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Picard Groups, Grothendieck Rings, and Burnside Rings of Categories

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FOR SAUNDERS MAC LANE, ON HIS 90TH BIRTHDAY

We discuss the Picard group, the Grothendieck ring, and the Burnside ring of a symmetric monoidal category, and we consider examples from algebra, homological algebra, topology, and algebraic geometry.

In October, 1999, a small conference was held at the University of Chicago in honor of Saunders Mac Lane's 90th birthday. I gave a talk there based on a paper that I happened to have started writing the month before. This is that paper, but with the prefatory and concluding remarks addressed to Mac Lane and the rest of the audience at the talk.

Preface. According to Peter Freyd [13], "Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial." That was written early on, in 1966. I prefer an update of that quote: "Perhaps the purpose of categorical algebra is to show that which is formal is formally formal." It is by now abundantly clear that mathematics can be formal without being trivial. Categorical algebra allows one to articulate analogies and to perceive unexpected relationships between concepts in different branches of mathematics. For example, this talk will give an

[1] I thank Po Hu for spurring me to write these observations, and I thank Halvard Fausk and Gaunce Lewis for careful readings of several drafts and many helpful comments. I thank Madhav Nori and Hyman Bass for help with the ring theory examples and Peter Freyd, Michael Boardman, and Neil Strickland for facts about cancellation phenomena in topology. I thank Fabien Morel for many interesting discussions of examples in algebraic geometry.

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answer to the following riddle: "How is a finitely generated projective R-module like a wedge summand of a finite G-CW spectrum?[2]"

1. INTRODUCTION

The classical Picard group $\mathrm{Pic}(R)$ of a commutative ring R is the group of isomorphism classes of R-modules invertible under the tensor product. This group embeds in the group of units in the Grothendieck ring of finitely generated projective R-modules. By analogy, many other "Picard groups" have been defined in algebraic geometry and algebraic topology. Most such groups are examples of the Picard group $\mathrm{Pic}(\mathscr{C})$ of a closed symmetric monoidal category $\mathscr{C}$. The notion of a symmetric monoidal category was formulated by Mac Lane [31] in 1963, long before others were aware of the utility of such a common language for thinking about categories with products (such as Cartesian products, tensor products, smash products, etc.). The definition of $\mathrm{Pic}(\mathscr{C})$ was pointed out by Hovey *et al.* [21, p. 108],[3] but there were many precursors. When $\mathscr{C}$ has finite coproducts, $\mathrm{Pic}(\mathscr{C})$ maps naturally to the group of units in the Grothendieck ring $K(\mathscr{C})$ of dualizable objects of $\mathscr{C}$.

One of the goals of this paper is to advertise the general theory of duality in symmetric monoidal categories, which has still not been fully exploited. We show that there is an Euler characteristic homomorphism of rings χ from $K(\mathscr{C})$ to the commutative ring $R(\mathscr{C})$ of self-maps of the unit object of $\mathscr{C}$. Moreover, χ factors as the composite of a quotient homomorphism of rings $K(\mathscr{C}) \longrightarrow A(\mathscr{C})$ and a monomorphism $\chi\colon A(\mathscr{C}) \longrightarrow R(\mathscr{C})$, where $A(\mathscr{C})$ is a ring that we call the Burnside ring of $\mathscr{C}$. When $\mathscr{C}$ is triangulated, we shall prove in the sequel [35] that χ is additive on exact triangles, which makes $A(\mathscr{C})$ relatively computable. This is a good example of a result that is formal but surprisingly non-trivial. These definitions and observations give a common way of thinking about some basic structure that arises in several branches of mathematics.

The framework sheds light on and is in part motivated by equivariant stable homotopy theory. If G is a compact Lie group and $\mathscr{C} = HoG\mathscr{S}$ is the stable homotopy category of G-spectra, then $A(\mathscr{C})$ is the Burnside ring $A(G)$ and $\chi\colon A(\mathscr{C}) \longrightarrow R(\mathscr{C})$ is the standard isomorphism from $A(G)$ to the zeroth equivariant stable homotopy group of spheres. In another sequel [11], Fausk, Lewis, and I will calculate $\mathrm{Pic}(HoG\mathscr{S})$ in terms of $\mathrm{Pic}(A(G))$.

[2] R is a commutative ring; G is a compact Lie group.

[3] Page 108 is the last page of ref. [21]: this paper can be viewed as a continuation of that one.

I conjecture that $\chi\colon A(\mathscr{C}) \longrightarrow R(\mathscr{C})$ is also an isomorphism when $\mathscr{C}$ is the $\mathbb{A}^1$-stable homotopy category of Morel and Voevodsky. Po Hu [26] has made significant progress on the calculation of $\mathrm{Pic}(\mathscr{C})$ in this case.

2. DUALITY AND THE DEFINITION OF PICARD GROUPS

We shall build up structure on $\mathscr{C}$ as we need it, and we begin by assuming that $\mathscr{C}$ is a closed symmetric monoidal category with unit object S, product $\wedge$, and internal hom functor F. We will later assume that $\mathscr{C}$ has finite coproducts and will denote the coproduct by $\vee$. Our interest is in categories with far more structure, such as the stable homotopy categories described axiomatically in [21].

The chosen notations will be congenial to the algebraic topologist, who will think of $\mathscr{C}$ as the stable homotopy category $\mathrm{Ho}\mathscr{S}$ with its smash products and function spectra, the unit object being the sphere spectrum and coproducts being wedges. There are many generalizations of this example in classical and equivariant stable homotopy theory, and many more in such modern refinements of stable homotopy theory as [9].

The algebraist will prefer to think of $\mathscr{C}$ as the category $\mathscr{M}_R$ of modules over a commutative ring R under $\otimes$ and Hom, with unit object R and coproduct $\oplus$. The homological algebraist will prefer to replace $\mathscr{M}_R$ by the derived category $\mathscr{D}_R$ and might want to generalize to differential graded modules over a differential graded commutative R-algebra (see e.g. [29]).

Actually, in algebra, restriction to the commutative case is rather unnatural. A more elaborate definitional framework, working with suitable monoidal, not just symmetric monoidal, categories would allow for Picard groups of bimodules over associative algebras and their derived analogues. The latter have been introduced and studied by Miyachi and Yekutieli [37, 47] and by Rouquier and Zimmermann [43], as a follow-up to Rickard's work on tilting complexes [41, 42]. The derived Picard group of a commutative k-algebra A in those papers is not the same as our $\mathrm{Pic}(\mathscr{D}_A)$ since the former is defined in terms of A-bimodules, whereas $\mathrm{Pic}(\mathscr{D}_A)$ is defined in terms of left A-modules.[4]

The algebraic geometer will think of $\mathscr{C}$ as the category $sh(X)$ of sheaves of modules over a scheme X under the tensor product and internal Hom, with unit object the structure sheaf $\mathscr{O}_X$. A more recent example in algebraic geometry is the $\mathbb{A}^1$-stable homotopy category of Morel and Voevodsky [39], which is closely analogous to the initial examples from stable homotopy theory in topology and is one of our motivating examples.

The notion of a "strongly dualizable" (or "finite") object in $\mathscr{C}$ was defined in [30, III.1.1]; we shall abbreviate by calling such objects

[4] These may be viewed as "central A-bimodules," whose left and right actions agree.

"dualizable." An early definition of this type was given by Dold and Puppe [8], but essentially the same definition also appears in the literature of algebraic geometry [3] and there are many precursors. The simplest of the many equivalent forms of the definition is as follows. In any closed symmetric monoidal category, we have unit and counit isomorphisms $S \wedge X \cong X$ and $X \cong F(S, X)$ and a pairing

$$\wedge : F(X, Y) \wedge F(X', Y') \longrightarrow F(X \wedge X', Y \wedge Y'). \tag{2.1}$$

Define

$$\nu : F(X, Y) \wedge Z \longrightarrow F(X, Y \wedge Z) \tag{2.2}$$

by replacing Z by $F(S, Z)$ and applying the pairing (2.1). Define the dual of X to be $DX = F(X, S)$.

Definition 2.3. An object X of $\mathscr{C}$ is *dualizable* if the canonical map

$$\nu : DX \wedge X \longrightarrow F(X, X)$$

is an isomorphism in $\mathscr{C}$. When X is dualizable, we define the "coevaluation map" $\eta : S \longrightarrow X \wedge DX$ to be the composite

$$S \xrightarrow{\iota} F(X, X) \xrightarrow{\nu^{-1}} DX \wedge X \xrightarrow{\gamma} X \wedge DX,$$

where ι is adjoint to the identity map of X and γ is the natural commutativity isomorphism given by the symmetric monoidal structure. Note that we have an evaluation map $\varepsilon : DX \wedge X \longrightarrow S$ for any object X.

The following examples already answer our riddle: finitely generated projective R-modules and wedge summands of finite G-CW spectra are the dualizable objects in their ambient symmetric monoidal categories.

Example 2.4. Let R be a commutative ring. It is an exercise to show that an R-module M is dualizable if and only if M is finitely generated and projective. Indeed, if ν is an isomorphism, then the resulting description of the identity map $M \longrightarrow M$ gives a recipe for presenting M as a direct summand of a finitely generated free R-module, and the converse is even easier.

Example 2.5. (i) A spectrum X (in the sense of algebraic topology) is dualizable in $\mathrm{Ho}\mathscr{S}$ if and only if it is a wedge summand of a finite CW spectrum [36, XVI.7.4]. The cited result proves this more generally for G-spectra in the equivariant stable homotopy category $\mathrm{Ho}G\mathscr{S}$ for any compact Lie group G. In fact, a wedge summand of a finite CW spectrum is itself a finite CW spectrum (e.g., [13, 4.5]), but that is not true equivariantly.

(ii) The characterization in (i) is axiomatized by [21, 2.1.3], which gives the analogous conclusion in any "unital algebraic stable homotopy category". Such a category has a set $\mathscr{G}$ of dualizable small generators, and an object X is dualizable if and only if it is in the thick subcategory generated by $\mathscr{G}$, namely the smallest subcategory of $\mathscr{C}$ that is closed under cofibrations and retracts and contains $\mathscr{G}$.

The following characterizations of dualizable objects are proven in [30, III.1.6]; other characterizations are given in [21, 2.1.3].

Theorem 2.6. *Fix objects X and Y of $\mathscr{C}$. The following are equivalent.*

(i) *X is dualizable and Y is isomorphic to DX.*

(ii) *There are maps $\eta\colon S \longrightarrow X \wedge Y$ and $\varepsilon\colon Y \wedge X \longrightarrow S$ such that the composites*

$$X \cong S \wedge X \xrightarrow{\eta \wedge id} X \wedge Y \wedge X \xrightarrow{id \wedge \varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{id \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge id} S \wedge Y \cong Y$$

are identity maps.

(iii) *There is a map $\eta\colon S \longrightarrow X \wedge Y$ such that the composite*

$$\mathscr{C}(W \wedge X, Z) \xrightarrow{(-)\wedge Y} \mathscr{C}(W \wedge X \wedge Y, Z \wedge Y) \xrightarrow{(id \wedge \eta)^*} \mathscr{C}(W, Z \wedge Y)$$

is a bijection for all objects W and Z of $\mathscr{C}$.

(iv) *There is a map $\varepsilon\colon Y \wedge X \longrightarrow S$ such that the composite*

$$\mathscr{C}(W, Z \wedge Y) \xrightarrow{(-)\wedge X} \mathscr{C}(W \wedge X, Z \wedge Y \wedge X) \xrightarrow{(id \wedge \varepsilon)_*} \mathscr{C}(W \wedge X, Z)$$

is a bijection for all objects W and Z of $\mathscr{C}$.

Here the adjoint $\tilde{\varepsilon}\colon Y \longrightarrow DX$ of a map ε satisfying (ii) or (iv) is an isomorphism under which the given map ε corresponds to the canonical evaluation map $\varepsilon : DX \wedge X \longrightarrow S$. We also have the following observations [30, II, Sect. 1].

Proposition 2.7. *If X and Y are dualizable, then DX and $X \wedge Y$ are dualizable and the canonical map $\rho\colon X \longrightarrow DDX$ is an isomorphism. Moreover, the map ν of (2.2) is an isomorphism if either X or Z is dualizable, and* *the map $\wedge$ of (2.1) is an isomorphism if both X and X' are dualizable or if both X and Y are dualizable.*

We have the following definition and observation [21, A.2.8].

DEFINITION 2.8. An object X of $\mathscr{C}$ is *invertible* if there is an object Y and an isomorphism $X \wedge Y \cong S$.

LEMMA 2.9. *If X is invertible with inverse Y, then X and Y satisfy the equivalent conditions of Theorem* 2.6.

Proof. Since the functor $(-) \wedge Y$ on $\mathscr{C}$ is an equivalence of categories, any isomorphism $\eta\colon S \longrightarrow X \wedge Y$ satisfies condition (iii) of Theorem 2.6. ∎

Following, [21, A.2.7] we make the following definition. Henceforward, we assume that there is only a set of isomorphism classes of dualizable objects in $\mathscr{C}$.

DEFINITION 2.10. Define the *Picard group* $\mathrm{Pic}(\mathscr{C})$ to be the set of isomorphism classes $[X]$ of invertible objects X with product and inverses defined by

$$[X][Y] = [X \wedge Y] \qquad \text{and} \qquad [X]^{-1} = [DX].$$

As is easily seen, $\mathrm{Pic}(\mathscr{C})$ is a well-defined Abelian group with identity element $[S]$.

EXAMPLE 2.11. By Lemma 2.9 and Example 2.4, an invertible R-module is finitely generated and projective. By [2, Sect. 5.4], it follows that M is invertible if and only if it is finitely generated projective of rank one. This shows that $\mathrm{Pic}(\mathscr{M}_R)$ coincides with $\mathrm{Pic}(R)$ as defined classically. In fact, for any scheme X, our $\mathrm{Pic}(sh(X))$ is isomorphic to $\mathrm{Pic}(X)$ as defined classically [17, II.6.12]; see [10].

EXAMPLE 2.12. The Picard groups of the derived categories $\mathscr{D}_R$ and of the analogous derived categories of sheaves of modules have been calculated by Fausk [10].

EXAMPLE 2.13. The Picard group $\mathrm{Pic}(Ho\mathscr{S})$ of the stable homotopy category is just $\mathbb{Z}$, the sphere spectra being the only invertible spectra [19, 46]. One can construct localizations of $Ho\mathscr{S}$ with respect to homology theories, and the problem of computing the resulting Picard groups is non-trivial. The Picard groups of $K(n)$-local spectra are studied in [19, 24], and the Picard groups of $E(n)$-local spectra are studied in [22].

We shall return to the study of $\mathrm{Pic}(\mathscr{C})$ for a general stable homotopy category $\mathscr{C}$ in [11], where $\mathrm{Pic}(HoG\mathscr{S})$ is computed. The category $\mathrm{Ho}G\mathscr{S}$ is constructed so as to invert the one-point compactifications S^V of real representations V, but we shall see that inverting the S^V has the effect of inverting other G-spectra as well.

EXAMPLE 2.14. Hu [25] has begun the study of $\mathrm{Pic}(\mathscr{C})$ when $\mathscr{C}$ is the $\mathbb{A}^1$-stable homotopy category of Morel and Voevodsky [39] by finding a surprising variety of exotic invertible elements of $\mathscr{C}$. Here again, $\mathscr{C}$ is constructed so as to invert certain canonical spheres, and Hu's examples show that many other varieties are also inverted. A complete computation is not yet in sight.

3. THE GROTHENDIECK AND UNIT ENDOMORPHISM RINGS OF $\mathscr{C}$

We now bring Grothendieck rings into the picture, and we add the assumption that $\mathscr{C}$ has finite coproducts. We write $*$ for the coproduct of the empty set of objects; it is an initial object of $\mathscr{C}$.

DEFINITION 3.1. Define $K(\mathscr{C})$, or better $K_0(\mathscr{C})$, to be the Grothendieck ring associated to the semi-ring $\mathrm{Iso}(\mathscr{C})$ of isomorphism classes of dualizable objects of $\mathscr{C}$, with $\vee$ as addition and $\wedge$ as multiplication; $[*]$ and $[S]$ are the 0 and 1. Let $\alpha\colon \mathrm{Iso}(\mathscr{C}) \longrightarrow K(\mathscr{C})$ be the canonical map of semi-rings.

The following definition and observation explain when α is injective.

DEFINITION 3.2. Dualizable objects X and Y are *stably isomorphic* if there is a dualizable object Z and an isomorphism $X \vee Z \cong Y \vee Z$. The category $\mathscr{C}$ satisfies the *cancellation property* if stably isomorphic dualizable objects are isomorphic.

Remark 3.3. In the topological examples, the notion of stable isomorphism must not be confused with the totally different notion of stable homotopy equivalence. When $\mathscr{C}$ is the stable homotopy category, the cancellation property and the structure of $K(\mathscr{C})$ have been studied extensively by Freyd [13–16] and Margolis [34]. Cancellation does not hold in general, but only due to mixing of primes. Cancellation does hold for the stable homotopy category after localization or completion at a prime p, as a consequence of a unique decomposition theorem expressing any finite p-local or p-complete spectrum as a finite wedge of indecomposable p-local or p-complete spectra. An inspection of the proofs shows that these results remain valid for the stable homotopy category of G-spectra for any compact Lie group G.

PROPOSITION 3.4. *Dualizable objects X and Y are stably isomorphic if and only if $\alpha[X]=\alpha[Y]$, hence $\alpha\colon Iso(\mathscr{C})\longrightarrow K(\mathscr{C})$ is an injection if and only if $\mathscr{C}$ satisfies the cancellation property.*

COROLLARY 3.5. *$\alpha[X]$ is a unit of $K(\mathscr{C})$ if and only if there is a dualizable object Y such that $X\wedge Y$ is stably isomorphic to S.*

Let $R^{\times}$ denote the group of units of a commutative ring R.

PROPOSITION 3.6. *α restricts to a homomorphism $\beta\colon Pic(\mathscr{C})\longrightarrow K(\mathscr{C})^{\times}$, and β is a monomorphism if stably isomorphic invertible objects are isomorphic.*

The last condition is much weaker than the general cancellation property. For example, cancellation usually does not hold in $\mathscr{M}_R$, but, as pointed out to me by Madhav Nori, it is known to hold on invertible R-modules.

PROPOSITION 3.7. *Stably isomorphic invertible modules M and N over a commutative ring R are isomorphic.*

Proof. Adding a suitable finitely generated projective module to a given isomorphism if necessary, we have $M\oplus F\cong N\oplus F$ for some finitely generated free R-module F. Applying the determinant functor gives an isomorphism $M\cong N$. ∎

We have the following commutative diagram, in which the horizontal arrows are inclusions:

$$\begin{tikzcd}
\mathrm{Pic}(\mathscr{C}) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Iso}(\mathscr{C}) \arrow[d, "\alpha"] \\
K(\mathscr{C})^{\times} \arrow[r] & K(\mathscr{C}).
\end{tikzcd}$$

PROPOSITION 3.8. *Let $\mathscr{C}=\mathscr{M}_R$ for a commutative ring R. Then the diagram just displayed is a pullback in which β is a monomorphism.*

Proof. Here $K(\mathscr{C})=K_0(R)$. To show that the diagram is a pullback, we must show that if P is a finitely generated projective R-module such that $\alpha[P]$ is a unit, then P is invertible. There are finitely generated projective R-modules P' and Q such that $(P\otimes P')\oplus Q\cong R\oplus Q$. This implies that the localization of $P \otimes P'$ at any prime ideal is free of rank one, so that $P \otimes P'$ has rank one. But then $P \otimes P'$, hence also P, is invertible. Proposition 3.7 gives that β is a monomorphism. ∎

The proofs above don't generalize, but the results might.

Problem 3.9. Find general conditions on $\mathscr{C}$ that ensure that the diagram above is a pullback in which β is a monomorphism.

Now assume further that the category $\mathscr{C}$ is additive, so that $\vee$ is its biproduct; it follows that the functor $\wedge$ is bilinear. We bring another ring into the picture, the unit endomorphism ring $R(\mathscr{C})$.

Definition 3.10. Define $R(\mathscr{C})$ to be the commutative ring $\mathscr{C}(S, S)$ of endomorphisms of S, with multiplication given by the $\wedge$-product of maps or, equivalently, by composition of maps. Then $\mathscr{C}(X, Y)$ is an $R(\mathscr{C})$-module and composition is $R(\mathscr{C})$-bilinear, so that $\mathscr{C}$ is enriched over $\mathscr{M}_{R(\mathscr{C})}$.

Definition 3.11. Define a functor $\pi_0 \colon \mathscr{C} \longrightarrow \mathscr{M}_{R(\mathscr{C})}$ by letting $\pi_0(X) = \mathscr{C}(S, X)$, so that $\pi_0(S) = R(\mathscr{C})$, and observe that π_0 is a lax symmetric monoidal functor under the natural map

$$\phi \colon \pi_0(X) \otimes_{R(\mathscr{C})} \pi_0(Y) \longrightarrow \pi_0(X \wedge Y)$$

induced by $\wedge$. Say that X is a *Künneth object* of $\mathscr{C}$ if X is dualizable and ϕ is an isomorphism when $Y = DX$.

The adjoint of $\pi_0(\varepsilon) \circ \phi \colon \pi_0(DX) \otimes_{R(\mathscr{C})} \pi_0(X) \longrightarrow \pi_0(S)$ is a natural map $\delta \colon \pi_0(DX) \longrightarrow D(\pi_0(X))$ of $R(\mathscr{C})$-modules. By [30, III.1.9], we have the following result relating Künneth objects of $\mathscr{C}$ to dualizable $R(\mathscr{C})$-modules.

Proposition 3.12. *Let X be a Künneth object of $\mathscr{C}$. Then $\pi_0(X)$ is a finitely generated projective $R(\mathscr{C})$-module, $\delta \colon \pi_0(DX) \longrightarrow D(\pi_0(X))$ is an isomorphism, and $\phi \colon \pi_0(X) \otimes_{R(\mathscr{C})} \pi_0(Y) \longrightarrow \pi_0(X \wedge Y)$ is an isomorphism for all objects Y.*

We shall return to the study of Künneth objects and the functor π_0 in [11], where the relationship between Künneth objects of $\mathscr{C}$ and finitely generated projective $R(\mathscr{C})$-modules is made considerably more precise.

In many of our examples, we have been considering morphisms of degree zero in triangulated categories. The notion of a Künneth object is sensitive to the grading. Definitions 3.10 and 3.11 make sense for graded morphisms in $\mathscr{C}$. Here $R(\mathscr{C})$ is a graded commutative ring, the theory of triangulated categories giving rise to the usual signs in the commutativity relation, and of course we replace the notation $\pi_0(X)$ by $\pi_*(X)$ in Definition 3.11.

EXAMPLE 3.13. (i) In the derived category $\mathscr{D}_R$ with morphisms of degree zero, where $R(\mathscr{D}_R) = R$, $\Sigma^n R$ is not a Künneth object unless $n = 0$. However, in the derived category $\mathscr{D}_R^*$ of R-chain complexes and $\mathbb{Z}$-graded morphisms, where again $R(\mathscr{D}_R^*) = R$ $(= \mathrm{Ext}_R^*(R, R))$, all $\Sigma^n R$, $n \in \mathbb{Z}$, are Künneth objects.

(ii) Similarly, in the stable homotopy category $\mathrm{Ho}\mathscr{S}$ with morphisms of degree zero, where $R(\mathrm{Ho}\mathscr{S}) = \mathbb{Z}$, S^n is not a Künneth object unless $n = 0$. In the stable homotopy category $\mathrm{Ho}^*\mathscr{S}$ with $\mathbb{Z}$-graded morphisms, where $R(\mathrm{Ho}^*\mathscr{S}) = \pi_*(S)$, all S^n, $n \in \mathbb{Z}$, are Künneth objects.

(iii) The equivariant stable homotopy category $\mathrm{Ho}G\mathscr{S}$ admits both a $\mathbb{Z}$-graded version $\mathrm{Ho}^*G\mathscr{S}$ and an $RO(G)$-graded version $\mathrm{Ho}^\bullet G\mathscr{S}$. Just as nonequivariantly, all S^n, $n \in \mathbb{Z}$, are Künneth objects in $\mathrm{Ho}^*G\mathscr{S}$. For $\alpha = V - W \in RO(G)$, there is a sphere G-spectrum $S^\alpha = S^{V-W}$. If $\dim V^H - \dim W^H = n$ for all (closed) subgroups H of G and some integer n independent of H, then results of tom Dieck and Petrie [5, 7] imply that S^α is also a Künneth object in $\mathrm{Ho}^*G\mathscr{S}$; see [11]. All S^α are Künneth objects in $\mathrm{Ho}^\bullet G\mathscr{S}$, where $R(\mathrm{Ho}^\bullet G\mathscr{S}) = \pi_\bullet^G(S)$. Here the signs in the graded commutativity must be interpreted as units in $\pi_0^G(S)$.

4. EULER CHARACTERISTICS AND THE BURNSIDE RING

In the previous example, $\pi_0^G(S)$, which by definition is the ring of endomorphisms of the sphere G-spectrum in $\mathrm{Ho}G\mathscr{S}$, is isomorphic to the Burnside ring $A(G)$. When G is finite, $A(G)$ is the Grothendieck ring of the semi-ring of finite G-sets, and this isomorphism was first observed by Segal [44]. For a general compact Lie group G, tom Dieck defined $A(G)$ and proved this isomorphism [4, 5]. The variant of tom Dieck's argument presented in [30] readily generalizes to give a definition of $A(\mathscr{C})$ and a monomorphism $A(\mathscr{C}) \longrightarrow R(\mathscr{C})$ for any stable homotopy category $\mathscr{C}$.

We first define traces and Euler characteristics, and for this we do not require our closed symmetric monoidal category $\mathscr{C}$ to have coproducts.

DEFINITION 4.1. Define the *Euler characteristic* $\chi(X) \in R(\mathscr{C})$ of a dualizable object X to be the map

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\varepsilon} S.$$

More generally, define the *trace* $\chi(f)$ of a map $f: X \longrightarrow X$ to be the composite

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\mathrm{id} \wedge f} DX \wedge X \xrightarrow{\varepsilon} S.$$

Traces and Euler characteristics are suitably natural in $\mathscr{C}$, by [30, III.7.7].

PROPOSITION 4.2. *Let $\Phi: \mathscr{C} \longrightarrow \mathscr{C}'$ be a strong symmetric monoidal functor between closed symmetric monoidal categories with unit objects S and S'. For an endomorphism f of a dualizable object X of $\mathscr{C}$, $\chi(\Phi f): S' \longrightarrow S'$ agrees with $\Phi\chi(f)$ on $S' \cong \Phi S$. In particular, $\chi(\Phi X)$ agrees with $\Phi\chi(X)$.*

A still more general definition of trace maps is possible and useful [30, III.7.1; 35, Sect. 1]. One can study analogues of the Lefschetz fixed point theorem starting from these trace maps, but we shall restrict attention to the Euler characteristic. In algebraic settings, the same notion is sometimes referred to as the rank [1, 18, 45], and here again it is unnatural to restrict to the commutative case.

Euler characteristics enjoy the following basic properties. We again assume that $\mathscr{C}$ is additive.

PROPOSITION 4.3. *$\chi(X \vee Y) = \chi(X) + \chi(Y)$, $\chi(X \wedge Y) = \chi(X)\,\chi(Y)$, $\chi(*) = 0$, $\chi(S) = 1$, and $\chi(DX) = \chi(X)$.*

Proof. The easy proofs are explicit or implicit in [8, 4.7] or. [30, III Sect. 7]. As pointed out to me by Gaunce Lewis and Halvard Fausk, $\chi(DX) = \chi(X)$ since the following diagram is seen to commute by use of the first diagram in the proof of [30, III.1.2]:

$$\begin{tikzcd}
S \arrow[r, "\eta"] \arrow[d, "\eta"'] & X \wedge DX \arrow[r, "\gamma"] & DX \wedge X \arrow[d, "\varepsilon"] \arrow[dll, "id \wedge \rho"'] \\
DX \wedge DDX \arrow[r, "\gamma"'] & DDX \wedge DX \arrow[r, "\varepsilon"'] & S.
\end{tikzcd}$$

∎

Remark 4.4. Suppose that X has a diagonal map $\Delta: X \longrightarrow X \wedge X$ and a projection $\pi: X \longrightarrow S$ such that $(\mathrm{id} \wedge \pi) \circ \Delta: X \longrightarrow X \wedge S \cong X$ is the identity map. Then $\chi(X) = \pi \circ \tau$, where the "transfer" τ is defined to be the composite

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\mathrm{id} \wedge \Delta} DX \wedge X \wedge X \xrightarrow{\varepsilon \wedge \mathrm{id}} X.$$

In the equivariant context, this factorization has proven to be a powerful computational tool.

The additivity on coproducts implies that $\chi(X) = \chi(Y)$ if X and Y are stably isomorphic. This allows the following definition.

DEFINITION 4.5. Define $\chi\colon K(\mathscr{C}) \longrightarrow R(\mathscr{C})$ to be the ring homomorphism obtained by universality from the semi-ring homomorphism $\chi\colon \mathrm{Iso}(\mathscr{C}) \longrightarrow R(\mathscr{C})$ that sends $[X]$ to $\chi(X)$. Define the *Burnside ring* $A(\mathscr{C})$ to be the quotient ring of $K(\mathscr{C})$ obtained by identifying two elements if they have the same Euler characteristic; equivalently, $A(\mathscr{C})$ is the image of χ. Write $\chi\colon A(\mathscr{C}) \longrightarrow R(\mathscr{C})$ for the resulting monomorphism of rings.

PROPOSITION 4.6. *For a commutative ring R, $A(\mathscr{M}_R)$ is the subring of R generated by its idempotent elements.*

Proof. Up to terminology, this is stated without proof by Bass [1, 2.11]. Fausk and Bass showed me the following quick argument. By Hattori [18, Ex. 6], if P is a finitely generated projective R-module of rank n, then $\chi(P)$ is multiplication by n. (Hattori assumes that R is Noetherian, but he doesn't use that hypothesis). If $\mathrm{Spec}(R)$ is connected, then every finitely generated projective R-module is of rank n for some n [2, II, Sect. 5.3] and the result follows. By consideration of products of rings, this implies the result when $\mathrm{Spec}(R)$ has finitely many open and closed components, as always holds if R is finitely generated. By Proposition 4.2, Euler characteristics are natural with respect to homomorphisms of rings. We may identify R with the colimit of its finitely generated subrings, and $K_0(R)$ is the colimit of K_0 applied to these subrings. The general case follows. ∎

We assume henceforward that $\mathscr{C}$ is a triangulated category whose triangulation is "compatible" with its symmetric monoidal structure. A first definition of what compatibility means is given in [21, App. A], but we have in mind the more structured definition that is given in the sequel [35]. In this case additive inverses are already present in the image of $\mathrm{Iso}(\mathscr{C}) \longrightarrow R(\mathscr{C})$, which therefore coincides with $A(\mathscr{C})$. That is, $A(\mathscr{C})$ is a quotient ring of the semi-ring $\mathrm{Iso}(\mathscr{C})$. Note that ΣX is dualizable if and only if X is dualizable.

LEMMA 4.7. $\chi(\Sigma^n X) = (-1)^n\,\chi(X)$*; in particular,* $\chi(\Sigma X) = -\chi(X)$.

Proof. With $S^n = \Sigma^n S$, we have $\Sigma^n X \cong X \wedge S^n$. The result follows from the multiplicativity formula for χ and the fact that $\chi(S^n)$ is the transposition map

$$\gamma\colon S \cong S^n \wedge S^{-n} \longrightarrow S^{-n} \wedge S^n \cong S,$$

which is multiplication by $(-1)^n$ in any symmetric monoidal category with compatible triangulation. ∎

The fact that $\chi(DX) = \chi(X)$ implies the following observation.

LEMMA 4.8. *Every unit* $[X]$ *of the ring* $A(\mathscr{C})$ *satisfies* $[X]^2 = 1$.

We must still explain why we call $A(\mathscr{C})$ the Burnside ring of $\mathscr{C}$.

EXAMPLE 4.9. Let G be a compact Lie group and let $\mathscr{C} = \mathrm{Ho}G\mathscr{S}$ be the stable homotopy category of G-spectra. Then, by definition, $R(\mathscr{C}) = \pi_0^G(S)$, where S is the sphere G-spectrum. By [30, V.2.12], we can define the Burnside ring of G by $A(G) = A(\mathscr{C})$. When G is finite, $A(G)$ is isomorphic to the classical Burnside ring of finite G-sets, as we shall explain in Example 4.17.

Now [30, V.2.11] gives the following version of the cited isomorphism of Segal [44] and tom Dieck [4, 5].

THEOREM 4.10. *Let* $\mathscr{C} = HoG\mathscr{S}$. *Then*

$$\chi\colon A(G) = A(\mathscr{C}) \longrightarrow R(\mathscr{C}) = \pi_0^G(S)$$

is an isomorphism of rings.

We offer the following conjecture.

Conjecture 4.11. The analogue of Theorem 4.10 holds for the $\mathbb{A}^1$-stable homotopy category $\mathscr{C}$ of Morel and Voevodsky (for a given ground field k). Precisely, we have defined a monomorphism of rings $\chi\colon A(\mathscr{C}) \longrightarrow R(\mathscr{C})$, and we conjecture that it is an isomorphism.

Remark 4.12. When char $k \neq 2$, Morel [38] has conjectured that $R(\mathscr{C})$ is isomorphic to the Grothendieck–Witt ring $GW(k)$, and he has constructed a split monomorphism $GW(k) \longrightarrow R(\mathscr{C})$. He has also proven[5] that this monomorphism factors through $A(\mathscr{C})$. Thus, if his conjecture is true, then so is ours.

Of course, $A(\mathscr{C})$ always gives a lower bound on the size of $R(\mathscr{C})$. The force of the definition of the Burnside ring comes from the following additivity theorem, which makes $A(\mathscr{C})$ a reasonably computable object. We shall prove this result, together with a substantial generalization, in the sequel [35]. In fact, the definition given there of "compatibility" between the triangulation and symmetric monoidal structure on $\mathscr{C}$ is designed to ensure the truth of this result.

THEOREM 4.13. *Let* $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ *be a distinguished triangle. Then*

$$\chi(Y) = \chi(X) + \chi(Z).$$

[5] Private communication.

EXAMPLE 4.14. When $\mathscr{C} = \mathrm{Ho}\mathscr{S}$, the theorem implies that χ is just the classical Euler characteristic on finite CW spectra.

In the triangulated context, we have another candidate for the Grothendieck ring of the category $\mathscr{C}$.

DEFINITION 4.15. Define $K'(\mathscr{C})$ to be the quotient of $K(\mathscr{C})$ by the subgroup generated by the elements $[Y]-[X]-[Z]$ for all exact triangles $X \to Y \to Z \to \Sigma X$. The compatibility of $\wedge$ with the triangulation ensures that the cited subgroup is an ideal, so that $K'(\mathscr{C})$ is a quotient ring of $K(\mathscr{C})$.

COROLLARY 4.16. *The quotient map $K(\mathscr{C}) \to A(\mathscr{C})$ factors through $K'(\mathscr{C})$.*

EXAMPLE 4.17. Let G be a compact Lie group and $\mathscr{C} = \mathrm{Ho}G\mathscr{S}$. Write $[G/H]$ for the element of $K'(\mathscr{C})$ or $A(\mathscr{C})$ represented by the suspension G-spectrum of G/H_+, where H is a closed subgroup of G and the $+$ denotes adjunction of a disjoint basepoint. We take one H from each conjugacy class of subgroups. There are wedge summands of finite G-CW spectra that are not themselves finite G-CW spectra; their isomorphism classes, together with the $[G/H]$, generate $K'(\mathrm{Ho}G\mathscr{S})$. The $[G/H]$ generate a subring, which is isomorphic to the Euler ring $U(G)$ introduced by tom Dieck [5, Sect. 5.4]. When G is finite, $U(G) \cong A(G)$. However, a transfer argument using Remark 4.4 shows that $\chi(\Sigma^\infty G/H_+) = 0$ unless H has finite index in its normalizer. Some further argument shows that $A(G)$ is the free Abelian group generated by the remaining [G/H]; see [30, III.8.3, V.2.6]. It is remarkable that the cited wedge summands make no contribution: as we have defined it, $A(G)$ is a quotient of $K'(\mathscr{C})$, but it turns out to be a quotient of $U(G)$; see [30, V.2.12]. It is unclear whether or not such a simplification occurs more generally in the context of the unital algebraic stable homotopy categories described in Example 2.5(ii).

Conclusion. This paper is a very modest example of a kind of mathematics new to the last half of the 20th century. A great deal of modern mathematics would quite literally be unthinkable without the language of categories, functors, and natural transformations that was introduced by Eilenberg and MacLane in 1945. It was perhaps inevitable that some such language would have appeared eventually. It was certainly not inevitable that such an early systematization would have proven so remarkably durable and appropriate; it is hard to imagine that this language will ever be supplanted. Its introduction heralded the present golden age of mathematics.

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