Measure, randomness and sublocales

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ABSTRACT

This paper investigates aspects of measure and randomness in the context of locale theory (point-free topology). We prove that every measure (σ-continuous valuation) μ, on the σ-frame of opens of a fitted σ-locale X, extends to a measure on the lattice of all σ-sublocales of X (Theorem 1). Furthermore, when μ is a finite measure with $\mu(X) = M$, the σ-locale X has a smallest σ-sublocale of measure M (Theorem 2). In particular, when μ is a probability measure, X has a smallest σ-sublocale of measure 1. All σ prefixes can be dropped from these statements whenever X is a strongly Lindelöf locale, as is the case in the following applications. When μ is the Lebesgue measure on the Euclidean space $\mathbb{R}^n$, Theorem 1 produces an isometry-invariant measure that, via the inclusion of the powerset $\mathcal{P}(\mathbb{R}^n)$ in the lattice of sublocales, assigns a weight to every subset of $\mathbb{R}^n$. (Contradiction is avoided because disjoint subsets need not be disjoint as sublocales.) When μ is the uniform probability measure on Cantor space $\{0, 1\}^\omega$, the smallest measure-1 sublocale, given by Theorem 2, provides a canonical *locale of random sequences*, where randomness means that *all* probabilistic laws (measure-1 properties) are satisfied.

1. Introduction

Assuming the Axiom of Choice (AC), it is not possible to define a nontrivial measure on all subsets of the Euclidean space $\mathbb{R}^n$ that is invariant under the Euclidean group of isometries. By partitioning the interval [0, 1] into countably many (essentially) intertranslatable pieces, Vitali [24] showed that, in dimension 1, the existence of such a measure is ruled out by countable additivity. In dimension 3, the Banach–Tarski theorem [4] provides a striking demonstration of the impossibility of finite additivity: a solid sphere can be decomposed into finitely many disjoint pieces, which can then be reassembled to produce two solid spheres, each congruent to the original. The price one pays for the existence of such examples is that, to work with measure consistently, one needs to define it to apply only to special subsets, declared *measurable*. The examples of Vitali and of Banach and Tarski make use of *nonmeasurable* sets, constructed using uncountable instances of AC. The application of AC in such examples is necessary. Solovay [22] showed that the measurability of every subset of $\mathbb{R}^n$ is consistent if AC is weakened to Dependent Choice (DC).[1] Nevertheless, ZFC remains the preferred metatheory of mathematics, and hence it is standard to restrict considerations of measure to measurable subsets.

This paper proposes a new way of approaching the problem of measuring subsets. Rather than restricting measure to special measurable subsets, we instead enlarge the collection of subsets by viewing the notion of subset as a special case of a more general notion of "part" of $\mathbb{R}^n$. Under our approach, it is possible in ZF+DC, hence in ZFC, to define a consistent measure on all "parts" of $\mathbb{R}^n$. In particular, every subset is assigned a measure. The usual contradictions are avoided by the following fact. The different pieces in the partitions defined by Vitali and by Banach and Tarski are deeply intertangled with each other. According to our notion of "part", two such intertangled pieces are not disjoint from each other, so additivity

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[1] Solovay's result assumes the consistency of ZFC + the existence of an inaccessible cardinal.

does not apply. An intuitive explanation for the failure of disjointness is that, although two such pieces share no point in common, they nevertheless overlap on the topological "glue" that bonds neighbouring points in $\mathbb{R}^n$ together.

Our approach would be of limited interest were our notion of "part" an unnatural one, manufactured specifically for the purpose of constructing a measure. But, in fact, our notion of "part" is a natural and established one. We view spaces of interest (such as $\mathbb{R}^n$) as *locales*, and the notion of "part" is given by the standard notion of *sublocale*, introduced by Isbell in [10]. Locales (essentially) generalise topological spaces, by replacing the lattice of open sets with an abstract lattice whose elements are called *opens*. There is no requirement that opens be sets of points, nor even that a nontrivial locale have any points at all. Every topological space determines a locale, by taking the open sets as the lattice of opens. However, when a space is viewed as a locale, the notion of *sublocale* gives rise to new "parts" of spaces that are not merely subsets, and need not be determined by their points. This fact was emphasised by Isbell in [10], where it is even advertised in the title.

For technical reasons, our results are developed for σ-*locales*, a generalisation of locales. We assume given a σ-locale X together with a measure[2] μ defined on the lattice $\mathcal{O}(X)$ of opens of X. Using the standard technique of outer measure, we extend the measure μ to a function μ^* on all sublocales. Our first main result, Theorem 1, states that, under a mild separation condition,[3] the function μ^* is a measure on the lattice $\mathcal{S}(X)$ of all σ-sublocales. In the case that X is strongly Lindelöf (see Definition 3.3), the σ prefixes can be dropped from this statement, and μ^* is a measure on the lattice of all sublocales. As Example 4.7, we show that Theorem 1 does indeed give rise to an isometry-invariant measure on the lattice of sublocales of the Euclidean space $\mathbb{R}^n$, and hence assigns values to all subsets.

The introduction of nonstandard "parts" of a space, not corresponding to subsets, constitutes a somewhat drastic revision of a fundamental notion. Nevertheless, it is not an unreasonable revision. Locales can be viewed as implementing a notion of topology where opens correspond to *observable properties* of spaces,[4] and sublocales can be understood intuitively as enforcing new relationships between such observable properties. Furthermore, the inclusion of new "parts" has positive mathematical consequences. One example of such a consequence is Isbell's observation that every locale has a smallest dense sublocale [10]. As the second contribution of this paper, we apply Theorem 1 to obtain another interesting consequence of the inclusion of new nonstandard "parts" of spaces. We show that adopting sublocale as the notion of "part" suggests a resolution of one of the conundrums of probability theory: the question of how to make sense of the notion of "randomness".

When one generates a random sequence $\alpha \in \mathbf{2}^\omega$ (where $\mathbf{2} = \{0, 1\}$), for example by tossing a fair coin *ad infinitum*, one expects α to satisfy any given probabilistic law. That is, given a subset $L \subseteq \mathbf{2}^\omega$ which has measure 1 in the uniform probability measure on $\mathbf{2}^\omega$, then, according to experience, one expects that α will be found inside of L. However, the family $\mathcal{L}$ of all measure 1 subsets of $\mathbf{2}^\omega$ has empty intersection, and so no sequence at all satisfies every probabilistic law. We call this fact the *paradox of randomness*.[5] It necessitates the introduction of the notion of "almost surely" in probability theory, and apparently prevents one from using the satisfaction of all probabilistic laws as the defining property of a canonical notion of "random sequence".

Of course, many consistent notions of random sequence have been proposed in the literature. The paradox is circumvented by defining a random sequence to be one that satisfies just those probabilistic laws contained in a chosen countable subfamily $\mathcal{L}' \subseteq \mathcal{L}$. The subfamily $\mathcal{L}'$ is typically specified using computability-theoretic (e.g., [17]) or logical (e.g., [22,18]) definability criteria. Since there are many (reasonable) possibilities for this, a spectrum of different notions of randomness arises. The area is now a mature one, see [19,8] for recent textbook surveys, and has important applications. Solovay's notion of randomness, based on definability in a model of set theory, was used to prove the result, referred to above, that it is consistent with ZF+DC that all subsets of $\mathbb{R}$ are Lebesgue measurable [22]. Computability-theoretic definitions of randomness provide a tool for classifying and comparing non-computable sequences [19,8]. Nevertheless, the fact that there is no single canonical choice of $\mathcal{L}'$ makes it debatable whether any of the competing theories provides a good abstraction for describing the fundamental stochastic notion of randomness from empirical experience.

The second contribution of the present paper is to provide a resolution of the paradox that does lead to a canonical notion of randomness. We avoid the arbitrariness in the choice of $\mathcal{L}'$, by retaining the idea that a random sequence should satisfy *all* measure 1 properties. No contradiction arises because we use this criterion to define a "random part" of $\mathbf{2}^\omega$ that is nontrivial even though it (necessarily) contains no individual sequence. Once again, this is achieved by interpreting $\mathbf{2}^\omega$ as a locale, and taking "part" to mean sublocale.

For the technical development, we assume a finite measure μ on the lattice $\mathcal{O}(X)$ of opens of a σ-locale X.[6] Let M be the finite value $\mu(X)$. By Theorem 1, μ extends to a measure μ^* on all σ-sublocales of X. Our second main result, Theorem 2, states that X has a smallest σ-sublocale of measure M under μ^*. We call this uniquely determined σ-sublocale $\mathbf{Ran}(\mu)$. In the case that μ is a probability measure, the σ-sublocale $\mathbf{Ran}(\mu)$ carves out the "part" of X given by the intersection (as a σ-sublocale) of all probabilistic laws. Since the σ-sublocale $\mathbf{Ran}(\mu)$ itself has measure 1, it is nontrivial, and one can informally think of all the probabilistic weight of X as being concentrated within $\mathbf{Ran}(\mu)$, which we therefore interpret as the σ-sublocale of "μ-random elements" of X. The motivating case of random sequences is incorporated by taking μ to be

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[2] By "measure", we mean σ-continuous valuation. This is the natural notion of measure in this context, see Section 2.

[3] The σ-locale X is assumed to be *fitted*, see Definition 4.3.

[4] See [1,21,23] for further elaboration of this view.

[5] This terminology is introduced for rhetorical effect in the present paper. Other "paradoxes of randomness" have been identified elsewhere, e.g., [6].

[6] Once again, all σ prefixes can be dropped when X is strongly Lindelöf.

the uniform probability measure on Cantor space $\mathbf{2}^\omega$, in which case the above prescription gives rise to a canonical *locale of random sequences*, **Ran**. This locale is discussed in detail as Example 6.5.

The paper is structured as follows. Section 2 recalls basic properties of measures (σ-continuous valuations) on lattices, and Section 3 reviews the theory of (σ-)locales insofar as we need it. Theorem 1, which asserts the existence of a measure on all σ-sublocales, is stated in Section 4, and its proof is given in Section 5. In Section 6, we prove Theorem 2, the existence random σ-sublocales, and explore some of its consequences. In Section 7, we discuss possible extensions of this work, including the question of obtaining constructive analogues of our results. The metatheory, for the present paper, is classical. Everything goes through comfortably within ZF+DC.

2. Measures on lattices

Throughout, we work with non-strict partial orders $(L, \leq)$, i.e., we assume reflexivity $x \leq x$. We write L^{op} for the opposite partial order $(L, \geq)$.

A *lattice* $(L, \leq)$ is a partially ordered set with finite joins (we write $\bot$ for the least element, and $x \vee y$ for binary join) and finite meets (we write $\top$ for the maximum element, and $x \wedge y$ for binary meet). We say that L is σ-*complete* if it has countable joins. We write $\bigvee_{i \geq 0} x_i$ for the join of a sequence $(x_i)_{i \geq 0}$. For L to be σ-complete, it suffices that $\bigvee_{i \geq 0} x_i$ exist whenever $(x_i)_{i \geq 0}$ is *ascending*, i.e., $i \leq j$ implies $x_i \leq x_j$. We say that L is *complete* if it has arbitrary joins, denoted $\bigvee$. If L is complete then it also has arbitrary meets, denoted $\bigwedge$.

A subset $D \subseteq L$ is said to be *directed* if it is nonempty, and, for all $x, y \in D$, there exists $z \in D$ with $x \leq z \geq y$. A subset $C \subseteq L$ is *filtered* if it is directed in L^{op}. A *filter* in L is a filtered subset $F \subseteq L$ that is also upward closed: if $x \in F$ and $x \leq y$ then $y \in F$. In a lattice, an upward closed subset is a filter if and only if it is closed under finite meets (and so, in particular, contains $\top$). For any $x \in L$, we write $\uparrow x$ for the *principal filter* $\{y \in L \mid x \leq y\}$. We write $\mathcal{F}il(L)$ for the set of filters in L.

A lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds for all x, y, z (equivalently if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for all x, y, z). Two elements x, y are *disjoint* in a lattice if $x \wedge y = \bot$. They are *complements* if they are disjoint and also $x \vee y = \top$. In a distributive lattice, every element x has at most one complement x^c. A distributive lattice in which every element has a complement is called a *Boolean algebra*.

A σ-*frame* is a σ-complete lattice satisfying the infinitary distributive law:

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i) \tag{1}$$

of finite meet over countable joins. The finite distributivity law is a special case, hence every σ-frame is distributive. A *frame* is a complete lattice satisfying Eq. (1) for arbitrary joins. A *(σ-)coframe* is a lattice L that is a (σ-)frame under its opposite order L^{op}.

A "measure" assigns a possibly infinite weight to each element of a lattice in a way compatible with the lattice structure. We use $[0, \infty]$ to denote the interval of non-negative real numbers together with infinity.

Definition 2.1. A *measure* on a σ-complete lattice L is a function $\mu : L \to [0, \infty]$ satisfying:

$$\mu(\bot) = 0 \tag{2}$$

$$x \leq y \implies \mu(x) \leq \mu(y) \tag{3}$$

$$\mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y) \tag{4}$$

$$(x_i)_{i \geq 0} \text{ ascending} \implies \mu\left(\bigvee_{i \geq 0} x_i \right) = \sup_{i \geq 0} \mu(x_i) \tag{5}$$

We say that μ is a *probability measure* if $\mu(\top) = 1$. It is *finite* if $\mu(\top) < \infty$. It is σ-*finite* if there exists a countable $(u_i)_{i \geq 0}$ with $\bigvee_{i \geq 0} u_i = \top$ and $\mu(u_i) < \infty$, for every i. Trivially, probability implies finite implies σ-finite.

Condition (4) in Definition 2.1 is called *modularity*, and condition (5) is called σ-*continuity*.

Our use of the term "measure" is nonstandard. Traditionally, our "measures" are called "σ-continuous valuations", and the term "measure" is reserved for the more restrictive context in which L is required to be a σ-complete Boolean algebra. Nevertheless, it is standard that σ-continuous valuations on a σ-complete Boolean algebra coincide with measures in the usual sense. Given this, our preference is to generalise the more concise "measure" terminology to the wider context. For the record, we briefly review, the connection between measures as we have defined them, and the traditional definition in terms of countable additivity. The routine proofs are omitted.

Definition 2.2. A function $\mu : L \to [0, \infty]$ is said to be *countably additive* if for any (possibly empty) countable family $\{x_i\}_{i \in I}$ of pairwise disjoint elements it holds that

$$\mu\left(\bigvee_{i \in I} x_i \right) = \sum_{i \in I} \mu(x_i).$$

Proposition 2.3. *If L is a distributive σ-complete lattice then any measure μ on L is countably additive.*

Proposition 2.4. *If L is a σ-complete Boolean algebra then $\mu : L \to [0, \infty]$ is a measure if and only if it is countably additive.*

In manipulating measures, we perform arithmetic in $[0, \infty]$. To avoid ambiguity, we adopt the convention that we only write a subtraction $e - e'$ in cases in which we know that $e' \leq e$ and $e' < \infty$.

Lemma 2.5. *If μ is a measure on a σ-complete lattice L then, for all x, y, y' with $y' \leq y$ and $\mu(y') < \infty$, we have:*

$$\mu(x \wedge y) - \mu(x \wedge y') \leq \mu(y) - \mu(y'). \tag{6}$$

Proof. We have:

$$\mu(y') + \mu(x \wedge y) = \mu(y' \vee (x \wedge y)) + \mu(y' \wedge x \wedge y) \leq \mu(y) + \mu(y' \wedge x),$$

where the equality is by (4), and the inequality by (3), using $y' \vee (x \wedge y) \leq y$ and $y' \wedge x \wedge y = x \wedge y'$. The required inequality (6), is just a rearrangement, valid because $\mu(y')$ and (hence) $\mu(x \wedge y')$ are finite. $\square$

A trivial reformulation of (6) is: if $x' \leq x, y' \leq y$ and $\mu(x \wedge y') < \infty$ then

$$\mu(x' \wedge y) - \mu(x' \wedge y') \leq \mu(x \wedge y) - \mu(x \wedge y'). \tag{7}$$

The next lemma bounds the difference in this inequality.

Lemma 2.6. *If μ is a measure on a σ-complete lattice L then, for all x, x', y, y' with $x' \leq x, y' \leq y, \mu(x') < \infty$ and $\mu(x \wedge y') < \infty$, we have:*

$$\mu(x \wedge y) - \mu(x \wedge y') \leq \mu(x' \wedge y) - \mu(x' \wedge y') + \mu(x) - \mu(x'). \tag{8}$$

Proof. We have:

$$\mu(x \wedge y) - \mu(x \wedge y') \leq \mu(x \wedge y) - \mu(x' \wedge y') \qquad \text{because } x' \leq x$$
$$\leq \mu(x' \wedge y) - \mu(x' \wedge y') + \mu(x) - \mu(x'),$$

where the second inequality is because

$$\mu(x \wedge y) \leq \mu(x' \wedge y) + \mu(x) - \mu(x'),$$

by Lemma 2.5. $\square$

Definition 2.7. We say that a measure μ on L is *stably σ-continuous* if it satisfies

$$(y_i)_{i \geq 0} \text{ ascending } \implies \mu \left(x \wedge \bigvee_{i \geq 0} y_i \right) = \sup_{i \geq 0} \mu(x \wedge y_i).$$

Lemma 2.8. *Any measure on a σ-frame is stably σ-continuous.*

Proof. Trivial, by (1) and (5). $\square$

Lemma 2.9. *Any finite measure μ on a σ-complete lattice is stably σ-continuous.*

Proof. Let $(y_i)_{i \geq 0}$ be ascending. We show that the equality below holds for an arbitrary measure μ with $\mu(\bigvee_{i \geq 0} y_i) < \infty$.

$$\mu \left(x \wedge \bigvee_{i \geq 0} y_i \right) = \sup_{i \geq 0} \mu(x \wedge y_i) \tag{9}$$

Only the $\leq$ inequality is in question. For any $\epsilon > 0$, there exists n such that $\mu(\bigvee_{i \geq 0} y_i) - \mu(y_n) < \epsilon$, by (5), since both terms are finite. Then,

$$\mu \left(x \wedge \bigvee_{i \geq 0} y_i \right) - \mu(x \wedge y_n) \leq \mu \left(\bigvee_{i \geq 0} y_i \right) - \mu(y_n) < \epsilon,$$

by Lemma 2.5. $\square$

Lemma 2.10. *Let μ be a measure on a σ-complete lattice L, and suppose there exists an ascending chain $(u_i)_{i \geq 0}$ in L satisfying:*

$$\mu(u_i) < \infty \tag{10}$$

$$\bigvee_{i \geq 0} x \wedge u_i = x \tag{11}$$

$$(y_j)_{j \geq 0} \text{ ascending } \implies u_i \wedge \bigvee_{j \geq 0} y_j = \bigvee_{j \geq 0} u_i \wedge y_j. \tag{12}$$

Then μ is stably σ-continuous.

Note that the conditions in the lemma imply that μ is σ-finite, and are satisfied by every finite measure.

Proof. We need to prove $\mu(x \wedge \bigvee_{i\geq 0} y_i) = \sup_{i\geq 0} \mu(x \wedge y_i)$. Note that (10) implies that $\mu(\bigvee_{i\geq 0} u_j \wedge y_i) \leq \mu(u_j) < \infty$. Then:

$$\mu\left(x \wedge \bigvee_{i\geq 0} y_i\right) = \mu\left(\bigvee_{j\geq 0}\left(u_j \wedge x \wedge \bigvee_{i\geq 0} y_i\right)\right) \qquad \text{by (11)}$$

$$= \sup_{j\geq 0} \mu\left(u_j \wedge x \wedge \bigvee_{i\geq 0} y_i\right) \qquad \text{by (5)}$$

$$= \sup_{j\geq 0} \mu\left(x \wedge \bigvee_{i\geq 0}(u_j \wedge y_i)\right) \qquad \text{by (12)}$$

$$= \sup_{j\geq 0}\sup_{i\geq 0} \mu(u_j \wedge x \wedge y_i) \qquad \text{by (9) since } \mu\left(\bigvee_{i\geq 0} u_j \wedge y_i\right) < \infty$$

$$= \sup_{i\geq 0} \mu\left(\bigvee_{j\geq 0}(u_j \wedge x \wedge y_i)\right) \qquad \text{by (5)}$$

$$= \sup_{i\geq 0} \mu(x \wedge y_i) \qquad \text{by (11)}. \quad \square$$

In the literature on valuations, see, e.g., [15,2,13], one often meets a stronger continuity requirement than σ-continuity.

Definition 2.11. If L is a complete lattice with measure μ, we say that μ is *continuous* if, for every directed $D \subseteq L$, it holds that $\mu(\bigvee D) = \sup_{x\in D} \mu(X)$.

The notion of continuous measure will not play any independent role in this paper. However, under an important condition, which will be useful later, full continuity is a consequence of σ-continuity.

Definition 2.12. We say that a σ-complete lattice L satisfies the *countable subcover property* if, for every subset $I \subseteq L$, there exists countable $J \subseteq I$ such that, for every $x \in I$, it holds that $x \leq \bigvee J$.

We omit the proof of the routine proposition below.

Proposition 2.13. *Let L be a σ-complete lattice with the countable subcover property.*

1. *L is a complete lattice.*
2. *Every measure on L is continuous.*

While full continuity is a distraction in this paper, the dual concept of *cocontinuity* will prove important.

Definition 2.14. If L is a complete lattice with measure μ, we say that μ is *cocontinuous on a filtered subset $C \subseteq L$* if it holds that $\mu(\bigwedge C) = \inf_{x\in C} \mu(x)$. We say that μ is *cocontinuous* if it is cocontinuous on every filtered subset.

The reason for introducing cocontinuity for a single C, is that we shall meet instances in which cocontinuity fails in general, but holds when restricted to filtered subsets C *of finite measure*, that is, C for which there exists $x \in C$ with $\mu(x) < \infty$. Note that C is of finite measure if and only if $\inf_{x\in C} \mu(x) < \infty$.

3. Locales and σ-locales

Definition 3.1. A *locale X* is given by a frame $\mathcal{O}(X)$ whose elements are called *opens*. A *continuous map $f : X \longrightarrow Y$* between locales is given by a function $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ that preserves finite meets and arbitrary joins.

The motivating examples of locales are topological spaces. Any topological space X determines a locale, where $\mathcal{O}(X)$ is defined to be the lattice of open subsets of X ordered by inclusion. If Y is also a topological spaces then any continuous function $f : X \to Y$ determines a continuous map of locales, because joins and finite meets of open sets are given by unions and finite intersections respectively, hence f^{-1} preserves them. In the case that Y is a sober space, the continuous functions from X to Y as spaces are in one-to-one correspondence with continuous maps from X to Y qua locales.

Because the investigations in this paper concern measures, which are defined using a σ-continuity condition that refers only to countable joins, it is natural to weaken the completeness requirement on $\mathcal{O}(X)$ to σ-completeness. (We discuss this weakening further in Section 7.)

Definition 3.2. A σ-*locale X* is given by a σ-frame $\mathcal{O}(X)$ whose elements are called *opens*. A σ-*continuous map $f : X \longrightarrow Y$* between σ-locales is given by a function $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ that preserves finite meets and countable joins.

Clearly every locale is a σ-locale and every continuous map of locales is a σ-continuous map of σ-locales. Also, any measurable space X determines a σ-locale, by defining $\mathcal{O}(X)$ to be the σ-algebra of measurable subsets of X. If Y is another measurable space then any measurable function $f : X \to Y$ determines a σ-continuous map of σ-locales. In the case that Y is "sober" in the sense of [3], measurable functions from X to Y are in one-to-one correspondence with σ-continuous maps. More generally, any σ-complete-Boolean algebra is a σ-locale, and any homomorphism of σ-complete-Boolean algebras is the inverse image function of a σ-continuous map of σ-locales.

In this paper, we shall work throughout with σ-locales, and we take care to apply the σ prefix when necessary. However, most concrete examples of σ-locales we shall encounter will in fact be locales, and so we shall drop the σ prefix when discussing them. In fact, such examples will always be instances of a general and common situation in which localic and σ-localic concepts coincide, described in Proposition 3.4 below.

Definition 3.3.

1. A *base* in a σ-locale X is given by a subset $B \subseteq \mathcal{O}(X)$, the *basic opens*, such that every $U \in \mathcal{O}(X)$ arises as a countable join of basic opens.
2. A σ-locale X is said to be *countably based* if it has a countable base B.
3. A σ-locale X is said to be *strongly Lindelöf* if $\mathcal{O}(X)$ satisfies the countable subcover property (Definition 2.12).

We omit the routine proof of the result below.

Proposition 3.4.

1. *Every countably based σ-locale X is strongly Lindelöf.*
2. *Every strongly Lindelöf σ-locale is a locale.*
3. *Every σ-continuous map between strongly Lindelöf locales is a continuous map of locales.*

As is standard with locales, we think of σ-locales as generalised spaces. Even though the σ-frame $\mathcal{O}(X)$ of opens of a σ-locale X is an abstract lattice, we shall adopt notation that encourages us to think about its elements as if X were an ordinary space and the opens of were ordinary subsets. We thus write $\emptyset$ for the least element of $\mathcal{O}(X)$, and use $\cup$ and $\bigcup$ for finite and countable joins, which we refer to as unions. Similarly, we write X for the top element in $\mathcal{O}(X)$, and use $\cap$ for finite meets, which we refer to as intersections.

Next we introduce the notion of σ-sublocale of X. This is our notion of "part" of X, which, as discussed in Section 1, plays a crucial role in this paper.

Definition 3.5. A σ-continuous map $f : X \longrightarrow Y$ between σ-locales is an *embedding* if the function f^{-1} is surjective.

The embeddings are exactly the regular monomorphisms (equalisers) in the category of σ-continuous maps between σ-locales. Regular subobjects of a σ-locale X provide a well-behaved notion of "part" of X, the σ-sublocales of X. Concretely, a σ-sublocale Y is conveniently presented as a congruence relation $=_Y$ on $\mathcal{O}(X)$, where the intuition is that $U =_Y V$ means that the opens U and V coincide on the part Y of X.

Definition 3.6. A σ-*sublocale* Y of a σ-locale X is presented by means of an equivalence relation $=_Y$ on $\mathcal{O}(X)$, satisfying the congruence properties:

$$U =_Y V \quad \text{and} \quad U' =_Y V' \quad \Longrightarrow \quad U \cap U' =_Y V \cap V$$
$$U_i =_Y V_i, \quad \text{for all } i \geq 0 \quad \Longrightarrow \quad \bigcup_{i \geq 0} U_i =_Y \bigcup_{i \geq 0} V_i.$$

The congruence $=_Y$ determines $\mathcal{O}(Y) := \mathcal{O}(X)/=_Y$, with a canonical embedding $e : Y \longrightarrow X$ defined by $e^{-1}(U) = [U]_Y$, where $[U]_Y$ is the equivalence class of U in $\mathcal{O}(X)/=_Y$. We write $Y \subseteq X$ to say that Y is a σ-sublocale of X. Also, given σ-sublocales Y and Z of X, we write $Y \subseteq Z$ if:

$$\text{for all } U, V \in \mathcal{O}(X), \quad U =_Z V \quad \Longrightarrow \quad U =_Y V.$$

We write $\mathcal{S}(X)$ for the set of σ-sublocales of X, partially ordered by $\subseteq$.

The next few results survey, without proof, various properties of σ-sublocales, all of which are standard. Corresponding results for sublocales can be found in, e.g., [7,10,11]. For the case of σ-(sub)locales, one can either adapt the proofs in [10] (which carry over), or refer to the more general treatment in [16], where σ-locales are subsumed as the $\kappa = \aleph_1$ (sic) instance of κ-locales (there treated purely algebraically as κ-frames).

Proposition 3.7. *For any σ-locale X, the partial order $\mathcal{S}(X)$ is a coframe, hence complete and distributive. The supremum $\bigvee_{i \in I} Y_i$ of a family of σ-sublocales is defined by:*

$$U =_{\bigvee_{i \in I} Y_i} V \quad \Leftrightarrow \quad \text{for all } i \in I, \quad U =_{Y_i} V,$$

i.e., the intersection of all $=_{Y_i}$ relations. The meet $=_{\bigcap_{i \in I} Y_i}$ is the smallest congruence relation that contains every $=_{Y_i}$.

In the proposition above we are tacitly introducing some of the notation we use for the lattice structure on σ-sublocales. In general, we write X for the top element of $\mathcal{S}(X)$, and $\bigcap$ for arbitrary meets. For joins in $\mathcal{S}(X)$, we write $\emptyset$ for the least element, $\cup$ for finite joins, and $\bigvee$ for infinite joins. The reason for the nonuniformity in the notation with respect to infinite meets and joins is that, because $\mathcal{S}$ is a coframe, we have the infinite distributivity law

$$Y \cup \bigcap_i Z_i = \bigcap_i Y \cup Z_i,$$

but not, in general, its dual (1). The rationale behind our use of set-theoretic notation for some of the lattice structure in $\mathcal{O}(X)$ and $\mathcal{S}(X)$ is to use it as a mnemonic for distributivity properties. We write $\bigcup$ for countable joins over which finite meets distribute, as happens in $\mathcal{O}(X)$. Likewise, we write $\bigcap$ for arbitrary meets over which finite joins distribute, as happens in $\mathcal{S}(X)$.[7]

Definition 3.8. Every $W \in \mathcal{O}(X)$ determines two associated σ-sublocales, denoted W and W^c, defined by:

$$U =_W V \iff U \cap W = V \cap W,$$
$$U =_{W^c} V \iff U \cup W = V \cup W.$$

A σ-sublocale $Y \subseteq X$ is said to be *open* (respectively *closed*) if it is equal to W (respectively W^c) for some (necessarily unique) $W \in \mathcal{O}(X)$.

Proposition 3.9.

1. *For any $W \in \mathcal{O}(X)$, the associated open and closed σ-sublocales, W and W^c, are complements in the lattice $\mathcal{S}(X)$.*
2. *The inclusion of opens as open σ-sublocales embeds $\mathcal{O}(X)$ in $\mathcal{S}(X)$ by a function that preserves countable joins and finite meets.*
3. *The function $W \mapsto W^c$ embeds $\mathcal{O}(X)^{op}$ in $\mathcal{S}(X)$ by a function that maps countable joins (respectively finite meets) in $\mathcal{O}(X)$ to countable meets (respectively finite joins) in $\mathcal{S}(X)$.*

The careful reader may be worried by the potential ambiguity arising from notational identification of the open $W \in \mathcal{O}(X)$ with the open σ-sublocale $W \in \mathcal{S}(X)$. Proposition 3.9.2 almost shows that this notation is harmless. There remains the niggle that one writes $\bigcup_i U_i$ for a countable join of opens in $\mathcal{O}(X)$. But, if one considers $\bigcup_i U_i$ as a σ-sublocale, the notation suggests, by our mnemonic convention, a distributivity property that is not immediately a consequence of the coframe structure of $\mathcal{S}(X)$. Nevertheless, by Proposition 3.10.2 below, this distributivity property does hold in $\mathcal{S}(X)$.

Proposition 3.10.
1. *For an open $U \in \mathcal{O}(X)$ and a sequence σ-sublocales $Y_i \in \mathcal{S}(X)$, it holds in $\mathcal{S}(X)$ that:*

$$U \cap \bigvee_{i \geq 0} Y_i = \bigvee_{i \geq 0} U \cap Y_i.$$

2. *For a σ-sublocale $Y \in \mathcal{S}(X)$ and a sequence of opens $U_i \in \mathcal{O}(X)$, it holds in $\mathcal{S}(X)$ that:*

$$Y \cap \bigvee_{i \geq 0} U_i = \bigvee_{i \geq 0} Y \cap U_i.$$

Accordingly, it is consistent with our mnemonic use of set-theoretic notation to write $\bigcup_i U_i$ for the countable join in $\mathcal{S}(X)$ of a sequence of open sublocales U_i. It follows also that finite meets distribute over the countable join of $Y \cap U_i$, so we may also write $\bigcup_i Y \cap U_i$ for such joins.

Proposition 3.11. *Let $f : X \longrightarrow Y$ be a σ-continuous map between σ-locales. There exists a unique function $f^{-1} : \mathcal{S}(Y) \to \mathcal{S}(X)$ that preserves arbitrary meets and finite joins, and also maps open sublocales in Y to their inverse images under $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$.*

In other words, inverse-image functions on opens extend to inverse-image functions on arbitrary σ-sublocales. (To define $f^{-1} : \mathcal{S}(Y) \to \mathcal{S}(X)$ concretely, for $Z \in \mathcal{S}(Y)$, define $=_{f^{-1}(Z)}$ to be the finest congruence relation on $\mathcal{O}(X)$ for which $f^{-1}(U) =_{f^{-1}(Z)} f^{-1}(V)$ whenever $U =_Z V$.)

Proposition 3.12. *Let Y be a σ-sublocale of X.*

1. *The function*

$$[U]_Y \mapsto Y \cap U : \mathcal{O}(Y) \to \mathcal{S}(X),$$

preserves countable joins, finite nonempty meets and is one-to-one (hence order reflecting).

---

[7] Our notation has one possibly misleading feature: it is joins in $\mathcal{S}(X)$ that correspond to set-theoretic unions of "points" in X, whereas meets in $\mathcal{S}(X)$ do not correspond to intersections of "points". Nevertheless, the mnemonic device of using set-theoretic notation to indicate distributivity properties seems helpful enough to outweigh such concerns. (There are also deeper reasons why it is reasonable to view meets of sublocales as intersections.)

2. *Given $Z \in \mathcal{S}(Y)$, define $Z_X \in \mathcal{S}(X)$ by:*

$$U =_{Z_X} V \iff [U]_Y =_Z [V]_Y.$$

Then the function

$$Z \mapsto Z_X : \mathcal{S}(Y) \to \mathcal{S}(X)$$

preserves finite joins, nonempty meets, is one-to-one, and its image is the down-closure of Y in $\mathcal{S}(X)$.

Statement 1 says that the subset $\{Y \cap U \mid U \in \mathcal{O}(X)\} \subseteq \mathcal{S}(X)$ is isomorphic to $\mathcal{O}(Y)$. Thus, the relation $U =_Y V$ is equivalent to the equality $Y \cap U = Y \cap V$ in $\mathcal{S}(X)$. This substantiates the intuition we gave for $=_Y$ before Definition 3.6. Statement 2 implements the transitivity of the sublocale relationship. In particular, we have that $\mathcal{S}(Y)$ is isomorphic to $\{Z \in \mathcal{S}(X) \mid Z \subseteq Y\}$.

We conclude our discussion of σ-sublocales, by comparing with ordinary sublocales of a locale. Without reviewing the definition of sublocale (see, e.g., [10,11]), we assert:

Proposition 3.13. *If X is a strongly Lindelöf σ-locale, then the σ-sublocales of X are exactly the sublocales of X qua locale. Thus $\mathcal{S}(X)$, as defined above, is the lattice of sublocales of X.*

Next, we recall some separation properties associated with σ-locales. For these, we give definitions that are appropriate to σ-locales (rather than to locales). In each case, it holds that the definition given below coincides with the standard definition for locales in the case of strongly Lindelöf σ-locales.

Definition 3.14. Let X be a σ-locale.

1. X is *zero dimensional* if it has a base of clopens. (A *clopen* is an open that has a complement in $\mathcal{O}(X)$, equivalently an open whose associated open sublocale is also closed.)
2. X is *regular* if every $U \in \mathcal{O}(X)$ arises as a countable join of opens from the set

$$\{V \in \mathcal{O}(X) \mid \text{there exists closed } C \text{ with } V \subseteq C \subseteq U\}.$$

Finally, we recall a concept that, in this paper, will prove useful mainly by its absence. Write 1 for the *terminal* locale: $\mathcal{O}(1) = \{\emptyset, \{*\}\}$ ordered by inclusion.

Definition 3.15. A *point* in a σ-locale X is a σ-continuous map from 1 to X.

4. Measures on σ-sublocales

By a *measure* on a σ-locale X, we mean a measure μ on the σ-frame $\mathcal{O}(X)$. We use such a measure to define an "outer measure" μ^* on the lattice $\mathcal{S}$ of σ-sublocales.

Definition 4.1. For a σ-sublocale $Y \in \mathcal{S}(X)$, the *open neighbourhood filter* $\mathcal{N}(Y)$ is the filter on $\mathcal{O}(X)$ defined by:

$$\mathcal{N}(Y) = \{U \in \mathcal{O}(X) \mid Y \subseteq U\}.$$

Let μ be a measure on a σ-locale X. Define a function $\mu^* : \mathcal{S}(X) \to [0, \infty]$ by:

$$\mu^*(Y) = \inf_{U \in \mathcal{N}(Y)} \mu(U).$$

We call μ^* the *outer measure* associated with μ.

In measure theory, there is a general notion of outer measure in which only the $\geq$ direction of the modularity equation (4) is required to hold. It is not hard to demonstrate that μ^* is always an outer measure in this general sense. Rather than pursuing this direction, we proceed directly to a more interesting result: under a mild separation condition on X, the outer measure μ^* is in fact a measure (Theorem 1 below).

Proposition 4.2. *The following conditions on a σ-locale X are equivalent.*

1. *For every sublocale $Y \subseteq X$, it holds that $Y = \bigcap \mathcal{N}(Y)$ in $\mathcal{S}(X)$.*
2. *For every closed sublocale $Y \subseteq X$, it holds that $Y = \bigcap \mathcal{N}(Y)$ in $\mathcal{S}(X)$.*

Moreover, these conditions hold if X is regular (Definition 3.14.2).

Definition 4.3. We say that a σ-locale X is *fitted* [10] if either of the equivalent conditions of Proposition 4.2 holds.

Theorem 1. *Let μ be a measure on a fitted σ-locale X.*

1. *The outer measure μ^* is a measure on $\mathcal{S}(X)$.*
2. *μ^* is cocontinuous on finite-measure filtered subsets of $\mathcal{S}(X)$.*
3. *If μ is σ-finite then μ^* is stably σ-continuous.*

We observe also that if μ is σ-finite then so is μ^*. Similarly, μ^* is finite when μ is. These statements are not included in the theorem since trivial.

The proof of Theorem 1 is given in Section 5 below. Before-hand, we present a series of examples clarifying the theorem. The first three show that none of the conditions in the theorem can be dropped. The fourth considers the motivating example of the Lebesgue measure on Euclidean space, discussed in Section 1. In all examples, the σ-locales considered are countably based, hence we freely adopt localic rather than σ-localic terminology.

Example 4.4. The outer measure μ^* can fail to be a measure for non-fitted σ-locales. Let S be the Sierpinski locale, where $\mathcal{O}(\mathsf{S}) = \{\emptyset, \{1\}, \{0, 1\}\}$ ordered by inclusion $\subseteq$. (These are the open sets of the Sierpinski topology on $\{0, 1\}$.) Define μ by $\mu(\emptyset) = 0$ and $\mu(\{1\}) = \mu(\{0, 1\}) = 1$. The lattice $\mathcal{S}(\mathsf{S})$ is isomorphic to $(\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}, \subseteq)$, and we have $\mu^*(\{0\}) + \mu^*(\{1\}) = 1 + 1 = 2$. But $\mu^*(\{0, 1\}) + \mu^*(\emptyset) = 1$. So the $\leq$ direction of the modularity equation (4) fails. (In Section 7, we revisit the case of non-fitted σ-locales.)

Example 4.5. Cocontinuity can fail for filtered subsets that do not have finite measure. Let N be the locale of discrete natural numbers, where $\mathcal{O}(\mathsf{N})$ is the powerset $\mathcal{P}(\mathbb{N})$. Define μ to be the cardinality function $\mu(U) = |U|$. Then μ is a σ-finite measure on N. Consider the filter $\mathcal{F}$ of cofinite subsets of $\mathbb{N}$ (i.e., sets with finite complement). This does not have finite measure, so it holds that $\inf\{\mu^*(X) \mid X \in \mathcal{F}\} = \infty$. But $\mu^*(\bigcap \mathcal{F}) = \mu^*(\emptyset) = 0$.

Example 4.6. Stable σ-continuity can fail for non-σ-finite measures. Let Q be the locale of rational numbers, where $\mathcal{O}(\mathsf{Q})$ is the lattice of open sets of the subspace $\mathbb{Q}$ of the Euclidean space $\mathbb{R}$. Define μ by: $\mu(U) = \infty$ if $U \neq \emptyset$, and $\mu(\emptyset) = 0$. This is a measure, but not σ-finite. Let D be the smallest dense sublocale of Q [10]. For $i \geq 0$, let Y_i be the smallest sublocale containing the points $\{q_1, \ldots, q_i\}$, where $(q_i)_{i \geq 1}$ is an enumeration of the rationals. It then holds that $\mu^*(D \cap \bigvee_{i \geq 0} Y_i) = \mu^*(D \cap \mathsf{Q}) = \mu^*(D) = \infty$. But $\sup_{i \geq 0} \mu^*(D \cap Y_i) = \sup_{i \geq 0} \mu^*(\emptyset) = 0$.

Example 4.7. Let R^n be the locale for the n-dimensional Euclidean space. Thus $\mathcal{O}(\mathsf{R}^n)$ is the lattice of open subsets of $\mathbb{R}^n$. Let λ_n be the n-dimensional Lebesgue measure restricted to $\mathcal{O}(\mathsf{R}^n)$. Theorem 1 extends λ_n to a measure λ_n^* on $\mathcal{S}(\mathsf{R}^n)$. We show that this measure is invariant under Euclidean transformations of $\mathbb{R}^n$. Let t be an element of the Euclidean group on $\mathbb{R}^n$. Then t is continuous and preserves measure of opens, that is $\lambda_n(t^{-1}(U)) = \lambda_n(U)$. Consider the extension of the inverse-image function t^{-1} to the lattice $\mathcal{S}(\mathsf{R}^n)$ (as in Proposition 3.11). Then, for any sublocale $Y \subseteq \mathsf{R}^n$, we have that

$$\lambda_n^*(Y) = \inf_{U \in \mathcal{N}(Y)} \lambda_n(U)$$

$$= \inf_{U \in \mathcal{N}(Y)} \lambda_n(t^{-1}(U)) \qquad \text{measure-preservation on opens}$$

$$\geq \lambda_n^*\left(\bigcap_{U \in \mathcal{N}(Y)} t^{-1}(U)\right) \qquad \text{monotonicity of } \lambda_n^*$$

$$= \lambda_n^*\left(t^{-1}\left(\bigcap \mathcal{N}(Y)\right)\right) \qquad t^{-1} \text{ preserves } \bigcap$$

$$= \lambda_n^*(t^{-1}(Y)) \qquad \mathsf{R}^n \text{ regular hence fitted.}$$

Thus, writing s for the inverse of t, also $\lambda_n^*(t^{-1}(Y)) \geq \lambda_n^*(s^{-1}(t^{-1}(Y))) = \lambda_n^*(Y)$. That is, $\lambda_n^*(Y) = \lambda_n^*(t^{-1}(Y))$, establishing invariance.

The function that maps any subset $Z \subseteq \mathbb{R}^n$ to the smallest sublocale of R^n containing every point in Z embeds the powerset $\mathcal{P}(\mathbb{R}^n)$ in $\mathcal{S}(\mathsf{R}^n)$. This embedding maps unions in $\mathcal{P}(\mathbb{R}^n)$ to joins in $\mathcal{S}(\mathsf{R}^n)$, but does not preserve finite intersections. Via the embedding, the measure λ_n^* assigns a value $\lambda_n^*(Z)$ to every subset Z of $\mathbb{R}^n$. When Z is Lebesgue measurable, the value $\lambda_n^*(Z)$ agrees with the usual measure (because the Lebesgue measure is outer regular). We thus have a measure on sublocales, which is preserved under isometries, assigns values to all subsets of $\mathbb{R}^n$, and agrees with the Lebesgue measure on measurable subsets. There is no contradiction because disjointness of sublocales is a stronger property than disjointness of sets. In particular, when disjoint subsets are sufficiently intertangled, they are not disjoint as sublocales. For example, $\mathbb{Q}^n \cap (\mathbb{R}^n - \mathbb{Q}^n)$ is empty in $\mathcal{P}(\mathbb{R}^n)$, but not in $\mathcal{S}(\mathsf{R}^n)$, cf. [10]. The same situation arises with a Banach–Tarski decomposition of a solid ball.

We end this section with a simple consequence of Theorem 1, whose straightforward proof we omit. Let Y be a σ-sublocale of X. Recall, from Proposition 3.12, that $\mathcal{O}(Y)$ is isomorphic to $\{Y \cap U \in \mathcal{S}(X) \mid U \in \mathcal{O}(X)\}$ and $\mathcal{S}(Y)$ is isomorphic to $\{Z \in \mathcal{S}(X) \mid Z \subseteq Y\}$.

Corollary 1. *Let μ be a measure on a fitted σ-locale X. Let Y be a σ-sublocale of X. We write $\mu\restriction_Y$ for the restriction of μ^* to $\{Y \cap U \mid U \in \mathcal{O}(X)\}$, and we write $\mu^*\restriction_Y$ for the restriction of μ^* to $\{Z \in \mathcal{S}(X) \mid Z \subseteq Y\}$. Then $\mu\restriction_Y$ is a measure on Y, and $\mu^*\restriction_Y$ is the induced measure on $\mathcal{S}(Y)$.*

5. Proof of Theorem 1

This entire section, which is purely technical, is devoted to the proof of Theorem 1. Readers who are mainly interested in the high-level development are recommended to skip to Section 6.

Throughout this section, let X be a fixed σ-locale, and μ a measure on X. The assumption that X is fitted will be introduced only when needed.

In order to show that μ^* is a measure on $\mathcal{S}(X)$, it would be relatively straightforward to give direct proofs of σ-continuity (5) and of the $\geq$ inequality of the modularity equation (4). (Conditions (2) and (3) are trivial.) This is to be expected, since these are the usual properties enjoyed by outer measures. However, the $\leq$ inequality of (4) presents more of a challenge. To address this, we structure the proof as follows. First, we define an outer measure μ° on the lattice $\mathcal{F}il(\mathcal{O}(X))$ of filters of opens, ordered by reverse inclusion. The easy description of the lattice structure on $\mathcal{F}il(\mathcal{O}(X))$ means that it is not hard to show that μ° is a measure (Proposition 5.2). The idea is then to use the measure properties of μ° on $\mathcal{F}il(\mathcal{O}(X))$ to show that μ^* on $\mathcal{S}(X)$ is a measure, using the function $\bigcap : \mathcal{F}il(\mathcal{O}(X)) \to \mathcal{S}(X)$, which is surjective when X is fitted. The crux of the proof is to show that, when $\mu^\circ(\mathcal{F})$ is finite, for $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$, it holds that $\mu^*(\bigcap \mathcal{F}) = \mu^\circ(\mathcal{F})$ (Proposition 5.6.2). This is proved using Proposition 5.5, which defines a σ-sublocale $r(\mathcal{F}) \subseteq \bigcap \mathcal{F}$ whose outer measure satisfies $\mu^*(r(\mathcal{F})) = \mu^\circ(\mathcal{F})$ by construction.

We now proceed with the details. As stated above, we order the set $\mathcal{F}il(\mathcal{O}(X))$ of filters of opens by reverse inclusion $\supseteq$. We omit the proof of the next result, which is standard, cf. [11, Corollary 2.11].

Proposition 5.1. $(\mathcal{F}il(\mathcal{O}(X)), \supseteq)$ *is a coframe with joins given by*

$$\bigvee_{i \in I} \mathcal{F}_i = \bigcap_{i \in I} \mathcal{F}_i,$$

and meets given by

$$\bigwedge_{i \in I} \mathcal{F}_i = \{U_1 \cap \cdots \cap U_n \mid n \geq 0, \forall j \in \{1, \ldots, n\}. \exists i \in I. U_j \in \mathcal{F}_i\}.$$

Note that we use the neutral lattice notation of $\bigwedge$ and $\bigvee$ for meets and joins in $\mathcal{F}il(\mathcal{O}(X))$, irrespective of distributivity properties. This is for two reasons: first, we do not think of filters of opens as "parts" of X, so set-theoretic notation would be misleading; and, second, it helps readability to have a notational distinction between operations in $\mathcal{F}il(\mathcal{O}(X))$ and in $\mathcal{S}(X)$.

We define an outer measure $\mu^\circ : \mathcal{F}il(\mathcal{O}(X)) \to [0, \infty]$, analogously to the definition of μ^* on $\mathcal{S}(X)$:

$$\mu^\circ(\mathcal{F}) = \inf_{U \in \mathcal{F}} \mu(U).$$

Note that, by definition of μ^*, we have that $\mu^*(Y) = \mu^\circ(\mathcal{N}(Y))$.

Proposition 5.2. *The function μ° is a cocontinuous measure on $\mathcal{F}il(\mathcal{O}(X))$.*

Proof. Conditions (2) and (3) are trivial.

For modularity (4), we verify the more difficult $\leq$ inequality:

$$\mu^\circ(\mathcal{F}) + \mu^\circ(\mathcal{G}) \leq \mu^\circ(\mathcal{F} \vee \mathcal{G}) + \mu^\circ(\mathcal{F} \wedge \mathcal{G}).$$

This is only in question if $\mu^\circ(\mathcal{F} \vee \mathcal{G}) < \infty$, in which case all terms are finite. So, for any $\epsilon > 0$, there exist $W \in \mathcal{F} \vee \mathcal{G}$ and $Z \in \mathcal{F} \wedge \mathcal{G}$ with

$$\mu(W) < \mu^\circ(\mathcal{F} \vee \mathcal{G}) + \epsilon/2 \tag{13}$$
$$\mu(Z) < \mu^\circ(\mathcal{F} \wedge \mathcal{G}) + \epsilon/2. \tag{14}$$

By the definition of the lattice structure on $\mathcal{F}il(\mathcal{O}(X))$, we have that $W \in \mathcal{F} \cap \mathcal{G}$, and $Z = U \cap V$ for some $U \in \mathcal{F}$ and $V \in \mathcal{G}$. Thus $W \cap U \in \mathcal{F}$ and $W \cap V \in \mathcal{G}$. Whence:

$$\begin{aligned}
\mu^\circ(\mathcal{F}) + \mu^\circ(\mathcal{G}) &\leq \mu(W \cap U) + \mu(W \cap V) && \text{by definition of } \mu^\circ \\
&= \mu(W \cap (U \cup V)) + \mu(W \cap U \cap V) && \text{by (4) for } \mu \\
&\leq \mu(W) + \mu(Z) && \text{by (3), since } Z = U \cap V \\
&< \mu^\circ(\mathcal{F} \vee \mathcal{G}) + \mu^\circ(\mathcal{F} \wedge \mathcal{G}) + \epsilon && \text{by (13) and (14).}
\end{aligned}$$

For σ-continuity (5), let $(\mathcal{F}_i)_{i \geq 0}$ be an ascending chain of filters. We need to verify that

$$\mu^\circ\left(\bigvee_{i \geq 0} \mathcal{F}_i\right) \leq \sup_{i \geq 0} \mu^\circ(\mathcal{F}_i). \tag{15}$$

Again, this is only in question if $\sup_{i \geq 0} \mu^\circ(\mathcal{F}_i)$ is finite, in which case every $\mu^\circ(\mathcal{F}_i)$ is finite. Take any $\epsilon > 0$. For every $i \geq 0$, there exists $U_i \in \mathcal{F}_i$ with:

$$\mu(U_i) < \mu^\circ(\mathcal{F}_i) + 2^{-(i+1)}\epsilon. \tag{16}$$

We show, by induction on $n \geq 0$, that

$$\mu \left(\bigcup_{0 \leq i \leq n} U_i \right) < \mu^\circ(\mathcal{F}_n) + (1 - 2^{-(n+1)})\epsilon. \tag{17}$$

When $n = 0$ this is just (16) in the case $i = 0$. If $n > 0$, then we have

$$\mu \left(\bigcup_{0 \leq i \leq n} U_i \right) = \mu \left(\left(\bigcup_{0 \leq i \leq n-1} U_i \right) \cup U_n \right)$$

$$= \mu \left(\bigcup_{0 \leq i \leq n-1} U_i \right) + \mu(U_n) - \mu \left(\left(\bigcup_{0 \leq i \leq n-1} U_i \right) \cap U_n \right) \tag{18}$$

$$< \mu^\circ(\mathcal{F}_{n-1}) + (1 - 2^{-n})\epsilon + \mu^\circ(\mathcal{F}_n) + 2^{-(n+1)}\epsilon - \mu^\circ(\mathcal{F}_{n-1}) \tag{19}$$

$$= \mu^\circ(\mathcal{F}_n) + (1 - 2^{-(n+1)})\epsilon.$$

Here, Eq. (18) is by the modularity (4) of μ. The inequality (19) is derived using: the induction hypothesis, inequality (16) in the case $i = n$, and the inequality

$$\mu^\circ(\mathcal{F}_{n-1}) \leq \mu \left(\left(\bigcup_{0 \leq i \leq n-1} U_i \right) \cap U_n \right),$$

which holds because $\mu(\bigcup_{0 \leq i \leq n-1} U_i) \cap U_n \in \mathcal{F}_{n-1}$, since $U_{n-1} \in \mathcal{F}_{n-1}$, and $U_n \in \mathcal{F}_n \subseteq \mathcal{F}_{n-1}$. This completes the proof of (17).

To finish the proof of (15), we observe that $\bigcup_{i \geq 0} U_i \in \mathcal{F}_n$, for every n. Hence $\bigcup_{i \geq 0} U_i \in \bigvee_{i \geq 0} \mathcal{F}_i$. Thus:

$$\mu^\circ \left(\bigvee_{i \geq 0} \mathcal{F}_i \right) \leq \mu \left(\bigvee_{i \geq 0} U_i \right)$$

$$= \sup_{n \geq 0} \mu \left(\bigvee_{0 \leq i \leq n} U_i \right) \qquad \text{by (5) for } \mu$$

$$\leq \sup_{n \geq 0} (\mu^\circ(\mathcal{F}_n) + (1 - 2^{-(n+1)})\epsilon) \qquad \text{by (17)}$$

$$\leq \sup_{i \geq 0} \mu^\circ(\mathcal{F}_i) + \epsilon.$$

This establishes the σ-continuity property (5) of μ°, and so completes the proof that μ° is a measure.

For cocontinuity, let $\mathcal{C}$ be a filtered subset of $\mathcal{F}il(\mathcal{O}(X))$. We need to show that:

$$\inf\{\mu^\circ(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\} \leq \mu^\circ \left(\bigwedge \mathcal{C} \right).$$

Suppose that $\mu^\circ(\bigwedge \mathcal{C})$ is finite. Take any $\epsilon > 0$. There exists $W \in \bigwedge \mathcal{C}$ with

$$\mu(W) < \mu^\circ \left(\bigwedge \mathcal{C} \right) + \epsilon.$$

Then $W = U_1 \cap \cdots \cap U_n$ where, for each i with $1 \leq i \leq n$, we have $U_i \in \mathcal{F}_i$ for some $\mathcal{F}_i \in \mathcal{C}$. Since $\mathcal{C}$ is filtered, there exists $\mathcal{G} \in \mathcal{C}$ such that $\mathcal{G} \supseteq \mathcal{F}_i$ and hence $U_i \in \mathcal{G}$, for each i with $1 \leq i \leq n$. It follows that $W \in \mathcal{G}$. Thus we have:

$$\inf\{\mu^\circ(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C}\} \leq \mu^\circ(\mathcal{G}) \leq \mu(W) < \mu^\circ \left(\bigwedge \mathcal{C} \right) + \epsilon.$$

This completes the proof. □

Example 5.3. By Lemma 2.9, when μ is finite, μ° is stably σ-continuous. However, stable σ-continuity can fail for σ-finite μ. Let μ be the σ-finite cardinality measure on the locale of discrete natural numbers N, from Example 4.5. Let $\mathcal{F}$ be the filter of cofinite subsets of $\mathbb{N}$, and let $\mathcal{G}_i$ be the filter $\uparrow\{0, \ldots, i\}$. Then $\sup_i \mu^\circ(\mathcal{F} \wedge \mathcal{G}_i) = \sup_i \mu^\circ(\uparrow \emptyset) = 0$, but $\mu^\circ(\mathcal{F} \wedge \bigvee_{i \geq 0} \mathcal{G}_i) = \mu^\circ(\mathcal{F} \wedge \{\mathbb{N}\}) = \mu^\circ(\mathcal{F}) = \infty$.

Given finite-measure $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$, define a relation $=_{r(\mathcal{F})}$ on $\mathcal{O}(X)$ by:

$$U =_{r(\mathcal{F})} V \iff \forall \epsilon > 0. \exists W \in \mathcal{F}. \mu(W) < \infty \quad \text{and}$$

$$\mu(W \cap U) - \mu(W \cap U \cap V) < \epsilon \quad \text{and} \tag{20}$$

$$\mu(W \cap V) - \mu(W \cap U \cap V) < \epsilon. \tag{21}$$

Lemma 5.4. *If $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$ has finite measure then:*

$$U =_{r(\mathcal{F})} V \quad \text{iff} \quad \forall W \in \mathcal{F}. \ \mu(W) < \infty \ \text{implies}$$
$$(\mu(W \cap U) - \mu(W \cap U \cap V) \le \mu(W) - \mu^\circ(\mathcal{F}) \quad \text{and} \tag{22}$$
$$\mu(W \cap V) - \mu(W \cap U \cap V) \le \mu(W) - \mu^\circ(\mathcal{F})). \tag{23}$$

Proof. For the right-to-left implication, take any $\epsilon > 0$. Since $\mathcal{F}$ has finite measure, there exists $W \in \mathcal{F}$ such that $\mu(W) - \mu^\circ(\mathcal{F}) < \epsilon$. Inequations (20) and (21) then follow from (22) and (23).

For the converse, suppose $U =_{r(\mathcal{F})} V$. Take any $W \in \mathcal{F}$ with $\mu(W) < \infty$. We show that (22) holds. Take any $\epsilon > 0$. Because $U =_{r(\mathcal{F})} V$, there exists $W' \in \mathcal{F}$ with $\mu(W) < \infty$ such that

$$\mu(W' \cap U) - \mu(W' \cap U \cap V) < \epsilon.$$

By Lemma 2.5, we have:

$$\mu(W \cap W' \cap U) - \mu(W \cap W' \cap U \cap V) < \epsilon.$$

Whence, by Lemma 2.6:

$$\mu(W \cap U) - \mu(W \cap U \cap V) < \epsilon + \mu(W) - \mu(W \cap W').$$

But $W \cap W' \in \mathcal{F}$, so $\mu(W \cap W') \ge \mu^\circ(\mathcal{F})$. Thus

$$\mu(W \cap U) - \mu(W \cap U \cap V) < \mu(W) - \mu^\circ(\mathcal{F}) + \epsilon,$$

as required. □

Proposition 5.5. *Suppose $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$ has finite measure. Then $=_{r(\mathcal{F})}$ is a congruence relation on $\mathcal{O}(X)$. The associated sublocale $r(\mathcal{F})$ satisfies $r(\mathcal{F}) \subseteq \bigcap \mathcal{F}$ and $\mu^*(r(\mathcal{F})) = \mu^\circ(\mathcal{F})$.*

Proof. The relation $=_{r(\mathcal{F})}$ is trivially reflexive and symmetric. For transitivity, suppose $U =_{r(\mathcal{F})} V =_{r(\mathcal{F})} W$. Take any $\epsilon > 0$. Let $Y \in \mathcal{F}$ be such that $\mu(Y) < \mu^\circ(\mathcal{F}) + \epsilon/2$. We show that:

$$\mu(Y \cap U) - \mu(Y \cap U \cap W) < \epsilon \tag{24}$$
$$\mu(Y \cap W) - \mu(Y \cap U \cap W) < \epsilon. \tag{25}$$

For (24), we have by Lemma 5.4:

$$\mu(Y \cap U) - \mu(Y \cap U \cap V) < \epsilon/2 \tag{26}$$
$$\mu(Y \cap V) - \mu(Y \cap V \cap W) < \epsilon/2. \tag{27}$$

But then:

$$\mu(Y \cap U) - \mu(Y \cap U \cap W)$$
$$\le \mu(Y \cap V \cap U) - \mu(Y \cap U \cap V \cap W) + \mu(Y \cap U) - \mu(Y \cap U \cap V) \tag{28}$$
$$\le \mu(Y \cap V) - \mu(Y \cap V \cap W) + \mu(Y \cap U) - \mu(Y \cap U \cap V) \tag{29}$$
$$< \epsilon, \tag{30}$$

where (28) is by Lemma 2.6, (29) is by Lemma 2.5, and (30) is by (26) and (27). This establishes (24) and (25) is similar. Thus $=_{r(\mathcal{F})}$ is an equivalence relation.

To establish the congruence properties of $=_{r(\mathcal{F})}$, we first consider finite meets. Suppose $U =_{r(\mathcal{F})} V$ and $U' =_{r(\mathcal{F})} V'$. Let $W \in \mathcal{F}$ be such that $\mu(W) < \mu^\circ(\mathcal{F}) + \epsilon/2$. We show that:

$$\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V \cap V') < \epsilon \tag{31}$$
$$\mu(W \cap V \cap V') - \mu(W \cap U \cap U' \cap V \cap V') < \epsilon. \tag{32}$$

For (31), we have by Lemmas 2.5 and 5.4:

$$\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V) < \epsilon/2 \tag{33}$$
$$\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V') < \epsilon/2. \tag{34}$$

Hence:

$$\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V \cap V')$$
$$\le 2\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V \cap V') - \mu(W \cap U \cap U' \cap (V \cup V')) \tag{35}$$
$$= 2\mu(W \cap U \cap U') - \mu(W \cap U \cap U' \cap V) - \mu(W \cap U \cap U' \cap V') \tag{36}$$
$$< \epsilon. \tag{37}$$

Here (35) follows from the inclusion $W \cap U \cap U' \cap (V \cup V') \subseteq W \cap U \cap U'$, Eq. (36) is by the modularity of μ, and (37) follows from (33) and (34). This establishes (31) and (32) is similar.

To show congruence for finite joins. Suppose $U =_{r(\mathcal{F})} V$ and $U' =_{r(\mathcal{F})} V'$. Let $W \in \mathcal{F}$ be such that $\mu(W) < \mu^{\circ}(\mathcal{F}) + \epsilon/2$. We show that:

$$\mu(W \cap (U \cup U')) - \mu(W \cap (U \cup U') \cap (V \cup V')) < \epsilon \tag{38}$$

$$\mu(W \cap (V \cup V')) - \mu(W \cap (U \cup U') \cap (V \cup V')) < \epsilon. \tag{39}$$

For (38), we have by Lemma 5.4:

$$\mu(W \cap U) - \mu(W \cap U \cap V) < \epsilon/2 \tag{40}$$

$$\mu(W \cap U') - \mu(W \cap U' \cap V') < \epsilon/2 \tag{41}$$

Then:

$$\mu(W \cap (U \cup U')) - \mu(W \cap (U \cup U') \cap (V \cup V'))$$
$$\leq \mu((W \cap U) \cup (W \cap U')) - \mu((W \cap U \cap V) \cup (W \cap U' \cap V')) \tag{42}$$
$$= \mu(W \cap U) + \mu(W \cap U') - \mu(W \cap U \cap U')$$
$$\quad - \mu(W \cap U \cap V) - \mu(W \cap U' \cap V') + \mu(W \cap U \cap U' \cap V \cap V') \tag{43}$$
$$< \epsilon. \tag{44}$$

Here (42) follows from the inclusion

$$(W \cap U \cap V) \cup (W \cap U' \cap V') \subseteq W \cap (U \cup U') \cap (V \cup V'),$$

Eq. (43) holds by the modularity of μ, and (44) is a consequence of (40), (41) and the inclusion $W \cap U \cap U' \cap V \cap V' \subseteq W \cap U \cap U'$. This shows (38) and (39) is similar. So $=_{r(\mathcal{F})}$ is indeed a congruence with respect to finite joins.

For congruence with respect to countable joins, suppose that $U_i =_{r(\mathcal{F})} V_i$, for all $i \geq 0$. Take any $W \in \mathcal{F}$ with $\mu(W) < \infty$. We show that:

$$\mu\left(W \cap \bigcup_{i \geq 0} U_i\right) - \mu\left(W \cap \bigcup_{i \geq 0} U_i \cap \bigcup_{i \geq 0} V_i\right) \leq \mu(W) - \mu^{\circ}(\mathcal{F}) \tag{45}$$

$$\mu\left(W \cap \bigcup_{i \geq 0} V_i\right) - \mu\left(W \cap \bigcup_{i \geq 0} U_i \cap \bigcup_{i \geq 0} V_i\right) \leq \mu(W) - \mu^{\circ}(\mathcal{F}). \tag{46}$$

We show (45). By congruence for finite joins, for every $n \geq 0$, we have that $\bigcup_{0 \leq i \leq n} U_i =_{r(\mathcal{F})} \bigcup_{0 \leq i \leq n} V_i$. So, by Lemma 5.4, for every $n \geq 0$,

$$\mu\left(W \cap \bigcup_{0 \leq i \leq n} U_i\right) - \mu\left(W \cap \bigcup_{0 \leq i \leq n} U_i \cap \bigcup_{0 \leq i \leq n} V_i\right) \leq \mu(W) - \mu^{\circ}(\mathcal{F}). \tag{47}$$

Thus:

$$\mu\left(W \cap \bigcup_{i \geq 0} U_i\right) - \mu\left(W \cap \bigcup_{i \geq 0} U_i \cap \bigcup_{i \geq 0} V_i\right)$$
$$= \sup_{n \geq 0} \mu\left(W \cap \bigcup_{0 \leq i \leq n} U_i\right) - \mu\left(W \cap \bigcup_{i \geq 0} U_i \cap \bigcup_{i \geq 0} V_i\right) \tag{48}$$
$$\leq \sup_{n \geq 0} \mu\left(W \cap \bigcup_{0 \leq i \leq n} U_i\right) - \mu\left(W \cap \bigcup_{0 \leq i \leq n} U_i \cap \bigcup_{0 \leq i \leq n} V_i\right)$$
$$\leq \mu(W) - \mu^{\circ}(\mathcal{F}). \tag{49}$$

Here (48) is by the (stable by Lemma 2.8) σ-continuity of μ, and (49) follows from (47). This concludes the proof that $=_{r(\mathcal{F})}$ is a congruence.

Next we show that the sublocale $r(\mathcal{F})$ satisfies $r(\mathcal{F}) \subseteq \bigcap \mathcal{F}$. Suppose $W \in F$. We must show that $r(\mathcal{F}) \subseteq W$. Without loss of generality, we can assume that $\mu(W) < \infty$. (Otherwise, find any $W' \in \mathcal{F}$ with $\mu(W') < \infty$ and apply the argument below to derive that $r(\mathcal{F}) \subseteq W \cap W'$.) Suppose $U =_W V$, i.e., that $U \cap W = V \cap W$. Then

$$\mu(W \cap U) - \mu(W \cap U \cap V) = 0 = \mu(W \cap V) - \mu(W \cap U \cap V).$$

So, indeed $U =_{r(\mathcal{F})} V$, by definition of $=_{r(\mathcal{F})}$, since $\mu(W) < \infty$.

Because $r(\mathcal{F}) \subseteq W$, for every $W \in \mathcal{F}$, we have that $\mu^*(r(\mathcal{F})) \leq \mu^\circ(\mathcal{F})$. It remains to show that $\mu^*(r(\mathcal{F})) \geq \mu^\circ(\mathcal{F})$. Accordingly, let $V \in \mathcal{O}(X)$, be such that $r(\mathcal{F}) \subseteq V$. We need to show that:

$$\mu(V) \geq \mu^\circ(\mathcal{F}).$$

Because $r(\mathcal{F}) \subseteq V$, it holds that $V =_{r(\mathcal{F})} X$. Take any $\epsilon > 0$. By definition of $=_{r(\mathcal{F})}$, there exists finite measure $W \in \mathcal{F}$ such that $\mu(W \cap X) - \mu(W \cap X \cap V) < \epsilon$, i.e., $\mu(W) - \mu(W \cap V) < \epsilon$. By modularity, $\mu(V) > \mu(W \cup V) - \epsilon$. But $\mu(W \cup V) \geq \mu(W) \geq \mu^\circ(\mathcal{F})$, since $W \in \mathcal{F}$. Thus indeed:

$$\mu(V) \geq \mu^\circ(\mathcal{F}) - \epsilon.$$

This completes the proof. $\quad\square$

Proposition 5.6.

1. *The functions* $\mathcal{N} : \mathscr{S}(X) \to \mathcal{F}il(\mathcal{O}(X))$ *and* $\bigcap : \mathcal{F}il(\mathcal{O}(X)) \to \mathscr{S}(X)$ *are in the adjoint relationship:*

$$\mathcal{N}(Y) \supseteq \mathcal{F} \text{ iff } Y \subseteq \bigcap \mathcal{F}. \tag{50}$$

   Hence $\mathcal{N}$ *preserves arbitrary joins and* $\bigcap$ *preserves arbitrary meets. In addition,* $\bigcap$ *preserves finite joins.*
2. *For every finite measure filter* $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$, *it holds that* $\mu^*(\bigcap \mathcal{F}) = \mu^\circ(\mathcal{F})$.

Proof. For statement 1, the adjoint equivalence (50) and its consequences for join and meet preservation are trivial. To see that $\bigcap$ preserves finite joins, observe that, for any $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$, we have $\mathcal{F} = \bigwedge_{U \in \mathcal{F}} \uparrow U$. Thus:

$$
\begin{aligned}
\bigcap (\mathcal{F} \vee \mathcal{G}) &= \bigcap \left(\left(\bigwedge_{U \in \mathcal{F}} \uparrow U \right) \vee \left(\bigwedge_{V \in \mathcal{G}} \uparrow V \right) \right) \\
&= \bigcap \bigwedge_{U \in \mathcal{F}} \bigwedge_{V \in \mathcal{G}} (\uparrow U) \vee (\uparrow V) &&\mathcal{F}il(\mathcal{O}(X)) \text{ is a coframe} \\
&= \bigcap \bigwedge_{U \in \mathcal{F}} \bigwedge_{V \in \mathcal{G}} \uparrow (U \cup V) \\
&= \bigcap_{U \in \mathcal{F}} \bigcap_{V \in \mathcal{G}} \bigcap \uparrow (U \cup V) &&\bigcap \text{ preserves meets} \\
&= \bigcap_{U \in \mathcal{F}} \bigcap_{V \in \mathcal{G}} U \cup V \\
&= \bigcap_{U \in \mathcal{F}} U \ \cup \ \bigcap_{V \in \mathcal{G}} V &&\mathscr{S}(X) \text{ is a coframe} \\
&= \bigcap \mathcal{F} \ \cup \ \bigcap \mathcal{G}.
\end{aligned}
$$

For 2, for any $\mathcal{F} \in \mathcal{F}il(\mathcal{O}(X))$, trivially $\mu^*(\bigcap \mathcal{F}) \leq \mu^\circ(\mathcal{F})$. Moreover, when $\mathcal{F}$ has finite measure, we have $r(\mathcal{F}) \subseteq \bigcap \mathcal{F}$ and $\mu^*(r(\mathcal{F})) = \mu^\circ(\mathcal{F})$, by Proposition 5.5. Hence also $\mu^*(\bigcap \mathcal{F}) \geq \mu^\circ(\mathcal{F})$. $\quad\square$

Example 5.7. The equation $\mu^*(\bigcap \mathcal{F}) = \mu^\circ(\mathcal{F})$ can fail for non-finite-measure $\mathcal{F}$. Let μ be the σ-finite cardinality measure on the locale of discrete natural numbers N, from Examples 4.5 and 5.3, and let $\mathcal{F}$ be the filter of cofinite subsets of $\mathbb{N}$. Then $\mu^\circ(\mathcal{F}) = \infty$, but $\mu^*(\bigcap \mathcal{F}) = \mu^*(\emptyset) = 0$.

Proof of Theorem 1. We finally impose the assumption that X is fitted.

For statement 1, we need to show that μ^* is a measure. Properties (2) and (3) are trivial. For modularity, we must show that

$$\mu^*(Y) + \mu^*(Z) = \mu^*(Y \cup Z) + \mu^*(Y \cap Z).$$

If either of $\mu^*(Y)$ and $\mu^*(Z)$ is infinite then both sides are infinite, and we are done. Otherwise, both sides are finite, and we have:

$$
\begin{aligned}
\mu^*(Y) + \mu^*(Z) &= \mu^\circ(\mathcal{N}(Y)) + \mu^\circ(\mathcal{N}(Z)) &&\text{def. } \mu^* \\
&= \mu^\circ(\mathcal{N}(Y) \vee \mathcal{N}(Z)) + \mu^\circ(\mathcal{N}(Y) \wedge \mathcal{N}(Z)) &&\text{Proposition 5.2} \\
&= \mu^* \left(\bigcap (\mathcal{N}(Y) \vee \mathcal{N}(Z)) \right) + \mu^* \left(\bigcap (\mathcal{N}(Y) \wedge \mathcal{N}(Z)) \right) &&\text{Proposition 5.6.2} \\
&= \mu^* \left(\left(\bigcap \mathcal{N}(Y) \right) \cup \left(\bigcap \mathcal{N}(Z) \right) \right) + \mu^* \left(\left(\bigcap \mathcal{N}(Y) \right) \cap \left(\bigcap \mathcal{N}(Z) \right) \right) &&\text{Proposition 5.6.1} \\
&= \mu^*(Y \cup Z) + \mu^*(Y \cap Z) &&X \text{ fitted.}
\end{aligned}
$$

For σ-continuity, let $(Y_i)_{i \geq 0}$ be ascending. Then

$$
\begin{aligned}
\mu^* \left(\bigvee_{i \geq 0} Y_i \right) &= \mu^\circ \left(\mathcal{N} \left(\bigvee_{i \geq 0} Y_i \right) \right) && \text{def. } \mu^* \\
&= \mu^\circ \left(\bigvee_{i \geq 0} \mathcal{N}(Y_i) \right) && \text{Proposition 5.6.1} \\
&= \sup_{i \geq 0} \mu^\circ (\mathcal{N}(Y_i)) && \text{Proposition 5.2} \\
&= \sup_{i \geq 0} \mu^*(Y_i) && \text{def. } \mu^*.
\end{aligned}
$$

For statement 2, we must show cocontinuity on filtered subsets of finite measure. Let $\mathcal{C} \subseteq \mathcal{S}(X)$ be such a subset. Then:

$$
\begin{aligned}
\mu^* \left(\bigcap \mathcal{C} \right) &= \mu^* \left(\bigcap_{Y \in \mathcal{C}} \bigcap \mathcal{N}(Y) \right) && X \text{ fitted} \\
&= \mu^* \left(\bigcap \bigwedge_{Y \in \mathcal{C}} \mathcal{N}(Y) \right) && \text{Proposition 5.6.1} \\
&= \mu^\circ \left(\bigwedge_{Y \in \mathcal{C}} \mathcal{N}(Y) \right) && \text{Proposition 5.6.2, since finite} \\
&= \inf_{Y \in \mathcal{C}} \mu^\circ (\mathcal{N}(Y)) && \text{Proposition 5.2 (cocontinuity)} \\
&= \inf_{Y \in \mathcal{C}} \mu^*(Y) && \text{def. } \mu^*.
\end{aligned}
$$

For statement 3 suppose that μ is σ-finite. We use Lemma 2.10 to show that μ^* is stably σ-continuous. Let $(U_i)_{i \geq 0}$ be a sequence of finite measure opens with $\bigcup_{i \geq 0} U_i = X$. Without loss of generality, we can assume $(U_i)_{i \geq 0}$ is increasing. (Given a non-increasing sequence, one can take $(\bigcup_{0 \leq j \leq i} U_j)_{i \geq 0}$ instead.) Condition (10) holds by assumption. Condition (11) follows from Proposition 3.10.2, since, for any $Y \in \mathcal{S}(X)$,

$$
\bigcup_{i \geq 0} Y \cap U_i = Y \cap \bigcup_{i \geq 0} U_i = Y \cap X = Y.
$$

Condition (12) is an instance of Proposition 3.10.1. □

6. Random σ-locales

If μ is a measure on a fitted σ-locale X then, by Theorem 1, the outer measure μ^* is a measure on $\mathcal{S}(X)$. It thus makes sense to think of every σ-sublocale Y of X as being "measurable", which means that there is no need to introduce any formal notion of "measurability". Instead, we acknowledge universal measurability by referring to the value $\mu^*(Y)$ as *the measure of Y*. Nevertheless, it is convenient to continue to distinguish notationally between the original measure μ on $\mathcal{O}(X)$, and the derived measure μ^* on $\mathcal{S}(X)$. This section will mostly concern the case in which μ is a finite measure, in which case the measure μ^* is, by Theorem 1, stably σ-continuous and cocontinuous.

Theorem 2. *Let μ be a measure on a fitted σ-locale X with $\mu(X) = M < \infty$.*

1. *The σ-locale X has a smallest σ-sublocale of measure M. We write $\mathbf{Ran}(\mu)$ for this σ-sublocale.*
2. *$\mathbf{Ran}(\mu)$ is the intersection in $\mathcal{S}(X)$ of all open σ-sublocales of measure M.*

Proof. For 1, by the modularity of μ^*, and because $\mu(X) = M < \infty$, the set

$$
\{ Y \in \mathcal{S}(X) \mid \mu^*(Y) = M \}
$$

is a filter on $\mathcal{S}(X)$. Define

$$
\mathbf{Ran}(\mu) = \bigcap \{ Y \in \mathcal{S}(X) \mid \mu^*(Y) = M \}. \tag{51}
$$

By cocontinuity of μ^*, we have $\mu^*(\mathbf{Ran}(\mu)) = M$. It thus holds that $\mathbf{Ran}(\mu)$ is the smallest σ-sublocale of X of measure M.

For 2, since X is fitted, we have that $\mathbf{Ran}(\mu) = \bigcap \mathcal{N}(\mathbf{Ran}(\mu))$. But, since $\mu^*(\mathbf{Ran}(\mu)) = M$,

$$
\mathcal{N}(\mathbf{Ran}(\mu)) \subseteq \{ U \in \mathcal{O}(X) \mid \mu(U) = M \},
$$

and the converse inclusion holds by (51). □

The theorem characterises $\mathbf{Ran}(\mu)$ as a nontrivial σ-sublocale of "μ-random elements" in X. The intuition behind it is that a "randomly generated element" of X should land in any measure M "part" of X, i.e., be contained in every measure M σ-sublocale. It is then natural to think of the smallest measure M σ-sublocale as being a "part" of X consisting only of "random elements". The fact that $\mathbf{Ran}(\mu)$ itself has measure M implies nontriviality, and means that all the "weight" in X is attached to "random elements". In the case that $M = 1$ (i.e., μ is a probability measure), the fact that $\mathbf{Ran}(\mu)$ is contained within every measure 1 "part" means that "random elements" satisfy every "probabilistic law", cf. Section 1. Furthermore, statement 2 of Theorem 2 implies that $\mathbf{Ran}(\mu)$ is characterised by those "probabilistic laws" that are given by opens. This feature is further emphasised by the corollary below.

Corollary 2. *Let μ be a measure on a fitted σ-locale X with $\mu(X) = M < \infty$. Then the σ-sublocale $\mathbf{Ran}(\mu)$ of X is characterised by the following property:*

$$\text{for all } W \in \mathcal{O}(X), \quad \mathbf{Ran}(\mu) \subseteq W \quad \text{iff} \quad \mu(W) = M. \tag{52}$$

Proof. It is immediate from Theorem 2.1 that $\mathbf{Ran}(\mu)$ satisfies the bi-implication. Moreover, this property characterises $\mathbf{Ran}(\mu)$ because any σ-locale Y satisfying the right-to-left implication of

$$\text{for all } W \in \mathcal{O}(X), \quad Y \subseteq W \quad \text{iff} \quad \mu(W) = M.$$

is contained in the meet of all measure M open sublocales, hence $Y \subseteq \mathbf{Ran}(\mu)$, by Theorem 2.2. But the left-to-right implication implies $\mu^*(Y) = M$ hence $\mathbf{Ran}(\mu) \subseteq Y$ by Theorem 2.1. $\quad\square$

Next, we give a concrete description of $\mathbf{Ran}(\mu)$. Henceforth, in this section, let μ be a measure on a σ-locale X with $\mu(X) = M < \infty$. Define a relation $=_{\mathbf{Ran}(\mu)}$ on $\mathcal{O}(X)$ by:

$$U =_{\mathbf{Ran}(\mu)} V \quad \Leftrightarrow \quad \mu(U) = \mu(U \cap V) = \mu(V). \tag{53}$$

Proposition 6.1. *The relation $=_{\mathbf{Ran}(\mu)}$ is a congruence on $\mathcal{O}(X)$, and the sublocale it defines is $\mathbf{Ran}(\mu)$.*

Proof. By Lemma 5.4, $=_{\mathbf{Ran}(\mu)}$ coincides with $=_{r(\mathcal{F})}$, as defined in Section 5, in the special case $\mathcal{F} = \{X\}$. Hence, by Proposition 5.5, it is a congruence.

To show that $\mathbf{Ran}(\mu)$ is the sublocale associated with $=_{\mathbf{Ran}(\mu)}$, we note that $=_{\mathbf{Ran}(\mu)}$ also coincides with $=_{r(\mathcal{F}_M)}$, where $\mathcal{F}_M = \{W \in \mathcal{O}(X) \mid \mu(W) = M\}$, which is a filter because $\mu(X) = M$. In one direction, Lemma 5.4 gives that $U =_{r(\mathcal{F}_M)} V$ implies $U =_{\mathbf{Ran}(\mu)} V$, because $X \in \mathcal{F}_M$ and $\mu^\circ(\mathcal{F}_M) = M$. Conversely, if $U =_{\mathbf{Ran}(\mu)} V$, then, for any $W \in \mathcal{O}(X)$,

$$\mu(W \cap U) - \mu(W \cap U \cap V) \leq \mu(U) - \mu(U \cap V) = 0,$$

and similarly with U and V swapped, using Lemma 2.5 for the inequality. Thus $U =_{r(\mathcal{F}_M)} V$, again by Lemma 5.4.

By Proposition 5.5, $r(\mathcal{F}_M) \subseteq \bigcap \mathcal{F}_M$ and $\mu^*(r(\mathcal{F}_M)) = \mu^\circ(\mathcal{F}_M) = M$. These two statements say that $r(\mathcal{F}_M)$ satisfies the characterisation of Corollary 2. Hence $=_{r(\mathcal{F}_M)}$, that is $=_{\mathbf{Ran}(\mu)}$, is indeed the congruence relation defining the sublocale $\mathbf{Ran}(\mu)$. $\quad\square$

The definition of $=_{\mathbf{Ran}(\mu)}$ in (53) is clarified by the simple characterisation below, which observes that $U =_{\mathbf{Ran}(\mu)} V$ holds if and only if U and V have null (with respect to μ^*) symmetric difference in the lattice $\mathcal{S}(X)$.

Proposition 6.2. *$U =_{\mathbf{Ran}(\mu)} V$ holds if and only if $\mu^*((U \cup V) \cap (U \cap V)^c) = 0$.*

Recall that we write $(U \cap V)^c$ for the closed sublocale complementing $U \cap V$.

Proof. One shows easily that $U =_{\mathbf{Ran}(\mu)} V$ iff $\mu(U \cup V) - \mu(U \cap V) = 0$ iff $\mu^*((U \cup V) \cap (U \cap V)^c) = 0$. $\quad\square$

It follows from Proposition 6.2 that $\mathcal{O}(\mathbf{Ran}(\mu))$ is simply $\mathcal{O}(X)$ quotiented by the relation of null symmetric difference. The construction of the σ-frame $\mathcal{O}(\mathbf{Ran}(\mu))$ is thus reminiscent of the construction of measure algebras in measure theory, see, e.g., [9]. This is discussed further in Examples 6.5 and 6.6.

By Corollary 1, the measure μ^* restricts to give a measure $\mu \!\restriction_{\mathbf{Ran}(\mu)}$ on the σ-locale $\mathbf{Ran}(\mu)$. It follows easily that the σ-sublocale $\mathbf{Ran}(\mu \!\restriction_{\mathbf{Ran}(\mu)})$ of $\mathbf{Ran}(\mu)$ is just $\mathbf{Ran}(\mu)$ itself (by Corollary 1, $\mathbf{Ran}(\mu \!\restriction_{\mathbf{Ran}(\mu)})$ is itself a measure-1 σ-sublocale of X), thus $\mathbf{Ran}(\mu)$ is its own "random part". We use the property of a σ-locale being its own "random part" to define the notion of "random σ-locale" alluded to in the title of the section.

Proposition 6.3. *Let μ be a finite measure on a σ-locale X. Then the following are equivalent.*

1. *$\mathbf{Ran}(\mu) = X$.*
2. *For all $U, V \in \mathcal{O}(X)$, if $U \subseteq V$ and $\mu(U) = \mu(V)$ then $U = V$.*

Proof. For the $1 \implies 2$ implication, suppose $\mathbf{Ran}(\mu) = X$. Let $U, V \in \mathcal{O}(X)$, be such that $U \subseteq V$ and $\mu(U) = \mu(V)$. But then $\mu(V) = \mu(U) = \mu(U \cap V)$, i.e., $U =_{\mathbf{Ran}(\mu)} V$. Hence, $U = V$ by the assumption that $\mathbf{Ran}(\mu) = X$.

Conversely, suppose that 2 holds. We need to show that $X \subseteq \mathbf{Ran}(\mu)$, i.e., that $U =_{\mathbf{Ran}(\mu)} V$ implies $U = V$. Suppose $U =_{\mathbf{Ran}(\mu)} V$. Then $\mu(U) = \mu(U \cap V) = \mu(V)$. But then 2 implies that $U = U \cap V = V$. $\quad\square$

Definition 6.4. A σ-locale X with finite measure μ is said to be *random* if either of the equivalent conditions of Proposition 6.3 hold.

Example 6.5. Let λ be the uniform probability measure restricted to the open sets $\mathcal{O}(2^\omega)$ of Cantor space 2^ω (recall that $2 = \{0, 1\}$). This determines a countably based locale 2^ω. We write **Ran** for the smallest measure-1 sublocale of 2^ω, as given by Theorem 2. We call **Ran** *the locale of random sequences* and we propose it as a natural model for the phenomenon of randomness. We have already remarked that **Ran** is canonically determined as the intersection in $\mathcal{S}(2^\omega)$ of all probabilistic laws. We briefly outline some of its other features. A more thorough treatment of these and other properties is planned for a follow-up paper [20].

Pointlessness. The first fact to observe about the locale **Ran** is that it has no points. The points of 2^ω are just the elements of 2^ω. For any such point α, the set $2^\omega - \{\alpha\}$ is open with measure 1, and hence **Ran** $\subseteq 2^\omega - \{\alpha\}$, as sublocales. Thus no point of 2^ω is a point of **Ran**, hence **Ran** $\subseteq 2^\omega$ can have no points.

At first sight, this fact may seem to defeat the purpose of defining **Ran**. We are trying to model random sequences, but there are no random sequences! Nevertheless, it is not an unreasonable position that any completed infinite sequence is nonrandom. A random sequence is never experienced in its entirety. We only ever see its finite prefixes. The locale **Ran** provides a model of this phenomenon. Rather than trying to identify properties enjoyed by individual completed random sequences, which is the perspective taken in much of the literature on randomness, the locale **Ran** instead models collective properties of the space random sequences, which is nontrivial even though it contains no points. The next paragraph considers one such property.

Transformations on random sequences. A random sequence, given by a random generation process, may be transformed into another sequence by applying a sequence of operations to the finite outputs of the process. Any such transformation is necessarily continuous in the sense that a finite prefix of the result of the transformation depends only on a finite prefix of the input sequence. It makes sense to ask when such a transformation preserves the randomness of the sequence. The locale **Ran** gives a canonical answer to this question, since the continuous maps from **Ran** to **Ran** provide just such a notion of transformation. They can be shown to be in one-to-one correspondence with continuous, nonsingular[8] partial functions from 2^ω to 2^ω, with measure-1 G_δ domain, modulo almost-everywhere equality. A proof of this will appear in the follow-up paper [20].

Characterisation of **Ran***.* Since **Ran** is a sublocale of 2^ω, it is countably based and zero dimensional. Also, with the measure $\lambda\!\restriction_{\mathbf{Ran}}$ (see Corollary 1), the locale **Ran** is random in the sense of Definition 6.4. These properties, together with pointlessness, characterise **Ran**. In fact, even more holds. Let X be any countably-based regular locale without points, which is random relative to a probability measure μ on X. Then X is homeomorphic[9] to **Ran** via measure-preserving isomorphisms. This result, whose proof will be given in the follow-up paper [20], is reminiscent of the measure-algebra isomorphism theorem from measure theory, see, e.g., [9]. However, $\mathcal{O}(\mathbf{Ran})$ is not the measure algebra, which fails the characterisation through not being countably based if considered as a locale.

Example 6.6. The measure algebra does nonetheless fit into our framework. Let $B(2^\omega)$ be the σ-locale whose σ-frame $\mathcal{O}(B(2^\omega))$ of "opens" is defined to be the σ-algebra $\mathcal{B}or(2^\omega)$ of Borel subsets of 2^ω. Let β be the uniform probability measure on $\mathcal{B}or(2^\omega)$. Then, by the characterisation of $\mathcal{O}(\mathbf{Ran}(\beta))$ as $\mathcal{O}(B(2^\omega))$ modulo null symmetric difference, it holds that $\mathcal{O}(\mathbf{Ran}(\beta))$ is the usual measure algebra, cf. [9]. In this example, the σ-locale $B(2^\omega)$ is not a locale. However, its σ-sublocale **Ran**(β) is strongly Lindelöf, hence a locale (in fact, as is well known, it is a complete Boolean algebra). The example of $B(2^\omega)$ allows us to model random sequences in a way that permits general measurable transformations, rather than just continuous transformations, on such sequences. (Of course, it is a reasonable question to ask whether non-continuous transformations really make sense on sequences generated bit by bit.)

7. Discussion

As Example 6.6 emphasises, there is considerable overlap in the approach of this paper with an old foundational tradition in measure and probability theory of basing measure/probability on algebras of parts/events quotiented modulo nullsets, see, e.g., [14,5]. There are, however, two main novelties in our approach. The lesser novelty is that we focus on $(\sigma$-)frames rather than Boolean algebras, which makes continuity the fundamental notion, and allows the import of topological concepts (as in the characterisation of **Ran** given in Example 6.5). The greater novelty is that we show how such quotient algebras arise (via the characterisation of Theorem 2) as a consequence of extending notions from measure theory to a theory of "parts" generalising subsets (Theorem 1). For this, the dual perspective of viewing algebras as $(\sigma$-)locales and quotient algebras as $(\sigma$-)sublocales is essential. Potentially, this allows random spaces to be treated alongside, and as "parts" of, ordinary spaces, within the context of $(\sigma$-)locale theory, which has elsewhere been promoted, for other reasons, as a natural generalisation of the set-theoretic framework of topological spaces.

One point that deserves further explanation is our decision to base the technical development on σ-locales rather than locales. As well as being essential to the inclusion of Example 6.6 above, this choice avoids certain technical issues, which

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[8] A function is nonsingular if inverse image preserves nullsets.

[9] We use topological terminology for isomorphisms of locales.

turn out to be more problematic for locales. For example, given a finite measure μ on a locale X, in order for the σ-sublocale **Ran**(μ) to be a sublocale, it is necessary to assume that μ is continuous. Although this is a natural assumption in the context of locales, it then does not seem possible to restrict μ to obtain a continuous measure $\mu{\restriction}_Y$ on $\mathcal{O}(Y)$ for an arbitrary sublocale Y of X, in contrast to the situation for measures on σ-sublocales (Corollary 1).

One direction for generalising the work in this paper would be to change the definition of outer measure, by using Boolean combinations of open and closed σ-sublocales to determine the outer measure, rather than just open σ-sublocales, as at present. This definition, which agrees with the present one on fitted σ-locales, should allow Theorem 1 to generalise to arbitrary (non-fitted) σ-locales. As future work, it is planned to verify this claim, basing the proof on Joyal and Tierney's characterisation of the opposite of the lattice of sublocales via a universal property in the category of frames [12], as adapted to σ-sublocales and σ-frames by Madden [16].

One important aspect of (σ-)locale theory, which we have ignored in this paper, is that it is often viewed as providing a framework for obtaining constructively acceptable versions of results from classical topology (and other fields). The reason for this is that many theorems of locale theory, suitably formulated, are provable within an intuitionistic metatheory. Nevertheless, the constructivization of the results of the present paper is not entirely straightforward. In a constructive treatment of measures on (σ-)locales, even in the case of the localic real line R, one has to cope with the fact that the weights assigned to opens are not themselves (Dedekind) real numbers. For opens, this can be overcome by using generalised real numbers, approximated from below by rationals. However, if the measure on opens is to be extended to the lattice of σ-sublocales, then an even more complex notion of real number is required. We leave the development of such a constructive version of the results of this paper for future work.

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