REGULAR CATEGORIES

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INTRODUCTION

Decompositions of morphisms into mono- and epimorphisms occur in nearly all the examples which justify the very existence of category theory. Thus it is not surprising that they received attention very early, with the emergence of abelian categories and, in the non-abelian case, MacLane's 1948 paper. It seems much more surprising that further developments has to await more than a decade for the work of Isbell and Barr and others, and also that satisfactory ways to describe non-abelian algebraic phenomena (triples, monoids etc.) did not appear until about the same time, and do not use decompositions. It would seem that, in non-abelian situations, the apparent lack of good properties may have made the actual manipulation of mono-epimorphism decompositions seem unable to attain enough versatility to be of any use in proving things, so that other methods had to be devised.

All the same, decompositions are there, and as categories are expected to account for more and more phenomena it becomes more and more difficult and unnatural not to use them. This may be the basic reason why in the last decade more and more people have been talking decompositions, each time in a slightly different form, but with similar ideas in mind. Also, it is not a denigration of triples and/or monoids to say that by their very nature they cannot by themselves always account for algebraic phenomena with the desired combination of generality and precision that is necessary in some situations (VanOsdol's contribution to this volume is a case in point).

As far as algebraic situations are concerned, the consideration of regular categories may fill these needs very neatly. A regular category (also considered in Micheal Barr's part with weaker but essentially similar axioms) is a finitely complete category in which every morphism f has a decomposition $f = mp$ where m is a monomorphism and p a regular epimorphism (= a coequalizer), and where pullbacks carry regular epimorphisms (i.e. if $fg' = gf'$ is a pullback and f is a regular epimorphism, then so is f'). There is considerable evidence that regular categories can play with regard to non-abelian algebra the role that abelian categories play in abelian algebra. Examples include varieties (finitary or infinitary) as well as abelian categories, and regularity transfers well to categories of functors, algebras over a triple and sheaves. Just as abelian categories can account for all elementary aspects of life with modules (kernels, hom groups, exact sequences etc.), all elementary manipulations of subobjects and congruences that are possible in a variety are equally possible in any regular category. [This includes one more (slightly different) account of decompositions, subobjects and relations; but this time it seems that regular categories provide the right context for all this. Indeed all properties one would expect of a satisfactory account are obtained, and there is evidence that the axioms cannot be significantly weakened and still accomplish this.] The rest of the evidence is the behavior of sheaves in a a regular category, and the fact that they provide the adequate concept for generalizations of Mitchell's full embedding theorem.

This author's contribution is divided into three parts. The first part gives an account of decompositions and relations in a regular category, as well as the easier examples and transfer properties. In the second part are given necessary and sufficient conditions that directed colimits in a cocomplete regular category preserve monomorphisms and finite limits; directed colimits then show additional instances of good behavior. The last part deals with sheaves in suitable regular categories. More can be found in the introduction of each part.

All three parts have been written so that only a minimal knowledge of the bare essentials of category theory (a fraction of [31], and the definition of a triple) and universal algebra (available in [7], [32]) is necessary for the text. The notation and terminology are as in Mitchell [31] with the following exceptions. Diagrams are defined as functors from a small category. In order that the text make sense in everybody's set theory, in which there may not exist choice functions in classes, we have used the following conventions regarding existence statements: taking as example the existence of limits, if we merely wish to say that there exists a limit to every diagram in $\mathbf{C}$, we say "$\mathbf{C}$ is with limits"; if we wish to say that there is a function which selects a limit for every diagram in $\mathbf{C}$, we say "$\mathbf{C}$ has limits". Complete, cocomplete and well-powered are to be read as "has", not "is with". Of course this makes no difference if $\mathbf{C}$ is small; in general we have kept the selecting functions as inobstrusive as possible. Subobjects are defined as equivalence classes of monomorphisms (where the monomorphisms m and n are equivalent in case $m = ni$ for some isomorphism i). The equalizer $\mathrm{Equ}(f,g)$ is a subobject, and an element thereof is just an equalizer of f and g; similar conventions apply to intersections and dually to quotient-objects, coequalizers and cointersections. We start from definitions of images and unions which differ from Mitchell's as indicated in the text.

One of the changes in notation is not a trifle. Products when used as functors are denoted by $\prod$ (π for finite products). Thus $\prod_{i\in I} f_i$ denotes the morphism $\prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i$ induced by all $f_i : A_i \longrightarrow B_i$. To denote the morphism induced by all $f_i : A \longrightarrow B_i$, i.e. $A \longrightarrow \prod_{i\in I} B_i$, we use the notation $\mathop{\mathsf{X}}_{i\in I} f_i$ ($\times$ for finite products). This allows to denote coproducts by $\coprod$ and we think that the confusion it may create is less than that of having to contend with $(f_i)_{i\in I}$ instead of $\mathop{\mathsf{X}}_{i\in I} f_i$ in numerous proofs.

I. EXAMPLES AND ELEMENTARY PROPERTIES

This part is divided into six sections. Sections 1 and 2 contain definition and examples of regular categories. Decompositions of various kinds are investigated in section 1, paving the way for the definition of regular categories which begins section 2. In section 2, we also show that when $\mathcal{A}$ is a regular category and $\mathcal{I}$ is a small category, then the functor category $[\mathcal{I},\mathcal{A}]$ is regular (and the evaluation functor $[\mathcal{I},\mathcal{A}] \longrightarrow \mathcal{A}^{\mathcal{I}}$ preserves and reflects regular decompositions); a similar result is proved for $\mathcal{A}^{\mathbb{T}}$, when $\mathbb{T} = (T,\mu,\varepsilon)$ is a triple on $\mathcal{A}$ such that T preserves regular epimorphisms.

Sections 3,4 and 5 concern the calculus of subobjects, relations and congruences respectively, in a regular category.

The last section gives various properties of limits and colimits in a regular category, as well as completeness $\rightarrow$ cocompleteness implications. A synopsis of the main formulæ in the middle part will be found at the end of that section.

We have tried to make the exposition as careful as possible, especially in giving additional justifications for the definitions and ways of doing things. Relations, and congruences, are defined as subobjects of products, rather than pairs of morphisms, or kernel pairs. Factorization systems, on which the emphasis has been historically, are but briefly considered; the main reason, explained in more detail at the end of section 5, is that they would bring very little additional generality, and this, we think, is not justified by the examples. The one advantage of using factorization systems would be to explain the duality of sorts, which is very apparent throughout, between monomorphisms and regular epimorphisms, subobjects and congruences, etc.; however, no perfectly self-dual account can be given, because the duals of several important properties just do not hold in varieties.

All the results here have been announced in [14]. While they have not otherwise been published before under that form, there is little claim of originality that can be laid for the contents of sections 3-5, since these have been considered before, in part and under sundry guises, by a great many people (most notably, [26],[27], [19],[20],[28],[33],[34],[13],[24],[23],[5],[3],[1],[2],[9]); the part on congruences is certainly the least unoriginal: congruences have been considered before, e.g. as kernel pairs as in [25], but this does not allow for all the manipulations that are possible here, or at least not in a way which is both satisfactory and natural. Most of the references above have to do with factorization systems, which likewise takes care of section 1. Only the most glaring cases of overlap have been indicated in the text.

1. DECOMPOSITIONS.

1. Let $\mathbf{C}$ be any category and f be a morphism of $\mathbf{C}$. A <u>decomposition</u> of f (also known as a <u>mono-epi decomposition</u>, or <u>factorization</u>) is a pair (m,p) of a monomorphism m and epimorphism p such that $f = mp$; $\mathbf{C}$ is a <u>category with decompositions</u> in case this exists for every $f \in \mathbf{C}$.

In general a preorder (= reflexive and transitive relation) is defined on decompositions of a given $f \in \mathbf{C}$ as follows: if (m,p), (n,q) are decompositions of f, then $(m,p) \leq (n,q)$ if and only if there exists a morphism $u \in \mathbf{C}$ such that the following diagram

$$\begin{tikzcd}
 & \cdot \arrow[dr, "n"] & \\
\cdot \arrow[ur, "q"] \arrow[dr, "p"'] & & \cdot \\
 & \cdot \arrow[uu, "u"] \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes. Note that u is necessarily a bimorphism. When there exists a diagram as above in which u is an isomorphism, the decompositions (m,p) and (n,q) are <u>equivalent</u>; this happens if and only if $(m,p) \leq (n,q)$ and $(n,q) \leq (m,p)$.

Granted that f has decompositions, it is natural to look for decompositions of f which are maximal or minimal, or even greatest or least, under the preorder relation. Indeed the general decompositions are not good for much, and in nice categories every morphism has a decomposition with one or the other of these properties; for instance in an abelian category all the decompositions of a given morphism are equivalent, hence they are all greatest <u>and</u> least. This is a rather extreme situation, but in the category of sets, and in that of groupoids (= sets with one binary operation, which does not have to be nice), as well as in the category of all groups, every decomposition is also greatest and least. Anticipating a little, in a variety every morphism has a least decomposition (the obvious one); but it need not be greatest. In fact, in the variety of semigroups there are morphisms which do not have a greatest decomposition [22]. In other situations, such as in the category of all topological spaces, every continuous mapping $f: X \longrightarrow Y$ has a least decomposition $X \longrightarrow f(X) \longrightarrow Y$ (with the quotient topology on $f(X)$) and a greatest decomposition $X \longrightarrow f(X) \longrightarrow Y$ (with the subspace topology on $f(X)$) and they may be distinct; a similar result holds for Hausdorff topological spaces (with $\overline{f(X)}$ in the greatest decomposition), and also for partially ordered sets and order-preserving mappings, and for hypergroupoids (= sets with one multivalued binary operation); more topological examples can be found in [17]. In conclusion, in purely algebraic situations (sets with single-

-valued operations defined everywhere), we can expect least-decompositions (and should not expect the other kinds).

2. The classical approach to least decompositions is either to consider only certain decompositions and set forth axioms which among other things insure one of them will be least (that is using factorization systems); or to consider decompositions (m,p) with an additional condition on p. This is the way minimal decompositions arise: (m,p) is a minimal decomposition if and only if in every decomposition (n,q) of p, n is an isomorphism (then p is called extremal [20], [21]). Unfortunately, there is no known necessary and sufficient condition on p that (m,p) be a least decomposition for every m; but there are a great many sufficient conditions. That p be a retraction is one of them, although too strong to be generally useful in that context. Of more interest are the following:

p is <u>regular</u>, i.e. is a coequalizer (used in that sense in [25], [3] et al.);

p is <u>strict</u> (cf.[21]), i.e.(without the set-theoretical sophistication of [21]) a (small) cointersection of coequalizers;

p is <u>subregular</u> (called special in [18], fermé (closed) in [13]), i.e. whenever g has same domain as p, and $pu = pv$ implies $gu = gv$, then $g = tp$ for some (unique) t;

p is <u>strong</u> [2], i.e. $fp = mg$, with m a monomorphism, implies $g = tp$ for some (unique) t.

<u>Proposition</u> 1.1. Let p be an epimorphism. Each of the following conditions implies the next one: i) p is regular; ii) p is strict; iii) p is subregular; iv) p is strong; v) whenever (m,p) is a decomposition of f it is a least decomposition of f.

<u>Proof</u>. Trivially i) implies ii). To show that ii) implies

iii), let p be a cointersection of coequalizers $p_i \in \mathrm{Coequ}(a_i, b_i)$ $(i \in I)$, in particular p factors through every p_i $(p = t_i p_i)$. If $pu = pv$ implies $gu = gv$, then for every i we have $pa_i = t_i p_i a_i = t_i p_i b_i =$ $= pb_i$, so that $ga_i = gb_i$ and g factors through every p_i; hence it factors through their cointersection p, which shows that p is subregular. Next, assume that p is subregular and that $fp = mg$, where m is a monomorphism; then $pu = pv$ implies $mgu = fpu = fpv = mgv$ and $gu = gv$; hence $g = tp$ for some t. If finally p is strong, and (m,p), (n,q) are decompositions of f, then $nq = mp$ implies $q = tp$ for some t; then also $mp = ntp$, and $m = nt$; therefore $(m,p) \leq (n,q)$, which shows that (m,p) is a least decomposition of f.

A <u>decomposition</u> is called <u>regular</u> (<u>strict</u>, <u>subregular</u>, <u>strong</u>) when the epimorphism therein is regular (strict, subregular, strong). All are least decompositions, by 1.1. In the usual cases, all five concepts are equivalent, so that the initial choice of conditions is not of extreme importance; more precisely, we have the following results.

First, call $\mathbf{C}$ <u>regularly co-well-powered</u> if for each object $A \in \mathbf{C}$ there exists a set $\mathfrak{C}$ of regular epimorphisms of domain A, such that every regular epimorphism p of domain A is equivalent to some $q \in \mathfrak{C}$ (more precisely, in accordance with our conventions, there is a choice function F, such that $q = F(p)$ always serves).

<u>Proposition</u> 1.2. If $\mathbf{C}$ has coequalizers and is regularly co-well-powered, then strict and subregular are equivalent.

<u>Proof</u>. Let p be a subregular epimorphism. Let $(q_i)_{i \in I}$ be the family of all $q_i \in \mathfrak{C}$ such that p factors through q_i; we shall prove that p is a cointersection of $(q_i)_{i \in I}$. Let g have same domain as p and factor through every q_i. If $pu = pv$, then there is some $q \in \mathfrak{C}$ with $q \in \mathrm{Coequ}(u,v)$; in fact, $q = q_i$ for some i since p factors

$q \in \mathrm{Coequ}(u,v)$; then $g = sq$ for some s and $gu = squ = sqv = gv$. Since p is subregular, it follows that $g = tp$ for some (unique) t . Thus p is a cointersection of regular epimorphisms, i.e. is strict. The converse is part of 1.1.

Recall that a kernel pair of a morphism f is any pair (x,y) such that $fx = fy$ is a pullback (any two such are equivalent in the obvious sense).

Proposition 1.3. In a category with kernel pairs, regular, strict and subregular are equivalent ; in fact an epimorphism satisfying any of these conditions coequalizes his kernel pair(s).

Proof. Let p be a subregular epimorphism and (x,y) be a kernel pair of p. Let g be such that $gx = gy$. If $pu = pv$, then $u = xs$, $v = ys$ for some s (since $px = py$ is a pullback) and therefore $gu = gv$. Then g factors (uniquely) through p, which proves that $p \in \mathrm{Coequ}(x,y)$, in particular p is regular. The remaining implications follow from 1.1.

Proposition 1.4. If $\mathbf{C}$ is with regular decompositions (or with kernel pairs and subregular decompositions), then regular, strict, subregular, strong and extremal are equivalent.

Proof. Let p be an extremal epimorphism. There is a decomposition (n,q) of p with q regular (subregular); since p is extremal, it is equivalent to q hence also regular (subregular). The remaining implications follow from 1.1, 1.3.

The hypotheses of 1.2, 1.3, 1.4 are satisfied in any variety (see below).

3. Another way to obtain least decompositions is to deduce their existence from completeness or cocompleteness properties of the category.

Proposition 1.5. Let $\mathbf{C}$ be a well-powered category with intersections and decompositions. For every morphism $f \in \mathbf{C}$ there exists a least decomposition of f.

Proof. Since $\mathbf{C}$ is well-powered, there exists a set $(m_i, p_i)_{i \in I}$ of decompositions of f, such that each decomposition of f is equivalent to one of these (note that, when (m,p) is a decomposition of f, p is uniquely determined by m). Let m be an intersection of all m_i; since f factors through every m_i, we have $f = mp$ for some p. First we show that p is an epimorphism. Let (n,q) be a decomposition of p. Then (mn,q) is a decomposition of f; hence there exist an i and an isomorphism v with $mn = m_i v$, $p_i = vq$. We also have $m = m_i u_i$ for some u_i. If v' is the inverse of v, then $mnv'u_i = m_i u_i = m$, $m_i u_i nv' = mnv' = m_i$; hence $nv'u_i = 1$, $u_i nv' = 1$, and nv' is an isomorphism. Therefore n is an isomorphism and $p = nq$ is an epimorphism. Thus (m,p) is a decomposition of f. Now m factors through every m_i, and this implies that $(m,p) \leq (m_i, p_i)$ for all i; it follows that (m,p) is a least decomposition of f.

Similar but vastly more sophisticated results can be found in [21]. Our last result is due to Tierney (mentioned in [3]):

Proposition 1.6. Let $\mathbf{C}$ be a category with pullbacks and coequalizers of kernel pairs. Assume that pullbacks in $\mathbf{C}$ carry regular epimorphisms. Then every morphism of $\mathbf{C}$ has a regular decomposition.

Proof. Take $f \in \mathbf{C}$; there exists a kernel pair (x,y) of f and $p \in \mathrm{Coequ}(x,y)$; since $fx = fy$, we have $f = mp$ for some m and it suffices to show that m is a monomorphism. There exist pullbacks $mu = mv$, $pu' = up'$, $pv' = vq'$, $p'q'' = q'p''$; juxtaposing yields a pullback $(mp)(u'q'') = (mp)(v'p'')$, i.e. $f(u'q'') = f(v'p'')$; since $fx = fy$ is also a pullback, $px = py$ implies $pu'q'' = pv'p''$. But then $up'q'' = vq'p'' = vp'q''$; the hypothesis implies that p',q',q'' are epimorphisms, and it follows that $u = v$. Hence m is a monomorphism: $ma = mb$ implies $a = ux$, $b = vx$ for some x and $a = b$. Thus (m,p) is a regular decomposition of f .

4. The categories to be considered later are all with regular decompositions, not just least decompositions; this will provide us with more general factorization properties. Subregular decompositions would do just as well, but in the cases we are interested in, 1.4 makes them coincide with regular decompositions. On the other hand, strong decompositions are not quite strong enough for our purposes (see the end of section 5).

There are two basic examples of categories with regular decompositions. First are abelian categories (more generally, exact categories in the sense of [31]): the decompositions there are regular, since every epimorphism is then conormal, hence regular.

The other example is provided by varieties (finitary or infinitary). In a variety, every [homo]morphism has an obvious injective-surjective decomposition. To see that these are regular decompositions, we recall the construction of the pullback on $f : A \longrightarrow C$ and $g : B \longrightarrow C$ in a variety. Let $D = \{(a,b) \in A \pi B \ ; \ f(a) = g(b)\}$; this is a subalgebra of $A \pi B$ and therefore lies in the variety. The maps $x : (a,b) \longmapsto a$, $y : (a,b) \longmapsto b$ are homomorphisms such that $fx = gy$, and in fact $fx = gy$ is a pullback. If now f is a surjective homomorphism and we let $g = f$, then in the above D is but the congruence $\ker f$ induced by f; if h is any homomorphism such that $hx = hy$, then for every $(a,b) \in D$ we have $h(a) = h(x(a,b)) = h(y(a,b)) = h(b)$, in other words $\ker f \subseteq \ker h$, and it follows from the induced homomorphism theorem that h factors uniquely through f . This shows that $f \in \mathrm{Coequ}(x,y)$, so that f is regular.

Thus a variety has regular decompositions. Furthermore, every extremal epimorphism in a variety has a decomposition (m,p) with p surjective, in which m must be an isomorphism; it follows that extremal epimorphisms are surjective and by 1.1 regular, extremal, etc. are all equivalent to surjective.

It should also be noted that varieties satisfy the hypothesis of 1.6. Indeed consider the general pullback as above and assume that f is surjective. For each $b \in B$, there exists $a \in A$ with $f(a) = g(b)$, i.e. $(a,b) \in D$: this shows that y is surjective, and that pullbacks in a variety carry regular epimorphisms. (The same is true in abelian categories.)

All these properties of varieties are still true in any class of universal algebras which admits products and subalgebras.

5. We conclude with a few trivial results showing that in a category with regular decompositions, the regular decompositions behave just as well as the injective-surjective decompositions in a variety, and regular epimorphisms just as well as surjective mappings. The existence of regular decompositions is assumed throughout.

Proposition 1.7. Any two regular decompositions of the same morphism are equivalent.

Proof. By 1.1, both are least decompositions of that morphism.

Proposition 1.8. A morphism f is an isomorphism if and only if it is both a monomorphism and a regular epimorphism.

Proof. If f is both a monomorphism and a regular epimorphism, then $(f,1)$ is a decomposition of the extremal epimorphism f and so f must be an isomorphism. The converse is clear.

Proposition 1.9. Let $fa = bg$ be a commutative square and

(m,p), (n,q) be regular decompositions of f and g. There is a unique morphism t such that the following diagram commutes:

$$\begin{tikzcd}
\cdot \arrow[r, two heads, "p"] & \cdot \arrow[r, tail, "m"] & \cdot \\
\cdot \arrow[u, "a"] \arrow[r, two heads, "q"'] & \cdot \arrow[u, "t"'] \arrow[r, tail, "n"'] & \cdot \arrow[u, "b"']
\end{tikzcd}$$

Proof. Since m is a monomorphism and q is strong, $m(pa) = (bn)q$ implies $pa = tq$ for some unique t. Then $mtq = bnq$ shows that $mt = bn$ as well. (One may call t "induced on the image")

Proposition 1.10. If f and g are regular epimorphisms and fg is defined, then fg is a regular epimorphism. [We are in a category with regular decompositions; the result is not true in general.]

Proof. Let (m,p) be a regular decomposition of fg. Since g is strong, $mp = fg$ implies $p = tg$ for some t; note that t is an epimorphism and that also $f = mt$ (since $fg = mtg$). Hence (m,t) is a decomposition of f and since f is extremal this implies that m is an isomorphism. Hence fg is regular, like p.

Proposition 1.11. If fg is a regular epimorphism, then so is f.

Proof. Take regular decompositions (m,p) of f, (n,q) of g, and (k,r) of pn. Then (mk,rq) is a decomposition of fg, and since fg is (in particular) extremal, mk is an isomorphism. If v is the inverse isomorphism, then $mkv = 1$ shows that m is a retraction; but m is also a monomorphism, so that it is in fact an isomorphism. Hence f is regular, like p.

We have stated these results in the form we shall use later, but it is clear that 'strong' is the condition that makes them work (in fact they still hold if the category is only with strong decompositions and regular is replaced by strong everywhere). They imply that a category with regular decompositions ipso facto has a bifactorization system in the sense of [2], as well as a "bicategory" structure in the sense of [26],[27] (that is, if in fact the category has regular decompositions) and [24].

The last property is connected with products:

Proposition 1.12. Assume furthermore that the category has finite products and that pullbacks carry regular epimorphisms. Then every finite product of regular epimorphisms is a regular epimorphism.

Proof. It suffices to show that when f and g are regular epimorphisms then so is $f \pi g$. For this, we note that $f \pi g = (f \pi 1)(1 \pi g)$ and apply the hypothesis, 1.10 and the following

Lemma 1.13. Every diagram

$$\begin{tikzcd} A \pi B \arrow[r] \arrow[d, "1 \pi g"'] & B \arrow[d, "g"] \\ A \pi B' \arrow[r] & B' \end{tikzcd}$$

(where the horizontal maps are projections) is a pullback.

The proof of the lemma is left to the reader.

2. REGULAR CATEGORIES: DEFINITION AND EXAMPLES.

1. A regular category is a finitely complete category with regular decompositions, in which the following condition holds:

Pullback axiom: if $fg' = gf'$ is a pullback and f is a regular epimorphism, then f' is also a regular epimorphism.

Finite completeness implies that we could replace 'regular' by 'strict' or even by 'subregular' everywhere in the definition (by 1.3); in particular, all three conditions are equivalent in a regular category, and also are equivalent to 'strong' and to 'extremal' (by 1.4), although the last two would not give an equivalent definition. Finally, in a category with coequalizers, the existence of regular decompositions follows from the other axioms (1.6).

The two basic examples of regular categories are abelian categories and varieties of universal algebras (more generally, classes of universal algebras which admit products and subalgebras); as we have seen in the previous section. Of course the definition was calculated to include these examples. On the other hand the pullback axiom rules out the category of all topological spaces and similar examples (other than compact).

2. Additional examples of regular categories come from transfer theorems.

<u>Theorem</u> 2.1. Let $\mathfrak{X}$ be a small category and $\mathfrak{C}$ be a regular category. Then the functor category $[\mathfrak{X},\mathfrak{C}]$ is regular. Furthermore a morphism of $[\mathfrak{X},\mathfrak{C}]$ is a monomorphism (a regular epimorphism) if and only if it is a pointwise monomorphism (regular epimorphism).

<u>Proof</u>. Let η be a pointwise regular epimorphism of $\mathfrak{F} = [\mathfrak{X},\mathfrak{C}]$ and $\eta\alpha = \eta\beta$ be a pullback in $\mathfrak{F}$. For each $X \in \mathfrak{X}$, $\eta X . \alpha X = \eta X . \beta X$ is then a pullback in $\mathfrak{C}$ and since ηX is regular it follows (from 1.3) that $\eta X \in \mathrm{Coequ}_{\mathfrak{C}}(\alpha X, \beta X)$. Therefore $\eta \in \mathrm{Coequ}_{\mathfrak{F}}(\alpha,\beta)$ is a regular epimorphism. On the other hand a pointwise monomorphism is also a monomorphism.

If now η is an arbitrary morphism in $\mathfrak{F}$, we choose for each $X \in \mathfrak{X}$ a regular decomposition $(\mu X, \pi X)$ of ηX [this does not require that $\mathfrak{C}$ <u>have</u> regular decompositions since $\mathfrak{X}$ is small]. Put $\eta : F \longrightarrow G$ and let HX be the domain of μX. For each $f : X \longrightarrow Y$ we have a commutative diagram

$$\begin{array}{ccccc} FX & \xrightarrow{\pi X} & HX & \xrightarrow{\mu X} & GX \\ {\scriptstyle Ff}\downarrow & & & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\pi Y} & HY & \xrightarrow{\mu Y} & GY \end{array}$$

and by 1.9 there is a unique morphism $Hf : HX \longrightarrow HY$ which keeps the diagram commutative. Because of the uniqueness it is then clear that we now have defined a functor H. In addition, the diagram shows that μ, π are natural transformations. By the first part of the proof, we have obtained a regular decomposition (μ, π) of η.

If in the above η is a monomorphism, then π is a monomorphism, hence an isomorphism by 1.8, so that η is a pointwise monomorphism (like μ). [This can also be proved using pullbacks.] If in the above η is a regular epimorphism, then so is μ by 1.11, so that μ is an isomorphism and η is a pointwise regular epimorphism (like π). It is then clear that the pullback axiom, as well as finite completeness, are inherited by $\mathfrak{F}$ from $\mathfrak{C}$, so that $\mathfrak{F}$ is a regular category, q.e.d.

The second part of the statement can again be expressed as follows. It follows from the theorem that the product category $\mathfrak{C}^{\mathrm{Ob}\mathfrak{X}}$ (of all functors of the discrete category $\mathrm{Ob}\mathfrak{X}$ into $\mathfrak{C}$) is a regular category, with pointwise regular decompositions. Then the evaluation functor $[\mathfrak{X}, \mathfrak{C}] \longrightarrow \mathfrak{C}^{\mathrm{Ob}\mathfrak{X}}$ preserves and reflects regular decompositions.

Generally, a functor F between regular categories will be called <u>left exact</u> if it preserves finite limits (hence also monomorphisms) <u>right exact</u> if it preserves existing finite colimits (hence also regular epimorphisms), <u>exact</u> if it has both properties; <u>reflectively</u> left exact etc. are obtained by replacing "preserves" by "reflects" in the above. In particular, an exact (reflectively exact) functor preserves (reflects) regular decompositions. The terminology is close to Barr's [3], with slight modifications to fit abelian usage more closely (in spirit, at least). The evaluation functor in 2.1 is exact and reflectively exact. A similar result is true for algebras over a triple, except that right exactness cannot be expected:

<u>Theorem</u> 2.2. Let $\mathcal{C}$ be a regular category and $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\mathcal{C}$ such that T preserves regular epimorphisms. Then $\mathcal{C}^{\mathbb{T}}$ is a regular category and the forgetful functor $\mathcal{C}^{\mathbb{T}} \longrightarrow \mathcal{C}$ preserves and reflects regular decompositions.

<u>Proof</u>. First, recall that the objects of $\mathcal{C}^{\mathbb{T}}$ are pairs (A,a) with $A \in \mathcal{C}$, $a : TA \longrightarrow A \in \mathcal{C}$, $a \cdot \eta A = 1$, $a \cdot \mu a = a \cdot Ta$; a morphism $f : (A,a) \longrightarrow (B,b) \in \mathcal{C}^{\mathbb{T}}$ is a morphism $f : A \longrightarrow B \in \mathcal{C}$ such that $fa = b \cdot Tf$ (see [8]). We already know that $\mathcal{C}^{\mathbb{T}}$ is as complete as $\mathcal{C}$, in particular is finitely complete ([30]; also easy to see directly).

The theorem itself is proved much as 2.1. Let $f \in \mathcal{C}^{\mathbb{T}}$. If f is a monomorphism in $\mathcal{C}$, then it is one in $\mathcal{C}^{\mathbb{T}}$. Now assume that f is a regular epimorphism in $\mathcal{C}$. Let $fx = fy$ be a pullback in $\mathcal{C}^{\mathbb{T}}$; this yields a pullback $fx = fy$ in $\mathcal{C}$ and since f is regular we have $f \in \mathrm{Coequ}_{\mathcal{C}}(x,y)$; we now show that this is still true in $\mathcal{C}^{\mathbb{T}}$. Put $f : (A,a) \longrightarrow (B,b)$ and let $g : (A,a) \longrightarrow (C,c) \in \mathcal{C}^{\mathbb{T}}$ be such that $gx = gy$. Then $g = tf$ for some $t \in \mathcal{C}$. Furthermore,

$$c \cdot Tt \cdot Tf = c \cdot Tg = ga = tfa = tb \cdot Tf ;$$

since T preserves regular epimorphisms it follows that $c \cdot Tt = tb$, so that $t \in \mathcal{C}^{\mathbb{T}}$. Therefore $f \in \mathrm{Coequ}_{\mathcal{C}^{\mathbb{T}}}(x,y)$ and is a regular epimorphism in $\mathcal{C}^{\mathbb{T}}$.

Let now $f : (A,a) \longrightarrow (B,b) \in \mathcal{C}^{\mathbb{T}}$ be arbitrary. Let (m,p) be a regular decomposition of f in $\mathcal{C}$ and C be the domain of m. We obtain a commutative diagram:

$$\begin{tikzcd} TA \arrow[r, two heads, "Tp"] \arrow[d, "a"'] & TC \arrow[r, "Tm"] & TB \arrow[d, "b"] \\ A \arrow[r, two heads, "p"] & C \arrow[r, tail, "m"] & B \end{tikzcd}$$

There Tp is regular, hence strong, so that $m(pa) = (b \cdot Tm)Tp$ implies $pa = c \circ Tp$ for some unique $c : TC \longrightarrow C$; then also $mc = b \circ Tm$. Furthermore,

$$c \cdot \eta C \cdot p = c \cdot Tp \cdot \eta A = pa \cdot \eta A = p \ ,$$

$$c \cdot \mu C \circ TTp = c \cdot Tp \circ \mu A = pa \circ \mu A = pa \cdot Ta = c \circ Tp \circ Ta = c \circ Tc \circ TTp \ ;$$

since p, TTp are epimorphisms it follows that $c \circ \eta C = 1$, $c \cdot \mu C = c \cdot Tc$. Hence $(C, c) \in \mathcal{C}^{\mathbb{T}}$. Our diagram then shows that $m, p \in \mathcal{C}^{\mathbb{T}}$ and by the first part of the proof we have found a regular decomposition of f in $\mathcal{C}^{\mathbb{T}}$. Then the proof is completed as for 2.1.

That varieties are regular follows immediately from this theorem since the category of sets is regular and any triple thereon preserves regular epimorphisms since they are retractions.

One more transfer theorem (to sheaves) will be found in this volume. Of course it follows from 2.1 that presheaves in a regular category over any topological space or Grothendieck topology, form a regular category.

3. SUBOBJECTS; DIRECT AND INVERSE IMAGES.

Let $\mathcal{C}$ be a regular category.

1. Recall that a subobject of $A \in \mathcal{C}$ is a class of equivalent monomorphisms of codomain A. The subobject containing a monomorphism m is denoted by $\operatorname{Im} m$. A [partial] order relation between subobjects of A is defined by: $\operatorname{Im} m \leq \operatorname{Im} n$ if and only if $m = nt$ for some

$t \in \mathcal{G}$. The intersection of a family $(\underline{x}_i)_{i \in I}$ of subobjects of A is defined as usual and denoted by $\underline{x} = \bigwedge_{i \in I} \underline{x}_i$; it is a g.l.b., i.e. $\underline{y} \leq \underline{x}$ if and only if $\underline{y} \leq \underline{x}_i$ for all i . Note that $\mathcal{G}$ has finite intersections. On the other hand, we define the union of $(\underline{x}_i)_{i \in I}$ as a l.u.b. (when such exists), i.e. $\underline{x} = \bigvee_{i \in I} \underline{x}_i$ in case $\underline{y} \geq \underline{x}$ if and only if $\underline{y} \geq \underline{x}_i$ for all i . (This differs from Mitchell's definition [31], but we shall soon see (3.3, below) that in the regular category $\mathcal{G}$ the two definitions are equivalent.) There is a greatest subobject of A, namely $1 = \operatorname{Im} 1_A$.

Each morphism $f \in \mathcal{G}$ yields a subobject of its codomain: indeed all the monomorphisms m in the regular decompositions (m,p) of f form a subobject. We denote it by $\operatorname{Im} f$; if f is a monomorphism, this is indeed the subobject containing f ; in general, it is an image in the sense of [31], although in this case again regularity enables us to give a definition which works as well but is somewhat more natural.

2. Each morphism $f : A \longrightarrow B \in \mathcal{G}$ induces an '<u>inverse image</u>' map f^s defined as usual: if $\operatorname{Im} n$ is a subobject of B, then $f^s \operatorname{Im} n$ is well-defined by $f^s \operatorname{Im} n = \operatorname{Im} m$, where $fm = ng$ is a pullback. The general properties of inverse images can be found in [31] (or are easy to prove directly): $(1_A)^s$ is the identity, $(fg)^s = g^s f^s$; $f^s 1 = 1$; f^s is order-preserving, in fact preserves all existing intersections. (These properties would hold in any category with pullbacks.)

3. The existence of regular decompositions allows us to define direct images as well. If f is as above and $\operatorname{Im} m$ is a subobject of A , then $f_s \operatorname{Im} m$ is well-defined by: $f_s \operatorname{Im} m = \operatorname{Im} fm$. Equivalently, one may take a regular decomposition (n,q) of fm , and then $f_s \operatorname{Im} m = \operatorname{Im} n$.

If $\mathcal{G}$ is a variety, we know that every monomorphism is equivalent to precisely one inclusion map, so that the subobjects of $A \in \mathcal{G}$ may be identified with the subalgebras of A; when this is done it is easy to see that direct and inverse images of subobjects have their usual meaning. The same is true in an abelian category; then the direct and inverse image maps can be used for a form of diagram chasing with subobjects (the idea is due to Mac Lane [29]). This still works, to some extent, in regular categories [13] and we shall presently indicate all relevant properties.

First we show that our direct images are indeed satisfactory.

Proposition 3.1. $(1_A)_s = I$ and $(fg)_s = f_s g_s$.

Proof. The first assertion is trivial (I denotes the identity map). For the second, let $\operatorname{Im} m$ be a subobject such that $(fg)_s \operatorname{Im} m$ is defined. Take regular decompositions (n,q) of gm, (k,r) of fn, so that $g_s \operatorname{Im} m = \operatorname{Im} n$ and $f_s g_s \operatorname{Im} m = \operatorname{Im} k$. Then rq is a regular epimorphism (1.10); since $fgm = fnq = krq$, (k,rq) is a regular decomposition of fgm, whence $(fg)_s \operatorname{Im} m = \operatorname{Im} k = f_s g_s \operatorname{Im} m$.

Proposition 3.2. $f_s 1 = \operatorname{Im} f$; more generally, $f_s \operatorname{Im} g = \operatorname{Im} fg$.

Proof. This follows at once from 3.1.

Proposition 3.3. f_s is order-preserving, in fact preserves all existing unions.

Proof. Let m,n be monomorphisms with $\operatorname{Im} m \leq \operatorname{Im} n$, i.e. $m = nt$ for some t. Take regular decompositions (k,q) of fn, (ℓ,r) of qt. Then $fm = fnt = kqt = k\ell r$, so that $f_s \operatorname{Im} m = \operatorname{Im} k\ell \leq$ $\leq \operatorname{Im} k = f_s \operatorname{Im} n$. Hence f_s is order-preserving.

Now assume that $\underline{x} = \bigvee_{i \in I} \underline{x}_i$. By the above, $\underline{y} \geq f_s \underline{x}$ implies $\underline{y} \geq f_s \underline{x}_i$ for all i. Conversely, assume that $\underline{y} \geq f_s \underline{x}_i$ for all i. Put $\underline{y} = \operatorname{Im} n$, $\underline{x}_i = \operatorname{Im} m_i$; let (n_i, q_i) be a regular decomposition of fm_i and $fm = ng$ be a pullback. Then $\operatorname{Im} n \geq \operatorname{Im} n_i$, so that

$n_i = nt_i$ for some t_i . Then $fm_i = n_i q_i = nt_i q_i$, which in the pullback $fm = ng$ implies $m_i = mu_i$ for some u_i , i.e. $\underline{x}_i \leq \text{Im } m$. This holds for every i and therefore $\underline{x} \leq \text{Im } m$. It follows that $f_s\underline{x} \leq f_s\text{Im } m = \text{Im } fm = \text{Im } ng = n_s \text{Im } g \leq n_s 1 = \text{Im } n = \underline{y}$. Thus we have proved that $f_s\underline{x} = \bigvee_{i\in I} f_s\underline{x}_i$.

Corollary 3.4. $\text{Im } fg \leq \text{Im } f$, with equality if g is a regular epimorphism.

Proof. $\text{Im } fg = f_s\text{Im } g \leq f_s 1 = \text{Im } f$. If g is a regular epimorphism, then $\text{Im } g = 1$ and the equality holds.

4. We now investigate the relationships between direct and inverse images.

Proposition 3.5. Let $f : A \longrightarrow B$. For each subobject $\underline{y}$ of B, $f^s\underline{y}$ is the greatest subobject $\underline{x}$ of A such that $f_s\underline{x} \leq \underline{y}$.(In particular, $f^s\text{Im } f = 1$.)

Proof. Put $\underline{y} = \text{Im } n$ and let $fm = ng$ be a pullback (so that $f^s\underline{y} = \text{Im } m$). First, $f_s f^s \underline{y} = \text{Im } fm = \text{Im } ng \leq \text{Im } n = \underline{y}$. Next, let $\underline{x} = \text{Im } k$ be such that $f_s\underline{x} \leq \underline{y}$. Let (ℓ, p) be a regular decomposition of fk (so that $f_s\underline{x} = \text{Im } \ell$). Then $\text{Im } \ell \leq \text{Im } n$ and $\ell = nt$ for some t . This implies $fk = \ell p = ntp$ and, since $fm = ng$ is a pullback, $k = mu$ for some u . Hence $\underline{x} = \text{Im } k \leq \text{Im } m = f^s\underline{y}$.

Corollary 3.6. $f_s f^s \leq I$; $f^s f_s \geq I$; $f_s f^s f_s = f_s$; $f^s f_s f^s = f^s$.

Proof. The first two parts are immediate from 3.5. Next, $f_s f^s \leq I$ implies $(f_s f^s) f_s \leq f_s$, while $f^s f_s \geq I$ implies $f_s(f^s f_s) \geq f_s$; it follows that $f_s f^s f_s = f_s$. The last formula is proved similarly.

It follows from 3.6 that $f^s f_s$ is a closure operator on subobjects of the domain of f; in a variety, $f^s f_s \underline{x}$ is the subalgebra of all elements equivalent to elements of the subalgebra $\underline{x}$ modulo the congruence ker f induced by f; in an abelian category, one finds $f^s f_s \underline{x} = \underline{x} \vee \mathrm{Ker}\, f$. The other operator $f_s f^s$ is also a closure operator, but in the opposite order, and is given by:

<u>Proposition</u> 3.7. $f_s f^s \underline{x} = \underline{x} \wedge \mathrm{Im}\, f$.

<u>Proof</u>. Put $\underline{x} = \mathrm{Im}\, m$ and let (n,q) be a regular decomposition of f and $mn' = nm'$, $qm'' = m'q'$ be pullbacks. Then $\underline{x} \wedge \mathrm{Im}\, f =$ $= \mathrm{Im}\, mn'$. On the other hand, $m(n'q') = (nq)m''$ [$= fm''$] is a pullback, so that $f^s \underline{x} = \mathrm{Im}\, m''$. Now the pullback axiom implies that q' is a regular epimorphism, so that (mn', q') is a regular decomposition of fm'' and $f_s f^s \underline{x} = \mathrm{Im}\, mn' = \underline{x} \wedge \mathrm{Im}\, f$.

<u>Proposition</u> 3.8. If f is a regular epimorphism then $f_s f^s = I$ (and conversely).

<u>Proof</u>. If f is a regular epimorphism, then $\mathrm{Im}\, f = 1$ and the direct part follows at once from 3.7. If conversely $f_s f^s = I$, then $\mathrm{Im}\, f = f_s 1 = f_s f^s 1 = 1$.

<u>Proposition</u> 3.9. If f is a monomorphism then $f^s f_s = I$.

<u>Proof</u>. Let $\underline{x} = \mathrm{Im}\, m$ be such that $f_s \underline{x} = \mathrm{Im}\, fm$ is defined. It is easily seen that $fm = (fm)1$ is a pullback (since f is a monomorphism); hence $f^s f_s \underline{x} = \mathrm{Im}\, m = \underline{x}$. (This time, the converse does not generally hold.)

5. Except for 3.7, 3.8 we have not used the pullback axiom. In addition, strong decompositions could have been used instead of regular ones. This will no longer be the cases in the following sections, in which the pullback axiom is used through the (equivalent) pullback lemma which follows:

<u>Lemma</u> 3.10. Let $fh = gk$ be a pullback. Then $f^s \mathrm{Im}\, g = \mathrm{Im}\, h$ and $\mathrm{Im}\, f \wedge \mathrm{Im}\, g = \mathrm{Im}\, fh$ ($= \mathrm{Im}\, gk$).

Proof. We know that this is trivial when f and g are monomorphisms. In general, take regular decompositions (m,p) of f, (n,q) of g, and construct the following diagram, in which each square is a pullback:

$$\begin{tikzcd}
\cdot \arrow[r, two heads, "p"] & \circ \arrow[r, tail, "m"] & \cdot \\
\cdot \arrow[u, tail, "n''"] \arrow[r, two heads, "p'"] \arrow[ur, phantom, "PB"] & \circ \arrow[u, tail, "n'"] \arrow[r, tail, "m'"] \arrow[ur, phantom, "PB"] & \circ \arrow[u, tail, "n"'] \\
\cdot \arrow[u, two heads, "q''"] \arrow[r, two heads, "p''"] \arrow[ur, phantom, "PB"] & \cdot \arrow[u, two heads, "q'"] \arrow[r, tail, "m''"] \arrow[ur, phantom, "PB"] & \circ \arrow[u, two heads, "q"']
\end{tikzcd}$$

Monomorphisms and regular epimorphisms are in the diagram as indicated, due to the pullback axiom. Now juxtaposition yields a pullback $(mp)(n''q'') = (nq)(m''p'')$ which we may assume is $fh = gk$. Then $f^S \mathrm{Im}\, g = f^S \mathrm{Im}\, n = \mathrm{Im}\, n'' = \mathrm{Im}\, h$; also, $q'p''$ is a regular epimorphism, by 1.10, so that $\mathrm{Im}\, f \wedge \mathrm{Im}\, g = \mathrm{Im}\, m \wedge \mathrm{Im}\, n = \mathrm{Im}\, mn' = \mathrm{Im}\, fh$.

Additional properties of direct and inverse images (with respect to a definition of exact sequences which can be given in any regular category, and works to a certain extent) can be found in [13]; they constitute the extension, properly said, of Mac Lane's diagram chasing with subobjects to regular categories.

4. RELATIONS.

Let $\mathfrak{C}$ be a regular category.

1. If $A, B \in \mathfrak{C}$, a <u>relation</u> $\alpha : A \longrightarrow B$ is a subobject of $A \sqcap B$. In the abelian case these are known as additive relations and have been considered most notably in [33] and, using an axiomatic approach, in [28]; if $\mathfrak{C}$ is a variety, a relation is a binary relation which admits the operations.

In general, every morphism $f : A \longrightarrow B$ yields a monomorphism

$1_A \times f : A \longrightarrow A \pi B$ and a relation $\mathrm{Im}(1_A \times f)$ which can be identified with f since $\mathrm{Im}(1_A \times f) = \mathrm{Im}(1_A \times g)$ implies $f = g$ (more generally, $\mathrm{Im}(1_A \times f) \leq \mathrm{Im}(1_A \times g)$ implies $1_A \times f = (1_A \times g)t$ for some t, $f = gt$ and $1_A = 1_A t$, and $f = g$).

To each relation $\alpha : A \longrightarrow B$ corresponds a subobject $\mathrm{Im}\,\alpha = p_s\alpha$ of B (where $p : A \pi B \longrightarrow B$ is the projection); since $p_s\ \mathrm{Im}(1_A \times f) = \mathrm{Im}\, f$, the notation creates no confusion when α is a morphism. One may also write $\alpha = \mathrm{Im}(a \times b)$ (where a, b have codomains A, B respectively) and then $\mathrm{Im}\,\alpha = \mathrm{Im}\, b$.

For each relation $\alpha : A \longrightarrow B$ one has an <u>inverse</u> relation $\alpha^{-1} : B \longrightarrow A$, defined by: $\alpha^{-1} = t_s\alpha$, where $t : A \pi B \longrightarrow B \pi A$ "exchanges the components". Since $tt = 1$, $(\alpha^{-1})^{-1} = \alpha$. If $f : A \longrightarrow B$ is an isomorphism, then its inverse as a relation is $\mathrm{Im}(f \times 1_A) =$ $= \mathrm{Im}((f \times 1_A)f^{-1}) = \mathrm{Im}(1_B \times f^{-1}) = f^{-1}$ the inverse isomorphism. In general, if $\alpha = \mathrm{Im}(a \times b)$ then $\alpha^{-1} = \mathrm{Im}(b \times a)$.

We also have an order relation between relations $A \longrightarrow B$. It is clear (from 3.3) that the taking of images and inverses are order-preserving operations, and in fact preserve unions. In addition, taking inverses also preserves intersections of relations (since in the above t is an isomorphism).

2. Of course the most important operation on relations is composition, and as we discuss it we shall also give some justification for our definition of relations. There are indeed two ways of defining relations, the above and the definition in which a relation is simply a pair of morphisms $A \xleftarrow{a} R \xrightarrow{b} B$. The latter (used by [31],[25] and many others, with interesting viewpoints in [5]) allows to define an associative composition by pullbacks: i.e. if α is as above and $\beta : B \xleftarrow{b'} \cdot \xrightarrow{c} C$, then $\beta\alpha$ is given on the diagram next page, where the square is a pullback:

$$\begin{tikzcd}
A & & B & & C \\
& \cdot \arrow[ul, "a"] \arrow[ur, "b"'] \arrow[rr, phantom, "PB"] & & \cdot \arrow[ul, "b'"] \arrow[ur, "c"'] & \\
& & \cdot \arrow[ul, "x"] \arrow[ur, "y"'] & &
\end{tikzcd}$$

by: $\beta\alpha : A \xleftarrow{ax} \cdot \xrightarrow{cy} C$. On this definition, all properties and examples of compositions are easily available, that is, as long as one does not need, say, unions or intersections of relations; this is perfectly satisfactory, as long as extended manipulation of relations (of the kind to be found in the next part) is not needed. For example, an easy way to define intersections, or unions, of relations, using that definition, has yet to be found.

The main advantage of defining relations as subobjects is the precision one gains. In addition, unions and intersections come naturally, as well as everything in the previous section (at least, in a regular category). However, one needs a definition of composition. Composing by pullbacks has the inconvenient that when $a \times b$ and $b' \times c$ are monomorphisms in the diagram above, $ax \times cy$ need not be a monomorphism; hence if we wish to compose more than two relations, we must not assume that $a \times b$, $b' \times c$ are monomorphisms, and have to prove that composition by pullbacks yields a well-defined operation. (At the end of section 5 we show that this is not true unless the pullback axiom holds.) Another way of composing relations $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ is to use <u>Puppe's formula</u> [33]: $\beta\alpha = r_s(p^s\alpha \wedge q^s\beta)$, where p, q, r are the projections from $A \pi B \pi C$ to $A \pi B$, $B \pi C$, $A \pi C$ respectively. Unfortunately, it is far more cumbersome to manipulate than the definition by pullbacks.

Fortunately, in a regular category, the two definitions agree. In particular, composition by pullbacks is well-defined. Incidentally, this is the first significant consequence of the pullback axiom. We state it as:

Lemma 4.1. Let $\alpha = \mathrm{Im}(a \times b) : A \longrightarrow B$, $\beta = \mathrm{Im}(b' \times c) : B \longrightarrow C$ be relations in the [regular] category $\mathcal{G}$, where $a \times b$ and $b' \times c$ need not be monomorphisms, and $bx = b'y$ be a pullback. Then $\mathrm{Im}(ax \times cy) = r_s(p^s\alpha \wedge q^s\beta)$.

Proof. Let X, Y be the domains of $a \times b$, $b' \times c$. Consider the diagram

$$\begin{tikzcd}
 & & & A \pi C & & & \\
 & A \pi B & & A \pi B \pi C \arrow[u, "r"] \arrow[ll, "p"'] \arrow[rr, "q"] & & B \pi C & \\
X \arrow[ur, "a \times b"] & & X \pi C \arrow[ll, "p'"] \arrow[ur, "(a \times b)\pi 1_C"] & & A \pi Y \arrow[ul, "1_A \pi(b' \times c)"'] \arrow[rr, "q'"'] & & Y \arrow[ul, "b' \times c"'] \\
 & & & \cdot \arrow[ul, "x \times cy"] \arrow[ur, "ax \times y"'] & & &
\end{tikzcd}$$

where p, q, r, p', q' are the projections. We see that the diagram commutes. In fact, the left and right squares are pullbacks, by 1.13. The same is true of the middle square. Indeed, let $x' \times c'$, $a' \times y'$ be such that $((a \times b)\pi 1_C)(x' \times c') = (1_A \pi(b' \times c))(a' \times y')$. Projecting to A, B, C yields $ax' = a'$, $bx' = b'y'$, $c' = cy'$; since $bx = b'y$ is a pullback, we have $x' = xu$, $y' = yu$ for some u . Then $x' \times c' = $ $= xu \times cyu = (x \times cy)u$, $a' \times y' = axu \times yu = (ax \times y)u$. Furthermore, $(x \times cy)u = (x \times cy)v$, $(ax \times y)u = (ax \times y)v$ implies $xu = xv$, $yu = yv$ and $u = v$ since $bx = b'y$ is a pullback.

Thus our three squares are pullbacks. Then 3.10 (a consequence of the pullback axiom) yields $p^s\alpha = p^s\mathrm{Im}(a \times b) = \mathrm{Im}((a \times b)\pi 1_C)$, $q^s\beta = \mathrm{Im}(1_A \pi(b' \times c))$ and $p^s\alpha \wedge q^s\beta = \mathrm{Im}(((a \times b)\pi 1_C)(x \times cy)) =$ $= \mathrm{Im}(ax \times bx \times cy)$. Therefore $r_s(p^s\alpha \wedge q^s\beta) = \mathrm{Im}(r(ax \times bx \times cy)) =$

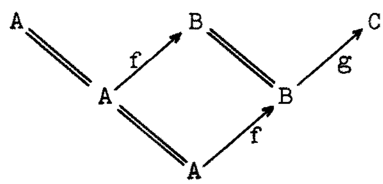
$= \operatorname{Im}(ax \times cy)$, q.e.d.

The relation $\operatorname{Im}(ax \times cy) = r_s(p^s\alpha \wedge q^s\beta)$ obtained in 4.1 is now defined to be the composition $\beta\alpha$ of α and β. It is easy to see that in the abelian case (in the case of a variety) it agrees with the usual composition of additive (binary) relations.

3. We now study the properties of that operation.

<u>Proposition</u> 4.2. The composition of relations agrees with that of morphisms.

<u>Proof</u>. In the diagram



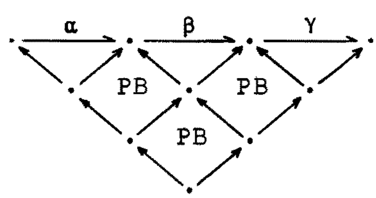
the square is a pullback.

<u>Proposition</u> 4.3. The composition of relations is order-preserving.

<u>Proof</u>. This means that $\alpha \leq \alpha'$ and $\beta \leq \beta'$ implies $\beta\alpha \leq \beta'\alpha'$ and is clear on Puppe's formula since direct images, inverse images and intersections are order-preserving.

<u>Proposition</u> 4.4. The composition of relations is associative.

<u>Proof</u>. Consider the diagram:



Juxtaposing the pullbacks yields pullbacks, and it follows that $\gamma(\beta\alpha)$

and $(\gamma\beta)\alpha$ are given by the same morphisms.

Identity elements are obtained by considering for each object A the diagonal $\Delta_A = 1_A \times 1_A$. Then $\epsilon_A = \mathrm{Im}\,\Delta_A$ serves (4.5 below).(Note that $\epsilon_A = \mathrm{Im}(1_A \times 1_A)$ can be identified with the morphism 1_A .) The notation ϵ means ϵ_A , where A is unspecified, unnamed or obvious.

Proposition 4.5. $\epsilon\alpha = \alpha$, $\beta\epsilon = \beta$, whenever the compositions are defined.

Proof. In the diagrams

the squares are pullbacks.

When $\mathfrak{a}$ is well-powered, 4.4 and 4.5 give us a new category, whose objects are those of $\mathfrak{a}$ and morphisms are relations (in the abelian case, see [33]; in the non-abelian case, see [1]).

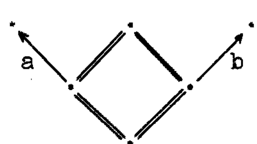
Proposition 4.6. $(\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$.

Proof. Compose by pullbacks and then watch the diagram in a mirror.

Our last result shows that any relation can be obtained from two morphisms by composition:

Proposition 4.7. If $\alpha = \mathrm{Im}(a \times b)$, then $\alpha = b\,a^{-1}$.

Proof. [a^{-1} is the inverse of a as a relation.] In the diagram



the square is a pullback.

4. Like morphisms, relations induce functions of subobjects. To see how this is defined, we start with a lemma:

Lemma 4.8. Let $\alpha = \mathrm{Im}(a \times b)$, $\beta = \mathrm{Im}(b' \times c)$ be such that $\beta\alpha$ is defined. Then $\mathrm{Im}\ \beta\alpha = c_s b'^s \mathrm{Im}\ \alpha$.

Proof. Let $bx = b'y$ be a pullback, so that $\beta\alpha = \mathrm{Im}(ax \times cy)$. By 3.10, $\mathrm{Im}\ y = b'^s \mathrm{Im}\ b = b'^s \mathrm{Im}\ \alpha$; hence $\mathrm{Im}\ \beta\alpha = \mathrm{Im}\ cy = c_s b'^s \mathrm{Im}\ \alpha$.

The lemma shows that, for a fixed α, $\mathrm{Im}\ \alpha\gamma$ depends solely upon $\mathrm{Im}\ \gamma$. In particular, if $\alpha : A \longrightarrow B$ and $\mathrm{Im}\ m$ is a subobject of A, then $\mathrm{Im}\ \alpha m$ does not depend on the choice of m in the subobject and hence may be denoted by $\alpha_s\ \mathrm{Im}\ m$, thereby defining α_s. The following properties are then immediate from the lemma:

Proposition 4.9. If $\alpha = \mathrm{Im}(a \times b)$ then $\alpha_s = b_s a^s$; $\mathrm{Im}\ \alpha\gamma = \alpha_s\ \mathrm{Im}\ \gamma$; if α is a morphism then α_s has the same meaning as before.

Further properties of this new kind of direct image are given by:

Proposition 4.10. a) α_s is order-preserving; b) $(\alpha\beta)_s = \alpha_s\beta_s$; c) $\varepsilon_s = I$; d) $\alpha_s 1 = \mathrm{Im}\ \alpha$; e) $\alpha \leq \beta$ implies $\alpha_s \leq \beta_s$.

Proof. a) and c) are clear from 4.9. Also, $(\alpha\beta)_s \mathrm{Im}\ m = \mathrm{Im}\ \alpha\beta m = \alpha_s \mathrm{Im}\ \beta m = \alpha_s\beta_s \mathrm{Im}\ m$, which proves b). If $\alpha = \mathrm{Im}(a \times b)$ then $\alpha_s 1 = b_s a^s 1 = b_s 1 = \mathrm{Im}\ b = \mathrm{Im}\ \alpha$, which proves d). Finally, assume that $\alpha \leq \beta$; write $\alpha = \mathrm{Im}(a \times b)$, $\beta = \mathrm{Im}(a' \times b')$, where $a \times b$ and $a' \times b'$ are monomorphisms, so that $a \times b = (a' \times b')t$ for some t. By 3.6, $t_s t^s \leq I$; hence $\alpha_s = b_s a^s = b'_s t_s t^s a'^s \leq b'_s a'^s = \beta_s$.

We may also define inverse images by: $\alpha^s = (\alpha^{-1})_s$. The following properties are then immediate from 4.9, 4.10:

Proposition 4.11. a) if α is a morphism, then α^s has the same meaning as before; b) if $\alpha = \mathrm{Im}(a \times b)$ then $\alpha^s = a_s b^s$; c) $(\alpha^{-1})^s = \alpha_s$, $(\alpha^{-1})_s = \alpha^s$; d) α^s is order-preserving; e) $(\alpha\beta)^s = \beta^s\alpha^s$; f) $\epsilon^s = I$; g) $\alpha \leq \beta$ implies $\alpha^s \leq \beta^s$.

In the case when $\mathcal{C}$ is a variety, let $A, B \in \mathcal{C}$ and R be a subalgebra of $A \pi B$. If $a : R \longrightarrow A$, $b : R \longrightarrow B$ are defined by: $a : (x,y) \longmapsto x$, $b : (x,y) \longmapsto y$, then the relation α which corresponds to R is $\mathrm{Im}(a \times b)$. If S is a subalgebra (= subobject) of A, then we can interpret $\alpha_s S$ as follows: first $a^s S = \{(x,y) \in R ; x \in S\}$ and then $\alpha_s S = b_s a^s S = \{ y \in B ; (x,y) \in R$ for some $x \in S \}$. Thus $\alpha_s S$ has the usual meaning. The same is true for inverse images.

Note that 4.9, 4.10, 4.11 extend to relations the properties of direct and inverse images under morphisms, except when it comes to preserving unions or intersections. The case of a variety shows that one cannot expect properties of that kind since e.g. direct images under a relation do not preserve unions of subalgebras (unless they happen to be set-theoretical unions).

5. We now give criteria to recognize morphisms among relations.

Proposition 4.12. Let $\alpha : A \longrightarrow B$ be a relation. The following are equivalent:

i) α is a morphism;

ii) $\alpha\alpha^{-1} \leq \epsilon_B$ and $\alpha^{-1}\alpha \geq \epsilon_A$;

iii) $\alpha\alpha^{-1} \leq \epsilon_B$ and $\alpha^s 1 = 1$.

Proof. If first $\alpha = f$ is a morphism, then $f^{-1} = \mathrm{Im}(f \times 1_A)$ and since $1_A 1_A = 1_A 1_A$ we conclude that $f f^{-1} = \mathrm{Im}(f \times f) = \mathrm{Im}\, \Delta_B f$. Hence $f f^{-1} \leq \mathrm{Im}\, \Delta_B = \epsilon_B$. On the other hand, let $fx = fy$ be a pullback, so that $f^{-1} f = \mathrm{Im}(x \times y)$. Since $f 1_A = f 1_A$, we have $1_A = xt = yt$ for some t, and therefore $\Delta_A = (x \times y)t$; hence $f^{-1} f \geq \mathrm{Im}\, \Delta_A = \epsilon_A$. Thus

i) implies ii). It is clear that ii) implies iii).

Conversely, assume that iii) holds. Put $\alpha = \mathrm{Im}(a \times b)$, where $a \times b$ is a monomorphism. Then $\mathrm{Im}\, a = \mathrm{Im}\, \alpha^{-1} = \alpha^s 1 = 1$, so that a is a regular epimorphism. On the other hand, let $ax = ay$ be a pullback. Then $\mathrm{Im}(bx \times by) = \alpha\alpha^{-1} \leq \mathrm{Im}\, \Delta_B$, whence $bx \times by = \Delta_B t$ for some t, i.e. $bx = t = by$. Since $ax = ay$, it follows that $(a \times b)x = (a \times b)y$ and $x = y$. But $a \in \mathrm{Coequ}(x,y)$, since it is a regular epimorphism, and therefore a is an isomorphism. Hence $\alpha = ba^{-1}$ is a morphism.

Proposition 4.13. Let f be a morphism. Then f is a monomorphism if and only if $f^{-1}f = \epsilon$, and a regular epimorphism if and only if $ff^{-1} = \epsilon$.

Proof. First, f is a monomorphism if and only if $f1 = f1$ is a pullback, if and only if $f^{-1}f = \mathrm{Im}\, \Delta$. If f is a regular epimorphism, then $ff^{-1} = \mathrm{Im}(f \times f) = \mathrm{Im}\, \Delta f = \mathrm{Im}\, \Delta$; if conversely $ff^{-1} = \epsilon$, then $\mathrm{Im}\, f = f_s 1 = f_s f^s 1 = \epsilon_s 1 = 1$ and f is a regular epimorphism.

Note that the first half of 3.6, as well as 3.8, 3.9 follow from these results (in fact, 4.13 is more accurate that 3.8-3.9).

The importance of 4.13 is that in a regular category it provides the only criterion for recognizing monomorphisms that can be manipulated in much the same way kernels are manipulated in abelian categories. This has a great deal to do with the nature of the proofs in the next part.

In an abelian category, much better criterions exist to recognize morphisms: for instance, $\alpha^s 1 = 1$, $\alpha_s 0 = 0$ [33],[29]. This criterion is still valid in the variety of all groups, and in that of rings, but not in general (even if there is a zero object; monoids, even sets with a base point, provide easy counterexamples). In certain cases (semigroups, with perhaps an identity and/or a zero), ii) may be weakened into: $\alpha_s\alpha^s \leq I$, $\alpha^s\alpha_s \geq I$ [an unpublished result of the author, which brings some improvements to [13]].

6. Another type of direct or inverse image is finally obtained from any morphism $f : A \longrightarrow B$, using relations $\alpha : A \longrightarrow A$ or $\beta : B \longrightarrow B$. Namely, we define $f_r\alpha = f\,\alpha\,f^{-1} : B \longrightarrow B$, and $f^r\beta = f^{-1}\beta\,f : A \longrightarrow A$.

A direct study of these maps is not necessary since they can be reduced to ordinary direct or inverse images. Define a functor $\tilde{\ } : \mathcal{G} \longrightarrow \mathcal{G}$ by: $\tilde{A} = A \,\pi\, A$, $\tilde{f} = f \,\pi\, f$. This functor is nice; namely:

Proposition 4.14. The tilda functor is left exact and preserves regular decompositions.

Proof. It already preserves monomorphisms, regular epimorphisms (by 1.12) and finite products. Finally, let $m \in \mathrm{Equ}(f,g)$. Then $\tilde{f}\tilde{m} = \tilde{g}\tilde{m}$ Also, $\tilde{f}(h \times k) = \tilde{g}(h \times k)$ implies: $fh = gh$, $fk = gk$; $h = mt$, $k = mu$ for some t,u; and $h \times k = \tilde{m}(t \times u)$. Since $\tilde{m}$ is a monomorphism it follows that $\tilde{m} \in \mathrm{Equ}(\tilde{f},\tilde{g})$. Thus $\tilde{\ }$ preserves equalizers; this completes the proof.

Using the tilda functor, we can reduce f_r, f^r to previously studied direct or inverse images; namely:

Proposition 4.15. $f_r = \tilde{f}_s$, $f^r = \tilde{f}^s$.

Proof. Take $\alpha = \mathrm{Im}(a \times b) : A \longrightarrow A$. We also have $f = \mathrm{Im}(1 \times f)$, $f^{-1} = \mathrm{Im}(f \times 1)$ and since $1a = a1$, $b1 = 1b$ and $11 = 11$ are pullbacks it follows that $f\,\alpha\,f^{-1} = \mathrm{Im}(fa \times fb) = \mathrm{Im}\,\tilde{f}(a \times b) = \tilde{f}_s\alpha$.

Let now $\beta = \mathrm{Im}(c \times d) : B \longrightarrow B$. Let $fc' = cf'$, $fd' = dg'$ and $f'g'' = g'f''$ be pullbacks, so that $f^{-1}\beta\,f = \mathrm{Im}(c'g'' \times d'f'')$. We now consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\Delta"] & \tilde{X} \arrow[r, "c \sqcap d"] & \tilde{B} \\ T \arrow[u, "f'g''"] \arrow[r, tail, "g'' \times f''"'] & Y \sqcap Z \arrow[u, "f' \sqcap g'"'] \arrow[r, "c' \sqcap d'"'] & \tilde{A} \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

where X, Y, Z, T are the respective domains of $c \times d$, c', d', $g'' \times f''$. The left square is a pullback. Indeed $(f' \sqcap g')(u \times v) = \Delta w$ implies $f'u = w = g'v$, $u = g''t$ and $v = f''t$ for some t and $u \times v = (g'' \times f'')t$, $w = f'g''t$; the factorization is unique since $g'' \times f''$ is a monomorphism (as $f'g'' = g'f''$ is a pullback). In addition, the right-hand square is also a pullback, since products preserve pullbacks. It follows that $\tilde{f}\,((c' \sqcap d')(g'' \times f'')) = ((c \sqcap d)\Delta)(f'g'')$ is a pullback; i.e. $\tilde{f}\,(c'g'' \times d'f'') = (c \times d)(f'g'')$ is a pullback. Then 3.10 yields $\tilde{f}^s \beta = \tilde{f}^s \,\mathrm{Im}(c \times d) = \mathrm{Im}(c'g'' \times d'f'') = f^r \beta$, q.e.d.

In particular, it follows from 4.15 that f_r (f^r) preserves existing unions (intersections), a fact which would not be easy to prove directly.

7. With all this we can account for just about all the elementary properties of relations and morphisms which hold in, say, a variety. The other model of relation theory known to us (Mac Lane's [ ]) is definitely oriented towards the abelian case and therefore the above does not fit into it. For one thing, even when $\mathcal{C}$ is well-powered and the subobjects of a given $A \in \mathcal{C}$ do form a lattice (i.e. when $\mathcal{C}$ has finite unions), then this lattice need not be modular. For another, it is not true that $\alpha\alpha^{-1}\alpha = \alpha$ always holds, although this is the case in an abelian category, and also in the varieties of groups and of rings, and, in general, when either α or α^{-1} is a morphism; counterexamples can easily be found with sets. If allowances are made for this, a number of results in [28] will still hold in our situation (though not the finer ones). Except for the characterization of morphisms, the more elementary aspects are saved.

5. CONGRUENCES.

1. Throughout, $\mathcal{C}$ denotes a given regular category.

If $f : A \longrightarrow B$ is a morphism, the relation $f^{-1}f : A \longrightarrow A$ is the congruence induced by f ; it will be denoted by $\ker f$. One can calculate $f^{-1}f$ by pullbacks: if $fx = fy$ is a pullback then $\ker f = f^{-1}f = \mathrm{Im}(x \times y)$; note that $x \times y$ is a monomorphism since $fx = fy$ is a pullback. Of course (x,y) is the kernel pair of f . A congruence is any relation of the form $\ker f$. We note that, if (m,p) is a regular decomposition of f , then f and p have same kernel pair, so that $\ker f = \ker p$; hence any congruence is induced by some regular epimorphism.

For instance, ϵ_A is a congruence on A (since $\epsilon_A = \ker 1_A$); it follows from 4.12 that it is in fact the least congruence on A. There also exists a greatest congruence on A. Indeed $\mathcal{C}$ is finitely complete, so that there exists a null object N of $\mathcal{C}$ (such that for every $X \in \mathcal{C}$ there exists precisely one morphism $n_X : X \longrightarrow N$) : namely, the limit of the empty diagram. If $f : A \longrightarrow B$ is any morphism of domain A , then $n_B f = n_A$, so that $n_A^{-1} n_A = f^{-1} n_B^{-1} n_B f \geq f^{-1}f$; thus $\upsilon_A = \ker n_A$ is the greatest congruence on A . In fact, we see that $n_A p = n_A q$, where $p,q : A \pi A \longrightarrow A$ are the projections, is a pullback; therefore $\upsilon_A = \mathrm{Im}\ \tilde{1}_A$ is the greatest subobject of $A \pi A$.

In a variety, the definition we used for $\ker f$ yields the congruence induced by f in the usual sense. In an abelian category, there is no need to use congruences because the congruence $\ker f$ induced by f gives us no more information than the kernel $\mathrm{Ker}\ f$ of f . Precisely:

Proposition 5.1. Assume that $\mathcal{C}$ is an abelian category. Then:

a) $\operatorname{Ker} f = (\ker f)_s 0$; b) if α, β are congruences on $A \in \mathcal{C}$ with $\alpha_s 0 = \beta_s 0$, then $\alpha = \beta$.

Remark. In other words, each of Ker f , ker f is completely determined by the other. This expresses the well-known fact that in an abelian group a congruence is completely determined by the class of the identity element.

Proof. For a), take $k \in \operatorname{Ker} f$; since $fk = 00$ is a pullback we see that $\operatorname{Ker} f = f^s 0$. We have to relate this to $(\ker f)_s 0$, which can be written as $y_s x^s 0$, where $fx = fy$ is a pullback (so that $\ker f = \operatorname{Im}(x \times y)$). First, $f_s y_s x^s 0 = f_s x_s x^s 0 \leq f_s 0 = 0$, so that $y_s x^s 0 \leq f^s 0 = \operatorname{Ker} f$. Conversely, $f0 = fk$ implies $k = yt$, $xt = 0$ for some t ; then $x_s \operatorname{Im} t = 0$, whence $\operatorname{Im} t \leq x^s 0$ and $\operatorname{Ker} f = \operatorname{Im} k = y_s \operatorname{Im} t \leq y_s x^s 0$. (In fact, we see that a) will hold even if $\mathcal{C}$ is not abelian, as long as every object of $\mathcal{C}$ has a least subobject 0).

If now α and β are congruences on the same object with $\alpha_s 0 = \beta_s 0$, then we can write $\alpha = \ker f$, $\beta = \ker g$, where f and g are [regular] epimorphisms; by part a) , we then have $\operatorname{Ker} f = \operatorname{Ker} g$, and this implies that $f = tg$ for some isomorphism t , whence $\ker f = \ker g$. This completes the proof.

In the two basic examples of regular categories, there is another property related to congruences. We know that in a variety any relation [admitting the operations] which is reflexive, symmetric and transitive, is a congruence. In general, a congruence α in any regular category is reflexive (i.e. $\varepsilon \leq \alpha$); symmetric (i.e. $\alpha^{-1} = \alpha$), for if $fx = fy$ is a pullback, then so is $fy = fx$, so that $x \times y$ and $y \times x$ are equivalent monomorphisms and $(\ker f)^{-1} = \operatorname{Im}(y \times x) = \operatorname{Im}(x \times y) = \ker f$; and transitive (i.e. $\alpha\alpha \leq \alpha$; equivalently, since α is reflexive,

$\alpha\alpha = \alpha$), since $\alpha = \ker f = f^{-1}f$ for some regular epimorphism f and then $\alpha\alpha = f^{-1}ff^{-1}f = f^{-1}f$ by 4.13 . However, the converse just might not be true. The condition

(L) Every relation $A \longrightarrow A$ which is reflexive, symmetric and transitive is a congruence

will be called Lawvere's condition; it is equivalent to one of the conditions in Lawvere's theorem characterizing finitary varieties [25] . We have seen that it holds in any variety; in addition:

Proposition 5.2. Every abelian category satisfies Lawvere's condition.

Proof. Let α be a reflexive, symmetric and transitive relation on the object A of an abelian category $\mathcal{C}$; pick a monomorphism $k \in \alpha_s 0$ and an exact sequence $0 \longrightarrow \cdot \xrightarrow{k} \cdot \xrightarrow{f} \cdot \longrightarrow 0$; it suffices to prove that $\alpha = \ker f$. If $\mathcal{C}$ is the category of all R-modules, where R is some ring, then since $\mathcal{C}$ is a variety α is a congruence and then $\alpha = \ker f$ is true by 5.1. In the general case we observe that the conditions on α and f and the conclusion that $\alpha = \ker f$ can be expressed in terms of finitely many objects and morphisms of $\mathcal{C}$; hence Mitchell's full embedding theorem can be used to go back to the particular case of R-modules.

2. In a regular category, congruences are manipulated much as in a variety.

Proposition 5.3. $\ker fg \geq \ker g$, with equality if f is a monomorphism.

Proof. $\ker fg = g^{-1}f^{-1}fg \geq g^{-1}g = \ker g$ by 4.12 ; if f is a monomorphism, the equality follows from 4.13.

A converse of 5.3 is the following "induced homomorphism theorem":

Proposition 5.4. If $\ker f \leq \ker g$ and f is a regular epimorphism, then $g = tf$ for some t; t is a monomorphism if and only if $\ker f = \ker g$.

Proof. Let $fx = fy$, $gx' = gy'$ be pullbacks; then $\mathrm{Im}(x \times y) \leq \mathrm{Im}(x' \times y')$, which implies (since $x \times y$, $x' \times y'$ are monomorphisms), $x \times y = (x' \times y')u$ for some u. Hence $gx = gx'u = gy'u = gy$; since $f \in \mathrm{Coequ}(x,y)$, it follows that $g = tf$ for some (unique) t. If furthermore $\ker f = \ker g$, then, by 4.13:

$$t^{-1}t = f f^{-1} t^{-1} t f f^{-1} = f g^{-1} g f^{-1} = f f^{-1} f f^{-1} = \epsilon$$

and t is a monomorphism; the converse follows from 5.3.

A first consequence of 5.4 is that, if f and g both are regular epimorphisms, then $\ker f = \ker g$ implies $g = tf$, where t is an isomorphism by 1.11, 1.8; if conversely f and g are equivalent, then $\ker f = \ker g$ by 5.3. Hence there is a one-to-one correspondance between the regular quotient-objects of a given $A \in \mathcal{C}$ and the congruences on A.

In particular

Corollary 5.5. A well-powered regular category is also regularly co-well-powered.

The next basic operations are direct and inverse images of congruences under morphisms.

Proposition 5.6. For any morphism f and congruence α, $f^r\alpha$ (if defined) is a congruence. Namely, $f^r \ker g = \ker gf$. In particular, $f^r\epsilon = \ker f$.

Proof. $f^r \ker g = f^{-1}g^{-1} g f = \ker gf$.

Predictably, direct images do not work so well. However:

Proposition 5.7. Let f be a morphism and α be a congruence on the domain of f. If f is a regular epimorphism and $\alpha \geq \ker f$, then $f_r\alpha$ is a congruence.

Proof. Put $\alpha = \ker g$. By 5.4, $g = tf$ for some t. By 4.13, $f_r\alpha = f\,g^{-1}g\,f^{-1} = f\,f^{-1}t^{-1}t\,f\,f^{-1} = t^{-1}t = \ker t$.

Proposition 5.8. $f_r \ker f \leq \epsilon$, $f_r\epsilon \leq \epsilon$; in each case, the equality holds if and only if f is a regular epimorphism.

Proof. We always have $f_r\epsilon \leq f_r \ker f = f\,f^{-1}\,f\,f^{-1} \leq \epsilon$ by 4.12. By 4.13, $f_r\epsilon = f\,f^{-1} = \epsilon$ if and only if f is a regular epimorphism, and then $f_r \ker f = \epsilon$. If conversely $f_r \ker f = \epsilon$ then let (m,p) be a regular decomposition of f; then $\operatorname{Im} \Delta = \epsilon \leq f_r \ker f \leq \operatorname{Im} \tilde{f} = \operatorname{Im} \tilde{m}$, so that $\Delta = \tilde{m}(u \times v)$ for some u,v; then $mu = 1$ and m (and hence f) is a regular epimorphism. [One can also show that $f\,f^{-1}f = f$ for all f, so that $f_r \ker f = f_r\epsilon$.]

This last proof shows that in general $f_r\alpha$ cannot be a congruence if f is not a regular epimorphism.

3. If $\mathcal{C}$ is regularly co-well-powered, then for each congruence α on $A \in \mathcal{C}$ we can select a regular epimorphism f with $\ker f = \alpha$ and thus select an object of $\mathcal{C}$ (the codomoain of f) which may be called A/α. If $\mathcal{C}$ is not regularly co-well-powered then one may still be willing to use the notation A/α despite the fact that it can only denote an object defined up to isomorphism only. At any rate we may now wonder if the isomorphisms theorems which hold in a variety can still be formulated in a regular category. The answer is yes, although the first two ($\operatorname{Im} f \cong A/\ker f$ and $(A/\alpha)/(\beta/\alpha) \cong A/\beta$) are now devoid of mathematical content (i.e. are trivial), the only difficulty being to set-up the obvious appropriate definitions. The last one, however, is still of interest.

First, let α be a congruence on $A \in \mathcal{G}$. If $m : B \longrightarrow A$ is a monomorphism, then the restriction $\alpha_{|B}$ of α to B may be defined by: $\alpha_{|B} = m^r\alpha$. The extension of B under α may be defined as the domain $\alpha(B)$ of a monomorphism n such that $\mathrm{Im}\; n = \alpha_s\; \mathrm{Im}\; m$. (In case $\mathcal{G}$ is a variety, this will yield the usual restriction, and the union of all classes modulo α of the elements of B, respectively). The last isomorphism theorem may then be stated as:

<u>Proposition</u> 5.9. Let $m : B \longrightarrow A$ be a monomorphism and α be a congruence on A. Put $\alpha' = \alpha_{|B}$, $C = \alpha(B)$, $\alpha'' = \alpha_{|C}$. Then $B/\alpha' \cong C/\alpha''$.

<u>Proof</u>. Let C be the domain of a monomorphism n such that $\mathrm{Im}\; n = \alpha_s\; \mathrm{Im}\; m$; then $\alpha'' = n^r\alpha$, $\alpha' = m^r\alpha$. First we note that $\mathrm{Im}\; n \geq \varepsilon_s\; \mathrm{Im}\; m = \mathrm{Im}\; m$, so that $m = nt$ for some (monomorphism) t. Let p, q be regular epimorphisms such that $\alpha' = \ker p$, $\alpha'' = \ker q$ (so that the codomains of p, q may be denoted by B/α', C/α''). We have $\ker qt = t^r n^r \alpha = m^r\alpha = \ker p$; hence by 5.4 $qt = up$ for some monomorphism $u : B/\alpha' \longrightarrow C/\alpha''$. We want u to be an isomorphism, and it suffices to show that it is a regular epimorphism.

First

$$\mathrm{Im}\; \alpha'' t = \mathrm{Im}\; n^{-1}\alpha nt = \mathrm{Im}\; n^{-1}\alpha m = n^s\; \mathrm{Im}\; \alpha m = n^s n_s 1 = 1 .$$

Hence $q^s\; \mathrm{Im}\; qt = \mathrm{Im}\; q^{-1}qt = \mathrm{Im}\; \alpha'' t = 1$. Since q is a regular epimorphism, it follows that $\mathrm{Im}\; qt = q_s q^s\; \mathrm{Im}\; qt = q_s 1 = 1$. i.e. qt is a regular epimorphism. By 1.9, so is u, q.e.d.

In a variety, 5.9 reduces to the usual isomorphism theorem (as stated e.g. in [ ]); e.g. in the variety of groups it means that $HK/K \cong H/H \cap K$ whenever H, K are subgroups of a group G with K normal.

The "correspondance theorem" which is sometimes included in the first isomorphism theorem, also retains some interest; in this case, it says that a regular epimorphism $f : A \longrightarrow B$ induces a one-to-one correspondance, which is order-preserving (both ways) between the congruences on B and the congruences on A that contain $\gamma = \ker f$. The maps are of course f_r, f^r (5.6, 5.7) ; for each congruence β on B, $f_r f^r \beta = \beta \wedge \operatorname{Im} \tilde{f} = \beta$; for each congruence $\alpha \geq \gamma$ on A, $f^r f_r \alpha = f^{-1} f \alpha f^{-1} f = \gamma\alpha\gamma$ is $\geq \alpha$ since $\gamma \geq \epsilon$, and $\leq$ $\alpha\alpha\alpha = \alpha$ since $\gamma \leq \alpha$; thus this, too, extends to regular categories.

4. We now have completed the basic study of subobjects, relations and congruences in a regular category, and shall give some evidence that the same body of properties would not be kept if the axioms of regular categories were substantially weakened. We do not have a precise necessary and sufficient condition to that effect, but can make the following remarks.

The assumption of finite completeness cannot be weakened since we need products to describe relations and pullbacks to compose them, as well as for inverse images and intersections. (One should note that it <u>can</u> be somewhat weakened if no hard manipulation of relations is necessary, as, for instance, in Barr's contribution to this volume.)

The assumption that there exist regular decompositions is mild, in view of 1.6. Yet there is no doubt that if one needs only "nice" decompositions it would be possible (hence, preferable) to start with strong decompositions; the greater part of section 3 would still hold (even without the pullback axiom). However, if we wish to account for basic algebraic phenomena, we certainly cannot overlook the induced homomorphism theorem (first part of 5.4). This does not have to be formulated with congruences (kernel pairs will do as nicely) but no matter what formulation is chosen, the property requires that our chosen epimorphisms be at least subregular. This means that we have to start with regular decompositions (1.3). For this reason, nothing can be gained by considering factorization systems (still granted that we want a suitable categorical description of basic algebraic phenomena), since the decompositions therein will have to be regular and then no gain of generality will occur.

The pullback axiom now has the effect of ruling out a number of topological examples and is more difficult to justify at this level. To do this, we shall refer the reader to the discussion in paragraph 4.2 and show that, in a category with regular decompositions, in which the pullback axiom does not hold, the composition of relations by pullbacks is not well-defined: that is, if $bx = b'y$ is a pullback, $\mathrm{Im}(ax \times cy)$ does not depend solely upon $\mathrm{Im}(a \times b)$ and $\mathrm{Im}(b' \times c)$. To see this, let $pf' = fp'$ be a pullback, where p is a regular epimorphism and p' is not; p' has a regular decomposition (n,q) in which n is not an isomorphism. Let $\alpha = \epsilon = \mathrm{Im}(1 \times 1)$, $\beta = f^{-1} = \mathrm{Im}(f \times 1)$; note that we also have $\alpha = \mathrm{Im}(p \times p)$. Calculating $\beta\alpha$ by pullbacks, from $\alpha = \mathrm{Im}(1 \times 1)$, yields $\mathrm{Im}(f \times 1) = \beta$. If we use $\alpha = \mathrm{Im}(p \times p)$, we obtain $\mathrm{Im}(pf' \times p') = \mathrm{Im}(fp' \times p') = \mathrm{Im}((f \times 1)nq) = \mathrm{Im}((f \times 1)n)$, and this is a different relation since otherwise n would have to be an isomorphism.

Of course there are other approaches to algebraic phenomena and we certainly do not advocate the above as a panacea. Yet in some cases regular categories have a definite advantage in both generality and precision.

6. LIMITS AND COLIMITS IN A REGULAR CATEGORY.

1. In a variety limits and colimits can be constructed in terms of elements. In this section we give similar constructions in an arbitrary regular category $\mathcal{C}$ and give a number of related facts and applications.

For the notation, we know that a limit or colimit is the limit or colimit of a diagram (=functor) $\mathcal{D} : I \longrightarrow \mathcal{C}$, where I is a small category; for all objects $i \in I$ and morphisms $m \in I$, we write D_i for $\mathcal{D}(i)$ and D_m for $\mathcal{D}(m)$; morphisms of diagrams over I will be denoted by $\mathfrak{a} = (a_i)_{i\in I}$ or by similar notations; instead of using constant diagrams (which is cumbersome) we denote the limit of $\mathcal{D}$ by $(L,(\ell_i)_{i\in I})$, and similarly for the colimits, when they exist, and use a similar notation for compatible (cocompatible) families (= morphisms from (to) constant diagrams).

2. First of all, in any (= not necessarily regular) complete category, there already is an "elementary" construction of limits. Namely, notation being as above, let $P = \prod_{i\in I} D_i$ be the product, with projections $p_i : P \longrightarrow D_i$; let $k \in \bigwedge_{m\in I} \mathrm{Equ}(p_j, D_m p_i$, where in the intersection $m : i \longrightarrow j$; put $k : L \longrightarrow P$ and $\ell_i = p_i k$. Then $(L,(\ell_i)_{i\in I})$ is a limit of $\mathcal{D}$ [9].

If $\mathcal{C}$ is regular (and complete), we have the following property:

Proposition 6.1. With the same notation, let $(A,(a_i)_{i\in I})$ be a compatible family for $\mathcal{D}$ inducing $a : A \longrightarrow L$. Then $\ker a = \bigwedge_{i\in I} \ker a_i$. In particular, $\bigwedge_{i\in I} \ker \ell_i = \varepsilon$.

Proof. We first prove the property in case I is discrete; i.e. $(a_i)_{i\in I}$ is just a family of morphisms $A \longrightarrow D_i$ and $a = \chi_{i\in I}\, a_i$.

Let $a_i x_i = a_i y_i$ be a pullback, so that $\ker a_i = \mathrm{Im}(x_i \times y_i)$; similarly, $\ker a = \mathrm{Im}(x \times y)$, where $ax = ay$ is a pullback; we have to show that $\mathrm{Im}(x \times y) = \bigwedge_{i \in I} \mathrm{Im}(x_i \times y_i)$. First, $a_i = p_i a$ [where p_i is the projection from the product] so that $\ker a \leq \ker a_i$ for all i . Conversely, assume that $\mathrm{Im}(u \times v) \leq \mathrm{Im}(x_i \times y_i)$ for all i. Since $x_i \times y_i$ is a monomorphism, we then have $u \times v = (x_i \times y_i)t_i$ for some t_i . It follows that $p_i au = a_i x_i t_i = a_i y_i t_i = p_i av$ for all i, whence $au = av$; hence $u \times v = (x \times y)t$ for some t and $\mathrm{Im}(u \times v) \leq \mathrm{Im}(x \times y)$. Thus the formula is proved in that case.

In the general case, we have (keeping the same notation) $a_i = \ell_i a = p_i ka$ for all i , so that $ka = \mathop{\times}_{i \in I} a_i$. Hence

$$\bigwedge_{i \in I} \ker a_i = \ker \mathop{\times}_{i \in I} a_i = \ker ka = \ker a \ .$$

In any regular category we also have the following description of equalizers:

Lemma 6.2. Let m be a monomorphism. The following are equivalent: i) $m \in \mathrm{Equ}(f,g)$; ii) $\mathrm{Im}(m \times m) = g^{-1}f \wedge \epsilon$; iii) $\mathrm{Im}\, m = \Delta^s(g^{-1}f)$. In particular $\mathrm{Equ}(f,g) = \Delta^s(g^{-1}f)$.

Proof. Let $fx = gy$ be a pullback; then $x \times y$ is a monomorphism and calculating $g^{-1}f$ by pullbacks yields $g^{-1}f = \mathrm{Im}(x \times y)$. If $(x \times y)k = \Delta n$ is another pullback, then $\mathrm{Im}(n \times n) = \mathrm{Im}\, \Delta n = g^{-1}f \wedge \epsilon$ and $\mathrm{Im}\, n = \Delta^s(g^{-1}f)$. We now show that $n \in \mathrm{Equ}(f,g)$. First, $xk = yk = n$, so that $fn = fxk = gyk = gn$; also, n is a monomorphism. Further assume that $fh = gh$. Then $h = xt = yt$ in the first pullback, whence $\mathrm{Im}(h \times h) \leq \mathrm{Im}(x \times y)$; since also $\mathrm{Im}(h \times h) = \mathrm{Im}\, \Delta h \leq \epsilon$, it follows that $\mathrm{Im}(h \times h) \leq \mathrm{Im}(n \times n)$; since $n \times n$ is a monomorphism, we conclude that $h \times h = (n \times n)u$, and $h = nu$ for some (unique) u . Thus $n \in \mathrm{Equ}(f,g)$.

Now, for any monomorphism m , each of i), ii), iii) is equivalent to $\operatorname{Im} m = \operatorname{Im} n$, as readily seen, so that these conditions are equivalent.

3. We now turn to colimits. In this case, we can give a new (= not due to [9]) construction. It is based upon the following lemma:

Lemma 6.3. $cf = cg$ if and only if $\operatorname{Im}(f \times g) \leq \ker c$.

Proof. Let $cx = cy$ be a pullback. Then $cf = cg$ is successively equivalent to: $f = xt$, $g = yt$ for some t ; $f \times g = (x \times y)t$ for some t ; $\operatorname{Im}(f \times g) \leq \operatorname{Im}(x \times y) = \ker c$.

Proposition 6.4. Let: $\mathcal{D} : I \longrightarrow \mathcal{C}$ be a diagram over the small category I ; $S = \coprod_{i \in I} D_i$ be a coproduct, with injections $n_i : D_i \longrightarrow S$; $c : S \longrightarrow C$ be a morphism. Then $(C, (cn_i)_{i \in I})$ is a colimit of $\mathcal{D}$ if and only if c is a regular epimorphism and $\ker c$ is the least congruence on S containing $\operatorname{Im}(n_i \times n_j D_m)$ for all $m : i \longrightarrow j \in I$.

Remark. The result has to be stated that way since we cannot be sure that there will be a least congruence containing all $\operatorname{Im}(n_i \times n_j D_m)$ (even if the coproduct exists); in fact we shall use 6.4 later to produce such least congruences.

Proof. First, assume that $(C, (cn_i)_{i \in I})$ is a colimit of $\mathcal{D}$. Then in particular it is a cocompatible family, so that $cn_i = cn_j D_m$ whenever $m : i \longrightarrow j \in I$; hence $\ker c$ contains all $\operatorname{Im}(n_i \times n_j D_m)$, by the lemma. If furthermore $\ker f$ is a congruence on S with that property, then $fn_i = fn_j D_m$ for all m , i.e. $(fn_i)_{i \in I}$ is a cocompatible family; therefore there exists t such that $fn_i = tcn_i$ for all i ; then $f = tc$ and $\ker c \leq \ker f$. Finally, let (m, p) be a regular decomposition of c . Since $\ker c = \ker p$ contains all $\operatorname{Im}(n_i \times n_j D_m)$, we conclude as above that $p = tc$ for some t ; it follows that $tm = 1$ and the monomorphism m is in fact a coretraction.

However, $uc = vc$ implies $ucn_i = vcn_i$ for all i and $u = v$, so that c is an epimorphism.

Conversely, assume that c satisfies the conditions in the statement. Then, first, $cn_i = cn_j D_m$ for all $m \in I$, so that $(cn_i)_{i \in I}$ is a cocompatible family. Any other cocompatible family $(f_i)_{i \in I}$ will induce a morphism f from the coproduct, with $f_i = fn_i$ for all i ; by cocompatibility we see that $fn_i = fn_j D_m$ for all m, so that $\ker f$ contains all $\mathrm{Im}(n_i \times n_j D_m)$; hence $\ker f \geq \ker c$, and it follows from 5.4 that $f = tc$ for some t. Then $f_i = tcn_i$ and this factorization is clearly unique, which shows that $(C, (cn_i)_{i \in I})$ is indeed a colimit of $\mathfrak{D}$ and completes the proof.

A similar, but simpler, result exists for coequalizers:

Proposition 6.5. $c \in \mathrm{Coequ}(f,g)$ if and only if c is a regular epimorphism and $\ker c$ is the least congruence containing $\mathrm{Im}(f \times g)$.

Proof. Easy enough, using 6.3, 5.4.

In general we also have a connection between colimits and unions:

Proposition 6.6. Let $\mathfrak{D} : I \longrightarrow \mathfrak{C}$ be a diagram with a colimit $(C, (c_i)_{i \in I})$; let $(A, (a_i)_{i \in I})$ be a cocompatible family inducing $a : C \longrightarrow A$. Then $\mathrm{Im}\, a = \bigvee_{i \in I} \mathrm{Im}\, a_i$. In particular $\bigvee_{i \in I} \mathrm{Im}\, c_i = 1$.

Proof. For all i, $a_i = ac_i$, so that $\mathrm{Im}\, a_i \leq \mathrm{Im}\, a$. Conversely let m be a monomorphism such that $\mathrm{Im}\, a_i \leq \mathrm{Im}\, m$ for all i. Then for each i we have $a_i = mt_i$ for some t_i, and since $(a_i)_{i \in I}$ is a cocompatible family and m is a monomorphism we see that $(t_i)_{i \in I}$ is cocompatible. Hence there exists a morphism t with $t_i = tc_i$ for all i, and since $ac_i = mt_i = mtc_i$ for all i we have $a = mt$ and $\mathrm{Im}\, a \geq \mathrm{Im}\, m$. Therefore $\mathrm{Im}\, a = \bigvee_{i \in I} \mathrm{Im}\, a_i$.

4. From these results we see that in a regular category there

are implications between the existence of certain limits and colimits.

Proposition 6.7. A well-powered regular category with coproducts has intersections.

Proof. By well-powered-ness we can produce for each object A a partially ordered set A_s which we could call set of all subobjects of A. By 6.6, A_s is a [small] complete $\vee$-semilattice with a greatest element 1, hence is a complete lattice. Thus we have intersections.

Intersections of congruences can also be obtained as follows. If we assume that $\mathfrak{A}$ is complete, then we always have an intersection for any family of congruences, and the resulting relation is a congruence, by 6.1 (for $\bigwedge_{i\in I} \ker f_i$ is then equal to the congruence $\ker \bigtimes_{i\in I} f_i$).

More interesting is the following result:

Proposition 6.8. Let $\mathfrak{A}$ be a regular category in which Lawvere's condition holds. Further assume either that $\mathfrak{A}$ is regularly co-well-powered and has intersections, or that $\mathfrak{A}$ has unions and that inverse images preserve directed unions. Then, for each relation $\alpha : A \longrightarrow A$ there exists a least congruence on A containing α.

Proof. First let $(\gamma_i)_{i\in I}$ be any family of congruences having an intersection $\gamma = \bigwedge_{i\in I} \gamma_i$. We have $\epsilon \leq \gamma$ since each γ_i is reflexive; also $\gamma^{-1} = (\bigwedge_{i\in I}\gamma_i)^{-1} = \bigwedge_{i\in I} \gamma_i^{-1} = \gamma$ since each γ_i is symmetric; and $\gamma\gamma \leq \gamma_i\gamma_i = \gamma_i$ for each i, so that $\gamma\gamma \leq \gamma$ and γ is transitive; therefore γ is a congruence. Thus, under Lawvere's condition, every existing intersection of congruences is a congruence. Under the first set of further assumptions it now suffices to take the intersection of all congruences on A that contain α.

Under the second set of assumptions, we first let $\beta = \epsilon \vee \alpha \vee \alpha^{-1}$.

Note that $\alpha \leq \beta$ and that β is reflexive and symmetric. We now define β^n by successive compositions: $\beta^{n+1} = \beta^n\beta$ and let $\gamma = \bigvee_{i\in I} \beta^n$; we note that $\beta^n \leq \beta^{n+1}$, since β is reflexive, so that this is a directed union. We claim that γ is the least congruence containing α. Since every congruence which contains α also contains β, and all β^n, it suffices to show that γ is a congruence. It is clear that γ is reflexive, and symmetric (since by induction, all β^n are symmetric). For the transitivity, we use the assumption that inverse images preserve directed unions.[This condition will be called (C_3') in next chapter and the proof of the following facts can be found there in detail.] When applied to inverse images under monomorphisms, it means that intersection with a fixed subobject distributes directed unions. By directedness the same is true for finite intersections in general. Thus we see from Puppe's formula that the composition of relations also distributes directed unions. It follows that $\gamma\gamma = \bigvee_{m,n>0} \beta^m\beta^n = \gamma$, and γ is transitive, which completes the proof.

Corollary 6.9. Let $\mathfrak{a}$ be a regular category satisfying either of the following conditions: i) $\mathfrak{a}$ is complete and regularly co-well-powered; ii) $\mathfrak{a}$ has intersections, is regularly co-well-powered and satisfies Lawvere's condition; iii) $\mathfrak{a}$ has unions, satisfies Lawvere's condition and inverse images in $\mathfrak{a}$ preserve directed unions. Then $\mathfrak{a}$ has coequalizers; if $\mathfrak{a}$ has coproducts(finite coproducts), then $\hat{\mathfrak{a}}$ is cocomplete (finitely cocomplete).

Proof. The conclusion of 6.8 will hold in either case and then it follows from 6.5 that $\mathfrak{a}$ has coequalizers.

Synopsis of definitions and formulæ

1 , 1_A : identity morphism, also greatest subobject

$\underline{x} \wedge \underline{y}$, $\bigvee_{i \in I} \underline{x}_i$: l.u.b. of families of subobjects

$\underline{x} \wedge \underline{y}$, $\bigwedge_{i \in I} \underline{x}_i$: g.l.b. (intersections) of subobjects

Im m : subobject containing the monomorphism m

Im f : Image of f (= Im m , if (m,p) is a regular decomposition of f.

$f^s\underline{x}$: inverse image of subobject $\underline{x}$ under f

$f_s\underline{x}$: direct image of subobject $\underline{x}$ under f

$$\begin{cases} (1_A)^s\underline{x} = \underline{x} \;,\; (fg)^s\underline{x} = g^s f^s \underline{x} \;,\; f^s 1 = 1 \\ f^s(\bigwedge_{i \in I} \underline{x}_i) = \bigwedge_{i \in I} f^s \underline{x}_i \;,\; \underline{x} \leq \underline{y} \Longrightarrow f^s\underline{x} \leq f^s\underline{y} \end{cases}$$

$$\begin{cases} (1_A)_s\underline{x} = \underline{x} \;,\; (fg)_s\underline{x} = f_s g_s \underline{x} \\ f_s 1 = \mathrm{Im}\, f \;,\; f_s\, \mathrm{Im}\, g = \mathrm{Im}\, fg \leq \mathrm{Im}\, f \\ f_s(\bigvee_{i \in I} \underline{x}_i) = \bigvee_{i \in I} f_s \underline{x}_i \;,\; \underline{x} \leq \underline{y} \Longrightarrow f_s\underline{x} \leq f_s\underline{y} \end{cases}$$

$$\begin{cases} \underline{x} \leq f^s\underline{y} \Longleftrightarrow f_s\underline{x} \leq \underline{y} \\ f_s f^s \underline{y} \leq \underline{y} \;,\; f^s f_s \underline{x} \geq \underline{x} \\ f_s f^s f_s = f_s \;,\; f^s f_s f^s = f^s \\ f_s f^s \underline{x} = \underline{x} \wedge \mathrm{Im}\, f \\ f \;\; \text{reg.epi} \Longrightarrow f_s f^s \underline{x} = \underline{x} \\ f \;\; \text{mono} \Longrightarrow f^s f_s \underline{y} = \underline{y} \end{cases}$$

α, β, γ : relations

ε , ε_A : ("equality" , "diagonal") least congruence on A

υ . υ_A : = $1_{\tilde{A}}$, greatest congruence on A

Im α = Im a if α = Im(a × b)

α^{-1} : inverse of α ; α^{-1} = Im(b × a) when α = Im(a × b)

$\tilde{A} = A \,\pi\, A$, $\tilde{f} = f \,\pi\, f$: tilda functor, preserves limits and regular decompositions

ker f : congruence induced by f , = $f^{-1}f$ = Im(x × y) where fx = fy is a pullback

$$\begin{cases} \epsilon^{-1} = \epsilon \ , \ \alpha \leq \beta \Longrightarrow \alpha^{-1} \leq \beta^{-1} \ , \ (\alpha^{-1})^{-1} = \alpha \\ (\bigvee_{i\in I} \alpha_i)^{-1} = \bigvee_{i\in I} \alpha_i^{-1} \ , \ (\bigwedge_{i\in I} \alpha_i)^{-1} = \bigwedge_{i\in I} \alpha_i^{-1} \end{cases}$$

$$\begin{cases} \mathrm{Im}(\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I} \mathrm{Im}\, \alpha_i \ , \ \alpha \leq \beta \Longrightarrow \mathrm{Im}\, \alpha \leq \mathrm{Im}\, \beta \end{cases}$$

$$\begin{cases} \beta\alpha = r_s(p^s\alpha \wedge q^s\beta) \quad \text{(Puppe's formula)} \\ \epsilon\alpha = \alpha \ , \ \beta\epsilon = \beta \ , \ (\alpha\beta)\gamma = \alpha(\beta\gamma) \\ \alpha \leq \alpha' \ , \ \beta \leq \beta' \Longrightarrow \alpha\beta \leq \alpha'\beta' \ ; \ (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} \\ \alpha = \mathrm{Im}(a \times b) \Longrightarrow \alpha = ba^{-1} \end{cases}$$

$\alpha_s\underline{x}$: direct image of subobject $\underline{x}$ under relation α

$\alpha^s\underline{x}$: inverse image of subobject $\underline{x}$ under relation α

$$\begin{cases} \epsilon_s\underline{x} = \underline{x} \ , \ (\alpha\beta)_s\underline{x} = \alpha_s\beta_s\underline{x} \ , \ \alpha_s 1 = \mathrm{Im}\, \alpha \\ \underline{x} \leq \underline{y} \Longrightarrow \alpha_s\underline{x} \leq \alpha_s\underline{y} \ , \ \alpha \leq \beta \Longrightarrow \alpha_s\underline{x} \leq \beta_s\underline{x} \\ \mathrm{Im}\, \alpha\beta = \alpha_s \mathrm{Im}\, \beta \ , \ \alpha = \mathrm{Im}(a \times b) \Longrightarrow \alpha_s = b_s a^s \end{cases}$$

$$\begin{cases} \alpha^s = (\alpha^{-1})_s \ , \ \alpha_s = (\alpha^{-1})^s \\ \epsilon^s\underline{x} = \underline{x} \ , \ (\alpha\beta)^s\underline{x} = \beta^s\alpha^s\underline{x} \ , \ \alpha = \mathrm{Im}(a \times b) \Longrightarrow \alpha^s = a_s b^s \\ \underline{x} \leq \underline{y} \Longrightarrow \alpha^s\underline{x} \leq \alpha^s\underline{y} \ , \ \alpha \leq \beta \Longrightarrow \alpha^s\underline{x} \leq \beta^s\underline{x} \end{cases}$$

$$\begin{cases} ff^{-1} \leq \epsilon \ , \ f^{-1}f \geq \epsilon \\ f \ \text{ reg. epi} \Longleftrightarrow ff^{-1} = \epsilon \\ f \ \text{ mono} \Longleftrightarrow f^{-1}f = \epsilon \end{cases}$$

$f_r\alpha$: direct image of relation α under morphism f

$f^r\alpha$: inverse image of relation α under morphism f

$$\begin{cases} f_r\alpha = f\alpha f^{-1} = \tilde{f}_s\alpha \ , \ f^r\alpha = f^{-1}\alpha f = \tilde{f}^s\alpha \\ f^r \ker g = \ker gf \geq \ker f \\ f^r\epsilon = \ker f \ , \ f_r \ker f \leq \epsilon \end{cases}$$

II. DIRECTED COLIMITS IN REGULAR CATEGORIES

Our first result gives necessary and sufficient conditions, of an elementary nature, that directed colimits in a given cocomplete regular category be exact. In the abelian case, Grothendieck showed that the subobject condition

A.B.5: $\underline{x} \wedge (\bigvee_{i \in I} \underline{y}_i) = \bigvee_{i \in I} (\underline{x} \wedge \underline{y}_i)$ whenever $(y_i)_{i \in I}$ is directed

is necessary and sufficient [15],[31] . In the case of a regular category, the necessary and sufficient condition comes in three parts:

(C_3') Inverse images preserve directed unions of subobjects;

(C_3'') A directed union of congruences is a congruence;

(C_3''') If $(X_i)_{i \in I}$ is the family of objects of a monic direct system [= in which all morphisms $X_i \longrightarrow X_j$ $(i \leq j)$ are monomorphisms], there exists a family of monomorphisms $X_i \longrightarrow C$ (not necessarily a cocompatible family).

In the abelian case, (C_3'') and (C_3''') evaporate. The remaining condition (C_3') is still stronger that A.B.5 (though no harder to verify on the examples): the extra strength is used in the proof to manipulate relations (which are not needed in the abelian case). It implies (C_3'') when Lawvere's condition on congruences holds; and (C_3''') holds whenever coproduct injections are monomorphisms, so these are fairly mild conditions.

The proof occupies most of this part. It is somewhat technical; also, unlike what happens in the abelian case, preservation of finite limits has to be established, and even though it implies preservation of monomorphisms, the latter has to be shown first anyway. In the we obtain additional results showing that when directed colimits are exact they show additional good behavior: for instance, Gray's condition $\mathfrak{F}_2$ holds.

A cocomplete regular category in which directed colimits are exact is called a C_3 regular category. It is called C_4 if in addition it is complete and satisfies Gray's condition $\mathfrak{F}_1$ [10],[31], which is the same as Grothendieck's condition A.B.6 [15]. In the last section we show that in a C_4 regular category any product of directed colimits can be rewritten as a directed colimit of products, provided that all direct systems under consideration are monic. The last restriction can be lifted if furthermore the category is [regularly] C_1^*, i.e. any product of regular epimorphisms is a regular epimorphism. Of course all these conditions hold in a C_4, C_1^* abelian category, as well as in any finitary variety.

All these results are taken from [14]. References such as I.x.y refer to result x.y in part I above; we use the same conventions as in that part. Throughout, I will also denote a directed preordered set. A direct system $\mathfrak{X}$ over I is a functor of domain I, and we write X_i for $\mathfrak{X}(i)$ $(i \in I)$ and $x_{ij} : X_i \longrightarrow X_j$ $(i \leq j)$, the objects and morphisms in the system. We denote $\varinjlim \mathfrak{X} = \varinjlim_{i \in I} X_i$ by X and $X_i \longrightarrow X$ by x_i. Similar conventions apply to direct systems $\mathfrak{Y}, \mathfrak{Z}$, etc.

It is suggested that the reader be well-acquainted with the techniques developed in the first part before reading the proofs which follow.

1. THE MAIN THEOREM: DIRECT PART.

1. In this part we let $\mathcal{C}$ be C_3 regular category, i.e. we assume that $\mathcal{C}$ is cocomplete and that directed colimits are exact. Note that for each directed preordered set I the functor category $[I,\mathcal{C}]$ is regular, by I.2.1, with pointwise decompositions, finite limits and colimits, so that in particular it makes sense to say that the colimit functor is exact. We shall show that (C_3'), (C_3''), (C_3''') hold in $\mathcal{C}$.

2. Let $f : A \longrightarrow B \in \mathcal{C}$ and $(\underline{x}_i)_{i\in I}$ be a directed family of subobjects of B. We may define $i \leq j$ if and only if $\underline{x}_i \leq \underline{x}_j$ and then I becomes a directed preordered set; a direct system $\mathfrak{X} : I \longrightarrow \mathcal{C}$ is then constructed as follows. Since I is a set we can select for each $i \in I$ a monomorphism m_i with $\operatorname{Im} m_i = \underline{x}_i$. Let X_i be the domain of m_i. If $i \leq j$, then $\operatorname{Im} m_i \leq \operatorname{Im} m_j$ and $m_i = m_j x_{ij}$ for some unique $x_{ij} : X_i \longrightarrow X_j$; by the uniqueness, it is clear that we now have a direct system. In addition, we have a [pointwise] monomorphism $\mathfrak{m} = (m_i)_{i\in I} : \mathfrak{X} \longrightarrow B$. Since $\mathcal{C}$ is C_3, the induced morphism $m : X \longrightarrow B$ is a monomorphism; by I.6.6, $\operatorname{Im} m = \bigvee_{i\in I} \operatorname{Im} m_i = \bigvee_{i\in I} \underline{x}_i$.

For each i, $f^s \underline{x}_i = \operatorname{Im} n_i$, where $f n_i = m_i g_i$ is a pullback. As above, there is a direct system $\mathfrak{Y} : I \longrightarrow \mathcal{C}$ with a [pointwise] monomorphism $\mathfrak{n} = (n_i)_{i\in I} : \mathfrak{Y} \longrightarrow A$ (note that $i \leq j$ implies $f^s \underline{x}_i \leq$ $\leq f^s \underline{x}_j$); the induced morphism $n : Y \longrightarrow A$ is a monomorphism and satisfies $\operatorname{Im} n = \bigvee_{i\in I} f^s \underline{x}_i$.

We also have a morphism $(g_i)_{i\in I} : \mathfrak{Y} \longrightarrow \mathfrak{X}$. Since $f\mathfrak{n} = \mathfrak{m}\mathfrak{g}$ is a pullback and $\mathcal{C}$ is C_3, the colimit square $fn = mg$ is also a pullback; hence $\bigvee_{i\in I} f^s \underline{x}_i = \operatorname{Im} n = f^s \operatorname{Im} m = f^s(\bigvee_{i\in I} \underline{x}_i)$. Therefore (C_3') holds.

3. The verification of (C_3'') is similar. Let $(\alpha_i)_{i\in I}$ be a directed family of congruences on $A \in \mathcal{C}$. Write $\alpha_i = \operatorname{Im}(x_i \times y_i) = \ker p_i$,

where p_i is a regular epimorphism and $p_i x_i = p_i y_i$ is a pullback, so that $x_i \times y_i$ is a monomorphism. A direct system $\mathfrak{X}$ is constructed as above, so that $(x_i \times y_i)_{i \in I}$ is a monomorphism $\mathfrak{X} \longrightarrow A \pi A$. In addition, $i \leq j$ implies $\ker p_i = \alpha_i \leq \alpha_j = \ker p_j$, so that by the induced homomorphism theorem (I.5.4) we have $p_j = b_{ij} p_i$ for some unique b_{ij}. From this we obtain a direct system $\mathfrak{B}$ such that $(p_i)_{i \in I}$ is a regular epimorphism $A \longrightarrow \mathfrak{B}$. Since $p_i x_i = p_i y_i$ is a pullback, we obtain at the colimit a pullback $px = py$. There $x \times y$ is also the colimit of $(x_i \times y_i)_{i \in I}$ and by I.6.6 $\operatorname{Im}(x \times y) =$
$= \bigvee_{i \in I} \operatorname{Im}(x_i \times y_i) = \bigvee_{i \in I} \alpha_i$. It follows that $\bigvee_{i \in I} \alpha_i = \ker p$ is a congruence.

4. The verification of (C_3''') is less straightforward. It follows from the slightly more general result:

Lemma 1.1. Let $\mathfrak{C}$ be a finitely complete category which has directed colimits that preserve monomorphisms. If $\mathfrak{X}$ is a monic direct system in $\mathfrak{C}$ over I, then every morphism $x_i : X_i \longrightarrow X$ is a monomorphism.

Proof. The proof is immediate if I happens to be a directed $\wedge$-semilattice. In that case there is for each $i \in I$ a direct system $\mathfrak{Y} : I \longrightarrow \mathfrak{C}$, defined by: $Y_j = X_{i \wedge j}$, $y_{jk} = x_{i\wedge j, i \wedge k}$ $(j \leq k)$. Also there is a monomorphism $(x_{i\wedge j, j})_{j \in I} : \mathfrak{Y} \longrightarrow \mathfrak{X}$; we claim its colimit is precisely x_i. First note that (up to isomorphism) $Y = X_i$, with $y_j = x_{i \wedge j, i}$; then, for all $j \in I$, $x_i\, y_j = x_i\, x_{i\wedge j, i} = x_{i \wedge j} =$
$= x_j\, x_{i \wedge j, j}$ – which proves the claim. By the hypothesis on $\mathfrak{C}$, x_i is then a monomorphism.

If now I is arbitrary, then we come back to the case of a directed $\wedge$-semilattice as follows. First we find the semilattice. For each $k \in I$, let S_k be the set of all intersections of finitely many subobjects of X_k of the form $\operatorname{Im} x_{ik}$ $(i \leq k)$. Note that S_k is an

$\wedge$-semilattice. If $k \leq \ell$ in I, a map $s_{k\ell} : S_k \longrightarrow S_\ell$ is defined as follows. Since $x_{k\ell} : X_k \longrightarrow X_\ell$ is a monomorphism, we do not need regular decompositions to have direct images under $x_{k\ell}$; in addition, direct images under $x_{k\ell}$ preserve intersections (for finite intersections: if $mn' = nm'$ is a pullback, then so is $(x_{k\ell}m)n' = (x_{k\ell}n)m'$). Hence $(x_{k\ell})_s$ restricts to a mapping $s_{k\ell} : S_k \longrightarrow S_\ell$, which is in fact an injective homomorphism of $\wedge$-semilattices. Furthermore, x_{kk} is the identity, hence so is s_{kk}; if $k \leq \ell \leq m$ in I, then $x_{km} = x_{\ell m}x_{k\ell}$, hence $(x_{km})_s = (x_{\ell m})_s(x_{k\ell})_s$ and $s_{km} = s_{\ell m}s_{k\ell}$; in other words, we now have a direct system of $\wedge$-semilattices; this yields an $\wedge$-semilattice $S = \varinjlim S_k$, which comes with injective homomorphisms $s_k : S_k \longrightarrow S$ such that $S = \bigcup_{k\in I} s_k(S_k)$.

An order-preserving map $i \longmapsto \hat{i}$, $I \longrightarrow S$, is defined by: $\hat{i} = s_i(\mathrm{Im}\ x_{ii}) = s_i(1)$; it is order-preserving since $i \leq j$ implies $\hat{i} = s_j(s_{ij}(\mathrm{Im}\ x_{ii})) = s_j(\mathrm{Im}\ x_{ij}) \leq s_j(\mathrm{Im}\ x_{jj})$ since each s_j is order-preserving. The image $\hat{I} = \{\hat{i} ; i \in I\}$ is cofinal in S since for each $u \in S$ we have $u \in s_k(S_k)$ for some k and therefore $u \leq \hat{k}$. It follows that S is directed.

For each $u \in S$, select $k \in I$ with $u \in s_k(S_k)$ and a monomorphism $y_{uk} : Y_u \longrightarrow X_k$ such that $u = s_k(\mathrm{Im}\ y_{uk})$ [it is easy to see that a different choice only replaces Y_u by an isomorphic object]. Now assume that $u \leq v$ in S and that $y_{v\ell} : Y_v \longrightarrow X_\ell$ has been selected for v (so that $v = s_\ell(\mathrm{Im}\ y_{v\ell})$). Since I is directed, we have $k \leq m$, $\ell \leq m$ for some $m \in I$; then also

$$s_m(\mathrm{Im}\ x_{km}y_{uk}) = s_m(s_{km}(\mathrm{Im}\ y_{uk})) = s_k(\mathrm{Im}\ y_{uk}) = u$$

and similarly $v = s_m(\mathrm{Im}\ x_{\ell m}y_{v\ell})$. Now s_m is an injective homomorphism and therefore reflects order [$s_m(a) \leq s_m(b)$ implies $s_m(a) = s_m(a) \wedge s_m(b) = s_m(a \wedge b)$ and $a = a \wedge b \leq b$]; hence, $u \leq v$ implies $\mathrm{Im}\ x_{km}y_{uk} \leq \mathrm{Im}\ x_{\ell m}y_{v\ell}$ and there exists a unique $y_{uv} : Y_u \longrightarrow Y_v$ such that $x_{km}y_{uk} = x_{\ell m}y_{v\ell}y_{uv}$. Note that y_{uv} is a monomorphism. If furthermore $m \leq n$ in I, then

$$x_{kn}y_{uk} = x_{mn}x_{km}y_{uk} = x_{mn}x_{\ell m}y_{v\ell}y_{uv} = x_{\ell n}y_{v\ell}y_{uv} \ ;$$

by the uniqueness, y_{uv} would be the same if we had started from $n \geq k$, instead of m. Since I is directed, it follows that y_{uv} does not depend on the choice of m (as long as m is large enough). If now $u = v$, then $k = \ell$, $y_{uk} = y_{v\ell}$ and since $x_{km}y_{uk} = x_{\ell m}y_{v\ell}1$ it follows that $y_{uu} = 1$. If $u \leq v \leq w$ in S, and we have selected $y_{wm} : Y_w \longrightarrow X_m$ for w and chosen n large enough, then

$$x_{mn}y_{wm}y_{uw} = x_{km}y_{uk} = x_{\ell n}y_{v\ell}y_{uv} = x_{mn}y_{wm}y_{vw}y_{uv}$$

shows that $y_{uw} = y_{vw}y_{uv}$. Therefore we have another monic direct system $\mathfrak{Y} : S \longrightarrow \mathfrak{C}$.

From $\mathfrak{Y}$ we obtain a direct system $\mathfrak{Y}' : I \longrightarrow \mathfrak{C}$, defined by $Y'_i = Y_{\hat{i}}$, $y'_{ij} = y_{\hat{i}\hat{j}}$ $(i \leq j)$; we claim that it is isomorphic to $\mathfrak{X}$. To see this, take $i \in I$. To $\hat{i} \in S$ we have associated $y_{\hat{i}k} : Y_{\hat{i}} \longrightarrow X_k$ (with $\hat{i} = s_k(\operatorname{Im} y_{\hat{i}k})$); we cannot assume that $k = i$ since it may happen that $\hat{i} = \hat{j}$ with $i \neq j$; but we may assume that $i \leq k$, for then we have seen that $\hat{i} \in s_k(S_k)$. Then

$$s_k(\operatorname{Im} y_{\hat{i}k}) = \hat{i} = s_i(\operatorname{Im} x_{ii}) = s_k(s_{ik}(\operatorname{Im} x_{ii})) = s_k(\operatorname{Im} x_{ik})$$

shows that $\operatorname{Im} y_{\hat{i}k} = \operatorname{Im} x_{ik}$ and therefore there is an isomorphism $a_i : Y_{\hat{i}} \longrightarrow X_i$ such that $y_{\hat{i}k} = x_{ik}a_i$. If $i \leq j$ in I and we have selected $y_{\hat{j}\ell} : Y_{\hat{j}} \longrightarrow X_\ell$ for $\hat{j}$ and chosen m large enough, then

$$x_{jm}x_{ij}a_i = x_{km}x_{ik}a_i = x_{km}y_{\hat{i}k} = x_{\ell m}y_{\hat{j}\ell}y_{\hat{i}\hat{j}} = x_{\ell m}x_{j\ell}a_jy_{\hat{i}\hat{j}} = x_{jm}a_jy_{\hat{i}\hat{j}}$$

(since $\hat{i} \leq \hat{j}$), so that $x_{ij}a_i = a_jy_{\hat{i}\hat{j}}$. Therefore $(a_i)_{i\in I} : \mathfrak{Y}' \longrightarrow \mathfrak{X}$ is an isomorphism.

Since $\hat{I}$ is cofinal in S, it is clear that the obvious morphism $\mathfrak{Y}' \longrightarrow \mathfrak{Y}$ induces an isomorphism at the colimits. Now S is a directed

$\wedge$-semilattice, and it follows from the first part of the proof that $y_f : Y_f \longrightarrow Y$ is a monomorphism. Using the isomorphisms $X \cong Y' \cong Y$ we conclude that $x_i : X_i \longrightarrow X$ is a monomorphism, q.e.d.

5. We have now proved the direct part of the main theorem in this part, namely:

Theorem 1.2. A cocomplete regular category is C_3 if and only if it satisfies (C_3'), (C_3'') and (C_3''') .

2. CONVERSE: PRESERVATION OF MONOMORPHISMS.

1. We now assume that $\mathfrak{a}$ is a cocomplete regular category which satisfies (C_3'), (C_3'') and (C_3''') and begin with a few easy consequences of (C_3') .

Proposition 2.1. Under (C_3'), finite intersections of subobjects and composition of relations distribute directed unions.

Proof. If first m is a monomorphism, then it follows from the definitions (or from I.3.7) that $m_s m^s \underline{x} = \operatorname{Im} m \wedge \underline{x}$ for all $\underline{x}$. If now $(\underline{x}_i)_{i\in I}$ is a directed family of subobjects of the codomain of m, then, by (C_3') and I.3.3,

$$\operatorname{Im} m \wedge (\bigvee_{i\in I} \underline{x}_i) = m_s(m^s(\bigvee_{i\in I} \underline{x}_i)) = m_s(\bigvee_{i\in I} m^s\underline{x}_i) = \\ = \bigvee_{i\in I} m_s m^s \underline{x}_i = \bigvee_{i\in I} (\operatorname{Im} m \wedge \underline{x}_i) \ .$$

This shows that intersections by a fixed subobject distributes directed unions. If now $(\underline{y}_j)_{j\in J}$ is another directed family, then, applying this to each $\underline{x}_i$ and then to $\bigvee_{j\in J} \underline{y}_j$, we obtain:

$$\bigvee_{\substack{i\in I\\ j\in J}} (\underline{x}_i \wedge \underline{y}_j) = \bigvee_{i\in I} (\bigvee_{j\in J} (\underline{x}_i \wedge \underline{y}_j)) = \bigvee_{i\in I} (\underline{x}_i \wedge (\bigvee_{j\in J} \underline{y}_j)) = (\bigvee_{i\in I} \underline{x}_i) \wedge (\bigvee_{j\in J} \underline{y}_j),$$

which proves the first assertion. The second assertion is them immediate on Puppe's formula.

Corollary 2.2. If Lawvere's condition (L) holds, then (C_3') implies (C_3'') .

Proof. Let $(\alpha_i)_{i\in I}$ be a directed family of congruences [with $I \neq \emptyset$] and $\alpha = \bigvee_{i\in I} \alpha_i$. It is clear that α is reflexive and symmetric; in view of (L) it suffices to prove that α is transitive (i.e. $\alpha\alpha \leq \alpha$). By 2.1, $\alpha\alpha = \bigvee_{j,k\in I} \alpha_j\alpha_k$. Now $\bigvee_{i\in I} \alpha_i\alpha_i \leq \bigvee_{j,k\in I} \alpha_j\alpha_k$ since the index set on the left is smaller; but the converse inequality holds since $(\alpha_i)_{i\in I}$ is directed. The transitivity of α then follows from that of each α_i .

It follows from 2.2 and I.5.2 that (C_3'') is superfluous in case G is abelian.

2. We now start a closer study of direct systems.

Lemma 2.3. Let $a_i : X_i \longrightarrow A$ $(i \in I)$ be a cocompatible family for the direct system $\mathfrak{X}\,[:I \longrightarrow G]$, inducing $a : X \longrightarrow A$. Then $a = \bigvee_{i\in I} a_i x_i^{-1}$.

Proof. First, $i \leq j$ implies $a_i x_i^{-1} = a_j x_{ij} x_{ij}^{-1} x_j^{-1} \leq a_j x_j^{-1}$; hence $(a_i x_i^{-1})_{i\in I}$ is a directed family of relations. Hence

$$\begin{aligned}(\bigvee_{i\in I} a_i x_i^{-1})(\bigvee_{j\in J} a_j x_j^{-1})^{-1} &= \bigvee_{i,j\in I} a_i x_i^{-1} x_j a_j^{-1} \leq \\ &\leq \bigvee_{k\in I} a_k x_k^{-1} x_k a_k^{-1} = \bigvee_{k\in I} a\, x_k x_k^{-1} x_k x_k^{-1} a^{-1} = \\ &= \bigvee_{k\in I} a\, x_k x_k^{-1} a^{-1} = \bigvee_{k\in I} (a x_k)(a x_k)^{-1} \leq \epsilon\ .\end{aligned}$$

Since also $(\bigvee_{i\in I} a_i x_i^{-1})^s 1 = \bigvee_{i\in I} (x_i)_s a^s 1 = \bigvee_{i\in I} \operatorname{Im} x_i = 1$ by I.6.6 , it follows from I.4.12 that $b = \bigvee_{i\in I} a_i x_i^{-1}$ is a morphism. To show that $b = a$, we note that, since I is directed, $b = \bigvee_{j\geq i} a_j x_j^{-1}$, for every $i \in I$; hence

$$bx_i = (\bigvee_{j\geq i} a_j x_j^{-1})x_i = \bigvee_{j\geq i} a_j x_j^{-1} x_i = \bigvee_{j\geq i} a_j x_j^{-1} x_j x_{ij} \geq$$
$$\geq \bigvee_{j\geq i} a_j x_{ij} = \bigvee_{j\geq i} a_i = a_i = ax_i \;;$$

since bx_i and ax_i are morphisms, this implies $bx_i = ax_i$; it holds for every i , hence $b = a$.

Corollary 2.4. If in 2.3 each a_i is a monomorphism, then a is a monomorphism.

Proof. $\ker a = (\bigvee_{i\in I} a_i x_i^{-1})^{-1}(\bigvee_{j\in I} a_j x_j^{-1}) \leq \bigvee_{k\in I} x_k a_k^{-1} a_k x_k^{-1} \leq \varepsilon$.

3. We now establish progressively stronger results. The next one already uses the full strength of the hypothesis.

Lemma 2.5. If $\mathfrak{X}$ is a monic direct system, then each x_i is a monomorphism.

Proof. Let $C = \coprod_{i\in I} X_i$ be the coproduct, with injections $m_i : X_i \longrightarrow C$; by (C_3'''), each m_i is a monomorphism. It follows from I.6.4 that there is a regular epimorphism $c : C \longrightarrow X$ such that $x_i = cm_i$ for all i , and that $\ker c$ is the least congruence on C that contains every $\mathrm{Im}(m_i \times m_j x_{ij})$ with $i \leq j$.

Let $\mathfrak{F}$ be the set of all finite subsets of $\{(i,j) \in I \,\Pi\, I \,;\, i \leq j\}$ [= of the preorder relation on I] . For each $F \in \mathfrak{F}$, the subdiagram of $\mathfrak{X}$ consisting of all X_i with only those x_{ij} with $(i,j) \in F$ has a colimit in $\mathfrak{C}$; again by I.6.4, there exists a least congruence α_F on C containing all $\mathrm{Im}(m_i \times m_j x_{ij})$ with $(i,j) \in F$. From that property it is clear that $F \subseteq G$ implies $\alpha_F \leq \alpha_G$, so that $(\alpha_F)_{F\in\mathfrak{F}}$ is a directed family of congruences. By (C_3''), $\alpha = \bigvee_{F\in\mathfrak{F}} \alpha_F$ is a congruence. Now the "least" property of α_F implies that $\alpha_F \leq \ker c$ for every F , so that $\alpha \leq \ker c$; the converse inequality follows from the similar property of $\ker c$, since α is a congruence. We conclude that $\ker c = \bigvee_{F\in\mathfrak{F}} \alpha_F$. Now if we can prove that $m_i^r\, \alpha_F \le \epsilon$ for all i and F, it will follows that

$$\ker x_i = \ker cm_i = m_i^r \ker c = m_i^r\Big(\bigvee_{F\in\mathfrak{F}} \alpha_F\Big) = \bigvee_{F\in\mathfrak{F}} m_i^r\, \alpha_F \le \epsilon ,$$

and the lemma will be proved.

For each $i \in I$, $F \in \mathfrak{F}$, there is a $t \in I$ with $i \le t$ and $j \le t$, $k \le t$ for all $(j,k) \in F$ (since F is finite). Consider the diagram:

$$\begin{tikzcd}
X_j \arrow[r] \arrow[rr, bend left, "m_j"] \arrow[dr, "x_{jt}"'] & \coprod_{j\le t} X_j \arrow[r] \arrow[d, "f"] & \Big(\coprod_{j\le t} X_j\Big)\sqcup\Big(\coprod_{j\not\le t} X_j\Big) = C \arrow[d, "f\sqcup 1 = h"] \\
& X_t \arrow[r, "g"'] & (X_t)\sqcup\Big(\coprod_{j\not\le t} X_j\Big)
\end{tikzcd}$$

where g as well as all unnamed maps are coproduct injections, and f is induced by all x_{jt} , $j \le t$, so that the diagram commutes for every $j \le t$.

If $(j,k) \in F$, then $j \le t$, $k \le t$ and we see on the diagram that

$$hm_j = gx_{jt} = gx_{kt}x_{jk} = hm_k x_{jk} \quad ;$$

then it follows from I.6.3 that $\mathrm{Im}(m_j \times m_k x_{jk}) \le \ker h$; therefore $\alpha_F \le \ker h$. On the other hand, g is a monomorphism (since m_t is a monomorphism), and so is x_{it} . Hence

$$m_i^r\, \alpha_F \le m_i^r \ker h = \ker hm_i = \ker gx_{it} = \epsilon ,$$

which completes the proof.

The next result gives one of the nice properties of directed colimits in our situation.

<u>Proposition</u> 2.6. For any direct system $X : I \longrightarrow \mathcal{C}$ [where $\mathcal{C}$ is a C_3 regular category], $\ker x_i = \bigvee_{j\ge i} \ker x_{ij}$ for every $i \in I$.

Proof. It is based on another construction of directed colimits which is somewhat more 'set-like'. First $j \leq k$ implies $\ker x_{ij} \leq \leq \ker x_{jk}x_{ij} = \ker x_{ik}$; it follows that $(\ker x_{ij})_{j \geq i}$ is a directed family of congruences, so that by (C_3'') $\alpha_i = \bigvee_{j \geq i} \ker x_{ij}$ is a congruence for every $i \in I$.

Put $\alpha_i = \ker p_i$, where $p_i : X_i \longrightarrow Y_i$ is a regular epimorphism. If $i \leq j$, then by (C_3'):

$$\ker p_j x_{ij} = \tilde{x}^s_{ij}(\bigvee_{k \geq j} \ker x_{jk}) = \bigvee_{k \geq j} \ker x_{ik} = \ker p_i$$

since I is directed; by I.5.4, $p_j x_{ij} = y_{ij} p_i$ for some unique $y_{ij} : Y_i \longrightarrow Y_j$, and y_{ij} is a monomorphism. The uniqueness implies that we now have a monic direct system $\mathfrak{Y} : I \longrightarrow \mathfrak{a}$. We now prove that the morphism $(p_i)_{i \in I} : \mathfrak{X} \longrightarrow \mathfrak{Y}$ induces an isomorphism on the colimits. First it is clear that $(y_i p_i)_{i \in I}$ is a cocompatible family for $\mathfrak{X}$. If $(a_i)_{i \in I}$ is any cocompatible family for $\mathfrak{X}$, then $i \leq j$ implies $\ker x_{ij} \leq \ker a_j x_{ij} = \ker a_i$; therefore $\ker p_i = \alpha_i \leq \ker a_i$; therefore $a_i = b_i p_i$ for some unique b_i ; the uniqueness easily implies that $(b_i)_{i \in I}$ is a cocompatible family for $\mathfrak{Y}$ [equivalently, one may use the induced homomorphism theorem in $[I,\mathfrak{a}]$]. This yields a morphism b unique such that $b_i = by_i$ for all i . We see that $a_i = by_i p_i$ for all i, and the uniqueness of b in this factorization follows from the other uniquenesses. Thus $(y_i p_i)_{i \in I}$ is a colimit of $\mathfrak{X}$, and there is an isomorphism t such that $tx_i = y_i p_i$ for all i.

Now y_i is a monomorphism, by 2.5, so that $\ker x_i = \ker tx_i = = \ker p_i = \bigvee_{j \geq i} \ker x_{ij}$, q.e.d.

4. We now give a lemma which is crucial for the next three proofs of preservation properties.

Lemma 2.7. Let $a_i : A_i \longrightarrow A$ be a family of morphisms such that $(\operatorname{Im} a_i)_{i\in I}$ is directed and $\bigvee_{i\in I} \operatorname{Im} a_i = 1$. Then $\bigvee_{i\in I} \operatorname{Im} \tilde{a}_i = 1$.

Proof. If $p : A \pi A \longrightarrow A$, $p_i : A_i \pi A \longrightarrow A_i$ are the first projections, then $p(a_i \pi 1_A) = a_i p_i$ is a pullback, so that, by I.3.10, $\operatorname{Im}(a_i \pi 1_A) = p^s \operatorname{Im} a_i$. Hence it follows from (C_3') that $\bigvee_{i\in I} \operatorname{Im}(a_i \pi 1_A) = 1$. Similarly, $\bigvee_{j\in I} \operatorname{Im}(1_{A_i} \pi a_j) = 1$ for each $i \in I$. Since $a_i \pi a_j = (a_i \pi 1_A)(1_{A_i} \pi a_j)$, it follows that

$$\bigvee_{i,j\in I} \operatorname{Im}(a_i \pi a_j) = \bigvee_{i\in I}\Big(\bigvee_{j\in I} (a_i \pi 1_A)_s \operatorname{Im}(1_{A_i} \pi a_j)\Big) =$$
$$= \bigvee_{i\in I}\Big((a_i \pi 1_A)_s\Big(\bigvee_{j\in I} \operatorname{Im}(1_{A_i} \pi a_j)\Big)\Big) =$$
$$= \bigvee_{i\in I} \operatorname{Im}(a_i \pi 1_A) = 1 .$$

Now if (m_i, p_i) is a regular decomposition of a_i for every i, we have $\operatorname{Im}(a_i \pi a_j) = \operatorname{Im}(m_i \pi m_j)$ since by I.1.12 finite products preserve decompositions. If $\operatorname{Im} a_i \le \operatorname{Im} a_k$, $\operatorname{Im} a_j \le \operatorname{Im} a_k$, then m_i, m_j factor though m_k and therefore $\operatorname{Im}(a_i \pi a_j) \le \operatorname{Im}(a_k \pi a_k)$. It follows that $(\operatorname{Im} \tilde{a}_i)_{i\in I}$ is cofinal in $(\operatorname{Im}(a_i \pi a_j))_{i,j\in I}$. The result follows.

We now are in position to prove that directed colimits in $\mathcal{G}$ preserve monomorphisms.

Let $\mathfrak{m} = (m_i)_{i\in I} : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a [pointwise] monomorphism of direct systems over I, and $m : X \longrightarrow Y$ be induced by $\mathfrak{m}$. By 2.7, $\bigvee_{i\in I} \operatorname{Im} \tilde{x}_i = 1$. Hence it follows from 2.6 that:

$$\ker m = \ker m \wedge \Big(\bigvee_{i\in I} \operatorname{Im} \tilde{x}_i\Big) = \bigvee_{i\in I}(\ker m \wedge \operatorname{Im} \tilde{x}_i) =$$
$$= \bigvee_{i\in I} (\tilde{x}_i)_s\, \tilde{x}_i^s \ker m = \bigvee_{i\in I} (\tilde{x}_i)_s \ker m x_i =$$
$$= \bigvee_{i\in I} (\tilde{x}_i)_s \ker y_i m_i = \bigvee_{i\in I} (\tilde{x}_i)_s\, \tilde{m}_i^s \ker y_i =$$
$$= \bigvee_{i\in I} (\tilde{x}_i)_s\, \tilde{m}_i^s\Big(\bigvee_{j\ge i} \ker y_{ij}\Big) = \bigvee_{j\ge i\in I} (\tilde{x}_i)_s\, \tilde{m}_i^s \ker y_{ij} =$$
$$= \bigvee_{j\ge i\in I} (\tilde{x}_i)_s \ker y_{ij} m_i = \bigvee_{j\ge i\in I} (\tilde{x}_i)_s \ker m_j x_{ij} =$$

$$= \bigvee_{j \geq i \in I} (\tilde{x}_i)_s \ker x_{ij} = \bigvee_{i \in I} (\tilde{x}_i)_s (\bigvee_{j \geq i} \ker x_{ij}) =$$
$$= \bigvee_{i \in I} (\tilde{x}_i)_s \ker x_i \leq \epsilon \quad ,$$

which proves that m is a monomorphism.

3. CONVERSE: PRESERVATION OF FINITE LIMITS.

1. Directed colimits already preserve finite colimits (and regular epimorphisms), hence to prove exactness it now suffices to show that they preserve finite limits. One may consider this section as the proof of the converse proper, the previous section (including preservation of monomorphisms) containing only lemmas. We successively prove that directed colimits preserve equalizers, and finite products.

2. Let $\mathcal{E} \xrightarrow{m} \mathcal{X} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{Y}$ be an equailizer diagram in $[I, \mathcal{G}]$; well, a cockney equalizer, what; we want to show that the colimit diagram $E \xrightarrow{m} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is an equalizer diagram (in $\mathcal{G}$). By the previous section we know that m is a monomorphism; also, $fm = gm$. We shall use the description of $\mathrm{Equ}(f,g)$ given by I.6.2 and hence try to prove that $\mathrm{Im}(m \times m) = g^{-1}f \wedge \epsilon$. Since $\mathrm{Im}\, m \leq \mathrm{Equ}(f,g)$, we already know that $\mathrm{Im}(m \times m) \leq g^{-1}f \wedge \epsilon$.

For each $i \in I$, we have, by 2.6 and (C_3') :

$$\tilde{x}_i^s (g^{-1}f \wedge \epsilon) = \tilde{x}_i^s (g^{-1}f) \wedge \tilde{x}_i^s \epsilon = x_i^{-1}g^{-1} f x_i \wedge x_i^{-1}x_i =$$
$$= g_i^{-1}y_i^{-1}y_i f_i \wedge x_i^{-1}x_i =$$
$$= (\bigvee_{j \geq i} g_i^{-1}y_{ij}^{-1}y_{ij}f_i) \wedge (\bigvee_{k \geq i} x_{ik}^{-1}x_{ik}) =$$
$$= \bigvee_{j,k \geq i} (g_i^{-1}y_{ij}^{-1}y_{ij}f_i \wedge x_{ik}^{-1}x_{ik}) =$$
$$= \bigvee_{t \geq i} (g_i^{-1}y_{it}^{-1}y_{it}f_i \wedge x_{it}^{-1}x_{it}) \quad \text{[by directedness]} =$$
$$= \bigvee_{t \geq i} (x_{it}^{-1}g_t^{-1}f_t x_{it} \wedge x_{it}^{-1}x_{it}) =$$

$$= \bigvee_{t\geq i} \tilde{x}_{it}^{s}(g_t^{-1}f_t \wedge \epsilon) = \bigvee_{t\geq i} \tilde{x}_{it}^{s} \operatorname{Im}(m_t \times m_t)$$

since $m_t \in \operatorname{Equ}(f_t, g_t)$. Therefore

$$\begin{aligned}(g^{-1}f \wedge \epsilon)\wedge \operatorname{Im} \tilde{x}_i &= (\tilde{x}_i)_s\, \tilde{x}_i^{s}(g^{-1}f \wedge \epsilon) = \bigvee_{t\geq i} (\tilde{x}_i)_s\, \tilde{x}_{it}^{s} \operatorname{Im}(m_t \times m_t) = \\ &= \bigvee_{t\geq i} (\tilde{x}_t)_s(\tilde{x}_{it})_s(\tilde{x}_{it})^{s} \operatorname{Im}(m_t \times m_t) \leq \\ &\leq \bigvee_{t\geq i} (\tilde{x}_t)_s \operatorname{Im}(m_t \times m_t) = \bigvee_{t\geq i} \operatorname{Im}(x_t m_t \times x_t m_t) \leq \\ &\leq \operatorname{Im}(m \times m)\end{aligned}$$

since $\operatorname{Im} x_t m_t = \operatorname{Im} me_t \leq \operatorname{Im} m$. Then it follows from (C_3') and 2.7 that $g^{-1}f \wedge \epsilon = \bigvee_{i\in I} ((g^{-1}f \wedge \epsilon)\wedge \operatorname{Im} \tilde{x}_i) \leq \operatorname{Im}(m \times m)$, q.e.d.

3. We now turn to the preservation of finite products. First we claim that it suffices to prove that the functor $-\pi A : \mathfrak{G} \longrightarrow \mathfrak{G}$ preserves directed colimits, for every $A \in \mathfrak{G}$. This will indeed yield a natural isomorphism $\varinjlim(\mathfrak{X}\,\pi A) \cong (\varinjlim \mathfrak{X})\,\pi A$ for every direct system $\mathfrak{X}$, hence, for any two $\mathfrak{X}, \mathfrak{Y} \in [I,\mathfrak{G}]$, natural isomorphisms

$$\varinjlim_{(i,j)\in I\pi I} (X_i \,\pi\, Y_j) \cong \varinjlim_{i\in I} (X_i \,\pi\, \varinjlim \mathfrak{Y}) \cong \varinjlim \mathfrak{X} \,\pi\, \varinjlim \mathfrak{Y} ;$$

and since I is directed, the diagonal is cofinal in $I \pi I$, so that there is a natural isomorphism

$$\varinjlim(\mathfrak{X}\,\pi\,\mathfrak{Y}) = \varinjlim_{i\in I} (X_i \,\pi\, Y_i) \cong \varinjlim_{(i,j)\in I\pi I} (X_i \,\pi\, Y_j) .$$

4. Now our functor $-\pi A$ preserves pullbacks (as readily seen) and regular decompositions (by I.1.12). Also, by I.1.13, I.3.10, for any f, $\operatorname{Im}(f \,\pi\, 1_A) = p^S \operatorname{Im} f$, where p is a projection, so that our functor also preserves directed unions of subobjects, by (C_3') .

Then let $\mathfrak{X}$ be a direct system (over I); let $\mathfrak{Y} = \mathfrak{X}\,\pi A$, so that $Y_i = X_i \,\pi A$ etc. It is clear that $(x_i \,\pi\, 1_A)_{i\in I}$ is a cocompatible family for $\mathfrak{Y}$; hence there is a morphism $t : Y \longrightarrow X\,\pi A$ such that $x_i \,\pi\, 1_A = ty_i$ for all i . Clearly t is natural in $\mathfrak{X}$; we want to show that it is an isomorphism.

Since our functor preserves directed unions of subobjects, $\bigvee_{i\in I} \operatorname{Im} x_i = 1$ implies $\bigvee_{i\in I} \operatorname{Im}(x_i \pi 1_A) = \operatorname{Im}(1 \pi 1_A) = 1$. Hence

$$\operatorname{Im} t = t_s(\bigvee_{i\in I} \operatorname{Im} y_i) = \bigvee_{i\in I} \operatorname{Im} ty_i = \bigvee_{i\in I} \operatorname{Im}(x_i \pi 1_A) = 1$$

and t is a regular epimorphism.

On the other hand, our functor preserves pullbacks, hence also congruences, as well as directed unions of subobjects, and therefore $\ker x_i = \bigvee_{j\geq i} \ker x_{ij}$ (2.6) implies $\ker(x_i \pi 1_A) = \bigvee_{j\geq i} \ker(x_{ij} \pi 1_A)$. Hence

$$\begin{aligned}
\ker t &= \ker t \wedge (\bigvee_{i\in I} \operatorname{Im} \tilde{y}_i) = \bigvee_{i\in I}(\ker t \wedge \operatorname{Im} \tilde{y}_i) = \\
&= \bigvee_{i\in I} (\tilde{y}_i)_s (\tilde{y}_i)^s \ker t = \bigvee_{i\in I} (\tilde{y}_i)_s \ker ty_i = \\
&= \bigvee_{i\in I} (\tilde{y}_i)_s \ker(x_i \pi 1_A) = \\
&= \bigvee_{i\in I} (\tilde{y}_i)_s (\bigvee_{j\geq i} \ker(x_{ij} \pi 1_A)) = \bigvee_{i\in I} (\tilde{y}_i)_s (\bigvee_{j\geq i} \ker y_{ij}) = \\
&= \bigvee_{i\in I} (\tilde{y}_i)_s \ker y_i \leq \varepsilon .
\end{aligned}$$

Thus t is also a monomorphism. Therefore it is an isomorphism, q.e.d.

The proof of the theorem is now complete.

4. ADDITIONAL PROPERTIES OF DIRECTED COLIMITS.

1. We now let $\mathcal{G}$ be a C_3 regular category. From the property 2.6 that for any direct system $\mathfrak{X} : I \longrightarrow \mathcal{G}$, $\ker x_i = \bigvee_{j\geq i} \ker x_{ij}$ for all i , it is easy to derive a number of additional properties.

First we have a very 'set-like' result, which complement the construction in the proof of 2.6 and could also be used in the last part of the proof above.

Proposition 4.1. Let $\mathfrak{X}$ be a direct system over I in a C_3 regular category, and $(a_i)_{i\in I}$ a cocompatible family for $\mathfrak{X}$ inducing at the colimit a morphism a. Then:

i) a is a regular epimorphism if and only if $\bigvee_{i\in I} \operatorname{Im} a_i = 1$;

ii) a is a monomorphism if and only if $\ker a_i = \bigvee_{j\geq i} \ker x_{ij}$ for for every i;

iii) a is an isomorphism if and only if both conditions hold.

Proof. First $\operatorname{Im} a = \bigvee_{i\in I} \operatorname{Im} a_i$ by I.6.6, which proves i) . In view of 2.6, ii) says that a is a monomorphism if and only if $\ker a_i = \ker x_i$ for every i. Since $a_i = ax_i$ this is certainly necessary. If conversely $\ker a_i = \ker x_i$ for every i, then the familiar argument

$$\begin{aligned}\ker a = \bigvee_{i\in I}(\ker a \wedge \operatorname{Im} \tilde{x}_i) &= \bigvee_{i\in I} (\tilde{x}_i)_s \ker ax_i = \\ &= \bigvee_{i\in I} (\tilde{x}_i)_s \ker x_i \leq \epsilon\end{aligned}$$

shows that a is a monomorphism. Finally, iii) follows from i) and ii).

Then we have two equalizer properties.

Proposition 4.2. In a C_3 regular category, Gray's condition $\mathfrak{F}_2$ holds; in other words, for every direct system $\mathfrak{X}$, $x_i f = x_i g$ implies $\bigvee_{j\geq i} \operatorname{Equ}(x_{ij}f, x_{ij}g) = 1$.

Proof. It follows from I.6.2 that

$$\begin{aligned}\bigvee_{j\geq i} \operatorname{Equ}(x_{ij}f, x_{ij}g) &= \bigvee_{j\geq i} \Delta^s (g^{-1}x_{ij}^{-1}x_{ij}f) = \Delta^s(g^{-1}(\bigvee_{j\geq i} x_{ij}^{-1}x_{ij})f) = \\ &= \Delta^s(g^{-1}x_i^{-1}x_i f) = \operatorname{Equ}(x_i f, x_i g) = 1 .\end{aligned}$$

Proposition 4.3. Let $\mathfrak{X}$ be a direct system over I in a C_3 regular category, and $f,g : \lim \mathfrak{X} \longrightarrow A$. Then $\operatorname{Equ}(f,g) = \bigvee_{i\in I}(x_i)_s \operatorname{Equ}(fx_i, gx_i)$.

Proof. Take $m \in \mathrm{Equ}(f,g)$, $n \in \mathrm{Equ}(fx_i,gx_i)$. Since $fx_i n = gx_i n$, there is a commutative square $x_i n = mt$; we claim it is in fact a pullback. Assume that $x_i a = mb$. Then $fx_i a = fmb = gmb = gx_i a$, so that $a = nu$ for some u; also, $mtu = x_i nu = x_i a = mb$, and $b = tu$. This factorization is unique since n is a monomorphism.

Then it follows that $(x_i)^s \mathrm{Equ}(f,g) = \mathrm{Equ}(fx_i,gx_i)$. Hence $(x_i)_s \mathrm{Equ}(fx_i,gx_i) = (x_i)_s(x_i)^s \mathrm{Equ}(f,g) = \mathrm{Equ}(f,g) \wedge \mathrm{Im}\, x_i$. The result follows since $\bigvee_{i\in I} \mathrm{Im}\, x_i = 1$.

2. Finally we show that additional conditions insure additional good behavior of directed colimits, with regards to [not necessarily finite] products. These are, first, Gray's condition $\mathfrak{F}_1$:

$\mathfrak{F}_1$: if $((\underline{x}_i)_{i\in I_\lambda})_{\lambda\in\Lambda}$ is a non-empty family of [non-empty] directed families of subobjects of the same $A \in \mathfrak{G}$, then

$$\bigwedge_{\lambda\in\Lambda}\Big(\bigvee_{i\in I_\lambda} \underline{x}_i\Big) = \bigvee_{\tau\in T}\Big(\bigwedge_{\lambda\in\Lambda} \underline{x}_{\tau\lambda}\Big)$$

where $T = \prod_{\lambda\in\Lambda} I_\lambda$.

This condition is formulable in any category with intersections. We shall always assume in the above that the sets I_λ are pairwise disjoint, and write $\tau\lambda$ instead of τ_λ (to avoid seventh order subscripts). A complete C_3 regular category satisfying $\mathfrak{F}_1$ is called a C_4 regular category. Examples include of course C_4 abelian categories, and finitary varieties, in which $\mathfrak{F}_1$ becomes the familiar $\cap$-$\cup$ distributivity (in its general form). We note that $\mathfrak{F}_1$ implies A.B.5 but not (C_3') ; yet the axioms of C_4 regular categories become redundant in yet another way, since by I.6.9 cocompleteness can be replaced by the existence of coproducts under either (L) or minor size restrictions (see also the results in [2] for other implications, under stronger size restrictions).

The other condition is that $\mathcal{C}$ be C_1^*, i.e. any product of regular epimorphisms is a regular epimorphism. The finite version of that condition would evaporate, by I.1.12. The condition itself holds in any variety.

Using these conditions, we have:

Theorem 4.4. Let $\mathcal{C}$ be a C_4 regular category and $(\mathfrak{X}^\lambda)_{\lambda\in\Lambda}$ be a non-empty family of direct systems $\mathfrak{X}^\lambda : I_\lambda \longrightarrow \mathcal{C}$. The morphisms $x_\tau' = \prod_{\lambda\in\Lambda} x_{\tau\lambda}$, $\tau \in T = \prod_{\lambda\in\Lambda} I_\lambda$, induce a natural monomorphism:

$$t : \varinjlim_{\tau\in T} \prod_{\lambda\in\Lambda} X_{\tau\lambda}^\lambda \longrightarrow \prod_{\lambda\in\Lambda} \varinjlim \mathfrak{X}^\lambda$$

which is in fact an isomorphism if all $\mathfrak{X}^\lambda$ are monic, or if $\mathcal{C}$ is C_1^*.

Of course we assume that the I_λ are pairwise disjoint, which allows us to write X_i etc. instead of X_i^λ $(i \in I_\lambda)$. Also note that under the coordinatewise preorder T is a directed preordered set; a direct system $\mathfrak{X} : T \longrightarrow \mathcal{C}$ is defined by $X_\tau = \prod_{\lambda\in\Lambda} X_{\tau\lambda}$, $x_{\sigma\tau} = \prod_{\lambda\in\Lambda} x_{\sigma\lambda,\tau\lambda}$ $(\sigma \leq \tau)$, giving the new [directed] colimit that appears in the theorem.

3. We begin the proof with the following generalization of 2.7:

Lemma 4.5. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a non-empty family of objects of $\mathcal{C}$ and, for each λ, $(f_i)_{i\in I_\lambda}$ be a family of morphisms of codomain A such that $(\operatorname{Im} f_i)_{i\in I_\lambda}$ is directed with $\bigvee_{i\in I_\lambda} \operatorname{Im} f_i = 1$. For each $\tau\in T$, let $f_\tau = \prod_{\lambda\in\Lambda} f_{\tau\lambda}$. If all f_i are monomorphisms, or if $\mathcal{C}$ is C_1^*, then $\bigvee_{\tau\in T} \operatorname{Im} f_\tau = 1$.

Proof. We consider first the case when all f_i are monomorphisms. For each $i \in I_\mu$, $\tau \in T$, put $f_i : Y_i \longrightarrow A_\mu$ and consider the diagram:

$$\begin{tikzcd}
 & A = A_\mu \pi (\prod_{\lambda\neq\mu} A_\lambda) \arrow[r] & A_\mu \\
 & Y_{\tau\mu} \pi (\prod_{\lambda\neq\mu} A_\lambda) \arrow[u, "g_{\tau\mu} = f_{\tau\mu}\pi 1"'] \arrow[r] \arrow[ur, phantom, "PB"] & Y_{\tau\mu} \arrow[u, "f_{\tau\mu}"'] \\
Y_{\tau\mu}\pi(\prod_{\lambda\neq\mu} Y_{\tau\lambda}) \arrow[uur, "f_\tau"] \arrow[ur, "h_{\tau,\mu} = 1\pi(\prod_{\lambda\neq\mu} f_{\tau\lambda})"'] & &
\end{tikzcd}$$

where the horizontal maps are projections. The diagram commutes, in fact the square is a pullback (I.1.13), and the new maps $g_{\tau\mu}$, $h_{\tau,\mu}$ are monomorphisms. We note that the square still serves if $\tau\mu$ is replaced by $i \in I_\mu$ (and then we denote $f_i \pi 1$ by g_i , instead of $g_{\tau\mu}$).

We see on the diagram that f_τ factors through all $g_{\tau\mu}$ $(\mu \in \Lambda)$. In fact it is an intersection of that family. Indeed let $u : Z \longrightarrow A$ factor through every $g_{\tau\mu}$ (with τ given); write $u = \bigtimes_{\lambda\in\Lambda} u_\lambda$, $u_\lambda : Z \longrightarrow A$. For each μ, $u = g_{\tau\mu} v_\mu$ for some $v_\mu = w_\mu \times (\bigtimes_{\lambda\neq\mu} w_{\mu\lambda})$; note that $w_\mu : Z \longrightarrow Y_{\tau\mu}$, $w_{\mu\lambda} : Z \longrightarrow A_\lambda$ if $\lambda \neq \mu$. Since $u = g_{\tau\mu} v_\mu$ we see that $u_\mu = f_{\tau\mu} w_\mu$, $u_\lambda = w_{\mu\lambda}$ if $\lambda \neq \mu$. Hence $v_\mu = w_\mu \times (\bigtimes_{\lambda\neq\mu} u_\lambda) = w_\mu \times (\bigtimes_{\lambda\neq\mu} f_{\tau\lambda} w_\lambda) = h_{\tau,\mu} \cdot (\bigtimes_{\lambda\in\Lambda} w_\lambda) = h_{\tau,\mu} w$, say . Therefore $u = g_{\tau\mu} v_\mu = g \quad h_{\tau,\mu} w = f_\tau w$, i.e. u factors through f_τ . Thus we do have an intersection and it follows that $\mathrm{Im}\, f_\tau = \bigwedge_{\lambda\in\Lambda} \mathrm{Im}\, g_{\tau\lambda}$.

On the other hand, let $p_\mu : A \longrightarrow A_\mu$ be the projection. For each $i \in I_\mu$ the pullback above (with $\tau\mu$ replaced by i) yields $\mathrm{Im}\, g_i = p_\mu^s \mathrm{Im}\, f_i$; by (C_3^*) and the hypothesis, it follows that $\bigvee_{i\in I_\mu} \mathrm{Im}\, g_i = 1$ for each $\mu \in \Lambda$. Then, by $\mathfrak{F}_1$:

$$\bigvee_{\tau\in T} \mathrm{Im}\, f_\tau = \bigvee_{\tau\in T} (\bigwedge_{\lambda\in\Lambda} \mathrm{Im}\, g_{\tau\lambda}) = \bigwedge_{\lambda\in\Lambda} (\bigvee_{i\in I_\lambda} \mathrm{Im}\, g_i) = 1 .$$

This takes care of the case when all f_i are monomorphisms. In the general case, we also assume that G is C_1^*, so that products preserve regular decompositions. Thus we can select for each i a regular decomposition (m_i, p_i) of f_i and obtain a regular decomposition (m_τ, p_τ) of f_τ, with $m_\tau = \prod_{\lambda\in\Lambda} m_{\tau\lambda}$. Since $\operatorname{Im} m_i = \operatorname{Im} f_i$ for every i, it follows from the above that $\bigvee_{\tau\in T} \operatorname{Im} m_\tau = 1$ and the result again holds since $\operatorname{Im} f_\tau = \operatorname{Im} m_\tau$.

4. We now prove the theorem. We want to show that the morphism t induced by all $x'_\tau = \prod_{\lambda\in\Lambda} x_{\tau\lambda}$ is a monomorphism and in some cases an isomorphism. We use 4.1. If all x_i are monomorphisms (i.e. if all $\mathfrak{X}^\lambda$ are monic), or if G is C_1^*, then the lemma applies to $((x_i)_{i\in I_\lambda})_{\lambda\in\Lambda}$ and yields $\bigvee_{\tau\in T} \operatorname{Im} x'_\tau$, so that t is a regular epimorphism.

We now show that t is a monomorphism, without using C_1^*. First note that the result is trivial when all $\mathfrak{X}^\lambda$ are monic, for then all x'_τ are monomorphisms; in that case, the proof of the theorem is over.

In the general case, take $\tau\in T$. For each λ and each $i\in I_\lambda$ with $i \geq \tau\lambda$, select monomorphisms $m_{\tau\lambda} : K_{\tau\lambda} \longrightarrow X_{\tau\lambda} \pi X_{\tau\lambda}$, $m_{\tau\lambda,i} : K_{\tau\lambda,i} \longrightarrow X_{\tau\lambda} \pi X_{\tau\lambda}$ such that $\ker x_{\tau\lambda} = \operatorname{Im} m_{\tau\lambda}$, $\ker x_{\tau\lambda,i} = \operatorname{Im} m_{\tau\lambda,i}$. Since $(\operatorname{Im} m_{\tau\lambda,i})_{i\in I}$ is a directed family of subobjects, there is a monic direct system $\mathcal{K}^\lambda$ over $\{i\in I_\lambda ; i \geq \tau\lambda\}$ with objects $K_{\tau\lambda,i}$. Since $\operatorname{Im} m_{\tau\lambda,i} \leq \operatorname{Im} m_{\tau\lambda}$, we have a clearly cocompatible family of monomorphisms $K_{\tau\lambda,i} \longrightarrow K_{\tau\lambda}$; since in fact $\operatorname{Im} m_{\tau\lambda} = \bigvee_{i\geq\tau\lambda} \operatorname{Im} m_{\tau\lambda,i}$, the induced monomorphism $\varinjlim \mathcal{K}^\lambda \longrightarrow K_{\tau\lambda}$ is an isomorphism, by 4.1.

Since we have already proved the theorem in the case of monic direct systems, we can apply it to the family $(\mathcal{K}^\lambda)_{\lambda\in\Lambda}$; we obtain an isomorphism

$$\varinjlim_{\sigma\in\Sigma} \prod_{\lambda\in\Lambda} K_{\tau\lambda,\sigma\lambda} \cong \prod_{\lambda\in\Lambda} K_{\tau\lambda} ,$$

where $\Sigma = \prod_{\lambda\in\Lambda} \{i \in I_\lambda ; i \geq \tau\lambda\} = \{\sigma \in T ; \sigma \geq \tau\}$, induced by all $\prod_{\lambda\in\Lambda}$ of monomorphisms $K_{\tau\lambda,\sigma\lambda} \longrightarrow K_{\tau\lambda}$. It follows that the morphism induced to the colimit by the cocompatible family $(\prod_{\lambda\in\Lambda} m_{\tau\lambda,\sigma\lambda})_{\sigma\in\Sigma}$ is equivalent (as a monomorphism) to $\prod_{\lambda\in\Lambda} m_{\tau\lambda}$; then, by I.6.6,

$$\bigvee_{\sigma\geq\tau} (\mathrm{Im} \prod_{\lambda\in\Lambda} m_{\tau\lambda,\sigma\lambda}) = \mathrm{Im} \prod_{\lambda\in\Lambda} m_{\tau\lambda} .$$

We now remember that $x_{\tau\sigma} = \prod_{\lambda\in\Lambda} x_{\tau\lambda,\sigma\lambda}$, $x'_\tau = \prod_{\lambda\in\Lambda} x_{\tau\lambda}$; since products preserve kernels, hence also congruences, we have in fact proved that $\ker x'_\tau = \bigvee_{\sigma\geq\tau} \ker x_{\tau\sigma}$. Then it follows from 4.1 that t is a monomorphism, and this completes the proof of the theorem.

III. SHEAVES IN REGULAR CATEGORIES

In this part we study categories $\mathfrak{F}(X,\mathcal{C})$ of sheaves in a regular category $\mathcal{C}$ (which we assume is at least C_4) over an arbitrary topological space or Grothendieck topology X.

Our first result is the existence of the associated sheaf of a presheaf, i.e. $\mathfrak{F}(X,\mathcal{C})$ is coreflective in the category $\mathcal{P}(X,\mathcal{C})$ of presheaves. The previous results of that kind are due to Gray [10],[11] (see also [31]), and, in the exact case, to Heller and Rowe [16], extending an older result of Grothendieck [15] . Gray's proof of existence is similar to that of the adjoint functor theorem (except that solution sets are neatly bypassed). Heller and Rowe's proof is constructive: one basic construction is iterated to build increasingly sheaf-like presheaves, which eventually terminates at the associated sheaf. In either case, some restriction is made on the size of the category one works in, such as being locally small or having a set of generators. We prove that, when $\mathcal{C}$ is a C_4 regular category, and X is any Grothendieck topology, Heller and Rowe's construction terminates after two steps; this answers a conjecture made by Gray [12] and in particular proves the existence of the associated sheaf. The assumption on $\mathcal{C}$ has more cocompleteness than in the other results, but involves no size restriction, and the result is new even in the case of a C_4 abelian category.

Under the same hypothesis, we show that $\mathfrak{F}(X,\mathcal{C})$ is a C_3 regular category. If furthermore $\mathcal{C}$ is C_1^* and X is a topological space, then the stalk functor reflects isomorphisms [which means that it is cotripleable, as VanOsdol pointed out to us] so that all finite limits, regular decompositions and colimits in $\mathfrak{F}(X,\mathcal{C})$ can safely be computed on the stalks.

Related and additional results will be found in VanOsdol's contribution to this volume. We owe much to VanOsdol, for discussions, and e.g. suggesting that 3.1 below might hold and bring an answer to Gray's conjecture. In addition, the results in section 4 were first proved by him in the case of varieties [35].

Our exposition follows that of [14], except for the inclusion of the details of Heller and Rowe's construction and the rather straightforward extension to Grothendieck topologies at the beginning [independently suggested by VanOsdol and Heller].

1. GROTHENDIECK TOPOLOGIES AND SHEAVES.

In this section we recall the basic definitions concerning Grothendieck topologies and sheaves thereon, and set forth some notation.

1. A Grothendieck topology X is a small category $\mathfrak{U}(X)$ together with a set $\mathfrak{C}$ of coterminal families of morphisms of $\mathfrak{U}(X)$ ('coverings') satisfying the following conditions (in which $\mathfrak{C}(U)$ denotes the set of all coverings of [codomain] U):

i) $\{\alpha\} \in \mathfrak{C}$ for every isomorphism $\alpha \in \mathfrak{U}(X)$;

ii) if $(\alpha_i)_{i\in I} \in \mathfrak{C}(U)$, $\alpha_i : U_i \longrightarrow U$, and for every $i \in I$ $(\beta_{ij})_{j\in J_i} \in \mathfrak{C}(U_i)$, then $(\alpha_i\beta_{ij})_{j\in J_i,\, i\in I} \in \mathfrak{C}(U)$;

iii) if $(\alpha_i)_{i\in I} \in \mathfrak{C}(U)$ and $\beta : V \longrightarrow U \in \mathfrak{U}(X)$, then for each i there exists a pullback $\beta\alpha_i' = \alpha_i\gamma_i$, and $(\alpha_i')_{i\in I} \in \mathfrak{C}(V)$.

The prime example of a Grothendieck topology is given by any topology in the usual sense, i.e. the family $\mathfrak{U}(X)$ of all open subsets of a topological space X. Then $\mathfrak{U}(X)$ is made into a category in the obvious way (the morphisms being all inclusion maps between objects, i.e. elements, of $\mathfrak{U}(X)$), and $\mathfrak{C}(U)$ is the set of all families of inclusion maps $U_i \longrightarrow U$ $(i \in I)$ such that $U_i \in \mathfrak{U}(X)$ and $\bigcup_{i \in I} U_i = U$. Any small regular category $\mathfrak{a}$ provides another example (which will not be used here): let $\mathfrak{U} = \mathfrak{a}$ and $\mathfrak{C}$ be the set of all families that consist of just one regular epimorphism. If (C_3') holds in $\mathfrak{a}$, another $\mathfrak{C}$ is defined by: $(\alpha_i)_{i \in I} \in \mathfrak{C}$ if and only if $(\operatorname{Im} \alpha_i)_{i \in I}$ is directed and $\bigvee_{i \in I} \operatorname{Im} \alpha_i = 1$. More examples can be found e.g. in [6].

2. If X is any Grothendieck topology, then, for each $U \in \mathfrak{U}(X)$, $\mathfrak{C}(U)$ can be made into a directed preordered set as follows. If $\mathcal{C} = (\alpha_i)_{i \in I}$, $\alpha_i : U_i \longrightarrow U$ and $\mathcal{D} = (\beta_j)_{j \in J}$, $\beta_j : V_j \longrightarrow U$ are in $\mathfrak{C}(U)$, say that $\mathcal{D}$ refines $\mathcal{C}$, and write $\mathcal{C} \leq \mathcal{D}$, in case there exists a mapping $\psi : J \longrightarrow I$ and morphisms $\psi_j : V_j \longrightarrow U_{\psi j}$ such that $\beta_j = \alpha_{\psi j} \psi_j$ for every $j \in J$. If $\mathcal{D} \leq \mathcal{E} = (\gamma_k)_{k \in K}$, with $\chi : K \longrightarrow J$ and morphisms χ_k serving in the definition (i.e., $\gamma_k = \beta_{\chi k} \chi_k$ for all k), then $\omega = \psi\chi : K \longrightarrow I$ and $\omega_k = \psi_{\chi k}\chi_k$ are such that $\gamma_k = \beta_{\chi k} \chi_k = \alpha_{\psi\chi k} \psi_{\chi k} \chi_k = \alpha_{\omega k} \omega_k$, and therefore $\mathcal{C} \leq \mathcal{E}$, which shows that $\leq$ is transitive; it is clearly reflexive. (In fact, we have implicitly defined morphisms in $\mathfrak{C}(U)$ and made it into a category).

To show directedness, start with $\mathcal{C}$ and $\mathcal{D}$ as above (except that $\mathcal{D}$ need not refine $\mathcal{C}$). By iii) there is for each i and j a pullback $\alpha_i \beta'_{ij} = \beta_j \alpha'_{ij}$. The family $\gamma_{ij} : U_i * V_j \longrightarrow U$ $((i,j) \in I \pi J)$ defined by $\gamma_{ij} = \beta_j \alpha'_{ij}$ is a covering of U by ii), since by iii) $(\alpha'_{ij})_{i \in I}$ is in $\mathfrak{C}(V_j)$ for every j; we denote it by $\mathcal{C} * \mathcal{D}$. [Other notations for $U_i * V_j$ are $U_i \times_U V_j$ and $U_i \cap V_j$. The notation $\mathcal{C} * \mathcal{D}$ is legitimate since in the small category $\mathfrak{U}(X)$ we can once and for all make a selection of pullbacks which covers all existence cases postulated by iii).] To see that $\mathcal{C} \leq \mathcal{C} * \mathcal{D}$ it suffices to consider the projection $\psi : I \pi J \longrightarrow I$ and define $\psi_{ij} = \beta'_{ij} : U_i * V_j \longrightarrow U_i$. Similarly, $\mathcal{D} \leq \mathcal{C} * \mathcal{D}$. Thus, for each $U \in \mathfrak{U}(X)$ we now have a [non-empty, by i)] directed preordered set $\mathfrak{C}(U)$.

If $\mathfrak{C} = (\alpha_i)_{i\in I} \in \mathfrak{C}(U)$ and $\beta : V \longrightarrow U$, the covering postulated by iii) will be denoted by $\mathfrak{C} * V$ (when there can be no confusion on β). We note that in the above $\mathfrak{C} * \mathfrak{D}$ is obtained by composing (as in ii)) $\mathfrak{D}$ and all $\mathfrak{C} * V_j$ (and also by composing $\mathfrak{C}$ and all $\mathfrak{D} * U_i$).

3. Let X be a Grothendieck topology and $\mathfrak{G}$ be any category which has products. A presheaf on X with values in $\mathfrak{G}$ is a contravariant functor $P : \mathfrak{U}(X) \longrightarrow \mathfrak{G}$; these form a category $\mathcal{P}(X,\mathfrak{G})$.

For each $P \in \mathcal{P}(X,\mathfrak{G})$ and $\mathfrak{C} \in \mathfrak{C}(U)$ we have a canonical diagram

$$P(U) \xrightarrow{u} P(\mathfrak{C}) \underset{g}{\overset{f}{\rightrightarrows}} P(\mathfrak{C}*\mathfrak{C})$$

defined as follows. Put $\mathfrak{C} = (\alpha_i)_{i\in I}$, $\alpha_i : U_i \longrightarrow U$ and let $\alpha_j \xi_{jk} = \alpha_k \eta_{jk}$ be the pullbacks defining $\mathfrak{C} * \mathfrak{C}$. Then $P(\mathfrak{C}) = \prod_{i\in I} P(U_i)$ and $P(\mathfrak{C}*\mathfrak{C}) = \prod_{j,k\in I} P(U_j * U_k)$. The morphisms u,f,g are induced by the obvious 'restriction maps', namely: $u = u^P_{\mathfrak{C}} = \mathop{\times}_{i\in I} P(\alpha_i)$, $f = f^P_{\mathfrak{C}} = \prod_{j\in I}(\mathop{\times}_{k\in I} P(\xi_{jk}))$, $g = g^P_{\mathfrak{C}} = \prod_{k\in I}(\mathop{\times}_{j\in I} P(\eta_{jk}))$; if we use the letter π to denote any projection from a product, as we shall do from here on, e.g. $\pi_i : P(\mathfrak{C}) \longrightarrow P(U_i)$, then we see that $\pi_i u = P(\alpha_i)$, $\pi_{jk} f = P(\xi_{jk})\pi_j$, $\pi_{jk} g = P(\eta_{jk})\pi_k$. The reader should verify that $fu = gu$.

The presheaf P is called a monopresheaf if u is a monomorphism for all $\mathfrak{C}$, and a sheaf if $u \in \mathrm{Equ}(f,g)$ for all $\mathfrak{C}$. The category of all sheaves on X with values in $\mathfrak{G}$ (a full subcategory of $\mathcal{P}(X,\mathfrak{G})$) will be denoted by $\mathfrak{F}(X,\mathfrak{G})$. It is defined in terms of products and equalizers, and since these commute with limits and limits in the functor category $\mathcal{P}(X,\mathfrak{G})$ are eveluated pointwise, it follows that $\mathfrak{F}(X,\mathfrak{G})$ is a complete subcategory of $\mathcal{P}(X,\mathfrak{G})$ (i.e. admits all existing limits).

2. THE HELLER AND ROWE CONSTRUCTION OF THE ASSOCIATED SHEAF.

1. Let X be any Grothendieck topology and $\mathfrak{G}$ be a complete category. Then we know that $\mathfrak{F}(X,\mathfrak{G})$ is a complete subcategory of $\mathfrak{P}(X,\mathfrak{G})$ and one may feel that it will take very little for $\mathfrak{F}(X,\mathfrak{G})$ to be coreflexive in $\mathfrak{P}(X,\mathfrak{G})$. A look at the existing results of that sort shows that this first impression may be misleading. It takes a complete well-powered category $\mathfrak{G}$ having directed colimits and satisfying Gray's condition $\mathfrak{F}_1,\mathfrak{F}_2$, for the existence of associated sheaves to be established by a reasonably short argument, similar to the proof of the adjoint functor theorem [10], [31]. A more explicit construction was given, when $\mathfrak{G}$ is a complete exact category having a projective generator and directed colimits which are exact, by Heller and Rowe [16]; in this construction, a presheaf P' is explicitely constructed from any given presheaf P , and when the construction is repeated sufficiently many times (by ordinal induction) it eventually terminates at the associated sheaf.

It was conjectured by Gray in [12] that in most good categories Heller and Rowe's construction should yield the associated sheaf in two steps. We shall prove this is indeed the case when $\mathfrak{G}$ is a regular C_4 category. First, we recall Heller and Rowe's construction; in what follows, $\mathfrak{G}$ is a complete category having directed colimits and X is any Grothendieck topology; $P \in \mathfrak{P}(X,\mathfrak{G})$ is given.

2. For each $\mathfrak{C} \in \mathfrak{C}(U)$, we have a canonical diagram

$$P(U) \xrightarrow{u} P(\mathfrak{C}) \underset{g}{\overset{f}{\rightrightarrows}} P(\mathfrak{C} * \mathfrak{C}) \ ;$$

let $u^*_{\mathfrak{C}} : E_{\mathfrak{C}}(U) \longrightarrow P(\mathfrak{C}) \in Equ_{\mathfrak{G}}(f,g)$. Since $fu = fu$, we have $u_{\mathfrak{C}} = u^*_{\mathfrak{C}}\, c_{\mathfrak{C}}(U)$ for some unique $c_{\mathfrak{C}}(U) : P(U) \longrightarrow E_{\mathfrak{C}}(U)$.

We now organize the objects $E_{\mathfrak{C}}(U)$ into a direct system over

$\underline{\mathfrak{S}(U)}$. Let $\mathcal{C}, \mathcal{D} \in \mathfrak{S}(U)$ satisfy $\mathcal{C} \leq \mathcal{D}$. Put $\mathcal{C} = (\alpha_i)_{i\in I}$, $\mathcal{D} = (\beta_j)_{j\in J}$; then there exist a mapping $\psi : J \longrightarrow I$ and morphisms ψ_p $(p \in J)$ such that $\beta_p = \alpha_{\psi p}\psi_p$ for all p . Also, let $\alpha_j \xi_{jk} = \alpha_k \eta_{jk}$, $\beta_p \xi_{pq} = \beta_q \eta_{pq}$ $(j,k \in I\ ,\ p,q \in J\ )$ be pullbacks; for each $p,q \in J$, $\alpha_{\psi p}\, \psi_p\, \xi_{pq} = \alpha_{\psi q}\, \psi_q\, \eta_{pq}$, so that there exists a unique morphism ψ_{pq} such that $\psi_p\, \xi_{pq} = \xi_{\psi p, \psi q}\, \psi_{pq}$, $\psi_p\, \eta_{pq} = \eta_{\psi p, \psi q}\, \psi_{pq}$ (that is, $\mathcal{C} * \mathcal{C} \leq$ $\leq \mathcal{D} * \mathcal{D}$). We can then define morphisms $P'(\psi) : P(\mathcal{C}) \longrightarrow P(\mathcal{D})$, $P''(\psi) : P(\mathcal{C} * \mathcal{C}) \longrightarrow P(\mathcal{D} * \mathcal{D})$ by:

$$P'(\psi) = \prod_{i\in I} \Big(\bigtimes_{p\in\psi^{-1}i} P(\psi_p)\Big)$$

$$P''(\psi) = \prod_{\substack{j\in I\\ k\in I}} \Big(\bigtimes_{\substack{p\in\psi^{-1}j\\ q\in\psi^{-1}k}} P(\psi_{pq})\Big)$$

(so that $\pi_p P'(\psi) = P(\psi_p)\pi_{\psi p}$, $\pi_{pq} P''(\psi) = P(\psi_{pq})\pi_{\psi p, \psi q}$).

<u>Lemma</u> 2.1. $f_{\mathcal{D}}\, P'(\psi) = P''(\psi)\, f_{\mathcal{C}}$, $g_{\mathcal{D}}\, P'(\psi) = P''(\psi)\, g_{\mathcal{C}}$ and furthermore $P'(\psi)\, u^*_{\mathcal{C}}$ depends only on $\mathcal{C}$ and $\mathcal{D}$ and not on the choice of ψ .

<u>Proof</u>. For each $p,q \in J$, composing $f_{\mathcal{D}}\, P'(\psi)$ and $P''(\psi)\, f_{\mathcal{C}}$ with π_{pq} yields, respectively, $P(\xi_{pq})\, P(\psi_p)\, \pi_{\psi p}$ and $P(\psi_{pq})\, P(\xi_{\psi p, \psi q})\, \pi_{\psi p}$; due to the relation $\psi_p\, \xi_{pq} = \xi_{\psi p, \psi q}\, \psi_{pq}$ above, these are always equal, which proves the first formula. The second one is proved similarly. For the last part, let $\psi' : J \longrightarrow I$, ψ'_p $(p\in J)$ be another mapping and family of morphisms such that $\beta_p = \alpha_{\psi' p}\, \psi'_p$ for all p .

Since $\alpha_{\psi p}\, \psi_p = \alpha_{\psi' p}\, \psi'_p$, we have $\psi_p = \xi_{\psi p, \psi' p}\, \psi''_p$, $\psi'_p = \eta_{\psi p, \psi' p}\, \psi''_p$ for some ψ''_p . Hence for each $p \in J$

$$\pi_p\, P'(\psi)\, u^*_{\mathcal{C}} = P(\psi_p)\, \pi_{\psi p}\, u^*_{\mathcal{C}} = P(\psi''_p)\, P(\xi_{\psi p, \psi' p})\, \pi_{\psi p}\, u^*_{\mathcal{C}} =$$

$$= P(\psi''_p)\, \pi_{\psi p, \psi' p}\, f_{\mathcal{C}}\, u^*_{\mathcal{C}} =$$

$$= P(\psi''_p)\ \pi_{\psi p,\psi' p}\ g_C\ u^*_C = \cdots = \pi_p\ P'(\psi')\ u^*_C\ .$$

It follows that $P'(\psi)\ u^*_C = P'(\psi')\ u^*_C$, which completes the proof.

It follows from the lemma that $f_{\mathcal{D}}\ P'(\psi)\ u^*_C = P''(\psi)\ f_C\ u^*_C = P''(\psi)\ g_C\ u^*_C = g_{\mathcal{D}}\ P'(\psi)\ u^*_C$; therefore there exists a morphism $E_{C\mathcal{D}} : E_C(U) \longrightarrow E_{\mathcal{D}}(U)$ unique such that $P'(\psi)\ u^*_C = u^*_{\mathcal{D}}\ E_{C\mathcal{D}}(U)$, i.e. the following diagram commutes:

$$\text{(1)}\qquad \begin{array}{ccccc} E_C(U) & \rightarrowtail & P(C) & \rightrightarrows & P(C * C) \\ {\scriptstyle E_{C\mathcal{D}}(U)}\big\downarrow & & \big\downarrow{\scriptstyle P'(\psi)} & & \big\downarrow{\scriptstyle P''(\psi)} \\ E_{\mathcal{D}}(U) & \rightarrowtail & P(\mathcal{D}) & \rightrightarrows & P(\mathcal{D} * \mathcal{D}) \end{array}$$

Furthermore the last part of the lemma shows that $E_{C\mathcal{D}}(U)$ depends only on C and $\mathcal{D}$ and not on the choice of ψ . $\square$

In case $C = \mathcal{D}$ we may choose for ψ the identity on I and for ψ_i the identity morphisms and then it is clear that $P'(\psi)$ is the identity and so is $E_{CC}(U)$. If also $C \leq \mathcal{D} \leq \mathcal{E}$ in $\mathfrak{C}(U)$, with $\mathcal{E} = (\gamma_z)_{z\in K}$, and $\chi : K \longrightarrow J$, χ_z $(z \in K)$ are such that $\gamma_z = \beta_{\chi z}\chi_z$ for all z , then, with ψ as above, we can define $\omega = \psi\chi : K \longrightarrow I$ and $\omega_z = \psi_{\chi z}\chi_z$, and see that $\gamma_z = \alpha_{\omega z}\ \omega_z$ for all $z \in K$ (this is how we showed the transitivity of $\leq$); furthermore,

$$\pi_z\ P'(\omega) = P(\omega_z)\ \pi_{\omega z} = P(\chi_z)P(\psi_{\chi z})\ \pi_{\psi\chi z} =$$
$$= P(\chi_z)\ \pi_{\chi z}\ P'(\psi) = \pi_z\ P'(\chi)\ P'(\psi)$$

for all z , so that $P'(\omega) = P'(\chi)P'(\psi)$; it follows that $E_{C\mathcal{E}}(U) = E_{\mathcal{D}\mathcal{E}}(U)\ E_{C\mathcal{D}}(U)$. Hence we now have a direct system over $\mathfrak{C}(U)_-$.

3 We now let $E(U) = \varinjlim E_C(U)$; it comes with maps $p_C(U) : E_C(U) \longrightarrow E(U)$. We also have a morphism $P(U) \longrightarrow E(U)$; indeed, $C \leq \mathcal{D}$ in $\mathfrak{C}(U)$ implies (keeping the same notation as before)

$\pi_p\, P'(\psi)\, u_{\mathfrak{C}} = P(\psi_p)P(\alpha_{\psi p}) = P(\beta_p) = \pi_p\, u_{\mathfrak{D}}$, whence $P'(\psi)\, u_{\mathfrak{C}} = u_{\mathfrak{D}}$ and $E_{\mathfrak{C}\mathfrak{D}}(U)c_{\mathfrak{C}}(U) = c_{\mathfrak{D}}(U)$; since $\mathfrak{C}(U)$ is directed, we conclude that $c(U) = p_{\mathfrak{C}}(U)\, c_{\mathfrak{C}}(U)$ does not depend on $\mathfrak{C}$.

Finally, E is made into a presheaf as follows. Let $\gamma : W \longrightarrow U \in \mathfrak{U}(X)$. Then $E(\gamma) : E(W) \longrightarrow E(U)$ is induced by the restriction maps of P in the following manner. For each $\mathfrak{C} = (\alpha_i)_{i\in I} \in \mathfrak{C}(U)$, let $\alpha_i\gamma_i' = \gamma\alpha_i'$ be pullbacks, so that $\mathfrak{C} * W = (\alpha_i')_{i\in I} \in \mathfrak{C}(W)$. We then have a morphism

$$h' = h_{\mathfrak{C}}'(\gamma) = \prod_{i\in I} P(\gamma_i') \ : P(\mathfrak{C}) \longrightarrow P(\mathfrak{C} * W) \ .$$

Also, let $\alpha_j\xi_{jk} = \alpha_k\eta_{jk}$, $\alpha_j'\xi_{jk}' = \alpha_k'\eta_{jk}'$ be pullbacks. We then have, for each $j,k \in I$, a morphism γ_{jk}'' induced by the γ's , such that $\gamma_j'\xi_{jk}' = \xi_{jk}\gamma_{jk}''$, $\gamma_k'\eta_{jk}' = \eta_{jk}\gamma_{jk}''$. This yields a morphism

$$h'' = h_{\mathfrak{C}}''(\gamma) = \prod_{j,k\in I} P(\gamma_{jk}'') \ : P(\mathfrak{C} * \mathfrak{C}) \longrightarrow P((\mathfrak{C} * W)*(\mathfrak{C} * W)) \ .$$

(By definition, $\pi_{jk}h'' = P(\gamma_{jk}'')\pi_{jk}$, $\pi_i h' = P(\gamma_i')\pi_i$.)

For each j,k,

$$\pi_{jk}\, h''\, f_{\mathfrak{C}} = P(\gamma_{jk}'')P(\xi_{jk})\, \pi_j = P(\xi_{jk}')P(\gamma_j')\, \pi_j = \pi_{jk}\, f_{\mathfrak{C}*W}\, h' \ ,$$ so

that $h''\, f_{\mathfrak{C}} = f_{\mathfrak{C}*W}\, h'$. Similarly, $h''\, g_{\mathfrak{C}} = g_{\mathfrak{C}*W}\, h'$. Therefore $f_{\mathfrak{C}*W}\, h'\, u_{\mathfrak{C}}^* = h''\, f_{\mathfrak{C}}\, u_{\mathfrak{C}}^* = h''\, g_{\mathfrak{C}}\, u_{\mathfrak{C}}^* = g_{\mathfrak{C}*W}\, h'\, u_{\mathfrak{C}}^*$ and there exists a morphism $E_{\mathfrak{C}}(\gamma) : E_{\mathfrak{C}}(U) \longrightarrow E_{\mathfrak{C}*W}(W)$ induced on equalizers, such that the following diagram commutes:

$$(2) \qquad \begin{array}{ccccc} E_{\mathfrak{C}}(U) & \rightarrowtail & P(\mathfrak{C}) & \rightrightarrows & P(\mathfrak{C} * \mathfrak{C}) \\ {\scriptstyle E_{\mathfrak{C}}(\gamma)}\downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h''} \\ E_{\mathfrak{C}*W}(W) & \rightarrowtail & P(\mathfrak{C}*W) & \rightrightarrows & P((\mathfrak{C}*W)*(\mathfrak{C}*W)) \end{array}$$

Now assume that $\mathfrak{C} \leq \mathfrak{D}$ in $\mathfrak{C}(U)$. With the notation as before, we have a mapping $\psi : J \longrightarrow I$ and morphisms ψ_p such that $\beta_p = \alpha_{\psi p}\psi_p$

for all p. We also have pullbacks $\beta_p \gamma'_p = \gamma \beta'_p$ yielding a covering $\mathcal{D} * W = (\beta'_p)_{p \in J} \in \mathfrak{C}(W)$, and maps $h'_{\mathcal{D}}(\gamma) : P(\mathcal{D}) \longrightarrow P(\mathcal{D} * W)$, $E_{\mathcal{D}}(\gamma) : E_{\mathcal{D}}(U) \longrightarrow E_{\mathcal{D}*W}(W)$. From $\alpha_{\psi p} \psi_p \gamma'_p = \gamma \beta'_p$ we also obtain for each morphism p a morphism ψ'_p such that $\psi_p \gamma'_p = \gamma'_{\psi p} \psi'_p$ and $\beta'_p = \alpha'_{\psi p} \psi'_p$; in particular this shows that $\mathcal{C} * W \leq \mathcal{D} * W$, and yields a map $P'(\psi') : P(\mathcal{C} * W) \longrightarrow P(\mathcal{D} * W)$. Now for each p,

$$\pi_p P'(\psi') h'_{\mathcal{C}} = P(\psi') P(\gamma'_{\psi p}) \pi_{\psi p} = P(\gamma'_p) P(\psi_p) \pi_{\psi p} = \pi_p h'_{\mathcal{D}} P'(\psi) ,$$

which shows that $P'(\psi') h'_{\mathcal{C}} = h'_{\mathcal{D}} P'(\psi)$. Thus every face of the following diagram

$$(3) \qquad \begin{tikzcd}
 & E_{\mathcal{D}}(U) \arrow[rr, tail, "u^*_{\mathcal{D}}"] \arrow[dd, "E_{\mathcal{D}}(\gamma)"'] & & P(\mathcal{D}) \arrow[dd, "h'_{\mathcal{D}}(\gamma)"] \\
E_{\mathcal{C}}(U) \arrow[ur, "E_{\mathcal{C}\mathcal{D}}(U)"] \arrow[rr, tail, "u^*_{\mathcal{C}}"] \arrow[dd, "E_{\mathcal{C}}(\gamma)"'] & & P(\mathcal{C}) \arrow[ur, "P'(\psi)"] \arrow[dd, "h'_{\mathcal{C}}(\gamma)"' near start] & \\
 & E_{\mathcal{D}*W}(W) \arrow[rr, tail, "u^*_{\mathcal{D}*W}" near start] & & P(\mathcal{D}*W) \\
E_{\mathcal{C}*W}(W) \arrow[ur, "E_{\mathcal{C}*W,\mathcal{D}*W}(W)"] \arrow[rr, tail, "u^*_{\mathcal{C}*W}"'] & & P(\mathcal{C}*W) \arrow[ur, "P'(\psi')"'] &
\end{tikzcd}$$

commutes except perhaps for the left face. But then this face commutes too, since $u^*_{\mathcal{D}*W}$ is a monomorphism; i.e. $E_{\mathcal{D}}(\gamma) E_{\mathcal{C}\mathcal{D}}(U) = E_{\mathcal{C}*W,\mathcal{D}*W}(W) E_{\mathcal{C}}(\gamma)$.

It follows that $p_{\mathcal{C}*W}(W)\, E_{\mathcal{C}}(\gamma) = p_{\mathcal{D}*W}(W)\, E_{\mathcal{D}}(\gamma)\, E_{\mathcal{C}\mathcal{D}}(U)$, i.e. we have shown that $(p_{\mathcal{C}*W}(W)\, E_{\mathcal{C}}(\gamma))_{\mathcal{C} \in \mathfrak{C}(U)}$ is a cocompatible family; hence it induces a morphism $E(\gamma) : E(U) \longrightarrow E(W)$, unique such that the following diagram commutes:

$$(4) \qquad \begin{tikzcd}
E(U) \arrow[d, "E(\gamma)"'] & E_{\mathcal{C}}(U) \arrow[l, "p_{\mathcal{C}}(u)"'] \arrow[r, tail, "u^*_{\mathcal{C}}"] \arrow[d, "E_{\mathcal{C}}(\gamma)"'] & P(\mathcal{C}) \arrow[d, "h'_{\mathcal{C}}(\gamma)"] \\
E(W) & E_{\mathcal{C}*W}(W) \arrow[l, "p_{\mathcal{C}*W}(W)"] \arrow[r, tail, "u^*_{\mathcal{C}*W}"'] & P(\mathcal{C}*W)
\end{tikzcd}$$

$\gamma = 1_U$ then $\alpha_i 1 = \gamma\alpha_i$ is a pullback for every i, in other words $\mathbf{C} * W = \mathbf{C}$ and $h' = 1$, for every $\mathbf{C} \in \mathfrak{C}(U)$; hence $E_{\mathbf{C}}(1_U) = 1$ and (going to the colimit) $E(1_U) = 1_{E(U)}$. If γ is arbitrary and $\delta : Z \longrightarrow W \in \mathfrak{U}(X)$, then to construct $E(\gamma)$, $E(\delta)$, we take pullbacks $\gamma\alpha_i' = \alpha_i\gamma_i'$, $\delta\alpha_i'' = \alpha_i'\delta_i'$ for each $\mathbf{C} = (\alpha_i)_{i\in I} \in \mathfrak{C}(U)$; juxtaposition yields pullbacks $(\gamma\delta)\alpha_i'' = \alpha_i(\gamma_i'\delta_i')$, which means that $\gamma_i'\delta_i' = (\gamma\delta)_i'$ and $(\mathbf{C} * W) * Z = \mathbf{C} * Z$. Hence all faces of the following diagram commute except perhaps the bottom face:

(5)
$$\begin{tikzcd}
 & P(\mathbf{C} * W) \arrow[dr, "h'(\delta)"] & \\
P(\mathbf{C}) \arrow[ur, "h'(\gamma)"] \arrow[rr, "h'(\gamma\delta)"] & & P(\mathbf{C} * Z) \\
 & E_{\mathbf{C}*W}(W) \arrow[uu, "u^*_{\mathbf{C}*W}"'] \arrow[dr, "E_{\mathbf{C}*W}(\gamma)"] & \\
E_{\mathbf{C}}(U) \arrow[uu, "u^*_{\mathbf{C}}"] \arrow[ur, "E_{\mathbf{C}}(\gamma)"] \arrow[rr, "E_{\mathbf{C}}(\gamma\delta)"'] & & E_{\mathbf{C}*Z}(Z) \arrow[uu, "u^*_{\mathbf{C}*Z}"']
\end{tikzcd}$$

then the bottom face commute anyway, since $u^*_{\mathbf{C}*Z}$ is a monomorphism. This shows that $E_{\mathbf{C}}(\gamma\delta) = E_{\mathbf{C}*W}(\delta)\ E_{\mathbf{C}}(\gamma)$. Hence

$$E_{\mathbf{C}}(\gamma\delta)\ p_{\mathbf{C}(U)} = p_{\mathbf{C}*Z}\ E_{\mathbf{C}}(\gamma\delta) = p_{\mathbf{C}*Z}(U)\ E_{\mathbf{C}*W}(\delta)\ E_{\mathbf{C}}(\gamma) =$$
$$= E(\delta)\ p_{\mathbf{C}*W}(U)\ E_{\mathbf{C}}(\gamma) = E(\delta)\ E(\gamma)\ p_{\mathbf{C}(U)}$$

for all $\mathbf{C}$, and $E(\gamma\delta) = E(\delta)E(\gamma)$. Therefore E is indeed a presheaf. We state this with two other properties of E :

4. Proposition 2.2. Let $\mathfrak{G}$ be a complete category having directed colimits, and X be any Grothendieck topology. Then for each $P \in \mathcal{P}(X,\mathfrak{G})$ the above construction yields a presheaf E and a morphism $c : P \longrightarrow E$ such that every morphism a of P to a sheaf factors uniquely through c ($a = tc$ for some t).

Proof. We already know that E is a presheaf, and have morphisms $c(U) : P(U) \longrightarrow E(U)$ for each $U \in \mathfrak{U}(X)$. Let $\gamma : W \longrightarrow U \in \mathfrak{U}(X)$. Keeping the notation as before, we have for each $\mathbf{C} \in \mathfrak{C}(U)$ the following diagram, in which, by definition of the various maps under consideration, every triangle and square commutes except possibly for the three squares fanning out of $P(\gamma)$:

(6)
$$\begin{tikzcd}
E(U) \arrow[dd, "E(\gamma)"'] & & E_C(U) \arrow[ll, "p_C(U)"'] \arrow[dr, "u^*_C"] \arrow[dd, "E_C(\gamma)"] & \\
& P(U) \arrow[ul, "c(U)"] \arrow[ur, "c_C(U)"] \arrow[rr, "u_C"] \arrow[dd, "P(\gamma)"] & & P(C) \arrow[dd, "h'_C(\gamma)"] \\
E(W) & & E_{C*W}(W) \arrow[ll, "p_{C*W}(W)"'] \arrow[dr, "u^*_{C*W}"] & \\
& P(W) \arrow[ul, "c(W)"] \arrow[ur, "c_{C*W}(W)"] \arrow[rr, "u_{C*W}"'] & & P(C*W)
\end{tikzcd}$$

Now, for each $i \in I$,

$$\pi_i \, h' \, u_C = P(\gamma_i')P(\alpha_i) = P(\alpha_i')P(\gamma) = \pi_i \, u_{C*W} \, P(\gamma) \quad ,$$

so that the front square commutes. Since u^*_{C*W} is a monomorphism, it follows that the diagonal square also commutes; and then the left square commutes. This shows that $c = (c(U))_{U \in \mathfrak{U}(X)}$ is a morphism of presheaves. $\square$

We now let $a : P \longrightarrow F$ be a morphism from P to a sheaf F. Take $U \in \mathfrak{U}(X)$, $C \in \mathfrak{C}(U)$. From $C = (\alpha_i)_{i \in I}$, $\alpha_i : U_i \longrightarrow U$ we obtain a diagram: Vor (1)

(7)
$$\begin{tikzcd}
F(U) \arrow[rr, tail, "u^F"] & & F(C) \arrow[r, shift left, "f^F"] \arrow[r, shift right, "g^F"'] & F(C*C) \\
& E_C(U) \arrow[ul, dashed, "t_C(U)"'] \arrow[dr, "u^*_C"] & & \\
P(U) \arrow[uu, "a(U)"] \arrow[ur, "c_C(U)"] \arrow[rr, "u^P"'] & & P(C) \arrow[uu, "a'_C(U)"'] \arrow[r, shift left, "f^P"] \arrow[r, shift right, "g^P"'] & P(C*C) \arrow[uu, "a''_C(U)"']
\end{tikzcd}$$

where $a'_C(U) = a' = \prod_{i \in I} a(U_i)$, $a''_C(U) = a'' = \prod_{j,k \in I} a(U_j * U_k)$ and $t_C(U)$ will be presently constructed. For each j,k, (2)

$$\pi_{jk} f^F a' = F(\xi_{jk}) a(U_j) \pi_j = a(U_j * U_k) P(\xi_{jk}) \pi_j = \pi_{jk} a'' f^P \quad ,$$

so that $f^F a' = a'' f^P$. Similarly, $g^F a' = a'' g^P$. Since F is a sheaf, there is a morphism $t_{\mathcal{C}}(U) : E_{\mathcal{C}}(U) \longrightarrow F(U)$ (induced on equalizers) such that $a' u^*_{\mathcal{C}} = u^F_{\mathcal{C}}\, t_{\mathcal{C}}(U)$; since u^F is a monomorphism, we also have $t_{\mathcal{C}}(U) c_{\mathcal{C}}(U) = a(U)$.

If $\mathcal{C} \leq \mathcal{D}$ in $\mathfrak{C}(U)$, then we obtain a diagram

(8)
$$\begin{tikzcd}
 & F(U) \arrow[rr, tail, "u^F_{\mathcal{D}}"] & & F(\mathcal{D}) \\
F(U) \arrow[ur, equal] \arrow[rr, tail, "u^F_{\mathcal{C}}"] & & F(\mathcal{C}) \arrow[ur, "F'(\psi)"] & \\
 & E_{\mathcal{D}}(U) \arrow[uu, "t_{\mathcal{D}}(U)"] \arrow[rr, tail, "u^*_{\mathcal{D}}"] & & P(\mathcal{D}) \arrow[uu, "a'_{\mathcal{D}}"'] \\
E_{\mathcal{C}}(U) \arrow[uu, "t_{\mathcal{C}}(U)"] \arrow[ur, "E_{\mathcal{C}\mathcal{D}}(U)"] \arrow[rr, tail, "u^*_{\mathcal{C}}"'] & & P(\mathcal{C}) \arrow[uu, "a'_{\mathcal{C}}"] \arrow[ur, "F'(\psi)"'] &
\end{tikzcd}$$

in which all squares commute except perhaps for the left and right squares. With the notation as above,

$$\pi_p\, a'_{\mathcal{D}}\, P'(\psi) = a(V_p) P(\psi_p) \pi_{\psi p} = F(\psi_p) a(U_{\psi p}) \pi_{\psi p} = \pi_p F'(\psi) a'_{\mathcal{C}}$$

for all p, so that the square on the right commutes also. Since $u^F_{\mathcal{D}}$ is a monomorphism, it follows that the square on the left commutes, i.e. $t_{\mathcal{D}}(U)\, E_{\mathcal{C}\mathcal{D}}(U) = t_{\mathcal{C}}(U)$.

In other words, $(t_{\mathcal{C}}(U))_{\mathcal{C} \in \mathfrak{C}(U)}$ is a cocompatible family. Therefore it induces a morphism $t(U) : E(U) \longrightarrow F(U)$, such that $t_{\mathcal{C}}(U) = t(U)\, p_{\mathcal{C}}(U)$ for all $\mathcal{C}$. Note that

$$t(U) c(U) = t(U)\, p_{\mathcal{C}}(U)\, c_{\mathcal{C}}(U) = t_{\mathcal{C}}(U)\, c_{\mathcal{C}}(U) = a(U) .$$

To prove the existence of the factorization, we only have to show that t is a morphism of presheaves.

Let $\gamma : W \longrightarrow U \in \mathfrak{U}(X)$. We have induced morphisms $h'^P(\gamma) : P(\mathcal{C}) \longrightarrow P(\mathcal{C} * W)$, $h'^F(\gamma) : F(\mathcal{C}) \longrightarrow F(\mathcal{C} * W)$. For each i ,

$$\pi_i h'^F a'_{\mathbb{C}} = F(\gamma'_i)a(U_i)\pi_i = a(U_i * W)P(\gamma'_i)\pi_i = \pi_i a'_{\mathbb{C}*W}\, h'^P$$

so that $h'^F a'_{\mathbb{C}} = a'_{\mathbb{C}*W} h'^P$. Similarly, $u^F_{\mathbb{C}*W}\, F(\gamma) = h'^F\, u^F_{\mathbb{C}}$. Since diagrams (4) and (7) commute, this yields a diagram

(9)

$$\begin{tikzcd}
 & F(U) \arrow[rr, tail, "u^F_{\mathbb{C}}"] \arrow[dd, "F(\gamma)"] & & F(\mathbb{C}) \arrow[dd, "h'^F"] & \\
E(U) \arrow[ur, "t(U)"] \arrow[dd, "E(\gamma)"'] & & E_{\mathbb{C}}(U) \arrow[ll, "p_{\mathbb{C}}(U)"'] \arrow[ul, "t_{\mathbb{C}}(U)"'] \arrow[rr, tail, "u^*_{\mathbb{C}}"] \arrow[dd, "E_{\mathbb{C}}(\gamma)"] & & P(\mathbb{C}) \arrow[ul, "a'_{\mathbb{C}}"'] \arrow[dd, "h'^P"] \\
 & F(W) \arrow[rr, tail, "u^F_{\mathbb{C}*W}"] & & F(\mathbb{C}*W) & \\
E(W) \arrow[ur, "t(W)"] & & E_{\mathbb{C}*W}(W) \arrow[ll, "p_{\mathbb{C}*W}(U)"] \arrow[ul, "t_{\mathbb{C}*W}(W)"'] \arrow[rr, tail, "u^*_{\mathbb{C}*W}"'] & & P(\mathbb{C}*W) \arrow[ul, "a'_{\mathbb{C}*W}"']
\end{tikzcd}$$

which commutes except perhaps for the two squares that connect $F(\gamma)$ to $E(\gamma)$ and $E_{\mathbb{C}}(\gamma)$. First we see that the latter does commute, since $u^F_{\mathbb{C}*W}$ is a monomorphism. This yields $t(W)E(\gamma)p_{\mathbb{C}}(U) = F(\gamma)t(U)p_{\mathbb{C}}(U)$; since this is true for all $\mathbb{C}$, it follows that $t(W)E(\gamma) = F(\gamma)t(U)$, i.e. the diagram commutes, and we have shown that t is a morphism of presheaves. ○

The uniqueness of the factorization will follow from a description of $u^E_{\mathbb{C}}$, namely $u^E_{\mathbb{C}}\, p_{\mathbb{C}}(U) = c'_{\mathbb{C}}(U)\, u^*_{\mathbb{C}}$ for all $\mathbb{C} \in \mathfrak{C}(U)$. (4)

To see that, set, as usual, $\mathbb{C} = (\alpha_i)_{i\in I}$, $\alpha_i : U_i \longrightarrow U$; we remember that $E(\alpha_i)$ is induced by $E_{\mathbb{C}}(\alpha_i) : E_{\mathbb{C}}(U) \longrightarrow E_{\mathbb{C}*U_i}(U_i)$, which is in turn induced by $h' : P(\mathbb{C}) \longrightarrow P(\mathbb{C}*U_i)$; the pullbacks we need to describe h' are precisely $\alpha_i \xi_{ij} = \alpha_j \eta_{ij}$, so that $\mathbb{C} * U_i = (\xi_{ij})_{j\in I}$ and $h' = \prod_{j\in I} P(\eta_{ij})$. We now consider the following diagram, in which the outer square and bottom triangle are already known to commute:

(10)
$$\begin{tikzcd}
E_{C}(U) \arrow[rr, tail, "u^*_{C}"] \arrow[dd, "E_{C}(\alpha_i)"'] & & P(C) \arrow[dl, "\pi_i"'] \arrow[dd, "h'"] \\
& P(U_i) \arrow[dl, "c_{C*U_i}(U_i)"'] \arrow[dr, "u^P_{C*U_i}"] & \\
E_{C*U_i}(U_i) \arrow[rr, tail, "u^*_{C*U_i}"'] & & P(C*U_i)
\end{tikzcd}$$

Note that the triangle on the right need not commute; however, for each $j \in I$:

$$\pi_j\, u^P_{C*U_i}\, \pi_i u^*_C = P(\xi_{ij})\ \pi_i\ u^*_C = \pi_{ij}\ f^P_C\ u^*_C \ ,$$
$$\pi_j\, h'\ u^*_C = P(\eta_{ij})\ \pi_j\, u^*_C = \pi_{ij}\ g^P_C\ u^*_C \ ,$$

and it follows that $u^P_{C*U_i}\,\pi_i\ u^*_C = h'\ u^*_C$. Since $u^*_{C*U_i}$ is a monomorphism, it follows that the upper left triangle commutes, i.e. $E_C(\alpha_i) = c_{C*U_i}\,\pi_i\ u^*_C$. Hence

$$c(U_i)\ \pi_i\ u^*_C = p_{C*U_i}\, c_{C*U_i}\,\pi_i\ u^*_C = p_{C*U_i}\, E_C(\alpha_i) = E(\alpha_i)\ p_C \ .$$

Applying $\underset{i\in I}{\times}$ to both sides yields the desired formula $c'_C(U)\ u^*_C = u^E_C\ p_C(U)$, i.e. the following diagram commutes:

(11)
$$\begin{tikzcd}
E(U) \arrow[r, "u^E_C"] & E(C) \\
E_C(U) \arrow[u, "p_C(U)"] \arrow[r, "u^*_C"'] & P(C) \arrow[u, "c'_C(U)"']
\end{tikzcd}$$

Let now $t_1, t_2 : E \longrightarrow F$ be morphisms such that $t_1 c = t_2 c$, where F is a sheaf (in fact, the uniqueness still holds if F is only a monopresheaf). For each $C \in \mathfrak{C}(U)$, we have $t_1' c' = t_2' c'$ and hence

$$u^F_C\ t_1(U)\ p_C(U) = t_1'\ u^E_C\ p_C(U) = t_1'\ c'\ u^*_C =$$
$$= t_2'\ c'\ u^*_C = \dots = u^F_C\ t_2(U)\ p_C(U) \ ;$$

since this holds for all C and u^F is a monomorphism, it follows that $t_1(U) = t_2(U)$, whence $t_1 = t_2$; this completes the proof. $\square$

Corollary 2.3. Further assume that directed colimits in $\mathfrak{A}$ preserve (pointwise) monomorphisms. If in 2.2 a is a pointwise monomorphism, then so is t.

Proof. Then $a'_{\mathfrak{C}}(U)$ is a monomorphism for all $\mathfrak{C} \in \mathfrak{C}(U)$; hence it is clear on diagram (7) that $t_{\mathfrak{C}}(U)$ is also a monomorphism. By the hypothesis, so is $t(U)$.

3. THE CASE OF A C_4 REGULAR CATEGORY.

1. We now assume that $\mathfrak{A}$ is a C_4 regular category.

Lemma 3.1. For any presheaf P , E is a monopresheaf.

Proof. Take $\mathfrak{C} \in \mathfrak{C}(U)$ $[\mathfrak{C} = (\alpha_i)_{i\in I}, \alpha_i : U_i \longrightarrow U]$; we want to show that $u^E_{\mathfrak{C}}$ is a monomorphism. Now $E(\mathfrak{C}) = \prod_{i\in I} \varinjlim E_{\mathfrak{C}_i}(U_i)$ and we can apply theorem II.4.4, which in this situation says that the morphisms $p'_\tau = \prod_{i\in I} p_{\tau i}(U_i)$, $\tau \in T = \prod_{i\in I} \mathfrak{C}(U_i)$, induce a monomorphism

$$t : \varinjlim_{\tau\in T} \prod_{i\in I} E_{\tau i}(U_i) \longrightarrow E(\mathfrak{C}) \quad .$$

For each $\tau \in T$, put $\tau i = (\beta_p)_{p\in J_i}$, where we assume that the sets J_i are pairwise disjoint and disjoint from I ; then $\mathfrak{C}_\tau = (\alpha_i\beta_p)_{p\in J_i, i\in I}$ is in $\mathfrak{C}(U)$. We also put $\bigcup_{i\in I} J_i = J$ and $\alpha_i\beta_p = \gamma_p : V_p \longrightarrow U$. We now interrupt the proof to observe:

Lemma 3.2. For every $\mathfrak{C} \in \mathfrak{C}(U)$, $\{\mathfrak{C}_\tau \,;\, \tau \in T\}$ is a cofinal subset of $\mathfrak{C}(U)$.

Proof of 3.2. Take $\mathfrak{D} \in \mathfrak{C}(U)$. Define $\tau \in T$ by: $\tau i = \mathfrak{D} * U_i \in \mathfrak{C}(U_i)$.

Then $\mathcal{C}_\tau = \mathcal{D} * \mathcal{C}$, so that $\mathcal{D} \leq \mathcal{C}_\tau$.

We now resume the proof of 3.1. Let π_o be the projection

$$\pi_o : P(\mathcal{C}_\tau * \mathcal{C}_\tau) = \prod_{p,q\in J} P(V_p * V_q) \longrightarrow \prod_{i\in I} \prod_{p,q\in J_i} P(V_p * V_q) = \prod_{i\in I} P(\tau i * \tau i)$$

(note that $\bigcup_{i\in I} J_i \pi J_i \subseteq J \pi J$). The pullbacks $\gamma_p \xi_{pq} = \gamma_q \eta_{pq}$ used in evaluating $\tau i * \tau i$ also serve for $\mathcal{C}_\tau * \mathcal{C}_\tau$; hence $\pi_{pq} \pi_o f^P_{\mathcal{C}_\tau} = P(\xi_{pq}) = \pi_{pq} f^P_{\tau i}$ for all $p,q \in J_i$, whence $\pi_o f^P_{\mathcal{C}_\tau} = \prod_{i\in I} f^P_{\tau i}$. Thus we have a commutative diagram (where u_τ will presently be constructed):

(12)
$$\begin{tikzcd}[column sep=large]
\prod_{i\in I} E_{\tau i}(U_i) \arrow[r, tail, "\prod_{i\in I} u^*_{\tau i}"] & \prod_{i\in I} P(\tau i) \arrow[r, "\prod_{i\in I} f^P_{\tau i}"] & \prod_{i\in I} P(\tau i * \tau i) \\
E_{\mathcal{C}_\tau}(U_i) \arrow[u, dashed, tail, "u_\tau"] \arrow[r, tail, "u^*_{\mathcal{C}_\tau}"'] & P(\mathcal{C}_\tau) \arrow[u, equal] \arrow[r, "f^P_{\mathcal{C}_\tau}"'] & P(\mathcal{C}_\tau * \mathcal{C}_\tau) \arrow[u, "\pi_o"']
\end{tikzcd}$$

There is a similar commutative diagram with g's instead of f's . Now, products preserve equalizers, and hence there is a morphism u_τ induced on equalizers, such that $u^*_{\mathcal{C}_\tau} = (\prod_{i\in I} u^*_{\tau i}) \, u_\tau$. Note that u_τ is a monomorphism.

We now prove that $p'_\tau u_\tau = u^E_{\mathcal{C}} \, p_{\mathcal{C}_\tau}$. First, note that $\mathcal{C}_\tau * U_i \leq \tau i$; more precisely, take pullbacks $\gamma_p \alpha'_{pi} = \alpha_i \gamma'_{ip}$ $(p \in J)$, so that $\mathcal{C}_\tau * U_i = (\gamma'_{ip})_{p\in J}$; let $\psi : J_i \longrightarrow J$ be the inclusion ; if $p \in J_i$ (so that $\psi p = p$), then $\gamma_p 1 = \alpha_i \beta_p$ and the pullback yield a morphism ψ_p such that $\alpha'_{pi} \psi_p = 1$ and $\beta_p = \gamma'_{ip} \psi_p$. We now have a three-dimensional diagram (next page) in which, of the six areas indicated, area 6 commutes trivially, and areas 2 , 5 and 3 commute by definition of the various E maps therein (see diagrams (4) and (1)). For each $p \in J_i$,

$$\pi_p \, P'(\psi) \, h' = P(\psi_p) P(\alpha'_{pi}) \, \pi_p = \pi_p \quad ,$$

and it follows that area (1) also commutes:

(13)
$$\begin{tikzcd}
E(U) \arrow[rr, "E(\alpha_i)"] & & E(U_i) & & \\
E_{C_\tau}(U) \arrow[u, "p_{C_\tau}(U)"] \arrow[rr, "u_\tau\ (5)"] \arrow[drr, "E_{C_\tau}(\alpha_i)"'] \arrow[dddd, tail, "u^*_{C_\tau}"'] & & \prod_{i\in I} E_{\tau i}(U_i) \arrow[u, hook] \arrow[rr, "(6)\ \pi_i"] & & E_{\tau i}(U_i) \arrow[ull, "p_{\tau i}(U_i)"'] \arrow[dddd, tail, "u^*_{\tau i}"] \\
 & (2) & E_{C_\tau * U_i}(U_i) \arrow[u, tail, "p_{C_\tau * U_i}(U_i)\ (4)"] \arrow[urr, "E_{C_\tau * U_i,\tau i}(U_i)"'] \arrow[d, tail, "u^*_{C_\tau * U_i}"] & (3) & \\
 & & P(C_\tau * U_i) \arrow[drr, "P'(\psi)"] & & \\
P(C_\tau) = \prod_{i\in I} P(\tau i) \arrow[urr, "h'"] \arrow[rrrr, "\pi_i"'] \arrow[urrrr, phantom, "(1)"] & & & & P(\tau i)
\end{tikzcd}$$

Finally, $u^*_{\tau i}\ \pi_i\ u_\tau = \pi_i\ u^*_{C_\tau}$ by definition of u_τ ; since $u^*_{\tau i}$ is a monomorphism, it follows that area (4) commutes. Hence the whole diagram is commutative, in particular

$$p_{\tau i}(U_i)\ \pi_i\ u_\tau = E(\alpha_i)\ p_{C_\tau}(U_i)\ .$$

Applying $\underset{i\in I}{\mathsf{X}}$ to both sides yields the desired formula $p'_\tau u_\tau = u^E_C\ p_{C_\tau}$.

We now take directed colimits (over T). In view of 3.2, this sends the commutative square below left to the commutative square below right:

(14)
$$\begin{tikzcd}
E(U) \arrow[r, "u^E_C"] & E(C) \\
E_{C_\tau}(U) \arrow[u, "p_{C_\tau}"] \arrow[r, tail, "u_\tau"'] & \prod_{i\in I} E_{\tau i}(U_i) \arrow[u, "p'_\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}
E(U) \arrow[r, "u^E_C"] \arrow[d, equal] & E(C) \\
E(U) \arrow[r, tail, "u"'] & \varinjlim_{\tau\in T} \prod_{i\in I} E_{\tau i}(U_i) \arrow[u, tail, "t"']
\end{tikzcd}$$

where t is a monomorphism and so is $u = \varinjlim u_\tau$. Then u^E_C is a monomorphism, q.e.d.

Lemma 3.3. If P is a monopresheaf, then E is a sheaf.

Proof. When P is a monopresheaf, $u^P_{\mathcal{C}} = u^*_{\mathcal{C}}\, c_{\mathcal{C}}(U)$ shows that $c_{\mathcal{C}}(U)$ is a monomorphism; hence $c(U) = \varinjlim c_{\mathcal{C}}(U)$ is also a monomorphism. In other words, $c : P \longrightarrow E$ is a monomorphism. Then looking at the commutative diagram (11), where $c'_{\mathcal{C}}(U)$ and $u^*_{\mathcal{C}}$ are monomorphisms, shows that $p_{\mathcal{C}}(U)$ is a monomorphism. This shows that every direct system $E_{-}(U) : \mathfrak{C}(U) \longrightarrow \mathfrak{G}$ is monic.

We now start the proof as for lemma 3.1; this time, by the above, theorem II.4.4 tells us that t is an isomorphism.

Given $j, k \in I$, certain relations exist between the coverings of U_j, U_k we already have and coverings of $U_j * U_k$ that arise from these. For each $p \in J_j$, $q \in J_k$, consider the diagram in $\mathfrak{U}(X)$:

$$(15) \qquad \begin{tikzcd}
 & & U & & \\
 & U_j \arrow[ur, "\alpha_j"] & \text{PB} & U_k \arrow[ul, "\alpha_k"'] & \\
V_p \arrow[ur, "\beta_p"] \arrow[uurr, bend left, "\gamma_p"] & \text{PB} & U_j * U_k \arrow[ul, "\xi_{jk}"] \arrow[ur, "\eta_{jk}"'] & \text{PB} & V_q \arrow[ul, "\beta_q"'] \arrow[uull, bend right, "\gamma_q"'] \\
 & V_p * U_k \arrow[ul, "\alpha'_{pk}"] \arrow[ur, "\beta'_{pk}"'] & \text{PB} & U_j * V_q \arrow[ul, "\beta'_{jq}"] \arrow[ur, "\alpha'_{jq}"'] & \\
 & & V_p * V_q \arrow[ul, "\xi'_{pq}"] \arrow[ur, "\eta'_{pq}"'] \arrow[uull, bend left, "\xi_{pq}"] \arrow[uurr, bend right, "\eta_{pq}"'] & &
\end{tikzcd}$$

in which each square is a pullback, arranged so that juxtaposition yields the previous pullbacks $\gamma_p \alpha'_{pk} = \alpha_k \gamma'_{pk}$, $\alpha_j \gamma'_{jq} = \gamma_q \alpha'_{jq}$, $\gamma_p \xi_{pq} = \gamma_q \eta_{pq}$. We remember that $\tau j = (\beta_p)_{p \in J_j}$, $\tau k = (\beta_q)_{q \in J_k}$; the diagram yields a covering $\tau j * (U_j * U_k) = (\beta'_{pk})_{p \in J_j}$, for which we shall abuse the notation by calling it $\tau j * U_k$. Similarly we obtain

a covering $\tau k * U_j\, [= \tau k *(U_j * U_k)] = (\beta'_{jq})_{q\in J_k}$ also in $\mathfrak{C}(U_j * U_k)$. The diagram finally yields a covering $\tau j * \tau k\, [= (\tau j * (U_j * U_k)) * (\tau k * (U_j * U_k))]$ which refines both $\tau j * U_k$ and $\tau k * U_j$; it is given by:
$\tau j * \tau k = (\gamma_{pq})_{p\in J_j,\, q\in J_k}$, where $\gamma_{pq} = \beta'_{pk}\xi'_{pq} = \beta'_{jq}\eta'_{pq}$; it is this finer covering we need for the proof. That $\tau j * \tau k \geq \tau j * U_k$ is seen more precisely by considering the projection $\chi : J_j \sqcap J_k \longrightarrow J_j$ and morphisms $\chi_{pq} = \xi'_{pq}$.

Then we have a commutative diagram:

$$
\begin{tikzcd}
E(U_j*U_k) \arrow[dr, phantom, "\textcircled{1}"] & E_{\tau j*\tau k}(U_j*U_k) \arrow[l, "p_{\tau j*\tau k}"'] \arrow[r, "u^*_{\tau j*\tau k}"] \arrow[dr, phantom, "\textcircled{2}"] & P(\tau j*\tau k) \\
 & E_{\tau j*U_k}(U_j*U_k) \arrow[ul, "p_{\tau j*U_k}"] \arrow[u, "E_{\tau j*U_k,\tau j*\tau k}(U_j*U_k)"'] \arrow[r, "u^*_{\tau j*U_k}"'] & P(\tau j*U_k) \arrow[u, "P'(\chi)"'] \\
E(U_j) \arrow[uu, "E(\xi_{ij})"'] \arrow[ur, phantom, "\textcircled{3}"] & E_{\tau j}(U_j) \arrow[l, "p_{\tau j}"'] \arrow[u, "E_{\tau j}(\xi_{jk})"'] \arrow[r, "u^*_{\tau j}"'] \arrow[ur, phantom, "\textcircled{4}"] & P(\tau j) \arrow[u, "h'_{\tau j}(\xi_{jk})"'] \\
E(U) \arrow[u, "E(\alpha_j)"'] \arrow[ur, phantom, "\textcircled{5}"] & E_{\mathfrak{C}_\tau}(U) \arrow[l, "p_{\mathfrak{C}_\tau}"] \arrow[u, "\pi_j u_\tau"'] \arrow[r, "u^*_{\mathfrak{C}_\tau}"'] \arrow[ur, phantom, "\textcircled{6}"] & P(\mathfrak{C}_\tau) \arrow[u, "\pi_j"'] \arrow[d, "f^P_{\mathfrak{C}_\tau}"] \\
 & & P(\mathfrak{C}_\tau*\mathfrak{C}_\tau)
\end{tikzcd}
\tag{16}
$$

where area 1 commutes trivially, areas 2, 3, 4 commute by definition of the E maps therein (see diagrams (1) and (4), and areas 5 and 6 commute because diagram (13) commutes.

We now merge areas 1 - 3, areas 2 - 4, and take products (over j,k for the top row, over j for the middle row). This yields the next commutative diagram, in which $\bar{u} = \underset{i\in I}{\times} u^*_{\tau i}$, $p'_\tau = \prod_{i\in I} p_{\tau i}$ (as defined before), $p''_\tau = \prod_{j,k\in I} p_{\tau j*\tau k}$ and $\bar{\bar{u}} = \prod_{j,k\in I} u^*_{\tau j*\tau k}$.

All four are monomorphisms (p' and p'' because in each monic direct system $E\_(\mathbf{W}) : \mathfrak{C}(\mathbf{W}) \longrightarrow \mathfrak{a}$ the morphisms p are monomorphisms, as observed at the beginning of the proof). In addition, using the definition of $P'(\chi)$ and $h'_{\tau j}(\xi_{jk})$, we see on diagram (15) that

$$\pi_{pq}\ P'(\chi)h'_{\tau j}(\xi_{jk}) = P(\xi'_{pq})P(\alpha'_{pk})\pi_p = P(\xi_{pq})\pi_p = \pi_{pq}\ f^P_{\mathbf{C}_\tau}$$

for all $p \in J_j$, $q \in J_k$; therefore the vertical map (top right) on the new diagram (17) below is $f^P_{\mathbf{C}_\tau}$. The diagram:

(17)
$$\begin{tikzcd}
E(\mathbf{C}*\mathbf{C}) & \prod\limits_{j,k\in I} E_{\tau j*\tau k}(U_j*U_k) \arrow[l, tail, "p''_\tau"'] \arrow[r, tail, "\bar{\bar{u}}"] & \prod\limits_{j,k\in I} P(\tau j*\tau k) \\
E(\mathbf{C}) \arrow[u, "f^E_{\mathbf{C}}"] & \prod\limits_{i\in I} E_{\tau i}(U_i) \arrow[l, tail, "p'_\tau"'] \arrow[r, tail, "\bar{u}"] \arrow[u] & P(\mathbf{C}_\tau) \arrow[u, "f^P_{\mathbf{C}_\tau}"'] \\
E(U) \arrow[u, "u^E_{\mathbf{C}}"] & E_{\mathbf{C}_\tau}(U) \arrow[l, tail, "p_{\mathbf{C}_\tau}"] \arrow[r, tail, "u^*_{\mathbf{C}_\tau}"'] \arrow[u, "u_\tau"'] & P(\mathbf{C}_\tau) \arrow[u, equal] \arrow[d, "f^P_{\mathbf{C}_\tau}"] \\
 & & \prod\limits_{j,k\in I} P(\tau j*\tau k) = P(\mathbf{C}_\tau*\mathbf{C}_\tau)
\end{tikzcd}$$

Since we went up in the coverings to $\tau j * \tau k$, which is "symmetric in j and k", we obtain, by working on the other side of diagram (15), a commutative diagram which is the same as (17) but with g's instead of f's (and a different unnamed morphism in the middle column); all other morphisms will remain unchanged. Now p''_τ and $\bar{\bar{u}}$ are monomorphisms and when we consider both diagrams it is evident that $\mathrm{Equ}(f^E_{\mathbf{C}}p'_\tau, g^E_{\mathbf{C}}p'_\tau) = \mathrm{Equ}(f^P_{\mathbf{C}_\tau}\bar{u}, g^P_{\mathbf{C}_\tau}\bar{u})$.

This in turn implies that $\mathrm{Equ}(f^E_{\mathbf{C}}p'_\tau, g^E_{\mathbf{C}}p'_\tau) = \mathrm{Im}\ u_\tau$. To see this, we first note that $u^E_{\mathbf{C}}$ is a monomorphism, by 3.1, and the diagram then shows that u_τ is a monomorphism. Next,

$$f^P_{\mathbf{C}_\tau} \bar{u}\, u_\tau = f^P_{\mathbf{C}_\tau} u^*_{\mathbf{C}_\tau} = g^P_{\mathbf{C}_\tau} u^*_{\mathbf{C}_\tau} = g^P_{\mathbf{C}_\tau} \bar{u}\, u_\tau \quad .$$

Finally, $f^P_{\mathbf{C}_\tau} \bar{u}\, a = g^P_{\mathbf{C}_\tau} \bar{u}\, a$ implies $\bar{u}\, a = u^*_{\mathbf{C}_\tau} x = \bar{u}\, u_\tau\, x$ for some x, whence $a = u_\tau x$ for some x. Hence $u_\tau \in \mathrm{Equ}(f^P_{\mathbf{C}_\tau} \bar{u}, g^P_{\mathbf{C}_\tau} \bar{u}) = \mathrm{Equ}(f^E_{\mathbf{C}} p'_\tau, g^E_{\mathbf{C}} p'_\tau)$

If now we go to the directed colimit over T, then we saw at the beginning of the proof that the morphisms p'_τ induce an isomorphism; so do the $p_{\mathbf{C}_\tau}$, by 3.2; hence it follows from II.4.3 that

$$\begin{aligned}\mathrm{Equ}(f^E_{\mathbf{C}}, g^E_{\mathbf{C}}) &= \bigvee_{\tau\in T} (p'_\tau)_s\, \mathrm{Equ}\,(f^E_{\mathbf{C}} p'_\tau, g^E_{\mathbf{C}} p'_\tau) = \\ &= \bigvee_{\tau\in T} (p'_\tau)_s\, \mathrm{Im}\, u_\tau = \bigvee_{\tau\in T} \mathrm{Im}\, p'_\tau u_\tau = \bigvee_{\tau\in T} \mathrm{Im}\, u^E_{\mathbf{C}} p_{\mathbf{C}_\tau} = \\ &= (u^E_{\mathbf{C}})_s \bigvee_{\tau\in T} \mathrm{Im}\, p_{\mathbf{C}_\tau} = \mathrm{Im}\, u^E_{\mathbf{C}} \quad .\end{aligned}$$

Since $u^E_{\mathbf{C}}$ is a monomorphism, this proves that E is a sheaf.

Hence we have proved:

<u>Theorem</u> 3.4. If $\mathcal{G}$ is a C_4 regular category, then for any Grothendieck topology X, $\mathfrak{F}(X,\mathcal{G})$ is coreflective in $\mathcal{P}(X,\mathcal{G})$, and Heller and Rowe's construction yields the coreflection in at most two steps.

2. We denote by $\hat{}\; : \mathcal{P}(X,\mathcal{G}) \longrightarrow \mathfrak{F}(X,\mathcal{G})$ the coreflection. In order to obtain a well-defined functor, we take the functor obtained by applying twice the clearly functorial Heller and Rowe construction, and amend it (i.e. change it by a natural isomorphism) so that $\hat{P}$ is obtained in one step from P if P is a monopresheaf, and $\hat{P} = P$ if P is a sheaf [no such fuss is necessary with a stronger set theory].

Since $\mathcal{G}$ is C_4, 2.3 holds, so that $\hat{}$ preserves monomorphisms when their codomain is a sheaf. We use this to prove:

<u>Theorem</u> 3.5. If $\mathcal{G}$ is a C_4 regular category, then for any Grothendieck topology X, $\mathfrak{F}(X,\mathcal{G})$ is a C_3 regular category.

<u>Proof</u>. First the functor category $\mathcal{P}(X,\mathcal{G})$ is regular by I.2.1,

with pointwise regular decompositions. It is also complete and cocomplete, like $\mathfrak{G}$, and since everything in $\mathfrak{P}(X,\mathfrak{G})$ works pointwise, including subobjects and their intersections and unions, $\mathfrak{P}(X,\mathfrak{G})$ is in fact a C_4 regular category. The coreflective subcategory $\mathfrak{F}(X,\mathfrak{G})$ inherits completeness and cocompleteness from $\mathfrak{P}(X,\mathfrak{G})$.

The existence of regular decompositions in $\mathfrak{F}(X,\mathfrak{G})$ then follows from I.1.6. However, it is interesting (and necessary) to see what they look like. First monomorphisms in the finitely complete category $\mathfrak{F}(X,\mathfrak{G})$ can be characterized by their kernel pairs, which are the same in $\mathfrak{F} = \mathfrak{F}(X,\mathfrak{G})$ as in $\mathfrak{P} = \mathfrak{P}(X,\mathfrak{G})$, and it follows that the monomorphisms of $\mathfrak{F}$ coincide with the pointwise monomorphisms of $\mathfrak{F}$. The regular epimorphisms are given by:

Proposition 3.6. Let $f \in \mathfrak{F}$ have the regular decomposition (m,p) in $\mathfrak{P}$. Then f is a regular epimorphism (in $\mathfrak{F}$) if and only if $\hat{m}$ is an isomorphism.

Proof of 3.6. First, assume that $\hat{m}$ is an isomorphism. Then $a,b \in \mathfrak{F}$, $af = bf$ implies $am = bm$, $a\hat{m} = \hat{a}\hat{m} = \hat{b}\hat{m} = b\hat{m}$ and $a = b$, so that f is an epimorphism in $\mathfrak{F}$. Now let $fx = fy$ be a pullback and $g \in \mathfrak{F}$ be such that $gx = gy$. Since, in $\mathfrak{P}$, $px = py$ is also a pullback (as m is a monomorphism) and p is a regular epimorphism, we have $g = tp$ for some t. Then also $g = \hat{t}\hat{p}$ and since $\hat{m}$ is an isomorphism g factors through f, uniquely since f is an epimorphism. This shows that $f \in \mathrm{Coequ}_{\mathfrak{F}}(x,y)$ and hence that f is a regular epimorphism.

Conversely assume that f is a regular epimorphism. Let M be the domain of m and $c : M \longrightarrow \hat{M}$ be the coreflection. Then $m = \hat{m}c$, $\hat{p} = cp$. Since $\hat{c} = 1$ is an isomorphism, and c is a monomorphism (since $m = \hat{m}c$, or for the more general reason that M is a monopresheaf), the first part of the proof shows that $\hat{p}$ is a regular epimorphism of $\mathfrak{F}$. On the other hand, $\hat{m}$ is a monomorphism, by 2.3. Hence

$(\hat{m},\hat{p})$ is a regular decomposition of f and since f is a regular epimorphism it follows that $\hat{m}$ is an isomorphism.

We now interrupt the proof of the theorem to show:

Proposition 3.7. Let $\mathcal{G}$ be a C_4 regular category, and X be any Grothendieck topology. Then the coreflection $\mathcal{P}(X,\mathcal{G}) \longrightarrow \mathfrak{F}(X,\mathcal{G})$ is exact.

Proof. We already know that it preserves colimits. On the other hand, it is obtained by applying twice Heller and Rowe's construction which, being defined in terms of products, equalizers and directed colimits of $\mathcal{G}$, commutes with finite limits. It is therefore an exact functor. [This provides an alternate proof of 3.6 above.]

To prove that $\mathfrak{F}(X,\mathcal{G})$ is regular, it now suffices to prove that it satisfies the pullback axiom. First we establish the following particular case:

Lemma 3.8. Let

$$\begin{array}{ccc} M & \xrightarrow{m} & G \\ r\uparrow & & \uparrow q \\ N & \xrightarrow[n]{} & K \end{array}$$

be a pullback in $\mathcal{P}(X,\mathcal{G})$, where G,K are sheaves and m, n monomorphisms. If $\hat{m}$ is an isomorphism, then so is $\hat{n}$.

Proof. Then M, N are monopresheaves and Heller and Rowe's construction gives their coreflections in one step; for M, it is given on the commutative diagram (18) below, obtained from (7), where the notation is as usual and we recall that $\hat{m}_C(U)$ is induced on equalizers (considering the similar diagram with g's instead of f's) and in turn induces $\hat{m}(U)$ when we take directed colimits over $\mathfrak{C}(U)$:

(18)

$$\begin{tikzcd}
G(U) \arrow[rr, tail, "u^G_{C}"] & & G(C) \arrow[r, "f^G_{C}"] & G(C*C) \\
\hat{M}(U) \arrow[u, "\hat{m}(U)"] & & & \\
& E_{C}(U) \arrow[ul, "p_{C}(U)"'] \arrow[uul, "\hat{m}_{C}(U)"'] \arrow[r, tail, "u^*_{C}"] & M(C) \arrow[uu, "m'(U)"] \arrow[r, "f^M_{C}"] & M(C*C) \arrow[uu, "m''(U)"] \\
M(U) \arrow[uuu, "m(U)"] \arrow[uu, "c^M(U)"] \arrow[ur, "c^M_{C}(U)"] \arrow[urr, tail, "u^M_{C}"'] & & &
\end{tikzcd}$$

With this and the similar diagram for n, we obtain a diagram:

(19)

$$\begin{tikzcd}
& G(U) \arrow[rrr, tail, "u^G_{C}"] & & & G(C) \arrow[rr, "f^G_{C}"] & & G(C*C) \\
M(U) \arrow[ur, "m(U)"] \arrow[rr, tail, "c^M_{C}(U)"] \arrow[rrr, tail, "u^M_{C}"'] & & E_{C}(U) \arrow[ul, "\hat{m}_{C}(U)"'] \arrow[r, tail, "u^*_{C}"] & M(C) \arrow[ur, "m'"] \arrow[rr, "f^M_{C}"] & & M(C*C) \arrow[ur, "m''"] & \\
& K(U) \arrow[uu, "q(U)"] \arrow[rrr, tail, "u^K_{C}"'] & & & K(C) \arrow[uu, "q'"'] \arrow[rr, "f^K_{C}"'] & & K(C*C) \arrow[uu, "q''"'] \\
N(U) \arrow[uu, "r(U)"] \arrow[ur, "n(U)"] \arrow[rr, tail, "c^N_{C}(U)"] \arrow[rrr, tail, "u^N_{C}"'] & & F_{C}(U) \arrow[uu, "r_{C}(U)"] \arrow[ul, "\hat{n}_{C}(U)"'] \arrow[r, tail, "u^*_{C}"] & N(C) \arrow[uu, "r'"] \arrow[ur, "n'"'] \arrow[rr, "f^N_{C}"] & & N(C*C) \arrow[uu, "r''"] \arrow[ur, "n''"'] &
\end{tikzcd}$$

where F is used instead of E in the construction of $\hat{N}$ and $r_{C}(U)$ is induced on equalizers (considering the similar diagram with g's instead of f's); the diagram commutes, since all faces already commute except perhaps two of the squares containing $r_{C}(U)$, and these commute since u^G_{C} and u^*_{C} are monomorphisms. In addition, since products and equalizers preserve pullbacks, we see that $m'r' = q'n'$, $m''r'' = q''n''$ are pullbacks, and then $q(U)\ \hat{n}_{C}(U) = \hat{m}_{C}(U)\ r_{C}(U)$ is a pullback, too.

Therefore $\operatorname{Im} \hat{n}_{C}(U) = q(U)^s \operatorname{Im} \hat{m}_{C}(U)$. Now we assume that $\hat{m}$ is an isomorphism, and apply II.4.1. Since $\hat{m}(U)$ is induced by the by the $\hat{m}_C(U)$, and is a regular epimorphism, we have $\bigvee_{C\in\mathfrak{C}(U)} \mathrm{Im}\ \hat{m}_C(U)$ $= 1$; by (C_3') it follows that $\bigvee_{C\in\mathfrak{C}(U)} \mathrm{Im}\ \hat{n}_C(U) = 1$, and the monomorphism $\hat{n}(U)$ induced by all $\hat{n}_C(U)$ is also a regular epimorphism, hence is an isomorphism, q.e.d.

With this lemma, it is easy to show that the pullback axiom holds in $\mathfrak{F}(X,G)$. Let $fg' = gf'$ be a pullback in $\mathfrak{F}$, where f is a regular epimorphism . Let (m,p) be a regular decomposition of f in $\mathfrak{P}$, and $mg_1' = gm'$, $pg_1'' = g_1'p'$ be pullbacks in $\mathfrak{P}$. Juxtaposition yields a pullback $(mp)g_1'' = g(m'p')$ which we may assume to be $fg' = gf'$; then $f' = m'p'$. Since $\mathfrak{P}$ is regular, (m',p') is a regular decomposition of f' in $\mathfrak{P}$. Now $\hat{m}$ is an isomorphism, by 3.6 ; by the lemma, $\hat{m}'$ is also an isomorphism, and then f' is a regular epimorphism in $\mathfrak{F}$. Thus we have proved that $\mathfrak{F} = \mathfrak{F}(X,G)$ is a regular category.

That it is a C_3 regular category is easily deduced from 3.7. First we have seen that it is cocomplete. Now let $\mathfrak{D}$ be a finite diagram in $[I,\mathfrak{F}]$, where I is a directed preordered set; we want to prove that $\varinjlim_{\mathfrak{F}} \lim_{\mathfrak{F}} \mathfrak{D} = \lim_{\mathfrak{F}} \varinjlim_{\mathfrak{F}} \mathfrak{D}$, where $\lim_{\mathfrak{F}}$ and $\varinjlim_{\mathfrak{F}}$ are limit and directed colimit functors, respectively, relative to $\mathfrak{F}$. We know that $\varinjlim_{\mathfrak{F}} = (\varinjlim_{\mathfrak{P}})\hat{}$, and that $\lim_{\mathfrak{F}}$ and $\lim_{\mathfrak{P}}$ are the same as long as we consider diagrams of sheaves only. Hence it follows from 3.7 that

$$\varinjlim_{\mathfrak{F}} \lim_{\mathfrak{F}} \mathfrak{D} = \varinjlim_{\mathfrak{F}} \lim_{\mathfrak{P}} \mathfrak{D} = (\varinjlim_{\mathfrak{P}} \lim_{\mathfrak{P}} \mathfrak{D})\hat{} = (\lim_{\mathfrak{P}} \varinjlim_{\mathfrak{P}} \mathfrak{D})\hat{} =$$
$$= \lim_{\mathfrak{F}} (\varinjlim_{\mathfrak{P}} \mathfrak{D})\hat{} = \lim_{\mathfrak{F}} \varinjlim_{\mathfrak{F}} \mathfrak{D} ,$$

since $\mathfrak{P}$ is a C_3 [in fact C_4] regular category. The proof of the theorem is now complete.

3. The reader will no doubt have noticed that the proofs in this section make very little use of the techniques developed in part I. Of course this is more apparent than real since we did use them to establish theorem II.4.4 on which these proofs hang. All the same this

leaves a possibility that regularity is not needed for these results, if we start from a category in which the conclusion of theorem II.4.4 (in the non-C_1^* cases) holds. In view of I.1.6, this simply means (if we still assume cocompleteness or a minimum thereof) that we wish to go on without the pullback axiom.

In this case, we exclude the proof that the pullback axiom holds in $\mathfrak{F}(X,\mathfrak{G})$; temporarily excluding the manipulation of subobjects at the end of the proof of 3.3, we see that all other proofs go through, if we assume that directed colimits are exact in $\mathfrak{G}$. Now what we temporarily excluded is just a manipulation of subobjects and therefore should not require the pullback axiom. Indeed for this we need only assume that the equalizer property II.4.3 holds in $\mathfrak{G}$ and that there are well-behaved images in $\mathfrak{G}$; for this we can assume that $\mathfrak{G}$ has regular decompositions (it suffices to assume that $\mathfrak{G}$ has coequalizers), or even that $\mathfrak{G}$ has strong decompositions (for then most of the results in section I.3 go through). In addition to this, and the property expressed by II.4.4 (in the non-C_1^* case), the assumptions on $\mathfrak{G}$ are: completeness, exact directed colimits. In the results, the contents of 3.4, 3.5, 3.7 are (except for the regularity of $\mathfrak{F}(X,\mathfrak{G})$) all saved. In other words, one still needs decompositions, but not the pullback axiom nor full cocompleteness.

The same remarks apply to the results of the next section, even though we formulate them for regular categories; however, Gray's condition $\mathfrak{F}_2$ is no longer a consequence of II.4.2 and must be added to the irregular hypotheses on $\mathfrak{G}$ [there is no need to add $\mathfrak{F}_1$ as it can be seen that the conclusion of II.4.4 is stronger]; one also needs the full strength of II.4.4 (= including the C_1^* case).

4. STALK PROPERTIES.

1. We now assume that X is a topological space (not an arbitrary Grothendieck topology). We still let $\mathcal{G}$ be a C_4 regular category.

For each presheaf $P \in \mathcal{P}(X,\mathcal{G})$, the stalk P_x of P at $x \in X$ is $\varinjlim_{U \ni x} P(U)$. These can be used to define a stalk functor S of $\mathcal{P}(X,\mathcal{G})$ into the product category $\mathcal{G}^X$ (i.e. the functor category $[X,\mathcal{G}]$ in which X now denotes the obvious discrete category); namely, S sends P onto $(P_x)_{x\in X}$, and similarly for morphisms. Since $\mathcal{G}$ is, in particular, C_3 , the stalk functor is exact.

We now observe that $\mathcal{G}$ satisfies all the axioms for an $\mathfrak{I}$-category as defined in [31] (see also [10],[11]) except for being locally small: in particular $\mathfrak{I}_1$ is part of the hypothesis and $\mathfrak{I}_2$ follows from II.4.2. Thus a great number, in fact most, of the results in [31] (in the non-abelian case) hold in our situation (the major exception being the existence of the associated sheaf, which we obtained previously). Specifically, we need to know that P and its associated sheaf have the same stalks (i.e. there is a natural isomorphism $S(P) \cong S(\hat{P})$); also, for each presheaf P , the presheaf $\overline{P}$ defined by: $\overline{P}(U) = \prod_{x\in U} P_x$, whose restriction maps are projections, is a sheaf, and a morphism of presheaves is defined by $m_P(U) = \times_{x\in U} P_{U,x} : P(U) \longrightarrow \overline{P}(U)$, and is a monomorphism if P is a monopresheaf. Then S is still exact on $\mathfrak{I}(X,\mathcal{G})$.

2. More can be proved if furthermore $\mathcal{G}$ is C_1^* . The basic result is the lemma which follows. We are indebted to VanOsdol for the remark that it shows $\mathfrak{I}(X,\mathcal{G})$ is cotripleable under $\mathcal{G}^X$ (more precisely, S is cotripleable).

<u>Lemma</u> 4.1. If $\mathcal{G}$ is C_1^* , then S reflects isomorphisms.

Proof. We have to show that if $f : F \longrightarrow G$ is a morphism of sheaves and $f_x : F_x \longrightarrow G_x$ is an isomorphism for all $x \in X$, then f is an isomorphism. From f we obtain a commutative diagram

$$\begin{tikzcd} \overline{F} \arrow[r, "\overline{f}"] & \overline{G} \\ F \arrow[u, tail, "m_F"] \arrow[r, "f"'] & G \arrow[u, tail, "m_G"'] \end{tikzcd} \tag{20}$$

(where $\overline{f}(U) = \prod_{x\in U} f_x$), where m_F, m_G are [pointwise] monomorphisms. On this diagram it is clear that, if $\overline{f}$ is a monomorphism (e.g. if $S(f)$ is an isomorphism) then f is also a monomorphism. Then the lemma follows at once from the more general fact that, when f is a monomorphism, (20) is a pullback. We now prove this property.

First G is C_1^* and so it follows from II.4.4 that an isomorphism

$$t_F(U) : \varinjlim_{\tau\in T} \prod_{x\in U} F(\tau x) \longrightarrow \prod_{x\in U} \varinjlim_{x\in V\subseteq U} F(V) = \overline{F}(U)$$

is induced by all $\qquad p_\tau^F = \prod_{x\in U} F_{\tau x, x} : \prod_{x\in U} F(\tau x) \longrightarrow \prod_{x\in U} F_x = \overline{F}(U)$

$(\tau \in T)$; there $T = \prod_{x\in U} \{ V ;\ x \in V \subseteq U,\ V \text{ open}\}$, in other words T is the set of all mappings τ which to every $x \in U$ assign an open neighborhood $\tau x \subseteq U$ of x. From here on, we identify each $\tau \in T$ and the corresponding open covering $(\tau x)_{x\in U}$ of U. A similar description of $\overline{G}(U)$ can be given, in terms of morphisms p_τ^G (with the same T).

For each $\tau \in T$ we have a commutative diagram

$$\begin{tikzcd} F(U) \arrow[r, tail, "u_\tau^F"] \arrow[d, "f(U)"'] & F(\tau) \arrow[r, "f_\tau^F"] \arrow[d, "f'_\tau(U)"] & F(\tau * \tau) \arrow[d, "f''_\tau(U)"] \\ G(U) \arrow[r, tail, "u_\tau^G"'] & G(\tau) \arrow[r, "f_\tau^G"'] & G(\tau * \tau) \end{tikzcd} \tag{21}$$

(where $f'_\tau(U) = \prod_{x\in U} f(\tau x)$ etc.); we now assume that f is a monomorphism, so that we have monomorphisms in the diagram as indicated. There is a similar diagram, with f^F_τ , f^G_τ replaced by g^F_τ , g^G_τ . We now claim that the left square in this diagram is a pullback. Assume indeed that $f'a = u^G b$. Then $f''f^F a = f^G f' a = f^G u^G b = g^G u^G b = \ldots = f'' g^F a$; since F is a sheaf, and f'' is a monomorphism, it follows that $a = u^F c$ for some c ; then also $u^G b = f'a = f'u^F c = u^G f(U)c$ and $b = f(U)c$. The factorization is unique since, say, u^G is a monomorphism.

Taking directed colimits over T yields a pullback to which we attach the isomorphisms $t_F(U)$, $t_G(U)$ to obtain the diagram below:

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d, "f(U)"'] \arrow[dr, phantom, "\mathrm{PB}"] & \underrightarrow{\lim}\, F(\tau) \arrow[r, "t_F(U)", "\cong"'] \arrow[d] & \overline{F}(U) \arrow[d, "\overline{f}(U)"] \\ G(U) \arrow[r] & \underrightarrow{\lim}\, G(\tau) \arrow[r, "t_G(U)", "\cong"'] & \overline{G}(U) \end{tikzcd} \tag{22}$$

The top row yields a morphism $F(U) \longrightarrow \overline{F}(U)$ which is the colimit of $p^F_\tau\, u^F_\tau$. Now

$$p^F_\tau\, u^F_\tau = \Big(\prod_{x\in U} F_{\tau x,x}\Big)\Big(\bigtimes_{x\in U} F_{U,\tau x}\Big) = \bigtimes_{x\in U} F_{U,x} = m_F(U) \ .$$

Hence the composite morphism in the top row is just $m_F(U)$. The bottom row similarly yields $m_G(U)$. Hence if we forget the middle column in (22), the resulting pullback is but (20) evaluated at U. It follows that (20) is a pullback, which completes the proof of the lemma.

The obvious application of the lemma is as follows. Let $\mathfrak{D}$ be a diagram of sheaves and $(a_i)_{i\in I}$ be a cocompatible family for $\mathfrak{D}$ (in $\mathfrak{F}(X,\mathfrak{A})$). Assume that $((a_i)_x)_{i\in I}$ is a colimit of $\mathfrak{D}_x$ (in $\mathfrak{A}$) for every $x \in X$. If a is the morphism induced to the colimit by the cocompatible family $(a_i)_{i\in I}$ then since S preserves colimits a_x is an isomorphism for every x ; hence a is an isomorphism, and $(a_i)_{i\in I}$ is in fact a colimit of $\mathfrak{D}$. This is expressed by saying that "colimits can safely be computed on the stalks" [we borrowed the expression from Van Osdol]. The same argument applies to anything that is preserved by S, which includes finite limits, and regular decompositions. Thus:

<u>Theorem</u> 4.2. Let $\mathcal{G}$ be a C_4, C_1^* regular category. For any topological space X, all colimits, finite limits and regular decompositions in $\mathfrak{F}(X,\mathcal{G})$ can safely be computed on the stalks.

REFERENCES

[1] BARR, M.: Relational algebras. Reports of the Midwest Category Seminar IV, 39-55. Springer Lecture Notes 137 (1970).

[2] BARR, M.: Factorizations, generators and rank. (Preprint)

[3] BARR, M.: Non-abelian full embedding, I. (Preprint)

[4] BARR, M.: Non-abelian full embedding, II. (Preprint) (For the last two references, see also the announcement in the Reports of the Midwest Category Seminar V, 205-208.)

[5] BENABOU, J.: Introduction to bicategories. Reports of the Midwest Category Seminar I, 1-77. Springer Lecture Notes 47 (1967).

[6] BUCUR and DELEANU: Introduction to the theory of categories and functors. John Wiley and Sons, 1968.

[7] COHN, P.M.: Universal Algebra. Harper and Row, 1965.

[8] ECKMANN, B. [editor]: Seminar on triples and categorical Homology theory. Springer Lecture Notes 80 (1969).

[9] FOLKS, The: Folk theorems.(Unpublished)(we hope)

[10] GRAY, J.S.: Sheaves with values in a category. Notes, Columbia University, 1962.

[11] GRAY, J.S.: Sheaves with values in a category. Topology 3 (1965) 1-18.

[12] GRAY, J.S.: Review of [16], MR 26 (1963) #1887.

[13] GRILLET, P.A.: Morphismes spéciaux et décompositions. C. R. Acad. Sci. Paris 266 (1968) sér.A, 397-398; Quelques propriétés des catégories non-abéliennes, ibid. 550-552; La suite exacte d'homologie dans une catégorie non-abélienne, ibid. 604-606.

[14] GRILLET, P.A.: Directed colimits and sheaves in some non-abelian categories. Reports of the Midwest Category Seminar V, 36-69. Springer Lecture Notes 195 (1971).

[15] GROTHENDIECK, A.: Sur quelques points d'Algèbre homologique. Tohoku Math. J. 9 (1957) 119-221.

[16] HELLER, A. and ROWE, K.A.: On the category of sheaves. Amer. J. Math. 84 (1962) 205-216.

[17] HERRLICH, H.: Topologische reflexionen und Coreflexionen. Springer Lecture Notes 78 (1968).

[18] HILTON, P.: Catégories non-abéliennes. Notes, Université de Montréal (1964).

[19] ISBELL, J.R.: Some remarks concerning categories and subspaces. Canad. J. Math. 9 (1957) 563-577.

[20] ISBELL, J.R.: Subobjects, adequacy, completeness and categories of algebras. Rozprawy Mat. 36 (1964).

[21] ISBELL, J.R.: Structure of categories. Bull. Amer. Math. Soc. 72 (1966) 619-655.

[22] ISBELL, J.R. and HOWIE, J.M.: Epimorphisms and dominions,II. J. Algebra 6 (1967) 7-21.

[23] KELLY, G.M.: Monomorphisms, epimorphisms and pullbacks. J. Austral. Math. Soc. 9 (1969) 124-142.

[24] KENNISON, J.F.: Full reflective subcategories and generalized coverings. Ill. J. Math. 12 (1968) 353-365.

[25] LAWVERE, F.V.: Functorial semantics of algebraic theories. Doct. Diss., Columbia University (1963).

[26] MAC LANE, S.: Groups, categories and duality. Proc. Nat. Acad. Sci. U.S.A. 34 (1948) 263-267.

[27] MAC LANE, S.: Duality for groups. Bull. Amer. Math. Soc. 56 (1950) 485-516.

[28] MAC LANE, S.: An algebra of additive relations. Proc. Nat. Acad. Sci. U.S.A. 47 (1961) 1043-1051.

[29] MAC LANE, S.: Homology. Springer, 1963.

[30] MANES, E.G.: A triple miscellany: some aspects of the theory of algebras over a triple. Doct. Diss., Wesleyan University (1967)

[31] MITCHELL, B.: Theory of categories. Academic Press, 1965.

[32] PIERCE, R.S.: Introduction to the theory of abstract algebras. Holt, Rinehart and Winston, 1968.

[33] PUPPE, D.: Korrespondenzen in Abelschen Kategorien. Math. Ann. 148 (1962) 1-30.

[34] SEMADENI, Z.: Projectivity, injectivity and duality. Rozprawy Mat. 35 (1963).

[35] VAN OSDOL, D.H.: Sheaves of algebras (to appear).