THE TROPICAL NULLSTELLENSATZ FOR CONGRUENCES

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1. Introduction

Tropical algebraic geometry originally developed as the study of certain geometric objects, called *amoebas*, that arose as the logarithmic images of complex varieties. These amoebas possessed semi-linear "skeletons" in which was encoded much of the geometric content of the original complex variety. The skeletons could also be recovered as limits of logarithmic (or valuative) maps of the original complex varieties, and soon the skeletons came to be called *tropical varieties*. A unifying characteristic of tropical varieties is their inherently piecewise-linear nature. One benefit of this linearity has been their ability to translate algebro-geometric questions into questions of combinatorics, graph theory and semi-linear algebra. Tropical algebraic geometry has been used to compute enumerative invariants of real and complex projective spaces ([BM09],[BM07]), as well as to give a new proof of the classical Brill-Noether theorem ([CDPR12]). As the field has developed, the number of algebro-geometric objects with tropical analogues has multiplied. There are now tropical analogues of the Grassmannian ([SS04]), the Riemann-Roch theorem ([BN07],[GK08]), the Torelli theorem ([BMV11],[CV10],[Cha12]), and the Hurwitz numbers ([CJM10]).

There are many ongoing programs to establish a foundational theory of tropical algebraic geometry. Some approaches are fairly toric in nature, involving polyhedra and balancing conditions (e.g., [Mik06]), while others are more valuative or categorical (e.g., [IKR09]-[IKR12]). There is also the related work on Berkovich spaces and $\mathbf{F}_1$-geometry (e.g., [Tak10]) and the nascent generalization to so-called congruence schemes (e.g., [Dei13]). For a general overview of tropical algebraic geometry, see [Gat06], [Spe05], or [SS09]. The current paper differs from the aforementioned programs in that it takes a straightforward approach. We begin with the goal of understanding geometric subsets of tropical n-space at which a collection of pairs of tropical polynomials agree; e.g., subsets of the form $\{\mathbf{a} \mid f(\mathbf{a}) = g(\mathbf{a})\}$ for some tropical polynomials f, g. In the classical setting such a locus is equivalent to the vanishing locus of the ideal $\langle f - g \rangle$. Since subtraction does not exist for tropical polynomials, however, we are forced to deal directly with the pair (f, g) (or, in general, a collection of such pairs). In order to have a coherent theory, we are led to consider algebraic objects on semirings that behave both like ideals and equivalence relations. Fortunately such a structure already exists, in the form of a "congruence" on a semiring. The focus of the current paper is an algebro-geometric study of the properties of congruences and their associated congruence varieties, with the goal of obtaining an analogue of Hilbert's Nullstellensatz.

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We note that the congruence varieties studied here do not exactly coincide with the tropical objects studied elsewhere, the latter of which are generally defined as the corner (or bend) loci of tropical polynomials. In the current setting it appears classical tropical objects may actually play the role of divisors. This topic will be taken up in a future paper. We also note that our Nullstellensatz differs from both [Izh05] and [SI07], which are concerned with vanishing loci.

The current paper is structured as follows. Section 2 is devoted to establishing the algebraic foundation on which we will be working, focusing especially on congruences. A *congruence* on a semiring R is a subsemiring of $R \times R$ that also defines an equivalence relation on R. The bulk of this section is devoted to establishing their basic properties, such as induced quotient semirings and prime congruences. In Section 3 we focus our attention on the $\mathbf{T}[\mathbf{x}]$, the semiring of tropical polynomials. We establish the fundamental correspondence between congruences on $\mathbf{T}[\mathbf{x}]$ and subsets of $\mathbf{T}^n$: to each congruence E on $\mathbf{T}[\mathbf{x}]$ we associate its *congruence variety*, $\mathbf{V}(E) \subseteq \mathbf{T}^n$, consisting of those points $\mathbf{a}$ for which $f(\mathbf{a}) = g(\mathbf{a})$ for every $(f, g) \in E$; and conversely, to any subset $S \subseteq \mathbf{T}^n$ we associate a congruence, $\mathbf{E}(S)$, consisting of all pairs (f, g) for which $f(\mathbf{a}) = g(\mathbf{a})$ for every $\mathbf{a} \in S$. As in classical algebraic geometry, two questions naturally arise. First, when is the set $\mathbf{V}(E)$ nonempty? And secondly, what is the algebraic structure of $\mathbf{E}(\mathbf{V}(E))$? Answers to these questions represent analogues of the weak and strong forms of Hilbert's Nullstellensatz, respectively. We lay the foundations for those results by first analyzing the case when $\mathbf{V}(f, g)$ is empty, and also the case when $\mathbf{V}(f, g) = \mathbf{T}^n$. The analysis of the latter involves an auxiliary notion of "saturation" for tropical polynomials, a property which we use to algebrize the condition that two tropical polynomials define the same tropical *function*. This ultimately leads to the following description of $\mathbf{E}(\mathbf{V}(E))$ in the special case of the trivial congruence Δ:

Theorem 1. *The congruence $\mathbf{E}(\mathbf{T}^n)$ is the intersection of all prime congruences on $\mathbf{T}[\mathbf{x}]$.*

Section 4 is devoted to answering the question of when $\mathbf{V}(E)$ is empty for a finitely generated congruence E. The analysis proceeds by a reduction to the principally generated case, together with an auxiliary notion of "flatness" for congruences on $\mathbf{T}[\mathbf{x}]$. This result is the following weak form of the Tropical Nullstellensatz for Congruences:

Theorem 2. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$. Then $\mathbf{V}(E)$ is empty if and only if there exists $h \in \mathbf{T}[\mathbf{x}]$ with nonzero constant term such that $(h, \epsilon \odot h) \in E$ for some $\epsilon > 0$ (equivalently, all $\epsilon \in \mathbf{R}$).*

The remainder of the paper is devoted to understanding the algebraic structure of $\mathbf{E}(\mathbf{V}(E))$. The first issue is to find a candidate for $\mathbf{E}(\mathbf{V}(E))$, which in the classical case is the radical ideal. It turns out to be a slightly more complicated object in the current setting. First, given a congruence E on $\mathbf{T}[\mathbf{x}]$ we define a set E_+, consisting of all pairs (f, g) for which the following condition is satisfied: there exist $r \in \mathbf{T}[\mathbf{x}], \epsilon \in \mathbf{T}\backslash\{1_\mathbf{T}\}$ and $N > 0$ such that

$$(1_\mathbf{T}, \epsilon) \times ((f \oplus g)^{\odot N} \oplus r, 0_\mathbf{T}) \times (f, g) \in E.$$

Here we are using the so-called *twisted product* (introduced in Section 2), defined by $(a, b) \times (c, d) = (ac + bd, ad + bc)$. The functor $\mathbf{V}$ behaves well with respect to twisted products, in that $\mathbf{V}((h, k) \times (f, g)) = \mathbf{V}(h, k) \cup \mathbf{V}(f, g)$. This property, together with Theorem 2 and a few observations about relations of the form $(f, g)^{\odot n}$ and $(f, g)^{\times n}$, naturally lead one to consider the set defined above. One always has $E \subseteq E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$ and $\mathbf{V}(E) = \mathbf{V}(E_+)$. Moreover, with this new notation Theorem 2 can be restated as follows:

Theorem 3. *If E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, then $\mathbf{V}(E)$ is empty if and only if E_+ is improper.*

We begin our final analysis of when (f, g) is contained in $\mathbf{E}(\mathbf{V}(E))$ by considering the special case in which one function is the zero polynomial. We pursue a strategy analogous to the famous "Raboniwitsch trick," attempting to "invert" the function f by adjoining the relation $(1_{\mathbf{T}}, y \odot f)$ to the congruence E, working now on the semiring $\mathbf{T}[\mathbf{x}, y]$. The resulting congruence on $\mathbf{T}[\mathbf{x}, y]$ has empty associated congruence variety, and so Theorem 3 applies. After "clearing denominators," we arrive at the following result:

Theorem 4. *If E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, then $(f, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$ if and only if $(f, 0_{\mathbf{T}}) \in E_+$.*

The general case is more complicated, as there is an additional phenomenon that arises in the tropical setting not seen in the classical setting. By way of example, suppose one knew $(f \oplus t, t) \in \mathbf{E}(\mathbf{V}(E))$ for every $t > 1_{\mathbf{T}}$, i.e., that $f(\mathbf{a}) \leq t$ for every $\mathbf{a} \in \mathbf{V}(E)$. By continuity one would then have $f(\mathbf{a}) \leq 1_{\mathbf{T}}$, and hence $(f \oplus 1_{\mathbf{T}}, 1_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. In other words, the congruence $\mathbf{E}(\mathbf{V}(E))$ is somehow "closed under limits," a property not shared by congruences in general. With this example in mind, we proceed to define a set $(E_+)^{\lim}$, consisting of all *limit relations* of *ascending chains* in E_+. This set might not be transitively closed, so we let $\hat{E}$ denote its transitive closure. (See Section 5 for complete details.) We are ultimately able to prove the following strong form of the Tropical Nullstellensatz for Congruences:

Theorem 5. *If E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, then $\mathbf{E}(\mathbf{V}(E)) = \hat{E}$.*

The proof proceeds by reducing the general case of $(f, g) \in \mathbf{E}(\mathbf{V}(E))$ to the special case when $g = 1_{\mathbf{T}}$, using a second application of the Rabinowitsch trick to "invert" the function g. We then use Theorem 3 to prove $(f \oplus t, t) \in E_+$ and $(f \oplus t^{\odot -1}, f) \in E_+$ for every $t > 1_{\mathbf{T}}$. The result ultimately follows.

2. Semirings and Congruences

2.1. Semirings.

The main algebraic structures we will be dealing with are semirings. Recall that a **semiring** $(R, +, \cdot)$ is a nonempty set R together with two binary operations, satisfying the following:

(i) $(R, +)$ is a commutative monoid with identity element 0;
(ii) $(R, \cdot)$ is a monoid with identity element $1 \neq 0$;
(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$;
(iv) $0 \cdot a = 0 = a \cdot 0$ for all $a \in R$.

A **morphism** $f : R \to S$ of semirings is a morphism of sets compatible with the semiring structures, i.e., compatible with the operations and mapping 0_R to 0_S and 1_R to 1_S. A semiring R is **commutative** if the monoid $(R, \cdot)$ is commutative. All semirings we consider will be commutative. A **subsemiring** of a semiring R is a subset that inherits the structure of a semiring from R.

Example 2.1. The set $\mathbf{F}_1 = \{0, 1\}$ is a semiring under the usual addition and multiplication, with the notable exception that $1 + 1 = 1$. This is sometimes called the **boolean semiring** and denoted $\mathbf{B}$; see [Gol92, pg. 12]. We use the $\mathbf{F}_1$-notation here to more closely parallel the theory of $\mathbf{F}_1$-geometry currently being developed by Berkovich and others; e.g., see [Tak10].

A semiring R is **additively idempotent** if the addition operation is idempotent, i.e., if $a + a = a$ for every $a \in R$. Additively idempotent semirings are precisely $\mathbf{F}_1$-semialgebras. Indeed, the only nontrivial requirement for $\mathbf{F}_1$ to act on a semiring R is that $a = 1_{\mathbf{F}_1} \cdot a = (1_{\mathbf{F}_1} + 1_{\mathbf{F}_1}) \cdot a = a + a$ for all $a \in R$. Observe that $\mathbf{F}_1$ is the initial object in the category of additively idempotent semirings: given any additively idempotent semiring R, the unique morphism $\mathbf{F}_1 \to R$ is the map sending $0_{\mathbf{F}_1}$ to 0_R and $1_{\mathbf{F}_1}$ to 1_R.

Example 2.2. The set $\mathbf{N} \cup \{\infty\}$ is an additively idempotent semiring under the operations of minimum and addition, where minimum plays the role of the additive operation and addition the role of the multiplicative operation. The additive identity is ∞, while the multiplicative identity is the natural number 0. This is a subsemiring of $(\mathbf{R} \cup \{\infty\}, \min, +)$, the so-called **optimization algebra**; see [Gol92, pp. 15-16]. Along similar lines, the set $\mathbf{N} \cup \{-\infty\}$ is an additively idempotent semiring under the operations of maximum and addition. It is a subsemiring of $(\mathbf{R} \cup \{-\infty\}, \max, +)$, which we will call the **tropical semiring** and denote by $\mathbf{T}$. We will generally denote the operations by $\oplus$ and $\odot$ when working in $\mathbf{T}$. Note that the additive identity $0_{\mathbf{T}}$ is the element $-\infty$, and the multiplicative identity $1_{\mathbf{T}}$ is the real number 0. (In some texts, the semiring $(\mathbf{R} \cup \{-\infty\}, \max, +)$ is called the **schedule algebra** or $(\max, +)$-**algebra**; the term tropical semiring is then reserved for $(\mathbf{N} \cup \{\infty\}, \min, +)$.

2.2. Congruences.

As mentioned in the introduction, we are ultimately interested in subsets of $\mathbf{T}^n$ where collections of pairs of tropical polynomials agree; e.g., subsets of the form $\{\mathbf{a} \in \mathbf{T}^n \mid f(\mathbf{a}) = g(\mathbf{a})\}$ for some tropical polynomials $f, g \in \mathbf{T}[\mathbf{x}]$. If we were working in a ring, such a locus would be equivalent to the vanishing locus of the ideal $\langle f - g \rangle$. When working in a semiring, however, we cannot reduce our original locus to the vanishing set of such an ideal. Instead, we must work directly with the original locus. This suggests that instead of studying ideals in R, we should instead study subsets $E \subseteq R \times R$ given by "relations." Fortunately, such an algebraic structure already exists.

Definition 2.3. A **congruence** on a semiring R is a subset $E \subseteq R \times R$ with the following properties:

- (E1) For every $a \in R$ we have $(a, a) \in E$;
- (E2) If $(a, b) \in E$, then $(b, a) \in E$;
- (E3) If $(a, b), (b, c) \in E$, then $(a, c) \in E$;
- (I1) If $(a, b), (a', b') \in E$, then $(a + a', b + b') \in E$; and
- (I2) If $(a, b), (a', b') \in E$, then $(aa', bb') \in E$.

In other words, a congruence is a subring of $R \times R$ that also defines an equivalence relation on R.

Remark 2.4. Condition (I2) can be replaced with a condition similar to that in the definition of an ideal. For notational convenience, define the **twisted product** on $R \times R$ as

$$(a, b) \times (c, d) := (ac + bd, ad + bc).$$

This is the product one would expect to encounter if pairs (a, b) were represented as differences $a - b$. Indeed, observe that $(a - b)(c - d) = (ac + bd) - (ad + bc)$. This product is commutative and behaves well with respect to addition. One readily verifies it is equivalent in the above definition to replace condition (I2) with the following:

(I2)' If $(a, b) \in E$, then $(c, d) \times (a, b) \in E$ for every $c, d \in R$.

As we will see later, twisted products behave well with respect to congruence varieties.

Remark 2.5. When working in the category of rings, there is an inclusion-preserving bijection between ideals and congruences, via the associations

$$\mathbf{E}(I) = \{(a, b) \mid a - b \in I\} \subseteq R \times R, \quad \mathbf{I}(E) = \{a - b \mid (a, b) \in E\} \subseteq R.$$

The **kernel** of a semiring morphism $f : R \to S$ is the (proper) congruence

$$\ker f = \{(a, b) \in R \times R \mid f(a) = f(b)\}.$$

A morphism f is **injective** if $\ker f = \Delta_R$.

2.3. **Quotient Semirings.**

If E is a congruence on a semiring R, then the set R/E of all equivalence classes induced by E is a semiring, with addition and multiplication defined in the obvious way, i.e., by $[a]+[b] = [a+b]$ and $[a][b] = [ab]$. This semiring is called the **quotient** (or **factor**) **semiring** of R by E. We always have a canonical surjective semiring morphism $\pi_E : R \to R/E$, given by $r \mapsto [r]$, with $\ker \pi_E = E$. By construction, there is a natural bijection between proper congruences on R and surjective semiring morphisms from R. Moreover, under this bijection containment of congruences corresponds to factorization of semiring morphisms. As a corollary, there is a natural inclusion-preserving bijection between congruences on a quotient semiring R/E and congruences on R that contain E.

Example 2.6. Define a map $\mathbf{T} \to \mathbf{F}_1$ by sending $0_{\mathbf{T}}$ to $0_{\mathbf{F}_1}$ and all other t to $1_{\mathbf{F}_1}$. This is easily seen to be a surjective semiring morphism; the key observation is that if $a, b \neq 0_{\mathbf{T}}$, then $a \oplus b \neq 0_{\mathbf{T}} \neq a \odot b$. The corresponding congruence is

$$E_{\mathbf{F}_1} = (\mathbf{T} \times \mathbf{T}) \backslash \{(1_{\mathbf{T}}, 0_{\mathbf{T}}), (0_{\mathbf{T}}, 1_{\mathbf{T}})\}.$$

We claim this is the unique nontrivial proper congruence on $\mathbf{T}$. It is clearly maximal among all proper congruences on $\mathbf{T}$. Suppose E is any nontrivial proper congruence on $\mathbf{T}$. As E is nontrivial, there exist some distinct $r, s \in \mathbf{T}$ with $(r, s) \in E$. As E is proper, we must also have $r, s \neq 0_{\mathbf{T}}$, i.e., r and s are real numbers. (Every $r \in \mathbf{T} \backslash \{0_{\mathbf{T}}\}$ corresponds to a real number, and so has a tropical multiplicative inverse – its usual additive inverse. If $(r, 0_{\mathbf{T}}) \in E$ for some $r \neq 0_{\mathbf{T}}$, then $(1_{\mathbf{T}}, 0_{\mathbf{T}}) = (r^{\odot -1}, 0_{\mathbf{T}}) \times (r, 0_{\mathbf{T}}) \in E$, and hence E is improper.) Without loss of generality, suppose $r < s$ (as real numbers). Then $(1_{\mathbf{T}}, s \odot r^{\odot -1}) = (r^{\odot -1}, 0_{\mathbf{T}}) \times (r, s) \in E$. It follows that $(1_{\mathbf{T}}, s \odot r^{\odot -1})^{\odot n} = (1_{\mathbf{T}}, (s \odot r^{\odot -1})^{\odot n}) \in E$ for every $n \in \mathbf{N}$. Similarly, $(1_{\mathbf{T}}, (r \odot s^{\odot -1})^{\odot n}) \in E$ for every $n \in \mathbf{N}$. Now fix any $t \in \mathbf{T} \backslash \{0_{\mathbf{T}}\}$. Then t

is a real number, so by the usual Archimedean property of $\mathbf{R}$ there exists some $n \in \mathbf{N}$ with $|t| < n(s-r) = (s \odot r^{\odot -1})^{\odot n}$. If $t \geq 0 = 1_{\mathbf{T}}$, then observe that we have

$$(t, (s \odot r^{\odot -1})^{\odot n}) = (1_{\mathbf{T}}, (s \odot r^{\odot -1})^{\odot n}) \oplus (t,t) \in E.$$

By transitivity, it follows that $(1_{\mathbf{T}}, t) \in E$. On the other hand, if $t \leq 0$, then observe that we have

$$(1_{\mathbf{T}}, t) = (1_{\mathbf{T}}, (r \odot s^{\odot -1})^{\odot n}) \oplus (t,t) \in E.$$

We thus have $(1_{\mathbf{T}}, t) \in E$ for every $t \in \mathbf{T} \backslash \{0_{\mathbf{T}}\}$, and hence $E_{\mathbf{F}_1} \subseteq E$. As $E_{\mathbf{F}_1}$ is maximal, equality holds.

2.4. **Generators for Congruences.**

Given any subset $\mathcal{S} \subseteq R \times R$, the congruence **generated by** $\mathcal{S}$ is the smallest congruence on R that contains $\mathcal{S}$. We denote this congruence by $\langle \mathcal{S} \rangle$. If we wish to emphasize the semiring on which this is a congruence, we write $\langle \mathcal{S} \rangle_R$. Since an arbitrary intersection of congruences is a congruence, we can express $\langle \mathcal{S} \rangle$ as the intersection of all congruences containing $\mathcal{S}$.

Example 2.7. Let R be any semiring, and let $\mathrm{ev}_0 : R[x] \to R$ be the evaluation-at-zero morphism. We claim $\ker \mathrm{ev}_0 = \langle (x, 0) \rangle$. Indeed, first notice that we certainly have $(x, 0) \in \ker \mathrm{ev}_0$, and hence $\langle (x, 0) \rangle \subseteq \ker \mathrm{ev}_0$. Conversely, it is straightforward to verify that $(f(x), f(0)) \in \langle (x, 0) \rangle$ for every $f \in R[x]$. From this it follows that if $f(0) = g(0)$, then by transitivity (of $(f(x), f(0)), (g(0), g(x)) \in \langle (x, 0) \rangle$) we have $(f(x), g(x)) \in \langle (x, 0) \rangle$, and hence $\ker \mathrm{ev}_0 \subseteq \langle (x, 0) \rangle$.

Unlike the situation with ideals in rings, a general member of a finitely generated congruence E cannot necessarily be expressed as a sum of multiples of the generators of E. We can say something similar, though.

Proposition 2.8. *Suppose R is a semiring and $\mathcal{S} \subseteq R \times R$ satisfies properties (E1), (E2), (I1), and (I2). Then $\langle \mathcal{S} \rangle$ is the transitive closure of $\mathcal{S}$, i.e.,*

$$\langle \mathcal{S} \rangle = \{(a, c) \in R \times R \mid (a, b), (b, c) \in \mathcal{S} \textit{ for some } b \in R\}.$$

The analogous statement holds when property (I2) is replaced by property (I2)'.

Proof. Let E denote the transitive closure of $\mathcal{S}$. Observe first that $\mathcal{S} \subseteq E$: if $(a, b) \in \mathcal{S}$ is any element, then $(b, b) \in \mathcal{S}$ (by property (E1)), so $(a, b) \in E$. Now, the set $\langle \mathcal{S} \rangle$ is, by definition, the smallest set containing $\mathcal{S}$ that satisfies properties (E1)-(E3), (I1), (I2). As the set E was constructed simply by adding elements required by property (E3), we automatically have $E \subseteq \langle \mathcal{S} \rangle$. Thus, to prove equality it suffices to prove E already satisfies properties (E1)-(E3), (I1),(I2). We verify each in turn.

Property (E1) is simply the requirement that E contain Δ_R, which is automatic from the fact that $\mathcal{S} \subseteq E$ and $\mathcal{S}$ satisfies (E1). Now suppose $(a, c) \in E$, so that $(a, b), (b, c) \in \mathcal{S}$ for some $b \in R$. As $\mathcal{S}$ satisfies (E2) we also have $(c, b), (c, a) \in \mathcal{S}$, and hence $(c, a) \in E$. Thus, E satisfies (E2). Property (E3) holds by construction. For the final two properties, suppose $(a, c), (a', c') \in E$, so that $(a, b), (b, c), (a', b'), (b', c') \in \mathcal{S}$ for some $b, b' \in R$. As $\mathcal{S}$ satisfies (I1), we have $(a+a', b+b'), (b+b', c+c') \in \mathcal{S}$, and hence $(a+a', c+c') \in E$. Thus, E satisfies

(I1). As $\mathcal{S}$ is now assumed to satisfy property (I2), we have $(a,b)\cdot(a',b') = (aa',bb') \in \mathcal{S}$ and $(b,c)\cdot(b',c') = (bb',cc') \in \mathcal{S}$, hence $(aa',cc') \in E$. Thus, E satisfies property (I2).

To prove the analogous statement with property (I2) replaced by property (I2)', first note that the proofs that properties (E1)-(E3) and (I1) held for E carry over without change. To verify property (I2)' holds for E, suppose $(a,c) \in E$, and so $(a,b),(b,c) \in \mathcal{S}$ for some $b \in R$. Let $(h,k) \in R \times R$ be any pair. By property (I2)' on $\mathcal{S}$, we have $(h,0)\times(a,b) = (ah,bh) \in \mathcal{S}$ and $(h,0)\times(b,c) = (bh,ch) \in \mathcal{S}$, hence $(ah,ch) \in E$. Similarly, $(0,k)\times(a,b) = (bk,ak) \in \mathcal{S}$ and $(0,k)\times(b,c) = (ck,bk) \in \mathcal{S}$, hence $(ak,ck) \in E$. By property (I1) on E, it follows that $(ah,ch)+(ck,ak) = (ah+ck, ak+ch) = (h,k)\times(a,c) \in E$. Thus, E satisfies property (I2)'. ☐

Corollary 2.9. *Suppose E is a finitely generated congruence on a semiring R, say $E = \langle (a_1,b_1),\ldots,(a_k,b_k)\rangle$. Then the elements in E are precisely those of the form*

$$\left(\sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{m}}\mathbf{b}^{\mathbf{n}}, \sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} s_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{m}}\mathbf{b}^{\mathbf{n}} \right),$$

where all but finitely many $r_{\mathbf{m},\mathbf{n}}, s_{\mathbf{m},\mathbf{n}} \in R$ are zero and

$$\sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{n}}\mathbf{b}^{\mathbf{m}} = \sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} s_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{n}}\mathbf{b}^{\mathbf{m}}.$$

Proof. Let $\mathcal{S} \subseteq R \times R$ denote the set of all elements of the form

$$\left(\sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{m}}\mathbf{b}^{\mathbf{n}}, \sum_{\mathbf{m},\mathbf{n}\in\mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \mathbf{a}^{\mathbf{n}}\mathbf{b}^{\mathbf{m}} \right),$$

where $r_{\mathbf{m},\mathbf{n}} \in R$, all but finitely many of which are zero. This is precisely the smallest subset of $R\times R$ that contains $\{(a_1,b_1),\ldots,(a_k,b_k)\}$ and satisfies properties (E1),(E2),(I1), and (I2). By Proposition 2.8, it follows that E is the transitive closure of $\mathcal{S}$. The result follows. ☐

2.5. **Restriction and Extension of Congruences.**

Suppose S is a subsemiring of R. Given a congruence F on S, define its **extension** to R to be the congruence $\langle F\rangle_R$. Similarly, given a congruence E on R, define its **restriction** to S to be $E|_S = E \cap (S\times S)$. Note that $E|_S$ is indeed a congruence on S: Properties (E1)-(E3) for $E|_S$ follow immediately from the corresponding properties for E, while properties (I1) and (I2) follow from the corresponding properties for E together with the fact that $S\times S$ is a subsemiring of $R\times R$.

Lemma 2.10. *Suppose $T \subseteq S \subseteq R$ are inclusions of semirings, $\mathcal{S} \subseteq S \times S$ is any subset, and E is a congruence on R. Then:*

(i) $\langle\langle\mathcal{S}\rangle_S\rangle_R = \langle\mathcal{S}\rangle_R$;
(ii) $(E|_S)|_T = E|_T$;
(iii) $(\langle\mathcal{S}\rangle_R)|_S \supseteq \langle\mathcal{S}\rangle_S$; *and*
(iv) $\langle E|_S\rangle_R \subseteq E$.

Proof. For (i), observe that the inclusions $\mathcal{S} \subseteq \langle \mathcal{S} \rangle_S \subseteq \langle \mathcal{S} \rangle_R$ induce the inclusion $\langle \mathcal{S} \rangle_R \subseteq \langle \langle \mathcal{S} \rangle_S \rangle_R \subseteq \langle \mathcal{S} \rangle_R$. For (ii), simply observe that $(S \times S) \cap (T \times T) = T \times T$, hence $(E|_S)|_T = (E \cap (S \times S)) \cap (T \times T) = E \cap (T \times T) = E|_T$. For (iii), the inclusion $\langle \mathcal{C} \rangle_S \subseteq \langle \mathcal{S} \rangle_R$ induces the inclusion $\langle \mathcal{S} \rangle_S \subseteq (\langle \mathcal{S} \rangle_R)\,|_S$. For (iv), the inclusion $E|_S \subseteq E$ induces the inclusion $\langle E|_S \rangle_R \subseteq E$. □

Example 2.11. The containments in (iii) and (iv) can be proper. Indeed, if we consider the inclusion $\mathbf{Z} \subset \mathbf{Q}$ and the ideal $\langle 2 \rangle \subset \mathbf{Z}$, then we see that $\langle 2 \rangle_{\mathbf{Q}} = \mathbf{Q}$ and so $(\langle 2 \rangle_{\mathbf{Q}})\,|_{\mathbf{Z}} = \mathbf{Z} \supsetneq \langle 2 \rangle$. For an example of proper containment in (iv), consider the congruence $E = \langle (x, 0_{\mathbf{T}}) \rangle$ on $\mathbf{T}[\mathbf{x}]$. This congruence consists of all pairs (f, g) of tropical polynomials with the same constant term. It follows that $E|_{\mathbf{T}} = \Delta_{\mathbf{T}}$, and hence $\langle E|_{\mathbf{T}} \rangle_{\mathbf{T}[\mathbf{x}]} = \Delta_{\mathbf{T}[\mathbf{x}]} \subsetneq E$.

Proposition 2.12. *If E is a congruence on a semiring R, then there is a canonical semiring isomorphism*

$$R[x]/\langle E \rangle_{R[x]} \cong (R/E)\,[x].$$

Proof. Extend the surjective semiring morphism $\pi_E : R \to R/E$ to a surjective semiring morphism $\tilde{\pi}_E : R[x] \to (R/E)\,[x]$ in the natural way, i.e., by defining $\tilde{\pi}_E \left(\sum_d r_d x^d \right) = \sum_d \pi_E(r_d) x^d$. Observe that, by construction,

$$\begin{aligned} \ker \tilde{\pi}_E &= \left\{ \left(\sum_d r_d x^d, \sum_d s_d x^d \right) \; : \; \sum_d \pi_E(r_d) x^d = \sum_d \pi_E(s_d) x^d \right\} \\ &= \left\{ \left(\sum_d r_d x^d, \sum_d s_d x^d \right) \; : \; \pi_E(r_d) = \pi_E(s_d) \text{ for every } d \right\} \\ &= \left\{ \left(\sum_d r_d x^d, \sum_d s_d x^d \right) \; : \; (r_d, s_d) \in E \text{ for every } d \right\}. \end{aligned}$$

In particular, we certainly have $E \subseteq \ker \tilde{\pi}_E$ (via pairs of constant polynomials). Moreover, any congruence on $R[x]$ that contains the set E must contain the above collection of relations, by virtue of the defining properties of a congruence on $R[x]$. Thus, $\ker \tilde{\pi}_E$ is the smallest congruence on $R[x]$ that contains E, and hence $\ker \tilde{\pi}_E = \langle E \rangle_{R[x]}$. □

2.6. **Prime and Maximal Congruences.**

We say a proper congruence E on a semiring R is **prime** if whenever $(ac, bc) \in E$ for some $a, b, c \in R$, either $(a, b) \in E$ or $(c, 0) \in E$. We say a semiring R is an **integral domain** if whenever $ac = bc$ for some $a, b, c \in R$, either $a = b$ or $c = 0$. As in the case with rings, a proper congruence E on a semiring R is prime if and only if the factor semiring R/E is an integral domain. By a standard Zorn's lemma argument, it can be shown that every proper congruence on a semiring is contained in a maximal congruence. Maximal congruences are closely related to fields.

Proposition 2.13. [Gol03, Prop 7.7] *If a semiring R has no nontrivial proper congruences, then either $R \cong \mathbf{F}_1$ or R is a field.*

Corollary 2.14. *A proper congruence E on a semiring R is maximal if and only if the factor semiring R/E is isomorphic to either $\mathbf{F}_1$ or a field.*

Corollary 2.15. *A proper congruence E on an additively idempotent semiring R is maximal if and only if the factor semiring R/E is isomorphic to $\mathbf{F}_1$.*

Proof. We always have the semiring surjection $\pi : R \to R/E$. It follows that

$$1_{R/E} = \pi(1_R) = \pi(1_R + 1_R) = \pi(1_R) + \pi(1_R) = 1_{R/E} + 1_{R/E}.$$

If R/E is a field, then this implies $1_{R/E} = 0_{R/E}$, a contradiction. $\square$

As both $\mathbf{F}_1$ and fields are integral domains, it follows that every maximal congruence on a semiring is prime, and hence every proper congruence is contained in a prime congruence.

Example 2.16. By Example 2.6, there are precisely two proper congruences on $\mathbf{T}$: the trivial congruence (with factor semiring $\mathbf{T}$) and the congruence $E_{\mathbf{F}_1}$ (with factor semiring $\mathbf{F}_1$). As $\mathbf{T}$ is an integral domain (a semifield, even), the trivial congruence is prime. Thus, $\mathbf{T}$ has two prime congruences, exactly one of which is maximal.

Example 2.17. Consider the additively idempotent semiring $(\mathbf{N} \cup \{\infty\}, \min, +)$. By Corollary 2.15 the maximal congruences correspond to semiring surjections $\phi : \mathbf{N} \cup \{\infty\} \to \mathbf{F}_1$. Such a semiring surjection must satisfy $\phi(\infty) = 0_{\mathbf{F}_1}, \phi(0) = 1_{\mathbf{F}_1}$, and $\phi(n) = n\phi(1) = \phi(1)$ for every positive $n \in \mathbf{N}$. (Recall that addition in $\mathbf{F}_1$ is idempotent.) It follows that ϕ is entirely determined by $\phi(1)$, and each of the two possibilities is equally valid. If $\phi(1) = 0_{\mathbf{F}_1}$, then $\phi(n) = 0_{\mathbf{F}_1}$ for every nonzero $n \in \mathbf{N} \cup \{\infty\}$. This map corresponds to the maximal congruence $\langle(1, \infty)\rangle$. If, on the other hand, $\phi(1) = 1_{\mathbf{F}_1}$, then and $\phi(n) = 1_{\mathbf{F}_1}$ for every $n \in \mathbf{N}$. This map corresponds to the maximal congruence $\langle(0, 1)\rangle$. (One can show these are the only nontrivial prime congruences.)

Example 2.18. Consider the additively idempotent semiring $(\mathbf{N} \cup \{-\infty\}, \max, +)$. As in the previous example, maximal congruences correspond to semiring surjections onto $\mathbf{F}_1$. In this case, however, there is only one semiring surjection $\phi : \mathbf{N} \cup \{-\infty\} \to \mathbf{F}_1$. Indeed, we must have $\phi(-\infty) = 0_{\mathbf{F}_1}, \phi(0) = 1_{\mathbf{F}_1}$, and $\phi(1) = \phi(\max\{0, 1\}) = \phi(0) + \phi(1) = 1_{\mathbf{F}_1} + \phi(1) = 1_{\mathbf{F}_1}$. It follows that $\phi(n) = 1_{\mathbf{F}_1}$ for all $n \in \mathbf{N}$, and hence ϕ is completely determined (and corresponds to the maximal congruence $\langle(0, 1)\rangle$). (One can show this is the only nontrivial prime congruence.)

3. Tropical Functions

3.1. **Congruences and Tropical Sets.**

We now restrict our attention to the $\mathbf{T}$-semialgebra $\mathbf{T}[\mathbf{x}]$. To any subset $S \subseteq \mathbf{T}^n$ we can associate a congruence

$$\mathbf{E}(S) = \{(f, g) \mid f(\mathbf{a}) = g(\mathbf{a}) \text{ for all } \mathbf{a} \in S\} \subseteq \mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}].$$

This association is clearly inclusion-reversing. Conversely, to any subset $E \subseteq \mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}]$ we can associate a set

$$\mathbf{V}(E) = \{\mathbf{a} \in \mathbf{T}^n \mid f(\mathbf{a}) = g(\mathbf{a}) \text{ for all } (f, g) \in E\} \subseteq \mathbf{T}^n.$$

We call $\mathbf{V}(E)$ the **congruence variety** associated to E. This association is also inclusion-reversing. It is clear that for any congruence E on $\mathbf{T}[\mathbf{x}]$ one has $E \subseteq \mathbf{E}(\mathbf{V}(E))$ and $\mathbf{V}(\mathbf{E}(\mathbf{V}(E))) = \mathbf{V}(E)$.

Proposition 3.1. *If $E = \langle (f_1, g_1), \ldots, (f_k, g_k) \rangle_{\mathbf{T}[\mathbf{x}]}$, then*

$$\mathbf{V}(E) = \{\mathbf{a} \in \mathbf{T}^n \mid f_i(\mathbf{a}) = g_i(\mathbf{a}) \text{ for } i = 1, \ldots, k\}.$$

Proof. Let $\mathcal{S} \subseteq \mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}]$ denote the set of all elements of the form

$$\left(\bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}}, \bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \right),$$

where $r_{\mathbf{m},\mathbf{n}} \in \mathbf{T}[\mathbf{x}]$, all but finitely many of which are zero. As we saw in Corollary 2.9, E is the transitive closure of $\mathcal{S}$. We claim $\mathbf{V}(E) = \mathbf{V}(\mathcal{S})$. The containment $\mathbf{V}(E) \subseteq \mathbf{V}(\mathcal{S})$ is immediate from the inclusion $\mathcal{S} \subseteq E$. For the reverse containment, suppose $\mathbf{a} \in \mathbf{V}(\mathcal{S})$. Take any $(f, g) \in E$, so that $(f, h), (h, g) \in \mathcal{S}$ for some $h \in \mathbf{T}[\mathbf{x}]$. Then $f(\mathbf{a}) = h(\mathbf{a})$ and $h(\mathbf{a}) = g(\mathbf{a})$ (since $\mathbf{a} \in \mathbf{V}(\mathcal{S})$), hence $f(\mathbf{a}) = g(\mathbf{a})$. Thus, $\mathbf{a} \in \mathbf{V}(E)$, and so $\mathbf{V}(E) = \mathbf{V}(\mathcal{S})$.

We next claim $\mathbf{V}(\mathcal{S}) = \{\mathbf{a} \in \mathbf{T}^n \mid f_i(\mathbf{a}) = g_i(\mathbf{a}) \text{ for } i = 1, \ldots, k\}$. We obviously have $\mathbf{V}(\mathcal{S}) \subseteq \{\mathbf{a} \in \mathbf{T}^n \mid f_i(\mathbf{a}) = g_i(\mathbf{a}) \text{ for } i = 1, \ldots, k\}$. Conversely, suppose $\mathbf{a} \in \mathbf{T}^n$ satisfies $f_i(\mathbf{a}) = g_i(\mathbf{a})$ for $i = 1, \ldots, k$. Take any element

$$\left(\bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}}, \bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \right)$$

in $\mathcal{S}$. As $f_i(\mathbf{a}) = g_i(\mathbf{a})$ for every i, we have

$$\bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}}(\mathbf{a}) \odot \mathbf{f}^{\odot \mathbf{m}}(\mathbf{a}) \odot \mathbf{g}^{\odot \mathbf{n}}(\mathbf{a}) = \bigoplus_{\mathbf{m},\mathbf{n} \in \mathbf{Z}^k_{\geq 0}} r_{\mathbf{m},\mathbf{n}}(\mathbf{a}) \odot \mathbf{g}^{\odot \mathbf{m}}(\mathbf{a}) \odot \mathbf{f}^{\odot \mathbf{n}}(\mathbf{a}).$$

Thus $\mathbf{a} \in \mathbf{V}(\mathcal{S})$, as desired. $\square$

The functor $\mathbf{V}$ behaves well with respect to twisted products:

Lemma 3.2. *For any $(f, g), (f', g') \in \mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}]$,*

$$\mathbf{V}((f, g) \times (f', g')) = \mathbf{V}(f, g) \cup \mathbf{V}(f', g').$$

Proof. As $(f, g) \times (f', g') \in \langle (f, g) \rangle$, we automatically have $\mathbf{V}(f, g) \subseteq \mathbf{V}((f, g) \times (f', g'))$; similarly, $\mathbf{V}(f', g') \subseteq \mathbf{V}((f, g) \times (f', g'))$. Conversely, suppose $\mathbf{a} \in \mathbf{V}((f, g) \times (f', g'))$. The equality

$$((f \odot f') \oplus (g \odot g'))(\mathbf{a}) = ((f \odot g') \oplus (g \odot f'))(\mathbf{a})$$

is equivalent to the equality

$$\max\{f(\mathbf{a}) + f'(\mathbf{a}), g(\mathbf{a}) + g'(\mathbf{a})\} = \max\{f(\mathbf{a}) + g'(\mathbf{a}), g(\mathbf{a}) + f'(\mathbf{a})\}.$$

Relabeling if necessary, we may assume $f(\mathbf{a}) \geq g(\mathbf{a})$ and $f'(\mathbf{a}) \geq g'(\mathbf{a})$. The above equality then reduces to

$$f(\mathbf{a}) + f'(\mathbf{a}) = \max\{f(\mathbf{a}) + g'(\mathbf{a}), g(\mathbf{a}) + f'(\mathbf{a})\}.$$

We claim that either $f(\mathbf{a}) = g(\mathbf{a})$ or $f'(\mathbf{a}) = g'(\mathbf{a})$. Suppose not, so that $f(\mathbf{a}) > g(\mathbf{a})$ and $f'(\mathbf{a}) > g'(\mathbf{a})$. Note that, in particular, we must then have $f(\mathbf{a}), f'(\mathbf{a}) > 0_{\mathbf{T}} = -\infty$. It follows, then, that $f(\mathbf{a}) + f'(\mathbf{a}) > f(\mathbf{a}) + g'(\mathbf{a})$ and $f(\mathbf{a}) + f'(\mathbf{a}) > g(\mathbf{a}) + f'(\mathbf{a})$, and so the equality above cannot be satisfied. Thus, we must have either $f(\mathbf{a}) = g(\mathbf{a})$ or $f'(\mathbf{a}) = g'(\mathbf{a})$, and hence $\mathbf{a} \in \mathbf{V}(f, g) \cup \mathbf{V}(f', g')$, as desired. $\square$

Lemma 3.3. *The congruence* $\mathbf{E}(\mathbf{T}^n)$ *is prime.*

Proof. Suppose $(h \odot f, h \odot g) \in \mathbf{E}(\mathbf{T}^n)$ for some $h, f, g \in \mathbf{T}[\mathbf{x}]$. If h is the zero polynomial, then it is the zero function on $\mathbf{T}^n$, and so $(h, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{T}^n)$. Suppose now h is not the zero polynomial. Then we have $h(\mathbf{a}) > 0_{\mathbf{T}}$ for all $\mathbf{a} \in \mathbf{R}^n$, and hence $h(\mathbf{a})$ has a (tropical) multiplicative inverse for every $\mathbf{a} \in \mathbf{R}^n$. As $h(\mathbf{a}) \odot f(\mathbf{a}) = h(\mathbf{a}) \odot g(\mathbf{a})$ for all $\mathbf{a}$, this implies $f(\mathbf{a}) = g(\mathbf{a})$ for all $\mathbf{a} \in \mathbf{R}^n$. Now take any $\mathbf{a} \in \mathbf{T}^n \backslash \mathbf{R}^n$, and without loss of generality suppose $\mathbf{a}$ is of the form $\mathbf{a} = (a_1, \dots, a_m, 0_{\mathbf{T}}, \dots, 0_{\mathbf{T}})$ with $a_i > 0_{\mathbf{T}}$ for $i = 1, \dots, m$. (Let $m = 0$ denote the case where $\mathbf{a}$ is the tropical origin.) Then note that for sufficiently negative $M \in \mathbf{R}$ we have $f(\mathbf{a}) = f(a_1, \dots, a_m, M, \dots, M) = g(a_1, \dots, a_m, M, \dots, M) = g(\mathbf{a})$. We therefore have $f(\mathbf{a}) = g(\mathbf{a})$ for all $\mathbf{a} \in \mathbf{T}^n$, and hence $(f, g) \in \mathbf{E}(\mathbf{T}^n)$. $\square$

We call the $\mathbf{T}$-semialgebra $\mathbf{T}[\mathbf{x}]/\mathbf{E}(\mathbf{T}^n)$ the **semiring of tropical polynomial functions**, denoted $\mathrm{Poly}[\mathbf{T}^n]$. By the above lemma, it is an integral domain.

As in classical algebraic geometry, it is natural to ask in what sense the functors $\mathbf{V}$ and $\mathbf{E}$ establish a correspondence between subsets of $\mathbf{T}^n$ and congruences on $\mathbf{T}[\mathbf{x}]$. In particular, two natural questions arise. First, when is the set $\mathbf{V}(E)$ nonempty? And secondly, what is the algebraic structure of $\mathbf{E}(\mathbf{V}(E))$? Answers to these questions represent tropical analogues of the weak and strong forms of Hilbert's Nullstellensatz, respectively. We lay the foundations for these results by analyzing two simple cases: first, the case when $\mathbf{V}(f, g)$ is empty; and second, the case when $\mathbf{V}(f, g) = \mathbf{T}^n$.

3.2. **Tropical Polynomials that Never Agree.**

This section can be viewed as laying the foundation for the weak form of the Tropical Nullstellensatz for Congruences. First, a simple lemma:

Lemma 3.4. *Suppose* $f \in \mathbf{T}[\mathbf{x}]$ *and* $\mathbf{a}, \mathbf{b} \in \mathbf{T}^n$*. If* $a_i \leq b_i$ *for all* i*, then* $f(\mathbf{a}) \leq f(\mathbf{b})$*.*

Proof. Say $f(\mathbf{x}) = \oplus_{\mathbf{d}}(t_{\mathbf{d}} \odot \mathbf{x}^{\mathbf{d}}) = \max_{\mathbf{d}}\{t_{\mathbf{d}} + \mathbf{d} \cdot \mathbf{x}\}$. If $\mathbf{a}, \mathbf{b} \in \mathbf{T}^n$ are such that $a_i \leq b_i$ for every i, then $\mathbf{d} \cdot \mathbf{a} \leq \mathbf{d} \cdot \mathbf{b}$ for every $\mathbf{d} \in \mathbf{N}^n$. Thus, $f(\mathbf{a}) \leq f(\mathbf{b})$. $\square$

For reasons that will become clear when we prove the weak form of the Tropical Nullstellensatz for Congruences, we now analyze the slightly more general situation in which a pair of functions (f, g) agrees at most in $\mathbf{T}^n \backslash \mathbf{R}^n$.

Lemma 3.5. *Suppose* $f, g \in \mathbf{T}[\mathbf{x}]$ *are such that* $f(\mathbf{a}) \neq g(\mathbf{a})$ *for every* $\mathbf{a} \in \mathbf{R}^n$*. Then either* $f(\mathbf{a}) \leq g(\mathbf{a})$ *for every* $\mathbf{a} \in \mathbf{T}^n$*, or* $f(\mathbf{a}) \geq g(\mathbf{a})$ *for every* $\mathbf{a} \in \mathbf{T}^n$*. In both cases, equality occurs only when* $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$*.*

Proof. The result is trivially true if either f or g is the zero polynomial, so we may assume f and g are nonzero. We now proceed by induction on the dimension, n. First suppose $n = 1$. Observe that $f_{|\mathbf{R}}$ and $g_{|\mathbf{R}}$ are continuous, piecewise-linear real-valued functions. By hypothesis, these functions are never equal, so by the usual Intermediate Value Theorem we must either have $f(a) < g(a)$ for all $a \in \mathbf{R}$, or $f(a) > g(a)$ for all $a \in \mathbf{R}$. We claim that whichever inequality holds, the same inequality (though no longer strict) must hold for $f(0_{\mathbf{T}})$ and $g(0_{\mathbf{T}})$.

Say $f(x) = \oplus_{i\geq 0}(c_i \odot x^{\odot i}) = \max_{i\geq 0}\{c_i + ix\}$ and $g(x) = \oplus_{i\geq 0}(d_i \odot x^{\odot i}) = \max_{i\geq 0}\{d_i + ix\}$. Let m_f (resp. m_g) be the least index i for which $c_i > 0_{\mathbf{T}}$ (resp. $d_i > 0_{\mathbf{T}}$). Then there exists some real number M such that $f(a) = c_{m_f} + m_f a$ and $g(a) = d_{m_g} + m_g a$ for all $a \in [-\infty, M] \subset \mathbf{T}$. If $m_f, m_g > 0$, then $f(0_{\mathbf{T}}) = 0_{\mathbf{T}} = g(0_{\mathbf{T}})$. If $m_f = 0$ and $m_g > 0$, then $f(a) = c_0 > g(a)$ for $a \in \mathbf{R}$ sufficiently negative, and $f(0_{\mathbf{T}}) = c_0 > 0_{\mathbf{T}} = g(0_{\mathbf{T}})$. Similarly, if $m_g = 0$ and $m_f > 0$, then $f(a) < d_0 = g(a)$ for $a \in \mathbf{R}$ sufficiently negative, and $f(0_{\mathbf{T}}) = 0_{\mathbf{T}} < d_0 = g(0_{\mathbf{T}})$. Finally, if both $m_f, m_g = 0$, then $f(a) = c_0 = f(0_{\mathbf{T}})$ and $g(a) = d_0 = g(0_{\mathbf{T}})$ for $a \in \mathbf{R}$ sufficiently negative. Thus, in every case we either have $f(0_{\mathbf{T}}) = 0_{\mathbf{T}} = g(0_{\mathbf{T}})$ or the relationship between $f(0_{\mathbf{T}})$ and $g(0_{\mathbf{T}})$ is the same as the relationship between $f(a)$ and $g(a)$ for $a \in \mathbf{R}$ sufficiently negative. We've therefore proven the statement in the case $n = 1$.

Suppose now $n > 1$, and that the statement of the lemma holds in dimensions less than n. The functions $f_{|\mathbf{R}^n}$ and $g_{|\mathbf{R}^n}$ are continuous, piecewise-linear real-valued functions that are never equal, so by the Intermediate Value Theorem we either have $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{R}^n$, or $f(\mathbf{a}) > g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{R}^n$. For each $i = 1, \ldots, n$, let $H_i = \{\mathbf{a} \in \mathbf{T}^n \mid a_i = 0_{\mathbf{T}}\} \subset \mathbf{T}^n$. Then $H_i \cong \mathbf{T}^{n-1}$, and $f_{|H_i}, g_{|H_i}$ are never equal on $\mathbf{R}^{n-1} \subset H_i$. By induction, for each i we must have either $f(\mathbf{a}) \leq g(\mathbf{a})$ for every $\mathbf{a} \in H_i$, or $f(\mathbf{a}) \geq g(\mathbf{a})$ for every $\mathbf{a} \in H_i$, with equality only when $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$. As $H_i \cap \mathbf{R}^n \neq \emptyset$ for each i, all of these inequalities must agree. As $\mathbf{R}^n \cup \bigcup_{i=1}^n H_i = \mathbf{T}^n$, it follows that either $f(\mathbf{a}) \leq g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$, or $f(\mathbf{a}) \geq g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$, again with equality only when $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$. Thus, the result holds for the general case. $\square$

Corollary 3.6. *Suppose $f, g \in \mathbf{T}[\mathbf{x}]$ are such that $f(\mathbf{a}) \neq g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$. Then either $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$, or $f(\mathbf{a}) > g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$.*

Lemma 3.7. *Suppose $f, g \in \mathbf{T}[\mathbf{x}]$ are such that $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{R}^n$. Then there exists some real number $\epsilon > 0$ such that $f(\mathbf{a}) + \epsilon \leq g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$, with equality only when $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$.*

Proof. The result is trivially true if f is the zero polynomial, so we may assume f is nonzero. We now proceed by induction on the dimension, n. We first consider the case when $n = 1$. Note that if $f(0_{\mathbf{T}}) = 0_{\mathbf{T}} = -\infty$, then we always have $f(0_{\mathbf{T}}) + \epsilon \leq g(0_{\mathbf{T}})$ regardless of ϵ. On the other hand, suppose $f(0_{\mathbf{T}}) > 0_{\mathbf{T}}$ (and hence also $g(0_{\mathbf{T}}) > 0_{\mathbf{T}}$). As f and g are piecewise-linear (with finitely many pieces), there is a real number $M \in \mathbf{R}$ such that f and g are linear on $[-\infty, M] \subset \mathbf{T}$. Under our current assumption that $f(0_{\mathbf{T}}), g(0_{\mathbf{T}}) > 0_{\mathbf{T}}$, it follows that f and g must actually be constant on $[-\infty, M]$. In particular, $0_{\mathbf{T}} < f(0_{\mathbf{T}}) = f(M) < g(M) = g(0_{\mathbf{T}})$, and so there certainly exists a real number $\epsilon_1 > 0$ such that $f(0_{\mathbf{T}}) + \epsilon_1 < g(0_{\mathbf{T}})$.

Now, as the functions $f_{|\mathbf{R}}$ and $g_{|\mathbf{R}}$ are real-valued and piecewise-linear (with finitely many pieces) with $f(a) < g(a)$ always, it follows that

$$\epsilon' := \inf_{a\in\mathbf{R}}(g(a) - f(a)) = \min_{a\in\mathbf{R}}(g(a) - f(a)) > 0.$$

Any real number $0 < \epsilon < \min\{\epsilon', \epsilon_1\}$ now satisfies $f(a)+\epsilon \leq g(a)$ for all $a \in \mathbf{T}$, with equality only when $f(a) = 0_{\mathbf{T}} = g(a)$. We've thus proven the statement for the case $n = 1$.

Suppose now $n > 1$, and that the statement of the lemma holds in dimensions less than n. For each $i = 1, \dots, n$, let $H_i = \{\mathbf{a} \in \mathbf{T}^n \mid a_i = 0_{\mathbf{T}}\} \subset \mathbf{T}^n$. Then $H_i \cong \mathbf{T}^{n-1}$, and $f_{|_{H_i}}, g_{|_{H_i}}$ are tropical polynomial functions that satisfy $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{R}^{n-1} \subset H_i$. By induction, for each i there is some real number $\epsilon_i > 0$ such that $f(\mathbf{a}) + \epsilon_i \leq g(\mathbf{a})$ for every $\mathbf{a} \in H_i$, with equality only when $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$. Now observe that the functions $f_{|_{\mathbf{R}^n}}, g_{|_{\mathbf{R}^n}}$ are real-valued and piecewise-linear (with finitely many pieces), with $f(\mathbf{a}) < g(\mathbf{a})$ always. It follows that

$$\epsilon' := \inf_{\mathbf{R}^n}(g(\mathbf{a}) - f(\mathbf{a})) = \min_{\mathbf{R}^n}(g(\mathbf{a}) - f(\mathbf{a})) > 0.$$

Any real number $0 < \epsilon < \min\{\epsilon', \epsilon_1, \dots, \epsilon_n\}$ now satisfies $f(\mathbf{a}) + \epsilon \leq g(\mathbf{a})$ for all $\mathbf{a} \in \mathbf{T}^n$, with equality only when $f(\mathbf{a}) = 0_{\mathbf{T}} = g(\mathbf{a})$, as desired. □

Corollary 3.8. *Suppose $f, g \in \mathbf{T}[\mathbf{x}]$ are such that $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$. Then there exists some real number $\epsilon > 0$ such that $f(\mathbf{a}) + \epsilon < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$.*

Corollary 3.9. *Suppose $f, g \in \mathbf{T}[\mathbf{x}]$ satisfy $\mathbf{V}(f, g) \subseteq \mathbf{T}^n \backslash \mathbf{R}^n$, and without loss of generality assume g is nonzero. Then $(g, \epsilon \odot g) \in \langle (f, g), \mathbf{E}(\mathbf{T}^n) \rangle$ for some real number $\epsilon > 0$.*

Proof. By Lemmas 3.5 and 3.7, there exists $\epsilon > 1_{\mathbf{T}}$ such that $f(\mathbf{a}) + \epsilon \leq g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$. As tropical addition corresponds to real maximum, it follows that $g \oplus (\epsilon \odot f)$ and g define the same polynomial functions, i.e., $(g \oplus (\epsilon \odot f), g) \in \mathbf{E}(\mathbf{T}^n)$. As $(\epsilon \odot f, \epsilon \odot g) = (\epsilon, \epsilon) \odot (f, g) \in \langle (f, g) \rangle$, we also have $(g \oplus (\epsilon \odot f), \epsilon \odot g) = (g, g) \oplus (\epsilon \odot f, \epsilon \odot g) \in \langle (f, g) \rangle$. By transitivity, then, we have $(g, \epsilon \odot g) \in \langle (f, g), \mathbf{E}(\mathbf{T}^n) \rangle$. □

3.3. **Tropical Polynomials that Always Agree.**

We next analyze the situation in which two tropical polynomials agree everywhere, and hence define the same function on $\mathbf{T}^n$. This section can be viewed as laying the groundwork for the strong form of the Tropical Nullstellensatz for Congruences. Given a tropical polynomial $f \in \mathbf{T}[\mathbf{x}]$, write $[f]$ for the equivalence class of tropical polynomials that define the same function on $\mathbf{T}^n$, i.e., let $[f]$ denote the image of f in $\mathrm{Poly}[\mathbf{T}^n] = \mathbf{T}[\mathbf{x}]/\mathbf{E}(\mathbf{T}^n)$. We will identify two "extremal" representatives in $[f]$, which we refer to as the *desaturated* and *saturated* representatives of $[f]$.

3.3.1. *Desaturated Polynomials.* The "minimal" representative of $[f]$ is the unique tropical polynomial f_{dsat} whose coefficients are each minimal among all tropical polynomials representing the function $[f]$. (As tropical addition corresponds to real max, this is well defined, and can be computed independently for each coefficient.) We call f_{dsat} the **desaturated representative** of $[f]$. This polynomial can be explicitly computed as follows. First choose any representative polynomial for $[f]$, say $f(\mathbf{x}) = \bigoplus_I a_I \odot \mathbf{x}^{\odot I} = \max_I\{a_I + I \cdot \mathbf{x}\}$. The graph of $[f]$ is then given by the upper convex hull of the union of the hyperplanes $\{z = a_I + I \cdot \mathbf{x}\} \subset \mathbf{T}^{n+1}$. The projection of this graph onto $\mathbf{T}^n$ gives a finite decomposition of $\mathbf{T}^n$ into closed n-dimensional polyhedral regions, in the interior of each of which $[f]$ is given by a single affine linear equation (one of the monomial terms in the tropical polynomial). The boundaries of these regions are where two (or more) of the monomial terms restrict to the same affine-linear function. In particular, the union of the codimension-one faces is precisely the double-max locus of f. We say the monomials corresponding to the top-dimensional polyhedra in this decomposition **contribute** to the function $[f]$. For each index $\mathbf{d} \in \mathbf{Z}^n_{\geq 0}$, we set $\underline{a}_{\mathbf{d}} = a_{\mathbf{d}}$ if the $\mathbf{d}^{\text{th}}$ monomial contributes to $[f]$, and $\underline{a}_{\mathbf{d}} = 0_{\mathbf{T}}$ otherwise. Then

$$f_{\text{dsat}}(\mathbf{x}) = \bigoplus_{\mathbf{d}} \underline{a}_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}.$$

If $g(\mathbf{x}) = \bigoplus_{\mathbf{d}} b_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$ is any other tropical polynomial with $[g] = [f]$, then $b_{\mathbf{d}} \geq \underline{a}_{\mathbf{d}}$ for every index $\mathbf{d}$, with equality for those indices that contribute to $[f]$. The desaturation of f is the unique tropical polynomial with this property. We say f is desaturated if $f = f_{\text{dsat}}$.

Example 3.10. Fix some $s > t > 1_{\mathbf{T}}$, and consider the tropical polynomial $f(x, y) = x^{\odot 3} \oplus (x \odot y) \oplus (s \odot y^{\odot 2}) \oplus t$. One can check $f_{\text{dsat}}(x, y) = x^{\odot 3} \oplus (s \odot y^{\odot 2}) \oplus t$.

Lemma 3.11. *If $f(\mathbf{x}) = \bigoplus_{\mathbf{d}} \mathbf{a}_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$ is desaturated, then $(f^{\odot m})_{dsat}(\mathbf{x}) = \bigoplus_{\mathbf{d}} \mathbf{a}_{\mathbf{d}}^{\odot m} \odot \mathbf{x}^{\odot m\mathbf{d}}$.*

Proof. In real operations, the polynomial function $[f^{\odot m}]$ is given by m times the function $[f]$. It therefore induces the same polyhedral decomposition of $\mathbf{T}^n$ as $[f]$, with the affine linear function on each cell equaling m times the affine linear function for $[f]$ on that cell. Thus, if $a_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$ is the monomial for $[f]$ on a given cell, then $a_{\mathbf{d}}^{\odot m} \odot \mathbf{x}^{\odot m\mathbf{d}}$ is the monomial for $[f^{\odot m}]$ on that cell. The result follows. □

3.3.2. *Saturated Polynomials.* The "maximal" representative of $[f]$ is the unique tropical polynomial f_{sat} whose coefficients are maximal among all tropical polynomials representing $[f]$. We call f_{sat} the **saturated representative** of $[f]$, and denote its coefficients by $\overline{a}_{\mathbf{d}}$. Note that if $\overline{a}_{\mathbf{d}} > 0_{\mathbf{T}}$, then there always exists some point $\mathbf{t} \in \mathbf{R}^n$ with $\overline{a}_{\mathbf{d}} \odot \mathbf{t}^{\odot \mathbf{d}} = f(\mathbf{t})$. If $g(\mathbf{x}) = \bigoplus_{\mathbf{d}} b_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$ is any other tropical polynomial with $[g] = [f]$, then $b_{\mathbf{d}} \leq \overline{a}_{\mathbf{d}}$ for every $\mathbf{d}$. The saturation of f is the unique tropical polynomial with this property. We say f is saturated if $f = f_{\text{sat}}$.

Before explaining how to explicitly calculate f_{sat}, we first introduce some notation. Let $\mathcal{C} = \{\mathbf{d}_1, \ldots, \mathbf{d}_k\}$ denote the set of indices contributing to $[f]$. For each $\mathbf{d} \in \mathcal{C}$, let $Z_{\mathbf{d}} \subseteq \mathbf{T}^n$ denote the closed n-dimensional polyhedral region on which $[f]$ is given by $\underline{a}_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$. For each convex combination $\sum_{i=1}^k \lambda_i \mathbf{d}_i$, let Z_λ denote the intersection of those polyhedral regions $Z_{\mathbf{d}_i}$ for which $\lambda_i > 0$, and define

$$\tilde{a}_\lambda = \bigodot_{i=1}^{k} \underline{a}_{\mathbf{d}_i}^{\odot \lambda_i}.$$

Lastly, considering $\mathcal{C}$ as a collection of lattice points in $\mathbf{R}^n$, let Δ denote the convex hull of $\mathcal{C}$. We can now give a recipe for computing f_{sat}.

Proposition 3.12. *Let $f \in \mathbf{T}[\mathbf{x}]$ be a tropical polynomial, and let $f_{sat}(\mathbf{x}) = \bigoplus_{\mathbf{d}} \overline{a}_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$. For each index $\mathbf{d}$, the following hold:*

i) if $\mathbf{d} \notin \Delta$, then $\overline{a}_{\mathbf{d}} = 0_{\mathbf{T}}$; and
ii) if $\mathbf{d} \in \Delta$, then $\overline{a}_{\mathbf{d}} \geq \tilde{a}_\lambda$ for any convex combination $\mathbf{d} = \sum_{i=1}^k \lambda_i \mathbf{d}_i$, with equality if $Z_\lambda \neq \emptyset$. Moreover, there exists at least one such convex combination with $Z_\lambda \neq \emptyset$.

Proof. Suppose $\mathbf{d}$ is any index such that $\overline{a}_{\mathbf{d}} > 0_{\mathbf{T}}$. Then there exists some $\mathbf{t}_0 \in \mathbf{R}^n$ with $\overline{a}_{\mathbf{d}} \odot \mathbf{t}_0^{\odot \mathbf{d}} = f(\mathbf{t}_0)$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the collection of those indices $\mathbf{d}_i$ for which $\mathbf{t}_0 \in Z_{\mathbf{d}_i}$, and let $\Delta' \subseteq \Delta$ be the convex hull of $\mathcal{C}'$. We first prove that every hyperplane in $\mathbf{R}^n$ containing $\mathbf{d}$ intersects Δ'. Let H be any hyperplane containing $\mathbf{d}$, with normal vector $\mathbf{v} \in \mathbf{R}^n$. Since the polyhedra $\{Z_{\mathbf{d}'} \mid \mathbf{d}' \in \mathcal{C}'\}$ give a local decomposition of $\mathbf{R}^n$ near $\mathbf{t}_0$, for sufficiently small $\mu > 0$ the vector $\mathbf{t}_0 + \mu\mathbf{v}$ is contained in (at least) one of the polyhedra $Z_{\mathbf{d}'}$, i.e., the direction $\mathbf{v}$ points into the region $Z_{\mathbf{d}'}$. By the definition of $\overline{a}_{\mathbf{d}}$, we must then have $\overline{a}_{\mathbf{d}} \odot (\mathbf{t}_0 + \mu\mathbf{v})^{\odot \mathbf{d}} \leq f(\mathbf{t}_0 + \mu\mathbf{v}) = \underline{a}_{\mathbf{d}'} \odot (\mathbf{t}_0 + \mu\mathbf{v})^{\odot \mathbf{d}'}$ for all sufficiently small $\mu > 0$, and consequently for all $\mu > 0$. In other words, the hyperplane $z = \overline{a}_{\mathbf{d}} + \mathbf{d} \cdot \mathbf{x}$ must lie beneath the hyperplane $z = \underline{a}_{\mathbf{d}'} + \mathbf{d}' \cdot \mathbf{x}$ in the direction $\mathbf{v}$ from the point $\mathbf{t}_0$. Now consider the sequence of points $\mathbf{x}_n = \mathbf{t}_0 + n\mathbf{v}$ for $n \in \mathbf{N}$. By the above discussion, we must have $\overline{a}_{\mathbf{d}} \odot (\mathbf{x}_n)^{\odot \mathbf{d}} \leq \underline{a}_{\mathbf{d}'} \odot (\mathbf{x}_n)^{\odot \mathbf{d}'}$ for every $n \in \mathbf{N}$, and hence (in real operations)

$$\overline{a}_{\mathbf{d}} + \mathbf{d} \cdot \mathbf{t}_0 + n(\mathbf{d} \cdot \mathbf{v}) \leq \underline{a}_{\mathbf{d}'} + \mathbf{d}' \cdot \mathbf{t}_0 + n(\mathbf{d}' \cdot \mathbf{v})$$

for all $n \in \mathbf{N}$. The coefficient $\overline{a}_{\mathbf{d}}$ is real (by hypothesis), as is the coefficient $\underline{a}_{\mathbf{d}'}$ (since $\mathbf{d}' \in \mathcal{C}$), and the point $\mathbf{t}_0 \in \mathbf{R}^n$, so the above inequality involves only real (finite) numbers. Since it holds for all n, we must therefore have $\mathbf{d} \cdot \mathbf{v} \leq \mathbf{d}' \cdot \mathbf{v}$. Repeating this argument in the direction $-\mathbf{v}$, we can similarly deduce there must exist some index $\mathbf{d}'' \in \mathcal{C}'$ with $\mathbf{d}'' \cdot \mathbf{v} \leq \mathbf{d} \cdot \mathbf{v}$. It follows that there exists $\lambda \in [0, 1]$ such that $(\lambda\mathbf{d}' + (1-\lambda)\mathbf{d}'') \cdot \mathbf{v} = \mathbf{d} \cdot \mathbf{v}$, i.e., $\lambda\mathbf{d}' + (1-\lambda)\mathbf{d}'' \in H$. Notice that $\mathbf{d}', \mathbf{d}'' \in \mathcal{C}' \subseteq \Delta'$, so by convexity we also have $\lambda\mathbf{d}' + (1-\lambda)\mathbf{d}'' \in \Delta'$. We've thus proven that every hyperplane containing $\mathbf{d}$ intersects Δ'. As Δ' is convex, this implies $\mathbf{d} \in \Delta' \subseteq \Delta$. This proves (i). Note that it also proves that if $\overline{a}_{\mathbf{d}} > 0_{\mathbf{T}}$, then $\mathbf{d}$ can be written as a convex combination $\mathbf{d} = \sum_i \lambda_i \mathbf{d}_i$ with $Z_\lambda \neq \emptyset$.

Conversely, suppose $\mathbf{d} \in \Delta$ and write $\mathbf{d}$ as convex combination $\mathbf{d} = \sum_i \lambda_i \mathbf{d}_i$. We claim that we must then have $\overline{a}_{\mathbf{d}} \geq \tilde{a}_\lambda$. Indeed, an inequality $\overline{a}_{\mathbf{d}} < \tilde{a}_\lambda$ would imply the existence of a point $\mathbf{t} \in \mathbf{T}^n$ with $\tilde{a}_\lambda \odot \mathbf{t}^{\odot \mathbf{d}} > f(\mathbf{t})$. But then we would have

$$f(\mathbf{t}) < \tilde{a}_\lambda \odot \mathbf{t}^{\odot \mathbf{d}} = \bigodot_{i=1}^{k} \left(\underline{a}_{\mathbf{d}_i} \odot \mathbf{t}^{\odot \mathbf{d}_i}\right)^{\odot \lambda_i} = \sum_{i=1}^{k} \lambda_i (\underline{a}_{\mathbf{d}_i} + \mathbf{d}_i \cdot \mathbf{t}) \leq \max_{1 \leq i \leq k} (\underline{a}_{\mathbf{d}_i} + \mathbf{d}_i \cdot \mathbf{t}) = f(\mathbf{t}),$$

a contradiction. Thus, we must have $\overline{a}_{\mathbf{d}} \geq \tilde{a}_\lambda > 0_{\mathbf{T}}$. By our argument above, it then follows that $\mathbf{d}$ can be written as a convex combination $\mathbf{d} = \sum_i \lambda_i \mathbf{d}_i$ with $Z_\lambda \neq \emptyset$. Note that since Z_λ is nonempty, it must contain a point $\mathbf{t}_0 \in \mathbf{R}^n$ (by considering the construction of the polyhedral decomposition of $\mathbf{T}^n$). Then observe that

$$\tilde{a}_\lambda \odot \mathbf{t}_0^{\odot \mathbf{d}} = \bigodot_{i=1}^{k} \left(\underline{a}_{\mathbf{d}_i} \odot \mathbf{t}_0^{\odot \mathbf{d}_i}\right)^{\odot \lambda_i} = \bigodot_{i=1}^{k} f(\mathbf{t}_0)^{\odot \lambda_i} = f(\mathbf{t}_0).$$

As $\mathbf{t}_0 \in \mathbf{R}^n$, this is an equality of real numbers, and hence implies $\overline{a}_{\mathbf{d}} \leq \tilde{a}_\lambda$. Thus, equality holds. $\square$

Remark 3.13. Note that for $\mathbf{d} \in \mathcal{C}$ the above proposition guarantees $\overline{a}_{\mathbf{d}} = \underline{a}_{\mathbf{d}}$, as expected.

Example 3.14. Fix some $t > 1_{\mathbf{T}}$, and consider the desaturated tropical polynomial $f(x) = x^{\odot 2} \oplus t$. By the above proposition, one easily computes $f_{\text{sat}}(x) = x^{\odot 2} \oplus \left(t^{\odot 1/2} \odot x\right) \oplus t$.

Example 3.15. Fix some $s > t > 1_{\mathbf{T}}$, and consider the desaturated tropical polynomial $f(x, y) = x^{\odot 3} \oplus (s \odot y^{\odot 2}) \oplus t$. Proposition 3.12 gives a straightforward recipe for computing the coefficients of f_{sat}. For example, to compute the coefficient of $x \odot y$ in f_{sat}, we first observe that $(1, 1) = \frac{1}{6}(0, 0) + \frac{1}{3}(3, 0) + \frac{1}{2}(0, 2)$, and that $Z_{(0,0)} \cap Z_{(3,0)} \cap Z_{(0,2)} \neq \emptyset$. It follows that the coefficient of $x \odot y$ in f_{sat} is $t^{\odot 1/6} \odot 1_{\mathbf{T}}^{\odot 1/3} \odot s^{\odot 1/2}$. The other coefficients are found similarly. In the end, one computes $f_{\text{sat}}(x, y) = x^{\odot 3} \oplus (t^{\odot 1/3} \odot x^{\odot 2}) \oplus (t^{\odot 1/6} \odot s^{\odot 1/2} \odot x \odot y) \oplus (s \odot y^{\odot 2}) \oplus (t^{\odot 2/3} \odot x) \oplus (t^{\odot 1/2} \odot s^{\odot 1/2} \odot y) \oplus t$.

Proposition 3.16. *Suppose $f \in \mathbf{T}[\mathbf{x}]$ is nonzero, and $k = |\mathcal{C}|$ is the number of monomials that contribute to $[f]$. Then $(f^{\odot m})_{sat} \odot f^{\odot m}$ is saturated for every $m \geq k$.*

Proof. First observe that every coefficient in $(f_{\text{dsat}}^{\odot m})_{\text{sat}} \odot f_{\text{dsat}}^{\odot m}$ is less than or equal to the corresponding coefficient in $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$, and hence the latter polynomial is automatically saturated if the former is. Without loss of generality, we may therefore assume f is desaturated. Next, observe that by Lemma 3.11 we have

$$\left((f^{\odot m})_{\text{sat}} \odot f^{\odot m}\right)_{\text{dsat}} = (f^{\odot 2m})_{\text{dsat}} = \oplus_{\mathbf{d}} \underline{a}_{\mathbf{d}}^{\odot 2m} \odot \mathbf{x}^{\odot 2m\mathbf{d}},$$

and hence the convex hull of the Newton polytope for $((f^{\odot m})_{\text{sat}} \odot f^{\odot m})_{\text{dsat}}$ is precisely $2m\Delta$, where Δ is the convex hull of the Newton polytope for f_{dsat}.

We now compute the coefficients in $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$ and prove they agree with those of its saturation. By Proposition 3.12 and the above observation, we may restrict our attention to indices $\mathbf{e} \in 2m\Delta$. Fix any index $\mathbf{e} \in 2m\Delta$. By Proposition 3.12, the coefficient of $\mathbf{x}^{\odot \mathbf{e}}$ in $((f^{\odot m})_{\text{sat}} \odot f^{\odot m})_{\text{sat}}$ equals $\bigodot_i \underline{a}_{\mathbf{d}_i}^{\odot 2m\lambda_i}$ for any convex combination $\mathbf{e} = \sum_i \lambda_i(2m\mathbf{d}_i)$ with $Z_\lambda \neq \emptyset$. Fix some such convex combination. To compute the coefficient of $\mathbf{x}^{\odot \mathbf{e}}$ in $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$, first observe that (again by Proposition 3.12)

$$(f^{\odot m})_{\text{sat}} = \bigoplus_{\mathbf{d} \in m\Delta} b_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}},$$

where $b_{\mathbf{d}} = \bigodot_i \underline{a}_{\mathbf{d}_i}^{\odot m\mu_i}$ for any convex combination $\mathbf{d} = \sum_i \mu_i(m\mathbf{d}_i)$ with $Z_\mu \neq \emptyset$. Next observe that, by direct calculation,

$$f^{\odot m} = \bigoplus_{\mathbf{d}' \in m\Delta} c_{\mathbf{d}'} \odot \mathbf{x}^{\odot \mathbf{d}'},$$

where $c_{\mathbf{d}'} = \bigoplus_{\mathbf{n} \in \mathcal{S}_{\mathbf{d}'}} \underline{a}_{\mathbf{d}_i}^{\odot n_i}$ and where $\mathcal{S}_{\mathbf{d}'} = \{\mathbf{n} \in \mathbf{N}_0^k \mid \mathbf{d}' = \sum_i n_i \mathbf{d}_i, m = \sum_i n_i\}$. (Here $c_{\mathbf{d}'} = 0_{\mathbf{T}}$ if $\mathcal{S}_{\mathbf{d}'}$ is empty.) It follows that the coefficient of $\mathbf{x}^{\odot \mathbf{e}}$ in $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$ is

$$d_{\mathbf{e}} = \bigoplus_{(\mu, \mathbf{n}) \in \mathcal{T}_{\mathbf{e}}} \bigodot_{i=1}^{k} \underline{a}_{\mathbf{d}_i}^{\odot (m\mu_i + n_i)},$$

where $\mathcal{T}_{\mathbf{e}}$ is the collection of all pairs $(\mu, \mathbf{n}) \in [0,1]^k \times \mathbf{N}_0^k$ such that $\sum_i \mu_i = 1$, $Z_\mu \neq \emptyset$, $\sum_i n_i = m$, and $\mathbf{e} = \sum_i (m\mu_i + n_i)\mathbf{d}_i$, and where $d_{\mathbf{e}} = 0_{\mathbf{T}}$ if $\mathcal{T}_{\mathbf{e}}$ is empty.

We claim $\mathcal{T}_{\mathbf{e}}$ is not empty (and here we will use the hypothesis $m \geq k$). For each $i = 1, \ldots, k$, define ${}_0\tau_i = 2m\lambda_i$ and ${}_0n_i = 0$. Observe then that $\sum_{i=1}^k {}_0\tau_i = 2m \geq k$, so there must exist at least one index i_1 with ${}_0\tau_{i_1} \geq 1$. Now define

$${}_1n_i = \begin{cases} {}_0n_i, & \text{if } i \neq i_1 \\ {}_0n_i + 1, & \text{if } i = i_1 \end{cases}.$$

and

$${}_1\tau_i = \begin{cases} {}_0\tau_i, & \text{if } i \neq i_1 \\ {}_0\tau_i - 1, & \text{if } i = i_1 \end{cases}.$$

Observe that $2m\lambda_i = {}_1\tau_i + {}_1n_i$ for every i, $\sum_i {}_1n_i = 1$, and that $\sum_{i=1}^k {}_1\tau_i = 2m - 1 \geq k$. The last inequality guarantees that there again exists at least one index i_2 with ${}_1\tau_{i_2} \geq 1$. We thus inductively continue, at each stage $0 \leq l \leq m$ producing nonnegative real numbers ${}_l\tau_i$ and nonnegative integers ${}_ln_i$ with $2m\lambda_i = {}_l\tau_i + {}_ln_i$, $\sum_{i=1}^k {}_ln_i = l$, and $\sum_{i=1}^k {}_l\tau_i = 2m - l \geq k$. In particular, at stage m we produce nonnegative real numbers ${}_m\tau_i$ and nonnegative integers ${}_mn_i$ with $2m\lambda_i = {}_m\tau_i + {}_mn_i$, $\sum_{i=1}^k {}_mn_i = m$ and $\sum_{i=1}^k {}_m\tau_i = m$. In other words, if we let $\mu_i = {}_m\tau_i/m$ and $n_i = {}_mn_i$, then $\mathbf{e} = \sum_i (m\mu_i + n_i)\mathbf{d}_i$. Moreover, by construction we have $\mu_i > 0$ only if $\lambda_i > 0$, and hence $Z_\lambda \subseteq Z_\mu$, i.e., Z_μ must be nonempty. We've thus constructed a pair $(\mu, \mathbf{n}) \in \mathcal{T}_\mathbf{e}$, and hence proven $\mathcal{T}_\mathbf{e}$ is nonempty.

In fact, our construction has proven slightly more. We have actually proven the existence of a pair $(\mu, \mathbf{n}) \in \mathcal{T}_\mathbf{e}$ with $2m\lambda = m\mu + \mathbf{n}$. For this pair, we have

$$\bigodot_{i=1}^{k} \underline{a}_{\mathbf{d}_i}^{\odot(m\mu_i+n_i)} = \bigodot_{i=1}^{k} \underline{a}_{\mathbf{d}_i}^{\odot 2m\lambda_i},$$

and hence

$$d_\mathbf{e} \geq \bigodot_{i=1}^{k} \underline{a}_{\mathbf{d}_i}^{\odot 2m\lambda_i}.$$

Since the right-hand side represents the maximum possible value for the coefficient of $\mathbf{x}^{\odot\mathbf{e}}$ in any representative of $[(f^{\odot m})_{\text{sat}} \odot f^{\odot m}]$, we must actually have equality. Thus, $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$ is saturated. $\square$

Corollary 3.17. *Suppose $f \in \mathbf{T}[\mathbf{x}]$ is nonzero, and $k = |\mathcal{C}|$ is the number of monomials that contribute to $[f]$. Then for every $m \geq k$ and every choice of tropical polynomials $f_0, \ldots, f_m \in [f]$, the polynomial $(f_0^{\odot m})_{sat} \odot f_1 \odot \cdots \odot f_m$ is saturated.*

Proof. Fix some $f \in \mathbf{T}[\mathbf{x}]$, $m \geq k$, and $f_0, \ldots, f_m \in [f]$. First observe that, by applying Proposition 3.16 to the polynomial f_{dsat}, the polynomial $(f_{\text{dsat}}^{\odot m})_{\text{sat}} \odot f_{\text{dsat}}^{\odot m}$ is saturated. Since $(f_0^{\odot m})_{\text{sat}} = (f_{\text{dsat}}^{\odot m})_{\text{sat}}$, we thus also have that $(f_0^{\odot m})_{\text{sat}} \odot f_{\text{dsat}}^{\odot m}$ is saturated. Now observe that every coefficient in $f_{\text{dsat}}^{\odot m}$ is less than or equal to the corresponding coefficient in $f_1 \odot \cdots \odot f_m$, and hence every coefficient in $(f_0^{\odot m})_{\text{sat}} \odot f_{\text{dsat}}^{\odot m}$ is less than or equal to the corresponding coefficient in $(f_0^{\odot m})_{\text{sat}} \odot f_1 \odot \cdots \odot f_m$. Since they both define the same function on $\mathbf{T}^n$, and since $(f_0^{\odot m})_{\text{sat}} \odot f_{\text{dsat}}^{\odot m}$ is saturated (i.e., all coefficients are maximal), it follows that $(f_0^{\odot m})_{\text{sat}} \odot f_1 \odot \cdots \odot f_m$ is also be saturated. $\square$

Corollary 3.18. *Suppose $(f, g) \in \mathbf{E}(\mathbf{T}^n)$ and $k = |\mathcal{C}|$ is the number of monomials that contribute to $[f] = [g]$. Then $(f^{\odot m})_{sat} \odot f^{\odot m} = (f^{\odot m})_{sat} \odot g^{\odot m}$, for every $m \geq \max\{1, k\}$.*

Proof. Fix some $(f, g) \in \mathbf{E}(\mathbf{T}^n)$ and $m \geq \max\{1, k\}$. Then $g \in [f]$, so by the previous corollary it follows that $(f^{\odot m})_{\text{sat}} \odot f^{\odot m}$ and $(f^{\odot m})_{\text{sat}} \odot g^{\odot m}$ are both saturated. As they both define the same function on $\mathbf{T}^n$, it follows (by the uniqueness of saturated representatives) that they must be equal. $\square$

Example 3.19. Fix any $t > 1_\mathbf{T}$ and $s < t^{\odot 1/2}$, and consider the tropical polynomials $f(x) = x^{\odot 2} \oplus t$ and $g(x) = x^{\odot 2} \oplus (s \odot x) \oplus t$. By Example 3.14, we see that $f_{\text{sat}}(x) = g_{\text{sat}}(x) = x^{\odot 2} \oplus (t^{\odot 1/2} \odot x) \oplus t$, and hence $(f, g) \in \mathbf{E}(\mathbf{T}^n)$. As f is desaturated, the number of monomials that contribute to $[f]$ is two, and hence by the above corollary we must have

$f_{\text{sat}}^{\odot 2} \odot f^{\odot 2} = f_{\text{sat}}^{\odot 2} \odot g^{\odot 2}$. This is easily verified by direct computation. In fact, one can verify in this case that $f_{\text{sat}} \odot f = f_{\text{sat}} \odot g$.

We can now describe $\mathbf{E}(\mathbf{V}(E))$ in the special case of the trivial congruence $E = \Delta$:

Theorem 1. *The congruence $\mathbf{E}(\mathbf{T}^n)$ is the intersection of all prime congruences on $\mathbf{T}[\mathbf{x}]$.*

Proof. By Lemma 3.3, the congruence $\mathbf{E}(\mathbf{T}^n)$ is prime, and hence automatically contains the intersection of all prime congruences. Conversely, suppose P is any prime congruence on $\mathbf{T}[\mathbf{x}]$, suppose $(f, g) \in \mathbf{E}(\mathbf{T}^n)$, and suppose, by way of contradiction, $(f, g) \notin P$. By Corollary 3.18, we have $((f^{\odot m})_{\text{sat}} \odot f^{\odot m}, (f^{\odot m})_{\text{sat}} \odot g^{\odot m}) \in \Delta \subset P$ for any fixed $m \geq \max\{1, k\}$, where k is the number of monomials that contribute to $[f] = [g]$. By primeness (together with the supposition $(f, g) \notin P$) this implies $((f^{\odot m})_{\text{sat}}, 0_{\mathbf{T}}) \in P$. We then have $((f^{\odot m})_{\text{sat}}, f^{\odot m}) = (f^{\odot m}, f^{\odot m}) \oplus ((f^{\odot m})_{\text{sat}}, 0_{\mathbf{T}}) \in P$, hence by transitivity $(f^{\odot m}, 0_{\mathbf{T}}) \in P$, and hence by primeness $(f, 0_{\mathbf{T}}) \in P$. Similarly, $((f^{\odot m})_{\text{sat}}, g^{\odot m}) = (g^{\odot m}, g^{\odot m}) \oplus ((f^{\odot m})_{\text{sat}}, 0_{\mathbf{T}}) \in P$, hence $(g^{\odot m}, 0_{\mathbf{T}}) \in P$, hence $(g, 0_{\mathbf{T}}) \in P$. By transitivity, we then have $(f, g) \in P$, a contradiction. $\square$

4. The Weak Tropical Nullstellensatz for Congruences

We now aim to understand the general situation in which $\mathbf{V}(E)$ is empty. We begin by examining the case in which E is principally generated.

4.1. **The Principally Generated Case.**

Proposition 4.1. *Suppose $f, g \in \mathbf{T}[\mathbf{x}]$ and $\mathbf{V}(f, g) = \emptyset$. Then there exists $h \in \mathbf{T}[\mathbf{x}]$ with nonzero constant term such that $(h, \epsilon \odot h) \in \langle (f, g) \rangle$ for some real number $\epsilon > 0$.*

Proof. By Corollary 3.6, and without loss of generality, we may assume $f(\mathbf{a}) < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$. (Note that this implies g has nonzero constant term.) By Corollary 3.8, there then exists $\epsilon > 0$ such that $f(\mathbf{a}) + \epsilon < g(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}^n$. Notice that this implies $(g, g \oplus (\epsilon \odot f)) \in \mathbf{E}(\mathbf{T}^n)$. By Corollary 3.18, it then follows that $(g^{\odot m})_{\text{sat}} \odot g^{\odot m} = (g^{\odot m})_{\text{sat}} \odot (g \oplus (\epsilon \odot f))^{\odot m}$ for m sufficiently large. Now observe that

$$(\epsilon \odot f, g \oplus (\epsilon \odot f)) = (f, g) \oplus (\epsilon \odot f, \epsilon \odot f) \in \langle (f, g) \rangle,$$

and hence

$$\begin{aligned}((g^{\odot m})_{\text{sat}} \odot (\epsilon \odot f)^{\odot m}, (g^{\odot m})_{\text{sat}} \odot g^{\odot m}) &= ((g^{\odot m})_{\text{sat}} \odot (\epsilon \odot f)^{\odot m}, (g^{\odot m})_{\text{sat}} \odot (g \oplus (\epsilon \odot f))^{\odot m}) \\ &= ((g^{\odot m})_{\text{sat}}, (g^{\odot m})_{\text{sat}}) \odot (\epsilon \odot f, g \oplus (\epsilon \odot f))^{\odot m} \in \langle (f, g) \rangle.\end{aligned}$$

On the other hand, we also have

$$((g^{\odot m})_{\text{sat}} \odot (\epsilon \odot f)^{\odot m}, (g^{\odot m})_{\text{sat}} \odot (\epsilon \odot g)^{\odot m}) = ((g^{\odot m})_{\text{sat}} \odot \epsilon^{\odot m}, (g^{\odot m})_{\text{sat}} \odot \epsilon^{\odot m}) \odot (f, g)^{\odot m} \in \langle (f, g) \rangle,$$

and so by transitivity $((g^{\odot m})_{\text{sat}} \odot g^{\odot m}, \epsilon^{\odot m} \odot (g^{\odot m})_{\text{sat}} \odot g^{\odot m}) \in \langle (f, g) \rangle$. The result now follows, taking $h = (g^{\odot m})_{\text{sat}} \odot g^{\odot m}$. $\square$

The epsilon that occurs in the above situation is not special:

Lemma 4.2. *Suppose E is a congruence on $\mathbf{T}[\mathbf{x}]$ and $h \in \mathbf{T}[\mathbf{x}]$. If $(h, \epsilon \odot h) \in E$ for some real number $\epsilon > 0$, then $(h, \delta \odot h) \in E$ for every $\delta \in \mathbf{R}$.*

Proof. First consider the case of any $\delta \geq 0$. Choose any N such that $\epsilon^{\odot N} > \delta$. Notice that for every $n \geq 1$ one has

$$(\epsilon^{\odot(n-1)}, \epsilon^{\odot(n-1)}) \odot (h, \epsilon \odot h) = (\epsilon^{\odot(n-1)} \odot h, \epsilon^{\odot n} \odot h) \in E.$$

By transitivity and induction, it follows that $(h, \epsilon^{\odot n} \odot h) \in E$ for every $n \geq 1$. In particular, it follows that

$$(\delta \odot h, \epsilon^{\odot N} \odot h) = (h, \epsilon^{\odot N} \odot h) \oplus (\delta \odot h, \delta \odot h) \in E,$$

and so again by transitivity $(h, \delta \odot h) \in E$. Now suppose $\delta < 0$. Then $\delta^{\odot(-1)} > 0$, and so by the above argument $(h, \delta^{\odot(-1)} \odot h) \in E$. It immediately follows that

$$(h, \delta \odot h) = (\delta, \delta) \odot (\delta^{\odot(-1)} \odot h, h) \in E.$$

□

4.2. **Flatness.**

Proposition 4.1 suggests the possibility that $\mathbf{V}(E)$ is empty exactly when E contains a relation of the form $(h, \epsilon \odot h)$, with $h \in \mathbf{T}[\mathbf{x}]$ having nonzero constant term and $\epsilon \neq 1_{\mathbf{T}}$. We therefore consider a new property of congruences on $\mathbf{T}[\mathbf{x}]$ called "flatness." We begin by defining a related property that can be applied more generally to a congruence on any $\mathbf{T}$-semialgebra. We say a congruence E on a $\mathbf{T}$-semialgebra R is **flat over T** if multiplication by $\mathbf{T}$ is a faithful action on R/E, i.e., if whenever $(sr, tr) \in E$ for some $s, t \in \mathbf{T}$ one has either $s = t$ or $(r, 0_R) \in E$. We say R is **flat** over $\mathbf{T}$ if Δ_R is flat over $\mathbf{T}$. Flatness has many desirable properties. For instance, if R is flat over $\mathbf{T}$, then the structure morphism $i : \mathbf{T} \to R$ is injective.

Example 4.3. If R is a flat $\mathbf{T}$-semialgebra, then $R[x]$ is also a flat $\mathbf{T}$-semialgebra. Indeed, suppose $s \neq t \in \mathbf{T}$ and $f \in R[x]$ satisfies $sf = tf$. Writing $f(x) = \sum_d c_d x^d$, we must then have $\sum_d (sc_d)x^d = \sum_d (tc_d)x^d$, and hence $sc_d = tc_d$ for every d. As R is flat and $s \neq t$, this implies $c_d = 0_R$ for every d, and so $f = 0_{R[x]}$.

Example 4.4. By Example 2.6, the only congruences on $\mathbf{T}$ are Δ_T and $E_{\mathbf{F}_1}$. The former is flat but the latter is not, as $\mathbf{T}/E_{\mathbf{F}_1} \cong \mathbf{F}_1$, which is not flat over $\mathbf{T}$. Thus, there are no nontrivial flat congruences on $\mathbf{T}$.

Example 4.5. Suppose E is any congruence on $\mathbf{T}[\mathbf{x}]$, and suppose $(s \odot h, t \odot h) \in \mathbf{E}(\mathbf{V}(E))$ for some $s, t \in \mathbf{T}$ and $h \in \mathbf{T}[\mathbf{x}]$. Then $s \odot h(\mathbf{a}) = t \odot h(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{V}(E)$, and so if $s \neq t$ then we must have $h(\mathbf{a}) = 0_{\mathbf{T}}$ for every $\mathbf{a} \in \mathbf{V}(E)$, i.e., $(h, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. Thus, $\mathbf{E}(V(E))$ is always flat.

For each $\mathbf{T}$-semialgebra R, there is a canonical bijection between flat congruences on R and surjective $\mathbf{T}$-semialgebra morphisms from R to flat $\mathbf{T}$-semialgebras. By the usual Zorn's lemma argument, every flat congruence on a $\mathbf{T}$-semialgebra is contained in a maximal flat congruence. It is *not* the case that maximal flat congruences are necessarily maximal congruences, cf. Example 2.6. However, it is the case that maximal flat congruences are prime.

Proposition 4.6. *Suppose R is a flat $\mathbf{T}$-semialgebra with no nontrivial flat congruences. Then R is an integral domain.*

Proof. Suppose not, so that there exists some nonzero $a \in R$ and $b \neq c \in R$ with $ab = ac$. Then the congruence $E_a = \{(d,e) \in R \times R \mid ad = ae\}$ is nontrivial. By assumption, E_a cannot be flat, and so there must exist $s \neq t \in \mathbf{T}$ and $d \in R$ with $(sd, td) \in E_a$ and $(d, 0_R) \notin E_a$, i.e., $a(sd) = a(td)$ and $ad \neq 0_R$. However, the first equality implies $s(ad) = t(ad)$. As R is flat and $s \neq t$, this implies $ad = 0_R$, a contradiction. $\square$

It follows that every maximal flat congruence on a $\mathbf{T}$-semialgebra is prime, and hence every flat congruence is contained in a flat prime congruence.

Example 4.7. We claim that for every $\mathbf{a} \in \mathbf{T}^n$ the congruence $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$ is a maximal flat congruence on $\mathbf{T}[\mathbf{x}]$. Indeed, following the argument in Example 2.7, one can show $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$ is the kernel of the surjective semiring morphism $\mathrm{ev}_{\mathbf{a}} : \mathbf{T}[\mathbf{x}] \to \mathbf{T}$. It follows that $\mathbf{T}[\mathbf{x}]/\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle = \mathbf{T}[\mathbf{x}]/\ker \mathrm{ev}_{\mathbf{a}} \cong \mathbf{T}$, and hence $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$ is a maximal flat congruence on $\mathbf{T}[\mathbf{x}]$.

We now return to the special case of the $\mathbf{T}$-semialgebra $\mathbf{T}[\mathbf{x}]$. We say a congruence E on $\mathbf{T}[\mathbf{x}]$ is **weakly flat** if whenever $(h, \epsilon \odot h) \in E$ for some $h \in \mathbf{T}[\mathbf{x}]$ and $\epsilon \neq 1_{\mathbf{T}}$, h necessarily has zero constant term.

Proposition 4.8. *A finitely generated congruence on $\mathbf{T}[\mathbf{x}]$ is weakly flat if and only if it is contained in a flat congruence on $\mathbf{T}[\mathbf{x}]$.*

Proof. First suppose a finitely generated congruence E on $\mathbf{T}[\mathbf{x}]$ is contained in a flat congruence. Then E is contained in a maximal flat congruence, and hence in a flat, prime congruence P. Suppose $(h, \epsilon \odot h) \in E$ for some $h \in \mathbf{T}[\mathbf{x}]$ and $\epsilon \neq 1_{\mathbf{T}}$. Then $(h, \epsilon \odot h) \in P$, hence by primality and flatness $(h, 0_{\mathbf{T}}) \in P$. If a_0 is the constant term of h, then we also have $(a_0, 0_{\mathbf{T}}) \in P$. Again by flatness, this implies $a_0 = 0_{\mathbf{T}}$. Thus, E is weakly flat.

We prove the converse by induction on n, the number of variables. The case $n = 0$ is trivial. Suppose now the converse statement holds in n variables, for some $n \geq 0$. Suppose E is a weakly flat, finitely generated congruence on $\mathbf{T}[x_1, \ldots, x_{n+1}]$. Consider the collection of all $h \in \mathbf{T}[x_1, \ldots, x_{n+1}]$ for which $(h, \epsilon \odot h) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$ (hence all $\epsilon \neq 0_{\mathbf{T}}$). If there are no such h, then E is itself flat and we are done. So suppose such an h does exist. By assumption, it has zero constant term. Take any variable occurring in any monomial in h. Relabeling if necessary, we may assume this variable is x_{n+1}. Now consider the congruence $F = \langle E, (x_{n+1}, 0_{\mathbf{T}})\rangle$. By Example 2.7, F corresponds (via the morphism $\mathrm{ev}_{x_{n+1}=0_{\mathbf{T}}}$) to a (finitely generated) congruence $\overline{F}$ on $\mathbf{T}[x_1, \ldots, x_n]$. We claim $\overline{F}$ is also weakly flat. To see this, suppose $(H, \epsilon \odot H) \in \overline{F}$ for some $H \in \mathbf{T}[x_1, \ldots, x_n]$ and $\epsilon \neq 1_{\mathbf{T}}$ (hence all $\epsilon \neq 0_{\mathbf{T}}$). Choose generators for E, say $(f_1, g_1), \ldots, (f_k, g_k)$. Then there exist $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}, s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \in \mathbf{T}[x_1, \ldots, x_{n+1}]$, all but finitely many zero, such that

1) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot x_{n+1}^{\odot m'} \odot 0_{\mathbf{T}}^{\odot n'} = H;$

2) $\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot x_{n+1}^{\odot m'} \odot 0_{\mathbf{T}}^{\odot n'} = \epsilon \odot H$; and

3) $\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \odot x_{n+1}^{\odot n'} \odot 0_{\mathbf{T}}^{\odot m'} = \bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \odot x_{n+1}^{\odot n'} \odot 0_{\mathbf{T}}^{\odot m'}$.

In our notation here, $\tilde{\mathbf{m}} = (\mathbf{m}, m')$ and $\tilde{\mathbf{n}} = (\mathbf{n}, n')$. As $H \in \mathbf{T}[x_1, \ldots, x_n]$, the above conditions are equivalent to the following:

1)' $\bigoplus_{\mathbf{m},\mathbf{n}} r_{\mathbf{m},0,\mathbf{n},0} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = H$, and $r_{\mathbf{m},m',\mathbf{n},0} = 0_{\mathbf{T}}$ for $m' > 0$;

2)' $\bigoplus_{\mathbf{m},\mathbf{n}} s_{\mathbf{m},0,\mathbf{n},0} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = \epsilon \odot H$, and $s_{\mathbf{m},m',\mathbf{n},0} = 0_{\mathbf{T}}$ for $m' > 0$; and

3)' $\bigoplus_{\mathbf{m},\tilde{\mathbf{n}}} r_{\mathbf{m},0,\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \odot x_{n+1}^{\odot n'} = \bigoplus_{\mathbf{m},\tilde{\mathbf{n}}} s_{\mathbf{m},0,\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \odot x_{n+1}^{\odot n'}$.

Set $R_{\mathbf{m},\mathbf{n}} = \bigoplus_{m'} r_{\mathbf{m},m',\mathbf{n},0} \odot x_{n+1}^{m'}$ and $S_{\mathbf{m},\mathbf{n}} = \bigoplus_{m'} s_{\mathbf{m},m',\mathbf{n},0} \odot x_{n+1}^{m'}$. Properties (1)'-(3)' then yield

1)" $\bigoplus_{\mathbf{m},\mathbf{n}} R_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = H \oplus (x_{n+1} \odot A)$, for some $A \in \mathbf{T}[x_1, \ldots, x_{n+1}]$;

2)" $\bigoplus_{\mathbf{m},\mathbf{n}} S_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = (\epsilon \odot H) \oplus (x_{n+1} \odot B)$, for some $B \in \mathbf{T}[x_1, \ldots, x_{n+1}]$; and

3)" $\bigoplus_{\mathbf{m},\mathbf{n}} R_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} = \bigoplus_{\mathbf{m},\mathbf{n}} S_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}}$.

It follows we have $(H \oplus (x_{n+1} \odot A), (\epsilon \odot H) \oplus (x_{n+1} \odot B)) \in E$. Adding $h \odot (A \oplus B)$ to both sides then gives $(H \oplus (h \odot (A \oplus B)), (\epsilon \odot H) \oplus (h \odot (A \oplus B))) \in E$. As we also have $(h, \epsilon \odot h) \in E$, it follows that

$$(H \oplus (h \odot (A \oplus B)), \epsilon \odot (H \oplus (h \odot (A \oplus B))) \in E.$$

As E is weakly flat, the polynomial $H \oplus (h \odot (A \oplus B))$ must then have zero constant term, hence the polynomial H must have zero constant term. Thus, $\overline{F}$ is weakly flat. It now follows by induction that $\overline{F}$ is contained in a flat congruence on $\mathbf{T}[x_1, \ldots, x_n]$. This implies F must be contained in a flat congruence on $\mathbf{T}[x_1, \ldots, x_{n+1}]$. As $E \subseteq F$, the result follows. $\square$

4.3. **The General Case.**

We now proceed to reduce the general case (of a finitely generated congruence) to the principally generated case. The main step is the following proposition.

Proposition 4.9. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$ and $\mathbf{V}(E) \subseteq \mathbf{T}^n \backslash \mathbf{R}^n$. Then there exists a relation $(f, g) \in E$ with $\mathbf{V}(f, g) \subseteq \mathbf{T}^n \backslash \mathbf{R}^n$.*

To prove this proposition, we proceed in two steps. First we replace the given collection of generators with a "transitive chain" of generators that have essentially the same associated congruence variety as E.

Lemma 4.10. *Suppose $E = \langle (f_1, g_1), \dots, (f_k, g_k)\rangle$ is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$. Then there exist relations $(h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k) \in E$ such that*

$$\mathbf{V}((h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k)) \subseteq \mathbf{V}(E) \cup \bigcup_{i=1}^{k} \mathbf{V}(f_i, 0_{\mathbf{T}})$$

Proof. Define $h_0 = f_1 \odot \cdots \odot f_k$, and $h_i = g_i \bigodot_j f_j$, for each $i = 1, \dots, k$, where the product is over $1 \leq j \leq k$ with $j \neq i$. Then $(h_0, h_1) = (f_1, g_1) \bigodot_{j \geq 2} (f_j, f_j) \in F$, and for each $i = 1, \dots k-1$, $(h_i, h_{i+1}) = (g_i, f_i) \odot (f_{i+1}, g_{i+1}) \bigodot_j (f_j, f_j) \in F$, where the product is over $1 \leq j \leq k$ with $j \neq i, i+1$. Now suppose $\mathbf{a} \in \mathbf{V}((h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k))$, and suppose $f_j(\mathbf{a}) \neq 0_{\mathbf{T}}$ for every j. Then observe that for every i we have $h_0(\mathbf{a}) = h_i(\mathbf{a})$, which is the equality

$$f_1(\mathbf{a}) \odot \cdots \odot f_i(\mathbf{a}) \odot \cdots \odot f_k(\mathbf{a}) = f_1(\mathbf{a}) \odot \cdots \odot g_i(\mathbf{a}) \odot \cdots \odot f_k(\mathbf{a}).$$

As we're assuming all the $f_j(\mathbf{a})$ are nonzero, this equality implies $f_i(\mathbf{a}) = g_i(\mathbf{a})$. As this is true for every i, we have $\mathbf{a} \in \mathbf{V}(E)$. $\square$

We next replace this "transitive chain" of generators with a single relation having the same congruence variety.

Lemma 4.11. *Given $h_0, \dots, h_k \in \mathbf{T}[\mathbf{x}]$, we always have:*

i) $((h_0 \oplus \cdots \oplus h_k)^{\odot(k+1)}, h_0 \odot \cdots \odot h_k) \in \langle (h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k)\rangle$*; and*
ii) $\mathbf{V}((h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k)) = \mathbf{V}((h_0 \oplus \cdots \oplus h_k)^{\odot(k+1)}, h_0 \odot \cdots \odot h_k)$.

Proof. Let $E = \langle (h_0, h_1), (h_1, h_2), \dots, (h_{k-1}, h_k)\rangle$. First observe that, whenever $i_0, \dots, i_k$ are nonnegative integers that sum to $k+1$, we have

$$(h_0^{\odot i_0} \odot \cdots \odot h_k^{\odot i_k}, h_0 \odot \cdots \odot h_k) = \bigodot_{j=0}^{k} \bigodot_{l=1}^{i_l} (h_j, h_l) \in E.$$

As the addition in $\mathbf{T}[\mathbf{x}]$ is idempotent, it follows that

$$\begin{aligned} ((h_0 \oplus \cdots \oplus h_k)^{\odot(k+1)}, h_0 \odot \cdots \odot h_k) &= \left(\bigoplus_{i_0 + \cdots + i_k = k+1} h_0^{\odot i_0} \odot \cdots \odot h_k^{\odot i_k}, h_0 \odot \cdots \odot h_k \right) \\ &= \bigoplus_{i_0 + \cdots + i_k = k+1} (h_0^{\odot i_0} \odot \cdots \odot h_k^{\odot i_k}, h_0 \odot \cdots \odot h_k) \in E. \end{aligned}$$

Thus,

$$\mathbf{V}(E) \subseteq \mathbf{V}((h_0 \oplus \cdots \oplus h_k)^{\odot(k+1)}, h_0 \odot \cdots \odot h_k).$$

Now suppose $\mathbf{a} \in \mathbf{V}((h_0 \oplus \cdots \oplus h_k)^{\odot(k+1)}, h_0 \odot \cdots \odot h_k)$. Then

$$(k+1) \cdot \max\{h_0(\mathbf{a}), \dots, h_k(\mathbf{a})\} = h_0(\mathbf{a}) + \cdots h_k(\mathbf{a}),$$

which is only possible if $h_0(\mathbf{a}) = \cdots = h_k(\mathbf{a})$. Thus, $\mathbf{a} \in \mathbf{V}(E)$. $\square$

The proof of Proposition 4.9 is now nearly immediate:

Proof of Proposition 4.9. Apply Lemmas 4.10 and 4.11, together with the observation that one always has $\mathbf{V}(f, 0_{\mathbf{T}}) \subseteq \mathbf{T}^n \backslash \mathbf{R}^n$ for any $f \in \mathbf{T}[\mathbf{x}]$ (as all tropical functions are real-valued on $\mathbf{R}^n$). □

Corollary 4.12. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$ and $\mathbf{V}(E) \subseteq \mathbf{T}^n \backslash \mathbf{R}^n$. Then $(g, \epsilon \odot g) \in \langle E, \mathbf{E}(\mathbf{T}^n)\rangle$ for some nonzero $g \in \mathbf{T}[\mathbf{x}]$ and real number $\epsilon > 0$.*

Proof. This follows from Proposition 4.9 and Corollary 3.9. □

We can now prove the following weak form of the Tropical Nullstellensatz for Congruences:

Theorem 2. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$. Then $\mathbf{V}(E)$ is empty if and only if E is not weakly flat, i.e., there exists $h \in \mathbf{T}[\mathbf{x}]$ with nonzero constant term such that $(h, \epsilon \odot h) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$ (and hence all $\epsilon \neq 0_{\mathbf{T}}$).*

Proof. By Proposition 4.8, it is equivalent to prove $\mathbf{V}(E)$ is empty if and only if E is not contained in any flat congruence. We proceed by induction on the n, the number of variables. We first consider the case $n = 0$, where $\mathbf{T}^0$ is just a point and $\mathbf{T}[\emptyset] = \mathbf{T}$. By Example 2.6, the semiring $\mathbf{T}$ only has three congruences: the trivial congruence $\Delta_{\mathbf{T}}$, the maximal congruence $E_{\mathbf{F}_1}$, and the improper congruence $E_{\mathbf{T}}$. As the hypothesis of the theorem implies $\mathbf{V}(E)$ is empty, the only possible congruences for which the hypothesis of the theorem holds are $E_{\mathbf{F}_1}$ and $E_{\mathbf{T}}$. As $E_{\mathbf{F}_1}$ is maximal and not flat, it is not contained in any flat congruence. As $E_{\mathbf{T}}$ is improper, it is obviously not contained in any flat congruence. Thus, the statement holds for $n = 0$.

Now assume the statement holds when there are n or fewer variables, for some $n \geq 0$. Let E be a finitely generated congruence on $\mathbf{T}[x_1, \ldots, x_{n+1}]$ such that $\mathbf{V}(E)$ is empty and suppose, towards a contradiction, that E is contained in a flat congruence. Then E is contained in a flat, prime congruence P. By Theorem 1, we then also have $\langle E, \mathbf{E}(\mathbf{T}^{n+1})\rangle \subseteq P$. By Corollary 4.12 there exist nonzero $g \in \mathbf{T}[x_1, \ldots, x_{n+1}]$ and $\epsilon > 1_{\mathbf{T}}$ such that $(g, \epsilon \odot g) \in \langle E, \mathbf{E}(\mathbf{T}^{n+1})\rangle$. Then $(g, \epsilon \odot g) \in P$, so by primality and flatness we have $(g, 0_{\mathbf{T}}) \in P$. Let a_I be a nonzero coefficient of a monomial $\mathbf{x}^{\odot I}$ in g. As tropical addition is idempotent, it follows that $g(\mathbf{x}) \oplus (a_I \odot \mathbf{x}^{\odot I})$ equals $g(\mathbf{x})$, and hence $(a_I \odot \mathbf{x}^I, 0_{\mathbf{T}}) \in P$. Again by primality and flatness, this implies $(x_i, 0_{\mathbf{T}}) \in P$ for some index $i \in \{1, \ldots, n+1\}$. Relabeling if necessary, we may assume $i = n+1$.

For any congruence F on $\mathbf{T}[x_1, \ldots, x_{n+1}]$, denote by $\overline{F}$ the congruence on $\mathbf{T}[x_1, \ldots, x_n]$ corresponding to the partial evaluation morphism $\mathrm{ev}_{x_{n+1}=0}$. Observe that $\overline{E}$ is still finitely generated, $\overline{E} \subseteq \overline{P}$, and $\mathbf{V}_{\mathbf{T}^n}(\overline{E}) = \mathbf{V}_{\mathbf{T}^{n+1}}(E) \cap \mathbf{V}_{\mathbf{T}^n}(x_{n+1}, 0_{\mathbf{T}}) = \emptyset$. By induction, then, $\overline{E}$ cannot be contained in any flat, prime congruence. However, we claim the congruence $\overline{P}$ is still flat and prime. Indeed, as $(x_{n+1}, 0_{\mathbf{T}}) \in P$, we have $\mathbf{T}[x_1, \ldots, x_n]/\overline{P} \cong \mathbf{T}[x_1, \ldots, x_{n+1}]/P$. As P is flat and prime, the $\mathbf{T}$-semialgebra $\mathbf{T}[x_1, \ldots, x_{n+1}]/P$ is a flat integral domain, hence so is $\mathbf{T}[x_1, \ldots, x_n]/\overline{P}$, i.e., $\overline{P}$ is also flat and prime. This is a contradiction. □

Corollary 4.13. *If E is a finitely generated flat congruence on $\mathbf{T}[\mathbf{x}]$, then $\mathbf{V}(E)$ is nonempty.*

Corollary 4.14. *The finitely generated maximal flat congruences on $\mathbf{T}[\mathbf{x}]$ are precisely the congruences of the form $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$, for $\mathbf{a} \in \mathbf{T}^n$.*

Proof. The congruence $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$ is a maximal flat congruence on $\mathbf{T}[\mathbf{x}]$ by Example 4.7. Also note that $\mathbf{V}(\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle) = \{\mathbf{a}\}$ and $\mathbf{E}(\mathbf{V}(\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle)) = \langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$, the latter by Example 4.5 and the maximality of $\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$. Now suppose E is any finitely generated flat congruence on $\mathbf{T}[\mathbf{x}]$. By the previous corollary, there is some point $\mathbf{a} \in \mathbf{V}(E)$. It follows that

$$E \subseteq \mathbf{E}(\mathbf{V}(E)) \subseteq \mathbf{E}(\{\mathbf{a}\}) = \mathbf{E}(\mathbf{V}(\langle (x_1, a_1), \ldots, (x_n, a_n)\rangle)) = \langle (x_1, a_1), \ldots, (x_n, a_n)\rangle.$$

In particular, if E is a maximal flat congruence, then we must have $E = \langle (x_1, a_1), \ldots, (x_n, a_n)\rangle$. ☐

5. The Strong Tropical Nullstellensatz for Congruences

We now aim to determine the structure of $\mathbf{E}(\mathbf{V}(E))$. We begin by considering a reasonable candidate for $\mathbf{E}(\mathbf{V}(E))$.

5.1. A Candidate for $\mathbf{E}(\mathbf{V}(E))$.

5.1.1. *Powers of Relations and Unsatisfiable Relations.*

Classically, the Nullstellensatz states that $f \in \mathbf{I}(\mathbf{V}(I))$ iff $f \in \sqrt{I}$. Similarly, in the tropical situation it is clear that if $(f^{\odot n}, g^{\odot n}) \in E$ for some $n \geq 1$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. The same holds true for twisted powers: if $(f, g)^{\times n} \in E$ for some $n \geq 1$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. These two properties can be combined with the following easily-proven fact: if either $(f^{\odot n}, g^{\odot n}) \in E$ or $(f, g)^{\times n} \in E$ for some $n \geq 1$, then $((f \oplus g)^{\odot N}, 0_{\mathbf{T}}) \times (f, g) \in E$ for sufficiently large N. Note that if $((f \oplus g)^{\odot N}, 0_{\mathbf{T}}) \times (f, g) \in E$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$.

Another issue to consider is the existence of unsatisfiable relations. Recall Lemma 3.2, which states that $\mathbf{V}((h, k) \times (f, g)) = \mathbf{V}(h, k) \cup \mathbf{V}(f, g)$. It follows that if $\mathbf{V}(h, k) = \emptyset$, then $\mathbf{V}((h, k) \times (f, g)) = \mathbf{V}(f, g)$. In particular, if $(h, k) \times (f, g) \in E$ and $\mathbf{V}(h, k) = \emptyset$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$. By Theorem 2, a congruence has an empty associated congruence variety exactly when the congruence contains a relation of the form $(h, \epsilon \odot h)$, where $h \in \mathbf{T}[\mathbf{x}]$ has nonzero constant term and $\epsilon \neq 1_{\mathbf{T}}$. Rescaling if necessary, we may assume the constant term of h is $1_{\mathbf{T}}$, and so h is of the form $h = 1_{\mathbf{T}} \oplus r$ for some $r \in \mathbf{T}[\mathbf{x}]$. In particular, we see that if $(1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r, 0_{\mathbf{T}}) \times (f, g) \in E$ for some $r \in \mathbf{T}[\mathbf{x}]$ and $\epsilon \neq 1_{\mathbf{T}}$, then $(f, g) \in \mathbf{E}(\mathbf{V}(E))$.

Combining the previous two observations, to any congruence E on $\mathbf{T}[\mathbf{x}]$ we consider the collection E_+ of all pairs (f, g) for which the following condition holds: there exist $r \in \mathbf{T}[\mathbf{x}], \epsilon \neq 1_{\mathbf{T}}$, and $N > 0$ such that

$$(1_{\mathbf{T}}, \epsilon) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f, g) \in E.$$

By construction, we have $E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$. The set E_+ has many other nice properties.

Lemma 5.1. *For any congruence E on $\mathbf{T}[\mathbf{x}]$, we have $\mathbf{V}(E_+) = \mathbf{V}(E)$.*

Proof. Since $E \subseteq E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$, it follows that $\mathbf{V}(E) = \mathbf{V}(\mathbf{E}(\mathbf{V}(E))) \subseteq \mathbf{V}(E_+) \subseteq \mathbf{V}(E)$, and hence $\mathbf{V}(E_+) = \mathbf{V}(E)$. ☐

Lemma 5.2. *Suppose $(f,g) \in E_+$. Then there exist $r \in \mathbf{T}[\mathbf{x}]$ and $N > 0$ such that $(1_{\mathbf{T}}, \delta) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E$ for every $\delta \in \mathbf{R}$.*

Proof. Since $(f,g) \in E_+$, there exist $r \in \mathbf{T}[\mathbf{x}], \epsilon \neq 1_{\mathbf{T}}$, and $N > 0$ such that

$$(*) \qquad (1_{\mathbf{T}}, \epsilon) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E.$$

If $\epsilon = 0_{\mathbf{T}}$, then $(*)$ states $((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E$. Twisted multiplying by $(1_{\mathbf{T}}, \delta)$ then gives the desired result. Now suppose $\epsilon \neq 0_{\mathbf{T}}$. Observe that (twisted) multiplying $(*)$ by $(0_{\mathbf{T}}, \epsilon^{\odot -1})$ implies

$$(1_{\mathbf{T}}, \epsilon^{\odot -1}) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E,$$

and so $(*)$ holds with ϵ replaced by $\epsilon^{\odot -1}$. We may therefore assume $\epsilon > 1_{\mathbf{T}}$. We claim $(*)$ also holds when ϵ is replaced by $\epsilon^{\odot 2}$. Indeed, let $h = (f \oplus g)^{\odot N} \oplus r$, and observe that (since $\epsilon^{\odot 2} > \epsilon > 1_{\mathbf{T}}$) we have

$$\begin{aligned} h \odot (f \oplus (\epsilon^{\odot 2} \odot g)) &= h \odot (f \oplus (\epsilon \odot g) \oplus (\epsilon^{\odot 2} \odot g)) \sim_E h \odot ((\epsilon \odot f) \oplus g \oplus (\epsilon^{\odot 2} \odot g)) \\ &= h \odot ((\epsilon \odot f) \oplus (\epsilon^{\odot 2} \odot g)) = \epsilon \odot h \odot (f \oplus (\epsilon \odot g)) \\ &\sim_E \epsilon \odot h \odot ((\epsilon \odot f) \oplus g) = h \odot ((\epsilon^{\odot 2} \odot f) \oplus (\epsilon \odot g)) \\ &= h \odot ((\epsilon^{\odot 2} \odot f) \oplus f \oplus (\epsilon \odot g)) \sim_E h \odot ((\epsilon^{\odot 2} \odot f) \oplus (\epsilon \odot f) \oplus g) \\ &= h \odot ((\epsilon^{\odot 2} \odot f) \oplus g). \end{aligned}$$

Thus, we indeed have $(1_{\mathbf{T}}, \epsilon^{\odot 2}) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E$. Proceeding inductively, one can then easily show $(*)$ holds with ϵ replaced by $\epsilon^{\odot M}$ for any $M \geq 1$ (without changing the constant N).

Now take any $\delta \in \mathbf{R}$. Replacing ϵ with $\epsilon^{\odot M}$ for M sufficiently large, we may assume $\epsilon^{\odot -1} \leq \delta \leq \epsilon$. Observe that

$$(\epsilon^{\odot -1} \odot \delta, \epsilon^{\odot -1}) \times (1_{\mathbf{T}}, \epsilon) = ((\epsilon^{\odot -1} \odot \delta) \oplus 1_{\mathbf{T}}, \delta \oplus \epsilon^{\odot -1}) = (1_{\mathbf{T}}, \delta),$$

and so (twisted) multiplying $(*)$ by $(\epsilon^{\odot -1} \odot \delta, \epsilon^{\odot -1})$ implies

$$(1_{\mathbf{T}}, \delta) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g) \in E.$$

$\square$

Example 5.3. By definition and the above lemma, $(1_{\mathbf{T}}, 0_{\mathbf{T}}) \in E_+$ if and only if there exist $r \in \mathbf{T}[\mathbf{x}]$ and $N > 0$ such that $(1_{\mathbf{T}}, \epsilon) \times ((1_T \oplus 0_{\mathbf{T}})^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (1_{\mathbf{T}}, 0_{\mathbf{T}}) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$ (equivalently, all $\epsilon \in \mathbf{R}$). Simplifying, we see that E_+ is improper if and only if there exists $r \in \mathbf{T}[\mathbf{x}]$ such that $(1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r, 0_{\mathbf{T}}) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$ (equivalently, all $\epsilon \in \mathbf{R}$).

Example 5.4. Let E be a congruence on $\mathbf{T}[\mathbf{x}]$ and $f \in \mathbf{T}[\mathbf{x}]$ be a fixed polynomial. If $(f, 0_{\mathbf{T}}) \in E_+$, then there exist $r \in \mathbf{T}[\mathbf{x}]$ and $N > 0$ such that $(1_{\mathbf{T}}, \epsilon) \times (f^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f, 0_{\mathbf{T}}) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$. Taking $r' = r \odot f$ and $N' = N+1$, it follows that $(1_{\mathbf{T}}, \epsilon) \times (f^{\odot N'} \oplus r', 0_{\mathbf{T}}) \in E_+$. Conversely, if there exist $r' \in \mathbf{T}[\mathbf{x}]$ and $N' > 0$ such that $(1_{\mathbf{T}}, \epsilon) \times (f^{\odot N'} \oplus r', 0_{\mathbf{T}}) \in E_+$ for some $\epsilon \neq 1_{\mathbf{T}}$, then (twisted) multiplying by $(f, 0_{\mathbf{T}})$ implies $(1_{\mathbf{T}}, \epsilon) \times (f^{\odot N'} \oplus r', 0_{\mathbf{T}}) \times (f, 0_{\mathbf{T}}) \in E$, and hence $(f, 0_{\mathbf{T}}) \in E_+$. Thus, $(f, 0_{\mathbf{T}}) \in E_+$ if and only if there exist $r \in \mathbf{T}[\mathbf{x}]$ and $N > 0$ such that $(1_{\mathbf{T}}, \epsilon) \times (f^{\odot N} \oplus r, 0_{\mathbf{T}}) \in E$ for some $\epsilon \neq 1_{\mathbf{T}}$ (equivalently, all $\epsilon \neq 0_{\mathbf{T}}$).

Lemma 5.5. *For any congruence E on $\mathbf{T}[\mathbf{x}]$, the set E_+ contains E and has properties (E1), (E2) and (I2)'.*

Proof. First suppose $(f,g) \in E$. Then $(1_{\mathbf{T}}, 0_{\mathbf{T}}) \times (f \oplus g, 0_{\mathbf{T}}) \times (f,g) \in E$, and hence $(f,g) \in E_+$. Thus, E_+ contains E, and hence has property (E1). Property (E2) for E_+ follows trivially from property (E2) for E. Property (I2)' is also essentially immediate. Suppose $(f,g) \in E_+$, via $r \in \mathbf{T}[\mathbf{x}], \epsilon \neq 1_{\mathbf{T}}$, and $N > 0$. Take any $h, k \in \mathbf{T}[\mathbf{x}]$. Then observe that, as E has property (I2)', we have

$$((h \oplus k)^{\odot N}, 0_{\mathbf{T}}) \times (h,k) \times \big((1_{\mathbf{T}}, \epsilon) \times ((f \oplus g)^{\odot N} \oplus r, 0_{\mathbf{T}}) \times (f,g)\big) \in E,$$

hence

$$(1_{\mathbf{T}}, \epsilon) \times \big(((h \oplus k)^{\odot N} \odot (f \oplus g)^{\odot N}) \oplus ((h \oplus k)^{\odot N} \odot r), 0_{\mathbf{T}}\big) \times ((h,k) \times (f,g)) \in E,$$

and hence

$$(1_{\mathbf{T}}, \epsilon) \times \big(((h \odot f) \oplus (k \odot g) \oplus (k \odot f) \oplus (h \odot g))^{\odot N} \oplus r', 0_{\mathbf{T}}\big) \times ((h,k) \times (a,b)) \in E,$$

where $r' = (h \oplus k)^{\odot N} \odot r$. Noting that $(h,k) \times (f,g) = ((h \odot f) \oplus (k \odot g), (k \odot f) \oplus (h \odot g))$, we've thus proven $(h,k) \times (f,g) \in E_+$, and hence E_+ has property (I2)'.

In this new terminology, we can restate Theorem 2 as follows:

Theorem 3. *If E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, then $\mathbf{V}(E)$ is empty if and only if E_+ is improper.*

Proof. By Theorem 2, $\mathbf{V}(E)$ is empty if and only if there exists $h \in \mathbf{T}[\mathbf{x}]$ with nonzero constant term such that $(h, \epsilon \odot h) \in E$ for every $\epsilon \neq 0_{\mathbf{T}}$. Rescaling if necessary, we may assume the constant term of h is $1_{\mathbf{T}}$, so that $h = 1_{\mathbf{T}} \oplus r$ for some $r \in \mathbf{T}[\mathbf{x}]$. Then $(h, \epsilon \odot h) = (1_{\mathbf{T}}, \epsilon) \times (h, 0_{\mathbf{T}}) = (1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r, 0_{\mathbf{T}})$. The result now follows from Example 5.3. ☐

☐

5.1.2. *Limits and Chains of Relations.*

We begin with the following motivating example:

Example 5.6. Suppose E is a congruence on $\mathbf{T}[\mathbf{x}]$ and $f \in \mathbf{T}[\mathbf{x}]$ is a tropical polynomial such that $(f \oplus t, t) \in \mathbf{E}(\mathbf{V}(E))$ for every $t > 1_{\mathbf{T}}$, i.e., such that $f(\mathbf{a}) \leq t$ for every $\mathbf{a} \in \mathbf{V}(E)$ and $t > 1_{\mathbf{T}}$. It necessarily follows that $f(\mathbf{a}) \leq 1_{\mathbf{T}}$ for every $\mathbf{a} \in \mathbf{V}(E)$, and hence $(f \oplus 1_{\mathbf{T}}, 1_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. Similarly, if $(f \oplus s, f) \in \mathbf{E}(\mathbf{V}(E))$ for every $s < 1_{\mathbf{T}}$, then $f(\mathbf{a}) \geq s$ for every $\mathbf{a} \in \mathbf{V}(E)$ and $s < 1_{\mathbf{T}}$, hence $f(\mathbf{a}) \geq 1_{\mathbf{T}}$ for every $\mathbf{a} \in \mathbf{V}(E)$, and hence $(f \oplus 1_{\mathbf{T}}, f) \in \mathbf{E}(\mathbf{V}(E))$. If both of the aforementioned properties hold, then it follows by transitivity that $(f, 1_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$.

The above example suggests $\mathbf{E}(\mathbf{V}(E))$ is somehow "closed under limits" a property not shared by arbitrary congruences. (Indeed, the congruence $F = \bigcup_{t > 1_{\mathbf{T}}} \langle (x \oplus t, x) \rangle$ does not possess this property.) Let $\{f_i\}_i$ be a sequence tropical polynomials, and write $f_i(\mathbf{x}) = \bigoplus_{\mathbf{d}} c_{\mathbf{d}}^{(i)} \odot \mathbf{x}^{\odot \mathbf{d}}$. We say this sequence **converges** to a tropical polynomial $f(\mathbf{x}) = \bigoplus_{\mathbf{d}} c_{\mathbf{d}} \odot \mathbf{x}^{\odot \mathbf{d}}$ if in each degree $\mathbf{d}$ the sequence of coefficients $\{c_{\mathbf{d}}^{(i)}\}_i$ converges to $c_{\mathbf{d}}$ (in the usual sense of convergence for extended real numbers). We write $\lim_i f_i = f$ to denote such a convergence. Since real limits commute with maximum and addition, these tropical limits commute with tropical addition and multiplication. Also, if $\lim_i f_i = f$, then $\lim_i f_i(\mathbf{a}) = f(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{T}[\mathbf{x}]$. We say a sequence of relations $\{(f_i, g_i)\}_i$ **converges** to a relation (f, g) if $\lim_i f_i = f$ and $\lim_i g_i = g$. Given a subset E of $\mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}]$, we say a relation (f, g) is a **limit relation** of E if it is the limit of a sequence of relations $\{(f_i, g_i)\}_i$ contained in E. We write $\lim_i(f_i, g_i) = (f, g)$.

We would like to claim that if $\lim_i(f_i, g_i) = (f, g)$, then $\lim_i \mathbf{V}(f_i, g_i) = \mathbf{V}(f, g)$. However, it is not clear in what sense this latter limit is to be understood to converge, as we haven't specified our topology on $\mathbf{T}^n$ (and points in $\mathbf{T}^n$ can be infinitely far away from each other). There is one scenario, however, in which this limit can be easily defined. We say a sequence of relations $\{(f_i, g_i)\}_i$ is an **ascending chain** if $\langle(f_i, g_i)\rangle \subseteq \langle(f_{i+1}, g_{i+1})\rangle$ for every i. If $\{(f_i, g_i)\}_i$ is an ascending chain, then $\mathbf{V}(f_{i+1}, g_{i+1}) \subseteq \mathbf{V}(f_i, g_i)$ for every i, and so it is clear we should define $\lim_i \mathbf{V}(f_i, g_i) := \bigcap_i \mathbf{V}(f_i, g_i)$. We say a relation (f, g) is an **ascending chain limit** if it is a limit relation of a sequence that is an ascending chain. Note that if (f, g) is the limit of an ascending chain $\{(f_i, g_i)\}_i$, then $\mathbf{V}(f, g) = \lim_i \mathbf{V}(f_i, g_i)$. Denote by $E^{\lim}$ the collection of ascending chain limits of E.

Lemma 5.7. *For any subset $E \subseteq \mathbf{T}[\mathbf{x}] \times \mathbf{T}[\mathbf{x}]$, we have $\mathbf{V}(E) = \mathbf{V}(E^{\lim})$.*

Proof. As $E \subseteq E^{\lim}$, we certainly have $\mathbf{V}(E^{\lim}) \subseteq \mathbf{V}(E)$. On the other hand, suppose $(f, g) \in E^{\lim}$. Then (f, g) is the limit of an ascending chain $\{(f_i, g_i)\}_i$ in E. Since for each i we have $(f_i, g_i) \in E$, for each i we also have $\mathbf{V}(E) \subseteq \mathbf{V}(f_i, g_i)$. It follows that $\mathbf{V}(E) \subseteq \bigcap_i \mathbf{V}(f_i, g_i) = \mathbf{V}(f, g)$. As this is true for all $(f, g) \in E^{\lim}$, it follows that $\mathbf{V}(E) \subseteq \bigcap_{(f,g)\in E^{\lim}} \mathbf{V}(f, g) = \mathbf{V}(E^{\lim})$. ☐

Remark 5.8. Our need to consider ascending chain limits may stem from the fact that $\mathbf{T}[\mathbf{x}]$ is non-Noetherian.

We are now led to make the following definition. Given a congruence E on $\mathbf{T}[\mathbf{x}]$, let $\hat{E}$ denote the transitive closure $(E_+)^{\lim}$. In other words, a relation (f, g) is contained in $\hat{E}$ if and only if there exist ascending chains $\{(f_i, h_i)\}_i, \{(h'_i, g_i)\}_i$ contained in E_+ with $\lim_i f_i = f, \lim_i h_i = \lim_i h'_i$, and $\lim_i g_i = g$.

By construction, for every congruence E on $\mathbf{T}[\mathbf{x}]$ we have $E \subseteq E_+ \subseteq (E_+)^{\lim} \subseteq \hat{E}$. We also have $\mathbf{V}(E) = \mathbf{V}(E_+) = \mathbf{V}((E_+)^{\lim}) = \mathbf{V}(\hat{E})$, where the first equality holds by Lemma 5.1, the second by Lemma 5.7, and the third by construction. It follows that $\hat{E} \subseteq \mathbf{E}(\mathbf{V}(\hat{E})) = \mathbf{E}(\mathbf{V}(E))$. Our goal for the remainder of this paper is to prove the reverse containment. (In particular, this will also prove $\hat{E}$ is a congruence.) We proceed in three steps: first, by considering relations in $\mathbf{E}(\mathbf{V}(E))$ of the form $(f, 0_{\mathbf{T}})$; second, by considering relations of the form $(f, 1_{\mathbf{T}})$; and finally, by reducing the general case to the two previous cases. We will use a version of the famous "Rabinowitsch trick" in both the first and third steps. Throughout, E shall be assumed to be a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$.

5.2. **The Tropically Vanishing Case.**

In this section we determine what can be said about relations of the form $(p, 0_{\mathbf{T}})$ in $\mathbf{E}(\mathbf{V}(E))$. (We are now denoting our function by p to avoid notational confusion with the generators for E.) We pursue a strategy analogous to the standard proof of the classical Nullstellensatz, attempting to "invert" the function p by adjoining the relation $(1_{\mathbf{T}}, y \odot p)$ to the congruence E, working now on the semiring $\mathbf{T}[\mathbf{x}, y]$. Under the assumption that p vanishes identically on $\mathbf{V}(E)$, the resulting congruence F on $\mathbf{T}[\mathbf{x}, y]$ has empty tropical locus, and so Theorem 2 implies F is not weakly flat. After "clearing denominators" we will arrive at our result. Most of the actual work is contained in proving the following proposition, which describes the relations on the original ring $\mathbf{T}[\mathbf{x}]$ that appear in F:

Proposition 5.9. *Let $p \in \mathbf{T}[\mathbf{x}]$ be a fixed nonzero polynomial and $F = \langle E, (1_{\mathbf{T}}, y \odot p)\rangle_{\mathbf{T}[\mathbf{x},y]}$. For $a, b \in \mathbf{T}[\mathbf{x}]$, we have $(a, b) \in F$ if and only if $(a \odot p^{\odot N}, b \odot p^{\odot N}) \in E$ for some $N > 0$.*

The proof of this proposition relies on the following preliminary results. Fix a choice of generators for E, say $(f_1, g_1), \ldots, (f_k, g_k)$, and define

$$\mathcal{T}_E = \{(\mathbf{m}, \mathbf{n}) \mid \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = 0_{\mathbf{T}}\} \subseteq \mathbf{Z}^k_{\geq 0} \times \mathbf{Z}^k_{\geq 0}$$
$$I_E = \langle \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \mid (\mathbf{m}, \mathbf{n}) \in \mathcal{T}\rangle \subseteq \mathbf{T}[\mathbf{x}].$$

Recall that, in general, the **additive closure** of an ideal I in a semiring R is the ideal

$$I^+ = \{j \in R \mid j + i \in I \text{ for some } i \in I\}.$$

Lemma 5.10. *Following the notation above:*

i) $(j, 0_{\mathbf{T}}) \in E$ for every $j \in I_E^+$;
ii) $(j \oplus a, a) \in E$ for every $j \in I_E^+, a \in \mathbf{T}[\mathbf{x}]$;
iii) If $(j_1 \oplus a, j_2 \oplus a) \in E$ for some $j_1, j_2 \in I_E^+, a, b \in \mathbf{T}[\mathbf{x}]$, then $(a, b) \in E$.

Proof. First notice that if $i \in I_E$, say $i = \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathcal{T}_E} r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}}$, then

$$\begin{aligned}(i, 0_{\mathbf{T}}) &= \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathcal{T}_E} (r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}}, 0_{\mathbf{T}}) \\ &= \bigoplus_{(\mathbf{m},\mathbf{n}) \in \mathcal{T}_E} (r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}}, r_{\mathbf{m},\mathbf{n}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}}) \in E,\end{aligned}$$

as $\mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} = 0_{\mathbf{T}}$ for every $(\mathbf{m}, \mathbf{n}) \in \mathcal{T}_E$. Now suppose $j \in I_E^+$, so $j \oplus i \in I_E$ for some $i \in I_E$. By the above argument, we then have $(j \oplus i, 0_{\mathbf{T}}) \in E$. It then follows that $(j \oplus i, j) = (j, j) \oplus (j \oplus i, 0_{\mathbf{T}}) \in E$. By transitivity, we therefore have $(j, 0_{\mathbf{T}}) \in E$. This proves (i).

For (ii), suppose $j \in I_E^+$. By (i), we then have $(j, 0_{\mathbf{T}}) \in E$. It follows that $(j \oplus a, a) = (a, a) \oplus (j, 0_{\mathbf{T}}) \in E$ for every $a \in \mathbf{T}[\mathbf{x}]$. For (iii), suppose $(j_1 \oplus a, j_2 \oplus a) \in E$ for some $j_1, j_2 \in I_E^+, a, b \in \mathbf{T}[\mathbf{x}]$. By (ii), we then have $(j_1 \oplus a, a), (b, j_2 \oplus b) \in E$, and so by transitivity $(a, b) \in E$. □

Proof of Proposition 5.9. Fix $a, b \in \mathbf{T}[\mathbf{x}]$. First suppose $(a \odot p^{\odot N}, b \odot p^{\odot N}) \in E \subseteq F$ for some $N > 0$. Multiplying by $(y^{\odot N}, y^{\odot N}) \in F$, we then obtain $(a \odot (y \odot p)^{\odot N}, b \odot (y \odot p)^{\odot N}) \in F$. As $(1_{\mathbf{T}}, y \odot p) \in F$, we thus have $(a, b) \in F$.

Conversely, suppose $(a,b) \in F$. By Corollary 2.9, there exist $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}, s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \in \mathbf{T}[\mathbf{x},y]$, all but finitely many zero, such that:

1) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \odot 1_{\mathbf{T}}^{\odot m'} \odot (y \odot p)^{\odot n'} = a$;
2) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \odot 1_{\mathbf{T}}^{\odot m'} \odot (y \odot p)^{\odot n'} = b$; and
3) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}} \odot 1_{\mathbf{T}}^{\odot n'} \odot (y \odot p)^{\odot m'} = \bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}} \odot 1_{\mathbf{T}}^{\odot n'} \odot (y \odot p)^{\odot m'}$.

In our notation here, $\tilde{\mathbf{m}} = (\mathbf{m}, m')$ and $\tilde{\mathbf{n}} = (\mathbf{n}, n')$. As $a, b \in \mathbf{T}[\mathbf{x}]$, conditions (1) and (2) are equivalent (by considering coefficients of powers of y) to the following:

1a) $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} = 0_{\mathbf{T}}$ when $(\mathbf{m},\mathbf{n}) \notin \mathcal{T}_E, n' > 0$; and $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \in \mathbf{T}[\mathbf{x}]$ when $(\mathbf{m},\mathbf{n}) \notin \mathcal{T}_E, n' = 0$;
1b) $\displaystyle\bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} \left(\bigoplus_{m'} r_{\tilde{\mathbf{m}},\mathbf{n},0} \right) \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} = a$;
2a) $s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} = 0_{\mathbf{T}}$ when $(\mathbf{m},\mathbf{n}) \notin \mathcal{T}_E, n' > 0$; and $s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \in \mathbf{T}[\mathbf{x}]$ when $(\mathbf{m},\mathbf{n}) \notin \mathcal{T}_E, n' = 0$;
2b) $\displaystyle\bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} \left(\bigoplus_{m'} s_{\tilde{\mathbf{m}},\mathbf{n},0} \right) \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} = b$.

Write $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}(\mathbf{x},y) = \displaystyle\bigoplus_d {}_d r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}(\mathbf{x}) \odot y^{\odot d}$ and $s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}(\mathbf{x},y) = \bigoplus_d {}_d s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}(\mathbf{x}) \odot y^{\odot d}$. Matching coefficients of $y^{\odot e}$ in (1b) and (2b), we find that for every e we have

$$\bigoplus_{\mathbf{m},\mathbf{n}} \bigoplus_{n',d+m'=e} {}_d r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot p^{\odot m'} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}} = \bigoplus_{\mathbf{m},\mathbf{n}} \bigoplus_{n',d+m'=e} {}_d s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot p^{\odot m'} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}}.$$

It follows that for every e, the relation below is contained in E:

$$\left(\bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} \bigoplus_{n',d+m'=e} {}_d r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot p^{\odot m'} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}}, \bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} \bigoplus_{n',d+m'=e} {}_d s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot p^{\odot m'} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \right).$$

(Recall that every term in the sums with $(\mathbf{m},\mathbf{n}) \in \mathcal{T}_E$ is tropically zero.) Using properties (1),(2), this implies that for every e, the relation below is contained in E:

$$\left(\bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} {}_0 r_{\mathbf{m},e,\mathbf{n},0} \odot p^{\odot e} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}}, \bigoplus_{(\mathbf{m},\mathbf{n})\notin\mathcal{T}_E} {}_0 s_{\mathbf{m},e,\mathbf{n},0} \odot p^{\odot e} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \right).$$

Let $N > 0$ be large enough such that $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} = 0_{\mathbf{T}} = s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}$ whenever $m' \geq N$. For each e, multiply the above relation by $(p^{\odot(N-e)}, p^{\odot(N-e)}) \in E$. Summing the resulting relations over e and using properties (1) and (2), this exactly states $(a \odot p^{\odot N}, b \odot p^{\odot N}) \in E$. □

We can now prove the following special case of the strong form of the Tropical Nullstellensatz for Congruences:

Theorem 4. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$ and $p \in \mathbf{T}[\mathbf{x}]$. Then $(p, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$ if and only if $(p, 0_{\mathbf{T}}) \in E_+$.*

Proof. One direction is clear, as $E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$, so suppose $(p, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. Without loss of generality, we may assume p is nonzero. Define $F = \langle E, (1_{\mathbf{T}}, y \odot p)\rangle_{\mathbf{T}[\mathbf{x},y]}$. Then $\mathbf{V}_{\mathbf{T}^{n+1}}(F) = \emptyset$, so by Example 5.3 and Theorem 3 it follows that there exists $r \in \mathbf{T}[\mathbf{x}, y]$ such that $(1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r, 0) \in F$ for some $\epsilon \neq 1_{\mathbf{T}}$ (and hence all $\epsilon \in \mathbf{R}$). Write $r(\mathbf{x}, y) = \bigoplus_{0\le d\le M} a_d(\mathbf{x}) \odot y^{\odot d}$. Then $p^{\odot M} \odot r = \bigoplus_{0\le d\le M} \left(a_d \odot p^{\odot(M-d)}\right) \odot (y \odot p)^{\odot d}$, and so

$$\left(p^{\odot M} \odot r, \bigoplus_{0\le d\le M} \left(a_d \odot p^{\odot(M-d)}\right)\right) \in F.$$

Let $r' = \bigoplus_{0\le d\le M} a_d \odot p^{\odot(M-d)} \in \mathbf{T}[\mathbf{x}]$. Then by the above observation we have $(1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r', 0_{\mathbf{T}}) \in \tilde{E}$. By Proposition 5.9, it follows that $(1_{\mathbf{T}}, \epsilon) \times (p^{\odot N} \oplus r'', 0_{\mathbf{T}}) \in E$ for some $N \ge 1$ (where $r'' = p^{\odot N} \odot r' \in \mathbf{T}[\mathbf{x}]$). In light of Example 5.4, this proves $(p, 0_{\mathbf{T}}) \in E_+$. $\square$

Corollary 5.11. *Suppose E is a flat, finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, and $p \in \mathbf{T}[\mathbf{x}]$. Then $(p, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$ if and only if $(p^{\odot N}, 0_{\mathbf{T}}) \in E$ for some $N > 0$.*

Proof. One direction is obvious, so suppose $(p, 0_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. By Theorem 4, we have $(p^{\odot N} \oplus A, \epsilon \odot (p^{\odot N} \oplus A)) \in E$ for some $A \in \mathbf{T}[\mathbf{x}], N > 0$ and $\epsilon \neq 1_{\mathbf{T}}$. As E is flat, this implies $(p^{\odot N} \oplus A, 0_{\mathbf{T}}) \in E$, and hence $(p^{\odot N}, 0_{\mathbf{T}}) \in E$. $\square$

5.3. The Tropically Constant Case.

In this section we will determine what can be said when a polynomial is constant (and nonzero) on $\mathbf{V}(E)$. It suffices to consider the case when $(p, 1_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$. Note that in this case we have $p(\mathbf{a}) < t$ for every $t > 1_{\mathbf{T}}$, and so $(p \oplus t, t) \in \mathbf{E}(\mathbf{V}(E))$ for every $t > 1_{\mathbf{T}}$. Similarly, $(p \oplus s, p) \in \mathbf{E}(\mathbf{V}(E))$ for every $s < 1_T$. We are thus precisely in the situation of Example 5.6. Our main work will be in proving the following:

Proposition 5.12. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$, and suppose $p, q \in \mathbf{T}[\mathbf{x}]$ are such that $p(\mathbf{a}) < q(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{V}(E)$. Then $(p \oplus q, q) \in E_+$.*

The proof of this proposition will rely on the following general lemma (which holds in any additively idempotent semiring):

Lemma 5.13. *Let R be an additively idempotent semiring, and let m, n be fixed nonnegative integers. For sufficiently large N, and any $a, b \in R$, we have the following equality:*

$$(a+b)^N(a+b)^m a^n \left[a^N(a+b)^n a^m\right] + \left[b(a+b)^{N-1}(a+b)^m a^n\right].$$

Proof. First suppose $m \le n$. Observe that

$$\begin{aligned}
(a+b)^N(a+b)^m a^n &= (a+b)^{N-(n-m)}a^{n-m}(a+b)^n a^m \\
&= \left(a^N + (a^{N-1}b) + \cdots + (a^{n-m}b^{N-(n-m)})\right)(a+b)^n a^m \\
&= \left[a^N(a+b)^n a^m\right] + \left[\left((a^{N-1}b) + \cdots + (a^{n-m}b^{N-(n-m)})\right)(a+b)^n a^m\right].
\end{aligned}$$

Notice that the second term on the right-hand side of the above equation equals

$$\begin{aligned}
&b\left[\left(a^{N-1}+\cdots+(a^{n-m}b^{N-1-(n-m)})\right)(a+b)^n a^m\right]\\
&\quad = b\left[a^{N-1+n}+\cdots+(a^{n-m}b^{N-1+m})\right]a^m = b(a+b)^{N-1+m}a^{n-m}a^m\\
&\quad = b(a+b)^{N-1}(a+b)^m a^n.
\end{aligned}$$

This proves the statement in the case $m \leq n$. The case when $m > n$ is proved similarly. In that case, observe that since $(a+b)^N = a^N + \left(b(a+b)^{N-1}\right)$, we have

$$(*) \qquad (a+b)^N(a+b)^m a^n = \left[a^N(a+b)^m a^n\right] + \left[b(a+b)^{N-1}(a+b)^m a^n\right].$$

Notice that the first term on the right-hand side of the above equation equals

$$\begin{aligned}
&a^N(a+b)^m a^n = a^{N-(m-n)}(a+b)^{m-n}(a+b)^n a^m\\
&\quad = \left[a^N+(a^{N-1}b)+\cdots+(a^{N-(m-n)}b^{m-n})\right](a+b)^n a^m\\
&\quad = \left[a^N(a+b)^n a^m\right] + \left[b\left(a^{N-1}+\cdots(a^{N-(m-n)}b^{m-n-1})\right)(a+b)^n a^m\right]
\end{aligned}$$

We claim the second term on the right-hand side of the above equation is a subsum of the second term on the right-hand side of $(*)$. Indeed, observe first that

$$b\left(a^{N-1}+\cdots(a^{N-(m-n)}b^{m-n-1})\right)(a+b)^n a^m = b\left(a^{N-1+(m-n)}+\cdots+(a^N b^{m-n-1})\right)(a+b)^n a^n.$$

Then also observe that

$$\begin{aligned}
b(a+b)^{N-1}(a+b)^m a^n &= b(a+b)^{N-1+(m-n)}(a+b)^n a^n\\
&= b\left(a^{N-1+(m-n)}+\cdots+b^{N-1+(m-n)}\right)(a+b)^n a^n.
\end{aligned}$$

Thus, the claim also holds in the case $m > n$. $\square$

Proof of Proposition 5.12. Fix generators $(f_1, g_1), \ldots, (f_k, g_k)$ for E. Let $F = \langle E, (p \oplus q, p)\rangle$. By hypothesis, then, $\mathbf{V}(F) = \emptyset$, so by Theorem 3 there exists $r \in \mathbf{T}[\mathbf{x}]$ such that $(1_{\mathbf{T}}, \epsilon) \times (1_{\mathbf{T}} \oplus r, 0_{\mathbf{T}}) \in F$ for every $\epsilon \in \mathbf{R}$. Fix any $\epsilon \in \mathbf{R}$. Then there exist $r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}, s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \in \mathbf{T}[\mathbf{x}]$, all but finitely many zero, such that

1) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \odot (p\oplus q)^{\odot m'} \odot p^{\odot n'} = 1_{\mathbf{T}} \oplus r$;
2) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{m}} \odot \mathbf{g}^{\odot\mathbf{n}} \odot (p\oplus q)^{\odot m'} \odot p^{\odot n'} = \epsilon \odot (1_{\mathbf{T}} \oplus r)$; and
3) $\displaystyle\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}} \odot (p\oplus q)^{\odot n'} \odot p^{\odot m'} = \bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot\mathbf{n}} \odot \mathbf{g}^{\odot\mathbf{m}} \odot (p\oplus q)^{\odot n'} \odot p^{\odot m'}$.

Now we observe that, for N sufficiently large, we have

$$
\begin{aligned}
(p \oplus q)^{\odot N} \odot (1_{\mathbf{T}} \oplus r) &= (p \oplus q)^{\odot N} \odot \left(\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot (p \oplus q)^{\odot m'} \odot p^{\odot n'} \right) \\
&= \bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot (p \oplus q)^{\odot N} \odot (p \oplus q)^{\odot m'} \odot p^{\odot n'} \\
&= \left[p^{\odot N} \odot \left(\bigoplus_{\mathbf{m},\mathbf{n}} \left(\bigoplus_{m',n'} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot (p \oplus q)^{\odot n'} \odot p^{\odot m'} \right) \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \right) \right] \\
&\quad \bigoplus \left[q \odot (p \oplus q)^{\odot N-1} \odot \left(\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot (p \oplus q)^{\odot m'} \odot p^{\odot n'} \right) \right] \\
&\sim_E \left[p^{\odot N} \odot \left(\bigoplus_{\mathbf{m},\mathbf{n}} \left(\bigoplus_{m',n'} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot (p \oplus q)^{\odot n'} \odot p^{\odot m'} \right) \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \right) \right] \\
&\quad \bigoplus \left[q \odot (p \oplus q)^{\odot N-1} \odot \left(\bigoplus_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} r_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \odot (p \oplus q)^{\odot m'} \odot p^{\odot n'} \right) \right] \\
&= \left[p^{\odot N} \odot \left(\bigoplus_{\mathbf{m},\mathbf{n}} \left(\bigoplus_{m',n'} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot (p \oplus q)^{\odot n'} \odot p^{\odot m'} \right) \odot \mathbf{f}^{\odot \mathbf{n}} \odot \mathbf{g}^{\odot \mathbf{m}} \right) \right] \\
&\quad \bigoplus \left[q \odot (p \oplus q)^{\odot N-1} \odot (1_{\mathbf{T}} \oplus r) \right] \\
&\sim_E \left[p^{\odot N} \odot \left(\bigoplus_{\mathbf{m},\mathbf{n}} \left(\bigoplus_{m',n'} s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}} \odot (p \oplus q)^{\odot n'} \odot p^{\odot m'} \right) \odot \mathbf{f}^{\odot \mathbf{m}} \odot \mathbf{g}^{\odot \mathbf{n}} \right) \right] \\
&\quad \bigoplus \left[q \odot (p \oplus q)^{\odot N-1} \odot (1_{\mathbf{T}} \oplus r) \right]
\end{aligned}
$$

(Note that we used (1) for the first equality, Lemma 5.13 for the third equality, basic properties of congruences for the first equivalence, (3) for the fourth equality, and then again basic properties of equivalences for the second equivalence.) If we now add $q \odot (p \oplus q)^{\odot N-1} \odot \epsilon \odot (1_{\mathbf{T}} \oplus r)$ to both sides, and then use property (3) and Lemma 5.13 in reverse (on the sum involving $s_{\tilde{\mathbf{m}},\tilde{\mathbf{n}}}$), we obtain

$$
\begin{aligned}
&\left[(p \oplus q)^{\odot N} \odot (1_{\mathbf{T}} \oplus r)\right] \oplus \left[q \odot (p \oplus q)^{\odot N-1} \odot \epsilon \odot (1_{\mathbf{T}} \oplus r)\right] \\
&\qquad \sim_E \left[q \odot (p \oplus q)^{\odot N-1} \odot (1_{\mathbf{T}} \oplus r)\right] \oplus \left[(p \oplus q)^{\odot N} \odot \epsilon \odot (1_{\mathbf{T}} \oplus r)\right].
\end{aligned}
$$

This is equivalent to the statement

$$(1_{\mathbf{T}}, \epsilon) \times ((p \oplus q)^{\odot N-1} \oplus r', 0_{\mathbf{T}}) \times (p \oplus q, q) \in E,$$

where $r' = (p \oplus q)^{\odot N-1} \odot r$. Thus, $(p \oplus q, q) \in E_+$. □

Corollary 5.14. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$. Then $(p, 1_{\mathbf{T}}) \in \mathbf{E}(\mathbf{V}(E))$ if and only if for every $t > 1_{\mathbf{T}}$ we have $(p \oplus t, t) \in E_+$ and $(p \oplus t^{\odot -1}, p) \in E_+$.*

Proof. If $(p \oplus t, t) \in E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$, then $p(\mathbf{a}) \leq t$ for every $\mathbf{a} \in \mathbf{V}(E)$. If this holds for every $t > 1_{\mathbf{T}}$, then we must actually have $p(\mathbf{a}) \leq 1_{\mathbf{T}}$ for every $\mathbf{a} \in \mathbf{V}(E)$, i.e., $(p \oplus 1_{\mathbf{T}}, 1_{\mathbf{T}}) \in$

$\mathbf{E}(\mathbf{V}(E))$. Similarly, if $(p \oplus t^{\odot -1}, p) \in E_+ \subseteq \mathbf{E}(\mathbf{V}(E))$ for every $t > 1_\mathbf{T}$, then we must actually have $(p \oplus 1_\mathbf{T}, p) \in \mathbf{E}(\mathbf{V}(E))$. It then follows by transitivity that $(p, 1_\mathbf{T}) \in \mathbf{E}(\mathbf{V}(E))$.

Conversely, suppose $(p, 1_\mathbf{T}) \in \mathbf{E}(\mathbf{V}(E))$, and so $p(\mathbf{a}) = 1_\mathbf{T}$ for every $\mathbf{a} \in \mathbf{V}(E)$. Take any $t > 1_\mathbf{T}$. Then $p(\mathbf{a}) < t$ for every $\mathbf{a} \in \mathbf{V}(E)$, so by Proposition 5.12 we have $(p \oplus t, t) \in E_+$. Similarly, we have $t^{\odot -1} < p(\mathbf{a})$ for every $\mathbf{a} \in \mathbf{V}(E)$, so Proposition 5.12 also gives $(p \oplus t^{\odot -1}, p) \in E_+$. □

5.4. **The General Case.**

At last we consider the general case, and prove the following strong form of the Tropical Nullstellensatz for Congruences. The proof will involve a second use of the Rabinowitsch trick, this time to reduce the general case to the tropically constant case.

Theorem 5. *Suppose E is a finitely generated congruence on $\mathbf{T}[\mathbf{x}]$. Then $\mathbf{E}(\mathbf{V}(E)) = \hat{E}$.*

Proof. We have already seen $\hat{E} \subseteq \mathbf{E}(\mathbf{V}(E))$, so it only remains to prove the reverse containment. Take any $(p, q) \in \mathbf{E}(\mathbf{V}(E))$. If $(p, 0_\mathbf{T}) \in \mathbf{E}(\mathbf{V}(E))$, then we also have $(q, 0_\mathbf{T}) \in \mathbf{E}(\mathbf{V}(E)))$, and by Theorem 4 it then follows that $(p, 0_\mathbf{T}), (q, 0_\mathbf{T}) \in E_+ \subseteq \hat{E}$. Thus $(p, q) \in \hat{E}$.

Now suppose $(p, 0_\mathbf{T}) \notin \mathbf{E}(\mathbf{V}(E))$, and so also $(q, 0_\mathbf{T}) \notin \mathbf{E}(\mathbf{V}(E))$. Define $F = \langle E, (p \odot y, 1_\mathbf{T}) \rangle_{\mathbf{T}[\mathbf{x},y]}$. Then $(q \odot y, 1_\mathbf{T}) \in \mathbf{E}(\mathbf{V}(F))$, and so by Corollary 5.14 it follows that for every $t > 1_\mathbf{T}$ there exist $r_1, r_2 \in \mathbf{T}[\mathbf{x}, y]$ and $N > 0$ such that for every $\epsilon \neq 0_\mathbf{T}$ we have

$$(1_\mathbf{T}, \epsilon) \times (((q \odot y) \oplus t)^{\odot N} \oplus r_1, 0_\mathbf{T}) \times ((q \odot y) \oplus t, t) \in F \tag{1}$$

and

$$(1_\mathbf{T}, \epsilon) \times (((q \odot y) \oplus t^{\odot -1})^{\odot N} \oplus r_2, 0_\mathbf{T}) \times ((q \odot y) \oplus t^{\odot -1}, q \odot y) \in F. \tag{2}$$

Let us first analyze property (1) for a fixed $t > 1_\mathbf{T}, r_1 \in \mathbf{T}[\mathbf{x}, y], N > 0$ and $\epsilon \neq 1_\mathbf{T}$. Write $r_1(\mathbf{x}, y) = \bigoplus_{0 \leq d \leq M} c_d(\mathbf{x}) \odot y^{\odot d}$. Using the fact that $(p \odot y, 1_\mathbf{T}) \in F$, (twisted) multiplying the given relation by $(p^{\odot(M+N+1)}, 0_\mathbf{T})$ implies

$$(1_\mathbf{T}, \epsilon) \times \left(\left(p^{\odot M} \odot (q \oplus (t \odot p))^{\odot N} \right) \oplus r_1', 0_\mathbf{T} \right) \times (q \oplus (t \odot p), t \odot p) \in F,$$

where $r_1' = p^{\odot N} \odot \bigoplus_{0 \leq d \leq M} c_d(\mathbf{x}) \odot p^{\odot(M-d)} \in \mathbf{T}[\mathbf{x}]$. By Proposition 5.9, it follows that

$$(1_\mathbf{T}, \epsilon) \times \left(\left(p^{\odot M'} \odot (q \oplus (t \odot p))^{\odot N} \right) \oplus r_1'', 0_\mathbf{T} \right) \times (q \oplus (t \odot p), t \odot p) \in E$$

for some even larger M' (where $r_1'' = p^{\odot(M'-M)} \odot r_1' \in \mathbf{T}[\mathbf{x}]$). Twisted multiplying by $(p^{\odot N} \odot (q \oplus (t \odot p))^{\odot M'}, 0_\mathbf{T})$ if necessary (and replacing r_1'' by another $r_1''' \in \mathbf{T}[\mathbf{x}]$), we may assume $M' = N$. Multiplying by $(t^{\odot M'}, 0_\mathbf{T})$, we then have

$$(1_\mathbf{T}, \epsilon) \times \left((t \odot p)^{\odot M'} \odot (q \oplus (t \odot p))^{\odot M'} \oplus r_1'', 0_\mathbf{T} \right) \times (q \oplus (t \odot p), t \odot p) \in E. \tag{3}$$

By the identical argument with the roles of p and q reversed, we similarly have

$$(1_\mathbf{T}, \epsilon) \times \left((t \odot q)^{\odot M'} \odot (p \oplus (t \odot q))^{\odot M'} \oplus r_3'', 0_\mathbf{T} \right) \times (p \oplus (t \odot q), t \odot q) \in E. \tag{4}$$

By a similar argument beginning with property (2), we likewise have that

$$(1_{\mathbf{T}}, \epsilon) \times \left(p^{\odot M'} \odot ((t \odot q) \oplus p)^{\odot M'} \oplus r_2'', 0_{\mathbf{T}}\right) \times ((t \odot q) \oplus p, t \odot q) \in E. \tag{5}$$

Again, repeating the argument with the roles of p and q reversed implies

$$(1_{\mathbf{T}}, \epsilon) \times \left(q^{\odot M'} \odot ((t \odot p) \oplus q)^{\odot M'} \oplus r_4'', 0_{\mathbf{T}}\right) \times ((t \odot p) \oplus q, t \odot p) \in E. \tag{6}$$

Now observe that if we add relations (3) and (6), it follows that

$$(1_{\mathbf{T}}, \epsilon) \times \left(\left((t \odot p)^{\odot M'} \oplus q^{\odot M'}\right) \odot (q \oplus (t \odot p))^{\odot M'} \oplus r_5, 0_{\mathbf{T}}\right) \times (q \oplus (t \odot p), t \odot p) \in E. \tag{7}$$

Noting that $((t \odot p)^{\odot M'} \oplus q^{\odot M'}) \odot (q \oplus (t \odot p))^{\odot M'} = (q \oplus (t \odot p))^{\odot 2M'}$, it follows that (7) precisely implies $(q \oplus (t \odot p), t \odot p) \in E_+$. As t decreases to $1_{\mathbf{T}}$, these relations form an ascending chain with limit relation $(q \oplus p, p)$, and hence we therefore have $(q \oplus p, p) \in (E_+)^{\lim}$.

Similarly, adding (4) and (5) and following the same argument, we find that $(p \oplus (t \odot q), t \odot q) \in E_+$, and hence $(p \oplus q, q) \in (E_+)^{\lim}$. It thus follows that (p, q) is in the transitive closure of $(E_+)^{\lim}$, i.e., $(p, q) \in \hat{E}$. □

6. Conclusion

The Tropical Nullstellensatz for Congruences marks the beginning of a proper foundational theory. We have solved the problem of characterizing those congruences that arise naturally from subsets of $\mathbf{T}^n$. The next issue to consider is the characterization of those subsets of $\mathbf{T}^n$ that arise as congruence varieties. Beyond their piecewise linear nature, it appears congruence varieties may also possess certain geometric finiteness properties, the existence of which would imply algebraic finiteness properties for the associated congruences. This would be significant, given the non-Noetherian properties of congruences on $\mathbf{T}[\mathbf{x}]$ in general.

Another immediate issue to address is the integration of standard tropical varieties into this new setting. As mentioned in the introduction, congruence varieties do not exactly correspond to standard tropical varieties. Instead, it is beginning to appear that standard tropical varieties may correspond to divisors in the current theory. Although purely speculative at this point, this would go some ways to explaining some of the difficulty others have had in intersecting tropical varieties (e.g., [AR10], [Kat12], [OP10]). Note that it would be trivial to intersect a congruence variety with a standard tropical variety (given as the corner locus of a tropical function), simply by restricting the tropical function to the congruence variety (thus obtaining a new divisor on the congruence variety). A detailed study of divisors will be the focus of a future manuscript.

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