COEQUALIZERS IN CATEGORIES OF ALGEBRAS

by

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Introduction

It is well known [2,§6] that (inverse) limits in a category of algebras over \mathcal{A} - in particular, in the category $\mathcal{A}^{\mathbf{T}}$ of algebras over a triple $\mathbf{T} = (\mathbf{T}, \eta, \mu)$ on \mathcal{A} - can be calculated in \mathcal{A} . Despite the fact that such a statement is, in general, false for colimits (direct limits), a number of colimit constructions can be carried out in $\mathcal{A}^{\mathbf{T}}$ provided they can be carried out in \mathcal{A} and $\mathcal{A}^{\mathbf{T}}$ has enough coequalizers.

The coequalizers $\mathcal{A}^{\mathbf{T}}$ should have, at a minimum, are, as we shall see in §1, those of <u>reflexive</u> pairs: a pair

$$x \xrightarrow{f} y$$

of maps f, g in a category χ is <u>reflexive</u> if there is an χ -morphism

$$\Delta : Y \longrightarrow X$$

satisfying the identities

$$f \circ \Delta = id_{Y} = g \circ \Delta$$
.

(This terminology arises from the fact that, when $\chi = \sum = \{\text{sets}\}\$, (f,g) is reflexive if and only if the image of the induced function

$$x \xrightarrow{f,g} Y \times Y$$

contains the diagonal of Y x Y.)

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In §2 we give two criteria for $\mathcal{A}^{\mathbf{T}}$ to have coequalizers of reflexive pairs, neither of them necessary, of course. In §1, it will turn out, so long as $\mathcal{A}^{\mathbf{T}}$ has such coequalizers, that each functor

$$\mathcal{A}^{\mathsf{T}} : \mathcal{A}^{\mathsf{T}} \longrightarrow \mathcal{A}^{\mathsf{S}}$$
,

induced by a map of triples $\tau: S \to T$, has a left adjoint, that \mathcal{A}^T has coproducts if \mathcal{A} does, indeed, has all small colimits if \mathcal{A} has coproducts, and that \mathcal{A}^T has tensor products if \mathcal{A} does. These are, of course, known facts when $\mathcal{A} = \mathbb{S} = \{\text{sets}\}$; however, at the time of this writing, it is unknown, for example, whether the category of contramodules over an associative coalgebra, presented (in [1]) as \mathcal{A}^T with $\mathcal{A} = \{\text{ab. groups}\}$, has coequalizers of reflexive pairs.

§ 1. Constructions using coequalizers of reflexive pairs.

We begin with a lemma that will have repeated use. It concerns the following definition, which clarifies what would otherwise be a recurrent conceptual obscurity in the proofs of this section.

Let $U: X \to \mathcal{H}$ be a functor, let $X \in X$, and let $(f,g) = \{(f_i,g_i)/i \in I\}$ be a family of \mathcal{A} -morphisms

$$A_{i} \xrightarrow{f_{i}} UX \quad (i \in I). \tag{1.1}$$

An X-morphism p: X \rightarrow P is a coequalizer (rel. U) of the family of pairs (1.1) if

- 1) $\forall i \in I$, $Up \circ f_i = Up \circ g_i$, and
- 2) if $q: X \rightarrow Y$ satisfies $Uq \circ f_i = Uq \circ g_i$ ($\forall i \in I$),

then $\exists ! x : P \rightarrow Y \text{ with } q = x \cdot p$.

If $U = id_{\chi} : \chi \to \chi$, a coequalizer (rel. U) of the family (1.1) will be called simply a <u>coequalizer</u> of (1.1).

Lemma 1. If U has a left adjoint $F: \mathcal{A} \to X$ and \bar{f}_i , $\bar{g}_i: FA_i \to X$ are the X-morphisms corresponding to f_i , g_i by adjointness, then $p: X \to P$ is a coequalizer (rel. U) of (f,g) if and only if it is a coequalizer of (\bar{f},\bar{g}) . If U is faithful

and $\bar{f}_i, \bar{g}_i : X_i \to X$ are $\chi \to M$ -morphisms with $U\bar{f}_i = f_i$, $U\bar{g}_i = g_i$, then $p : X \to P$ is a coequalizer (rel U) of (f,g) if and only if it is a coequalizer of (\bar{f}, \bar{g}).

Proof. In the first case, the naturality of the adjunction isomorphisms yields

$$q \cdot \bar{f}_i = q \cdot \bar{g}_i \longleftrightarrow Uq \cdot f_i = Uq \cdot g_i$$

for every X -morphism q defined on X. In the second case, that relation follows from the faithfulness of U. Clearly, that relation is all the proof required.

Proposition 1. Let $S = (S, \eta^{\dagger}, \mu^{\dagger})$ and $T = (T, \eta, \mu)$ be triples on \mathcal{A} , suppose the natural transformation $\tau : S \to T$ is a map of triples from S to T, and let (A, α) be an S-algebra, (B, β) a T-algebra, $p : TA \to B$ a T-homomorphism from (TA, μ_A) to (B, β) , and $\iota = p \circ \eta_A : A \to B$. Then the following statements are equivalent.

1) p is a coequalizer of the pair

$$(TSA, \mu_{SA}) \xrightarrow{T(\tau_{A})} (TTA, \mu_{TA}) \xrightarrow{\mu_{A}} (TA, \mu_{A}) ;$$

$$T(\alpha)$$

2) p <u>is a coequalizer</u> (rel U^T) of the pair

$$A \longrightarrow A \longrightarrow A \longrightarrow A$$

3) ι is an s-homomorphism $(A,\alpha) \rightarrow (B,\beta \cdot \tau_B)$ making the composition

$$\mathcal{A}^{\mathbf{T}}((B,\beta),X)$$

$$\mathcal{A}^{\mathbf{S}}(\mathcal{A}^{\mathsf{T}}(B,\beta),\mathcal{A}^{\mathsf{T}}X)$$

$$\downarrow =$$

$$\mathcal{A}^{\mathbf{S}}((B,\beta\cdot\tau_{B}),\mathcal{A}^{\mathsf{T}}X)$$

$$\downarrow \iota$$

$$\mathcal{A}^{\mathbf{S}}((A,\alpha),\mathcal{A}^{\mathsf{T}}X)$$

a one-one correspondence, $\forall x \in \mathcal{A}^T$!.

<u>Proof.</u> The equivalence of statements 1) and 2) follows from Lemma 1, since $\mu_A \cdot T(\tau_A) \text{ is the } \mathcal{A}^T\text{-morphism corresponding to } \tau_A \text{ by adjointness and } \eta_A \cdot \alpha = T(\alpha) \cdot \eta_{SA} \text{ is the } \mathcal{A}\text{-morphism corresponding to } T(\alpha) \text{ by adjointness.}$

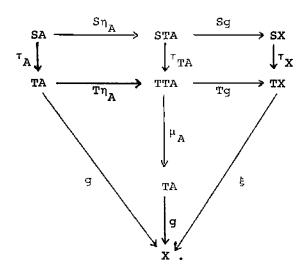
Next, if $g:TA\to X$ is a **T**-homomorphism from (TA,μ_A) to a **T**-algebra (X,ξ) , having equal compositions with τ_A and $\eta_A\cdot\alpha$, we show that $g\cdot\eta_A:A\to X$ is an **S**-morphism from (A,α) to $\mathcal{A}^T(X,\xi)=(X,\xi\cdot\tau_X)$, i.e., that

$$g \cdot \eta_A \cdot \alpha = \xi \cdot \tau_{_{\mathbf{Y}}} \cdot S(g \cdot \eta_A)$$
.

Clearly this requires only the proof of

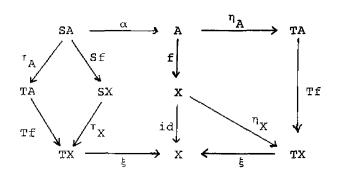
g .
$$\tau_A = \xi$$
 . τ_X . Sg . $S\eta_A$,

for which, consider the diagram



The upper squares commute because τ is natural, the left hand triangle, because μ_A · $T\eta_A = id_{TA}$, the right hand triangle, because g is a T-homomorphism.

Finally, given an \$\$-homomorphism $f: A \to X$ from (A,α) to $\mathcal{A}^\top(X,\xi) = (X,\xi \cdot \tau_X)$, it turns out that ξ . If is a \$\$T\$-homomorphism $(TA,\mu_A) \to (X,\xi)$ having equal compositions with τ_A and $\eta_A \cdot \alpha$. For, the diagram

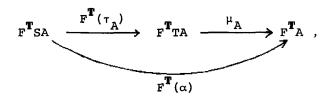


commutes, since τ is natural, f is an \$-homomorphism, η is natural, and $\xi\,\,\cdot\,\,\eta_{_{\bf X}}\,=\,id_{_{\bf X}}\,\,.$

These arguments form the core of a proof of Proposition 1.

Corollary 1. If $\mathcal{A}^{\mathbf{T}}$ has coequalizers of reflexive pairs, then each functor $\mathcal{A}^{\mathsf{T}}:\mathcal{A}^{\mathbf{T}}\to\mathcal{A}^{\mathbf{S}}$, induced by a triple map $\tau:\mathbf{S}\to\mathbf{T}$, has a left adjoint $\hat{\tau}$.

Proof. For each (A,α) $\in \mathcal{A}^{\mathbf{S}}$, the pair

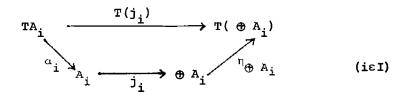


whose coequalizer, if any, is (by Proposition 1) the value $\hat{\tau}(A,\alpha)$ of $\hat{\tau}$ at (A,α) , is reflexive by virtue of

$$\Delta = \mathbf{F}^{\mathbf{T}}(\eta^{\dagger}_{\mathbf{A}}) .$$

Proposition 2. Let (A_i, α_i) (i ϵ I) be a family of T-algebras, and assume the coproduct $\bigoplus_{i \in I} A_i$ exists in \mathcal{A} , say with injections $j_i : A_i \to \bigoplus_i A_i$. Let $p : T(\bigoplus_i A_i) \to P$ be a T-homomorphism. Then the following statements are equivalent.

1) p is a coequalizer (rel UT) of the family of pairs



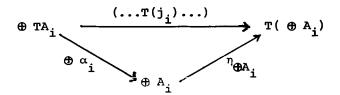
2) each map h_i = p · η_{⊕A_i} · j_i : A_i → P

is a T-homomorphism and the family (h_i)_{i∈I} serves to make P the coproduct in A^T

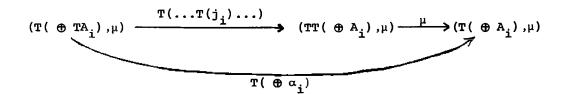
of (A_i)_{i∈I} ·

Moreover, if \oplus TA is available in $\mathcal A$, statements 1) and 2) are equivalent to each of the following statements about p:

3) p <u>is a coequalizer</u> (rel U^T) of the pair

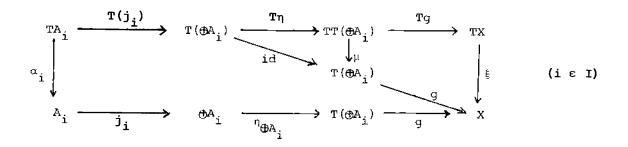


4) p is a coequalizer of the pair



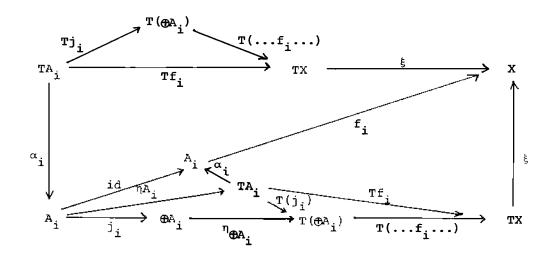
<u>Proof.</u> The equivalence of statements 1) and 3) is obvious. The equivalence of
3) with 4) is due to Lemma 1, since the top (bottom) maps correspond to each other by
adjointness.

Next, let $g: T(\oplus A_i) \to X$ be a **T**-homomorphism from $F^T(\oplus A_i)$ to (X,ξ) , having equal compositions with both components of all the pairs in 1). Then $g \cdot \eta_{\bigoplus A_i} \cdot j_i : A_i \to X \text{ is a T-homomorphism } (A_i,\alpha_i) \to (X,\xi), \text{ for all i, as is shown}$ by the commutativity of the diagrams



Finally, if $f_i:A_i\to X$ is a family of **T**-homomorphisms $(A_i,\alpha_i)\to (X,\xi)$, then the map $g=\xi + T(\dots f_i\dots):T(\oplus A_i)\to TX\xrightarrow{\xi} X$

is a **T**-homomorphism (the only one) having $g \cdot \eta_{\bigoplus A_i} \cdot j_i = f_i$, and, as the diagram below shows, has equal compositions with both members of all the pairs in 1).



This essentially concludes the proof of Proposition 2.

Corollary 2. If \mathcal{A}^{T} has coequalizers of reflexive pairs, and if \mathcal{A} has all small coproducts, then \mathcal{A}^{T} has all small colimits (direct limits).

<u>Proof.</u> In the first place, \mathcal{A}^T has small coproducts, because, given (A_i,α_i) \in $|\mathcal{A}^T|$ (i \in I), the pair

$$\mathbf{F}^{\mathbf{T}}(\oplus \mathbf{T} \mathbf{A}_{\underline{\mathbf{1}}}) \xrightarrow{\mathbf{F}^{\mathbf{T}}(\mathbf{T}(\mathbf{A}_{\underline{\mathbf{1}}}) \dots)} \mathbf{F}^{\mathbf{T}}(\mathbf{T}(\mathbf{A}_{\underline{\mathbf{1}}}) \xrightarrow{\mu} \mathbf{F}^{\mathbf{T}}(\mathbf{T}(\mathbf{A}_{\underline{\mathbf{1}}}) \xrightarrow{\mu} \mathbf{F}^{\mathbf{T}}(\mathbf{T}(\mathbf{A}_{\underline{\mathbf{1}}}) \xrightarrow{\mu} \mathbf{F}^{\mathbf{T}}(\mathbf{T}(\mathbf{A}_{\underline{\mathbf{1}}}))$$

whose coequalizer, according to Proposition 2, serves as coproduct in $\mathcal{A}^{\mathbf{T}}$ of the family $\{(\mathbf{A_i},\alpha_i) \ / \ i \in \mathbf{I}\}$, is reflexive by virtue of

$$\Delta = \mathbf{F}^{\mathbf{T}} (\oplus \eta_{\mathbf{A}_{\mathbf{i}}})$$
.

But then, having coproducts and coequalizers of reflexive pairs, $\mathcal{A}^{\, T}$ has all small colimits. Indeed, the pair

$$\underbrace{\bigoplus_{\delta \in \mathcal{A}} \mathcal{I}_{\mid}}^{D_{\text{dom } \delta}} \xrightarrow{(\dots j_{\text{cod } \delta} \dots D_{\delta} \dots D_{\delta} \dots D_{\delta} \dots D_{\delta} \dots D_{\delta}}_{(\dots j_{\text{dom } \delta} \dots D_{\delta} \dots D_$$

whose coequalizer is well known to serve as colimit of the functor $D:\mathcal{Q}\longrightarrow ?$, is reflexive by virtue of

$$\Delta = (\dots j_{id_i} \dots)_{i \in I} \mathcal{A}_I$$

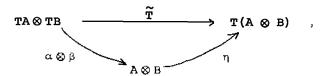
Remark. If \mathcal{H} is a monoidal category [0] and $\mathbf{T}=(\mathbf{T},\eta,\mu)$ is a suitable triple (meaning at least that $\mathbf{T}:\mathcal{H}\to\mathcal{H}$ is a monoidal functor [0], so that there are maps $\widetilde{\mathbf{T}}:\mathbf{T}\mathbf{A}\otimes\mathbf{T}\mathbf{B}\to\mathbf{T}(\mathbf{A}\otimes\mathbf{B})$ subject to conditions, and η is a monoidal natural transformation, as should probably be μ), then, given \mathbf{T} -algebras (\mathbf{A},α) , (\mathbf{B},β) , a coequalizer (rel $\mathbf{U}^{\mathbf{T}}$) of the pair

$$T(TA \otimes TB) \xrightarrow{T(\widetilde{T})} TT(A \otimes B) \xrightarrow{\mu} T(A \otimes B) ,$$

which is reflexive by virtue of

$$\Delta = \mathbf{T}(\eta_{A} \otimes \eta_{B}) ,$$

serves equally well as a coequalizer (rel ${ t U}^{f T}$) of the pair



and, if \mathcal{A} is closed monoidal [0], can be interpreted (in terms of "bilinear maps") as a tensor product, in $\mathcal{A}^{\mathbf{T}}$, of (A,α) and (B,β) . Such phenomena hope to be treated in detail elsewhere.

§2. Criteria for the existence of such coequalizers.

In view of §1, it behooves us to find workable sufficient conditions, on $\mathcal A$, on $\mathbf T$, or on both, that $\mathcal A^{\mathbf T}$ have coequalizers of reflexive pairs. The first such condition, though rather special, depends on knowing when coequalizers in $\mathcal A^{\mathbf T}$ can be calculated in $\mathcal A$.

Proposition 3. Let $T = (T, \eta, \mu)$ be a triple in \mathcal{A} , and let

$$(A,\alpha) \xrightarrow{f} (B,\beta)$$

be a pair of AT-morphisms. Assume

- 1) there is an A-morphism $p : B \to C$ which is a coequalizer (in A) of $A \xrightarrow{f} B$;
 - 2) Tp is a coequalizer of (Tf,Tg);
 - 3) TTp is epic.

Then: there is a map γ: TC → C, uniquely determined by the single requirement that

$$\begin{array}{ccc}
\mathsf{TB} & \xrightarrow{\mathsf{Tp}} & \mathsf{TC} \\
\downarrow^{\beta} & & \downarrow^{\gamma} \\
\mathsf{B} & \xrightarrow{\mathsf{P}} & \mathsf{C}
\end{array}$$

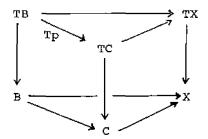
commute; (C,γ) is a T-algebra; and $p:(B,\beta)\to(C,\gamma)$ is a coequalizer in \mathcal{A}^T of (f,g).

<u>Proof.</u> The equations $p \circ \beta \circ Tf = p \circ f \circ \alpha = p \circ g \circ \alpha = p \circ \beta \circ Tg$, occurring because of assumption 1, force, because of assumption 2, a unique $\gamma : TC \to C$ with $\gamma Tp = p\beta$.

Then $\gamma \circ \eta_C \circ p = \gamma \circ Tp \circ \eta_B = p \circ \beta \circ \eta_B = p$, but p is epic (by 1) and so $\gamma \circ \eta_C = id_C$. Similarly, using assumption 3, the equation $\gamma \circ T\gamma = \gamma \circ \mu_C$ follows from the calculation

$$\gamma$$
 • T γ • TT p = γ • T p • T β = p • β • T β = p • p •

Finally, if $q: B \to X$ is an \mathcal{A}^T -morphism from (B,β) to (X,ξ) factoring through C, the factorization must be an \mathcal{A}^T -morphism, because the diagram



commutes everywhere else, and Tp is epic, by assumption 2.

Corollary 3. If \mathcal{A} has and T preserves coequalizers of reflexive pairs, then \mathcal{A}^{T} has coequalizers of reflexive pairs, and Corollaries 1 and 2 apply.

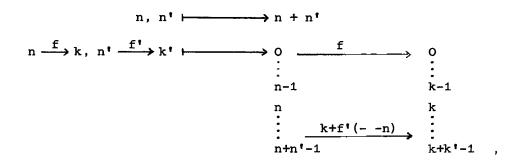
<u>Examples</u>. If **T** is an adjoint triple, then T preserves all coequalizers (and all colimits, in fact). Samples of such triples:

- a) $\mathbf{T} = (- \otimes \Lambda \text{ , id } \otimes \text{ u , id } \otimes \text{ m})$, where the ground category \mathcal{A} is $\{k\text{-modules}\}$ (k comm.) and Λ is an associative k-algebra with unit $u: k \to \Lambda$ and multiplication $m: \Lambda \otimes \Lambda \to \Lambda (\otimes = \otimes_k)$. $\mathcal{A}^{\mathbf{T}} = \Lambda\text{-modules}$.
- b) $T = (- \times 2, id \times (1 2), id \times (2 \times 2 2))$, where the ground category A is Cat and A is the p.o. set 0 + 1. Cat A = {categories with idempotent triples}. Where A is constructed like A is replacing A by the category A given by

$$|\Delta| = \{0,1,2,\ldots,n,\ldots\}$$

$$\Delta(n,k) = \text{order preserving maps } \{0\ldots n-1\} \rightarrow \{0\ldots k-1\} ,$$

with the obvious composition, $0: \mathbf{1} \to \Delta$ the inclusion of the object 0, $m: \Delta \times \Delta \to \Delta$ the functor given by



Cat so is {categories equipped with a triple}. Define $\tau: S \to T$ by crossing with the only functor $\Delta \to 2$ sending $n \neq 0$ to 1, and 0 to 0. Then Cat $\tau: Cat^T \to Cat^S$ is the functor interpreting an idempotent triple as a triple on the same category. These constructions and observations are all due to Lawvere. Since Cat has coequalizers and T is an adjoint triple, Cat τ has a left adjoint, by Corollary 1; roughly speaking, it assigns to a triple in a category, a best idempotent triple on an as closely related other category as possible.

- 2. Let \mathcal{A} be an additive category, let $m: G \times G \to G$ be an \mathcal{A} -morphism satisfying $m(m\times G) = m(G\times m)$, m(id,0) = id = m(0,id). Define a triple $\mathbf{T} = (-\times G, (id,0), (id\times m))$ on \mathcal{A} . Then \mathbf{T} preserves all coequalizers because $\mathbf{A} \times \mathbf{G} = \mathbf{A} \oplus \mathbf{G}$.
- 3. Any functor preserves split coequalizer systems [3]. In particular every triple does, and so Proposition 3 guarantees that coequalizers of $\mathbf{U}^{\mathbf{T}}$ -split pairs of $\mathcal{A}^{\mathbf{T}}$ -morphisms can be computed in \mathcal{A} , as was stated in greater generality in [2,\$6].

The other criterion involves images. We treat images axiomatically, in a manner suggestive of (and perhaps equivalent to) bicategories. Recall that 1, 2 and 3 are the categories depicted as the partially ordered sets

$$\begin{array}{rcl}
1 & = & \{0\} & , \\
2 & = & \{0 & +1\} & , \\
3 & = & \left\{0 & \xrightarrow{c=b \cdot a} 2\right\} & .
\end{array}$$

We will need the functors $\mathcal{Q} \xrightarrow{c} \mathcal{J}$ and $\mathcal{A} \xrightarrow{1} \mathcal{J}$ (whose values serve as their names). These induce functors $\mathcal{A}^c: \mathcal{A}^{\mathfrak{F}} \longrightarrow \mathcal{A}^{\mathfrak{F}}$ and $\mathcal{A}^1: \mathcal{A}^{\mathfrak{F}} \longrightarrow \mathcal{A}^{\mathfrak{F}} \cong \mathcal{A}$, for any category \mathcal{A} .

By an $\underline{\mathsf{image}}$ $\underline{\mathsf{factorization}}$ $\underline{\mathsf{functor}}$ for the category $\mathcal A$, we mean a functor

$$\mathcal{A}: \mathcal{A}^2 \longrightarrow \mathcal{A}^3$$

having the property

1) $\mathcal{A}^2 \xrightarrow{\mathcal{A}} \mathcal{A}^3 \xrightarrow{\mathcal{A}^c} \mathcal{A}^2 = \text{identity on } \mathcal{A}^2$, and three more properties which we state using the notations

$$\mathcal{A}^{1}(\mathcal{L}(f)) = I_{f}$$

$$\mathcal{L}_{f} = \cdot \xrightarrow{f_{a}} I_{f} \xrightarrow{f_{b}} \cdot :$$

2)
$$f \in \mathcal{A}^{2} | \Longrightarrow f_{a} \text{ is an epimorphism,}$$

$$f \in \mathcal{A}^2 \mid \Longrightarrow f_b \text{ is a monomorphism,}$$

4)
$$f \in \mathcal{A}^2 \longrightarrow (f_b)_a \text{ and } (f_a)_b \text{ are isomorphisms.}$$

A functor T preserves ℓ -images if there is a natural equivalence, whose composition with \mathcal{A}^{C} is the identity, between T³ \cdot ℓ and ℓ \cdot This entails, for each f ε $\mathcal{A}(A,B)$, an isomorphism $\iota_{f}: T(I_{f}) \to I_{Tf}$ making the triangle

commute.

A triple $T = (T, \eta, \mu)$ on \mathcal{A} preserves \mathcal{A} -images if the functor T does.

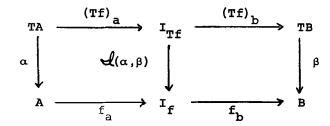
Lemma 2. If $\mathcal{A}:\mathcal{A}^3$ is an image factorization functor for \mathcal{A} and \mathbf{T} is a triple that preserves \mathcal{U} -images, then there is one and only one image factorization functor $\mathcal{A}^{\mathbf{T}}$ for $\mathcal{A}^{\mathbf{T}}$ with the property

$$(\mathbf{U}^{\mathbf{T}})^{\mathbf{3}} \cdot \mathbf{\mathcal{L}}^{\mathbf{T}} = \mathbf{\mathcal{L}}$$
.

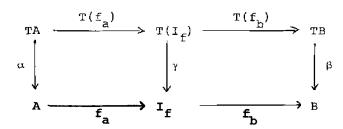
<u>Proof.</u> Given $f: A \to B$, an $\mathcal{H}^{\mathbf{T}}$ -morphism from (A,α) to (B,β) , the commutativity of the square

$$\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\alpha \downarrow & & \downarrow \beta \\
A & \xrightarrow{f} & B
\end{array}$$

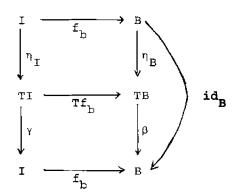
yields a commutative diagram



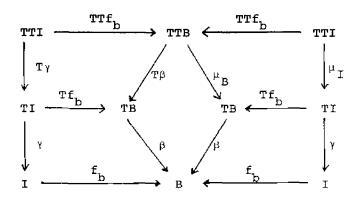
Combining this with the commutative diagram arising from the definition of "T preserves \mathcal{L} -images", we obtain a map $\gamma = \mathcal{L}(\alpha,\beta) \circ \iota_f : T(I_f) \to I_f$ making the diagram



commute. There is only one such map γ because $T(f_a)$ is epic and f_b is monic. We show now that (I_f, γ) is a **T**-algebra. Write simply $I = I_f$. $\gamma \circ \eta_I = id_I$ follows from the commutativity of



and the fact that f_b is monic. $\gamma \circ T\gamma = \gamma \circ \mu_I$ follows from the commutativity of



and the fact that f_b is monic. Since $(U^T)^{3} \circ \mathcal{L}^T = \mathcal{L}$, the axioms \mathcal{L}^T must satisfy, to be an image factorization functor, are easily verified. The uniqueness assertion is taken care of, essentially, by the obvious uniqueness of $\gamma: TI \to I$, subject to the commutativity relations expressed in the diagram

$$\begin{array}{ccccc}
\mathsf{TA} & \longrightarrow & \mathsf{TI} & \longrightarrow & \mathsf{TB} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{A} & \longrightarrow & \mathsf{I} & \longrightarrow & \mathsf{B}
\end{array}$$

Proposition 4. Let \mathscr{L} be an image factorization functor for \mathscr{A} , and let T be a triple on \mathscr{A} preserving \mathscr{L} -images. Assume \mathscr{A} has small products and is cowell-powered (or even just that the isomorphism classes of each class

$$\mathscr{A}$$
-epi (A) = {f/f : A + B, $f_b : I_f \xrightarrow{\cong} B$, B $\varepsilon \mid \mathscr{A} \mid$ }

constitute a set (isomorphisms that are the identity on A, of course). Then \mathcal{A}^{T} has all coequalizers.

Proof. Given a pair
$$(E,\varepsilon) \xrightarrow{f} (A,\alpha)$$
 of $\mathcal{A}^{\mathbf{T}}$ -morphisms, let
$$\mathcal{E}_{f,g} = \{h/h \ \varepsilon \mid \mathcal{A}^{\mathbf{T}} \mid , \ h : (A,\alpha) \to (X,\xi), \ hf = hg, \ h = h_a\} .$$

Observe that an isomorphism class of $\mathcal{E}_{f,g}$ in the sense of \mathcal{A}^T or in the sense of \mathcal{A} is the same thing, because T preserves \mathcal{A} -images, and the maps h, Th are epic. Pick representatives of the isomorphism classes of $\mathcal{E}_{f,g}$, say

$$h_i : (A,\alpha) \longrightarrow (X_i,\xi_i)$$
 (i ϵ I)

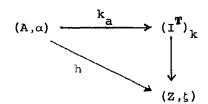
and form the induced map (an $\mathcal{A}^{\mathbf{T}}$ -morphism by [2, §6]

$$k = \langle ...h_{i}... \rangle : (A,\alpha) \longrightarrow (\Pi_{i}X_{j}, \Pi_{i}\xi_{i}) .$$

Then k_a is a coequalizer of (f,g) in $\mathcal{A}^{\mathbf{T}}$. Indeed, given $h:(A,\alpha)\to(Z,\xi)$ with hf=hg, we have $h_a\in\mathcal{E}_{f,g}$ and so $\exists i_0\in I$ with $h_a\stackrel{\sim}{=}h_i$. Then the composition

$$(\mathbf{I}^{\mathbf{T}})_{k} \xrightarrow{k_{b}} (\mathbf{I}_{\mathbf{i}} \mathbf{X}_{\mathbf{i}}, \mathbf{I}_{\mathbf{i}} \boldsymbol{\xi}_{\mathbf{i}}) \xrightarrow{pr_{\mathbf{i}_{o}}} (\mathbf{X}_{\mathbf{i}_{o}}, \boldsymbol{\xi}_{\mathbf{i}_{o}}) \xrightarrow{\cong} (\mathbf{I}^{\mathbf{T}})_{h} \xrightarrow{h_{b}} (\mathbf{Z}, \boldsymbol{\xi})$$

makes the triangle



commute, and since k_a is epic, it is the only such map. That $k_a \in \mathcal{E}_{f,g}$ is obvious, and completes the proof of the existence of coequalizers.

Remark. Much the same arguments prove, under the same hypotheses, that $\mathcal{A}^{\, T}$ has coequalizers of families of pairs of maps.

Example. $\mathcal{A} = \mathbb{S}$, $\mathcal{A} = \text{usual epic-monic factorization}$. Then any triple preserves \mathcal{A} -images (proof below), and consequently \mathbb{S}^{T} has coequalizers, and, by virtue of Corollary 2, all colimits. That \mathcal{A}^{T} has all limits if \mathcal{A} has is well known, and this then takes care of the completeness properties of varietal categories.

To see that every triple in \lesssim preserves images, it suffices to see that every triple in \lesssim preserves monomorphisms since the usual epic-monic factorization is determined to within isomorphism by the requirement that it be an epic-monic factorization, and every functor preserves epimorphisms, since they split. So let $\mathbf{T} = (\mathbf{T}, \eta, \mu)$ be a triple in \lesssim . The only monomorphisms of \mathbf{T} has a chance of not preserving are those that are not split, i.e., those with empty domain. Now if $\mathbf{T}(\emptyset)$ is \emptyset , To is surely monic. But if $\mathbf{T}\emptyset$ has at least one element, $\mathbf{T}\mathbf{f}$, which may be thought of as a \mathbf{T} -morphism from $\mathbf{F}^{\mathbf{T}}(\emptyset)$ to $\mathbf{F}^{\mathbf{T}}(\mathbf{n})$, admits a retraction, namely the extension to a \mathbf{T} -homomorphism from

phism of any function

$$n \rightarrow U^T F^T \emptyset = T \emptyset$$
.

That the composition on $F^T(\emptyset)$ is the identity is due to the fact that $F^T(\emptyset)$ is a left zero (is initial, is a copoint) in S^T .

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