *biclosed* category $\mathbb{C}$. Like in linear logic, any object of the category can play the role of $\perp$ — understood intuitively as the formula *False*. One shows that there exists a morphism

$$\partial_A \quad : \quad A \;\longrightarrow\; \perp \multimapinv (A \multimap \perp)$$

for every object A of the monoidal biclosed category $\mathbb{C}$, and that this morphism is natural in A. This does not come by chance: when the category $\mathbb{C}$ is symmetric monoidal closed, the two objects $\perp \multimapinv (A \multimap \perp)$ and $(A \multimap \perp) \multimap \perp$ coincide, and the map

$$\partial_A \quad : \quad A \;\longrightarrow\; (A \multimap \perp) \multimap \perp$$

is precisely the interpretation of the proof π that every formula A implies its double negation $(A \multimap \perp) \multimap \perp$ in intuitionistic linear logic.

The morphism ∂_A is constructed by a series of manipulations on the identity morphism:

$$id_{A \multimap \perp} \;:\; (A \multimap \perp) \;\longrightarrow\; (A \multimap \perp).$$

First, one applies the bijection (31) associated to the left closed structure, from right to left, in order to obtain the morphism:

$$A \otimes (A \multimap \perp) \;\longrightarrow\; \perp \tag{35}$$

Then, one applies the bijection (34) associated to the right closed structure, from left to right, in order to obtain the morphism:

$$\partial_A \;:\; A \;\longrightarrow\; \perp \multimapinv (A \multimap \perp).$$

When the category $\mathbb{C}$ is symmetric monoidal closed, the morphism ∂_A is alternatively constructed by precomposing the morphism (35) with the symmetry

$$\gamma_{A, A \multimap \perp} \;:\; (A \multimap \perp) \otimes A \;\longrightarrow\; A \otimes (A \multimap \perp)$$

so as to obtain the morphism

$$(A \multimap \perp) \otimes A \;\longrightarrow\; \perp.$$

then the bijection (31) from left to right:

$$\partial_A \;:\; A \;\longrightarrow\; (A \multimap \perp) \multimap \perp.$$

Exercise. Show that the morphism ∂_A is natural in A. ■

4.7 $*$-autonomous categories

A $*$-autonomous category is a symmetric monoidal closed category equipped with a dualizing object. A dualizing object $\perp$ is an object of the category $\mathbb{C}$ making the natural morphism constructed in Section 4.6:

$$\partial_A \;:\; A \longrightarrow (A \multimap \perp) \multimap \perp$$

an isomorphism, for every object A of the category $\mathbb{C}$.

The notion of dualizing object may be given a logical flavour. There is a governing principle in classical logic that the disjunction of a formula A and of its negation $\neg A$ is necessarily true. This principle called *Tertium non Datur* is supported by the idea that a formula is either true or false. This principle may be formulated in another way: every formula A is equivalent to its double negation $\neg\neg A$. This principle does not hold in intuitionistic logic: a formula A implies its double negation $\neg\neg A$, but the converse is not necessarily true. Indeed, the existence of a dualizing object $\bot$ in a symmetric monoidal closed category enables an interpretation of *classical* multiplicative linear logic and its involutive negation, instead of just *intuitionistic* multiplicative linear logic.

Exercise. Show that the object $\bot \multimap \bot$ is isomorphic to the unit object e in any $*$-autonomous category. ■

4.8 Linearly distributive categories

A linearly distributive category $\mathbb{C}$ is a monoidal category twice: once for the bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ with unit e and natural isomorphisms

$$\alpha^{\otimes}_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C),$$

$$\lambda^{\otimes}_{A} : e \otimes A \longrightarrow A, \qquad \rho^{\otimes}_{A} : A \otimes e \longrightarrow A,$$

and again for the bifunctor $\bullet : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ with unit u and natural isomorphisms

$$\alpha^{\bullet}_{A,B,C} : (A \bullet B) \bullet C \longrightarrow A \bullet (B \bullet C),$$

$$\lambda^{\bullet}_{A} : u \bullet A \longrightarrow A, \qquad \rho^{\bullet}_{A} : A \bullet u \longrightarrow A.$$

In order to distinguish them, the operations $\otimes$ and $\bullet$ are called “tensor product” and “cotensor product” respectively. The tensor product is required to distribute over the cotensor product by natural morphims

$$\delta^{L}_{A,B,C} : A \otimes (B \bullet C) \longrightarrow (A \otimes B) \bullet C,$$

$$\delta^{R}_{A,B,C} : (A \bullet B) \otimes C \longrightarrow A \bullet (B \otimes C).$$

These structure maps must satisfy a series of commutativity axioms: six pentagons and four triangles, which we review below.

The pentagons relate the distributions δ^L and δ^R to the associativity laws, and to themselves. We were careful to draw these pentagons in a uniform way. This presentation emphasizes the fact that the distributions are (lax) *associativity laws* between the tensor and the cotensor products. Consequently, each of the pentagonal diagram below is a variant of the usual pentagonal diagram for monoidal categories. Note that there are exactly $2^3 = 8$ different ways to combine four objects A, B, C, D by a tensor and a cotensor product. The two extremal cases (only tensors, only cotensors) are treated by the requirement that the tensor and cotensor products define monoidal categories. Each of the six remaining cases is treated by one pentagon below.

$$\begin{tikzcd}
 & (A \bullet B) \otimes (C \otimes D) \arrow[dr, "\delta^{R}"] & \\
((A \bullet B) \otimes C) \otimes D \arrow[ur, "\alpha^{\otimes}"] \arrow[d, "\delta^{R} \otimes D"'] & & A \bullet (B \otimes (C \otimes D)) \\
(A \bullet (B \otimes C)) \otimes D \arrow[rr, "\delta^{R}"] & & A \bullet ((B \otimes C) \otimes D) \arrow[u, "A \bullet \alpha^{\otimes}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \bullet B) \bullet (C \otimes D) \arrow[dr, "\alpha^\bullet"] & \\
((A \bullet B) \bullet C) \otimes D \arrow[ur, "\delta^R"] \arrow[d, "\alpha^\bullet \otimes D"'] & & A \bullet (B \bullet (C \otimes D)) \\
(A \bullet (B \bullet C)) \otimes D \arrow[rr, "\delta^R"] & & A \bullet ((B \bullet C) \otimes D) \arrow[u, "A \bullet \delta^R"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \bullet D) \arrow[dl, "\delta^L"'] \arrow[dr, "\alpha^\otimes"] & \\
((A \otimes B) \otimes C) \bullet D \arrow[d, "\alpha^\otimes \bullet D"'] & & A \otimes (B \otimes (C \bullet D)) \arrow[d, "A \otimes \delta^L"] \\
(A \otimes (B \otimes C)) \bullet D & & A \otimes ((B \otimes C) \bullet D) \arrow[ll, "\delta^L"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \otimes B) \bullet (C \bullet D) & \\
((A \otimes B) \bullet C) \bullet D \arrow[ur, "\alpha^\bullet"] & & A \otimes (B \bullet (C \bullet D)) \arrow[ul, "\delta^L"'] \\
(A \otimes (B \bullet C)) \bullet D \arrow[u, "\delta^L \bullet D"] & & A \otimes ((B \bullet C) \bullet D) \arrow[ll, "\delta^L"'] \arrow[u, "A \otimes \alpha^\bullet"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \bullet B) \otimes (C \bullet D) \arrow[dl, "\delta^L"'] \arrow[dr, "\delta^R"] & \\
((A \bullet B) \otimes C) \bullet D \arrow[d, "\delta^R \bullet D"'] & & A \bullet (B \otimes (C \bullet D)) \arrow[d, "A \bullet \delta^L"] \\
(A \bullet (B \otimes C)) \bullet D \arrow[rr, "\alpha^\bullet"] & & A \bullet ((B \otimes C) \bullet D)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \otimes B) \bullet (C \otimes D) & \\
((A \otimes B) \bullet C) \otimes D \arrow[ur, "\delta^R"] & & A \otimes (B \bullet (C \otimes D)) \arrow[ul, "\delta^L"'] \\
(A \otimes (B \bullet C)) \otimes D \arrow[u, "\delta^L \otimes D"] \arrow[rr, "\alpha^\otimes"] & & A \otimes ((B \bullet C) \otimes D) \arrow[u, "A \otimes \delta^R"']
\end{tikzcd}$$

The triangles relate the distributions to the units. Again, each triangle is a variant of the familiar diagram in monoidal categories, analyzed in Proposition 1.

$$\begin{tikzcd}
e \otimes (A \bullet B) \arrow[rr, "\delta^L"] \arrow[d, "\lambda^\otimes"'] & & (e \otimes A) \bullet B \arrow[d, "\lambda^\otimes \bullet B"] \\
A \bullet B & = & A \bullet B
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \bullet B) \otimes e \arrow[rr, "\delta^R"] \arrow[d, "\rho^\otimes"'] & & A \bullet (B \otimes e) \arrow[d, "A \bullet \rho^\otimes"] \\
A \bullet B & = & A \bullet B
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \bullet u) \arrow[rr, "\delta^L"] \arrow[d, "A \otimes \rho^\bullet"'] & & (A \otimes B) \bullet u \arrow[d, "\rho^\bullet"] \\
A \otimes B & = & A \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
(u \bullet A) \otimes B \arrow[rr, "\delta^R"] \arrow[d, "\lambda^\bullet \otimes B"'] & & u \bullet (A \otimes B) \arrow[d, "\lambda^\bullet"] \\
A \otimes B & = & A \otimes B
\end{tikzcd}$$

Exercise. Show that every monoidal category defines a linearly distributive category in which the tensor and cotensor products coincide. ■

4.9 Duality in linearly distributive categories

Let $\mathbb{C}$ be a linearly distributive category, formulated with the same notations as in Section 4.8. A right duality in $\mathbb{C}$ is the data of:

- an object A^*,
- two morphisms $ax^R_A : e \longrightarrow A^* \bullet A$ and $cut^R_A : A \otimes A^* \longrightarrow u$

for every object A of the category $\mathbb{C}$. The morphisms are required to make the diagrams

$$\begin{tikzcd}
A \otimes e \arrow[r, "A\otimes ax^R"] \arrow[dd, "\rho^{\otimes}"'] & A \otimes (A^* \bullet A) \arrow[d, "\delta^L"] \\
& (A \otimes A^*) \bullet A \arrow[d, "cut^R \bullet A"] \\
A & u \bullet A \arrow[l, "\lambda^{\bullet}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
e \otimes A^* \arrow[r, "ax^R \otimes A^*"] \arrow[dd, "\lambda^{\otimes}"'] & (A^* \bullet A) \otimes A^* \arrow[d, "\delta^R"] \\
& A^* \bullet (A \otimes A^*) \arrow[d, "A^* \bullet cut^R"] \\
A^* & A^* \bullet u \arrow[l, "\rho^{\bullet}"']
\end{tikzcd}$$

commute. To every morphism $f : A \longrightarrow B$ in the category $\mathbb{C}$, one associates the morphism $f^* : B^* \longrightarrow A^*$ constructed in the following way:

$$\begin{tikzcd}
B^* \arrow[d, "(\lambda^{\otimes})^{-1}"'] & A^* \bullet (A \otimes B^*) \arrow[r, "A^* \bullet (f \otimes B^*)"] & A^* \bullet (B \otimes B^*) \arrow[r, "A^* \bullet cut^R"] & A^* \bullet u \arrow[d, "\rho^{\bullet}"] \\
e \otimes B^* \arrow[r, "ax^R \otimes B^*"] & (A^* \bullet A) \otimes B^* \arrow[u, "\delta^R"] \arrow[r, "(A^* \bullet f) \otimes B^*"] & (A^* \bullet B) \otimes B^* \arrow[u, "\delta^R"'] & A^*
\end{tikzcd}$$

The coherence diagrams ensure that this operation on morphisms defines a contravariant functor

$$(A \mapsto A^*) \ : \ \mathbb{C}^{op} \longrightarrow \mathbb{C}.$$

Moreover, one shows that

Proposition 4 *In any linearly distributive category $\mathbb{C}$ with a right duality,*

- *the functor $(A \otimes -)$ is left adjoint to the functor $(A^* \bullet -)$,*
- *the functor $(- \bullet B)$ is right adjoint to the functor $(- \otimes B^*)$,*

for all objects A, B of the category. In particular, any such category is monoidal closed.

There is also a notion of left duality in a linearly distributive category $\mathbb{C}$, which is given by the data of:

- an object ${}^*A$,
- two morphisms $ax^L_A : e \longrightarrow A \bullet {}^*A$ and $cut^L_A : {}^*A \otimes A \longrightarrow u$

for every object A of the category $\mathbb{C}$. Just as in the case of a right duality, the morphisms are required to make the coherence diagrams

$$\begin{tikzcd}
e \otimes A \arrow[r, "ax^L \otimes A"] \arrow[dd, "\lambda^{\otimes}"'] & (A \bullet {}^*A) \otimes A \arrow[d, "\delta^R"] \\
 & A \bullet ({}^*A \otimes A) \arrow[d, "A \bullet cut^L"] \\
A & A \bullet u \arrow[l, "\rho^{\bullet}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{}^*A \otimes e \arrow[r, "{}^*A \otimes ax^L"] \arrow[dd, "\rho^{\otimes}"'] & {}^*A \otimes (A \bullet {}^*A) \arrow[d, "\delta^L"] \\
 & ({}^*A \otimes A) \bullet {}^*A \arrow[d, "cut^L \bullet {}^*A"] \\
{}^*A & u \bullet {}^*A \arrow[l, "\lambda^{\bullet}"']
\end{tikzcd}$$

commute.

Proposition 5 *In any linearly distributive category $\mathbb{C}$ with a left duality,*

- *the functor $(- \otimes B)$ is left adjoint to the functor $(- \bullet {}^*B)$,*
- *the functor $(A \bullet -)$ is right adjoint to the functor $({}^*A \otimes -)$,*

for all objects A, B of the category.

Exercise. Show that there is a natural isomorphism between A, ${}^*(A^*)$ and $({}^*A)^*$ in any linearly distributive category with a left and right duality. Hint: show that the bijections

$$\mathbb{C}(A, B) \;\cong\; \mathbb{C}(e, A^* \bullet B) \;\cong\; \mathbb{C}({}^*(A^*), B)$$

are natural in A and B. Deduce that there exists a natural isomorphism between A and ${}^*(A^*)$. Proceed similarly to establish the existence of a natural isomorphism between A and $({}^*A)^*$. ■

Exercise. Suppose that $\mathbb{C}$ is a linearly distributive category with a right duality. Deduce from the previous exercise, and some diagrammatic inspection, that there exists at most one left duality in the category $\mathbb{C}$, up to the expected notion of isomorphism between left dualities. ■

4.10 Symmetric linearly distributive categories

A symmetric linearly distributive category $\mathbb{C}$ is a linearly distributive category in which the two monoidal structures are symmetric, with symmetries given by natural isomorphisms:

$$\gamma^{\otimes}_{A,B} : A \otimes B \longrightarrow B \otimes A, \qquad \gamma^{\bullet}_{A,B} : A \bullet B \longrightarrow B \bullet A.$$

The symmetries and the distributions must make the diagram

$$\begin{tikzcd}
A \otimes (B \bullet C) \arrow[r, "A \otimes \gamma^{\bullet}"] \arrow[d, "\delta^L"'] & A \otimes (C \bullet B) \arrow[r, "\gamma^{\otimes}"] & (C \bullet B) \otimes A \arrow[d, "\delta^R"] \\
(A \otimes B) \bullet C \arrow[r, "\gamma^{\bullet}"] & C \bullet (A \otimes B) \arrow[r, "C \bullet \gamma^{\otimes}"] & C \bullet (B \otimes A)
\end{tikzcd}$$

commute, for all objects A, B, C.

Exercise. Show that every symmetric monoidal category defines a symmetric linearly distributive category in which the tensor and cotensor products coincide. ■

4.11 $*$-autonomous categories as linearly distributive categories

In a symmetric linearly distributive category, any right duality ($A \mapsto A^*$) induces a left duality ($A \mapsto {}^*A$) given by ${}^*A = A^*$ and the structure morphisms:

$$ax^L_A = \gamma^{\bullet}_{A^*,A} \circ ax^R_A, \qquad\qquad cut^L_A = cut^R_A \circ \gamma^{\otimes}_{A^*,A}.$$

We have seen in Section 4.9 (last exercise) that this defines the unique left duality in the category $\mathbb{C}$, up to the expected notion of isomorphism between left duality. In fact, Cockett and Seely prove that this provides another formulation

Proposition 6 (Cockett-Seely) *The three notions below coincide:*

- *$*$-autonomous categories,*
- *symmetric linearly distributive categories with a right duality,*
- *symmetric linearly distributive categories with a left duality.*

4.12 Notes and references

The notion of linearly distributive is introduced by Robin Cockett and Robert Seely in [15]. A coherence theorem for linearly distributive categories has been established by these two authors, in collaboration with Rick Blute and David Trimble [11]. The construction of the free linearly distributive category over a given category $\mathbb{C}$ (or more generally, a polygraph) is described in full details. The approach is based on the proof-net notation introduced by Jean-Yves Girard in linear logic [21]. The main difficulty is to describe properly the equality of proof-nets induced by the free linearly distributive category. An interesting conservativity result is established there: the canonical functor from a linearly distributive category to the free $*$-autonomous category over it, is a full and faithful embedding.

5 Adjunctions between monoidal categories

In this chapter and in the last one, we discuss one of the earliest and most debated questions of linear logic: what is a categorical model of linear logic? This topic is surprisingly subtle and interesting. A few months only after the introduction of linear logic, there was already a general agreement among specialists

- that the category of denotations $\mathbb{L}$ should be symmetric monoidal closed in order to interpret intuitionistic linear logic,
- that the category $\mathbb{L}$ should be $*$-autonomous in order to interpret classical linear logic,
- that the category $\mathbb{L}$ should be cartesian in order to interpret the additive connective &, and cocartesian in order to interpret the additive connective $\oplus$.

But difficulties (and possible disagreements) arose when people started to axiomatize the categorical properties of the exponential modality "!″. These categorical properties should ensure that the category $\mathbb{L}$ defines a modular invariant of proofs for the whole of linear logic. Several alternative definitions were formulated, each one adapted to a particular situation or philosophy: Seely categories, Lafont categories, Linear categories, etc.

Today, nearly twenty years after the formulation of linear logic, it seems that a consensus has finally emerged between these various definitions — around the notion of *symmetric monoidal adjunction.* It appears indeed that each of the axiomatizations of the exponential modality ! implements a particular recipe to produce a symmetric monoidal adjunction between the category of denotations $\mathbb{L}$ and a specific cartesian category $\mathbb{M}$, as depicted below.

$$\begin{tikzcd}
(\mathbb{M}, \times, e) \arrow[r, bend left, "(L{,}m)"] \arrow[r, phantom, "\perp"] & (\mathbb{L}, \otimes, 1) \arrow[l, bend left, "(M{,}n)"]
\end{tikzcd}$$

Our presentation in Chapter 7 of the categorical models of linear logic is thus regulated by the theory of monoidal categories, and more specifically, by the notion of symmetric monoidal adjunction. For that reason, we devote the present chapter to the elementary theory of monoidal categories and monoidal adjunctions, with an emphasis on the 2-categorical aspects of the theory:

- Sections 5.1— 5.6: we recall the notions of lax and colax monoidal functor, including the symmetric case, and the notion of monoidal natural transformation between such functors,
- Section 5.7— 5.8: after recalling the definition of a 2-category, we construct the 2-category **LaxMonCat** with monoidal categories as objects, lax monoidal functors as horizontal morphisms, and monoidal natural transformations as vertical morphisms,

- Sections 5.9— 5.13: the 2-categorical definition of adjunction is formulated in three different ways, and applied to the 2-category **LaxMonCat** in order to define the notion of monoidal adjunction,
- Section 5.14— 5.15: the notion of monoidal adjunction is characterized as an adjunction $\mathcal{F}_* \dashv \mathcal{F}^*$ between monoidal categories, in which the left adjoint functor $(\mathcal{F}_*, m)$ is strong monoidal.
- Section 5.16: in this last section, we explicate the notion of *symmetric* monoidal adjunction, and characterize it as a monoidal adjunction in which the left adjoint functor $(\mathcal{F}_*, m)$ is strong and symmetric.

The various categorical axiomatizations of linear logic: Lafont categories, Seely categories, Linear categories, and their relationship to monoidal adjunctions, are discussed thoroughly in the final Chapter.

5.1 Lax monoidal functors

A *lax monoidal functor* $(\mathcal{F}, m)$ between monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ is a functor $\mathcal{F} : \mathbb{C} \longrightarrow \mathbb{D}$ equipped with natural transformations

$$m^2_{A,B} : \mathcal{F}A \bullet \mathcal{F}B \longrightarrow \mathcal{F}(A \otimes B), \qquad m^0 : u \longrightarrow \mathcal{F}e,$$

making the three diagrams

$$\begin{tikzcd}
(\mathcal{F}A \bullet \mathcal{F}B) \bullet \mathcal{F}C \arrow[r, "\alpha^{\bullet}"] \arrow[d, "m \bullet \mathcal{F}C"'] & \mathcal{F}A \bullet (\mathcal{F}B \bullet \mathcal{F}C) \arrow[d, "\mathcal{F}A \bullet m"] \\
\mathcal{F}(A \otimes B) \bullet \mathcal{F}C \arrow[d, "m"'] & \mathcal{F}A \bullet \mathcal{F}(B \otimes C) \arrow[d, "m"] \\
\mathcal{F}((A \otimes B) \otimes C) \arrow[r, "\mathcal{F}\alpha^{\otimes}"] & \mathcal{F}(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}A \bullet u \arrow[r, "\rho^{\bullet}"] \arrow[d, "\mathcal{F}A \bullet m"'] & \mathcal{F}A & u \bullet \mathcal{F}B \arrow[r, "\lambda^{\bullet}"] \arrow[d, "m \bullet \mathcal{F}B"'] & \mathcal{F}B \\
\mathcal{F}A \bullet \mathcal{F}e \arrow[r, "m"] & \mathcal{F}(A \otimes e) \arrow[u, "\mathcal{F}\rho^{\otimes}"'] & \mathcal{F}e \otimes \mathcal{F}B \arrow[r, "m"] & \mathcal{F}(e \otimes B) \arrow[u, "\mathcal{F}\lambda^{\otimes}"']
\end{tikzcd}$$

commute in the category $\mathbb{D}$, for all objects A, B, C of the category $\mathbb{C}$.

A *strong monoidal functor* is defined as a lax monoidal functor whose mediating maps m^2 and m^0 are isomorphisms. A *strict monoidal functor* is a strong monoidal functor whose mediating maps are identities.

Remark. Here, we take the terminology advocated by Lack, which is based on the idea that lax monoidal functors are lax morphisms between algebras for a particular strict monad in the 2-category of categories: the monad which associates to a category its free symmetric monoidal category. But at the same time, we are careful to call *strong monoidal* functor what Lack would simply call monoidal functor.

Remark. We will encounter in Section 6.2, Chapter 7, one of the original motivations for the definition of lax monoidal functor, discussed by Jean Bénabou in [5]. The category $\mathbb{1}$ with one object and its identity morphism defines a monoidal category in a unique way. It appears then that a lax monoidal functor from this monoidal category $\mathbb{1}$ to a monoidal category $\mathbb{C}$ is essentially the same thing as a monoid in the category $\mathbb{C}$. As we will see, this has the remarkable consequence that monoids are preserved by lax monoidal functors, in a very strong sense.

5.2 Colax monoidal functors

The definition of a lax monoidal functor is based on a particular orientation of the mediating maps: from the object $\mathcal{F}A \bullet \mathcal{F}B$ to the object $\mathcal{F}(A \otimes B)$, and from the object u to the object $\mathcal{F}e$. Reversing the orientation leads to another notion of "lax" monoidal functor, explicated now. A *colax monoidal functor* $(\mathcal{F}, n)$ between monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ consists of a functor $\mathcal{F} : \mathbb{C} \longrightarrow \mathbb{D}$ and natural transformations

$$n^2_{A,B} : \mathcal{F}(A \otimes B) \longrightarrow \mathcal{F}A \bullet \mathcal{F}B \qquad\qquad n^0 : \mathcal{F}e \longrightarrow u$$

making the three diagrams

$$\begin{tikzcd}
\mathcal{F}((A \otimes B) \otimes C) \arrow[r, "\mathcal{F}\alpha^{\otimes}"] \arrow[d, "n"'] & \mathcal{F}(A \otimes (B \otimes C)) \arrow[d, "n"] \\
\mathcal{F}(A \otimes B) \bullet \mathcal{F}C \arrow[d, "n \bullet \mathcal{F}C"'] & \mathcal{F}A \bullet \mathcal{F}(B \otimes C) \arrow[d, "\mathcal{F}A \bullet n"] \\
(\mathcal{F}A \bullet \mathcal{F}B) \bullet \mathcal{F}C \arrow[r, "\alpha^{\bullet}"] & \mathcal{F}A \bullet (\mathcal{F}B \bullet \mathcal{F}C)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(A \otimes e) \arrow[r, "\mathcal{F}\rho^{\otimes}"] \arrow[d, "n"'] & \mathcal{F}A \\
\mathcal{F}A \bullet \mathcal{F}e \arrow[r, "\mathcal{F}A \bullet n"] & \mathcal{F}A \bullet u \arrow[u, "\rho^{\bullet}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(e \otimes B) \arrow[r, "\mathcal{F}\lambda^{\otimes}"] \arrow[d, "n"'] & \mathcal{F}B \\
\mathcal{F}e \bullet \mathcal{F}B \arrow[r, "n \bullet \mathcal{F}B"] & u \bullet \mathcal{F}B \arrow[u, "\lambda^{\bullet}"']
\end{tikzcd}$$

commute in the category $\mathbb{D}$, for all objects A, B, C of the category $\mathbb{C}$.

The notion of colax monoidal functor is slightly less familiar than its lax counterpart. It may be justified by the following observation.

Exercise. Show that every functor $\mathcal{F} : \mathbb{C} \longrightarrow \mathbb{D}$ between cartesian categories defines a colax monoidal functor $(\mathcal{F}, n)$ in a unique way. ■

The definition of colax monoidal functor leads to an alternative definition of strong monoidal functor, defined now as a *colax* monoidal functor whose mediating maps n^2 and n^0 are isomorphisms. We leave the reader to prove in the next exercise that this definition of strong monoidal functor is equivalent to the definition given in Section 5.1.

Exercise. Show that every colax monoidal functor $(\mathcal{F}, n)$ whose mediating morphisms n^2 and n^0 are isomorphisms, defines a lax monoidal functor $(\mathcal{F}, m)$ with mediating morphisms $m^2_{A,B}$ and m^0 the inverse of $n^2_{A,B}$ and n^0. ■

5.3 Natural transformations

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two functors between the same categories:

$$\mathbb{C} \longrightarrow \mathbb{D}.$$

We recall that a natural transformation

$$\theta : \mathcal{F} \Rightarrow \mathcal{G} : \mathbb{C} \longrightarrow \mathbb{D}$$

between the two functors $\mathcal{F}$ and $\mathcal{G}$ is a family $(\theta_A)_{A \in \mathbf{Ob}(\mathbb{C})}$ of morphisms of the category $\mathbb{D}$ indexed by the objects of the category $\mathbb{C}$, and making the diagram

$$\begin{tikzcd}
\mathcal{F}A \arrow[r, "\theta_A"] \arrow[d, "\mathcal{F}f"'] & \mathcal{G}A \arrow[d, "\mathcal{G}f"] \\
\mathcal{F}B \arrow[r, "\theta_B"] & \mathcal{G}B
\end{tikzcd}$$

commute in the category $\mathbb{D}$, for every morphism $f : A \longrightarrow B$ in the category $\mathbb{C}$.

5.4 Monoidal natural transformations (between lax functors)

We suppose here that $(\mathcal{F}, m)$ and $(\mathcal{G}, n)$ are lax monoidal functors between the same monoidal categories:

$$(\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u).$$

A *monoidal natural transformation*

$$\theta : (\mathcal{F}, m) \Rightarrow (\mathcal{G}, n) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

between the lax monoidal functors $(\mathcal{F}, m)$ and $(\mathcal{G}, n)$ is a natural transformation

$$\theta : \mathcal{F} \Rightarrow \mathcal{G} : \mathbb{C} \longrightarrow \mathbb{D}$$

between the underlying functors, making the two diagrams

$$\begin{tikzcd}
\mathcal{F}A \bullet \mathcal{F}B \arrow[r, "\theta_A \bullet \theta_B"] \arrow[d, "m"'] & \mathcal{G}A \bullet \mathcal{G}B \arrow[d, "n"] \\
\mathcal{F}(A \otimes B) \arrow[r, "\theta_{A \otimes B}"] & \mathcal{G}(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
u \arrow[r, phantom, "="] \arrow[d, "m"'] & u \arrow[d, "n"] \\
\mathcal{F}e \arrow[r, "\theta_e"] & \mathcal{G}e
\end{tikzcd}$$

commute, for all objects A and B of the category $\mathbb{C}$.

5.5 Monoidal natural transformations (between colax functors)

The definition of monoidal natural transformation formulated in Section 5.4 for lax monoidal functors is easily adapted to the colax situation. A *monoidal natural transformation*

$$\theta : (\mathcal{F}, m) \Rightarrow (\mathcal{G}, n) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

between two colax monoidal functors $(\mathcal{F}, m)$ and $(\mathcal{G}, n)$ is a natural transformation

$$\theta : \mathcal{F} \Rightarrow \mathcal{G} : \mathbb{C} \longrightarrow \mathbb{D}$$

between the underlying functors, making the two diagrams

$$\begin{tikzcd}
\mathcal{F}(A \otimes B) \arrow[r, "\theta_{A\otimes B}"] \arrow[d, "m"'] & \mathcal{G}(A \otimes B) \arrow[d, "n"] & & \mathcal{F}e \arrow[r, "\theta_e"] \arrow[d, "m"'] & \mathcal{G}e \arrow[d, "n"] \\
\mathcal{F}A \bullet \mathcal{F}B \arrow[r, "\theta_A \bullet \theta_B"] & \mathcal{G}A \bullet \mathcal{G}B & & u \arrow[r, phantom, "="] & u
\end{tikzcd}$$

commute, for all objects A and B of the category $\mathbb{C}$. We have seen in Section 5.2 that every functor $\mathcal{F}$ between cartesian categories is colax in a canonical way. We leave the reader to establish as an exercise that natural transformations between such functors are themselves monoidal.

Exercise. Suppose that $\theta : \mathcal{F} \Rightarrow \mathcal{G} : \mathbb{C} \longrightarrow \mathbb{D}$ is a natural transformation between two functors $\mathcal{F}$ and $\mathcal{G}$ acting on cartesian categories $\mathbb{C}$ and $\mathbb{D}$. Show that the natural transformation θ is monoidal between the functors $\mathcal{F}$ and $\mathcal{G}$ understood as colax monoidal functors. ■

5.6 Symmetric monoidal functors (lax and colax)

We suppose here that the two monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ are symmetric, with symmetries noted $\gamma^{\otimes}$ and $\gamma^{\bullet}$ respectively. A lax monoidal functor

$$(\mathcal{F}, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

is called *symmetric* when the diagram

$$\begin{tikzcd}
\mathcal{F}A \bullet \mathcal{F}B \arrow[r, "\gamma^{\bullet}"] \arrow[d, "m"'] & \mathcal{F}B \bullet \mathcal{F}A \arrow[d, "m"] \\
\mathcal{F}(A \otimes B) \arrow[r, "\mathcal{F}\gamma^{\otimes}"] & \mathcal{F}(B \otimes A)
\end{tikzcd}$$

commutes in the category $\mathbb{D}$ for all objects A, B of the category $\mathbb{C}$. Similary, a colax monoidal functor

$$(\mathcal{F}, n) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

is called *symmetric* when the diagram

$$\begin{tikzcd}
\mathcal{F}(A \otimes B) \arrow[r, "\mathcal{F}\gamma^{\otimes}"] \arrow[d, "n"'] & \mathcal{F}(B \otimes A) \arrow[d, "n"] \\
\mathcal{F}A \bullet \mathcal{F}B \arrow[r, "\gamma^{\bullet}"] & \mathcal{F}B \bullet \mathcal{F}A
\end{tikzcd}$$

commutes in the category $\mathbb{D}$ for all objects A, B of the category $\mathbb{C}$.

Exercise. We have seen in Section 5.2 that every functor $\mathcal{F}$ between cartesian categories lifts to a colax monoidal functor $(\mathcal{F}, n)$ in a unique way. Show that this colax monoidal functor is symmetric. ■

5.7 The language of 2-categories

In order to define the notion of *monoidal adjunction* between monoidal categories, we proceed in three stages:

- In this section, we recall the notion of 2-category,
- In Section 5.8, we construct the 2-category **LaxMonCat** with monoidal categories as objects, lax monoidal functors as horizontal morphisms, and monoidal natural transformations as vertical morphisms,
- In Section 5.10, we define what one means by an *adjunction* in a 2-category, and apply the definition to the 2-category **LaxMonCat** in order to define the notion of monoidal adjunction.

Basically, a 2-category $\mathcal{C}$ is a category in which the class $\mathcal{C}(A, B)$ of morphisms between two objects A and B is not a set, but a category. In other words, a 2-category is a category in which there exist morphisms $f : A \longrightarrow B$ between objects, and also morphisms $\alpha : f \Rightarrow g$ between morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow B$ with the same source and target. The underlying category is noted $\mathcal{C}_0$. The morphisms $f : A \longrightarrow B$ are called *horizontal morphisms*, and the morphisms $\alpha : f \Rightarrow g$ are called *vertical morphisms* or *cells*. They are generally represented as *2-dimensional arrows* between the *1-dimensional arrows* $f : A \longrightarrow B$ and $g : A \longrightarrow B$ of the underlying category $\mathcal{C}_0$:

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}$$

Cells may be composed "vertically" and "horizontally". We write

$$\beta * \alpha : f \Rightarrow h$$

for the *vertical composite* of two cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, which is represented diagrammatically as:

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\beta"] & B \arrow[r, phantom, "="] & A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & B
\end{tikzcd}$$

We write

$$\alpha_1 \circ \alpha_2 : f_2 \circ f_1 \Rightarrow g_2 \circ g_1$$

for the *horizontal composite* of two cells $\alpha_1 : f_1 \Rightarrow g_1$ and $\alpha_2 : f_2 \Rightarrow g_2$, represented diagrammatically as:

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f_1"] \arrow[r, bend right=50, "g_1"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & B \arrow[r, bend left=50, "f_2"] \arrow[r, bend right=50, "g_2"'] \arrow[r, phantom, "\Downarrow\alpha_2"] & C \arrow[r, phantom, "="] & A \arrow[r, bend left=50, "f_2\circ f_1"] \arrow[r, bend right=50, "g_2\circ g_1"'] \arrow[r, phantom, "\Downarrow\alpha_2\circ\alpha_1"] & C
\end{tikzcd}$$

The vertical and horizontal composition laws are required to define categories: they are associative and have identities:

- the vertical composition has an identity cell $1^f : f \Rightarrow f$ for every morphism f of the underlying category $\mathcal{C}_0$,
- the horizontal composition has an identity cell $1_A : id_A \Rightarrow id_A$ for every object A and associated identity morphism $id_A : A \longrightarrow A$ of the underlying category $\mathcal{C}_0$.

The interchange law asks that composing four cells

$$\alpha_1 : f_1 \Rightarrow g_1 \qquad \beta_1 : g_1 \Rightarrow h_1 \qquad \alpha_2 : f_2 \Rightarrow g_2 \qquad \beta_2 : g_2 \Rightarrow h_2$$

vertically then horizontally as

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) : f_2 \circ f_1 \Rightarrow h_2 \circ h_1$$

or horizontally then vertically as

$$(\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1) : f_2 \circ f_1 \Rightarrow h_2 \circ h_1$$

as in the diagram below

$$\begin{tikzcd}
A \arrow[r, bend left=60, "f_1"{name=F1}] \arrow[r, "g_1"{name=G1, description}] \arrow[r, bend right=60, "h_1"'{name=H1}] & B \arrow[r, bend left=60, "f_2"{name=F2}] \arrow[r, "g_2"{name=G2, description}] \arrow[r, bend right=60, "h_2"'{name=H2}] & C
\arrow[Rightarrow, from=F1, to=G1, "\alpha_1"]
\arrow[Rightarrow, from=G1, to=H1, "\beta_1"]
\arrow[Rightarrow, from=F2, to=G2, "\alpha_2"]
\arrow[Rightarrow, from=G2, to=H2, "\beta_2"]
\end{tikzcd}$$

makes no difference:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) \quad = \quad (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

Finally, two coherence axioms are required on the identities:

- $1^{f_2} \circ 1^{f_1} = 1^{f_2 \circ f_1}$ for every pair of morphisms $f_1 : A \longrightarrow B$ and $f_2 : B \longrightarrow C$ of the underlying category $\mathcal{C}_0$,
- the vertical identity 1^{id_A} associated to the identity morphism $id_A : A \longrightarrow A$ of the underlying category $\mathcal{C}_0$ coincides with the horizontal identity 1_A, for every object A.

Exercise. Show that every pair of morphisms $h_1 : A \longrightarrow B$ and $h_2 : C \longrightarrow D$ in a 2-category $\mathcal{C}$ defines a functor from the category $\mathcal{C}(B, C)$ to the category $\mathcal{C}(A, D)$ which transports every cell

$$\begin{tikzcd}
B \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] & C
\arrow[Rightarrow, from=F, to=G, "\alpha"]
\end{tikzcd}$$

to the cell

$$\begin{tikzcd}
A \arrow[r, bend left=50, "h_2 \circ f \circ h_1"{name=F}] \arrow[r, bend right=50, "h_2 \circ g \circ h_1"'{name=G}] & D
\arrow[Rightarrow, from=F, to=G, "1^{h_2} \circ \alpha \circ 1^{h_1}"]
\end{tikzcd}$$

■

5.8 The 2-category of monoidal categories and lax functors

We start by recalling a well-known property of category theory:

Proposition 7 *Categories, functors and natural transformations define a 2-category, noted **Cat**.*

Proof. The vertical composite $\theta * \zeta$ of two natural transformations

$$\zeta : \mathcal{F} \Rightarrow \mathcal{G} : \mathbb{C} \longrightarrow \mathbb{D} \quad \text{and} \quad \theta : \mathcal{G} \Rightarrow \mathcal{H} : \mathbb{C} \longrightarrow \mathbb{D}$$

is defined as the natural transformation

$$\theta * \zeta : \mathcal{F} \Rightarrow \mathcal{H} : \mathbb{C} \longrightarrow \mathbb{D}$$

with components

$$(\theta * \zeta)_A \qquad : \qquad \mathcal{F}A \xrightarrow{\zeta_A} \mathcal{G}A \xrightarrow{\theta_A} \mathcal{H}A.$$

The horizontal composite $\theta \circ \zeta$ of two natural transformations

$$\zeta : \mathcal{F}_1 \Rightarrow \mathcal{G}_1 : \mathbb{C} \longrightarrow \mathbb{D} \quad \text{and} \quad \theta : \mathcal{F}_2 \Rightarrow \mathcal{G}_2 : \mathbb{D} \longrightarrow \mathbb{E}$$

is defined as the natural transformation

$$\theta \circ \zeta : \mathcal{F}_2 \circ \mathcal{F}_1 \Rightarrow \mathcal{G}_2 \circ \mathcal{G}_1 : \mathbb{C} \longrightarrow \mathbb{E}$$

with components $(\theta \circ \zeta)_A$ the diagonal $\mathcal{F}_2\mathcal{F}_1A \longrightarrow \mathcal{G}_2\mathcal{G}_1A$ of the commutative square

$$\begin{tikzcd}
\mathcal{F}_2\mathcal{F}_1A \arrow[r, "\mathcal{F}_2\zeta_A"] \arrow[d, "\theta_{\mathcal{F}_1A}"'] & \mathcal{F}_2\mathcal{G}_1A \arrow[d, "\theta_{\mathcal{G}_1A}"] \\
\mathcal{G}_2\mathcal{F}_1A \arrow[r, "\mathcal{G}_2\zeta_A"] & \mathcal{G}_2\mathcal{G}_1A
\end{tikzcd}$$

We leave the reader to check as an exercise that the constructions just defined satisfy the axioms of a 2-category. □

The whole point of introducing the notion of 2-category in Section 5.7 is precisely that:

Proposition 8 *Monoidal categories, lax monoidal functors and monoidal natural transformations between lax monoidal functors define a 2-category, noted* ***LaxMonCat****.*

Proof. The composite of two lax monoidal functors

$$(\mathcal{F}, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u) \quad \text{and} \quad (\mathcal{G}, n) : (\mathbb{D}, \bullet, u) \longrightarrow (\mathbb{E}, \cdot, i)$$

is defined as the composite $\mathcal{G} \circ \mathcal{F}$ of the two underlying functors $\mathcal{F}$ and $\mathcal{G}$, equipped with the mediating maps:

$$\mathcal{G}\mathcal{F}A \cdot \mathcal{G}\mathcal{F}B \xrightarrow{n} \mathcal{G}(\mathcal{F}A \bullet \mathcal{F}B) \xrightarrow{\mathcal{G}m} \mathcal{G}\mathcal{F}(A \otimes B)$$

and

$$i \xrightarrow{n} \mathcal{G}u \xrightarrow{\mathcal{G}m} \mathcal{G}\mathcal{F}e.$$

The vertical and horizontal composition of monoidal natural transformations are defined just as in the 2-category **Cat**. We leave the reader to check that the vertical and horizontal composites of monoidal natural transformations define *monoidal* natural transformations, and from this, that the constructions satisfy the axioms of a 2-category. □

It is not difficult to establish in the same way that

Proposition 9 *Symmetric monoidal categories, symmetric lax monoidal functors and monoidal natural transformations between lax monoidal functors define a 2-category, noted* ***SymMonCat****.*

Proposition 10 *Symmetric monoidal categories, symmetric colax monoidal functors and monoidal natural transformations between colax monoidal functors define a 2-category, noted* ***SymColaxMonCat****.*

5.9 Adjunctions between functors

By definition, an adjunction is a triple $(\mathcal{F}_*, \mathcal{F}^*, \phi)$ consisting of two functors

$$\mathcal{F}_* : \mathbb{C} \longrightarrow \mathbb{D} \qquad\qquad \mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}$$

and a family of bijections

$$\phi_{A,B} \quad : \quad \mathbb{C}(A, \mathcal{F}^* B) \;\cong\; \mathbb{D}(\mathcal{F}_* A, B)$$

indexed by objects A of the category $\mathbb{C}$, and objects B of the category $\mathbb{D}$. The functor $\mathcal{F}_*$ is called *left adjoint* to the functor $\mathcal{F}^*$, and one writes

$$\mathcal{F}_* \dashv \mathcal{F}^*.$$

The family ϕ is required to be natural in A and B. This point is sometimes misunderstood, or simply forgotten. For that reason, we explain it briefly here. Suppose we have a morphism

$$h : A \longrightarrow \mathcal{F}^* B$$

in the category $\mathbb{C}$, and a pair of morphisms $h_A : A' \longrightarrow A$ in the category $\mathbb{C}$ and $h_B : B \longrightarrow B'$ in the category $\mathbb{D}$. The two morphisms h_A and h_B should be understood as *actions* on the morphism h, in the group-theoretic sense. Naturality means that the bijection ϕ preserves the actions by the morphisms of the categories $\mathbb{C}$ and $\mathbb{D}$ on the families of sets $\mathbb{C}(A, \mathcal{F}^* B)$ and $\mathbb{D}(\mathcal{F}_* A, B)$. More precisely, let the morphism

$$h' = \mathcal{F}^*(h_B) \circ h \circ h_A : A' \longrightarrow \mathcal{F}^* B'$$

denote the result of the action by h_A and h_B on the morphism h. The morphism h' is thus defined to make the diagram

$$\begin{tikzcd}
A \arrow[r, "h"] & \mathcal{F}^* B \arrow[d, "\mathcal{F}^* h_B"] \\
A' \arrow[u, "h_A"] \arrow[r, "h'"] & \mathcal{F}^* B'
\end{tikzcd}$$

commute in the category $\mathbb{C}$. Naturality in A and B means that the equality

$$\phi_{A',B'}(h') = h_B \circ \phi_{A,B}(h) \circ \mathcal{F}_*(h_A)$$

is satisfied, i.e. the diagram

$$\begin{tikzcd}
\mathcal{F}_*A \arrow[rr, "\phi_{A,B}(h)"] & & B \arrow[d, "h_B"] \\
\mathcal{F}_*A' \arrow[u, "\mathcal{F}_*h_A"] \arrow[rr, "\phi_{A',B'}(h')"] & & B'
\end{tikzcd}$$

commutes in the category $\mathbb{D}$.

5.10 Adjunctions in the language of 2-categories

The definition of adjunction between functors given in Section 5.9 may be reformulated using the language of 2-categories, in the following way. The translation is based on the observation:

- that an object A in the category $\mathbb{C}$ is the same thing as a functor $[A]$ from the category $\mathbb{1}$ (the category with one object equipped with its identity morphism) to the category $\mathbb{C}$,
- that a morphism $h : A \longrightarrow B$ in the category $\mathbb{C}$ is the same thing as a natural transformation $[h] : [A] \Rightarrow [B]$ between the functors representing the objects A and B,
- that the functor $[A] : \mathbb{1} \longrightarrow \mathbb{C}$ composed with the functor $\mathcal{F}_* : \mathbb{C} \longrightarrow \mathbb{D}$ coincides with the functor $[\mathcal{F}_*A] : \mathbb{1} \longrightarrow \mathbb{D}$:

  $$\mathcal{F}_* \circ [A] \quad = \quad [\mathcal{F}_*A]$$

  for every object A of the category $\mathbb{C}$. And similarly, that

  $$\mathcal{F}^* \circ [B] \quad = \quad [\mathcal{F}^*B]$$

  for every object B of the category $\mathbb{D}$.

Putting all this together, the adjunction $\phi_{A,B}$ becomes a bijection between the natural transformations

$$[A] \Rightarrow \mathcal{F}^* \circ [B] \quad : \quad \mathbb{1} \longrightarrow \mathbb{C}$$

and the natural transformations

$$\mathcal{F}_* \circ [A] \Rightarrow [B] \quad : \quad \mathbb{1} \longrightarrow \mathbb{C}.$$

Diagrammatically, the bijection $\phi_{A,B}$ defines a one-to-one relationship between the cells

$$\begin{tikzcd}
 & & \mathbb{C} \\
\mathbb{1} \arrow[urr, bend left, "{[A]}"] \arrow[drr, bend right, "{[B]}"'] & \Downarrow & \\
 & & \mathbb{D} \arrow[uu, "\mathcal{F}^*"']
\end{tikzcd}$$

and the cells

$$\begin{tikzcd} & \mathbb{C} \arrow[dd, "\mathcal{F}_*"] \\ \mathbb{1} \arrow[ur, bend left, "{[A]}"] \arrow[dr, bend right, "{[B]}"'] \arrow[r, phantom, "\Downarrow"] & \\ & \mathbb{D} \end{tikzcd}$$

in the 2-category **Cat**. Interestingly, it is possible to replace the category $\mathbb{1}$ by any category $\mathbb{E}$ in the bijection below. We leave the proof as exercise to the reader.

Exercise. Show that for every adjunction $(\mathcal{F}_*, \mathcal{F}^*, \phi)$ the family ϕ extends to a family (also noted ϕ) indexed by pairs of coinitial functors

$$A : \mathbb{E} \longrightarrow \mathbb{C} \qquad\qquad B : \mathbb{E} \longrightarrow \mathbb{D}$$

whose components $\phi_{A,B}$ define a bijection between the natural transformations

$$A \Rightarrow \mathcal{F}^* \circ B \quad : \quad \mathbb{E} \longrightarrow \mathbb{C}$$

and the natural transformations

$$\mathcal{F}_* \circ A \Rightarrow B \quad : \quad \mathbb{E} \longrightarrow \mathbb{C}.$$

Formulate accordingly the naturality condition on the extended family ϕ. ■

The discussion (and exercise) leads us to a very pleasant definition of adjunction in a 2-category. From now on, we suppose given a 2-category $\mathcal{C}$. An *adjunction* in the 2-category $\mathcal{C}$ is defined as a triple (f_*, f^*, ϕ) consisting of two morphisms

$$f_* : C \longrightarrow D \qquad\qquad f^* : D \longrightarrow C$$

and a family of bijections

$$\phi_{a,b} \quad : \quad \mathbb{C}(E, C)(a, f^* \circ b) \;\cong\; \mathbb{C}(E, D)(f_* \circ a, b)$$

indexed by pairs of coinitial morphisms

$$a : E \longrightarrow C \qquad\qquad b : E \longrightarrow D$$

in the 2-category $\mathcal{C}$. In that case, the morphism f_* is called *left adjoint* to the morphism f^* in the 2-category $\mathcal{C}$, and one writes

$$f_* \dashv f^*.$$

The family ϕ is required to be natural in a and b, in the following sense. Suppose that the bijection $\phi_{a,b}$ transports the cell θ to the cell $\zeta = \phi_{a,b}(\theta)$ — as depicted below.

$$\begin{tikzcd} & C & & & & C \arrow[dd, "f_*"] \\ E \arrow[ur, bend left, "a"] \arrow[dr, bend right, "b"'] \arrow[r, phantom, "\Downarrow\theta"] & & \arrow[r, mapsto, "\phi_{a,b}"] & {} & E \arrow[ur, bend left, "a"] \arrow[dr, bend right, "b"'] \arrow[r, phantom, "\Downarrow\zeta"] & \\ & D \arrow[uu, "f^*"'] & & & & D \end{tikzcd}$$

Suppose given a morphism $h : F \longrightarrow E$ and two cells

$$\alpha : a' \Rightarrow a \circ h : F \longrightarrow C \qquad\qquad \beta : b \circ h \Rightarrow b' : F \longrightarrow D$$

represented diagrammatically as:

$$\begin{tikzcd}
& \Downarrow\alpha & C \\
F \arrow[r, "h"] \arrow[urr, bend left=60, "a'"] \arrow[drr, bend right=60, "b'"'] & E \arrow[ur, bend left, "a"] \arrow[dr, bend right, "b"'] & \\
& \Downarrow\beta & D
\end{tikzcd}$$

Naturality in a and b means that the bijection $\phi_{a,b}$ preserves the actions of the cells α and β, in the following sense: the bijection $\phi_{a',b'}$ transports the cell θ' obtained by pasting together the three cells α, β, θ to the cell ζ' obtained by pasting together the three cells α, β, ζ — as depicted below.

$$\begin{tikzcd}
& \Downarrow\alpha & & C & & & \Downarrow\alpha & & C \arrow[dd, "f_*"] \\
F \arrow[r, dotted] \arrow[urrr, bend left=60, "a'"] \arrow[drrr, bend right=60, "b'"'] & E \arrow[urr, dotted, bend left, "a"] \arrow[drr, dotted, bend right, "b"'] & \Downarrow\theta & & \overset{\phi_{a',b'}}{\longmapsto} & F \arrow[r, dotted] \arrow[urrr, bend left=60, "a'"] \arrow[drrr, bend right=60, "b'"'] & E \arrow[urr, dotted, bend left, "a"] \arrow[drr, dotted, bend right, "b"'] & \Downarrow\zeta & \\
& \Downarrow\beta & & D \arrow[uu, "f^*"'] & & & \Downarrow\beta & & D
\end{tikzcd}$$

Exercise. Show that the definition of adjunction given in Section 5.9 coincides with the definition of adjunction in the 2-category **Cat**. Show moreover that the original formulation of naturality is limited to the instance in which $E = F$ is the category $\mathbb{1}$ with one object, and $h : F \longrightarrow E$ is the identity functor on that category. ■

5.11 Another formulation: the triangular identities

As just defined in Section 5.10, suppose given an adjunction (f_*, f^*, ϕ) in a 2-category $\mathcal{C}$. The two cells

$$\eta : id_C \Rightarrow f^* \circ f_* \qquad\qquad \epsilon : f_* \circ f^* \Rightarrow id_D$$

are defined respectively as the cells related to the vertical identity cells 1^{f_*} and 1^{f^*} by the bijections ϕ_{id_C,f_*} and ϕ_{f^*,id_D} — as depicted below.

$$\begin{tikzcd}
 & & C & & & & C \\
C \arrow[urr, bend left, "id_C"] \arrow[drr, bend right, "f_*"'] & \Downarrow\eta & & \overset{\phi_{id_C,f_*}}{\longmapsto} & C \arrow[urr, bend left, "id_C"] \arrow[drr, bend right, "f_*"'] & \Downarrow 1^{f_*} & \\
 & & D \arrow[uu, "f^*"'] & & & & D \arrow[from=uu, "f_*"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & C & & & & C \\
D \arrow[urr, bend left, "f^*"] \arrow[drr, bend right, "id_D"'] & \Downarrow 1^{f^*} & & \overset{\phi_{f^*,id_D}}{\longmapsto} & D \arrow[urr, bend left, "f^*"] \arrow[drr, bend right, "id_D"'] & \Downarrow\epsilon & \\
 & & D \arrow[uu, "f^*"'] & & & & D \arrow[from=uu, "f_*"]
\end{tikzcd}$$

This leads to a more concise (and equivalent) definition of adjunction in the 2-category $\mathcal{C}$. An adjunction is alternatively defined as a quadruple $(f_*, f^*, \eta, \epsilon)$ consisting of two morphisms:

$$f_* : C \longrightarrow D \qquad\qquad f^* : D \longrightarrow C$$

and two cells

$$\eta : id_C \Rightarrow f^* \circ f_* \qquad\qquad \epsilon : f_* \circ f^* \Rightarrow id_D$$

satisfying the so-called *triangular identities*:

$$(\epsilon \circ f_*) * (f_* \circ \eta) \quad = \quad 1^{f_*} \quad : \quad C \longrightarrow D$$

and

$$(f^* \circ \epsilon) * (\eta \circ f^*) \quad = \quad 1^{f^*} \quad : \quad D \longrightarrow C.$$

The morphisms $f^* \circ f_*$ and $f_* \circ f^*$ are called respectively the *monad* and the *comonad* of the adjunction. The cells η and ϵ are called respectively the *unit* of the monad $f^* \circ f_*$ and the *counit* of the comonad $f_* \circ f^*$.

Diagrammatically, the two triangular identities are represented as:

$$\begin{tikzcd}
 & \Downarrow\eta & & & & & & & & & \\
C \arrow[r, "f_*"] \arrow[rr, bend left=60, "id_C"] & D \arrow[r, dotted, "f^*"] \arrow[rr, bend right=60, "id_D"'] & C \arrow[r, "f_*"] & D & = & C \arrow[r, "f_*"] \arrow[rrr, bend left=60, "id_C"] & D \arrow[rrr, bend right=60, "id_D"'] & \Downarrow 1^{f_*} & C \arrow[r, "f_*"] & D \\
 & & \Downarrow\epsilon & & & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \Downarrow\eta & & & & & & & \\
D \arrow[r, "f^*"] \arrow[rr, bend right=60, "id_D"'] & C \arrow[r, dotted, "f_*"] \arrow[rr, bend left=60, "id_C"] & D \arrow[r, "f^*"] & C & = & D \arrow[r, "f^*"] \arrow[rrr, bend right=60, "id_D"'] & C \arrow[rrr, bend left=60, "id_C"] & \Downarrow 1^{f^*} & D \arrow[r, "f^*"] & C \\
 & \Downarrow\epsilon & & & & & & & &
\end{tikzcd}$$

We leave to the reader (exercise below) the proof that this formulation of adjunction coincides with the previous one.

Exercise. Show that the definition of adjunction based on triangular identities is equivalent to the definition of adjunction in a 2-category $\mathcal{C}$ formulated in Section 5.10. ■

5.12 A dual definition of adjunction

The definition of adjunction formulated in Section 5.11 is not only remarkable for its conciseness; it is also remarkable for its self-duality. Notice indeed that an adjunction $(f_*, f^*, \eta, \epsilon)$ in a 2-category $\mathcal{C}$ induces an adjunction

$$(f_*)^{op} \dashv (f^*)^{op}$$

between the morphisms

$$(f_*)^{op} : D \longrightarrow C \qquad (f^*)^{op} : C \longrightarrow D$$

in the 2-category $\mathcal{C}^{op}$ in which the direction of every morphism is reversed (but the direction of cells is maintained.)

From this, it follows mechanically that the original definition of adjunction formulated in Section 5.10 may be dualized! An adjunction in a 2-category $\mathbb{C}$ is thus alternatively defined as a triple (f_*, f^*, ψ) consisting of two morphisms

$$f_* : C \longrightarrow D \qquad f^* : D \longrightarrow C$$

and a family of bijections

$$\psi_{a,b} \quad : \quad \mathbb{C}(C,E)(a, b \circ f_*) \;\cong\; \mathbb{C}(D,E)(a \circ f^*, b)$$

indexed by pairs of cofinal morphisms

$$a : C \longrightarrow E, \qquad\qquad b : D \longrightarrow E$$

in the 2-category $\mathcal{C}$. The family ψ of bijections should be natural in a and b in a dualized sense of Section 5.10. Suppose that the bijection $\psi_{a,b}$ transports the cell θ to the cell $\zeta = \psi_{a,b}(\theta)$ — as depicted below.

$$\begin{tikzcd}
C \arrow[rrd, bend left, "a"] \arrow[dd, "f_*"'] & & & & & C \arrow[rrd, bend left, "a"] & & \\
& \Downarrow\theta & E & {} \arrow[r, mapsto, "\psi_{a,b}"] & {} & & \Downarrow\zeta & E \\
D \arrow[rru, bend right, "b"'] & & & & & D \arrow[rru, bend right, "b"'] \arrow[uu, "f^*"] & &
\end{tikzcd}$$

Suppose given a morphism $h : E \longrightarrow F$ and two cells

$$\alpha : a' \Rightarrow h \circ a : C \longrightarrow F \qquad\qquad \beta : h \circ b \Rightarrow b' : D \longrightarrow F$$

represented diagrammatically as:

$$\begin{tikzcd}
C \arrow[ddr, bend left, "a"] \arrow[ddrrr, bend left=50, "a'"] & & & \\
& & \Downarrow\alpha & \\
& E \arrow[rr, "h"] & & F \\
& & \Downarrow\beta & \\
D \arrow[uur, bend right, "b"'] \arrow[uurrr, bend right=50, "b'"'] & & &
\end{tikzcd}$$

Just as in Section 5.10, naturality in a and b means that the bijection $\psi_{a,b}$ preserves the actions of the cells α and β. Namely, the bijection $\psi_{a',b'}$ transports the cell θ' obtained by pasting together the three cells α, β, θ to the cell ζ' obtained by pasting together the three cells α, β, ζ — as depicted below.

$$\begin{tikzcd}
C \arrow[dddd, "f_*"'] \arrow[ddrr, bend left, dotted, "a"] \arrow[ddrrrr, bend left=50, "a'"] & & & & \\
& & & \Downarrow\alpha & \\
& \Downarrow\theta & E \arrow[rr, dotted] & & F \\
& & & \Downarrow\beta & \\
D \arrow[uurr, bend right, dotted, "b"'] \arrow[uurrrr, bend right=50, "b'"'] & & & &
\end{tikzcd}
\quad \xmapsto{\psi_{a',b'}} \quad
\begin{tikzcd}
C \arrow[ddrr, bend left, dotted, "a"] \arrow[ddrrrr, bend left=50, "a'"] & & & & \\
& & & \Downarrow\alpha & \\
& \Downarrow\zeta & E \arrow[rr, dotted] & & F \\
& & & \Downarrow\beta & \\
D \arrow[uuuu, "f^*"] \arrow[uurr, bend right, dotted, "b"'] \arrow[uurrrr, bend right=50, "b'"'] & & & &
\end{tikzcd}$$

It is thus possible to define an adjunction as a triple (f_*, f^*, ϕ) as in Section 5.10, or as a triple (f_*, f^*, ψ) as just done here. Remarkably, the two bijections ϕ and ψ are *compatible* in the following sense. Suppose given two cells

$$\theta_1 : f_* \circ a_1 \Rightarrow b_1 : E_1 \longrightarrow D \qquad \theta_2 : a_2 \circ f^* \Rightarrow b_2 : D \longrightarrow E_2$$

depicted as follows:

$$\begin{tikzcd}
& & C \\
E_1 \arrow[urr, bend left, "a_1"] \arrow[drr, bend right, "b_1"'] & \Downarrow\theta_1 & \\
& & D \arrow[uu, "f^*"']
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
C \arrow[dd, "f_*"'] \arrow[drr, bend left, "a_2"] & & \\
& \Downarrow\theta_2 & E_2 \\
D \arrow[urr, bend right, "b_2"'] & &
\end{tikzcd}$$

The equality

$$(\psi_{a_2,b_2}(\theta_2) \circ 1^{b_1}) * (1^{a_2} \circ \theta_1) \quad = \quad (1^{b_2} \circ \phi_{a_1,b_1}(\theta_1)) * (\theta_2 \circ 1^{a_1})$$

between cells $a_2 \circ a_1 \Rightarrow b_2 \circ b_1$ is then satisfied; diagrammatically speaking:

$$\begin{tikzcd}
& & C \arrow[rrd, bend left, "a_2"] & & & & & & C \arrow[rrd, bend left, "a_2"] \arrow[dd, dotted, "f_*"] & & \\
E_1 \arrow[urr, bend left, "a_1"] \arrow[drr, bend right, "b_1"'] & \Downarrow\theta_1 & & \Downarrow\psi(\theta_2) & E_2 & = & E_1 \arrow[urr, bend left, "a_1"] \arrow[drr, bend right, "b_1"'] & \Downarrow\phi(\theta_1) & & \Downarrow\theta_2 & E_2 \\
& & D \arrow[uu, dotted, "f^*"] \arrow[rru, bend right, "b_2"'] & & & & & & D \arrow[rru, bend right, "b_2"'] & &
\end{tikzcd}$$

Exercise. Deduce the triangular identities of Section 5.11 from the compatibility just mentioned between the bijections ϕ and ψ. ■

5.13 Monoidal adjunctions

Basically, the notion of *monoidal adjunction* is defined by applying one of the three equivalent definitions of adjunction of Sections 5.10, 5.11 and 5.12 to the 2-category **LaxMonCat** (defined in Section 5.8). However, starting from Section 5.10 and its definition based on triangular identities leads to a particularly simple definition, which we will use in Section 5.15. Suppose given a pair of lax monoidal functors:

$$(\mathcal{F}_*, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u) \qquad\qquad (\mathcal{F}^*, n) : (\mathbb{D}, \bullet, u) \longrightarrow (\mathbb{C}, \otimes, e).$$

Then, a monoidal adjunction between the lax monoidal functors

$$(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, n)$$

is simply an adjunction $(\mathcal{F}_*, \mathcal{F}^*, \eta, \epsilon)$ between the underlying functors

$$\mathcal{F}_* : \mathbb{C} \longrightarrow \mathbb{D} \qquad\qquad \mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}$$

whose natural transformations

$$\eta : id_C \Rightarrow \mathcal{F}^* \circ \mathcal{F}_* \qquad\qquad \epsilon : \mathcal{F}_* \circ \mathcal{F}^* \Rightarrow id_D$$

are *monoidal* in the sense of Section 5.4. We characterize the notion of monoidal adjunction later on in Section 5.15.

5.14 A duality between lax and colax monoidal functors

Suppose given a pair of monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ and a functor $\mathcal{F}_* : \mathbb{C} \longrightarrow \mathbb{D}$ left adjoint to a functor $\mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}$. Diagrammatically:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "\mathcal{F}_*"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "\mathcal{F}^*"]
\end{tikzcd}$$

In that situation,

Proposition 11 *Every lax monoidal structure* $(\mathcal{F}^*, p)$ *on the functor* $\mathcal{F}^*$ *induces a colax monoidal structure* $(\mathcal{F}_*, n)$ *on the functor* $\mathcal{F}_*$*, defined as follows:*

$$n^2_{A,B} : \quad \begin{tikzcd}
\mathcal{F}_*(A \otimes B) \arrow[d, "\mathcal{F}_*(\eta \otimes \eta)"'] & \mathcal{F}_*A \bullet \mathcal{F}_*B \\
\mathcal{F}_*(\mathcal{F}^*\mathcal{F}_*A \otimes \mathcal{F}^*\mathcal{F}_*B) \arrow[r, "\mathcal{F}_*p"] & \mathcal{F}_*\mathcal{F}^*(\mathcal{F}_*A \bullet \mathcal{F}_*B) \arrow[u, "\epsilon"']
\end{tikzcd}$$

$$n^0 : \quad \begin{tikzcd}
\mathcal{F}_*e \arrow[r, "\mathcal{F}_*p"] & \mathcal{F}_*\mathcal{F}^*u \arrow[r, "\epsilon"] & u.
\end{tikzcd}$$

Conversely, every colax monoidal structure $(\mathcal{F}_*, n)$ on the functor $\mathcal{F}_*$ induces a lax monoidal structure $(\mathcal{F}^*, p)$ on the functor $\mathcal{F}^*$, defined as follows:

$$p^2_{A,B} : \quad \begin{tikzcd}
\mathcal{F}^*A \otimes \mathcal{F}^*B \arrow[d, "\eta"'] & \mathcal{F}_*A \bullet \mathcal{F}_*B \\
\mathcal{F}^*\mathcal{F}_*(\mathcal{F}^*A \otimes \mathcal{F}^*B) \arrow[r, "\mathcal{F}^*n"] & \mathcal{F}^*(\mathcal{F}_*\mathcal{F}^*A \bullet \mathcal{F}_*\mathcal{F}^*B) \arrow[u, "\mathcal{F}^*(\epsilon \bullet \epsilon)"']
\end{tikzcd}$$

$$p^0 : \quad \begin{tikzcd}
e \arrow[r, "\eta"] & \mathcal{F}^*\mathcal{F}_*e \arrow[r, "\mathcal{F}^*n"] & \mathcal{F}^*u.
\end{tikzcd}$$

Moreover, the two functions $(p \mapsto n)$ and $(n \mapsto p)$ are inverse, and thus define a one-to-one relationship between the lax monoidal structures on the functor $\mathcal{F}^*$ and the colax monoidal structures on the functor $\mathcal{F}_*$.

Note that the colax monoidal structure n may be defined alternatively from the lax monoidal structure p as the unique family of morphisms making the diagrams

$$\begin{tikzcd}
A \otimes B \arrow[r, "\eta"] \arrow[d, "\eta \otimes \eta"'] & \mathcal{F}^*\mathcal{F}_*(A \otimes B) \arrow[d, "\mathcal{F}^*n"] \\
\mathcal{F}^*\mathcal{F}_*A \otimes \mathcal{F}^*\mathcal{F}_*B \arrow[r, "p"] & \mathcal{F}^*(\mathcal{F}_*A \bullet \mathcal{F}_*B)
\end{tikzcd}
\qquad
\begin{tikzcd}
e \arrow[r, "\eta"] \arrow[d, "p"'] & \mathcal{F}^*\mathcal{F}_*e \arrow[d, "\mathcal{F}^*n"] \\
\mathcal{F}^*u \arrow[r, phantom, "="] & \mathcal{F}^*u
\end{tikzcd}$$

commute for all objects A and B of the category $\mathbb{C}$. Conversely, the lax monoidal structure p may be defined from the colax monoidal structure n as the unique family of morphism making the diagrams

$$\begin{tikzcd}
\mathcal{F}_*(\mathcal{F}^*A \otimes \mathcal{F}^*B) \arrow[r, "n"] \arrow[d, "\mathcal{F}_*p"'] & \mathcal{F}_*\mathcal{F}^*A \bullet \mathcal{F}_*\mathcal{F}^*B \arrow[d, "\epsilon \otimes \epsilon"] \\
\mathcal{F}_*\mathcal{F}^*(A \bullet B) \arrow[r, "\epsilon"] & A \bullet B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_*e \arrow[r, phantom, "="] \arrow[d, "\mathcal{F}_*p"'] & \mathcal{F}_*e \arrow[d, "n"] \\
\mathcal{F}_*\mathcal{F}^*u \arrow[r, "\epsilon"] & u
\end{tikzcd}$$

commute for all objects A and B of the category $\mathbb{D}$.

5.15 A characterization of monoidal adjunctions

Here, we suppose given two monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ and a monoidal functor

$$(\mathcal{F}_*, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u).$$

We suppose moreover that, just as in Section 5.14, the functor $\mathcal{F}_*$ is left adjoint to a functor $\mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}$. We investigate now when the adjunction

$$\mathcal{F}_* \dashv \mathcal{F}^*$$

may be lifted to a monoidal adjunction

$$(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p). \qquad (36)$$

Obviously, this depends on the lax structure p chosen to equip the functor $\mathcal{F}^*$. By Proposition 11 (Section 5.14) every such lax structure p is associated in a one-to-one fashion to a colax structure n on the functor $\mathcal{F}_*$. Hence, the question becomes: when does a pair of lax and colax structures m and n on the functor $\mathcal{F}_*$ define a monoidal adjunction $(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p)$ by the bijection $n \mapsto p$? The answer to this question is remarkably simple. We leave the reader to establish as an exercise that:

Exercise. Show that

- the colax structure n is right inverse to the lax structure m iff the natural transformation η is monoidal from the identity functor on the category $\mathbb{C}$ to the lax monoidal functor $(\mathcal{F}^*, p) \circ (\mathcal{F}_*, m)$, and
- the colax structure n is left inverse to the lax structure m iff the natural transformation ϵ is monoidal from the lax monoidal functor $(\mathcal{F}_*, m) \circ (\mathcal{F}^*, p)$ to the identity functor on the category $\mathbb{D}$.

By the colax structure n is right inverse to the lax structure m, we mean that the morphisms

$$m^2_{A,B} \circ n^2_{A,B} : \qquad \begin{tikzcd} \mathcal{F}_*(A \otimes B) \arrow[r, "n"] & \mathcal{F}_*A \bullet \mathcal{F}_*B \arrow[r, "m"] & \mathcal{F}_*(A \otimes B) \end{tikzcd}$$

$$m^0 \circ n^0 : \qquad \begin{tikzcd} \mathcal{F}_*e \arrow[r, "n"] & u \arrow[r, "m"] & \mathcal{F}_*e \end{tikzcd}$$

coincide with the identity for every pair of objects A and B of the category $\mathbb{C}$. Similarly, by the colax structure n is left inverse to the lax structure m, we mean that the morphisms

$$n^2_{A,B} \circ m^2_{A,B} : \qquad \begin{tikzcd} \mathcal{F}_*A \bullet \mathcal{F}_*B \arrow[r, "m"] & \mathcal{F}_*(A \otimes B) \arrow[r, "n"] & \mathcal{F}_*A \bullet \mathcal{F}_*B \end{tikzcd}$$

$$n^0 \circ m^0 : \qquad \begin{tikzcd} u \arrow[r, "m"] & \mathcal{F}_*e \arrow[r, "n"] & u \end{tikzcd}$$

coincide with the identity for every pair of objects A and B of the category $\mathbb{C}$. ■

This leads to the following characterization of monoidal adjunctions, originally noticed by Max Kelly.

Proposition 12 *Suppose given two monoidal categories* $(\mathbb{C}, \otimes, e)$ *and* $(\mathbb{D}, \bullet, u)$ *and a lax monoidal functor*

$$(\mathcal{F}_*, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u).$$

Suppose that the functor $\mathcal{F}_*$ *is left adjoint to a functor*

$$\mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}.$$

Then, the adjunction

$$\mathcal{F}_* \dashv \mathcal{F}^*$$

lifts to a monoidal adjunction

$$(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p)$$

iff the lax monoidal functor $(\mathcal{F}_*, m)$ *is strong. In that case, the lax structure p is associated by the bijection of Proposition 11 to the colax structure* $n = m^{-1}$ *provided by the inverse of the lax structure m.*

In particular, the left adjoint functor $(\mathcal{F}_*, m)$ is strongly monoidal in every monoidal adjunction $(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p)$.

5.16 A characterization of symmetric monoidal adjunctions

A *symmetric monoidal adjunction* is defined in the same way as a monoidal adjunction, by applying the 2-categorical definition of adjunction to the 2-category **SymMonCat** formulated in Proposition 9 (Section 5.13). We explain briefly how the characterization of monoidal adjunctions formulated in Section 5.15 may be adapted to the symmetric case.

The 2-category **SymMonCat** has *symmetric* monoidal categories as objects, *symmetric* monoidal functors as horizontal morphisms, and monoidal natural transformations as vertical morphisms. So, a symmetric monoidal adjunction is simply a monoidal adjunction

$$(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p)$$

between two lax monoidal functors

$$(\mathcal{F}_*, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u) \qquad\qquad (\mathcal{F}^*, p) : (\mathbb{D}, \bullet, u) \longrightarrow (\mathbb{C}, \otimes, e)$$

in which:

- the two monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ are equipped with symmetries $\gamma^{\otimes}$ and $\gamma^{\bullet}$,
- the two lax monoidal functors $(\mathcal{F}_*, m)$ and $(\mathcal{F}^*, p)$ are symmetric in the sense of Section 5.6.

Symmetric monoidal adjunctions may be characterized in the same way as monoidal adjunctions, by observing that in Proposition 11, the lax monoidal functor $(\mathcal{F}^*, p)$ is symmetric iff the colax monoidal functor $(\mathcal{F}_*, n)$ is symmetric. This leads to the following variant of Proposition 12.

Proposition 13 *Suppose given two symmetric monoidal categories* $(\mathbb{C}, \otimes, e)$ *and* $(\mathbb{D}, \bullet, u)$ *and a symmetric lax monoidal functor*

$$(\mathcal{F}_*, m) : (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u).$$

Suppose that the functor $\mathcal{F}_*$ *is left adjoint to a functor*

$$\mathcal{F}^* : \mathbb{D} \longrightarrow \mathbb{C}.$$

Then, the adjunction

$$\mathcal{F}_* \dashv \mathcal{F}^*$$

lifts to a symmetric monoidal adjunction

$$(\mathcal{F}_*, m) \dashv (\mathcal{F}^*, p)$$

iff the lax monoidal functor $(\mathcal{F}_*, m)$ *is strong. In that case, the lax structure* p *is associated by the bijection of Proposition 11 to the colax structure* $n = m^{-1}$ *provided by the inverse of the lax structure* m.

5.17 Notes and references

The notion of adjunction was formulated for the first time in 1958 in an article by Daniel Kan [27]. The 2-categorical definition of adjunction was introduced by Ross Street in [35]. We do not introduce the notion of Kan extension in this chapter, although the trained reader will immediately recognize them hidden in our treatment of adjunctions performed in Section 5.10. The relationship between adjunctions and Kan extensions appears already in the original paper by Kan, as well as in Chapter 10 of MacLane's book [31].

6 Monoids and monads

In this chapter, we recall the definitions and main properties of monoids and monads. Once dualized and transformed as comonoids and comonads, the two notions play a central role in the definition of the various categorical models of linear logic given in Chapter 7 of the survey.

6.1 Monoids

A *monoid* in a monoidal category $(\mathbb{C}, \otimes, 1)$ is defined as a triple (A, m, u) consisting of an object A and two morphisms

$$\begin{tikzcd}
1 \arrow[r, "u"] & A & A \otimes A \arrow[l, "m"']
\end{tikzcd}$$

making the *associativity* diagram

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "m \otimes A"] \arrow[d, "\alpha"'] & & A \otimes A \arrow[d, "m"] \\
A \otimes (A \otimes A) \arrow[r, "A \otimes m"] & A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

and the two *unit* diagrams

$$\begin{tikzcd}
1 \otimes A \arrow[r, "u \otimes A"] \arrow[d, "\lambda"'] & A \otimes A \arrow[d, "m"] & A \otimes 1 \arrow[l, "A \otimes u"'] \arrow[d, "\rho"] \\
A \arrow[r, phantom, "="] & A \arrow[r, phantom, "="] & A
\end{tikzcd}$$

commute. A *monoid morphism*

$$f \quad : \quad (A, m_A, u_A) \longrightarrow (B, m_B, u_B)$$

between monoids (A, m_A, u_A) and (B, m_B, u_B) is defined as a morphism

$$f \quad : \quad A \longrightarrow B$$

between the underlying objects in the category $\mathbb{C}$, making the two diagrams

$$\begin{tikzcd}
1 \arrow[r, phantom, "="] \arrow[d, "u_A"'] & 1 \arrow[d, "u_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commute. A monoid defined in a symmetric monoidal category $(\mathbb{C}, \otimes, 1)$ is called *commutative* when the diagram

$$\begin{tikzcd}
A \otimes A \arrow[dd, "\gamma"'] \arrow[dr, "m"] & \\
& A \\
A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes.

Exercise. Show that one retrieves the usual notions of monoid, of commutative monoid and of monoid morphism when one applies these definitions to the monoidal category $(\mathbf{Set}, \times, 1)$ with sets as objects, functions as morphisms, cartesian product as tensor product, and terminal object as unit. ■

6.2 The category of monoids

One reason invoked by Jean Bénabou for introducing the notion of lax monoidal functor in [5] is its remarkable affinity with the traditional notion of monoid. This affinity is witnessed by the following *lifting* property. To every monoidal category $(\mathbb{C}, \otimes, 1)$, one associates the category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$

- with objects the monoids,
- with morphisms the monoid morphisms.

Then, every lax monoidal functor

$$(\mathcal{F}, n) \quad : \quad (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

induces a functor

$$\mathrm{Mon}(\mathcal{F}, n) \quad : \quad \mathrm{Mon}(\mathbb{C}, \otimes, e) \longrightarrow \mathrm{Mon}(\mathbb{D}, \bullet, u)$$

which transports a monoid (A, m_A, u_A) to the monoid $(\mathcal{F}A, m_{\mathcal{F}A}, u_{\mathcal{F}A})$ defined as follows:

$$m_{\mathcal{F}A} \; : \quad \mathcal{F}A \otimes \mathcal{F}A \xrightarrow{n^2} \mathcal{F}(A \otimes A) \xrightarrow{\mathcal{F}m_A} \mathcal{F}A$$

$$u_{\mathcal{F}A} \; : \quad u \xrightarrow{n^0} \mathcal{F}e \xrightarrow{u_A} \mathcal{F}A$$

We leave it to the reader to check that $\mathrm{Mon}(\mathcal{F}, n)$ does indeed define a functor. This may be established directly by simple diagram chasing, or more conceptually by completing the exercise below.

Exercise. Show that the category $\mathbb{1}$ consisting of one object and its identity morphism is monoidal. Show that a lax monoidal functor from the monoidal category $\mathbb{1}$ to a monoidal category $\mathbb{C} = (\mathbb{C}, \otimes, 1)$ is the same thing as a monoid in this category; and that the category $\mathbf{LaxMonCat}(\mathbb{1}, \mathbb{C})$ coincides with the category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$ with monoids as objects and monoid morphisms as morphisms. Deduce the existence of the functor $\mathrm{Mon}(\mathcal{F}, n)$ from 2-categorical considerations. ■

Note that the category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$ is not monoidal in general. However, the category becomes monoidal, even symmetric monoidal, when the underlying category $(\mathbb{C}, \otimes, 1)$ is symmetric monoidal.

Proposition 14 *Every symmetric monoidal category* $(\mathbb{C}, \otimes, 1)$ *induces a symmetric monoidal category* $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$ *with the monoid*

$$\begin{tikzcd} 1 \arrow[r, "id_1"] & 1 & 1 \otimes 1 \arrow[l, "\lambda=\rho"'] \end{tikzcd}$$

as monoidal unit, and the monoid $(A \otimes B, m_{A\otimes B}, u_{A\otimes B})$ *defined below*

$$u_{A\otimes B} \quad : \qquad 1 \xrightarrow{\rho^{-1}=\lambda^{-1}} 1 \otimes 1 \xrightarrow{u_A \otimes u_B} A \otimes B$$

$$\begin{tikzcd} m_{A\otimes B} \ : & (A \otimes B) \otimes (A \otimes B) \arrow[d, "\alpha"'] & (A \otimes A) \otimes (B \otimes B) \arrow[r, "m_A \otimes m_B"] & A \otimes B. \\ & A \otimes (B \otimes (A \otimes B)) \arrow[d, "A\otimes\alpha^{-1}"'] & A \otimes (A \otimes (B \otimes B)) \arrow[u, "\alpha"'] & \\ & A \otimes ((B \otimes A) \otimes B) \arrow[r, "A\otimes(\gamma\otimes B)"] & A \otimes ((A \otimes B) \otimes B) \arrow[u, "A\otimes\alpha"'] & \end{tikzcd}$$

as tensor product of two monoids (A, m_A, u_A) *and* (B, m_B, u_B)*. Moreover, the forgetful functor*

$$U \quad : \quad \mathrm{Mon}(\mathbb{C}, \otimes, 1) \longrightarrow (\mathbb{C}, \otimes, 1)$$

which transports a monoid (A, m, u) *to its underlying object* A *is symmetric and strict monoidal (that is, its coercion maps are provided by identities.)*

We have noted at the beginning of the section that every lax monoidal functor between monoidal categories

$$(\mathcal{F}, n) \quad : \quad (\mathbb{C}, \otimes, e) \longrightarrow (\mathbb{D}, \bullet, u)$$

lifts to a functor

$$\mathrm{Mon}(\mathcal{F}, n) \quad : \quad \mathrm{Mon}(\mathbb{C}, \otimes, e) \longrightarrow \mathrm{Mon}(\mathbb{D}, \bullet, u).$$

We have seen moreover in Proposition 14 that when the monoidal categories $(\mathbb{C}, \otimes, e)$ and $(\mathbb{D}, \bullet, u)$ are symmetric, they induce symmetric monoidal categories $\mathrm{Mon}(\mathbb{C}, \otimes, e)$. In that situation, and when the lax monoidal functor $(\mathcal{F}, n)$ is symmetric, the functor $\mathrm{Mon}(\mathcal{F}, n)$ lifts to a symmetric lax monoidal functor — equipped with the coercions n. This induces a commutative diagram of symmetric lax monoidal functors:

$$\begin{tikzcd} \mathrm{Mon}(\mathbb{C}, \otimes, e) \arrow[r, "\mathrm{Mon}(\mathcal{F},n)"] \arrow[d, "U"'] & \mathrm{Mon}(\mathbb{D}, \bullet, u) \arrow[d, "U"] \\ (\mathbb{C}, \otimes, e) \arrow[r, "(\mathcal{F},n)"] & (\mathbb{D}, \bullet, u) \end{tikzcd}$$

Exercise. Show that every commutative monoid in a symmetric monoidal category $(\mathbb{C}, \otimes, 1)$ lifts to a commutative monoid in the category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$. Conversely, show that every monoid in the category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$ is obtained in such a way. Conclude that the category $\mathrm{Mon}(\mathrm{Mon}(\mathbb{C}, \otimes, 1), \otimes, 1)$ is isomorphic (as a symmetric monoidal category) to the full subcategory of $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$ with commutative monoids as objects, equipped with the same monoidal structure as the surrounding category $\mathrm{Mon}(\mathbb{C}, \otimes, 1)$.
■

6.3 Comonoids

Every category $\mathbb{C}$ defines an opposite category $\mathbb{C}^{op}$ obtained by reversing the direction of every morphism in the category $\mathbb{C}$. The resulting category $\mathbb{C}^{op}$ has the same objects as the category $\mathbb{C}$, and satisfies

$$\mathbb{C}^{op}(A, B) \quad = \quad \mathbb{C}(B, A)$$

for every pair of objects A and B. A remarkable aspect of the theory of monoidal categories is its self-duality. Indeed, every monoidal category $(\mathbb{C}, \otimes, e)$ defines a monoidal category $(\mathbb{C}^{op}, \otimes, e)$ on the opposite category $\mathbb{C}^{op}$, with same tensor product and unit as in the original category $\mathbb{C}$.

From this, it follows that every notion formulated in the theory of "monoidal categories" may be dualized by reversing the direction of morphisms in the definition. This principle is nicely illustrated by the notion of comonoid, which is dual to the notion of monoid formulated in Section 6.1. Hence, a *comonoid* in a monoidal category $(\mathbb{C}, \otimes, 1)$ is defined as a triple (A, d, e) consisting of an object A and two morphisms

$$\begin{tikzcd} 1 & A \arrow[l, "e"'] \arrow[r, "d"] & A \otimes A \end{tikzcd}$$

making the *associativity* diagram

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "d"'] & A \otimes A \arrow[r, "d \otimes A"] & (A \otimes A) \otimes A \arrow[d, "\alpha"] \\ A \otimes A \arrow[rr, "A \otimes d"] & & A \otimes (A \otimes A) \end{tikzcd}$$

and the two *unit* diagrams

$$\begin{tikzcd} 1 \otimes A \arrow[d, "\lambda"'] & A \otimes A \arrow[l, "e \otimes A"'] \arrow[r, "A \otimes e"] & A \otimes 1 \arrow[d, "\rho"] \\ A \arrow[r, phantom, "="] & A \arrow[u, "d"'] \arrow[r, phantom, "="] & A \end{tikzcd}$$

commute. A *comonoid morphism*

$$f \quad : \quad (A, d_A, e_A) \longrightarrow (B, d_B, e_B)$$

is defined as a morphism

$$f \quad : \quad A \longrightarrow B$$

between the underlying objects in the category $\mathbb{C}$, making the two diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e_A"'] & B \arrow[d, "e_B"] \\
1 \arrow[r, phantom, "="] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "d_A"'] & B \arrow[d, "d_B"] \\
A \otimes A \arrow[r, "f \otimes f"] & B \otimes B
\end{tikzcd}$$

commute. A comonoid defined in a symmetric monoidal category $(\mathbb{C}, \otimes, 1)$ is called *commutative* when the diagram

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, equal] & A \otimes A \arrow[d, "\gamma"] \\
A \arrow[r, "d"] & A \otimes A
\end{tikzcd}$$

commutes.

6.4 Cartesian categories among monoidal categories

In a cartesian category, every object defines a comonoid. Conversely, it is useful to know when a monoidal category $(\mathbb{C}, \otimes, 1)$, in which every object defines a comonoid, is a cartesian category. This is precisely the content of the next proposition.

Proposition 15 *Let $(\mathbb{C}, \otimes, 1)$ be a monoidal category. The tensor unit is a terminal object and the tensor product is a cartesian product if and only if there exists two natural transformations d and e with components*

$$d_A : A \longrightarrow A \otimes A \qquad\qquad e_A : A \longrightarrow 1$$

such that:

1. *(A, d_A, e_A) is a comonoid for every object A,*
2. *the diagram*

$$\begin{tikzcd}
A \otimes B \arrow[r, "d_{A\otimes B}"] \arrow[d, equal] & (A \otimes B) \otimes (A \otimes B) \arrow[d, "(A\otimes e_B)\otimes(e_A\otimes B)"] \\
A \otimes B & (A \otimes 1) \otimes (1 \otimes B) \arrow[l, "\rho\otimes\lambda"']
\end{tikzcd} \tag{37}$$

   commutes for every pair of objects A and B,

3. *the component $e_1 : 1 \longrightarrow 1$ coincides with the identity morphism.*

Proof. The direction ($\Rightarrow$) is nearly immediate, and we leave it as exercise to the reader. We prove the other more difficult direction ($\Leftarrow$). We show that for every pair of objects A and B, the morphisms

$$\begin{aligned} \pi_1 \quad &: \quad A \otimes B \xrightarrow{A \otimes e_B} A \otimes 1 \xrightarrow{\rho} A \\ \pi_2 \quad &: \quad A \otimes B \xrightarrow{e_A \otimes B} 1 \otimes B \xrightarrow{\lambda} B \end{aligned}$$

define the two projections of a cartesian product. To that purpose, we need to show that for every two morphisms

$$f \quad : \quad X \longrightarrow A \qquad\qquad g \quad : \quad X \longrightarrow B$$

there exists a unique morphism

$$\langle f, g \rangle \quad : \quad X \longrightarrow A \otimes B$$

making the diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[r, "\langle f,g \rangle"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \otimes B \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & B
\end{tikzcd} \tag{38}$$

commute in the category $\mathbb{C}$. Existence follows easily from the definition of the morphism $\langle f, g \rangle$ as

$$\langle f, g \rangle \quad : \quad X \xrightarrow{d_X} X \otimes X \xrightarrow{f \otimes g} A \otimes B.$$

One establishes by an elementary diagram chasing that Diagram (38) commutes. Typically, the equality $\pi_1 \circ \langle f, g \rangle = f$ holds because the diagram

$$\begin{tikzcd}
X \arrow[r, "d_X"] & X \otimes X \arrow[r, "f \otimes g"] \arrow[d, "X \otimes e_X"'] \arrow[dr, phantom, "(b)"] & A \otimes B \arrow[d, "A \otimes e_B"] \\
\| \arrow[r, phantom, "(a)"] & X \otimes 1 \arrow[r, "f \otimes 1"] \arrow[d, "\rho"'] \arrow[dr, phantom, "(c)"] & A \otimes 1 \arrow[d, "\rho"] \\
X \arrow[r, phantom, "="] & X \arrow[r, "f"'] & A
\end{tikzcd}$$

(a) property of the comonoid X,
(b) g is a comonoid morphism,
(c) ρ is natural.

commutes. We prove uniqueness. Suppose that a morphism $h : X \longrightarrow A \otimes B$ makes the diagram

$$\begin{tikzcd}
 & & A \\
X \arrow[r, "h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \otimes B \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & B
\end{tikzcd} \tag{39}$$

commutes. In that case, a simple diagram chasing shows that the two diagrams below commute in the category $\mathbb{C}$.

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "h"'] \arrow[dr, phantom, "(a)"] & X \otimes X \arrow[d, "h \otimes h"] \\
A \otimes B \arrow[r, "d_{A \otimes B}"] \arrow[d, equal] \arrow[dr, phantom, "(b)"] & (A \otimes B) \otimes (A \otimes B) \arrow[d, "(A \otimes e_B) \otimes (e_A \otimes B)"] \\
A \otimes B & (A \otimes 1) \otimes (1 \otimes B) \arrow[l, "\rho \otimes \lambda"']
\end{tikzcd}$$

(a) naturality of d,
(b) Diagram (37).

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[dd, "\langle f,g \rangle"'] \arrow[d, phantom, "(c)"] & X \otimes X \arrow[d, "h \otimes h"] \arrow[ddl, "f \otimes g"] \\
 & (A \otimes B) \otimes (A \otimes B) \arrow[d, "(A \otimes e_B) \otimes (e_A \otimes B)"] \arrow[d, phantom, "(d)"'] \\
A \otimes B & (A \otimes 1) \otimes (1 \otimes B) \arrow[l, "\rho \otimes \lambda"']
\end{tikzcd}$$

(c) definition of $\langle f, g \rangle$
(d) Diagram (39)
and definition of π_1 and π_2.

From this follows that the two morphisms h and $\langle f, g \rangle$ coincide. We conclude that the tensor product is a cartesian product.

There only remains to show that the tensor unit is a terminal object. For every object A, there exists the morphism $e_A : A \longrightarrow 1$. We claim that e_A is the unique morphism from the object A to the object 1. Suppose that $f : A \longrightarrow 1$ is any such morphism. By naturality of e, the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e_A"'] & & 1 \arrow[dl, "e_1"] \\
& 1 &
\end{tikzcd}$$

commutes. From this and the hypothesis that e_1 is the identity morphism follows that the morphism f necessarily coincides with the morphism e_A. The tensor unit 1 is thus a terminal object of the category $\mathbb{C}$. This concludes the proof of Proposition 15. □

Note that we do not need to suppose that the category $(\mathbb{C}, \otimes, 1)$ is symmetric monoidal in Proposition 15, nor that every object A defines a *commutative* comonoid. However, the situation becomes slightly more conceptual when the category $(\mathbb{C}, \otimes, 1)$ is symmetric monoidal. In that case, indeed, the two endofunctors

$$X \mapsto X \otimes X \qquad X \mapsto 1$$

on the category $\mathbb{C}$ may be seen as lax monoidal endofunctors of the monoidal category $(\mathbb{C}, \otimes, 1)$. The coercions m of the functor $X \mapsto X \otimes X$ are defined in a similar fashion as the product of two monoids in Proposition 14:

$$m^0 \ : \qquad 1 \xrightarrow{\rho^{-1}=\lambda^{-1}} 1 \otimes 1$$

$$\begin{tikzcd}
m^2_{A,B} \ : & (A \otimes B) \otimes (A \otimes B) \arrow[d, "\alpha"'] & (A \otimes A) \otimes (B \otimes B) \\
& A \otimes (B \otimes (A \otimes B)) \arrow[d, "A\otimes\alpha^{-1}"'] & A \otimes (A \otimes (B \otimes B)) \arrow[u, "\alpha"'] \\
& A \otimes ((B \otimes A) \otimes B) \arrow[r, "A\otimes(\gamma\otimes B)"] & A \otimes ((A \otimes B) \otimes B) \arrow[u, "A\otimes\alpha"']
\end{tikzcd} \tag{40}$$

The coercion n of the functor $X \mapsto 1$ is defined as the identity $n^0 : 1 \longrightarrow 1$ and the morphism $n^2 = \lambda_1 = \rho_1 : 1 \otimes 1 \longrightarrow 1$. Note that the endofunctors $X \mapsto X \otimes X$ and $X \mapsto 1$ are strong and symmetric, but we do not care about this additional property here. The following result is folklore:

Corollary 16 *Let* $(\mathbb{C}, \otimes, 1)$ *be a symmetric monoidal category. The tensor unit is a terminal object and the tensor product is a cartesian product if and only if there exists two monoidal natural transformations d and e with components*

$$d_A : A \longrightarrow A \otimes A \qquad\qquad e_A : A \longrightarrow 1$$

defining a comonoid (A, d_A, e_A) *for every object A.*

Proof. The direction $(\Rightarrow)$ is easy, and left as exercise to the reader. The other direction $(\Leftarrow)$ established by applying Proposition 15. To that purpose, we show that Diagram (37) commutes for every pair of objects A and B, and that the component e_1

coincides with the identity. This is deduced by an elementary diagram chasing in which the monoidality of d and e is only used to ensure that $e_1 = id$ and that the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "d_A \otimes d_B"] \arrow[ddddd, "id"'] & (A \otimes A) \otimes (B \otimes B) \arrow[d, "\alpha"] \\
& A \otimes (A \otimes (B \otimes B)) \arrow[d, "A \otimes \alpha^{-1}"] \\
& A \otimes ((A \otimes B) \otimes B) \arrow[d, "A \otimes (\gamma \otimes B)"] \\
& A \otimes ((B \otimes A) \otimes B) \arrow[d, "A \otimes \alpha"] \\
& A \otimes (B \otimes (A \otimes B)) \arrow[d, "\alpha^{-1}"] \\
A \otimes B \arrow[r, "d_{A \otimes B}"] & (A \otimes B) \otimes (A \otimes B)
\end{tikzcd}$$

commutes for all objects A and B. □

Remark. Note that we do not require that the comonoid (A, d_A, e_A) is commutative in Corollary 16, and that we do not use in the proof the equality $d_1 = \lambda_1$, nor the property that the diagram

$$\begin{tikzcd}
& 1 \otimes 1 \arrow[dd, "\lambda = \rho"] \\
A \otimes B \arrow[ur, "e_A \otimes e_B"] \arrow[dr, "e_{A \otimes B}"'] & \\
& 1
\end{tikzcd}$$

commutes for all object A and B, although these two facts hold by monoidality of d and e.

6.5 The category of commutative comonoids

To every symmetric monoidal category $(\mathbb{C}, \otimes, 1)$, we associate the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$

- with commutative comonoids as objects,
- with comonoid morphisms as morphisms.

The category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ is symmetric monoidal with the monoidal structure defined in Proposition 14 in Section 6.2 dualized. We establish below that the tensor product is a cartesian product, and that the tensor unit is a terminal object in the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$. This folklore property is deduced from Proposition 15.

Corollary 17 *The category* $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ *is cartesian.*

Proof. Once dualized, Proposition 14 in Section 6.2 states that the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ is symmetric monoidal. By definition, every object A of the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ is a commutative comonoid $A = (A, d_A, e_A)$ of the underlying symmetric monoidal category $(\mathbb{C}, \otimes, 1)$. This commutative comonoid lifts to a commutative comonoid in the symmetric monoidal category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$. This is precisely the content (once dualized) of the exercise appearing at the end of Section 6.2. Similarly, every morphism

$$f : A \longrightarrow B$$

in the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ defines a comonoid morphism

$$f : (A, d_A, e_A) \longrightarrow (B, d_B, e_B)$$

in the underlying monoidal category $(\mathbb{C}, \otimes, 1)$. From this follows that f is a comonoid morphism

$$f : (A, d_A, e_A) \longrightarrow (B, d_B, e_B)$$

in the monoidal category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$ itself. This proves that d and e are natural transformations in the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$. Finally, the construction of the monoids 1 and $A \otimes B$ in Proposition 14 in Section 6.2 implies that, once dualized, Diagram (37) commutes for every pair of objects A and B, and that the morphism e_1 coincides with the identity. We apply Proposition 15 and conclude that in the category $\mathrm{CoMon}(\mathbb{C}, \otimes, 1)$, the tensor product is a cartesian product, and the tensor unit is a terminal object. □

Corollary 18 *A symmetric monoidal category* $(\mathbb{C}, \otimes, 1)$ *is cartesian iff the forgetful functor*

$$U \quad : \quad \mathrm{CoMon}(\mathbb{C}, \otimes, 1) \;\longrightarrow\; \mathbb{C}$$

defines an isomorphism of category.

Exercise. By isomorphism of category, we mean a functor U with an inverse, that is, a functor V such that the two composite functors $U \circ V$ and $V \circ U$ are the identity. Suppose that the functor (U, m) is strong monoidal and symmetric between symmetric monoidal categories — as this is the case in Corollary 18. Show that the inverse functor V lifts as a strong monoidal and symmetric functor (V, n) such that $(U, m) \circ (V, n)$ and $(V, n) \circ (U, m)$ are the identity functors, with trivial coercions. [Hint: use the fact that V is at the same time left and right adjoint to the functor U, with trivial unit η and counit ϵ, and apply Proposition 13 in Section 5.16, Chapter 5.] ■

Exercise. Establish the following universality property of the forgetful functor U above, understood as a symmetric and strict monoidal functor (U, p) whose coercion maps p are provided by identities. Show that for every colax monoidal functor

$$(\mathcal{F}, m) \quad : \quad (\mathbb{D}, \times, e) \;\longrightarrow\; (\mathbb{C}, \otimes, 1)$$

from a cartesian category $(\mathbb{D}, \times, e)$ to a symmetric monoidal category $(\mathbb{C}, \otimes, 1)$ there exists a unique symmetric colax monoidal functor

$$(\mathcal{G}, n) \quad : \quad (\mathbb{D}, \times, e) \;\longrightarrow\; \mathrm{CoMon}(\mathbb{C}, \otimes, 1)$$

making the diagram of symmetric colax monoidal functors

$$\begin{tikzcd}
(\mathbb{D}, \times, e) \arrow[r, "(\mathcal{G},n)"] \arrow[d, equal] & \mathrm{CoMon}(\mathbb{C}, \otimes, 1) \arrow[d, "(U,p)"] \\
(\mathbb{D}, \times, e) \arrow[r, "(\mathcal{F},m)"] & (\mathbb{C}, \otimes, 1)
\end{tikzcd}$$

commute. ■

6.6 Monads and comonads

A monad $T = (T, \mu, \eta)$ in a category $\mathbb{C}$ consists of a functor

$$T \quad : \quad \mathbb{C} \longrightarrow \mathbb{C}$$

and two natural transformations

$$\mu \ : \ T \circ T \ \Rightarrow \ T \qquad\qquad \eta \ : \ I \ \Rightarrow \ T$$

making the *associativity* diagram

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}$$

and the two *unit* diagrams

$$\begin{tikzcd}
IT \arrow[r, "\eta T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[d, equal] \\
T \arrow[r, phantom, "="] & T \arrow[r, phantom, "="] & T
\end{tikzcd}$$

commute, where I denotes the identity functor on the category $\mathbb{C}$.

Exercise. Show that the category **Cat**$(\mathbb{C}, \mathbb{C})$ of endofunctors on a category $\mathbb{C}$

- with functors $F : \mathbb{C} \longrightarrow \mathbb{C}$ as objects,
- with natural transformations $\theta : F \Rightarrow G$ as morphisms,

defines a strict monoidal category in which

- the product $F \otimes G$ of two functors is defined as their composite $F \circ G$,

- the unit e is defined as the identity functor on the category $\mathbb{C}$.

Show that a monad on the category $\mathbb{C}$ is the same thing as a monoid in the monoidal category $(\mathbf{Cat}(\mathbb{C},\mathbb{C}),\circ,I)$. ■

Dually, a comonad (K,δ,ϵ) in a category $\mathbb{C}$ consists of a functor

$$K \quad : \quad \mathbb{C} \longrightarrow \mathbb{C}$$

and two natural transformations

$$\delta \ : \ K \Rightarrow K \circ K \qquad\qquad \epsilon \ : \ K \Rightarrow I$$

making the *associativity* diagram

$$\begin{tikzcd}
K \arrow[r, "\delta"] \arrow[d, "\delta"'] & K^2 \arrow[d, "K\delta"] \\
K^2 \arrow[r, "\delta K"] & K^3
\end{tikzcd}$$

and the two *unit* diagrams

$$\begin{tikzcd}
IK & K^2 \arrow[l, "\epsilon K"'] \arrow[r, "K\epsilon"] & KI \\
K \arrow[u, phantom, "\|"] \arrow[r, phantom, "="] & K \arrow[u, "\delta"] \arrow[r, phantom, "="] & K \arrow[u, phantom, "\|"]
\end{tikzcd}$$

commute.

Exercise. Show that a comonad on a category $\mathbb{C}$ is the same thing as a comonoid in its monoidal category $(\mathbf{Cat}(\mathbb{C},\mathbb{C}),\circ,I)$ of endofunctors. ■

Exercise. Every object A in a monoidal category $(\mathbb{C},\otimes,e)$ defines a functor

$$X \mapsto A \otimes X \quad : \quad \mathbb{C} \longrightarrow \mathbb{C}.$$

Show that this defines a strong monoidal functor from the monoidal category $(\mathbb{C},\otimes,e)$ to its monoidal category $(\mathbf{Cat}(\mathbb{C},\mathbb{C}),\circ,I)$ of endofunctors. Deduce that every monoid (A,m,u) in the monoidal category $(\mathbb{C},\otimes,e)$ defines in this way a monad (T,μ,η) on the category $\mathbb{C}$; and dually, that every comonoid (A,d,e) defines in this way a comonad (K,δ,ϵ) on the category $\mathbb{C}$. ■

We have seen that a monad (resp. a comonad) over a category $\mathbb{C}$ is a monoid (resp. a comonoid) in the monoidal category $\mathbf{Cat}(\mathbb{C},\mathbb{C})$ of endofunctors and natural transformations. This leads to a generic notion of monad and comonad in a 2-category, developed in Section 6.9.

6.7 Monads and adjunctions

Every adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[rr, bend left, "\mathcal{F}_*"] \arrow[rr, phantom, "\perp"] & & \mathbb{D} \arrow[ll, bend left, "\mathcal{F}^*"] \end{tikzcd} \tag{41}$$

induces a monad (T, μ, η) on the category $\mathbb{C}$ and a comonad (K, δ, ϵ) on the category $\mathbb{D}$, in which the functors T and K are the composites:

$$T = \mathcal{F}^* \circ \mathcal{F}_* \qquad\qquad K = \mathcal{F}_* \circ \mathcal{F}^*$$

and the two natural transformations

$$\eta : 1_{\mathbb{C}} \Rightarrow \mathcal{F}^* \circ \mathcal{F}_* \qquad \epsilon : \mathcal{F}_* \circ \mathcal{F}^* \Rightarrow 1_{\mathbb{D}}$$

are constructed as explained in Section 5.11 of Chapter 5. Here, we use the notation $1_{\mathbb{C}}$ for the identity functor of the category $\mathbb{C}$. The two natural transformations μ and δ are then deduced from η and ϵ by composition:

$$\begin{array}{rcl} \mu = \mathcal{F}^* \circ \epsilon \circ \mathcal{F}_* & : & \mathcal{F}^* \circ \mathcal{F}_* \circ \mathcal{F}^* \circ \mathcal{F}_* \Rightarrow \mathcal{F}^* \circ \mathcal{F}_* \\ \delta = \mathcal{F}_* \circ \eta \circ \mathcal{F}^* & : & \mathcal{F}_* \circ \mathcal{F}^* \Rightarrow \mathcal{F}_* \circ \mathcal{F}^* \circ \mathcal{F}_* \circ \mathcal{F}^* \end{array}$$

We leave the reader to check that, indeed, we have defined a monad (T, μ, η) and a comonad (K, δ, ϵ). The proof follows from the triangular equalities formulated in Chapter 5 (Section 5.11). It may also be performed at a more abstract 2-categorical level, as will be explored in Section 6.9.

Conversely, given a monad (T, μ, η) on the category $\mathbb{C}$, does there exist an adjunction (41) whose induced monad on the category $\mathbb{C}$ coincides precisely with the monad (T, μ, η). The answer happens to be positive, and positive twice: there exists indeed two different canonical ways to construct such an adjunction, each one based on a specific category $\mathbb{C}_T$ and $\mathbb{C}^T$.

$$\begin{tikzcd} \mathbb{C} \arrow[rr, bend left, "\mathcal{F}_*"] \arrow[rr, phantom, "\perp"] & & \mathbb{C}_T \arrow[ll, bend left, "\mathcal{F}^*"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{C} \arrow[rr, bend left, "\mathcal{G}_*"] \arrow[rr, phantom, "\perp"] & & \mathbb{C}^T \arrow[ll, bend left, "\mathcal{G}^*"] \end{tikzcd}$$

The two categories are called:

- the Kleisli category $\mathbb{C}_T$ of the monad,
- the Eilenberg-Moore category $\mathbb{C}^T$ of the monad.

The interested reader will find the construction of the two categories $\mathbb{C}_T$ and $\mathbb{C}^T$ in any good textbook on category theory, like Saunders Mac Lane's monograph [31] Francis Borceux's Handbook of Categorical Algebra [13]. We will define them in turn here. Once dualized and adapted to comonads, the two categories $\mathbb{C}_T$ and $\mathbb{C}^T$ play indeed a central role in the semantics of proofs in linear logic, as will become clear in Chapter 7.

The Kleisli category $\mathbb{C}_T$ has

- the same objects as the category $\mathbb{C}$,
- the morphisms $A \longrightarrow B$ are the morphisms $A \longrightarrow TB$ of the category $\mathbb{C}$.

Composition is defined as follows. Given two morphisms

$$f \quad : \quad A \longrightarrow B \qquad\qquad g \quad : \quad B \longrightarrow C$$

in the category $\mathbb{C}_T$, understood as morphisms

$$f \quad : \quad A \longrightarrow TB \qquad\qquad g \quad : \quad B \longrightarrow TC$$

in the category $\mathbb{C}$, the morphism

$$g \circ f \quad : \quad A \longrightarrow C$$

in the category $\mathbb{C}_T$ is defined as the morphism

$$A \xrightarrow{\quad f \quad} TB \xrightarrow{\quad Tg \quad} TTC \xrightarrow{\quad \mu \quad} TC.$$

The identity on the object A is defined as the morphism

$$\eta_A \quad : \quad A \longrightarrow TA$$

in the category $\mathbb{C}$.

Exercise. Prove that the composition law defines indeed a category $\mathbb{C}_T$. In order to establish associativity of the composition law, one may consider the diagram

$$\begin{tikzcd}
 & & & T^3D \arrow[d, "T\mu"] \\
 & & T^2C \arrow[ur, "T^2h"] \arrow[d, "\mu"] & T^2D \arrow[d, "\mu"] \\
 & TB \arrow[ur, "Tg"] & TC \arrow[ur, "Th"] & TD \\
A \arrow[ur, "f"] & B \arrow[ur, "g"] & C \arrow[ur, "h"] & D
\end{tikzcd}$$

in the category $\mathbb{C}$, and check that the two morphisms from A to TD coincide. Note that we write T^2 and T^3 for the composite functors $T^2 = T \circ T$ and $T^3 = T \circ T \circ T$. ■

The functor

$$\mathbb{C}_T \xrightarrow{\qquad \mathcal{F}_* \qquad} \mathbb{C}$$

transports every object A of the Kleisli category $\mathbb{C}_T$ to the object TA of the category $\mathbb{C}$, and every morphism

$$f \quad : \quad A \longrightarrow B$$

in the category $\mathbb{C}_T$ understood as a morphism

$$f \quad : \quad A \longrightarrow TB$$

in the category $\mathbb{C}$, to the morphism

$$\mathcal{F}^*(f) \quad = \quad TA \xrightarrow{Tf} T^2B \xrightarrow{\mu} TB$$

in the category $\mathbb{C}$.

The functor

$$\mathbb{C} \xrightarrow{\mathcal{F}^*} \mathbb{C}_T$$

transports every object A of category $\mathbb{C}$ to the same object A of the Kleisli category $\mathbb{C}_T$; and every every morphism

$$f \quad : \quad A \longrightarrow B$$

in the category $\mathbb{C}$, to the morphism

$$\mathcal{F}_*(f) \quad : \quad A \xrightarrow{f} B \xrightarrow{\eta_B} TB$$

in the category $\mathbb{C}$, understood as a morphism $A \longrightarrow B$ in the category $\mathbb{C}_T$.

The category $\mathbb{C}^T$ has

- the algebras of the monad (T, μ, η) as objects,
- the algebra morphisms as morphisms.

An algebra of the monad (T, μ, η) is defined as a pair (A, h) consisting of an object A of the category $\mathbb{C}$, and a morphism

$$h \quad : \quad TA \longrightarrow A$$

making the two diagrams

$$\begin{tikzcd}
 & TA \arrow[dr, "h"] & \\
A \arrow[ur, "\eta_A"] & = & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "\mu_A"] \arrow[d, "Th"'] & TA \arrow[d, "h"] \\
TA \arrow[r, "h"] & A
\end{tikzcd}$$

commute in the category $\mathbb{C}$. An algebra morphism

$$f \quad : \quad (A, h_A) \longrightarrow (B, h_B)$$

is defined as a morphism $f : A \longrightarrow B$ between the underlying objects in the category $\mathbb{C}$, making the diagram

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "h_A"'] & TB \arrow[d, "h_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commute. The functor

$$\begin{tikzcd}\mathbb{C}^T \arrow[rr, "\mathcal{G}^*"] & & \mathbb{C}\end{tikzcd}$$

is called the *forgetful functor*. It transports every algebra (A, h) to the underlying object A, and every algebra morphism

$$f \quad : \quad (A, h_A) \longrightarrow (B, h_B)$$

to the underlying morphism $f : A \longrightarrow B$. The functor

$$\begin{tikzcd}\mathbb{C} \arrow[rr, "\mathcal{G}_*"] & & \mathbb{C}^T\end{tikzcd}$$

is called the *free functor*. It transports every object A to the algebra

$$\mu_A \quad : \quad T^2 A \longrightarrow TA$$

This algebra (TA, μ_A) is called the *free algebra* of the object A.

Every morphism $f : A \longrightarrow B$ of the category $\mathbb{C}$ is transported to the algebra morphism

$$Tf \quad : \quad (TA, \mu_A) \longrightarrow (TB, \mu_B).$$

Exercise. Check that the pair (TA, μ_A) defines indeed an algebra of the monad (T, μ, η); and that the morphism $Tf : TA \longrightarrow TB$ defines an algebra morphism between the free algebras (TA, μ_A) and (TB, μ_B). ■

It is folklore in category theory that:

- the adjunction $\mathcal{F}_* \dashv \mathcal{F}^*$ based on the Kleisli category $\mathbb{C}_T$ is *initial* among all the possible "factorizations" of the monad (T, δ, ϵ) as an adjunction,
- the adjunction $\mathcal{G}_* \dashv \mathcal{G}^*$ based on the Eilenberg-Moore category $\mathbb{C}^T$ is *terminal* among all the possible "factorization" of the monad (T, δ, ϵ) as an adjunction.

We will not develop this point here, although it is fundamental in this topic. The interested reader will find a nice exposition in Mac Lane's monograph [31].

6.8 Comonads and adjunctions

Because we are mainly interested here in the categorical semantics of linear logic, we will generally work with a comonad (K, δ, ϵ) on a given category $\mathbb{C}$, instead of a monad (T, μ, η). This does not matter really, since a comonad on the category $\mathbb{C}$ is simply a monad on the opposite category $\mathbb{C}^{op}$. Hence, the two constructions of a Kleisli category $\mathbb{C}_T$ and of an Eilenberg-Moore category $\mathbb{C}^T$ for a monad, dualize to:

- a co-Kleisli category $\mathbb{C}_K$,
- an Eilenberg-Moore category $\mathbb{C}^K$,

for the comonad (K, δ, ϵ), with the expected derived adjunctions:

$$\begin{tikzcd}
\mathbb{C}_K \arrow[r, bend left, "\mathcal{F}_*"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "\mathcal{F}^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C}^K \arrow[r, bend left, "\mathcal{G}_*"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "\mathcal{G}^*"]
\end{tikzcd}$$

The co-Kleisli category $\mathbb{C}_K$ has:

- the objects of the category $\mathbb{C}$ as objects,
- the morphisms $KA \longrightarrow B$ as morphisms $A \longrightarrow B$.

The Eilenberg-Moore category $\mathbb{C}^K$ has

- the coalgebras of the comonad (K, δ, ϵ) as objects,
- the coalgebra morphisms as morphisms.

A coalgebra of the comonad (K, δ, ϵ) is defined as a pair (A, h) consisting of an object A of the category $\mathbb{C}$, and a morphism

$$h \quad : \quad A \longrightarrow KA$$

making the two diagrams

$$\begin{tikzcd}
 & KA \arrow[dr, "\epsilon_A"] & \\
A \arrow[ur, "h"] \arrow[rr, phantom, "="] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "h"'] & KA \arrow[d, "\delta"] \\
KA \arrow[r, "Kh"] & K^2A
\end{tikzcd}$$

commute in the category $\mathbb{C}$. A coalgebra morphism

$$f \quad : \quad (A, h_A) \longrightarrow (B, h_B)$$

is defined as a morphism $f : A \longrightarrow B$ between the underlying objects in the category $\mathbb{C}$, making the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h_A"'] & B \arrow[d, "h_B"] \\
KA \arrow[r, "Kf"] & KB
\end{tikzcd}$$

commute.

6.9 Symmetric monoidal comonads (lax and colax)

The notion of symmetric monoidal comonad plays a central role in the definition of a linear category, the third axiomatization of linear logic presented in Section 7.4. In order to introduce the notion, we proceed as in Chapter 5 and start by providing a generic definition of comonad (k, δ, ϵ) over an object C in a 2-category $\mathcal{C}$. The 2-categorical definition of comonad generalizes the definition of comonad developed previously: a comonad in the sense of Section 6.6 is the same thing as a comonad in the 2-category **Cat** of categories, functors, and natural transformations. From this follows by analogy the definition of a (lax) symmetric monoidal comonad as a comonad in the 2-category **SymMonCat** of symmetric monoidal categories, lax monoidal functors, and monoidal natural transformations (introduced in Chapter 5, Section 5.8, Proposition 9).

Every object C in a 2-category $\mathcal{C}$ induces a strict monoidal category $\mathcal{C}(C, C)$ with objects the endomorphisms $f : C \longrightarrow C$, with morphisms the cells $f \Rightarrow g : C \longrightarrow C$ and with monoidal structure provided by horizontal composition in the 2-category $\mathcal{C}$. A *comonad* on the object C is defined as a comonoid of this monoidal category $\mathcal{C}(C, C)$. The definition may be explicated in the following way. A comonad on the object C is defined as a triple (k, ϵ, δ) consisting of a morphism

$$k : C \longrightarrow C$$

and two cells ϵ and δ:

$$\begin{tikzcd}
 & C \arrow[dr, "k"] \arrow[d, phantom, "\Uparrow \delta"] & \\
C \arrow[ur, "k"] \arrow[rr, "k"] \arrow[rr, bend right=60, "id_C"'] \arrow[rr, phantom, bend right=30, "\Downarrow \epsilon"] & {} & C
\end{tikzcd}$$

satisfying associativity:

$$\begin{tikzcd}
C \arrow[r, "k"] \arrow[dr, phantom, "\Downarrow \delta" near start] & C \arrow[d, "k"] \arrow[dl, phantom, "\Downarrow \delta" near end] & & C \arrow[r, "k"] \arrow[dr, "k"] & C \arrow[d, "k"] \arrow[dl, phantom, "\Downarrow \delta" near start] \\
C \arrow[u, "k"] \arrow[ur, "k"] \arrow[r, "k"'] & C \arrow[r, phantom, "="] & {} & C \arrow[u, "k"] \arrow[r, "k"'] \arrow[ur, phantom, "\Downarrow \delta" near start] & C
\end{tikzcd}$$

and the two unit laws:

$$\begin{tikzcd}
C \arrow[r, "k"] \arrow[rr, bend left=50, "k"] \arrow[r, bend right=70, "id_C"'] \arrow[r, phantom, bend right=35, "\Downarrow \epsilon"] \arrow[rr, phantom, bend left=25, "\Downarrow \delta"] & C \arrow[r, "k"] & C \arrow[r, phantom, "="] & C \arrow[r, bend left=50, "k"] \arrow[r, bend right=50, "k"'] \arrow[r, phantom, "\Downarrow 1^k"] & C \arrow[r, phantom, "="] & C \arrow[r, "k"] \arrow[rr, bend left=50, "k"] \arrow[rr, phantom, bend left=25, "\Downarrow \delta"] & C \arrow[r, "k"] \arrow[r, bend right=70, "id_C"'] \arrow[r, phantom, bend right=35, "\Downarrow \epsilon"] & C
\end{tikzcd}$$

A monad on the object C is defined in a similar fashion, as a monoid in the monoidal category $\mathcal{C}(C, C)$.

Exercise. Show that every adjunction $f_* \dashv f^*$ between morphisms $f_* : C \longrightarrow D$ and $f^* : D \longrightarrow C$ in a 2-category $\mathcal{C}$ induces a monad on the object C and a comonad on the object D. ■

Conversely, we have seen in Section 6.6 that every comonad over a category $\mathbb{C}$ is the comonad associated to two particular adjunctions:

1. an adjunction with the co-kleisli category $\mathbb{C}_K$,
2. an adjunction with the category of Eilenberg-Moore coalgebras $\mathbb{C}^K$.

This well-known fact about a comonad in the 2-category **Cat** is not true any more (or only half-true) for a comonad in the 2-category **SymMonCat**. Let us explain this point. There exists a forgetful 2-functor from the 2-category **SymMonCat** to the 2-category **Cat** which transports every comonad K in **SymMonCat** to a comonad UK in **Cat**. This comonad UK generates an adjunction with each of the two categories $\mathbb{C}_{UK}$ and $\mathbb{C}^{UK}$. It follows from a general 2-categorical argument developed by Stephen Lack in [29] that only the adjunction with the category $\mathbb{C}^{UK}$ of Eilenberg-Moore coalgebras lifts to an adjunction in **SymMonCat**.

$$\begin{tikzcd}
(\mathbb{C}^K, \otimes, 1) \arrow[rr, bend left, "(L,m)"] \arrow[rr, phantom, "\perp"] & & (\mathbb{C}, \otimes, 1) \arrow[ll, bend left, "(M,n)"]
\end{tikzcd}$$

In this symmetric monoidal adjunction, the category $\mathbb{C}^{UK} = \mathbb{C}^K$ is equipped with the symmetric monoidal structure:

$$\begin{tikzcd}
 & & & A \otimes B \arrow[d, "h_A \otimes h_B"] & \\
A \arrow[d, "h_A"] & \otimes & B \arrow[d, "h_B"] & KA \otimes KB \arrow[d, "m_{A,B}"] & 1 \arrow[d, "m^0"] \\
KA & & KB & K(A \otimes B) & K1
\end{tikzcd} \qquad (42)$$

On the other hand, the adjunction between $\mathbb{C}$ and its co-kleisli category $\mathbb{C}_!$ does not lift in general to a symmetric monoidal adjunction.

Dually, we may define a *colax* symmetric monoidal comonad as a comonad in the 2-category **SymColaxMonCat** of symmetric monoidal categories, *colax* monoidal functors, and monoidal natural transformations introduced in Proposition 10 (Chapter 5, Section 5.8). The same 2-categorical argument by Stephen Lack in [29] applies by duality, and shows that, dually to the previous case, only the adjunction with the co-kleisli category $\mathbb{C}_{UK}$ lifts to an adjunction in **SymColaxMonCat**.

$$\begin{tikzcd}
(\mathbb{C}_K, \otimes_K, 1) \arrow[rr, bend left, "{(L,m)}"] \arrow[rr, phantom, "\perp"] & & (\mathbb{C}, \otimes, 1) \arrow[ll, bend left, "{(M,n)}"]
\end{tikzcd}$$

The monoidal structure of the category $\mathbb{C}$ lifts to the co-kleisli category $\mathbb{C}_K$ of the colax symmetric monoidal comonad $((K, n), \delta, \epsilon)$ in the following way. Every pair of morphisms

$$f \ : \ A \longrightarrow A' \quad \text{and} \quad g \ : \ B \longrightarrow B'$$

in the category $\mathbb{C}_K$ is given by a pair of morphisms

$$f \ : \ KA \longrightarrow A' \quad \text{and} \quad g \ : \ KB \longrightarrow B'$$

in the category $\mathbb{C}$; the morphism $f \otimes g$ in the category $\mathbb{C}_K$ is defined as the morphism

$$f \otimes_K g \qquad : \qquad K(A \otimes B) \xrightarrow{n^2_{A,B}} (KA \otimes KB) \xrightarrow{f \otimes g} A' \otimes B'$$

in the category $\mathbb{C}$. This is precisely what happens in Section 7.3 with the colax symmetric monoidal comonad $((!, n), \delta, \epsilon)$ whose associated co-kleisli category $\mathbb{L}_!$ is symmetric monoidal, and in fact in that case, cartesian.

7 Categorical models of linear logic

We review here three alternative categorical semantics of linear logic: Lafont categories, Seely categories, and Linear categories. We show that, in each case, the axiomatization induces a *symmetric monoidal adjunction*

$$(L, m) \dashv (M, n)$$

between the symmetric monoidal closed category of denotations $\mathbb{L}$ and a specific cartesian category $\mathbb{M}$. The reader starting at this point will find the definition of a symmetric monoidal adjunction in Section 5.16 at the end of Chapter 5.

Definition 19 *A linear-non-linear adjunction is a symmetric monoidal adjunction between lax symmetric monoidal functors*

$$\begin{tikzcd}(\mathbb{M}, \times, e) \arrow[r, bend left, "(L{,}m)"] \arrow[r, phantom, "\perp"] & (\mathbb{L}, \otimes, 1) \arrow[l, bend left, "(M{,}n)"]\end{tikzcd} .$$

in which the category $\mathbb{M}$ *is equipped with a cartesian product* $\times$ *and a terminal object* e*.*

The notations L and M are mnemonics for *Linearization* and *Multiplication*. Informally, the functor M transports a *linear* proof — which may be used exactly once as hypothesis in a reasoning — to a *multiple* proof — which may be repeated or discarded. Conversely, the functor L transports a *multiple* proof to a *linear* proof — which may then be manipulated as a linear entity inside the symmetric monoidal closed category $\mathbb{L}$.

The exponential modality ! of linear logic is then interpreted as the *comonad* on the category $\mathbb{L}$ defined by composing the two functors of the adjunction:

$$! \quad = \quad L \circ M.$$

This factorization is certainly one of the most interesting aspects of the categorical semantics of linear logic; we will see in Section 7.1 one of its most remarkable effects. Each categorical semantics of linear logic provides a particular recipe to construct a cartesian category $(\mathbb{M}, \times, e)$ and a monoidal adjunction $(L, m) \dashv (M, n)$ from the symmetric monoidal category $(\mathbb{L}, \otimes, e)$:

- Lafont category: the category $\mathbb{M}$ is defined as the category $\mathrm{CoMon}(\mathbb{L}, \otimes, e)$ with commutative comonoids of the category $(\mathbb{L}, \otimes, e)$ as objects, and comonoid morphisms between them as morphisms,
- Seely category: the category $\mathbb{M}$ is defined as the co-kleisli category $\mathbb{L}_!$ associated to the comonad ! which equips the category $\mathbb{L}$ in the definition of a Seely category (here, one needs to replace Seely's original definition by the definition of a new-Seely category advocated by Bierman in [9]).
- Linear category: the category $\mathbb{M}$ is defined as the category $\mathbb{L}^!$ of Eilenberg-Moore coalgebras associated to the symmetric monoidal comonad ! which equips the category $\mathbb{L}$ in the definition of a Linear category.

We recall that by Proposition 13 (Chapter 5, Section 5.16) an adjunction between functors

$$L \dashv M$$

lifts to a symmetric monoidal adjunction

$$(L, m) \dashv (M, n)$$

iff the monoidal functor

$$(L, m) : (\mathbb{M}, \times, e) \longrightarrow (\mathbb{L}, \otimes, 1)$$

is symmetric and strong monoidal. The purpose of each axiomatization of linear logic is thus to provide what is missing (not much!) to be in such a situation.

- Lafont category: the category $\mathbb{M} = \mathrm{CoMon}(\mathbb{L}, \otimes, e)$ associated to a given symmetric monoidal category $(\mathbb{L}, \otimes, e)$ is necessarily cartesian; and the forgetful functor L from $\mathrm{CoMon}(\mathbb{L}, \otimes, e)$ to $(\mathbb{L}, \otimes, e)$ is strict monoidal and symmetric. Thus, the only task of Lafont's axiomatization is to ensure that the forgetful functor L has a right adjoint M.

- Seely category: given a comonad $(!, \epsilon, \delta)$ on the category $\mathbb{L}$, there exists a canonical adjunction $L \dashv M$ between the category $\mathbb{L}$ and its co-kleisli category $\mathbb{M} = \mathbb{L}_!$. Moreover, since the category $\mathbb{L}$ is supposed to be cartesian in the definition of a Seely category, its co-kleisli category $\mathbb{L}_!$ is necessarily cartesian. The only task of the axiomatization is thus to ensure that the functor L is strong monoidal and symmetric.

- Linear category: given a symmetric monoidal comonad $(!, \epsilon, \delta, p)$ on the symmetric monoidal category $(\mathbb{L}, \otimes, e)$, there exists a canonical symmetric monoidal adjunction $(L, m) \dashv (M, n)$ between the symmetric monoidal category $(\mathbb{L}, \otimes, e)$ and its category $\mathbb{M} = \mathbb{L}^!$ of Eilenberg-Moore coalgebras. The category $\mathbb{L}^!$ is equipped with the symmetric monoidal structure induced from $(\mathbb{L}, \otimes, e)$. The only task of the axiomatization is thus to ensure that this symmetric monoidal structure on the category $\mathbb{L}^!$ defines a cartesian category.

The notions of symmetric monoidal comonad, co-kleisli category, category of Eilenberg-Moore coalgebras, have been introduced in the course of Chapters 5 and 6.

7.1 The transmutation principle of linear logic

One fundamental principle formulated by Jean-Yves Girard in his original article on linear logic [21] states that the exponential modality ! transports (or *transmutes* in the language of alchemy) the additive connective & and its unit $\top$ into the multiplicative connective $\otimes$ and its unit 1. This means formally that there exists a pair of isomorphisms

$$!A \otimes !B \;\cong\; !(A \& B) \qquad\qquad 1 \;\cong\; !\top \tag{43}$$

for every formula A and B of linear logic.

Quite remarkably, the existence of these isomorphisms may be derived from purely categorical principles, starting from the slightly enigmatic factorization of the exponential modality as

$$! \quad = \quad L \circ M.$$

We find useful to start the section on this topic, because it demonstrates the beauty and elegance of categorical semantics. At the same time, this short discussion will provide us with a categorical explanation (instead of a proof-theoretic one) for the appearance of the isomorphisms (43) in any *cartesian* category of denotations $\mathbb{L}$ — and will clarify the intrinsic nature and properties of these isomorphisms.

In order to interpret the additive connective & and unit $\top$ of linear logic, we suppose from now on that the category of denotations $\mathbb{L}$ is cartesian, with:

- the cartesian product of a pair of objects A and B noted $A\&B$,
- the terminal object noted $\top$.

We have seen in Chapter 5 (exercises at the end of Section 5.2, Section 5.5 and Section 5.6) that

- every functor $\mathcal{F}$ between cartesian categories lifts as a symmetric and colax monoidal functor $(\mathcal{F}, j)$ in a unique way,
- every natural transformation between two such symmetric colax monoidal functors is monoidal.

From this, it follows that the adjunction

$$\begin{tikzcd}
\mathbb{M} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbb{L} \arrow[l, bend left, "M"]
\end{tikzcd}$$

lifts as a *symmetric* and *colax monoidal* adjunction:

$$\begin{tikzcd}
(\mathbb{M}, \times, e) \arrow[r, bend left, "(L{,}j)"] \arrow[r, phantom, "\perp"] & (\mathbb{L}, \&, \top) \arrow[l, bend left, "(M{,}k)"]
\end{tikzcd}$$

By this, we mean an adjunction in the 2-category **SymColaxMonCat** defined in Proposition 10 (Chapter 5, Section 5.8). Such an adjunction is characterized by Proposition 13 (Chapter 5, Section 5.16) as an adjunction in which the *right adjoint functor* (M, k) is strong monoidal and symmetric. By this slightly sinuous path, we get the well-known principle that right adjoint functors preserve limits (in that case, the cartesian products and the terminal object) modulo isomorphism.

Thus, taken separately, each of the two functors

$$(\mathbb{L}, \&, \top) \xrightarrow{(M,k)} (\mathbb{M}, \times, e) \xrightarrow{(L,m)} (\mathbb{L}, \otimes, e)$$

is strong monoidal and symmetric. From this follows that their composite

$$(!, p) \quad = \quad (L, m) \circ (M, k) \qquad : \qquad (\mathbb{L}, \&, \top) \longrightarrow (\mathbb{L}, \otimes, e)$$

is also strong monoidal and symmetric. By the definition of such a functor, the monoidal structure p defines a pair of isomorphisms

$$p^2_{A,B} \ : \ !A \otimes !B \xrightarrow{\cong} !(A \& B) \qquad\qquad p^0 \ : \ 1 \xrightarrow{\cong} !\top$$

natural in the objects A and B of the category $\mathbb{L}$, and satisfying the coherence conditions formulated in Chapter 5, Sections 5.1 and 5.6.

7.2 Lafont categories

A *Lafont category* is defined as a symmetric monoidal closed category $(\mathbb{L}, \otimes, 1)$ in which the forgetful functor

$$U \quad : \quad \mathrm{CoMon}(\mathbb{L}, \otimes, 1) \longrightarrow \mathbb{L}$$

has a right adjoint. The right adjunct functor ! is called a free construction, because it associates the *free* commutative comonoid $!A$ to any object A of the category $\mathbb{L}$.

Equivalently, a Lafont category is defined as a symmetric monoidal closed category $(\mathbb{L}, \otimes, 1)$ in which there exists a commutative comonoid

$$!A \quad = \quad (!A, d_A, e_A)$$

and a morphism

$$\epsilon_A \quad : \quad !A \longrightarrow A$$

for every object A of the category, satisfying the following universality property: for every commutative comonoid

$$X \quad = \quad (X, d, e)$$

and for every morphism

$$f \quad = \quad X \longrightarrow A$$

there exists a *unique* comonoid morphism

$$f^\dagger \quad : \quad (X, d, e) \longrightarrow (!A, d_A, e_A)$$

making the diagram

$$\begin{tikzcd} & !A \arrow[d, "\epsilon_A"] \\ X \arrow[ur, "f^\dagger"] \arrow[r, "f"'] & A \end{tikzcd}$$

commute in the category $\mathbb{L}$, noted !. Once dualized and specialized to commutative comonoids, Proposition 14 in Section 6.2 states that the forgetful functor U is strict monoidal and symmetric. It follows from Proposition 13 in Chapter 5, Section 5.16, that the adjunction $U \dashv !$ between the forgetful functor and the free construction lifts to a symmetric monoidal adjunction:

$$\begin{tikzcd}
(\mathrm{CoMon}(\mathbb{L},\otimes,1),\otimes,1) \arrow[rr, bend left, "{(L,m)}"] \arrow[rr, phantom, "\perp"] & & (\mathbb{L},\otimes,1) \arrow[ll, bend left, "{(M,n)}"]
\end{tikzcd}$$

in which L is the forgetful functor U from the category $\mathrm{CoMon}(\mathbb{L},\otimes,1)$ of commutative comonoids to the underlying symmetric monoidal category $(\mathbb{L},\otimes,1)$. Finally, we apply Corollary 17 in Section 6.5 and deduce that the category $\mathrm{CoMon}(\mathbb{L},\otimes,1)$ is cartesian.

This establishes that

Proposition 20 *Every Lafont category defines a linear-non-linear adjunction, and thus, a model of intuitionistic linear logic.*

Remark. One well-known limitation of this categorical axiomatization is that the exponential modality is necessarily interpreted as a free construction. This is often limiting, especially in game semantics, where several exponential modality may coexist on the same category $\mathbb{L}$, each of them expressing a particular duplication policy: repetitive vs. non repetitive, uniform vs. non uniform, etc. It is thus useful to notice that the category $\mathrm{CoMon}(\mathbb{L},\otimes,1)$ may be replaced by any *full* subcategory $\mathbb{M}$ closed under tensor product and containing the unit comonoid 1. A Lafont category is then defined as a symmetric monoidal closed category in which the (restriction of the) forgetful functor

$$U \quad : \quad \mathbb{M} \longrightarrow \mathbb{L}$$

has a right adjoint. As previously, this definition may be reformulated as a universality property of the morphism

$$\epsilon_A \quad : \quad !A \longrightarrow A$$

in which, this time, only the commutative comonoids (X,d,e) in the subcategory $\mathbb{M}$ are considered. We leave the reader check that Proposition 20 adapts smoothly. We will take advantage of this remark in Section 7.5, where we crossbreed the two definitions of Lafont and of Seely category.

7.3 Seely categories

A *Seely category* is defined as a symmetric monoidal closed category $(\mathbb{L},\otimes,1)$ with finite products (binary product noted $A\&B$ and terminal object noted $\top$) together with

1. a comonad $(!,\delta,\epsilon)$,
2. two natural isomorphisms

$$m^2_{A,B} \quad : \quad !A\otimes !B \cong !(A\&B) \qquad\qquad m^0 \quad : \quad 1 \cong !\top.$$

One asks moreover that the five coherence diagrams below commute in the category $\mathbb{L}$, for all objects A, B, C:

$$\begin{tikzcd}
!A \otimes !B \arrow[rr, "m"] \arrow[dd, "\delta_A \otimes \delta_B"'] & & !(A \& B) \arrow[d, "\delta_{A\&B}"] \\
& & !!(A \& B) \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\
!!A \otimes !!B \arrow[rr, "m"] & & !(!A \& !B)
\end{tikzcd} \tag{44}$$

$$\begin{tikzcd}
(!A \otimes !B) \otimes !C \arrow[r, "\alpha"] \arrow[d, "m \otimes !C"'] & !A \otimes (!B \otimes !C) \arrow[d, "!A \otimes m"] \\
!(A \& B) \otimes !C \arrow[d, "m"'] & !A \otimes !(B \& C) \arrow[d, "m"] \\
!((A \& B) \& C) \arrow[r, "!\alpha"] & !(A \& (B \& C))
\end{tikzcd} \tag{45}$$

$$\begin{tikzcd}
!A \otimes 1 \arrow[r, "\rho"] \arrow[d, "!A \otimes m"'] & !A & 1 \otimes !B \arrow[r, "\lambda"] \arrow[d, "m \otimes !B"'] & !B \\
!A \otimes !\top \arrow[r, "m"] & !(A \& \top) \arrow[u, "!\rho"'] & !\top \otimes !B \arrow[r, "m"] & !(\top \& B) \arrow[u, "!\lambda"']
\end{tikzcd} \tag{46}$$

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "\gamma"] \arrow[d, "m"'] & !B \otimes !A \arrow[d, "m"] \\
!(A \& B) \arrow[r, "!\gamma"] & !(B \& A)
\end{tikzcd} \tag{47}$$

By a general categorical fact explained in Sections 6.7 and 6.8, the comonad $(!, \delta, \epsilon)$ generates an adjunction

$$\begin{tikzcd}
\mathbb{L}_! \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbb{L} \arrow[ll, bend left, "M"]
\end{tikzcd} \tag{48}$$

between the co-kleisli category $\mathbb{L}_!$ of the comonad and the original category $\mathbb{L}$.

We would like to show that the adjunction (48) defines a linear-non-linear adjunction. By definition, the category $\mathbb{L}$ is cartesian. We have seen in Chapter 5, Section 5.2, that every functor ! between cartesian categories defines a colax monoidal functor $(!, n)$ in a unique way. The coercion is provided by

$$m^2_{A,B} \quad : \quad !(A \& B) \xrightarrow{\langle !\pi_1, !\pi_2 \rangle} !A \& !B$$

where π_1 and π_2 denote the two projections of the cartesian product, and $\langle -, - \rangle$ the pairing bracket; and by the unique morphism $m^0 \, :!\top \longrightarrow \top$ to the terminal object. We will see in Section 6.9 that the comonad $(!, \delta, \epsilon)$ itself defines a (colax) symmetric monoidal functor in the sense of Section 6.9. From all this, it follows that the co-kleisli category $\mathbb{L}_!$ is cartesian, with finite products $(\&, \top)$ inherited from the category $\mathbb{L}$. It is worth explaining here how the cartesian product & lifts from a bifunctor on the category $\mathbb{L}$ to a bifunctor on the category $\mathbb{L}_!$. Every pair of morphisms

$$f \; : \; A \longrightarrow A' \quad \text{and} \quad g \; : \; B \longrightarrow B' \tag{49}$$

in the category $\mathbb{L}_!$ may be seen alternatively as a pair of morphisms

$$f \; : !A \longrightarrow A' \quad \text{and} \quad g \; : !B \longrightarrow B'$$

in the category $\mathbb{L}$; the morphism $f \& g$ in the category $\mathbb{L}_!$ is then defined as the morphism

$$f \& g \qquad : \qquad !(A \& B) \xrightarrow{\langle !\pi_1, !\pi_2 \rangle} (!A \& !B) \xrightarrow{f \& g} A' \& B'$$

in the category $\mathbb{L}$. Since the co-kleisli category $\mathbb{L}_!$ is cartesian, there only remains to show that the adjunction (48) lifts to a symmetric monoidal adjunction

$$\begin{tikzcd}
(\mathbb{L}_!, \&, \top) \arrow[rr, bend left, "{(L,m)}"] \arrow[rr, phantom, "\perp"] & & (\mathbb{L}, \otimes, 1) \arrow[ll, bend left, "{(M,n)}"]
\end{tikzcd}$$

in order to obtain a linear-non-linear adjunction. By Proposition 13 of Section 5.16, Chapter 5, this reduces to showing that the functor L equipped with the family of isomorphisms m defines a strong monoidal functor. The main difficulty to achieve that purpose is to establish that the family of isomorphisms m is natural with respect to the category $\mathbb{L}_!$, and not only with respect to the category $\mathbb{L}$. The functor L transports every morphism

$$f \; : \; A \longrightarrow B$$

of the category $\mathbb{L}_!$, understood as a morphism $f \; :!A \longrightarrow B$ of the category $\mathbb{L}$, to the morphism

$$L(f) \; : !A \xrightarrow{\delta_A} !!A \xrightarrow{!f} !B$$

of the category $\mathbb{L}$. Thus, naturality of m with respect to the category $\mathbb{L}_!$ means that the following diagram

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "m"] \arrow[dd, "\delta\otimes\delta"'] & !(A \& B) \arrow[d, "\delta"] \\
 & !!(A \& B) \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\
!!A \otimes !!B \arrow[d, "!f \otimes !g"'] & !(!A \& !B) \arrow[d, "!(f \& g)"] \\
!A' \otimes !B' \arrow[r, "m"'] & !(A' \& B')
\end{tikzcd}$$

commutes in the category $\mathbb{L}$ for every pair of morphisms (49). This follows from the first coherence Diagram (47) of Seely categories, and from naturality of m with respect to the category $\mathbb{L}$, by decomposing the diagram in the following way:

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "m"] \arrow[dd, "\delta\otimes\delta"'] \arrow[ddr, phantom, "(1)"] & !(A \& B) \arrow[d, "\delta"] \\
 & !!(A \& B) \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\
!!A \otimes !!B \arrow[r, "m"] \arrow[d, "!f \otimes !g"'] \arrow[dr, phantom, "(2)"] & !(!A \& !B) \arrow[d, "!(f \& g)"] \\
!A' \otimes !B' \arrow[r, "m"'] & !(A' \& B')
\end{tikzcd}$$

(1) coherence Diagram (47),
(2) naturality of m with respect to $\mathbb{L}$.

This establishes the naturality of m with respect to the category $\mathbb{L}_!$. The last four coherence diagrams (45—47) of Seely categories then ensure that (L, m) defines a strong monoidal functor from the cartesian category $(\mathbb{L}_!, \&, \top)$ to the symmetric monoidal category $(\mathbb{L}, \otimes, 1)$. From this follows that

Proposition 21 *Every Seely category defines a linear-non-linear adjunction, and thus a model of intuitionistic linear logic with additives.*

Remark. Here, we call Seely category what Gavin Bierman calls a new-Seely category in his work on categorical models of linear logic [9]. See the end of the chapter for a discussion.

7.4 Linear categories

A *linear category* is defined as a symmetric monoidal closed category $(\mathbb{L}, \otimes, 1)$ together with

1. a symmetric monoidal comonad $((!, m), \delta, \epsilon)$,

2. two monoidal natural transformations d and e whose components

$$d_A \quad : \quad !A \longrightarrow !A \otimes !A \qquad\qquad e_A \quad : \quad !A \longrightarrow 1$$

form a commutative comonoid and are coalgebra morphisms from the free coalgebra $(!A, \delta_A)$,

3. whenever $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ is a coalgebra morphism between free coalgebras, then it is also a comonoid morphism.

By a general categorical property explained in Section 6.9, the (lax) symmetric monoidal comonad $((!, m), \delta, \epsilon)$ induces a symmetric monoidal adjunction

$$\begin{tikzcd}
(\mathbb{L}^!, \otimes, 1) \arrow[rr, bend left, "(L{,}m)"] \arrow[rr, phantom, "\perp"] & & (\mathbb{L}, \otimes, 1) \arrow[ll, bend left, "(M{,}n)"]
\end{tikzcd}$$

In order to prove that every linear category defines a linear-non-linear adjunction, there remains to show that the category $\mathbb{L}^!$ of Eilenberg-Moore coalgebras equipped with the tensor product inherited from the category $\mathbb{L}$, is cartesian. The proof is elementary but far from immediate. In particular, it does not seem to follow from general categorical properties. The proof is also difficult to find in the litterature, although it appears in Gavin Bierman's PhD thesis [8]. We give a variant of the proof here.

Proposition 22 *In a linear category $\mathbb{L}$, every coalgebra $h_A \; : !A \longrightarrow A$ defines a retraction:*

$$A \xrightarrow{h_A} !A \xrightarrow{\epsilon_A} A$$

making the diagram

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & !A \arrow[ddd, "d_A"] \\
!A \arrow[d, "d_A"'] & \\
!A \otimes !A \arrow[d, "\epsilon_A \otimes \epsilon_A"'] & \\
A \otimes A \arrow[r, "h_A \otimes h_A"] & !A \otimes !A
\end{tikzcd} \tag{50}$$

commute.

Proof. Any comonad $(!, \delta, \epsilon)$ has the property that the diagram

$$\begin{tikzcd}
!A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & !!A \arrow[d, "\delta_{!A}"] \\
!!A \arrow[r, "!\delta_A"'] & !!!A
\end{tikzcd}$$

commutes. This says that the morphism δ_A is coalgebraic from the free coalgebra $!A$ to the free coalgebra $!!A$. By Property 3. of linear categories, the morphism δ_A is also a comonoid morphism. This simply means that the diagram

$$\begin{tikzcd}
!A \arrow[r, "d_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \arrow[d, "\delta_A \otimes \delta_A"] \\
!!A \arrow[r, "d_{!A}"'] & !!A \otimes !!A
\end{tikzcd}$$

commutes. The diagram

$$\begin{tikzcd}
!A \arrow[r, "d_A"] \arrow[d, "\delta_A"'] & !A \otimes !A \\
!!A \arrow[r, "d_{!A}"'] & !!A \otimes !!A \arrow[u, "\epsilon_A \otimes \epsilon_A"']
\end{tikzcd} \tag{51}$$

is obtained by postcomposing the previous diagram with the morphism $\epsilon_A \otimes \epsilon_A$ and by applying the identity $\epsilon_A \circ \delta_A = id_{!A}$. It thus commutes.

From all this, and the commuting diagram below, we deduce that Diagram (50) commutes for every coalgebra h_A in a linear category.

$$\begin{tikzcd}
A \arrow[rr, "h_A"] \arrow[dd, "h_A"'] & & !A \arrow[dd, "\delta_A"] & = & !A \arrow[dddddd, "d_A"] \\
& (a) & & & \\
!A \arrow[rr, "!h_A"] \arrow[dd, "d_A"'] & & !!A \arrow[dd, "d_{!A}"] & & \\
& (b) & & (d) & \\
!A \otimes !A \arrow[rr, "!h_A \otimes !h_A"] \arrow[dd, "\epsilon_A \otimes \epsilon_A"'] & & !!A \otimes !!A \arrow[dd, "\epsilon_{!A} \otimes \epsilon_{!A}"] & & \\
& (c) & & & \\
A \otimes A \arrow[rr, "h_A \otimes h_A"'] & & !A \otimes !A & = & !A \otimes !A
\end{tikzcd}$$

(a)	property of the coalgebra h_A,	(b)	naturality of d,
(c)	naturality of ϵ,	(d)	Diagram (51) commutes.

This concludes the proof. □

Proposition 23 *Let* $(\mathbb{C}, \otimes, 1)$ *be a monoidal category, and suppose given a retraction*

$$A \xrightarrow{i} B \xrightarrow{r} A \quad = \quad A \xrightarrow{id_A} A \tag{52}$$

between an object A and a comonoid (B, d_B, e_B)*. Then, the object A induces a comonoid* (A, d_A, e_A) *in such a way that the morphism i becomes a comonoid morphism:*

$$(A, d_A, e_A) \xrightarrow{i} (B, d_B, e_B) \tag{53}$$

iff the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[ddd, "d_B"] \\
B \arrow[d, "d_B"'] & \\
B \otimes B \arrow[d, "r\otimes r"'] & \\
A \otimes A \arrow[r, "i\otimes i"] & B \otimes B
\end{tikzcd} \tag{54}$$

In that case, the comonoid (A, d_A, e_A) *is necessarily defined in the following way:*

$$\begin{aligned} A \xrightarrow{d_A} A \otimes A \quad &= \quad A \xrightarrow{i} B \xrightarrow{d_B} B \otimes B \xrightarrow{r\otimes r} A \otimes A \\ A \xrightarrow{e_A} 1 \quad &= \quad A \xrightarrow{i} B \xrightarrow{e_B} 1 \end{aligned} \tag{55}$$

Proof. The direction $(\Rightarrow)$ is nearly immediate. Suppose indeed that (A, d_A, e_A) defines a comonoid involved in a comonoid morphism (53). In that case, the two diagrams

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "d_A"'] & B \arrow[d, "d_B"] & & A \arrow[r, "i"] \arrow[d, "e_A"'] & B \arrow[d, "e_B"] \\
A \otimes A \arrow[r, "i\otimes i"] & B \otimes B & & 1 \arrow[r, phantom, "="] & 1
\end{tikzcd}$$

commute. The diagram below is then obtained by postcomposing the lefthand side with the morphism $r \otimes r$, and by applying the equality $r \circ i = id_A$:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "d_A"'] & B \arrow[d, "d_B"] \\
A \otimes A & B \otimes B \arrow[l, "r\otimes r"']
\end{tikzcd}$$

and is thus commuting. From this follows that the comonoid A is necessarily defined as in Equation (55). Moreover, Diagram (54) commutes simply because the morphism i is a comonoid morphism.

We prove the more difficult direction ($\Leftarrow$) and suppose that Diagram (54) commutes. We want to show that the triple (A, d_A, e_A) defined in Equation (55) satisfies the properties (associativity, units) of a comonoid. The two diagrams below are obtained by postcomposing part (a) of Diagram (55) with the morphism $B \otimes r$ and the morphism $r \otimes B$, and by applying the equality $r \circ i = id_A$:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[dd, "d_B"] & A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[dd, "d_B"] \\
B \arrow[d, "d_B"'] & & B \arrow[d, "d_B"'] & \\
B \otimes B \arrow[d, "r\otimes r"'] & B \otimes B \arrow[d, "B\otimes r"] & B \otimes B \arrow[d, "r\otimes r"'] & B \otimes B \arrow[d, "r\otimes B"] \\
A \otimes A \arrow[r, "i\otimes A"] & B \otimes A & A \otimes A \arrow[r, "A\otimes i"] & A \otimes B
\end{tikzcd} \tag{56}$$

For that reason, they both commute. Coassociativity of the triple (A, d_A, e_A) follows from the commuting diagram below.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, "i"] & B \arrow[rr, "d_B"] & \arrow[d, phantom, "(b)"] & B \otimes B \arrow[r, "r\otimes r"] & A \otimes A \arrow[d, "A\otimes i"] \\
B \arrow[d, "d_B"'] & B \arrow[rr, "d_B"] \arrow[d, "d_B"'] & \arrow[d, phantom, "(c)"] & B \otimes B \arrow[r, "r\otimes B"] \arrow[d, "B\otimes d_B"'] \arrow[dr, phantom, "(d)"] & A \otimes B \arrow[d, "A\otimes d_B"] \\
B \otimes B \arrow[d, "r\otimes r"'] \arrow[r, phantom, "(a)"] & B \otimes B \arrow[r, "d_B\otimes B"] \arrow[d, "B\otimes r"'] \arrow[dr, phantom, "(e)"] & (B \otimes B) \otimes B \arrow[r, "\alpha"] \arrow[d, "(B\otimes B)\otimes r"] \arrow[dr, phantom, "(f)"] & B \otimes (B \otimes B) \arrow[r, "r\otimes(B\otimes B)"] & A \otimes (B \otimes B) \arrow[d, "A\otimes(r\otimes r)"] \\
A \otimes A \arrow[r, "i\otimes A"'] & B \otimes A \arrow[r, "d_B\otimes A"'] & (B \otimes B) \otimes A \arrow[r, "(r\otimes r)\otimes A"'] & (A \otimes A) \otimes A \arrow[r, "\alpha"'] & A \otimes (A \otimes A)
\end{tikzcd}$$

(a)	lefthand side of Diagram (56),	(b)	righthand side of Diagram (56),
(c)	coassociativity of d_B,	(d)	bifunctoriality of $\otimes$,
(e)	bifunctoriality of $\otimes$,	(f)	naturality of α and bifunctoriality of $\otimes$.

Now, the diagram below commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[dd, "d_B"] \arrow[rr, phantom, "="] & & B \arrow[dd, equal] \\
B \arrow[d, "d_B"'] & & (b) & \\
B \otimes B \arrow[d, "r \otimes r"'] & B \otimes B \arrow[r, "e_B \otimes B"] \arrow[d, "B \otimes r"'] & 1 \otimes B \arrow[r, "\lambda_B"] \arrow[d, "1 \otimes r"] & B \arrow[d, "r"] \\
A \otimes A \arrow[r, "i \otimes A"'] & B \otimes A \arrow[r, "e_B \otimes A"'] & 1 \otimes A \arrow[r, "\lambda_A"'] & A
\arrow[from=2-1, to=3-2, phantom, "(a)"]
\arrow[from=3-2, to=4-3, phantom, "(c)"]
\arrow[from=3-3, to=4-4, phantom, "(d)"]
\end{tikzcd}$$

(a)	lefthand side of Diagram (56),	(b)	unit law of the comonoid (B, d_B, e_B),
(c)	bifunctoriality of $\otimes$,	(d)	naturality of λ.

The series of equalities follows:

$$A \xrightarrow{d_A} A \otimes A \xrightarrow{e_A \otimes A} 1 \otimes A \xrightarrow{\lambda_A} A \quad = \quad A \xrightarrow{i} B \xrightarrow{r} A \quad = \quad A \xrightarrow{id_A} A.$$

This establishes one of the two unit laws of the triple (A, d_A, e_A). The other unit law is established by a similar diagram chasing, involving this time the righthand side of Diagram (56). From all this, we conclude that Equation (55) defines a comonoid (A, d_A, e_A). It is not difficult to check that the morphism i in the retraction (52) is a comonoid morphism (53) since this is precisely what is stated by the commuting Diagram (54). This concludes the proof. □

We consider below the category $\mathbb{L}^!$ of Eilenberg-Moore coalgebras equipped with the monoidal structure defined in Equation (42) of Section 6.9. After the last two Propositions 22 and 23, it follows from Corollary 15 that:

Proposition 24 *Let* $(\mathbb{L}, \otimes, 1)$ *be a linear category. The category* $\mathbb{L}^!$ *of Eilenberg-Moore coalgebras equipped with the monoidal structure inherited from* $(\mathbb{L}, \otimes, 1)$ *is cartesian.*

Proof. Together, Proposition 22 and Proposition 23 imply that in a linear category $\mathbb{L}$, every coalgebra

$$h_A : A \longrightarrow !A$$

defines a comonoid $(A, \mathbf{d}_A, \mathbf{e}_A)$ equipped with the morphisms:

$$\begin{array}{rcl} A \xrightarrow{\mathbf{d}_A} A \otimes A & = & A \xrightarrow{h_A} !A \xrightarrow{d_A} !A \otimes !A \xrightarrow{\epsilon_A \otimes \epsilon_A} A \otimes A \\ A \xrightarrow{\mathbf{e}_A} 1 & = & A \xrightarrow{h_A} !A \xrightarrow{e_A} 1 \end{array} \tag{57}$$

In order to apply Corollary 15 in Section 6.5, one needs to show that $(A, \mathbf{d}_A, \mathbf{e}_A)$ is not only a comonoid in the category $(\mathbb{L}, \otimes, 1)$, but also a comonoid in the category $(\mathbb{L}^!, \otimes, 1)$

of Eilenberg-Moore coalgebras. This is far from obvious, at least for the morphism $\mathbf{d}_A$, because the morphism $\epsilon_A \otimes \epsilon_A$ is not a coalgebra morphism in general. Establishing the property amounts to showing that the two diagrams

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[dd, "h_A"'] & !A \arrow[r, "d_A"] & !A \otimes !A \arrow[r, "\epsilon_A \otimes \epsilon_A"] & A \otimes A \arrow[d, "h_A \otimes h_A"] \\
 & & & !A \otimes !A \arrow[d, "m"] \\
!A \arrow[r, "!h_A"] & !!A \arrow[r, "!d_A"] & !(!A \otimes !A) \arrow[r, "!(\epsilon_A \otimes \epsilon_A)"] & !(A \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] & !A \arrow[r, "e_A"] & 1 \arrow[d, "m"] \\
!A \arrow[r, "!h_A"] & !!A \arrow[r, "!e_A"] & !1
\end{tikzcd}$$

commute in the category $\mathbb{L}$. This is achieved by the following diagram pasting.

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[dr, "h_A"] \arrow[ddd, "h_A"'] \arrow[dddr, phantom, "(b)"] & !A \arrow[r, "d_A"] & !A \otimes !A \arrow[r, "\epsilon_A \otimes \epsilon_A"] \arrow[dr, phantom, "(a)"] & A \otimes A \arrow[d, "h_A \otimes h_A"] \\
 & !A \arrow[r, "d_A"] \arrow[dd, "\delta_A"] \arrow[ddr, phantom, "(c)"] & !A \otimes !A \arrow[d, "\delta_A \otimes \delta_A"'] \arrow[r, phantom, "="] \arrow[dr, phantom, "(d)"] & !A \otimes !A \arrow[d, equal] \\
 & & !!A \otimes !!A \arrow[r, "!\epsilon_A \otimes !\epsilon_A"] \arrow[d, "m"'] \arrow[dr, phantom, "(e)"] & !A \otimes !A \arrow[d, "m"] \\
!A \arrow[r, "!h_A"] & !!A \arrow[r, "!d_A"] & !(!A \otimes !A) \arrow[r, "!(\epsilon_A \otimes \epsilon_A)"] & !(A \otimes A)
\end{tikzcd}$$

(a) Diagram (50),
(b) $h_A : A \longrightarrow !A$ is a coalgebra,
(c) d_A is a coalgebra morphism,
(d) $(!, \delta, \epsilon)$ is a comonad,
(e) naturality of m.

$$\begin{tikzcd}
A \arrow[r, "h_A"] \arrow[d, "h_A"'] \arrow[dr, phantom, "(a)"] & !A \arrow[r, "e_A"] \arrow[d, "\delta_A"] \arrow[dr, phantom, "(b)"] & 1 \arrow[d, "m"] \\
!A \arrow[r, "!h_A"'] & !!A \arrow[r, "!e_A"'] & !1
\end{tikzcd}$$

(a) $h_A : A \longrightarrow !A$ is a coalgebra,
(b) e_A is a coalgebra morphism.

This establishes that the triple $(A, \mathbf{d}_A, \mathbf{e}_A)$ defines a comonoid in the category $\mathbb{L}^!$. Now we need to prove that every coalgebra morphism

$$f \;:\; A \longrightarrow B$$

is at the same time a comonoid morphism

$$f \;:\; (A, \mathbf{d}_A, \mathbf{e}_A) \longrightarrow (B, \mathbf{d}_B, \mathbf{e}_B).$$

Consider the diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h_A"'] & B \arrow[d, "h_B"] \\
!A \arrow[r, "!f"] \arrow[d, "d_A"'] & !B \arrow[d, "d_B"] \\
!A \otimes !A \arrow[r, "!f \otimes !f"] \arrow[d, "\epsilon_A \otimes \epsilon_A"'] & !B \otimes !B \arrow[d, "\epsilon_B \otimes \epsilon_B"] \\
A \otimes A \arrow[r, "f \otimes f"'] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "h_A"'] & & B \arrow[d, "h_B"] \\
!A \arrow[rr, "!f"] \arrow[dr, "e_A"'] & & !B \arrow[dl, "e_B"] \\
& 1 &
\end{tikzcd}$$

The top squares commute because f is a coalgebra morphism, and the other cells commute by naturality of d and e. This establishes that every coalgebra morphism is a comonoid morphism, or equivalently, that

$$\mathbf{d}_A \quad : \quad A \longrightarrow A \otimes A \qquad\qquad \mathbf{e}_A \quad : \quad A \longrightarrow 1$$

are natural transformations in the category $\mathbb{L}^!$ of Eilenberg-Moore coalgebras.

It remains to show that the two natural transformations $\mathbf{d}$ and $\mathbf{e}$ are monoidal. In fact, in order to apply Corollary 15 in Section 6.5, we only need to check that $\mathbf{e}_1$

coincides with the identity, and that the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "h_A \otimes h_B"] \arrow[dddddd, swap, "id"] & !A \otimes !B \arrow[r, "d_A \otimes d_B"] \arrow[dddddd, swap, "m"] & (!A \otimes !A) \otimes (!B \otimes !B) \arrow[r, "(\epsilon \otimes \epsilon) \otimes (\epsilon \otimes \epsilon)"] \arrow[d, swap, "\alpha"] & (A \otimes A) \otimes (B \otimes B) \arrow[d, "\alpha"] \\
& & !A \otimes (!A \otimes (!B \otimes !B)) \arrow[d, swap, "!A \otimes \alpha^{-1}"] & A \otimes (A \otimes (B \otimes B)) \arrow[d, "A \otimes \alpha^{-1}"] \\
& & !A \otimes ((!A \otimes !B) \otimes !B) \arrow[d, swap, "!A \otimes (\gamma \otimes !B)"] & A \otimes ((A \otimes B) \otimes B) \arrow[d, "A \otimes (\gamma \otimes B)"] \\
& & !A \otimes ((!B \otimes !A) \otimes !B) \arrow[d, swap, "!A \otimes \alpha"] & A \otimes ((B \otimes A) \otimes B) \arrow[dd, "A \otimes \alpha"] \\
& & !A \otimes (!B \otimes (!A \otimes !B)) \arrow[d, swap, "\alpha^{-1}"] & \\
& & (!A \otimes !B) \otimes (!A \otimes !B) \arrow[d, swap, "m \otimes m"] \arrow[dr, "(\epsilon \otimes \epsilon) \otimes (\epsilon \otimes \epsilon)"] & A \otimes (B \otimes (A \otimes B)) \arrow[d, "\alpha^{-1}"] \\
A \otimes B \arrow[r, swap, "h_{A \otimes B}"] & !(A \otimes B) \arrow[r, swap, "d_{A \otimes B}"] & !(A \otimes B) \otimes !(A \otimes B) \arrow[r, swap, "\epsilon \otimes \epsilon"] & (A \otimes B) \otimes (A \otimes B)
\end{tikzcd}$$

commutes. In this diagram, the left rectangle commutes, by Definition (42) of the tensor product $h_{A \otimes B}$ of the two Eilenberg-Moore coalgebras h_A and h_B, the middle rectangle commutes by monoidality of d, the right trapezium by naturality of associativity α and symmetry γ, and the triangle by monoidality of ϵ.

Now, the morphism

$$\mathbf{e}_1 \quad = \quad 1 \xrightarrow{h_1} !1 \xrightarrow{e_1} 1$$

coincides with the identity because $h_1 = m^0$ by Equation (42) in Section 6.9, and because the equality

$$1 \xrightarrow{h_1} !1 \xrightarrow{e_1} 1 \quad = \quad 1 \xrightarrow{id} 1$$

follows by monoidality of e. This concludes the proof: the category $\mathbb{L}^!$ of Eilenberg-Moore coalgebras equipped with the monoidal structure of Equation (42) of Section 6.9 is cartesian. □

Proposition 25 *Every linear category defines a linear-non-linear adjunction, and thus a model of intuitionistic linear logic.*

7.5 Lafont-Seely categories

We introduce below a fourth axiomatization of intuitionistic linear logic as so-called *Lafont-Seely* categories, which cross-breeds Lafont categories and Seely categories. The axiomatization is designed to be general and easy to check on concrete models of linear logic. Of Lafont categories, Lafont-Seely categories retain the simplicity: unlike Seely categories and linear categories, the axiomatization does not require that the modality ! defines a comonad — a property which is sometimes difficult to check in full detail, for instance in game-theoretic models. Of Seely categories, Lafont-Seely categories retain the generality: unlike Lafont categories, the axiomatization is not limited to free exponential modalities.

A *Lafont-Seely category* is defined as a symmetric monoidal closed category $(\mathbb{L}, \otimes, 1)$ with finite products (noted $A\&B$ and $\top$) together with the following data:

1. for every object A, a commutative comonoid

$$!A \quad = \quad (!A, d_A, e_A)$$

   with respect to the tensor product, and a morphism

$$\epsilon_A \quad : \quad !A \longrightarrow A$$

   satisfying the following universal property: for every morphism

$$f \quad : \quad !A \longrightarrow B$$

   there exists a unique comonoid morphism

$$f^{\dagger} \quad : \quad (!A, d_A, e_A) \longrightarrow (!B, d_B, e_B)$$

   making the diagram

$$\begin{tikzcd}
 & !B \arrow[d, "\epsilon_B"] \\
!A \arrow[ur, "f^{\dagger}"] \arrow[r, "f"'] & B
\end{tikzcd}$$

   commute,

2. for every pair of objects A and B, two comonoid isomorphisms between the commutative comonoids:

$$p^2_{A,B} \;:\; (!A, d_A, e_A) \otimes (!B, d_B, e_B) \;\overset{\cong}{\longrightarrow}\; (!(A\&B), d_{A\&B}, e_{A\&B})$$

$$p^0 \;:\; (1, \rho_1^{-1} = \lambda_1^{-1}, id_1) \;\overset{\cong}{\longrightarrow}\; (!\top, d_\top, e_\top)$$

Just like in the case of Lafont categories, every Lafont-Seely category defines a symmetric monoidal adjunction

$$\begin{tikzcd}
(\mathbb{M}, \otimes, 1) \arrow[rr, bend left, "(L{,}m)"] \arrow[rr, phantom, "\perp"] & & (\mathbb{L}, \otimes, 1) \arrow[ll, bend left, "(M{,}n)"]
\end{tikzcd}$$

in which:

- $\mathbb{M}$ is the full subcategory of $\mathrm{CoMon}(\mathbb{L}, \otimes, 1)$ whose objects are the commutative comonoids isomorphic (as comonoids) to a commutative comonoid of the form $(!A, d_A, e_A)$.
- the functor L is the restriction of the forgetful functor U from the cartesian category $\mathrm{CoMon}(\mathbb{L}, \otimes, 1)$ of commutative comonoids to the underlying symmetric monoidal category $(\mathbb{L}, \otimes, 1)$.

In addition, it follows easily from Corollary 17 in Section 6.5 that the category $\mathbb{M}$ equipped with the tensor product $\otimes$ and the tensor unit 1 is cartesian. This establishes that:

Proposition 26 *Every Lafont-Seely category $\mathbb{L}$ induces a linear-non-linear adjunction, and thus a model of intuitionistic linear logic with additives.*

7.6 Notes and references

In his original formulation, Seely defines a *Girard category* as a $*$-autonomous category $(\mathbb{L}, \otimes, 1)$ with finite products, together with

1. a comonad $(!, \delta, \epsilon)$,
2. for every object A, a comonoid $(!A, d_A, e_A)$ with respect to the tensor product,
3. two natural isomorphisms

$$m^2_{A,B} \quad : \quad !A \otimes !B \cong !(A \& B) \qquad\qquad m^0 \quad : \quad 1 \cong !\top$$

which transport the comonoid structure (A, Δ_A, u_A) of the cartesian product to the comonoid structure $(!A, d_A, e_A)$ of the tensor product, in the sense that the diagrams

$$\begin{tikzcd}
& !A \otimes !A \arrow[dd, "m"] \\
!A \arrow[ur, "d_A"] \arrow[dr, "!\Delta_A"'] & \\
& !(A \& A)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& 1 \arrow[dd, "m"] \\
!A \arrow[ur, "e_A"] \arrow[dr, "!u_A"'] & \\
& !\top
\end{tikzcd}$$

commute.

In Seely's axiomatization, linear logic is explicitly reduced to a decomposition of intuitionistic logic. To quote Seely in [34]: "what is really wanted [of a model of intuitionistic linear logic] is that the kleisli category associated to [the comonad] $(!, \delta, \epsilon)$ be cartesian closed, so the question is: what is the minimal condition on $(!, \delta, \epsilon)$ that guarantees this — ie. can we axiomatize this condition satisfactorily?"

A few years later, Benton, Bierman, de Paiva and Hyland [6, 26] reconsidered Seely's axioms from the point of view of linear logic, instead of intuitionistic logic. Surprisingly, they discovered that something is missing in Seely's axiomatization. More precisely, Bierman points out in [8, 9] that the interpretation of proofs in a Seely category is not necessarily invariant under cut-elimination. One main reason is that the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "f"] & !A \arrow[r, "\delta_A"] \arrow[d, "d_A"'] & !!A \arrow[r, "!g"] & !B \arrow[d, "d_B"] & \\
& !A\otimes!A \arrow[r, "\delta_A\otimes\delta_A"] & !!A\otimes!!A \arrow[r, "!g\otimes!g"] & !B\otimes!B \arrow[r, "h"] & C
\end{tikzcd} \tag{58}$$

which interprets the duplication of a proof

$$!g \circ \delta_A :!A \longrightarrow !B$$

inside a proof

$$h \circ d_B \circ !g \circ \delta_A \circ f : \Gamma \longrightarrow !C$$

does not need to commute in Seely's axiomatization. Bierman suggests to call *new-Seely* category any Seely category in which the adjunction between the original category $\mathbb{L}$ and its co-kleisli category $\mathbb{L}_!$ is symmetric monoidal. This amounts precisely to our definition of Seely category in Section 7.3. In that case, the category provides invariants of proofs, see Proposition 21. In particular, Diagram (58) is shown to commute by pasting the two diagrams below:

$$\begin{tikzcd}
\Gamma \arrow[r, "f"] & !A \arrow[r, "\delta_A"] \arrow[d, "d_A"'] & !!A \arrow[r, "!g"] \arrow[d, "d_{!A}"'] & !B \arrow[d, "d_B"] & \\
& !A\otimes!A \arrow[r, "\delta_A\otimes\delta_A"] & !!A\otimes!!A \arrow[r, "!g\otimes!g"] & !B\otimes!B \arrow[r, "h"] & C
\end{tikzcd}$$

The definition of linear-non-linear adjunction was introduced by Benton in [7] after discussions with Martin Hyland and Gordon Plotkin. Interestingly, the original definition of Nick Benton requires that the category $\mathbb{M}$ is cartesian-closed. People realized only later that this additional condition is not necessary in order to establish soundness: a cartesian category $\mathbb{M}$ is sufficient for that purpose.

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