

The tensor product of commutative monoids

We work throughout in the category **Cmd** of commutative monoids. In other words, all the monoids we meet are commutative, and consequently we say ‘monoid’ in place of the longer ‘commutative monoid’. However, occasionally we will emphasize the commutativity.

Almost every monoid A we meet will be written multiplicatively, that is we write

$$xy$$

for the compound of $x, y \in A$. In places we write something like

$$x \cdot y$$

if this helps to avoid confusion. The unit (neutral element) is 1 or 1_A if the subscript helps to avoid confusion.

One monoid that we use is written additively. This is the set \mathbb{N} of natural number under addition. As we will see, this plays a special role in the constructions we describe.

In the usual way for monoids B, C we write

$$[B, C]$$

for the set of morphism from B to C . Our first job is to turn this external set into an internal object of **Cmd**. To distinguish between the set and the monoid produce we introduce some notation.

0.1 DEFINITION. For commutative monoids B, C let

$$(B \Rightarrow C)$$

be the set $[B, C]$. For $g, h \in (B \Rightarrow C)$ let

$$g \cdot h : B \longrightarrow C$$

be the function given by

$$(g \cdot h)b = (gb)(hb)$$

for $b \in B$. Let

$$1_{B \Rightarrow C} : B \longrightarrow C$$

be the constant function given by

$$1_{B \Rightarrow C}b = 1_C$$

for $b \in B$. ■

It is routine to check that the constructed function $g \cdot h$ is a morphism, but this does depend on the commutativity of C . Thus for $b_1, b_2 \in B$ we have

$$\begin{aligned} (h \cdot g)(b_1 b_2) &= (h(b_1 b_2))(g(b_1 b_2)) \\ &= (hb_1)(hb_2)(gb_1)(gb_2) \\ &= (hb_1)(gb_1)(hb_2)(gb_2) = ((h \cdot g)b_1)((h \cdot g)b_2) \end{aligned}$$

where the third equality depends on the commutativity. A trivial exercise shows that the construction gives an associative and commutative operation on $(B \Rightarrow C)$, and the function $1_{B \Rightarrow C}$ is the unit of this operation.

0.2 LEMMA. *For each pair B, C of commutative monoids the set $(B \Rightarrow C)$ with the furnishings described above is a commutative monoid.*

This construction has the expected functorial properties. Let

$$B_2 \xrightarrow{k} B_1 \qquad C_1 \xrightarrow{l} C_2$$

be a pair of morphisms, and consider the square

$$\begin{array}{ccc} (B_1 \Rightarrow C_1) & \xrightarrow{l \circ -} & (B_1 \Rightarrow C_2) \\ \downarrow - \circ k & & \downarrow - \circ k \\ (B_2 \Rightarrow C_1) & \xrightarrow{l \circ -} & (B_2 \Rightarrow C_2) \end{array}$$

of functions. It is easy to check that each has the indicated source and target, and each is a morphism. Trivially the square commutes, since each $g \in (B_1 \Rightarrow C_2)$ is sent to $l \circ g \circ k$ by either path. We don't need to go into the details of these various calculations, but let us show that the full composite

$$l \circ - \circ k$$

is a morphism. In other words, let's show that

$$l \circ (g \cdot h) \circ k = (l \circ g \circ k) \cdot (l \circ h \circ k)$$

holds for all $g, h \in (B_1 \Rightarrow C_1)$. To do this we evaluate either side at an arbitrary $b \in B_2$. Thus we have

$$\begin{aligned} (l \circ (g \cdot h) \circ k)b &= l((g \cdot h)(kb)) \\ &= l((g(kb))(h(kb))) \\ &= (l(g(kb)))(l(h(kb))) \\ &= ((l \circ g \circ k)b)((l \circ h \circ k)b) = ((l \circ g \circ k) \cdot (l \circ h \circ k))b \end{aligned}$$

to give the required result.

These calculations show that the construction $(- \Rightarrow -)$ is functorial, contravariant in the first input and covariant in the second. In particular, by fixing the first input at an arbitrary monoid B we obtain an endofunctor $(B \Rightarrow -)$ on **Cmd**. Our main aim here is to produce a left adjoint

$$- \otimes B \dashv B \Rightarrow -$$

to this endofunctor. Of course, this will give the tensor product on **Cmd**.

For given monoids A, B, C we wish construct a certain monoid $A \otimes B$ and set up a bijective correspondence

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ [A \otimes B, C] & & [A, B \Rightarrow C] \\ g_b & \longleftarrow & \vdash g \end{array}$$

between the two sets of morphisms. To do this we first obtain a more amenable description of the morphisms g .

Recall that the cartesian produce $A \times B$ of monoids is constructed in the usual fashion as the set of ordered pairs (a, b) with the pointwise operation. This gives the categorical product in **Cmd**, and we may consider the morphisms

$$A \times B \longrightarrow C$$

to C . These are not the arrows that do the job we want done.

0.3 DEFINITION. For commutative monoids A, B, C a function

$$f : A \times B \longrightarrow C$$

is **balanced** (some would say bilinear) if

$$\begin{aligned} f(a_1 a_2, b) &= f(a_1, b) f(a_2, b) & f(a, b_1 b_2) &= f(a, b_1) f(a, b_2) \\ f(1_A, b) &= 1_C = f(a, 1_B) \end{aligned}$$

for all $a_1, a_2, a \in A, b_1, b_2, b \in B$.

Let

$$\mathbf{Bal}[A \times B, C]$$

be the set of all such balanced functions. ■

In other words, if we fix either or the inputs to f the the resulting 1-placed function is a morphism. However, the 2-placed function need not be a morphism (unless one of A or B is trivial).

Notice that we are not trying to produce a new category with balanced functions as the arrows. Indeed, this doesn't make sense since there is no obvious way to compose balanced functions. However, we can form a composite

$$A \times B \xrightarrow{f} C \xrightarrow{h} B$$

of a balanced function f and a morphism h to produce a balanced function.

Many internal adjunctions have some currying-uncurrying as the crucial trick. The one we are looking for here is no different.

0.4 LEMMA. *For commutative monoids A, B, C the two sets*

$$\mathbf{Bal}[A \times B, C] \quad [A, (B \Rightarrow C)]$$

are in bijective correspondence, where this is induced by the currying-uncurrying process.

Proof. In more detail we show that bijective correspondence is given by the two assignments

$$\begin{array}{ccc} f & \longmapsto & f^\bullet \\ \mathbf{Bal}[A \times B, C] & & [A, (B \Rightarrow C)] \\ g_\bullet & \longleftarrow & g \end{array}$$

where

$$g_\bullet(a, b) = gab \quad f^\bullet ab = f(a, b)$$

for each $a \in A, b \in B$. We verify several simple properties. For given f, g we need to show the following.

- (i) For each $a \in A$ the function $f^\bullet a : B \longrightarrow C$ is a morphism.
- (ii) The function $f^\bullet : A \longrightarrow (B \Rightarrow C)$ is a morphism.
- (iii) The function $g_\bullet : A \times B \longrightarrow C$ is a morphism.
- (iv) Both $f^\bullet_\bullet = f$ and $g_\bullet^\bullet = g$ hold.

For (i) we required

$$f^\bullet a(b_1, b_2) = (f^\bullet a b_1)(f^\bullet a b_2) \quad f^\bullet a 1_B = 1_C$$

for all $a \in A, b_1, b_2 \in B$. Once unravelled these are nothing more than the right balancing properties of f .

Requirement (ii) follows by the left balancing properties of f .

For (iii) we required

$$g_\bullet(a_1 a_2, b) = g_\bullet(a_1, b) g_\bullet(a_2, b) \quad g_\bullet(a, b_1 b_2) = g_\bullet(a, b_1) g_\bullet(a, b_2)$$

$$g_\bullet(1_A, b) = 1_C = g_\bullet(a, 1_B)$$

for all $a_1, a_2, \in A, b \in B$. The proof of these are a little more interesting.

Since g is a morphism we have

$$g(a_1 a_2) = (g a_1) \cdot (g a_2)$$

for each $a_1, a_2 \in A$. Thus for $b \in B$ we have

$$g_\bullet(a_1 a_2, b) = g(a_1 a_2) b = ((g a_1) \cdot (g a_2)) b = (g a_1 b)(g a_2 b) = g_\bullet(a_1, b) g_\bullet(a_2, b)$$

to give the left hand requirement.

For each $a \in A$ the value $g a$ is a morphism. Thus for $b_1, b_2 \in B$ we have

$$g_\bullet(a, b_1 b_2) = g a(b_1 b_2) = (g a b_1)(g a b_2) = g_\bullet(a, b_1) g_\bullet(a, b_2)$$

to give the right hand requirement.

The unit requirements are almost immediate.

The requirements (iv) are nothing more than currying-uncurrying. ■

Our main aim is to produce a monoid $A \otimes B$ such that the two sets

$$[A \otimes B, C] \quad [A, (B \Rightarrow C)]$$

are in bijective correspondence. Of course, we also want various functorial properties but we will find that these are automatic consequences of the constructions we use. By Lemma it suffices to produce $A \otimes B$ so that

$$[A \otimes B, C] \quad \mathbf{Bal}[A \times B, C]$$

are in bijective correspondence. To do that we modify $A \times B$ in two steps.

For each set X we may generate the free commutative monoid FX on X . We recall how this is done later. This gives us a bijective correspondence between

$$[FX, C] \quad \mathbf{Fun}[X, C]$$

where $\mathbf{Fun}[X, C]$ is the set of all functions from the set X to the monoid C . We apply this construction to the set $A \times B$ to obtain the first step $F(A \times B)$ towards $A \otimes B$. Since

$$\mathbf{Bal}[A \times B, C] \subseteq \mathbf{Fun}[A \times B, C]$$

the correspondence

$$[F(A \times B), C] \quad \mathbf{Fun}[A \times B, C]$$

isolates a certain set of morphisms $F(A \times B) \longrightarrow C$. We make good use of these.

For the second step we take a certain quotient

$$F(A \times B) \longrightarrow A \otimes B$$

that is, we defined $A \otimes B$ as a quotient of the free monoid.

Thus starting from a balanced function

$$A \times B \xrightarrow{g} C$$

we obtain a diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{g} & C \\ \downarrow \iota & & \downarrow id \\ F(A \times B) & \xrightarrow{g^*} & C \\ \downarrow q & & \downarrow id \\ A \otimes B & \xrightarrow{g^\#} & C \end{array}$$

where the morphisms g^* and $g^\#$ are generated by the freeness and the quotient properties, respectively. Here ι is the free insertion, q is the quotient morphism, and id is the identity morphism on C .

Let's look at these two steps in turn. We review the appropriate machinery as we go along.

For any set X the not-necessarily-commutative monoid freely generated by X is just the set of all words on X as an alphabet. The operation is just concatenation, and the insertion sends each letter x to the word x of length one. To convert this into the free commutative monoid we factor out a certain equivalence relation that makes the order of the letters in a word irrelevant. Put differently, we view each word as a multiset.

When we apply this construction to $A \times B$ we obtain a commutative monoid $F(A \times B)$ where a typical element is

$$(a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)$$

for $a_1, a_2, \dots, a_l \in A$ and b_1, b_2, \dots, b_l . However, the order of the components (a, b) is irrelevant.

For completeness here is the appropriate result.

0.5 LEMMA. Let A, B, C be commutative monoids. For each function

$$A \times B \xrightarrow{g} C$$

there is a unique morphism

$$F(A \times B) \xrightarrow{g^*} C$$

such that the triangle

$$\begin{array}{ccc} A \times B & \xrightarrow{g} & C \\ & \searrow \iota & \nearrow g^* \\ & F(A \times B) & \end{array}$$

commutes. The morphism g^* is given by

$$g((a_1, b_1)(a_2, b_2) \cdots (a_l, b_l)) = (g(a_1, b_1))(g(a_2, b_2)) \cdots (g(a_l, b_l))$$

for $a_1, a_2, \dots, a_l \in A$ and b_1, b_2, \dots, b_l .

For the second step we need to review how a quotient morphism can be produced. In fact, we use the general universal algebraic method (but done in a more useful way).

For a monoid D a **congruence** on D is an equivalence relation \approx which satisfies

$$\left. \begin{array}{l} d_1 \approx e_1 \\ d_2 \approx e_2 \end{array} \right\} \implies d_1 d_2 \approx e_1 e_2$$

for all $d_1, d_2, e_1, e_2 \in D$. Such congruences are easy to find. For each morphism

$$D \xrightarrow{h} C$$

the kernel $\ker(h)$ of h is the equivalence relation \cong given by

$$d \cong e \iff hd = he$$

for $d, e \in D$. Almost trivially, the kernel of a morphism is a congruence.

A slightly more taxing exercise is to show that each congruence on a monoid is the kernel of a quotient (surjective morphism).

Recall that each equivalence relation \approx on a set D splits the set into blocks, the equivalence classes of the relation. For each $d \in D$ let $[d]$ be the block which contains d , and let

$$D/\approx$$

be the set of all blocks. Let

$$\begin{array}{ccc} D & \xrightarrow{q} & D/\approx \\ d & \longmapsto & [d] \end{array}$$

be the associated surjection.

Now suppose \approx is a congruence on a monoid D . In this case we can furnish D/\approx as a monoid by setting

$$[d] \cdot [e] = [de]$$

for $d, e \in D$. Of course, there is some work to be done here. In particular, we must check that this operation is well-defined. However, that is easy to do, and we may then check that the surjection q is a morphism where $\ker(q)$ is the original congruence.

We compare binary relations on D in the usual way, by inclusion. In particular, we can compare an arbitrary relation on D with the kernel of a morphism from D . Let \sim be any such relation on D (where this need not have any special properties). Let h be a morphism from D and let \cong be the kernel $\ker(h)$. We say \sim is subsumed by h if

$$d \sim e \implies d \cong e$$

holds for each $d, e \in D$.

Each relation \sim on D generates a smallest congruence \approx on D . We simply take the intersection of all congruence relations above \sim . From this we see that h subsumes \sim precisely when it subsumes \approx .

The following result encapsulates the universality of the quotient construction.

0.6 LEMMA. *Let D be a monoid, let \sim be any relation on D , let \approx be the generated congruence on D , and let*

$$D \xrightarrow{q} D/\approx$$

be the associated quotient.

For each morphism

$$D \xrightarrow{h} C$$

which subsumes \sim there is a unique morphism

$$D/\approx \xrightarrow{h^\sim} C$$

such that the triangle

$$\begin{array}{ccc} D & \xrightarrow{h} & C \\ & \searrow q \quad \nearrow h^\sim & \\ & D/\approx & \end{array}$$

commutes. The morphism h^\sim is given by

$$h^\sim([d]) = hd$$

for each $d \in D$.

We combine these two constructions to produce the required tensor product.

Thus, as before, let A, B be arbitrary monoids, and let $F(A \times B)$ be the free (commutative) monoid on the set $A \times B$. We generate a certain congruence on this monoid, and use this to form a quotient.

0.7 DEFINITION. For a pair A, B of commutative monoids let \sim be the balancing relation on $F(A \times B)$ given by

$$(a_1 a_2, b) \sim (a_1, b)(a_2, b) \quad (a, b_1 b_2) \sim (a, b_1)(a, b_2)$$

$$(1_A, b) \sim (1_A, 1_B) \sim (a, 1_B)$$

for $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$.

Let \approx be the congruence on $F(A \times B)$ generated by \sim and let

$$A \otimes B = F(A \times B) / \approx$$

to obtain a monoid. ■

This is a rather small relation on $F(A \times b)$ for only a few words belong to its field, and these words never have more than two letters. However, this relation does generate a congruence \approx on $F(A \times B)$. The composite function $q \circ \iota$

$$\begin{array}{ccccc} A \times B & \xrightarrow{\iota} & F(A \times B) & \xrightarrow{q} & A \otimes B \\ (a, b) & \longmapsto & \iota(a, b) & \longmapsto & [(a, b)] \\ (a, b) & \longmapsto & & \longmapsto & a \otimes b \end{array}$$

first converts the letter (a, b) into the word of length one and then take the block in which this word lives, as in the middle line. This composite is usually written as in the bottom line. Notice that, in general, this composite is not a morphism, but it is a balance function. That of course, is the aim of the construction.

0.8 THEOREM. *Let A, B, C be commutative monoids. For each balanced function*

$$A \times B \xrightarrow{g} C$$

there is a unique morphism

$$F(A \times B) \xrightarrow{g_\otimes} C$$

such that the triangle

$$\begin{array}{ccc} A \times B & \xrightarrow{g} & C \\ & \searrow & \nearrow g_\otimes \\ & A \otimes B & \end{array}$$

commutes. The morphism g_\otimes is generated by

$$g_\otimes(a \otimes b) = g(a, b)$$

for $a \in A, b \in B$.

Proof. Starting from the (balanced) function

$$A \times B \xrightarrow{g} C$$

we apply Lemma 0.5 to obtain a commuting triangle

$$\begin{array}{ccc} A \times B & \xrightarrow{g} & C \\ & \searrow \iota \quad \nearrow g^* & \\ & F(A \times B) & \end{array}$$

where

$$g^*(a, b) = g(a, b)$$

for all $a \in A, b \in B$. Let \sim be the balancing relation on $F(A \times B)$, as given in Definition 0.7. We show that g^* subsumes \sim and then apply Lemma 0.6 to obtain the required morphism g_\otimes .

We require

$$g^*(a_1 a_2, b) = g^*((a_1, b)(a_2, b))$$

for $a_1, a_2 \in A, b \in B$. But

$$g^*(a_1 a_2, b) = g(a_1 a_2, b)$$

and, since g^* is a morphism, we have

$$g^*((a_1, b)(a_2, b)) = g^*(a_1, b)g^*(a_2, b) = g(a_1, b)g(a_2, b)$$

hence this requirement follows by one of the given balancing properties of g .

The other requirements follow in the same way. ■

Recall that our main aim here is to produce an adjoint pair of endofunctors on **Cmd**

$$- \otimes B \dashv B \Rightarrow -$$

for each monoid B . We have more or less achieved that aim.

0.9 THEOREM. *For each triple A, B, C of commutative monoids there is an inverse pair*

$$\begin{array}{ccc} f & \xrightarrow{\quad} & f^\sharp \\ [A \otimes B, C] & & [A, B \Rightarrow C] \\ g_\flat & \xleftarrow{\quad} & \vdash g \end{array}$$

of assignments given by

$$g_\flat(a \otimes b) = gab \quad f^\sharp ab = f(a \otimes b)$$

for $a \in A, b \in B$.

Proof. We set up the correspondence using $\mathbf{Bal}[A \times B, C]$ as an intermediary. In other words we consider

$$\begin{array}{ccccc} f & \xrightarrow{\quad} & f^\flat & \xrightarrow{\quad} & f^\sharp = f^{\flat\bullet} \\ [A \otimes B, C] & & \mathbf{Bal}[A \times B, C] & & [A, B \Rightarrow C] \\ g_\flat = g_{\bullet\otimes} & \xleftarrow{\quad} & g_\bullet & \xleftarrow{\quad} & \vdash g \end{array}$$

using the assignments of Lemma 0.4 and Theorem 0.8.

Thus starting from a morphism

$$A \otimes B \xrightarrow{f} C$$

we let

$$A \times B \xrightarrow{f^|} C$$

be the restriction (or composite)

$$A \times B \longrightarrow A \otimes B \xrightarrow{f} C$$

given by

$$f(a, b) = fa \otimes b$$

for each $a \in A, b \in B$. We then apply the currying of Lemma 0.4 to obtain f^\sharp where

$$f^\sharp a = f(a \otimes -) \quad f^\sharp ab = f(a \otimes b)$$

for each $a \in A, b \in B$.

Conversely, starting from a morphism

$$A \xrightarrow{g} B \Rightarrow C$$

we first uncurry to obtain a balance map

$$A \times B \xrightarrow{g_\bullet} C$$

and then apply the translation of Theorem 0.8 to obtain g_b where

$$g_b(a \otimes b) = g(a, b)$$

for each $a \in A, b \in B$. ■

Of course, in the description of g_b the input $a \otimes b$ is not a typical element of $A \otimes B$, but only a typical generator. More generally we have

$$g_b((a_1 \otimes b_1) \cdots (a_l \otimes b_l)) = g(a_1, b_1) \cdots g(a_l, b_l)$$

for $a_1, \dots, a_l \in A, b_1, \dots, b_l \in B$.

Notice also that Theorem 0.9 doesn't quite set up an adjunction $- \otimes B \dashv B \Rightarrow -$, for there are still some naturality conditions to be checked, However, these are routine consequences of the constructions used.

We could now show that the two constructs

$$\cdot \otimes \cdot \quad \cdot \Rightarrow \cdot$$

furnish **Cmd** as a symmetric monoidal closed category. Most of this is routine and doesn't give us much insight into the workings of the tensor product. However, these are one aspect that we should look at, namely the unit of the tensor construct.

We consider the natural numbers

$$(\mathbb{N}, +, 0)$$

as a commutative monoid. Here the operation is written additively, whereas all other operations are written multiplicatively. Thus we need to take a little care with the notation.

Let B, C be a pair of monoids, and think what a morphism

$$\mathbb{N} \xrightarrow{f} (B \Rightarrow C)$$

might be. We require

$$f(m + n) = (fm) \cdot (fn)$$

for all $m, n \in \mathbb{N}$. Thus

$$f(m + n)b = (fmb)(fnb)$$

for all $m, n \in \mathbb{N}$ and $b \in B$. A simple induction now gives the following.

0.10 LEMMA. *For each pair B, C of commutative monoids and morphism*

$$\mathbb{N} \xrightarrow{f} (B \Rightarrow C)$$

we have

$$fmb = (f1b)^m$$

for each $m \in \mathbb{N}, b \in B$.

This shows that the morphism f is determined by the morphism $f1 : B \longrightarrow C$, and goes a long way towards proving the following.

0.11 LEMMA. *For each pair B, C of commutative monoids there is a bijective correspondence*

$$\begin{array}{ccc} f & \longmapsto & f^+ \\ [N, (B \Rightarrow C)] & & [B, C] \\ g_- & \longleftarrow & \dashv g \end{array}$$

given by

$$g_-mb = (gb)^m \quad f^+b = f1b$$

for each $b \in B, m \in \mathbb{N}$.

Proof. Trivially, for each morphism

$$\mathbb{N} \xrightarrow{f} (B \Rightarrow C)$$

the function $f1$ is a morphism from B to C .

Given a morphism

$$B \xrightarrow{g} C$$

we must first check that for each $m \in \mathbb{N}$ the assignment

$$\begin{array}{ccc} B & \longrightarrow & C \\ b & \longmapsto & (gb)^m \end{array}$$

is a morphism from B to C , and then we must check that g^+ is a morphism from \mathbb{N} to $(B \Rightarrow C)$. These are straight forward (but depend crucially on the commutativity of C).

Finally, the inverse properties

$$f^+_- = f \quad g_-^+ = g$$

follow by a couple of simple calculations and a use of Lemma 0.10. ■

This result shows that B has the separation property required of $\mathbb{N} \otimes B$. Why can this be? Because the two monoids are isomorphic. Let's investigate this.

Consider the composite transforms

$$\begin{array}{ccccc} f & \longmapsto & f^\sharp & \longmapsto & f^{\sharp+} \\ [N \otimes B, C] & & [N, (B \Rightarrow C)] & & [B, C] \\ g_{-b} & \longleftarrow & g_- & \longleftarrow & g \end{array}$$

obtained by combining those of Theorem 0.9 and Lemma 0.11. These are generated by

$$g_{-b}(m \otimes b) = g_-(m, b) = (gb)^m \quad f^{\sharp+}bc = f^\sharp 1c = f(1 \otimes b)$$

for $m \in \mathbb{N}, b \in B$.

We use this correspondence for two cases. Firstly we set $C = \mathbb{N} \otimes B$ and take f to be the identity morphism on that C . Secondly, we take $C = B$ and take g to be the identity morphism on that C . This generates the following morphisms.

$$\begin{array}{ccc} b & \longmapsto & 1 \otimes b \\ B & & \mathbb{N} \otimes B \\ b^m & \longleftarrow & m \otimes b \end{array}$$

It turns out that these assignments are an inverse pair of isomorphism. This is, perhaps, a little surprising since it seem that not all elements of $\mathbb{N} \otimes B$ are considered.

The monoid $\mathbb{N} \otimes B$ is certainly generated by the elements $m \otimes b$. Thus a typical element has the form

$$\beta = (m(1) \otimes b_1) \cdots (m(l) \otimes b_l)$$

for $m(1), \dots, m(l) \in \mathbb{N}$ and $b_1, \dots, b_l \in B$. But remember that the map

$$m, b \longmapsto m \otimes b$$

is balanced. In particular

$$m \otimes b = (1 \otimes b)^m \quad (1 \otimes b_1) \cdots (1 \otimes b_l) = 1 \otimes b_1 \cdots b_l$$

for $m \in \mathbb{N}$ and $b, b_1, l, \dots, b_l \in B$. Using these (and the commutativity) we find that

$$\beta = (1 \otimes b)^m$$

where

$$b = b_1 \cdots b_l \quad m = m(1) + \cdots + m(l)$$

are the base and exponent.