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Taut Monads and $T0$-spaces

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1. Introduction

Dana Scott's founding paper establishing continuous lattices as the injective $T0$-spaces [22] appeared in the conference proceedings of a meeting at Dalhousie University. I was present at the meeting and enjoyed Scott's talk, but it did not occur to me that it might be related to what I was working on at the time, namely a comprehensive theory of relational models over a monad. I developed quite a bit of theory but, in the end, there appeared to be little point to it because – with the notable exception of Barr's theorem [2] that the relational models of the ultrafilter monad are topological spaces – the resulting categories of models were neither familiar nor interesting. To this day almost nothing has been written on relational models beyond Barr's paper; we are aware only of [5, 17, 12].

In this paper, relational models are resurrected, as a tool rather than as an end, to construct certain symmetric monoidal closed full subcategories of $T0$-spaces. Much remains to be done to better link the framework here with domain theory in general, the theory of continuous lattices [10] and with the work of Escardo and Flagg [7–9] in particular, but it is a beginning. Many of the facts about taut monads and the support topology for a relational model were worked out in the early 1970s.

There is much recent interest in monads in the functional programming community (see [4] and the references cited there). The paper [20] axiomatizes *collection monads* (generalizing sets, bags and lists) in connection with the specification and implementation of collection classes in a programming language. By definition, such monads are finitary. The *taut monads* to be introduced in this paper do not have a finitary restriction; as will be seen, the finitary taut monads are precisely the collection monads. The definition is simple. A functor is *taut* if it preserves inverse images, a natural

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transformation is *taut* if the naturality squares induced by monics are pullbacks and a (non trivial) monad (T, η, μ) is *taut* if T, η and μ are.

These more general taut monads also have a programming motivation. By adapting ideas of [3] we shall introduce the concept of a *support classifier* for a monad to express a functional program using the fold and map operators for the test "do the members of a given collection on X belong to a given subset of X?". It will be shown in Theorem 3.3 that a monad has a support classifier if and only if it is taut.

We presume some knowledge of category theory and monad theory. The book [1] contains all background needed, including image factorization systems and topological categories. (Note that for our factorization systems $(\mathscr{E}, \mathscr{M})$ we assume that morphisms in $\mathscr{M}$ are monic so that, necessarily, morphisms in $\mathscr{E}$ are epic). We shall use the notation $\mathscr{S}$ for the category of sets and (total) functions. We will provide here a few definitions, facts and clarifications to better orient the reader, starting with a lemma about pullbacks in a category.

Lemma 1.1. *The following statements hold about the following commutative diagram in an arbitrary category*:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d] \arrow[dr, phantom, "(A)"] & B \arrow[r, "b"] \arrow[d, "d"] \arrow[dr, phantom, "(B)"] & C \arrow[d] \\
D \arrow[r, "f"'] & E \arrow[r, "g"'] & F
\end{tikzcd}$$

1. *If (A) and (B) are pullbacks, so is the perimeter (A,B).*
2. *If (A,B) is a pullback and b,d are jointly monic then (A) is a pullback.*
3. *If b,d are jointly monic, f is split epic and (A,B) is a pullback then (B) is a pullback.*

Proof. Routine. $\square$

Definition 1.2. Taut functors and transformations. A pullback of $X \xrightarrow{f} Y \xleftarrow{i} B$ with i monic is called an *inverse image*. A functor which maps an inverse image to a pullback is said to be *taut*.

As g is monic if and only if the square

$$\begin{tikzcd}
Y \arrow[r, "id"] \arrow[d, "id"'] & Y \arrow[d, "g"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

is a pullback, every taut functor preserves monics. In particular, a taut functor preserves inverse images so that a composition of taut functors is taut.

A natural transformation is **taut** if each naturality square induced by a monic is a pullback.

We will use boldface letters such as $\mathbf{T}$ for a monad in a category $\mathscr{K}$, $\mathbf{T}=(T,\eta,\mu)$. A different monad might be written $\mathbf{W}=(W,\eta,\mu)$, just as one might denote two abelian groups as $(X,+)$, $(Y,+)$ changing only the underlying set, not the operation symbol.

We write the category of (Eilenberg–Moore [6]) $\mathbf{T}$-algebras as $\mathscr{K}^{\mathbf{T}}$. For (X,ξ) a $\mathbf{T}$-algebra, $\xi : TX \to X$ is called the **structure map**. The free $\mathbf{T}$-algebra generated by X in $\mathscr{K}$ is (TX,μ_X) and the unique $\mathbf{T}$-homomorphism $f^{\#} : (TX,\mu_X) \to (Y,\theta)$ extending $f : X \to (Y,\theta)$ is given by

$$f^{\#} = TX \xrightarrow{Tf} TY \xrightarrow{\theta} Y. \tag{1}$$

Two morphisms $\varphi,\psi : TX \to Y$ are said to "agree on generators" if $\varphi\eta_X = \psi\eta_X$. Two $\mathbf{T}$-homomorphisms which agree on generators are equal.

To verify that a particular (T,η,μ) constitutes a monad requires seven axioms (two to show T is a functor, two more to show η and μ are natural and yet three more axioms, one of which iterates T twice – one must prove that two paths beginning at $TTTX$ are equal. Since [18, Exercise 1.3.12, p. 32] we have championed supplementing the T,η,μ form with an equivalent definition of monad which replaces μ using the extension operator $\alpha^{\#}$ restricted to α of form $X \to TY$ which requires only three axioms and for which T is never iterated. We shall later also give extension-form definitions for Eilenberg–Moore algebras, monad maps and relational models. Though largely ignored in the early years, these axioms are frequently used in the more recent functional programming literature.

Definition 1.3. A **monad** (in extension form) is $\mathbf{T}=(T,\eta,(-)^{\#})$ where

$$\begin{aligned}(\eta_X)^{\#} &= id_{TX},\\ \alpha^{\#}\eta_X &= \alpha,\\ (\beta^{\#}\alpha)^{\#} &= \beta^{\#}\alpha^{\#}.\end{aligned}$$

It is an elementary exercise to prove that the definitions are equivalent using the following constructions. Given (T,η,μ), define $\alpha^{\#}=\mu_Y T\alpha$. Given $(T,\eta,(-)^{\#})$, T is a functor via $Tf=(\eta_Y f)^{\#}$ whence η is natural and $\mu_X=(id_{TX})^{\#}$ as suggested by (1) above.

(T,η,μ) is indispensible in theoretical situations but Definition 1.3 is more convenient in establishing that a particular construction constitutes a monad as in examples below.

A **T-algebra** is usually defined as (X,ξ) with X in $\mathscr{K}$ and $\xi : TX \to X$ satisfying the laws $\xi\eta_X = id_X$ and $\xi T\xi = \xi\mu_X$. The second law can be replaced with the following equivalent one which avoids iterating T:

$$\forall \alpha,\beta : Y \to TX,\ \xi\alpha = \xi\beta \ \Rightarrow\ \xi\alpha^{\#} = \xi\beta^{\#}. \tag{2}$$

The proof of equivalence is easy. See the remarks following the proof of Proposition 4.6 for another version of the axioms in which T is not iterated and which consists of two equations.

If $\mathbf{W},\mathbf{T}$ are monads in $\mathscr{K}$, a family $\lambda_X : WX \to TX$ (X in $\mathscr{K}$) is a **monad map** $\lambda : \mathbf{W} \to \mathbf{T}$ if $\lambda_X \eta_X = \eta_X$ and if for all $\alpha : X \to WY$ the following square commutes:

$$\begin{tikzcd}
WX \arrow[r, "\lambda_X"] \arrow[d, "\alpha^\#"'] & TX \arrow[d, "(X \xrightarrow{\alpha} WY \xrightarrow{\lambda_Y} TY)^\#"] \\
WY \arrow[r, "\lambda_Y"'] & TY
\end{tikzcd}$$

Such λ is a natural transformation and is a monad map $(W,\eta,\mu) \to (T,\eta,\mu)$ as defined, say, in [16].

If $\mathbf{T} = (T,\eta,(-)^\#)$ is a monad in $\mathscr{K}$ and $\iota_X : WX \to TX$ is a family of monics indexed by the objects of $\mathscr{K}$, W is a **submonad** of $\mathbf{T}$ if η_X factors through ι_X (defining η for the induced monad $\mathbf{W}$) and if for all $\alpha : X \to WY$, $(\iota_Y \alpha)^\#$ factors through ι_Y (defining $(-)^\#$ for $\mathbf{W}$).

Example 1.4. The identity monad **id**.

$TX = X$, $\eta_X = id_X = \mu_X$. An **id**-algebra has form (X, id_X). Thus $\mathscr{S}^{\mathbf{id}} = \mathscr{S}$.

Example 1.5. The power set monad $\mathbf{P} = (P,\eta,\mu)$ in $\mathscr{S}$.

Here $PX = 2^X$, the set of all subsets of X, $\eta_X\ x = \{x\}$, $\mu_X(\mathscr{A}) = \bigcup \mathscr{A}$. The extension is given by $\alpha^\#(A) = \bigcup_{a \in A} \alpha a$. $\mathscr{S}^{\mathbf{P}}$ is complete sup-semilattices with structure map $\xi : 2^X \to X$ the supremum map (see [18, Example 5.15]). Though the objects are complete lattices, the morphisms preserve only suprema. One can eliminate the bottom elements from the algebras by using the submonad of non-empty subsets. The partial order corresponding to ξ is $x \leqslant y \Leftrightarrow \xi\{x,y\} = y$. One can also redo the representation using $x \leqslant y \Leftrightarrow \xi\{x,y\} = x$ to obtain the respective categories of inf-semilattices. The sup- and inf-categories are isomorphic, of course. In Example 3.7 we will choose the inf version.

Example 1.6. Double-dualization monads.

Let $\mathscr{K}$ be a locally small category and let J be an object in $\mathscr{K}$ which has small powers. Then $\mathscr{K}(-,J) : \mathscr{K}^{op} \to \mathscr{S}$ has left adjoint $n \mapsto J^n$, giving rise to the **double-dualization monad** $\mathbf{D}_J = (D_J, ev, (-)^\#)$. The constructions are as follows:

$$\begin{aligned}
D_J X &= J^{\mathscr{K}(X,J)}, \\
X \xrightarrow{ev_X} J^{\mathscr{K}(X,J)} \xrightarrow{pr_g} J &= g, \\
J^{\mathscr{K}(X,J)} \xrightarrow{\alpha^\#} J^{\mathscr{K}(Y,\ J)} \xrightarrow{pr_g} J &= pr_{X \xrightarrow{\alpha} D_J Y \xrightarrow{pr_g} J}.
\end{aligned}$$

The first paper introducing monads into the theory of programming languages is [21]. Moggi regarded TX as the object of denotations of programs of type X. He called $\mathbf{D}_J$ the *continuations monad* with result set J because it provided a formal model of the continuation-passing style of denotational semantics.

These monads were introduced as strong monads over a symmetric monoidal closed category in [13]. We hope to exploit such strength in categories of $T0$ spaces in a later paper. The simple version just presented will suit the needs of this paper which are limited to Propositions 1.12 and 1.13.

Example 1.7. The contravariant double powerset monad.

This is the double-dualization monad $\mathbf{D}_2$ induced by $2=\{False, True\}$ in $\mathscr{S}$. We denote the monad as $\mathbf{P}^{-2}=(P^{-2},\eta,(-)^{\#})$ where $P^{-2}X=\{\mathscr{A}:\mathscr{A}\subset 2^X\}$, $\eta_X x=\{A\subset X: x\in A\}$ and $\alpha^{\#}\mathscr{A}=\{B\subset Y:\{x\in X:B\in\alpha x\}\in\mathscr{A}\}$. In particular, $P^{-2}f:P^{-2}X\to P^{-2}Y$ maps $\mathscr{A}$ to $\{B\subset Y:f^{-1}B\in\mathscr{A}\}$. The algebras of this monad are complete atomic Boolean algebras. The structure map $\xi:P^{-2}X\to X$ maps $\mathscr{A}$ to $\bigvee\{x:x$ is an atom, $\uparrow x\in\mathscr{A}\}$ where $\uparrow x=\{y:y\geqslant x\}$ [18, Example 5.17].

Example 1.8. The covariant double powerset monad.

This is the monad $\mathbf{P}^2=(P^2,\eta,(-)^{\#})$ given by $P^2X=P(PX)=\{\mathscr{A}:\mathscr{A}\subset 2^X\}$, $\eta_X\, x=\{\{x\}\}$, $\alpha^{\#}\mathscr{A}=\{\bigcup_{x\in A}B_x:A\in\mathscr{A},\ (B_x)\in\prod_{x\in A}\alpha_x\}$.

Example 1.9. The filter monad.

A non-empty collection $\mathscr{F}$ of subsets of X is a **filter** on X if the intersection of two sets in $\mathscr{F}$ is again in $\mathscr{F}$ and if any superset of an element of $\mathscr{F}$ is also in $\mathscr{F}$. A **principal** filter consists of the supersets of one set, that is has the form

$$prin(A)=\{B\subset X : A\subset B\},$$

where $prin(\emptyset)=2^X$ is the **improper** filter. All other filters are **proper**. $FX=\{\mathscr{F}\in P^{-2}X:\mathscr{F}$ is a filter$\}$ is a submonad of $\mathbf{P}^{-2}$ called, naturally enough, the **filter monad**. We denote this monad $\mathbf{F}$ henceforth. It was proved independently by [5, 23] that $\mathscr{S}^{\mathbf{F}}$ is the category of continuous lattices and morphisms which preserve directed suprema and arbitrary infima. The structure map $\xi:FX\to X$ of a continuous lattice is given by $\xi(\mathscr{F})=\bigvee_{F\in\mathscr{F}}\bigwedge F$.

Example 1.10. Submonads of the filter monad.

1. Proper filters constitute a submonad of $\mathbf{F}$.
2. Filters with non-empty intersection constitute a submonad of $\mathbf{F}$. We call this the **neighborhood monad** because the neighborhood filter of a point in a topological space provides a typical example. This monad was first noted in [19].

3. Ultrafilters constitute a submonad of $\mathbf{F}$. We denote this monad as $\boldsymbol{\beta}$. $\mathscr{S}^{\boldsymbol{\beta}}$ is the category of compact Hausdorff spaces and continuous maps (see [18, Theorem 5.29] or [11, Section 3.2]).

For any monad $\mathbf{T}$ in $\mathscr{S}$, if $f:X \to Y$ is monic, so is Tf. This is obvious if $X \neq \emptyset$ because every functor preserves split monics. The only case that needs proof, then, is $X = \emptyset$, $TX \neq \emptyset$. Choose any $\alpha : Y \to TX$. As $\alpha^{\#}Tf$ is a $\mathbf{T}$-homomorphism of the initial algebra $(T\emptyset, \mu_\emptyset)$, this map is $id_{T\emptyset}$ and so Tf is split monic. It follows that whenever $A \subset X$ then "$TA \subset TX$" in the sense that $TA \to TX$ is an isomorphism onto its image. In short, it makes sense, given $\omega \in TX$ and $A \subset X$, to ask whether or not $\omega \in TA$. If so, we say A is a **support** of ω. The set of all supports of ω will be written $supp(\omega)$.

For the filter monad, the filters in FX which are in FA are exactly those filters to which A belongs. This shows that any filter can arise as a set of supports. Conversely, we leave it as an exercise to prove that for A, B subsets of X with non-empty intersection, $T(A \cap B) = TA \cap TB$ for any endofunctor of $\mathscr{S}$ (construct a commutative diagram preserved by all functors which forces the pullback property of the intersection). [20, Example 4.3] provides an example of a monad for which $supp(\omega)$ is not always a filter.

There are two **trivial** monads in $\mathscr{S}$ whose functors are $TX = 1$ for all X, (1 being a one-element set) or, alternatively, $T\emptyset = \emptyset$, $TX = 1$ for $X \neq \emptyset$. Any other monad is called **non-trivial**. The following dates to [15].

Proposition 1.11. *For a monad $\mathbf{T}$ of $\mathscr{S}, \mathbf{T}$ is non-trivial $\Leftrightarrow$ η_X is monic for every X $\Leftrightarrow$ T is faithful $\Leftrightarrow$ there exists a $\mathbf{T}$-algebra with at least two elements.*

Proof. See [18, Proposition 5.2]. $\square$

The elements of a $\mathbf{T}$-algebra are in bijective correspondence with its homomorphisms from $(T1, \mu_1)$ whereas there is exactly one homomorphism from $(T\emptyset, \mu_\emptyset)$. Thus

$$\text{For a non-trivial monad } \mathbf{T} \text{ in } \mathscr{S},\ T\emptyset \to T1 \text{ is not surjective.} \tag{3}$$

We conclude the background review with an early folklore result of the Yoneda Lemma type first published by [13] in the more general context of strong monads in a category over a symmetric monoidal closed category. Proposition 1.13 says that an element in an abstract free algebra Tn corresponds to a representation as an actual n-ary operation on each algebra.

Proposition 1.12. *For any functor $T : \mathscr{S} \to \mathscr{S}$ and set J, there is a bijective correspondence between the set of natural transformations from T to the functor part D_J of the double-dualization monad $\mathbf{D}_J$ and the set of all functions from TJ to J.*

Proof. Map the natural transformation $\gamma: T \to D_J$ to $TJ \xrightarrow{\gamma_J} J^{J^J} \xrightarrow{pr_{id}} J$. Conversely, given $\xi: TJ \to J$, define $\gamma_X: TX \to J^{J^X}$ by $\gamma_X(\omega)(X \xrightarrow{f} J) = (TX \xrightarrow{Tf} TJ \xrightarrow{\xi} J)\omega =$ (by (1)) $f^{\#}\omega$. $\square$

Proposition 1.13. *Let* $\mathbf{T}$ *be a monad in* $\mathscr{S}$ *and let* J *be a set. Then there is a bijective correspondence between the set of monad maps from* $\mathbf{T}$ *to* $\mathbf{D}_J$ *and the set of all* $\xi: TJ \to J$ *for which* (J, ξ) *is a* $\mathbf{T}$*-algebra.*

Proof. Restrict the correspondence of the preceding proposition. $\square$

2. Taut monads

Informally, regard elements of TX as "$\mathbf{T}$-collections of elements of X" and say that *collection membership is decidable* if $\forall X\ \forall \omega \in TX\ \forall A \subset X$ if elementhood in A is decidable then whether or not all the members of ω belong to A is also decidable. In the next section we shall offer the notion of *support classifier* to formalize this definition. In this section we begin a study of taut monads. In Theorem 3.3 it will be shown that tautness is equivalent to the existence of a support classifier.

We start by repeating the definition given earlier.

Definition 2.1. A non-trivial monad (T, η, μ) is **taut** if T, η, μ are.

The following characterization is usually easier to use – there is only one thing to check and T is not iterated.

Proposition 2.2. $(T, \eta, (-)^{\#})$ *is taut if and only if it is non-trivial and the following taut extension law holds*: *in the diagram below with* m *monic*, *whenever* (A) *is a pullback so*, *too*, *is* (B):

$$\begin{tikzcd}
P \arrow[r, "\beta"] \arrow[d, "n"'] \arrow[dr, phantom, "\text{(A)}"] & TB \arrow[d, "Tm"] \arrow[dr, phantom, "\text{(B)}"] & TP \arrow[l, "\beta^{\#}"'] \arrow[d, "Tn"] \\
X \arrow[r, "\alpha"'] & TY & TX \arrow[l, "\alpha^{\#}"]
\end{tikzcd}$$

(Note: As both paths in (B) are $\mathbf{T}$-homomorphisms which agree on generators by (A), (B) always commutes.)

Proof. This implies Definition 2.1 as follows. If $X \xleftarrow{k} f^{-1}B \xrightarrow{g} B$ is the pullback of $X \xrightarrow{f} Y \xleftarrow{j} B$ with $j: B \subset Y$, set $\alpha = \eta_Y f$, $\beta = \eta_B g$ to see that T is taut. The tautness of

η then follows from the next proposition. Set $\alpha = id_{TY}$, $\beta = id_{TB}$ to see that μ is taut. Conversely, $\alpha^{\#} = \mu_Y T\alpha$, $\beta^{\#} = \mu_B T\beta$. $\square$

Proposition 2.3. *For any monad* $\mathbf{T}$ *in* $\mathscr{S}$, *if* T *is taut so is* η.

To see this, it is easily verified that for the two trivial monads, T and η are either both taut or both not taut. Otherwise, assume that $\mathbf{T}$ is non trivial. Let $A \subset X$ with inclusion $i: A \to X$ and suppose that $Ti\eta_A = \eta_X i$ is not a pullback. Equivalently (as Ti is monic), there exists $\omega \in TA$, $x \in X$ with $Ti\,\omega = \eta_X x$ but $x \notin A$. We seek a contradiction. As T is taut, we have a pullback

$$\begin{tikzcd} T\emptyset \arrow[r, "T!"] \arrow[d, "T!"'] & T1 \arrow[d, "Tx"] \\ TA \arrow[r, "Ti"'] & TX \end{tikzcd}$$

As $Tx\,\eta_1 = \eta_X\,x = Ti\,\omega$, it follows from the pullback property that η_1 factors through $T\emptyset$, and $T\emptyset = T1$. This contradicts (3) above.

Example 2.4. The filter monad is taut.

For the taut extension law, let $B \in \alpha^{\#}(\mathscr{F})$ so that $F_B = \{x \in X : B \in \alpha x\} \in \mathscr{F}$. By the pullback hypothesis, $F_B \subset P$ so $P \in \mathscr{F}$ as desired.

Lemma 2.5. *Let* $\lambda: \mathbf{W} \to \mathbf{T}$ *be a taut monad map. Then the following three statements hold.*

1. $\mathbf{W}$ *is taut if* $\mathbf{T}$ *is.*
2. *If* λ *is pointwise surjective*, $\mathbf{T}$ *is taut if* $\mathbf{W}$ *is.*
3. *Let* $\lambda_X = \iota_X\ \rho_X$ *be a surjective–injective factorization. Then* ι *and* ρ *are taut monad maps. In particular*, *the image submonad is taut.*

Proof. (1) For the taut extension law, let m be monic and, in the diagram below, let (A) be a pullback. To show: (B) is a pullback.

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d, "n"'] \arrow[dr, phantom, "\text{(A)}"] & WB \arrow[d, "Wm"] \arrow[dr, phantom, "\text{(B)}"] & WP \arrow[l, "\beta^{\#}"'] \arrow[d, "Wn"] \\ X \arrow[r, "\alpha"'] & WY & WY \arrow[l, "\alpha^{\#}"] \end{tikzcd}$$

As λ and $\mathbf{T}$ are taut, the perimeter (C, D) of the diagram below is a pullback because both (C) and (D) are.

$$\begin{tikzcd}
WP \arrow[r, "\lambda_P"] \arrow[d, "Wn"'] \arrow[dr, phantom, "\text{(C)}"] & TP \arrow[r, "(\lambda_B\beta)^\#"] \arrow[d, "Tn"] \arrow[dr, phantom, "\text{(D)}"] & TB \arrow[d, "Tm"] \\
WX \arrow[r, "\lambda_X"'] & TX \arrow[r, "(\lambda_Y\alpha)^\#"'] & TY
\end{tikzcd}$$

As λ is a monad map, (C, D) has the same perimeter as (B, E):

$$\begin{tikzcd}
WP \arrow[r, "\beta^\#"] \arrow[d, "Wn"'] \arrow[dr, phantom, "\text{(B)}"] & WB \arrow[r, "\lambda_B"] \arrow[d, "Wm"] \arrow[dr, phantom, "\text{(E)}"] & TB \arrow[d, "Tm"] \\
WX \arrow[r, "\alpha^\#"'] & WY \arrow[r, "\lambda_Y"'] & TY
\end{tikzcd}$$

Thus (B) is a pullback by Lemma 1.1.

(2) In the diagram of Proposition 2.2, let m be monic and let (A) be a pullback. We must show that (B) is also a pullback. Consider the diagram

$$\begin{tikzcd}
P \arrow[r, "\beta"] \arrow[d, "n"'] \arrow[dr, phantom, "\text{(A)}"] & TB \arrow[r, "j"] \arrow[d, "Tm"] \arrow[dr, phantom, "\text{(C)}"] & WB \arrow[r, "\lambda_B"] \arrow[d, "Wm"] \arrow[dr, phantom, "\text{(D)}"] & TB \arrow[d, "Tm"] \\
X \arrow[r, "\alpha"'] & TY \arrow[r, "i"'] & WY \arrow[r, "\lambda_Y"'] & TY
\end{tikzcd}$$

Here, choose i with $\lambda_Y\ i = id_{TY}$ since λ_Y is surjective. As (D) is a pullback, there exists j such that (C) commutes and $\lambda_B\ j = id_{TB}$. As (A, C, D) and (D) are pullbacks, (A, C) is a pullback. Now apply the taut extension law for $\mathbf{W}$:

$$\begin{tikzcd}
WP \arrow[r, "(j\beta)^\#"] \arrow[d, "Wn"'] \arrow[dr, phantom, "\text{(E)}"] & WB \arrow[r, "\lambda_B"] \arrow[d, "Wm"] \arrow[dr, phantom, "\text{(D)}"] & TB \arrow[d, "Tm"] \\
WX \arrow[r, "(i\alpha)^\#"'] & WY \arrow[r, "\lambda_Y"'] & TY
\end{tikzcd}$$

As D, E are pullbacks, so is (E, D). But we also have

$$\begin{tikzcd} WP \arrow[r, "\lambda_P"] \arrow[d, "Wn"'] \arrow[dr, phantom, "\text{(F)}"] & TP \arrow[r, "\beta^\#"] \arrow[d, "Tn"] \arrow[dr, phantom, "\text{(B)}"] & TB \arrow[d, "Tm"] \\ WX \arrow[r, "\lambda_X"'] & TX \arrow[r, "\alpha^\#"'] & TY \end{tikzcd}$$

The perimeters (E, D), (F, B) coincide because λ is a monad map and because $\lambda_B\ j\beta = \beta$, $\lambda_Y\ i\alpha = \alpha$. As λ_X is split epic and Tn is monic, if follows from Lemma 1.1 that (B) is a pullback as desired.

(3) That ι and ρ are monad maps is standard; see, e.g. [18, Exercise 3.4.3, p. 245]. Now use 1.1. □

3. The support classifier of a Taut monad

To motivate the concept of a support classifier via concepts of functional programming, we turn to Bird's 1987 paper on the theory of lists [3] which was written in the pre-monad days before Moggi's paper [21].

Let X^+ be the free semigroup generated by X of all finite, non-empty lists of elements of X. Say that a function $h: X^+ \to Y$ is a **homomorphism** if there is an associative operation $\oplus$ on X such that $h: X^+ \to (Y, \oplus)$ is a semigroup homomorphism. Bird argued that many algorithms have homomorphic components and that homomorphisms have a canonical form which can be implemented using functional operations given an algorithm for $\oplus$. For a small example of a homomorphism, let $Text = A^+$, $Line = (A - \{\backslash n\})^+$ where $\backslash n$ is the newline character. Let $deline : Line^+ \to Text$ convert a list of lines into single text by inserting newline characters between the lines. Then if $w \oplus v = w\backslash nv$, $(Text, \oplus)$ is a semigroup and *deline* is a homomorphism.

Bird proved the following *homomorphism lemma* [3, Lemma 1, p. 14]; A function $h: X^+ \to Y$ is a homomorphism if and only if for $f: X \to Y$, $f(x) = h(x)$,

$$h = (Fold \oplus)(Map\, f). \tag{4}$$

Here, for $f: X \to Y, Map\, f: X^+ \to Y^+$, $x_1 \cdots x_n \mapsto (f\,x_1) \cdots (f\,x_n)$ and $Fold \oplus : Y^+ \to Y$, $y_1 \cdots y_n \mapsto y_1 \oplus \cdots \oplus y_n$.

Now let $2 = \{True, False\}$, let A be a fixed subset of X and consider the function that tests if a list has all its members in A,

$$X^+ \xrightarrow{h} 2, \ h(x_1 \cdots x_n) = True \quad \Leftrightarrow \quad \{x_1, \ldots, x_n\} \subset A.$$

If $\wedge : 2 \times 2 \to 2$ is boolean *and*, it is clear that h is the homomorphism $h = (Fold \wedge)$ $(Map\, \chi_A)$ where $\chi_A(x) = True \Leftrightarrow x \in A$ is the characteristic function of A.

Let us now recast some of these constructions from the point of view that $TX = X^+$, $\eta_X(x) = x$, $\alpha^\#(x_1, \ldots, x_n) = \alpha(x_1) \cdots \alpha(x_n)$ (concatenation) is a monad. $Map\, f$ is just Tf.

It is well known that the algebras over this monad are precisely semigroups (see, e.g., [18, Example 1.4.7, p. 38]). $Fold \wedge : 2^+ \to 2$ is just the structure map of the semigroup $(2, \wedge)$ (as, indeed, $Fold \oplus$ is the structure map of the general semigroup $(Y, \oplus)$). The homomorphism lemma, then, is exactly the formula for **T**-homomorphic extension (1) above.

Notice that the Eilenberg–Moore structure map gives $Fold \oplus$ directly. The $\oplus$ depends on a particular choice of equational presentation for the **T**-algebras and implementation issues may be affected by this choice.

For a general monad **T**, $\wedge : 2 \times 2 \to 2$ cannot be expected to induce a **T**-algebra structure, so we must distill an abstract property of $(2, \wedge)$ for the list monad which provides a general axiom. The key observation is that for $i : A \subset X$ with characteristic function $\chi_A : X \to 2$, the square

$$\begin{tikzcd} A^+ \arrow[r] \arrow[d, "Map\ i"'] & 1 \arrow[d, "True"] \\ X^+ \arrow[r, "(\chi_A)^\#"'] & (2, \wedge) \end{tikzcd}$$

is a pullback. This is true because $(\chi_A)^\#(x_1, \ldots, x_n) = \chi_A(x_1) \wedge \cdots \wedge \chi_A(x_n)$.

It turns out that at most one **T**-algebra structure on 2 can have this property. We turn to the definition.

Definition 3.1. Let **T** be a monad in $\mathscr{S}$. A **support classifier** for **T** is a **T**-algebra $(2, \psi)$ such that for all sets X and subsets $i : A \subset X$ the following square is a pullback:

$$\begin{tikzcd} TA \arrow[r] \arrow[d, "Ti"'] & 1 \arrow[d, "True"] \\ TX \arrow[r, "(\chi_A)^\#"'] & (2, \psi) \end{tikzcd}$$

It is clear that $(2, \wedge)$ is a support classifier for the list monad. We think of ψ as a generalized "$Fold \wedge$". Thus for $\omega \in TX, A \in supp(\omega) \Leftrightarrow \psi\ T\chi_A$ ("mapping with χ_A and folding with $\wedge$") maps ω to $True$. This explains the term "support classifier".

Proposition 3.2. *If a support classifier exists it is unique.*

Proof. For any **T**-algebra (X, ξ), $\xi = (id_X)^\#$. As $\psi\ \eta_2 = id_2 = \chi_{\{True\}}$, it follows from the pullback property that $\psi = \chi_{T(True)}$. $\square$

We give the special notation $\mathscr{Z}$ for the image of $T(True) : T1 \to T2$. Thus $\psi = \chi_{\mathscr{Z}}$.

We are now ready to establish the relation between support classifiers and tautness. Recall that **F** is the filter monad.

Theorem 3.3. *The following statements about a monad* $\mathbf{T}$ *in* $\mathscr{S}$ *are equivalent*:
1. $\mathbf{T}$ *has a support classifier.*
2. *The set of supports of an element of* TX *is a filter and* $supp : \mathbf{T} \to \mathbf{F}$ *is a taut monad map.*
3. *There exists a taut monad map* $\mathbf{T} \to \mathbf{F}$.
4. $\mathbf{T}$ *is non-trivial and* T, η, μ *are taut*, *that is*, $\mathbf{T}$ *is a taut monad.*

Proof. $(1 \Rightarrow 2)$ Consider the double-dualization monad $\mathbf{D}_2$. The support classifier $(2, \psi)$ induces (Proposition 1.13) the monad map $\gamma : \mathbf{T} \to \mathbf{D}_2$ where $\gamma_X(\omega) : 2^X \to 2$, $\chi_A \mapsto (\chi_A)^{\#}(\omega)$ so that, thinking of elements of D_2X as families of subsets of X,

$$\begin{aligned}\gamma_X(\omega) &= \{A \subset X : \chi_A^{\#}(\omega) = True\} \\ &= \{A \subset X : \omega \in TA\} \text{ (support classifier)} \\ &= supp(\omega).\end{aligned}$$

To show that *supp* is filter-valued, it is most efficient to observe that T is taut. To this end, consider $f : X \to Y$, $i : B \subset Y$ and the diagram

$$\begin{tikzcd} T(f^{-1}B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(A)}"] & TB \arrow[r] \arrow[d, "Ti"] \arrow[dr, phantom, "\text{(B)}"] & 1 \arrow[d, "True"] \\ TX \arrow[r, "Tf"'] & TY \arrow[r, "(\chi_B)^{\#}"'] & (2, \psi) \end{tikzcd}$$

As $(\chi_B)^{\#}\, Tf = (\chi_B)^{\#}\, (\eta_Y f)^{\#} = ((\chi_B)^{\#}\, \eta_y\, f))^{\#} = (\chi_B f)^{\#} = (\chi_{f^{-1}B})^{\#}$, it follows from the definition of a support classifier that (A, B) and (B) are pullbacks. Thus (A) is a pullback and T is taut. As a special case of tautness, T preserves binary intersections (including empty ones) so *supp* is filter-valued. That *supp* is a taut transformation is immediate from the definition of “support”.

$(3 \Rightarrow 4)$ Use 2.5. Since there are continuous lattices with more than two elements, the same holds for $\mathbf{T}$-algebras, so $\mathbf{T}$ is non-trivial.

$(4 \Rightarrow 1)$ Define $\psi = \chi_{\mathscr{Z}} : T2 \to 2$. To show: $(2, \psi)$ is a support classifier. (D) is a pullback and, because η is taut, so is (C).

$$\begin{tikzcd} 1 \arrow[r, "\eta_1"] \arrow[d, "True"'] \arrow[dr, phantom, "\text{(C)}"] & T1 \arrow[r] \arrow[d, "T(True)"] \arrow[dr, phantom, "\text{(D)}"] & 1 \arrow[d, "True"] \\ 2 \arrow[r, "\eta_2"'] & T2 \arrow[r, "\psi"'] & 2 \end{tikzcd}$$

As 2 is a subobject classifier in $\mathscr{S}$, $\psi\, \eta_2 = id_2$. Now let α, $\beta : W \to T2$ with $\psi\, \alpha = \psi\, \beta$. To show that $(2, \psi)$ is a $\mathbf{T}$-algebra it must be proved that $\psi\, \alpha^{\#} = \psi\, \beta^{\#}$. Consider the diagram

$$\begin{tikzcd}
P_\alpha \arrow[r, "\gamma_\alpha"] \arrow[d, "j_\alpha"'] \arrow[dr, phantom, "\text{(E)}"] & T1 \arrow[r] \arrow[d, "T(True)"] \arrow[dr, phantom, "\text{(D)}"] & 1 \arrow[d, "True"] \\
W \arrow[r, "\alpha"'] & T2 \arrow[r, "\psi"'] & 2
\end{tikzcd}$$

in which (E) is a pullback by definition, and consider a similar diagram with $j_\beta : P_\beta \to W$, $\gamma_\beta : P_\beta \to T1$ induced by β. As $\psi\alpha = \psi\beta$, the two pullbacks (E,D) are equal so that $P_\alpha = P = P_\beta$, $j_\alpha = j = j_\beta$. As $\mathbf{T}$ is taut there are pullbacks

$$\begin{tikzcd}
TP \arrow[r, "\gamma_\alpha^\#{,}\,\gamma_\beta^\#"] \arrow[d, "Tj"'] & T1 \arrow[r] \arrow[d, "T(True)"] & 1 \arrow[d, "True"] \\
TW \arrow[r, "\alpha^\#{,}\beta^\#"'] & T2 \arrow[r, "\psi"'] & 2
\end{tikzcd}$$

so that $\psi\alpha^\# = \chi_{TP} = \psi\beta^\#$ as desired. Finally, let $i : A \subset X$. As T preserves the pullback of $X \xrightarrow{\chi_A} 2 \xleftarrow{True} 1$ there are pullbacks

$$\begin{tikzcd}
TA \arrow[r] \arrow[d, "Ti"'] & T1 \arrow[r] \arrow[d, "T(True)"] & 1 \arrow[d, "True"] \\
TX \arrow[r, "T_{\chi A}"'] & T2 \arrow[r, "\psi"'] & 2
\end{tikzcd}$$

As $\psi T\chi_A = (\chi_A)^\#$, we are done. $\square$

Corollary 3.4. *If* $\mathbf{T}$ *is a non-trivial taut monad and* L *is a continuous lattice then* $\xi : TL \to L$ *defined by*

$$\xi(\omega) = \bigvee_{F \in supp(\omega)} \bigwedge F$$

is a $\mathbf{T}$*-algebra. The support classifier arises this way from the continuous lattice* 2 *with* $False < True$.

Proof. The first statement just describes the functor from continous lattices that corresponds to the monad map *supp*. For $False < True$, $\xi(\omega) = True \Leftrightarrow \exists F \in supp(\omega)$ with $\bigwedge F = True \Leftrightarrow \{True\} \in supp(\omega) \Leftrightarrow \omega \in \mathscr{Z}$, so $\xi = \chi_{\mathscr{Z}}$ is the support classifier. $\square$

Proposition 3.5. *Let* $\mathbf{W}$, $\mathbf{T}$ *be non-trivial taut monads and let* $\lambda : \mathbf{W} \to \mathbf{T}$ *be a taut monad map. Then the induced functor* $\mathscr{S}^{\mathbf{T}} \to \mathscr{S}^{\mathbf{W}}$ *preserves the support classifier.*

Proof. Let $(2,\psi)$ be the support classifier of $\mathbf{T}$. As both squares below are pullbacks

$$\begin{tikzcd} W1 \arrow[r, "\lambda_1"] \arrow[d, "W(True)"'] & T1 \arrow[r] \arrow[d, "T(True)"] & 1 \arrow[d, "True"] \\ W2 \arrow[r, "\lambda_2"'] & T2 \arrow[r, "\psi"'] & 2 \end{tikzcd}$$

$(2,\psi\lambda_2)$ is the support classifier of $\mathbf{W}$. □

Proposition 3.6. *Let* $\mathbf{T}$ *be a non trivial taut monad and let* $supp = \iota\rho$ *with* $\rho : \mathbf{T} \to \mathbf{G}$ *pointwise surjective and* $\iota : \mathbf{G} \to \mathbf{F}$ *pointwise injective so that by Lemma* 2.5 *and Theorem* 3.3 ρ, ι *and* G *are taut. Then* ι *is the support map of* $\mathbf{G}$ *and* $\mathscr{S}^{\mathbf{G}}$ *is the variety of* $\mathbf{T}$*-algebras generated by the support classifier* $(2,\psi)$ *of* $\mathbf{T}$.

Proof. By the Birkhoff Theorem (see, e.g., [18]) , $\mathscr{S}^{\mathbf{G}}$ is a variety of $\mathbf{T}$-algebras whose inclusion functor $\Lambda : \mathscr{S}^{\mathbf{G}} \to \mathscr{S}^{\mathbf{T}}$ over $\mathscr{S}$ is given by $\Lambda(X, GX \xrightarrow{\xi} X) = (X, TX \xrightarrow{\rho X} GX \xrightarrow{\xi} X)$. For $\mathscr{F} \in GX$, $A \subset X$, $A \in \mathscr{F} \Leftrightarrow \mathscr{F} \in FA \Leftrightarrow \mathscr{F} \in GA$ (as ι is taut), so $\iota_X(\mathscr{F})$ is indeed the set of $\mathbf{G}$-supports of $\mathscr{F}$. $(2,\psi)$ is a $\mathbf{G}$-algebra by Proposition 3.5. It suffices to show that the free $\mathbf{G}$-algebra $(GX,\ \mu_X)$ generated by X is a subalgebra of a power of $(2,\psi)$. Consider the composition

$$TX \xrightarrow{\rho X} GX \xrightarrow{\iota_X} FX \xrightarrow{k} (2,\psi)^{2^X} \xrightarrow{pr_A} (2,\psi),$$

where k is the inclusion function from filters on X to families on X and $A \in 2^X$. We will show that $k\ \iota_X : (GX,\ \mu_X) \to (2,\psi)^{2^X}$ is a $\mathbf{G}$-homomorphism (since it is obviously monic). Equivalently, to show: $pr_A\ k\ \iota_X\ \rho_X$ is a $\mathbf{T}$-homomorphism. As $\rho_X\ \iota_X = supp_X$, $\rho_X\ \iota_X\ k\ \ pr_A = \chi_{TA} =$ (by support classifier) $(\chi_A)^{\#}$, and we are done. □

Example 3.7. The power set monad $\mathbf{P}$ of Example 1.5 is taut. For $A \in PX$, $supp(A) = \{B \subset X : A \subset B\}$ is the principal filter $prin(A)$ generated by A. It is easy to check that $prin : \mathbf{P} \to \mathbf{F}$ is a taut monad map. The corresponding support classifier is the infimum map $\wedge : 2 \times 2 \to 2$. For this reason it is convenient to (isomorphically) regard $\mathscr{S}^{\mathbf{P}}$ as complete inf- rather than sup-semilattices with the structure map being infimum. This turnaround problem – that no one choice, sup or inf works for all cases – has long been known, see e.g. [23, p. 391]. $\mathscr{S}^{\mathbf{P}}$ is the variety generated by the 2-element semilattice. This is well known (consider inf-semilattices as the modules over the Boolean semiring whence this result is standard module theory).

Example 3.8. $\mathbf{P}^{-2}$ is not taut. For there are only two complete atomic Boolean algebra structures on $2 = \{False, True\}$ namely $False < True$ and $True < False$. For either of these, the structure map of Example 1.7 is not $\chi_{\mathscr{Z}}$.

Example 3.9. $\mathbf{P}^2$ is taut. The taut extension law is easy. The support map is given by $supp(\mathscr{A}) = prin(\bigcup \mathscr{A})$.

We next compare taut monads to the *collection monads* of an earlier paper.

Definition 3.10 (Manes [20, Definition 4.1 and 4.8]). A **collection monad** is a monad $\mathbf{T}$ in $\mathscr{S}$ satisfying the following axioms.
1. $\mathbf{T}$ is non-trivial.
2. $\mathbf{T}$ is *finitary*, that is each $\omega \in TX$ has a finite support.
3. *Members exist*, that is, each $\omega \in TX$ has a minimum support called its set of *members*, written $mem_X(\omega)$.
4. $mem : \mathbf{T} \to \mathbf{P}$ is a monad map to the power set monad.

Proposition 3.11. *The collection monads are the finitary taut monads.*

Proof. First let $\mathbf{T}$ be a collection monad. It is obvious that for $\omega \in TX$, $A \subset X$, if $mem_X(\omega) \subset A$ then $\omega \in TA$, and this says that the monad map *mem* is taut. Conversely, let $\mathbf{T}$ be non-trivial, taut and finitary and let $\omega \in TX$. Since $supp(\omega)$ is a filter and contains a finite set, $\bigcap supp(\omega)$ provides $mem_X(\omega)$. Because the support filters are principal, *supp* factors via *mem* through the power set monad. □

Proposition 3.12. *Let* $\iota : \mathbf{G} \to \mathbf{F}$ *be a submonad of the filter monad. Then* ι (*and hence* $\mathbf{G}$) *are taut.* $\mathscr{S}^{\mathbf{G}}$ *is generated as a variety by its support classifier.*

Proof. $\mathbf{G}$ is non-trivial because every submonad of a non-trivial monad is (consider the η-monic characterization). For $A \subset X$ with inclusion $j : A \to X$ we must show that the following square is a pullback:

$$\begin{tikzcd} GA \arrow[r, "\iota_A"] \arrow[d, "Gj"'] & FA \arrow[d, "Fj"] \\ GX \arrow[r, "\iota_X"'] & FX \end{tikzcd}$$

We will give rather different proofs for the cases $A = \emptyset$, $A \neq \emptyset$. Start with the empty case. Let $\mathscr{F} \in GX \cap F\emptyset$, that is, $\mathscr{F} = 2^X$. We must show $G\emptyset \neq \emptyset$. Let $\mathbf{P}$ be the power set monad considered as a submonad of $\mathbf{F}$ via principal filters. $\mathbf{P}$ satisfies all of the axioms for a collection monad except for being finitary. But every submonad of a collection monad has taut inclusion [20, Theorem 5.1] and the proof of this does not use the finitary property, so this holds for $\mathbf{P}$. Thus $\mathbf{G} \cap \mathbf{P}$ is a taut submonad of $\mathbf{P}$. $2^X = prin(\emptyset) = Pi(\{\emptyset\}) \in PX$ so, from the pullback

$$\begin{tikzcd} G\emptyset \cap P\emptyset \arrow[r] \arrow[d] & P\emptyset \arrow[d] \\ GX \cap PX \arrow[r] & PX \end{tikzcd}$$

we see that $G\emptyset \neq \emptyset$ as required.

Now consider the case $A \neq \emptyset$. In that case choose a function $s : X \to A$ with $sj = id_A$. Let $\mathscr{F} \in GX$ with $\mathscr{F} \in FA$, that is, with $A \in \mathscr{F}$. We will show that $\mathscr{F} \wedge A = \{E \cap A : E \in \mathscr{F}\} \in GA$ since then $Gj(\mathscr{F} \wedge A) = \mathscr{F}$. It suffices to show that $\mathscr{F} \wedge A = (Gs)\mathscr{F}$ since the latter is in GA by construction. $(Gs)\mathscr{F} = \{C \subset A : s^{-1}C \in \mathscr{F}\}$. If $C \in (Gs)\mathscr{F}$ then $C = j^{-1}s^{-1}C = A \cap s^{-1}C \in \mathscr{F} \wedge A$. Conversely, if $E \in \mathscr{F}$ then $s^{-1}(E \cap A) = \{x \in X : sx \in E \cap A\} \supset E \cap A$ whereas $E \cap A \in \mathscr{F}$ because $A \in \mathscr{F}$, so $s^{-1}(E \cap A) \in \mathscr{F}$ as desired. The last statement is immediate from 3.6. $\square$

It is an open problem at this time whether or not any submonad of a taut monad is taut.

4. The Sierpinski object

Compact Hausdorff Objects are $\boldsymbol{\beta}$-algebras, the topology being represented by the convergence function $\xi : \beta X \to X$. More generally, any topological space is determined by its ultrafilter convergence relation $\xi\!\!\xi \subset \beta X \times X$ given by

$$\mathscr{U} \xi\!\!\xi\, x \quad \Leftrightarrow \quad \forall \text{ open } U \text{with } x \in U,\ U \in \mathscr{U}.$$

(Note: Throughout the paper we will use double-letter ligatures such as $\xi\!\!\xi$ to denote relations.) In [2], Michael Barr identified the axioms on $\xi\!\!\xi$ relative to an arbitrary monad in $\mathscr{S}$ that, for $\boldsymbol{\beta}$, produce exactly the category **Top** of topological spaces and continuous maps. In this section, we will review Barr's definition of relational **T**-models and show that among these models there is a natural generalization of the Sierpinski space. This machine will be used to generate full subcategories of $T0$-spaces in the next section.

Definition 4.1. T-models. If $T : \mathscr{S} \to \mathscr{S}$ is any functor, a T-**model** is a pair $(X, \xi\!\!\xi)$ with X a set and $\xi\!\!\xi : TX \to X$ a relation (i.e., $\xi\!\!\xi \subset TX \times X$). The category T-**Mod** has T-models as objects and morphisms $f : (X, \xi\!\!\xi) \to (Y, \theta\!\!\theta)$ all functions $f : X \to Y$ such that $Tf \times f$ maps $\xi\!\!\xi$ into $\theta\!\!\theta$.

Equivalently,

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\xi\!\!\xi"'] \arrow[dr, phantom, "\subset"] & TY \arrow[d, "\theta\!\!\theta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(specifically, a function such as f is a relation —xfy if $fx = y$—, $f\xi\!\!\xi$ and $\theta\!\!\theta Tf$ represent relation composition $x(SR)z$ if $\exists y\, xRy,\ ySz$, and the inclusion sign inside the diagram means $\omega\ (f\xi\!\!\xi)y \Rightarrow \omega(\theta\!\!\theta\ Tf)y$). That this forms a category is obvious.

T-**Mod** is a topological category. The initial lift of a single morphism $f : X \to (Y, \theta\!\!\theta)$ is $(X, \xi\!\!\xi)$ given by $\xi\!\!\xi = f^\circ\, \theta\!\!\theta\, Tf$ where the converse relation f° is defined by $y\ f^\circ x \Leftrightarrow$

xfy. The initial lift of a family is obtained by intersecting the initial lifts of the members. For the empty family, $\xi\!\!\!\xi = TX \times X$. It follows that final lifts exist and the constructions are easily given. The final lift of $(X, \xi\!\!\!\xi) \to Y$ is $(Y, (Tf \times f)\xi\!\!\!\xi)$. Take the union of these for a family, the empty case being the empty relation.

Definition 4.2. **T**-models. Let **T** be a monad of sets. A **T-model** is a T-model $(X, \xi\!\!\!\xi)$ satisfying the **reflexive law** $id_X \subset \xi\!\!\!\xi\,\eta_X$ and the **transitive law** $\xi\!\!\!\xi\; T(\xi\!\!\!\xi) \subset \xi\!\!\!\xi\,\mu_X$. Here, if $\xi\!\!\!\xi$ has projections $TX \xleftarrow{p} \xi\!\!\!\xi \xrightarrow{q} X$, $T(\xi\!\!\!\xi): TTX \to TX$ is defined as the image of $[Tp, Tq]: T\xi\!\!\!\xi \to TTX \times TX$. The category **T-Mod** of **T-models** is defined as the full subcategory of T-**Mod** of all **T**-models.

It is easy to check that **T-Mod** is initially (but not finally) closed in T-**Mod** and so is itself a topological category. In particular, small limits and colimits are constructed at the level of $\mathscr{S}$.

Example 4.3. Pre-ordered sets. Let **T** be the identity monad in $\mathscr{S}$. If (X, ξ) is a **T**-algebra then $\xi = id_X$, so $\mathscr{S}^{\mathbf{T}} \cong \mathscr{S}$. T-**Mod** is the category of sets equipped with an arbitrary binary relation and the reflexive and transitive laws have their usual meaning. Hence **T-Mod** is the category of pre-ordered (that is, reflexively- and transitively ordered) sets and monotone maps.

Example 4.4. $\boldsymbol{\beta}$-models. As already mentioned, [2] showed that $\boldsymbol{\beta}$-models are topological spaces with $\xi\!\!\!\xi$ the ultrafilter convergence relation.

The proof of the next lemma is safely left to the reader.

Lemma 4.5. *Let $\alpha\!\!\alpha: X \to Y$, $\beta\!\!\beta: Y \to Z$ be relations with respective projections*

$$X \xleftarrow{p} \alpha\alpha \xrightarrow{q} Y \xleftarrow{r} \beta\beta \xrightarrow{s} Z$$

Let $H, L: \mathscr{S} \to \mathscr{S}$ be functors and let $\zeta: H \to L$ be a natural transformation. Then the following hold.

1. $H(\beta\!\!\beta\,\alpha\!\!\alpha) \subset H(\beta\!\!\beta)H(\alpha\!\!\alpha)$
2. $H(\beta\!\!\beta\,\alpha\!\!\alpha) = H(\beta\!\!\beta)H(\alpha\!\!\alpha)$ *providing either $\alpha\!\!\alpha$ is a function or H preserves the pullback of q and r.*
3. $H(\alpha\!\!\alpha)^\circ = (H\alpha\!\!\alpha)^\circ$
4. *Although ζ is not a natural transformation on the category of sets and relations in general, the following inequality is always valid:*

$$\begin{tikzcd} HX \arrow[r, "\zeta_X"] \arrow[d, "H\alpha\!\!\alpha"'] \arrow[dr, phantom, "\subset"] & HY \arrow[d, "L\alpha\!\!\alpha"] \\ HY \arrow[r, "\zeta_Y"'] & LY \end{tikzcd}$$

Armed with the machinery of the previous lemma, we can characterize **T**-models without iterating T.

Proposition 4.6. *Let* $(T, \eta, (-)^{\#})$ *be a monad,* (X, ξ) *a* T*-model. Then* (X, ξ) *is a* **T**-*model if and only if for all functions* $\alpha : Y \to TX$, $\xi\, T(\xi\, \alpha) \subset \xi\, \alpha^{\#}$.

Proof. If (X, ξ) is a **T**-model then

$$
\begin{aligned}
\xi T(\xi\alpha) &\subset \xi T \xi T\alpha \text{ (the lemma)} \\
&\subset \xi \mu_X T\alpha \text{ (}\mathbf{T}\text{-model)} \\
&= \xi \mu_X T\alpha^{\#} T\eta_Y \text{ (monad)} \\
&= \xi \alpha^{\#} \mu_Y T\eta_Y \text{ (}\alpha^{\#}\ \mathbf{T}\text{-homomorphism)} \\
&= \xi \alpha^{\#} \text{ (monad).}
\end{aligned}
$$

Conversely, set $\alpha = id_{TX}$. Then

$$\xi\, T(\xi) = \xi\, T(\xi\alpha) \subset \xi\, \alpha^{\#} = \xi\, \mu_X. \qquad \square$$

Notice that a **T**-algebra is simply a **T**-model (X, ξ) for which ξ is a function. All graph inclusions mentioned so far are then equalities, so we have just given an *equational* extension-form axiomatization of a **T**-algebra (X, ξ) in which T is not iterated, namely $\xi\, \eta_X = id_X$ and $\forall \alpha : W \to TX$, $\xi T(\xi\, \alpha) = \xi\, \alpha^{\#}$.

We now turn to defining a "Sierpinski object" in **T-Mod**. For $\boldsymbol{\beta}$ we expect this to be the well-known Sierpinski space on $2 = \{False, True\}$ in which $\{True\}$ is open and $\{False\}$ is not. Every ultrafilter converges to *False* but *prin*(1) is the only ultrafilter converging to *True*. This motivates the following definition.

Definition 4.7. For any monad **T** of sets, the **Sierpinski object of T** is $S = (2, \sigma\sigma)$ where

$$\sigma\sigma = T2 \times \{False\} \cup \mathscr{Z} \times \{True\}$$

where $\mathscr{Z}$ is, recall, the image of $T(True)$ in $T2$.

Proposition 4.8. *If* **T** *is a monad in* $\mathscr{S}$ *and* T *is taut then the Sierpinski object of* **T** *is a* **T**-*model.*

Proof. Since $\eta_2(True) = T(True)\, \eta_1 \in \mathscr{Z}$, the reflexive law is obvious. For the transitive law, let $\bar{\omega} \in TT2$, $i \in 2$. We must show that $\bar{\omega}\, (\sigma\sigma\, T\, \sigma\sigma)\, i \Rightarrow \mu_2(\bar{\omega})\, \sigma\sigma\, i$. This is obvious if $i = False$ so we assume $i = True$. To show: $\mu_2(\bar{\omega}) \in \mathscr{Z}$. Writing the projections of $\sigma\sigma$ as $T2 \xleftarrow{p} \sigma\sigma \xrightarrow{q} 2$, there exists $v \in T\sigma\sigma$ with $(Tp)v = \bar{\omega}$, $(Tq)v \in \mathscr{Z}$. Write 2 as a coproduct of $\{False\}$, $\{True\}$ with respective injections t, u. By the definition of $\sigma\sigma$

we have

$$\begin{tikzcd}
Z \arrow[d, "Tu"'] \arrow[dr, phantom, "\text{(A)}"] & Z\times 1 \arrow[l, "pr_Z"'] \arrow[r, "!"] \arrow[d, "i"] \arrow[dr, phantom, "\text{(B)}"] & 1 \arrow[d, "u"] \\
T2 & \varpi \arrow[l, "p"] \arrow[r, "q"'] & 2
\end{tikzcd}$$

where (B) is a pullback. As T is taut, (C) is a pullback:

$$\begin{tikzcd}
TT1 \arrow[d, "TTu"'] & T(Z\times 1) \arrow[l, "Tpr_Z"'] \arrow[r, "T!"] \arrow[d, "Ti"] \arrow[dr, phantom, "\text{(C)}"] & Z \arrow[d, "Tu"] \\
TT2 & T\varpi \arrow[l, "Tp"] \arrow[r, "Tq"'] & T2
\end{tikzcd}$$

As $v \in T\varpi$ with $(Tq)v \in \mathscr{Z}$ and as $pr_{\mathscr{Z}}$ is an isomorphism there exists $\bar{v} \in T\mathscr{Z}$ with $(TTu)\bar{v} = (Tp)v = \bar{\omega}$. As μ is natural, we have

$$\begin{tikzcd}
TZ \arrow[r, "\mu_1"] \arrow[d, "TTu"'] & Z \arrow[d, "Tu"] \\
TT2 \arrow[r, "\mu_2"'] & T2
\end{tikzcd}$$

so that

$$\mu_2\bar{\omega} = (\mu_2 TTu)\bar{v} = Tu\mu_1(\bar{v}),$$

which is in $\mathscr{Z}$ as desired. $\square$

5. $T0$ spaces

Proposition 5.1. *Let $T : \mathscr{S} \to \mathscr{S}$ be a taut functor. Let* **Top** *be the category of topological spaces and continuous maps. Then there exists a canonical functor Φ :* **T-Mod** $\to$ **Top** *over $\mathscr{S}$ defined by $\Phi(X, \xi) = (X, \mathscr{T}_\xi)$ where the topology $\mathscr{T}_\xi$ of ξ-open sets is given by*

$$U \subset X \text{ is } \xi\text{-open} \;\Leftrightarrow\; \omega\,\xi\,x,\; x \in U \Rightarrow \omega \in TU.$$

Proof. That $\mathscr{T}_\xi$ is a topology is trivial since $T(U \cap V) = TU \cap TV$. Now let $f:(X,\xi) \to (Y,\theta)$ be a T-model map and let $V \in \mathscr{T}_\theta$. If $\omega\, \xi\, x \in f^{-1}(V)$ then $(Tf)\omega \in TV$. As T is taut, $\omega \in T(f^{-1}V)$, so f is continuous. $\square$

We call $\mathscr{T}_\xi$ the **support topology** because "a set is open if and only if it supports any ω ξ-related to one of its elements".

Example 5.2. Φ for $\boldsymbol{\beta}$-**Mod** is an isomorphism. The open sets of a $\boldsymbol{\beta}$-model are the usual ones since "a set is open if and only if it belongs to every ultrafilter converging in it" rephrases the fact (depending on a weakened axiom of choice) that a neighborhood filter is the intersection of its containing ultrafilters. Thus Φ is an isomorphism over $\mathscr{S}$.

Example 5.3. Φ for $\mathscr{S}^{\mathbf{F}}$ is the Scott topology.

To see this, let L be a continuous lattice with structure map $\xi(\mathscr{F}) = \bigvee_{F \in \mathscr{F}} \bigwedge F$. We shall show now that Φ assigns the Scott topology to $\mathscr{F}$-algebras. First suppose that U is ξ-open. If $x \leqslant y$ with $x \in U$ then as $\xi(prin(\{x,y\})) = x \wedge y = x \in U$, $prin(\{x,y\}) \in FU$, that is, $U \in prin(\{x,y\})$. This shows that $y \in U$ so that U is an upper set. Now let $x = \bigvee_{i \in I} x_i$ be a directed supremum and let $x \in U$. Then $\mathscr{F} = \{A \subset X : \exists i \in I,\ A \supset \uparrow x_i\}$ is a filter on X and $\xi(\mathscr{F}) = x \in U$ so $U \in \mathscr{F}$ and there exists $i \in I$ with $\uparrow x_i \subset U$. In particular $x_i \in U$. Thus U is Scott open. Conversely, suppose that U is Scott open and let $\xi\mathscr{F} \in U$. As this supremum is directed, there exists $F \in \mathscr{F}$ with $\wedge F \in U$. As U is an upper set, each element of F is in U so $F \subset U$ and $U \in \mathscr{F}$ as desired.

We shall return to the functor Φ after some generalities about cogenerators.

Definition 5.4. Cogenerating categories. Let $\mathscr{K}$ be a locally small category and let $(\mathscr{E}, \mathscr{M})$ be an image factorization system for $\mathscr{K}$. Let J be an object of $\mathscr{K}$ which has all small powers. The **cogenerating category** of J, denoted Cog_J, is the full subcategory of all X in $\mathscr{K}$ for which the "evaluation morphism" $ev_X : X \to J^{\mathscr{K}(X,J)}$ defined by $pr_f\, ev_X = f$ is a morphism in $\mathscr{M}$. We say that J is an $\mathscr{M}$-**cogenerator** of Cog_J.

Part (2) of the next result is a nice application of elementary monad theory.

Proposition 5.5. *In the context of the preceding definition, the following statements hold.*
1. $J \in Cog_J$.
2. *Cog_J is a full $\mathscr{E}$-reflective subcategory of $\mathscr{K}$. Indeed, if $ev_X = m\ e$ is the $\mathscr{E}$-$\mathscr{M}$ factorization of ev_X, e is the reflection of X in Cog_J.*
3. *X is in Cog_J if and only if X admits a morphism in $\mathscr{M}$ to a power of J.*

Proof. For the first statement, $pr_{id_J}\ ev_J = id_J$ shows that ev_J is split monic and hence is in $\mathscr{M}$ (since $\mathscr{M}$ contains all split monics for any image factorization system). For the second statement, recall the double-dualization monad $\mathbf{D}_J$ of Example 1.6. Observe that $ev: id \to D_J$ is the unit of this monad. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "e"] & \bar{X} \arrow[r, "m"] \arrow[dr, "ev_{\bar{X}}"'] & D_J X \\ & & D_J \bar{X} \arrow[u, "m^{\#}"'] \end{tikzcd}$$

Then $m^{\#}\ ev_{\bar{X}} = m \in \mathscr{M} \Rightarrow ev_{\bar{X}} \in \mathscr{M}$ since this law holds for any image factorization system. Thus $\bar{X} \in Cog_J$. For the universal property, observe that the perimeter of the diagram below commutes

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[dd, "f"'] & & \bar{X} \arrow[d, "m"] \arrow[ddll, "\psi"'] \\ & & D_J X \arrow[d, "D_J f"] \\ Y \arrow[rr, "ev_Y"'] & & D_J Y \end{tikzcd}$$

because ev is a natural transformation. The desired unique ψ then exists by diagonal fill-in. The last statement is clear since every full $\mathscr{E}$-reflective subcategory is closed under products and $\mathscr{M}$-subobjects. □

We are now ready for an important definition and theorem.

Definition 5.6. The $T0$ kernel. Let $\mathbf{T}$ be a taut monad of sets. Provide **T-Mod** with the image factorization system $\mathscr{E}$ = surjective maps, $\mathscr{M}$ = monic initial maps. The $T0$-**kernel** of $\mathbf{T}$, denoted $T0_{\mathbf{T}}$, is defined as Cog_S where S is the Sierpinski object of **T-Mod**.

Theorem 5.7. *In the context of the previous definition, the following hold.*
1. *The support topology functor* Φ : **T-mod** $\to$ **Top** *maps* $T0_{\mathbf{T}}$ *isomorphically onto a full subcategory of* $T0$*-spaces.* $\Phi(S)$ *is the Sierpinski space.*
2. $T0_{\mathbf{T}}$ *contains all discrete topological spaces.*
3. *As a category,* $T0_{\mathbf{T}}$ *is small bicomplete.*
4. *There is a natural closed monoidal structure on* $T0_{\mathbf{T}}$. *A map* $X \otimes Y \to Z$ *is continuous if and only if* $X \times Y \to Z$ *is separately continuous. The topology on the function space* $[X \to Y]$ *of all continuous functions from* X *to* Y *is at least as fine as pointwise convergence.*

Proof. If $\omega\ \sigma\sigma\ True$, then $\omega \in \mathscr{Z}$ by the definition of $\sigma\sigma$. This shows that $\{True\}$ is open in S. As $\eta_2(True)\ \sigma\sigma\ False$ but $\eta_2(True) \notin T\{False\}$ because η is taut, the only open neighborhood of $False$ is 2. Thus $\mathscr{T}_{\sigma\sigma}$ is the topology of the Sierpinski space.

Now let (X, ξ) be a T-Model. Let U be ξ-open with inclusion $i : U \to X$ and let $\chi_U : X \to 2$ be the characteristic function of U. Consider the square

$$\begin{tikzcd} TU \arrow[r, "T!"] \arrow[d, "Ti"'] & T_1 \arrow[d, "T\,(True)"] \\ TX \arrow[r, "T\chi U"'] & T_2 \end{tikzcd}$$

which is a pullback because T is taut. We next show that χ_U is a $\mathbf{T}$-model map. Suppose $\omega \, \xi \, x$. If $x \in U$ then $\omega \in TU$ so that $(T\chi_U)\omega \in \mathscr{T}$. Thus $(T\chi_U)\omega \, \sigma\sigma \, \chi_U x$ holds in this case. It also holds if $x \notin U$ because $(T\chi_U)\ \omega\ \sigma\sigma\ False$ holds for all ω. Conversely let $f : (X, \xi) \to S$ be admissible in **T-Mod**. Let $U = f^{-1}(True)$ so that $f = \chi_U$. We claim U is open. To this end, let $\omega \, \xi \, x$ with $x \in U$. By hypothesis, $(T\chi_U)\omega \, \sigma\sigma \, True$, so $(T\chi_U)\omega \in \mathscr{T}$. Thus, by the pullback property, $\omega \in TU$ as desired. We have shown, so far, that for $\mathbf{T}$-models generally, the S-valued morphisms are precisely the characteristic functions of open sets in the support topology.

Now let $(Y, \theta\theta) \in T0_{\mathbf{T}}$. By definition, the set of all $\mathbf{T}$-model maps from $(Y, \theta\theta)$ to S is an initial family. Claim:

For $f : (X, \xi) \to (Y, \theta\theta)$, (X, ξ) a $\mathbf{T}$-model, $(Y, \theta\theta) \in T0_{\mathbf{T}}$, f is admissible $\Leftrightarrow$ f is continuous.

The claim is true because

$$\begin{aligned} & f \text{ is a } \mathbf{T}\text{-map} \\ \Leftrightarrow\ & \forall \theta\theta\text{-open } U,\ \chi_U f \text{ is a } \mathbf{T}\text{-map} \\ \Leftrightarrow\ & \forall \theta\theta\text{-open } U,\ \chi_{f^{-1}U} \text{ is a } \mathbf{T}\text{-map} \\ \Leftrightarrow\ & \forall \theta\theta\text{-open } U,\ f^{-1}U \text{ is } \xi\text{-open.} \end{aligned}$$

In particular, the restriction of Φ to $T0_{\mathbf{T}}$ is a full functor. By virtue of being over $\mathscr{S}$, Φ is faithful. The values of Φ are $T0$-spaces because the maps ev_X are monic. To complete the proof of the first statement we need only show that if (X, ξ), $(Y, \theta\theta)$ in $T0_{\mathbf{T}}$ have the same topology then $\xi = \theta\theta$. But this is obvious since $(\chi_U : U \text{ open})$ is an initial family for both structures.

We turn to the proof of the second statement. Let X be a set. Let $\delta\delta_X$ be the smallest relation on X which is a $\mathbf{T}$-model, that is, let $f : (X, \delta\delta) \to (Y, \theta\theta)$ be an initial family as $(Y, \theta\theta)$ and f range over all $\mathbf{T}$-models and functions. [Note: It is not hard to show that $\delta\delta_X = \eta_X^\circ$)]. $\Phi(Y, \theta\theta)$ is discrete since all functions to S are continuous. In particular, $ev_{(X,\ \delta\delta_X)}$ is monic. Thus the reflection of $(X, \delta\delta_X)$ into $T0_{\mathbf{T}}$ of Proposition 5.5 has form $(X, \overline{\delta\delta_X})$ with $\overline{\delta\delta_X} \subset \delta\delta_X$. By the reflection property, all functions $(X, \overline{\delta\delta_X}) \to S$ are admissible, so $\Phi(X, \overline{\delta\delta_X})$ is discrete.

As **T-Mod** is (small) complete and co-complete, being topological over $\mathscr{S}$, and since $T0_{\mathbf{T}}$ is a full reflective subcategory, the latter is again complete and co-complete.

Warning: $T0_{\mathbf{T}}$ is closed under the limits of **T-Mod** but the colimits will not, in general, even be over $\mathscr{S}$.

We now prove the last statement. This follows from much more general principles. Consider an arbitrary category topological over $\mathscr{S}$, $\mathscr{A} \to \mathscr{S}$. We will write $\mathscr{A}$-objects as (X, α) where X is a set and α is the "$\mathscr{A}$-structure". Given (X, α), (Y, β) in $\mathscr{A}$, let $in_x : Y \to X \times Y$ be the function $y \mapsto (x, y)$ for each $x \in X$ and $in_y : X \to X \times Y$ similarly. Define $(X, \alpha) \otimes (Y, \beta) = (X \times Y, \alpha \otimes \beta)$ as the final lift of all in_x, in_y. Thus a function $f : X \times Y \to Z$ is admissible $(X, \alpha) \otimes (Y, \beta) \to (Z, \gamma)$ if and only if f is admissible in each variable separately. The unit object I is the 1-element set equipped with its "discrete" structure out of which all maps are admissible. That this gives a monoidal category is routine.

Define the function space $[(Y, \beta) \to (Z, \gamma)]$ as the set of all $\mathscr{A}$-admissible maps with the initial substructure of the power $(Z, \gamma)^Y$. Thus a morphism $\psi : (X, \alpha) \to [(Y, \beta) \to (Z, \gamma)]$ is admissible if and only if for all $y \in Y$, $\lambda_x \psi(x)(y) : (X, \alpha) \to (Z, \gamma)$ is admissible. The remaining details to establish the closed monoidal structure are routine.

To apply this to the context at hand, **T-Mod** is a closed monoidal category. As $T0_{\mathbf{T}}$ is a full subcategory closed under products and initial substructures, it is closed under the function spaces of **T-Mod**. The tensor product of two objects in $T0_{\mathbf{T}}$ is then obtained by reflecting the tensor product at the **T**-model level. By the universal property of reflections, maps out of the tensor product still correspond to separately continuous maps. $(1, \delta\delta_1)$ provides the unit for tensor. The remaining details that $T0_{\mathbf{T}}$ is a closed monoidal category are routine.

Finally, the projection functions $pr_y : [(Y, \theta) \to (Z, \zeta)] \to (Z, \zeta)$ are admissible, hence continuous, so the topology must be at least as fine as pointwise convergence. □

Example 5.8. Posets with upper topology.

We will show that $T0_{\mathbf{id}}$ is the category of posets and that Φ assigns the topology in which the open sets are the upper sets. As previously discussed, **id-Mod** is preordered sets. S is the poset $False < True$, providing that we interpret $\xi \subset X \times X$ as $\geqslant$ rather than $\leqslant$. It is then immediate that the open sets are the upper sets. Products in **id-Mod** are by pointwise order and initial substructures are the usual sub-preordered sets – the same preorder but restricted to the subset.

On the one hand, every sub-preordered set of a product of posets is a poset. Conversely, let $(X, \leqslant)$ be a poset. Let $\Omega(X, \leqslant)$ be the set of upper sets of $(X, \leqslant)$. Then $ev_{(X,\leqslant)} : X \to S^{\Omega(X,\leqslant)}$ is initial in **id-Mod**. To see this, let $f : (Y, \leqslant) \to X$ be such that $ev_{(X,\leqslant)} f$ is monotone and let $y_1 \leqslant y_2 \in (Y, \leqslant)$. By hypothesis, $pr_{\uparrow f y_1} ev_{(X,\leqslant)} f = \chi_{\uparrow f y_1} f = \chi_{f^{-1}(\uparrow f y_1)}$ is monotone so $f^{-1}(\uparrow f y_1)$ is an upper set which contains y_1, and hence y_2. But then $f y_1 \leqslant f y_2$, so f is monotone. To show $(X, \leqslant)$ is in $T0_{\mathbf{id}}$, we must also show that $ev_{(X,\leqslant)}$ is monic, that is, if $x_1 \neq x_2$ then some monotone

$f:(X,\leqslant)\to S$ exists with $fx_1\neq fx_2$. The following does the job:

$$fx=\begin{cases} \mathit{False} & x\leqslant x_1<x_2,\\ \mathit{True} & x\nleqslant x_1<x_2,\\ \mathit{False} & x_1\nleqslant x_2\geqslant x,\\ \mathit{True} & \text{otherwise.}\end{cases}$$

So far we have seen that $T0_{\mathbf{id}}$ is posets and that $\chi_U:(X,\leqslant)\to S$ forms an initial family as U ranges over the upper sets of $(X,\leqslant)$. Since $x\leqslant y\Leftrightarrow$ for all upper sets U, $\chi_U\,x\leqslant\chi_U\,y$ we see that $\leqslant$ is the specialization ordering of the space $\Phi(X,\leqslant)$. It is well known that not every $T0$-topology is a topology of upper sets [11, p. 45], so $T0_{\mathbf{id}}$ is not all $T0$-spaces. Another way to see this is to look at products since the upper sets of the coordinatewise ordering is a finer topology than pointwise convergence.

The closed monoidal structure is the usual cartesian closed one (observe that there is no distinction between separate and joint monotonicity for posets).

Example 5.9. $T0_{\boldsymbol{\beta}}$ is $T0$-spaces. For all topological spaces X, ev_X is an initial map (definition of continuity) and is monic exactly when X is $T0$. Notice that the function spaces have precisely the topology of pointwise convergence.

Example 5.10. $T0_{\mathbf{F}}$ is also $T0$-spaces. Topological spaces may be described by filter convergence. This amounts to a functor $\mathbf{Top}\to\mathbf{F}\text{-}\mathbf{Mod}$ over $\mathscr{S}$. It is well known that continuity is characterized by preserving filter convergence and that a family f_i is initial if and only if $\mathscr{F}\to x\Leftrightarrow f_i(\mathscr{F})\to f_i x$ for all i. Thus **Top** is an initially closed full subcategory of **F**-models.

If $S=(2,\sigma\sigma)$ is the Sierpinski object of **F**, $\sigma\sigma$ is the filter convergence relation of the Sierpinski space. Thus S belongs to **Top**. It is now clear that $T0_{\mathbf{F}}=T0_{\boldsymbol{\beta}}=T0$-spaces.

There are a number of further remarks to make here. Every $\boldsymbol{\beta}$-algebra (= compact Hausdorff space) is in **Top**. By contrast, no **F**-algebra (X,ξ) with more than two elements is the convergence relation of a topology. For suppose otherwise. Let $x\neq y$ in X, $z=\xi(\mathit{prin}(\{x,y\}))$. Assume, say, that $x\neq z$. Every neighborhood of z also contains x whereas the topology must be Hausdorff since filters converge uniquely, the desired contradiction.

At this point we should clarify some potential confusion about continuous lattices. One the one hand, a continuous lattice is an **F**-algebra so, as has just been established, its structure map, the lim-inf function $\xi(\mathscr{F})=\bigvee_{F\in\mathscr{F}}\bigwedge F$, cannot be the convergence relation of a topology. On the other hand, $T0_{\mathbf{F}}$ is the category of $T0$ spaces and continuous maps whose structure relations *are* filter convergence. In particular, the injective $T0$ spaces are [22] the continuous lattices. How can this apparent contradiction be resolved?

If X is a continuous lattice with Scott-open sets $\Omega(X)$ then, qua $T0$ space, the **F**-model structure relation $\xi:FX\to X$ makes the family of all $(\chi_U:X\to(2,\sigma\sigma)$ $(U\in$

$\Omega(X)$) initial. This is just the definition of $T0_{\mathbf{F}}$. Thus

$$\begin{aligned}\mathscr{F}\,\xi\, x &\Leftrightarrow \forall U \in \Omega(X)\ (F\chi_U)\mathscr{F}\,\sigma\sigma\chi_U(x)\\ &\Leftrightarrow \forall U \in \Omega(X)\ \{B\subset 2 : \chi_U^{-1}(B)\} \in \mathscr{F}\,\sigma\sigma\chi_U(x)\\ &\Leftrightarrow \forall U \in \Omega(X)\ x \in U \Rightarrow U \in \mathscr{F}\\ &\Leftrightarrow \forall U \in \Omega(X)\ \mathscr{F}\ \text{Scott-converges to}\ x.\end{aligned}$$

In short, it amounts to this: The category of continuous lattices and morphisms which preserve directed suprema is a full subcategory of $T0_{\mathbf{F}}$ consisting of the injective objects. The category of continuous lattices and morphisms which preserve directed suprema and arbitrary infima is the category of **F**-algebras. These **F**-algebras are *not* in $T0_{\mathbf{F}}$. For a continuous lattice, if ξ is the algebra structure map and if ξ is $T0_{\mathbf{F}}$ structure relation, it is well known that

$$\mathscr{F}\,\xi\, x \quad\Leftrightarrow\quad x \leqslant \xi(\mathscr{F}).$$

One might suspect that Scott continuity implies the preservation of the lim–inf operation, but this is not so. From [17, Theorem 6.3], the morphisms preserving algebra structure map (i.e. lim–inf) are precisely the perfect maps. Sierpinski-valued perfect maps are *compact* open sets and an open subset of an injective $T0$-space need not be compact.

Example 5.11. Let $\mathbf{P}_{Fin}$ be the finite subsets submonad of $\mathbf{P}$. This a submonad of $\mathbf{F}$ so is a taut monad. Let's explore the $T0$ kernel.

$T0_{\mathbf{P}_{Fin}}$ is subspaces of box-powers of S. Let (X, ξ) be an initial substructure of S^I, a typical object in $T0_{\mathbf{P}_{Fin}}$. For $J \subset I$, let B_J be the "box"

$$B_J = \prod_{i\in I} C_i, \quad C_i = \begin{cases} \{True\}, & i \in j,\\ \quad 2, & i \notin j.\end{cases}$$

The B_J form a base for the box topology on S so we must show that sets of form $X \cap B_J$ form a base for $\Phi(X, \xi)$. Write an element $x \in X$ as a tuple $x = (x_a : a \in I)$. By the definition of S and initial families in $\mathbf{P}_{Fin}$-**Mod**, we have

$$\begin{aligned}\{x_1,\ldots,x_n\}\,\xi\, x &\Leftrightarrow \forall a \in I\ \ \{x_{1a},\ldots,x_{na}\}\sigma\sigma\, x_a\\ &\Leftrightarrow \forall a \in I\ x_a = 1 \Rightarrow x_{1a} = \cdots = x_{na} = 1.\end{aligned}$$

It follows that U is ξ-open if and only if

$$x \in U \text{ and } \forall i = 1,\ldots,n,\ (x_{ia} = 1 \text{ whenever } x_a = 1) \Rightarrow \{x_1,\ldots,x_n\} \in U.$$

To see that $B_J \cap X$ is ξ-open, let $x_1,\ldots,x_n,\ x \in X,\ \{x_1,\ldots,x_n\}\,\xi\, x,\ x \in U,\ j \in J$. As $x_j = 1,\ x_{1j} = \cdots = x_{nj} = 1$. As $j \in J$ is arbitrary, $\{x_1,\ldots x_n\} \subset B_J$. Conversely, let U be ξ-open, $x \in U$. Let $J = \{i : x_i = 1\}$. Then $x \in B_J \cap X$. Let $y \in B_J \cap X$. Then $\{y\}\,\xi\, x$

because $x_i = 1 \Rightarrow i \in J \Rightarrow y_i = 1$. Thus $y \in U$ as desired. Note, in particular, that the product in this category of $T0$-spaces is the box product.

6. Monad maps and $T0$ spaces

In this section, we explore the effect of monad maps on the constructions of the preceding section.

Proposition 6.1. *Let* $\mathbf{W}, \mathbf{T}$ *be monads and let* $\lambda : \mathbf{W} \to \mathbf{T}$ *be a monad map. The following statements hold.*
1. $\Lambda : \mathbf{T}\text{-}\mathbf{Mod} \to \mathbf{W}\text{-}\mathbf{Mod}$, $(X, \xi) \mapsto (X, \xi\, \lambda_X)$ *is a well-defined functor over* $\mathscr{S}$.
2. Λ *preserves all initial families.*

Proof. $(\xi\, \lambda_X)\eta_X = \xi(\lambda_X \eta_X) = \xi\, \eta_X \supset id_X$ shows the reflexive law. The transitive law is delicate – somehow all three inequalities one needs manage to go in the right way. The details are as follows:

$$
\begin{aligned}
&(\xi\, \lambda_X) W(\xi\, \lambda_X) \\
&\quad \supset \xi\, \lambda_X W\xi\, W\lambda_X \quad \text{(Lemma 4.5 (1))} \\
&\quad \supset \xi\, T\xi\, \lambda_{TX} W\lambda_X \quad \text{(Lemma 4.5 (4))} \\
&\quad \supset \xi\, \mu_X \lambda_{TX} W\lambda_X \quad \text{(transitive law for } \xi\text{)} \\
&\quad = (\xi\, \lambda_X)\ \mu_X \quad (\mu\text{-form of } \lambda\text{-monad-map law}).
\end{aligned}
$$

This construction is functorial over $\mathscr{S}$ since λ is natural, so the first statement is proved.

For the second one, let $(X, TX \xrightarrow{\xi} X) \xrightarrow{f_i} (Y_i, TY_i \xrightarrow{\xi_i} Y_i)$ be initial in $\mathbf{T}$-**Mod** and let $\omega \in WX$, $x \in X$ be such that $(Wf_i)\omega(\xi_i\, \lambda_{X_i})\ f_i x$ for all i. Equivalently, $(\lambda_{X_i} Wf_i)\omega\, \xi_i\, f_i x$ for all $i \Leftrightarrow Tf_i\, \lambda_X \omega\, \xi_i\, f_i x$ for all i so that $\lambda_x \omega\, \xi\, x$ as ξ is an initial lift. But this is equivalent to $\omega(\lambda_X \xi)x$ as desired. $\square$

Proposition 6.2. *Let* $\mathbf{W}, \mathbf{T}$ *be monads and let* $\lambda : \mathbf{W} \to \mathbf{T}$ *be a taut monad map with* $\Lambda : \mathbf{T}\text{-}\mathbf{Mod} \to \mathbf{W}\text{-}\mathbf{Mod}$ *as in the preceding proposition. Then the following statements hold.*
1. *For* (X, ξ) *a* $\mathbf{T}$*-model,* $U \subset X$, U ξ*-open* $\Rightarrow$ $U(\xi\, \lambda_X)$*-open.*
2. *If* $\mathbf{W}, \mathbf{T}$ *are taut monads then* Λ *preserves the Sierpinski object and maps* $T0_{\mathbf{T}}$ *onto* $T0_{\mathbf{W}}$.

Proof. Let U be ξ-open. If $\omega(\xi\, \lambda_X)x \Leftrightarrow (\lambda_X \omega)\, \xi\, x$ and if $x \in U$ then $\lambda_X\, \omega \in TU$. As λ is taut, $\omega \in WU$. This shows U is $(\xi\, \lambda_X)$-open. If $(2, \sigma\sigma)$ is the Sierpinski object of $\mathbf{T}$ we must show $(2, \sigma\sigma\, \lambda_2)$ is the Sierpinski object for $\mathbf{W}$. For any ω in $W2$, $(\lambda_2 \omega)$ $\sigma\sigma$ *False*, so ω $(\sigma\sigma\lambda_2)$ *False*. Also, because λ is taut, ω $(\sigma\sigma\, \lambda_2)$*True* $\Leftrightarrow (\lambda_2 \omega)\, \sigma\sigma$

$True \Leftrightarrow \lambda_2\omega \in \mathscr{L}_T \Leftrightarrow \omega \in \mathscr{L}_W$. As Λ preserves initial families, initial substructures of powers of S are preserved so $T0_{\mathbf{T}}$ is mapped into $T0_{\mathbf{W}}$. If, in $T0_{\mathbf{W}}$, (E, ϑ) is an initial substructure of the $\mathbf{W}$-power $(\Lambda S)^I$, let ξ be the initial substructure of E in the $\mathbf{T}$-power S^I. Then, similarly, Λ maps (E, ξ) to (E, ϑ) so Λ is onto as claimed. $\square$

Corollary 6.3. *If* $\mathbf{T}$ *is any taut monad and if* (X, ξ) *is in the* $T0$ *kernel then there is at least one* $T0$ *topology on* X *with filter convergence relation* ϑ *such that* $\omega\, \xi\, x \Leftrightarrow (supp(\omega))\, \vartheta\, x$.

Example 6.4. $\boldsymbol{\beta}$ is a submonad of $\mathbf{F}$ and all such submonads are taut by Proposition 3.12 so the inclusion $\iota : \boldsymbol{\beta} \to \mathbf{F}$ is a taut monad map to which the preceding proposition applies. We saw earlier in Examples 5.3, 5.10 that if $\xi : FX \to X$ is the lim–inf operation of a continuous lattice, then ξ is *not* the filter convergence relation of a topology. But $\xi\iota_X$ – the restriction of ξ to ultrafilters – must be the convergence relation of a compact Hausdorff space. By the proposition, the compact T2 topology is finer than the Scott topology. This sounds like it should be the Lawson topology. This is indeed so. For details, see [23, 11].

Proposition 6.5. *Let* $\mathbf{W}, \mathbf{T}$ *be taut monads and let* $\lambda : \mathbf{W} \to \mathbf{T}$ *be a taut monad map with each* λ_X *surjective. Then the following statements hold.*
1. $\Lambda : \mathbf{T}\text{-}\mathbf{Mod} \to \mathbf{W}\text{-}\mathbf{Mod}$ *is a full subcategory.*
2. *For every T-model* (X, ξ), $\Phi(X, \xi) = \Phi(X, \xi\,\lambda_X)$ *are the same topology.*
3. Λ *maps* $T0_{\mathbf{T}}$ *isomorphically onto* $T0_{\mathbf{W}}$.

Proof. It is obvious that if $\xi\,\lambda_X = \vartheta\,\lambda_X$ then $\xi = \vartheta$, so to prove the first statement consider

$$\begin{tikzcd}
WX \arrow[r, "\lambda_X"] \arrow[d, "Wf"'] \arrow[dr, phantom, "\text{(A)}"] & TX \arrow[r, "\xi"] \arrow[d, "Tf"] & X \arrow[d, "f"] \\
WY \arrow[r, "\lambda_Y"'] & TY \arrow[r, "\vartheta"'] & Y
\end{tikzcd}$$

It is enough to show that if f is a $\mathbf{W}$-map $(X, \xi\,\lambda_X) \to (Y, \vartheta\,\lambda_Y)$ then it is a $\mathbf{T}$-map $(X, \xi) \to (Y, \vartheta)$. Let $\omega \in TX, y \in Y$ with $(Tf\omega)\ \vartheta y$. To show: $\exists x \in X,\ \omega\,\xi\,x$, $fx = y$. As λ_X is surjective there exists $\bar{\omega} \in WX$, $\lambda_X\bar{\omega} = \omega$. We have $(\lambda_Y Wf)\bar{\omega} = Tf\omega$ so $(Wf\bar{\omega})(\vartheta\,\lambda_Y)y$. As f is a $\mathbf{W}$-map, $\exists x \in X$ with $\omega\,\xi\,x$, $fx = y$ as desired.

In view of the preceding proposition, to establish the second statement we need only show that if $U \subset X$ is $(\xi\,\lambda_X)$-open then U is ξ-open. Let $\omega \in TX$, $x \in U$ with $\omega\,\xi\,x$. Let $\lambda_X\bar{\omega} = \omega$. As $\bar{\omega}(\xi\,\lambda_X)x$, $\bar{\omega} \in WU$. But then $\omega = \lambda_X\bar{\omega} \in TU$ as needed.

The third statement is obvious from the result just proved and the previous proposition. $\square$

Corollary 6.6. *Let* $\mathbf{T}$ *be a taut monad. Then there exists a taut submonad* $\mathbf{G}$ *of the filter monad* $\mathbf{F}$ *with* $T0_{\mathbf{T}}$, $T0_{\mathbf{G}}$ *isomorphic over* $\mathscr{S}$.

Proof. Immediate from the preceding, using Propositions 3.6, 3.12. $\square$

7. The open filter monad for $T0_T$

Day and Wyler [5, 23] (see also [7]) showed that continuous lattices and morphisms which preserve directed suprema and arbitrary infima are the algebras of the *open filter monad* in the category **T0** of $T0$ spaces. This monad assigns to a $T0$-space $(X, \Omega(X))$ the set $F^{\circ}X$ of all filters on the poset $\Omega(X)$ (i.e. non-empty families of open sets closed under finite intersections and supersets) with open base $\{\Box U : U \in \Omega(X)\}$ where $\Box U = \{\mathscr{F} : U \in \mathscr{F}\}$. The rest of the monad structure is defined analogously to $\mathbf{F}$ and all requisite morphisms are shown to be continuous (which we actually show in this section as a special case). We will use the interior symbol $–F^{\circ}$ as opposed to $F–$ to distinguish open filter monads from the filter monad in $\mathscr{S}$.

T0 is a poset-enriched category via the specialization ordering $x \leqslant y \Leftrightarrow x \in cls(y)$. This means that the set of continuous maps between two $T0$ spaces is a poset under the pointwise specialization order and composition on either side with a continuous map is a monotone operation. A monad $\mathbf{T}$ in a poset enriched category is said to be of **Kock–Zöberlein type** if $f \leqslant g \Rightarrow Tf \leqslant Tg$ and if $\eta_{TX} \leqslant T\eta_X$ for all X. The work of [14] when specialized from a more general 2-categorical setting to poset-enriched categories, shows that the algebras of a Kock–Zöberlein monad always form a (not necessarily full) subcategory of the base category; indeed, if (X, ξ) is an algebra then $\xi\eta_X = id_X$ whereas $\eta_X \xi \leqslant id_{TX}$, so that ξ is the right adjoint of η_X and hence is unique.

Escardó [7] took this further. Say that $f : X \to Y$ is a T-**embedding** if $Tf : TX \to TY$ has a reflective left adjoint $g : TY \to TX$, that is, $g(Tf) = id_{TY}$, $(Tf)g \leqslant id_{TX}$. Escardó showed that the subcategory of $\mathbf{T}$-algebras consists of the objects which are injective with respect to T-embeddings and that this specialized to Scott's theorem about injective $T0$ spaces for the open filter monad in **T0** which is indeed of Kock–Zöberlein type. In the bargain, he also gave a new proof that the algebras *are* continuous lattices based on this setup.

It is obvious that every submonad of the open filter monad (or any other Kock–Zöberlein monad) is Kock–Zöberlein. In [9] many examples of these are given.

It seems natural to try to extend this perspicuous theory of Kock–Zöberlein monads to a suitable open filter monad in $T0_{\mathbf{T}}$. Regrettably, very few such examples are known to us at this time. In the balance of this short section we define the open filter monad in $T0_{\mathbf{T}}$ and give a specific example of a taut monad $\mathbf{T}$ for which this open filter monad is not Kock–Zöberlein.

Definition 7.1. The open filter monad induced by a taut monad $\mathbf{T}$ in $\mathscr{S}$.

This monad is defined in the base category $T0_{\mathbf{T}}$. In this category the morphisms, which are **T**-model maps are also the same as continuous maps between $T0$ spaces because of Theorem 5.7. The monad is defined as follows:

$$F^{\circ}X = \{\mathscr{F} \subset 2^{\Omega(X)} : \mathscr{F} \text{ is a filter on } \Omega(X)\},$$

$$\eta_X(x) = \{U \in \Omega(X) : x \in U\},$$

$$\text{For } \alpha : X \to F^{\circ}X, \alpha^{\#}(\mathscr{F}) = \{V \in \Omega(Y) : \{x : V \in \alpha(x)\} \in \mathscr{F}\}.$$

Here $F^{\circ}X$ is an initial substructure of the categorical product $S^{\Omega(X)}$ where S is the Sierpinski object. Since α is continuous and for $V \in \Omega(Y)$, $pr_V\alpha = \chi_{\{x : V \in \alpha(x)\}}$, $\{x : V \in \alpha(x)\}$ is always in $\Omega(X)$ so the definition of $\alpha^{\#}$ makes sense. For $U \in \Omega(X)$, $pr_U\, \eta_X = \chi_U$, so η_X is continuous. Similarly, for $V \in \Omega(Y)$, $pr_V\alpha^{\#} = \Box\{x : V \in \alpha(x)\}$ is open in $F^{\circ}X$ because by 5.7 the topology is finer than pointwise convergence (and the next example shows that the standard open filter spaces (with base all $\Box U$ for $U \in \Omega(X)$) carry the topology of pointwise convergence). The remaining proof that this gives a monad is like the set case.

Example 7.2. The open filter monad in $T0_{\mathbf{F}}$

Let $F^{\circ}X$ be the set of open filters on a $T0$ space X, let $i : F^{\circ}X \to S^{\Omega(X)}$ be inclusion and let $\mathscr{T}$ be any topology on $F^{\circ}X$. Then

$$\forall U \in \Omega(X), \Box U \in \mathscr{T} \Leftrightarrow \forall U \in \Omega(X), \chi_{\Box U} : (F^{\circ}X, \mathscr{T}) \to S \text{ is continuous,}$$

$$\Leftrightarrow \forall U \in \Omega(X), pr_U i : F^{\circ}X \to S^{\Omega(X)} \to S \text{ is continuous.}$$

It follows immediately that the topology with all $\Box U$ as subbase (base, actually, since $\Box(U \cap V) = \Box U \cap \Box V$) is precisely the topology of pointwise convergence. Thus for **F** the filter monad in $\mathscr{S}$, the products in $T0_{\mathbf{F}} = \mathbf{T0}$ have the usual product topology of pointwise convergence and the open filter monad of the previous definition is the usual one.

Example 7.3. The open filter monad in $T0_{\mathbf{T}}$ need not be Kock–Zöberlein.

Take **T** to be the identity monad of Example 5.8 with $T0_{\mathbf{id}}$ the cartesian-closed category of posets and monotone maps as $T0$ spaces with the upper sets as the open sets. In this category, products are pointwise and initial substructures are the usual inherited partial orderings. Thus $F^{\circ}X$ has the usual inclusion ordering on filters. We have

$$\eta_{F^{\circ}X}(\mathscr{F}) = \{\bar{U} \in \Omega(F^{\circ}X) : \mathscr{F} \in \bar{U}\},$$

$$F(\eta_X)\mathscr{F} = \{\bar{U} \in \Omega(F^{\circ}X) : \eta_X^{-1}\bar{U} \in \mathscr{F}\}.$$

For fixed $\mathscr{F}$, consider $\bar{U} = \uparrow \mathscr{F} \in \Omega(F^{\circ}X)$. As $\mathscr{F} \in \bar{U}$, $\bar{U} \in \eta_{FX}(\mathscr{F})$. Now calculate

$$\begin{aligned}\eta_X^{-1}(\bar{U}) &= \{x \in X : \mathscr{F} \subset prin(x)\}\\ &= \left\{x \in X : x \in \bigcap \mathscr{F}\right\}\\ &= \bigcap \mathscr{F}\end{aligned}$$

If X is discrete, $\mathscr{F}$ can be any filter on X so $\bigcap \mathscr{F} \in \mathscr{F}$ is not always true. Thus $\bar{U} \in \eta_{F^{\circ}X}$ but $\bar{U} \notin F^{\circ}\eta_X$ so this open filter monad is not Kock–Zöberlein.

8. Conclusions

It has been shown that tautness, a category-theoretic constraint involving pullbacks, adequately axiomatizes monads (considered as collection data structures) for which collection membership is decidable by a functional algorithm. This is achieved in the form of a particular two-element Eilenberg–Moore algebra for the monad called a support classifier and this is unique when it exists. In their development of monad theory, the functional programming community has not yet looked at Eilenberg–Moore algebras. We have interpreted the structure map of such algebras as a fold operator.

Since the finitary taut monads are precisely the collection monads, taut monads are a first step in axiomatizing general collections (such as might be used in lazy evaluation). Possibly, a general collection monad should be defined as a taut monad for which each support filter is principal.

Taut monads admit a canonical monad map to the filter monad. This singles out the filter monad as having a universal role (we use the word “universal” in the sense of an object in which all similar objects can be embedded). This is interesting, since domain theorists have not spotlighted continuous lattices in this way among the many different types of domain that have been considered by theoretical computer scientists.

There is a plenitude of taut monads and it has been shown that these induce submonads of the filter monad so these abound too. Many monads are not taut, however – for example, those which do not have a two-element algebra (e.g. vector spaces over any field with more than two elements).

For the representation of taut monads in $T0$ spaces, much remains to be done.

It is remarkable that the support topology functor Φ from T-models to topological spaces exists for any taut functor T and gives the right answer both for continuous lattices and for general topological spaces. We have sought constructions of full subcategories of $T0$ spaces which are in turn full subcategories of **T-Mod** for a taut monad **T** in $\mathscr{S}$. This approach uses the long-dormant theory of relational models of a monad. For example, regarded as $T0$ spaces with the Scott topology, continuous lattices described by filter convergence are a full subcategory of the relational models of the filter monad. Nothing like this was noticed by Day [5] even though he considered relational models of that monad in his seminal paper.

We have not yet discovered a useful characterization of the categories $T0_{\mathbf{T}}$. That $\mathbf{T}$ can be chosen as a submonad of the filter monad is a first step but since the filter monad and its ultrafilter submonad both give rise to the same category, "submonad of the filter monad" is not the final invariant. Something about "reduced" submonads in which the Sierpinski object sits tightly will be necessary.

Further work is needed to fully understand the use of Kock–Zöberlein monads in connection with $T0_{\mathbf{T}}$. For example, this category might be poset-enriched using some natural order other than the specialization order, so that the counterexample of 7.3 may be misleading.

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