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The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT*

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Abstract. It is proved that for any ultrametric space (X, d), the set $\mathbf{L}(X)$ of its closed balls is a lattice $(\mathbf{L}(X), \cap, \sup, r(B))$. It is complete, atomic, tree-like, and real graduated. For any such lattice $(L, \wedge, \vee, r)$, the set $\mathbf{A}(L)$ of its atoms can be naturally equipped with an ultrametric $\Delta(x, y)$. These assignments are inverse of one another:

$$(\mathbf{A}(\mathbf{L}(X)), \Delta) = (X, d) \text{ and } (L, \wedge, \vee, r) = (\mathbf{L}(\mathbf{A}(L)), \cap, \sup, r(B)),$$

where the first equality means an isometry while the second one is a lattice isomorphism. A similar correspondence established for morphisms, shows that there is an isomorphism of categories. The category **ULTRAMETR** of ultrametric spaces and non-expanding maps is isomorphic to the category **LAT*** of complete, atomic, tree-like, real graduated lattices and isotonic, semi-continuous, non-extensive maps. We describe properties of the isomorphism functor and its relations to the categorical operations and action of other functors. Basic properties of a space (such as completeness, spherical completeness, total boundedness, compactness, etc.) are translated into algebraic properties of the corresponding lattice $\mathbf{L}(X)$.

1. Introduction

The goal of the paper is to connect two different and remote areas of mathematics — the theory of ultrametric spaces and lattice theory. Our Main Theorem states that a category **ULTRAMETR** of ultrametric spaces and non-expanding maps is isomorphic to a category **LAT*** of complete, atomic, tree-like, real graduated lattices and isotonic, semi-continuous, non-extensive maps (for definitions see below).

A metric space (X, d) is called *ultrametric* [7] (or *non-Archimedean* [4], or *isosceles* [8], [12]) if its metric satisfies the strong triangle axiom:

$$d(x, z) \leq \max[d(x, y), d(y, z)]. \tag{$*$}$$

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This inequality is equivalent to the following property: any three points x, y, and z are vertices of an isosceles triangle with the base being no greater than the sides. We denote this property by $(**)$.

Ultrametric spaces play an important role in many areas of mathematics such as number theory (rings $\mathbb{Z}_p$ and fields $\mathbb{Q}_p$ of p-adic numbers), algebra (non-Archimedean valued rings and fields), real analysis (the Baire space $B_{\aleph_o}$), complex analysis (rings $\mathfrak{M}(U)$ of meromorphic functions over an open region $U \subset \mathbb{C}$), general topology (generalized Baire spaces B_τ), computer linguistics (a set of words of computer language equipped with the metric inherited from $B_{\aleph_o}$), p-adic analysis (the ground field Ω_p), p-adic functional analysis (algebras of Ω-valued functions), the theory of p-adic analytic manifolds, the theory of topological groups, and so on .

Ultrametric spaces were described up to topological equivalence in [4], [24], up to uniform equivalence in [9], [15], [18] and up to isometry in [8] and [21]. For the last 15 years the theory of ultrametric spaces has found new close relations to geometry of Euclidean and Hilbert spaces [6], [12], [19], [21], theory of topological groups [8], [22] and Boolean algebras [15], [18], Lebesgue measure theory [16], [19] and computer science [3], [20], [23], category theory [9]–[11], [13], [14], [18], [23] and topoi [17], set theory and foundations [26].

The explanation of deep specific properties of ultrametric spaces is beyond the scope of this paper. For our goal here it is enough to know only their elementary metric properties. They are stated in the first Lemmas of Section 2 to get a reader out of necessity of reading any other paper on the subject. The well-known books [1], [2], [5] provide us with all required definitions and theorems from lattice theory, category theory and topology.

2. Spaces and lattices

Let (X, d) be an (ultra-)metric space. A set $B(a, s) = \{x \mid d(x, a) \leq s\}$ is called a *closed ball* of *nominal radius* s with a center located at a point a.

Lemma 1. *Any point of a ball is its center, i.e.,* $B(a, s) = B(x, s)$ *for any point* $x \in B(a, s)$.

An *actual radius* of a ball $B(a, s)$ is a number $r = \sup\{d(a, x) \mid x \in B(a, s)\}$. Clearly, $r \leq s$. Further we consider only actual radii, only closed balls, and only ultrametric spaces.

Lemma 2. *An actual radius of a ball is equal to its diameter, i.e.,* $r = \sup\{d(x, y) \mid x, y \in B\}$ *for any ball* B *lying in an ultrametric space.*

Lemma 3. *Any two balls in an ultrametric space are either disjoint or one is a subset of the other.*

Lemma 4. *If the balls $B(a,s)$ and $B(b,t)$ are disjoint, then $d(a,b) = d(x,y)$ for any $x \in B(a,s)$ and for any $y \in B(b,t)$.*

Proofs of these Lemmas follow directly from either the axiom $(*)$ or the isosceles property $(**)$ and are left to the reader.

Denote by $\mathbf{L}(X) = \{B(\alpha) \mid \alpha \in I\}$ the set of all closed balls of a space (X, d), supplemented with the empty set $\emptyset$ (we call it the *empty ball*) and the whole space X (in case $\operatorname{diam} X = \propto$), where I is an index set. Consider $\mathbf{L}(X)$ with an order relation $<$ induced by set-theoretic inclusion.

Lemma 5. *The set $\mathbf{L}(X)$ is a lattice, i.e.,*

- *for any two balls B and B', their meet is just their intersection, and is either $\emptyset$ or the smaller of the balls;*
- *for any two balls B and B', their join is either the greater of the balls (if one of them contains the other) or the ball of radius $r = d(x,y)$, where $x \in B$, $y \in B'$ with center located at any point of $B \cup B'$.*

Proof. Use Lemmas 3 and 4, respectively. □

Definition. A lattice L is called *tree-like* if the set $L \backslash \{0\}$ is a tree, i.e. it satisfies the following condition:

(t) for any element $l \in L \backslash \{0\}$, the set $[l)$ of elements which are greater than or equal to l is linearly ordered.

This property is equivalent, for lattices with least element 0, to the following:

(t') if elements l and m are not comparable, then $\inf[l, m] = 0$.

Definition. A lattice L is *real graduated* provided there is a real valued non-negative isotonic function $r \colon L \backslash \{0\} \to \mathbb{R}_+ \cup \{+\propto\}$, satisfying the following conditions:

(i) if x is an atom of L, then $r(x) = 0$;
(ii) if $l < m \in L$, then $r(l) < r(m)$;
(iii) for any $X' \subset \mathbf{A}(L)$, $r(\sup\{x \mid x \in X'\}) = \sup\{r(x \vee y) \mid x, y \in X'\}$, where $\mathbf{A}(L)$ is the set of atoms of L.

Note 1. For any real graduated lattice L, $r(l) = 0$ implies that l is an atom; thus, $\mathbf{A}(L) = r^{-1}(0)$.

Note 2. Let L be a tree-like, real graduated lattice and l, m, n be elements of L such that $0 \neq l < m$ and $l < n$. Then $r(m) < r(n)$ implies $m < n$; $r(n) < r(m)$ implies $n < m$, and $r(m) = r(n)$ implies $m = n$.

Definition (of the category **LAT***)**.** The objects of the category **LAT*** are the complete, atomic, tree-like, and real graduated lattices. Morphisms of **LAT*** are all isotonic, semi-continuous, non-extensive maps. The definition of morphisms of **LAT*** is given in Section 3 below.

Main Theorem. *The category* **ULTRAMETR** *of ultrametric spaces and non-expanding maps is isomorphic to the category* **LAT*** *of complete, atomic, tree-like, real graduated lattices and isotonic, semi-continuous, non-extensive maps.*

The proof of the Main Theorem is naturally divided in two parts: Theorem 1 on duality between spaces and lattices and Theorem 2 on morphisms.

Theorem 1. *For any ultrametric space* (X, d) *there is a lattice* $(\mathbf{L}(X), \sup, \cap, r(B))$ *in the category* **LAT*** *and for any lattice* $(L, \vee, \wedge, r)$ *in* **LAT*** *there exists an ultrametric space* $(\mathbf{A}(L), \Delta)$ *such that:*
- *the space* (X, d) *is isometric to the space* $\mathbf{A}(\mathbf{L}(X)), \Delta)$*;*
- *the lattice* $(L, \wedge, \vee, r)$ *is isomorphic to the lattice* $(\mathbf{L}(\mathbf{A}(L)), \cap, \sup, r(B))$*.*

The following Proposition contains half of Theorem 1.

Proposition 1. *For any ultrametric space* (X, d)*, the lattice* $(\mathbf{L}(X), \cap, \sup, r(B))$ *of its closed balls (equipped with a "radius of a ball" function* $r(B)$*) is a complete, atomic, tree-like, and real graduated lattice.*

The completeness of $\mathbf{L}(X)$ follows from the next Lemma.

Lemma 6. *An intersection of any family of balls is a ball.*

Proof. Denote by $\{B(\alpha) \mid \alpha \in I' \subseteq I\}$ a family of balls in (X, d) and let $B = \bigcap\{B(\alpha) \mid \alpha \in I'\}$. If $B = \emptyset$ or $B = \{x\}$ is a singleton, then the Lemma is trivial. Suppose there are $x \neq y \in B$. Then $r = \sup\{d(x, y) \mid x, y \in B\} > 0$. Since $B \subseteq B(\alpha)$ for any $\alpha \in I'$, $r = \operatorname{diam} B \leq \operatorname{diam} B(\alpha) = r(B(\alpha))$. Therefore, for any $x \in B$ and for any $\alpha \in I'$, the ball $B(x, r) \subseteq B(\alpha)$. Thus,

$$B(x, r) \subseteq \bigcap\{B(\alpha) \mid \alpha \in I'\} = B.$$

On the other hand, $B \subseteq B(x, r)$ by the definition of r. $\square$

Proof of Proposition 1. Thus, the property of a subset of (X, d) of being a ball is stable with respect to intersections. There is the least element $\emptyset$ in the lattice $\mathbf{L}(X)$. By Theorem 1.6 [1], $\mathbf{L}(X)$ is complete.

Clearly the singletons $\{x\}$ of a space X (= balls of zero radius) are the atoms of $\mathbf{L}(X)$. We denote them by the letters x, y, and z and identify them with the points of the initial space X. For any ball B in (X, d), $B = \sup\{x \mid x \in B\} = \sup\{x \mid x \leq B\}$. Thus, $\mathbf{L}(X)$ is an atomic lattice.

In view of (t') and Lemma 3, $\mathbf{L}(X)$ is tree-like.

Finally, $\mathbf{L}(X)$ is real graduated by a "radius of ball" function $r(B)$. (i) is obvious for non-isolated points and it follows from the actuality of radius for isolated points.

(ii) follows from the closedness of balls. For any $X' \subset X$, the set $\sup\{x \mid x \in X'\}$ is the smallest ball containing all points $x \in X'$. Due to the isosceles property of X, this is a ball $B(x, r)$ of radius $r = \operatorname{diam} X' = \sup\{d(x, y) \mid x, y \in X'\}$, and $d(x, y) = r(x \vee y)$. This proves (iii). $\square$

Suppose we are given a lattice $L \in \mathrm{Ob}(\mathbf{LAT^*})$. Denote its graduating function by r. Let $\mathbf{A}(L)$ be the set of atoms of L, and $<$ be the order relation induced by the lattice operations $\wedge$ and $\vee$. We define a real valued non-negative function $\Delta(x, y)$ on $\mathbf{A}(L) \otimes \mathbf{A}(L)$ as follows:

$$\Delta(x, y) = \inf\{r(l) \mid l > x, y\} = r(x \vee y). \qquad (***)$$

Proposition 2. *For any lattice L in $\mathbf{LAT^*}$, the space of its atoms, $(\mathbf{A}(L), \Delta)$, is an ultrametric space.*

Proof. The properties $\Delta(x, y) \geq 0$, $\Delta(x, x) = 0$, and $\Delta(x, y) = \Delta(y, x)$ are obvious. Suppose $\Delta(x, y) = 0$. Then, in view of Note 1, $x \vee y$ is an atom; thus, $x = y$.

To prove the strong triangle inequality denote by x, y, and z three different atoms of L in such a manner that $\Delta(x, y) \leq \Delta(x, z) \leq \Delta(y, z)$. If $\Delta(x, y) = \Delta(x, z) = \Delta(y, z)$, then a triangle $\{x, y, z\}$ is equilateral. Suppose $\Delta(x, y) = \Delta(x, z) \leq \Delta(y, z)$. Since $x < x \vee y$ and $x < x \vee z$, $\inf(x \vee y, x \vee z) \neq 0$. Thus, in view of the treelike-ness of L, $x \vee y$ and $x \vee z$ are comparable. The equality $r(x \vee y) = r(x \vee z)$ and Note 2 imply that $x \vee y = x \vee z$. Consequently, $z < x \vee y$ and $y \vee z \leq x \vee y$; therefore, $\Delta(y, z) \leq \Delta(x, y)$. Thus, $\Delta(x, y) = \Delta(x, z) = \Delta(y, z)$ so the triangle $\{x, y, z\}$ is equilateral again. Suppose finally that $\Delta(x, y) < \Delta(x, z) \leq \Delta(y, z)$. Following similar arguments we get $x < x \vee y$, $x < x \vee z$; therefore, $\inf(x \vee y, x \vee z) \neq 0$. The inequality $r(x \vee y) < r(x \vee z)$ and Note 2 imply that $x \vee y < x \vee z$. Hence, $y < x \vee z$, $y \vee z \leq x \vee z$, and thus $\Delta(y, z) \leq \Delta(x, z)$. Therefore, $\Delta(x, z) = \Delta(y, z)$, i.e., $\{x, y, z\}$ is an isosceles triangle with base being smaller than the sides. $\square$

Thus for any ultrametric space (X, d), Propositions 1 and 2 enable us to construct a lattice $\mathbf{L}(X)$ of its balls, a space $\mathbf{A}(\mathbf{L}(X))$ of its atoms, and to prove that (X, d) and $(\mathbf{A}(\mathbf{L}(X)), \Delta)$ are isometric. The proof is straightforward and is left to the reader. Conversely, for any lattice L from $\mathbf{LAT^*}$, we can construct a space $(\mathbf{A}(L), \Delta)$ of its atoms and a lattice $\mathbf{L}(\mathbf{A}(L))$ of its closed balls. However, to prove that these lattices are isomorphic we need two more delicate Lemmas. Their proofs are based on the continuity of r (property iii), which has not been used yet. Examples 1 and 2 below demonstrate its necessity.

Lemma 7. *For any element l of a lattice $(L, \wedge, \vee, r(l))$ in $\mathbf{LAT^*}$, the set $B(l) = \{x \in \mathbf{A}(L) \mid x \leq l\}$ is a ball of radius $r(l)$ in the space $\mathbf{A}(L)$.*

Proof. If l is an atom then the Lemma is trivial. Otherwise there exists an atom $a < l$, and $r(l) > 0$. Consider the ball $B(a, r) = \{x \in \mathbf{A}(L) \mid \Delta(a, x) \leq r\}$ of radius

$r = r(l)$ with center located at a. For any $x \in B(a,r)$ we have $\Delta(a,x) = r(a \vee x) \leq r$. Since $a < l$ Note 2 implies that $a \vee x \leq l$; thus, $x < l$. Therefore, $x \in B(l)$, and $B(a,r) \subseteq B(l)$. On the other hand, for $x \in B(l)$, $x < l$; thus, $a \vee x \leq l$, and hence, $\Delta(a,x) = r(a \vee x) \leq r$. This implies that $x \in B(a,r)$; hence, $B(l) \subseteq B(a,r)$. Thus, $B(l) = B(a,r)$. Finally, it follows from condition (iii), the definition of $B(l)$, and Lemma 2 that

$$\begin{aligned} r(l) &= \sup\{r(x \vee y) \mid x, y \leq l\} = \sup\{r(x \vee y) \mid x, y \in B(l)\} \\ &= \sup\{\Delta(x \vee y) \mid x, y \in B(l)\} = \operatorname{diam} B(l) = r(B(l)), \end{aligned}$$

i.e. the radius of the ball B equals $r(l)$. □

A converse Lemma is proved by a similar argument.

Lemma 8. *Let $(L, \wedge, \vee, r)$ be a lattice from the category **LAT*** and $\mathbf{A}(L)$ the space of its atoms. For any ball $B \subset \mathbf{A}(L)$ of radius $r(B)$ there exists an element $l(B) \in L$ such that $B = \{x \mid x \leq l(B)\}$, and $r(l(B)) = r(B)$.*

Proof. Let $B = B(a,r) = \{x \mid \Delta(a,x) \leq r\}$ be a ball of radius $r = r(B)$ in the space $\mathbf{A}(L)$. Since (L) is complete there exists an element $l(B) = \sup\{x \mid x \in B\}$. By property (iii) and Lemma 2,

$$\begin{aligned} r(l(B)) &= r(\sup\{x \mid x \in B\}) = \sup\{r(x \vee y) \mid x, y \in B\} \\ &= \sup\{\Delta(x,y) \mid x, y \in B\} = \operatorname{diam} B = r(B). \end{aligned}$$

For any atom $x \leq l(B)$, we have $a \vee x \leq l$, $r(a \vee x) \leq r(l) = r(B)$; therefore, $x \in B(a,r)$, i.e., $\{x \mid x \leq l(B)\} \subseteq B$. The reverse inclusion is obvious. Thus, $B = \{x \mid x \leq l(B)\}$, and $r(l(B)) = r(B)$. □

Proof of Theorem 1. Let $(L, \wedge, \vee, r(l)) \in \mathrm{Ob}(\mathbf{LAT^*})$. By Proposition 2, the space $(\mathbf{A}(L), \Delta(x,y))$ of its atoms is ultrametric. The lattice of balls of this space belongs to **LAT*** in view of Proposition 1. Lemmas 7 and 8 establish a bijection $id_L \colon L \leftrightarrow \mathbf{L}(\mathbf{A}(L))$, which preserves the equality $r(l) = r(B)$. Let $<$ be the order relation induced on L by the lattice operations. If elements l and m are not comparable, then in view of treelike-ness of L, $l \wedge m = 0$. Therefore, there is no atom x which is smaller than both l and m. Thus, the corresponding balls $B(l)$ and $B(m)$ are disjoint. If $l < m$, then for any x smaller than l, x is also smaller than m. Since L is complete and atomic, there is an atom $y < m$ such that y is not smaller than l. Thus, $B(l) \subset B(m)$. Hence, the ordered sets $(L, <)$ and $(\mathbf{L}(\mathbf{A}(L)), \subset)$ are order isomorphic; therefore, the lattices L and $\mathbf{L}(\mathbf{A}(L))$ are lattice isomorphic. The equality $r(l) = r(B)$ is already proved. □

Note 3. Two lattices L and L' are **LAT***-isomorphic if and only if they are lattice isomorphic and $r(l) = r(l')$ for any pair of corresponding elements l and l' (see Note 4 below).

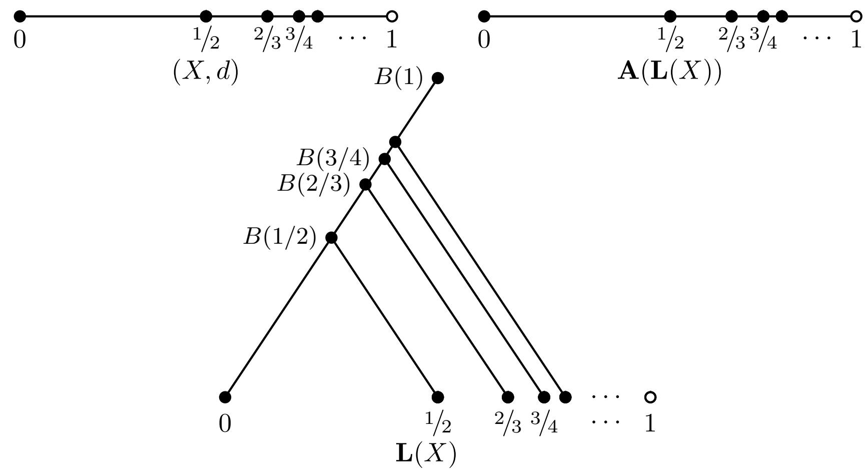

FIGURE 1

Example 1. Let $X = \{x_n = \frac{n}{n+1} \mid n \in \mathbb{N}\}$ be a sequence of rational numbers equipped with a metric $d(x, y) = \max(x, y)$ for any $x \neq y$. The lattice $\mathbf{L}(X)$ consists of the balls of radius 0 (= points of X), the balls $B(\frac{n}{n+1}) = \{x_k \mid k \leq n\}$ of radius $\frac{n}{n+1}$, the empty ball $\emptyset$, and the ball $B(1) = X$ of radius 1 (see Figure 1). Here a slant segment connecting two points a and b, where a is located higher than b, signifies that $a > b$ (throughout the paper we draw the graphs of lattices without the least element 0). Clearly $X = \mathbf{A}(\mathbf{L}(X))$. The space $Y = X \cup \{1\}$ has a similar lattice $\mathbf{L}(Y)$. It is clear that $\mathbf{L}(Y) = \mathbf{L}(X) \cup \{1\}$ and $Y = \mathbf{A}(\mathbf{L}(Y))$.

Example 2. Let L be a lattice similar to the previous ones with set of atoms $\mathbf{A}(L) = X \cup \{2\}$. The other elements of L are $l_{\frac{n}{n+1}} = \sup\{x_k \mid k \leq n\}$, $\emptyset$, and $l_2 = \sup\{X, 2\}$ = the greatest element of L. The value of the graduating function is $r(l_t) = t$. The graph of L, the set $\mathbf{A}(L)$, and the graph of the lattice $\mathbf{L}(\mathbf{A}(L))$ are shown in Figure 2. The lattice L does not satisfy (iii) for the set $X \subset \mathbf{A}(L)$, although it satisfies a weaker property: $r(l) = \sup\{r(x \vee y) \mid x, y < l\}$, where $x, y \in \mathbf{A}(L)$ for any non-atom $l \in L$. Thus, L doesn't belong to **LAT*** and the natural identification $id_L \colon L \to \mathbf{L}(\mathbf{A}(L))$ is not a lattice isomorphism.

3. Morphisms

To study metric spaces up to isometry we consider the category **METR** of all metric spaces and non-expanding maps as well as its full subcategory **ULTRA-METR** of ultrametric spaces and the same maps. Recall that a map $f \colon (X, d) \to$

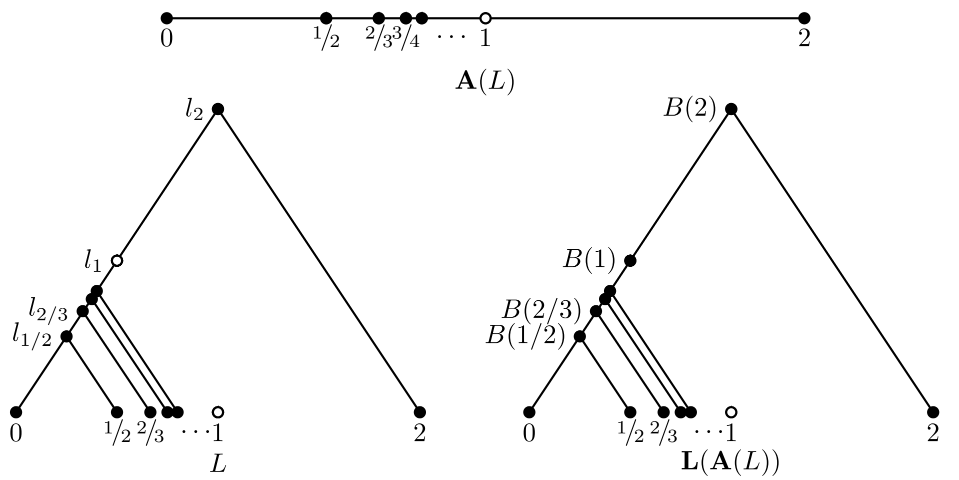

FIGURE 2

(X', d') is called *non-expanding* if it satisfies the inequality $d'(f(x), f(y)) \leq d(x, y)$ for any $x, y \in X$. The categories **METR**, $\mathbf{METR}_c$ (of metric spaces of diameter $\leq c$), and **METR*** (of metric spaces with a base point) are described in details in [10], [11], [14], [18], [23]. There it is proved that **ULTRAMETR** ($\mathbf{ULTRAMETR}_c$, **ULTRAMETR***) is closed in **METR**, ($\mathbf{METR}_c$, **METR***) with respect to base operations (sums and products, pull-backs and push-outs, limits of direct and inverse spectra, equalizers and co-equalizers and so on). Moreover, they are also closed under the actions of fundamental functors such as the completion functor, trimming functor, Hausdorff exponential functor, the functor of passing to a function space with the metric of uniform convergence, etc. (see Section 4 below and [10], [11], [18], [23]). To relate non-expanding maps of spaces to the maps of the corresponding lattices we introduce the following notion.

For any subset A in an ultrametric space X, a *rounding* (A) of a set A is the smallest ball containing A. By Lemma 6, such a ball does exist and it is equal to the intersection of all balls containing A. It also equals $B(x, \operatorname{diam} A)$ for any $x \in A$ (see Lemma 2).

The rounding operation induces a map $(\ )\colon P(X) \to \mathbf{L}(X)$ of the set of all subsets of X onto $\mathbf{L}(X)$ satisfying the following conditions:

C0. $\emptyset = (\emptyset)$;
C1. $A \leq (A)$; (extensive)
C2. $((A)) = (A)$; (idempotent)
C3. $A \leq C \Rightarrow (A) \leq (C)$. (isotonic)

Definition. Given a map $f\colon (X,d) \to (X',d')$ of ultrametric spaces, an induced map of lattices $f^*\colon \mathbf{L}(X) \to \mathbf{L}(X')$ is defined by the formula $f^*(B) = (f(B))$, where B is a ball in X and $(f(B))$ is the rounding of the set $f(B)$ in X'.

Definition (Morphisms of **LAT***)**.** Morphisms of **LAT*** are isotonic, semi-continuous, non-extensive maps, i.e. the maps $\varphi\colon (L,\wedge,\vee,r) \to (L,\wedge,\vee,r')$ satisfying the following conditions:

(1) $l < m$ implies $\varphi(l) \le \varphi(m)$;	(isotonic)
(2) $\varphi(l) = \sup\{\varphi(x) \mid x \le l\}$ for any $l \in L$;	(semi-continuous)
(3) $r'(\varphi(l)) \le r(l)$.	(non-extensive)

Note 4. Property (3) and Note 1 imply that $\varphi(x)$ is an atom in L' for any atom $x \in L$. If two lattices are isomorphic in **LAT*** then, in view of (3), $r(l) \equiv r'(')$.

Proposition 3. *For any non-expanding map $f\colon (X,d) \to (X',d')$ of ultrametric spaces, the induced map of lattices $f^*\colon \mathbf{L}(X) \to \mathbf{L}(X')$ is isotonic, semi-continuous, and non-extensive. It also satisfies that $(id_X)^* = id_{L(X)}$, $(f \cdot g)^* = f^* \cdot g^*$.*

Proof. $B < B'$ means that $B \subset B'$; hence, $f(B) \subseteq f(B')$. This implies that $(f(B)) \subseteq (f(B'))$, in view of C3, and proves (1). For any point $x \le B$, $x \in B$; hence, $f^*(x) = f(x) \in f(B) \subseteq (f(B)) = f^*(B)$. Thus, $f^*(x) \le f^*(B)$. Therefore, $\sup\{f^*(x) \mid x \le B\} \le f^*(B)$. On the other hand, the ball $\sup\{f^*(x) \mid x \le B\}$ contains the set $f(B)$; therefore, it contains the rounding $(f(B)) = f^*(B)$. Thus, $f^*(B) \le \sup\{f^*(x) \mid x \le B\}$. This proves (2). Finally,

$$\begin{aligned} r(B) &= \sup\{d(x,y) \mid x,y \in B\} \ge \sup\{d'(f(x),f(y)) \mid x,y \in B\} \\ &= \sup\{d'(f(x),f(y)) \mid f(x),f(y) \in f(B)\} = \sup\{d'(x',y') \mid x',y' \in f(B)\} \\ &= \operatorname{diam} f(B) = \operatorname{diam}(f(B)) = \operatorname{diam} f^*(B) = r'(f^*(B)). \end{aligned}$$

This proves (3). The equalities $(id_X)^* = id_{L(X)}$ and $(f\cdot g)^* = f^*\cdot g^*$ are obvious. □

Proposition 4. *For any isotonic, semi-continuous, non-extensive map $\varphi\colon (L,r) \to (L',r')$ of lattices from* **LAT****, the restriction $\varphi^* = \varphi|_{A(L)}$ is a non-expanding map $\varphi^*\colon (\mathbf{A}(L),\Delta) \to (\mathbf{A}(L'),\Delta')$ of ultrametric spaces such that $(id_L)^* = id_{A(L)}$, and $(\varphi \cdot \psi)^* = \varphi^* \cdot \psi^*$.*

Proof. A map $\varphi\colon L \to L'$ induces a map $\varphi^*\colon \mathbf{A}(L) \to \mathbf{A}(L')$ of the sets of atoms by the formula $\varphi^*(x) = \varphi(x)$. The ultrametrics Δ and Δ' are defined on $\mathbf{A}(L)$ and $\mathbf{A}(L')$ by $(***)$. For any $x,y \in \mathbf{A}(L)$ we have

$$\begin{aligned} \Delta(x,y) &= \inf\{r(l) \mid l > x,y\} \ge \inf\{r'(\varphi(l)) \mid l > x,y\} \\ &\ge \inf\{r'(l') \mid l' = \varphi(l) > \varphi(x),\varphi(y)\} \ge \inf\{r'(l') \mid l' > \varphi(x),\varphi(y)\} \\ &= \inf\{r'(l') \mid l' > \varphi^*(x),\varphi^*(y)\} = \Delta'(\varphi^*(x),\varphi^*(y)). \end{aligned}$$

The properties $(id_L)^* = id_{A(L)}$, and $(\varphi \cdot \psi)^* = \varphi^* \cdot \psi^*$ follow from the definition of φ^*. $\square$

Theorem 2. *For any non-expanding map of ultrametric spaces $f\colon (X,d) \to (X',d')$ there is a map of lattices $f^*\colon (\mathbf{L}(X), r) \to (\mathbf{L}(X'), r') \in \mathrm{Mor}(\mathbf{LAT^*})$ and for any map of lattices $\varphi\colon (L,\Delta) \to (L',\Delta') \in \mathrm{Mor}(\mathbf{LAT^*})$ there is a non-expanding map of spaces $\varphi^*\colon (\mathbf{A}(L),\Delta) \to (\mathbf{A}(L'),\Delta')$ such that the following diagrams commute:*

$$\begin{array}{ccc} (X,d) & \xrightarrow{id_X} & (\mathbf{A}(\mathbf{L}(X)),\Delta) \\ \downarrow f & & \downarrow f^{**} \\ (X',d') & \xrightarrow{id_{X'}} & (\mathbf{A}(\mathbf{L}(X')),\Delta') \end{array}$$

Diagram 1

$$\begin{array}{ccc} (L,r) & \xrightarrow{id_X} & (\mathbf{L}(\mathbf{A}(L)), r(B)) \\ \downarrow \varphi & & \downarrow \varphi^{**} \\ (L',r') & \xrightarrow{id_{X'}} & (\mathbf{L}(\mathbf{A}(L')), r'(B)) \end{array}$$

Diagram 2

Proof. We combine Propositions 3 and 4. Consider diagram 1. The map $f\colon (X,d) \to (X',d')$ induces the map of lattices $f^*\colon \mathbf{L}(X) \to \mathbf{L}(X')$ such that $f^*(x) = f(x)$ for any $x \in X$. Further, the map of lattices $f^*\colon \mathbf{L}(X) \to \mathbf{L}(X')$ generates the map $f^{**}\colon (\mathbf{A}(\mathbf{L}(X)),\Delta) \to (\mathbf{A}(\mathbf{L}(X')),\Delta')$ of the sets of their atoms and $f^{**}(x) = f^*(x)$ for any atom x of $\mathbf{L}(X)$. Thus, $f^{**}(x) = f(x)$ or, more precisely, $id_{X'} \cdot f = f^{**} \cdot id_X$, where $id_{X'}$ and id_X are the natural identities stated in Theorem 1.

Consider diagram 2. For any $l \in L$, Lemma 7 assigns a ball $B(l) = \{x \mid x < l\} \subset \mathbf{A}(L)$ of radius $r(l)$, i.e., an element of the lattice $\mathbf{L}(\mathbf{A}(L))$. The map φ^{**} takes this ball to the rounding $(\varphi^*(B(l)))$ of the set $\varphi^*(B(l))$ in the space $\mathbf{A}(L')$. The latter set is

$$\varphi^*(B(l)) = \{\varphi^*(x) \mid x \in B(l)\} = \{\varphi(x) \mid x < l\}.$$

Thus, we have

$$\begin{aligned} \varphi^{**}(B(l)) &= (\varphi^*(B(l)))_{A(L')} = (\{\varphi^*(x) \mid x \in B(l)\})_{A(L')} \\ &= \sup\{\varphi^{**}(x) \mid x < l\}_{L(A(L'))} = \sup\{\varphi(x) \mid x < l\}_{L'} = \varphi(l) = l', \end{aligned}$$

since the lattices L' and $\mathbf{L}(\mathbf{A}(L'))$ are complete and isomorphic, and $\varphi^{**}(x) = \varphi^*(x) = \varphi(x)$ for any atom $x \in L$. This means that $\varphi^{**}(x) \cdot id_L = id_{L'} \cdot \varphi$, where id_L and $id_{\prime}$ are the natural identities stated in Theorem 1. $\square$

Combining Theorems 1 and 2 and notion 14.1 [7] we get the Main Theorem.

Note 5. Since the rounding operation has Properties C0–C3, it is a *closure operation* in a sense of Moore-Birkhoff ([1], chapter 5). Property C3 is obviously equivalent to the inequality

C3$'$ $(A) \cup (C) \subseteq (A \cup C)$.

If the closure operator satisfies, instead of C3$'$, a stronger condition

C4. $(A) \cup (C) = (A \cup C)$,

it is called a *closure operation in sense of Kuratowski* [2]. Then it introduces a topology on the set X and is usually denoted by $[A]$. The Kuratowski Axiom $[A] \cup [C] = [A \cup C]$ means that the Kuratowski closure operator is a semi-lattice sup-homomorphism of the lattice $P(X)$ onto the lattice $Cl(X)$ of all closed subsets of a topological space X. Clearly, balls of an ultrametric space X are closed elements of the lattice $P(X)$ with respect to the rounding operation. This provides us with another proof of the completeness of $\mathbf{L}(X)$ (see [1], corollary to Theorem 5.4) and implies the following proposition.

Proposition 5. *The rounding operator is a semi-lattice homomorphism of the lattice $P(X)$ onto the lattice $\mathbf{L}(X)$, i.e., $(A \cup C) = (A) \vee (C)$.*

Example 3. Let $X = \{a, b, c\}$ be vertices of an isosceles triangle with sides $d(a, b) = 2$, $d(a, c) = d(b, c) = 3$. Then $\emptyset = (\emptyset) = (\{a, c\} \cap \{b\}) \neq (\{a, c\}) \cap (\{b\}) = X \cap \{b\} = \{b\}$. This shows that the rounding operation does not preserve intersections. Thus it is not a lattice homomorphism.

Example 4. Let $X' = \{a', b', c'\}$ be a similar triangle with sides $d(a', b) = d(a', c') = 2$, $d(b', c') = 1$ and $f\colon X \to X'$ be the map $f(k) = k'$. Then neither $X' = f^*(X) = f^*(b \vee c) \neq f^*(b) \vee f^*(c) = b' \vee c' = \{b', c'\}$, nor $\emptyset = f^*(\emptyset) = f^*((a \vee b) \cap c) \neq f^*(a \vee b) \cap f^*(c) = (f(\{a,\ b\})) \cap f(c) = (a' \vee b') \cap \{c'\} = X' \cap \{c'\} = \{c'\}$. Thus, f^* is neither a lattice homomorphism, nor a semi-lattice one.

Finally, properties C1–C3 mean that the rounding operation can be interpreted as a modal operator $\diamond$, 'possible that', in Modal Propositional Calculus. This will be considered elsewhere.

4. Properties of the isomorphism functors

Since the functors $\mathbf{A}$ and $\mathbf{L}$ establish an isomorphism of the categories, they enable us to translate metric properties of a space into algebraic properties of the corresponding lattice, and vice versa. Thus three directions for further research arise now.

Problem 1. Characterize metric, uniform, and topological properties of a given space $(X,\ d)$ in terms of algebraic properties of a lattice $\mathbf{L}(X)$. In particular, we have proved:

- An ultrametric space (X, d) is spherically complete if and only if any maximal chain C in $\mathbf{L}(X) \backslash \{\emptyset\}$ contains an atom.
- An ultrametric space (X, d) is complete iff any maximal chain C in $\mathbf{L}(X) \backslash \{\emptyset\}$ such that for any $\varepsilon > 0$, there is a ball $B \in C$ with $r(B) < \varepsilon$, contains an atom.

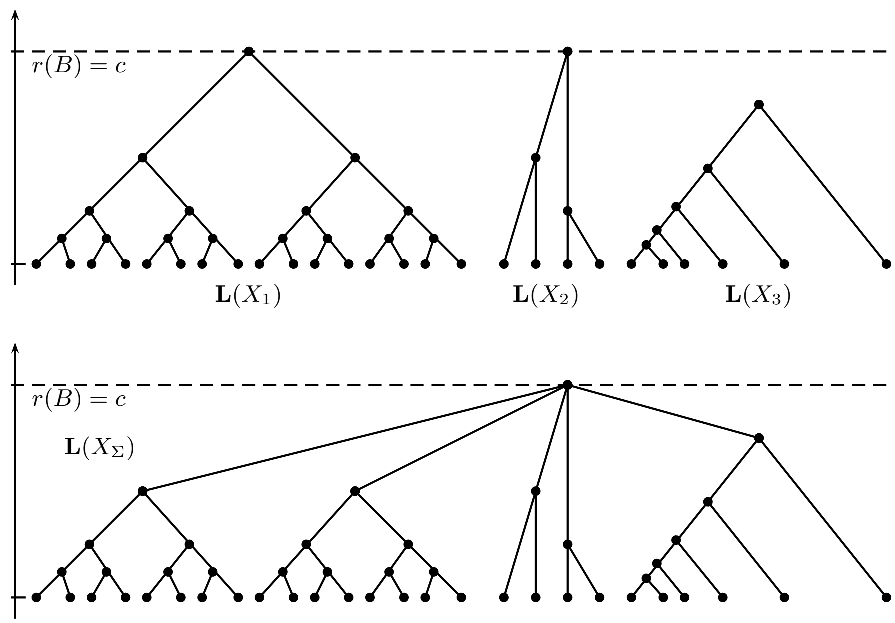

FIGURE 3

• An ultrametric space (X, d) is totally bounded iff, for any $\varepsilon > 0$, the horizontal $r(B) = \varepsilon$ intersects at most finitely many edges of the graph of $\mathbf{L}(X)$.

Combining the latter two we get an algebraic criterion for compactness of a space. To illustrate this, we present, above, the graph of $\mathbf{L}(\mathbb{Z}_p)$ for the ring of p-adic Hensel integers for $p = 2$ (see the upper left third in Figure 3, where each level of vertices is located twice as low as the previous one and $c = 1$).

Problem 2. Describe directly transformations of lattices $\mathbf{L}(X_\alpha)$ under the categorical operations on X_α. Exempli gratia,

• Consider the category $\mathbf{METR}_c$. A sum $m\Sigma(X_\alpha, d_\alpha)$ of spaces (X_α, d_α) in $\mathbf{METR}_c$ (a *metric sum*) is a space (X_Σ, d_Σ), where X_Σ is a sum of X_α in the category **SET** and $d_\Sigma(x_\alpha, y_\alpha) = d_\alpha(x_\alpha, y_\alpha)$, $d_\Sigma(x_\alpha, y_\beta) = c$ for $\alpha \neq \beta$, ([10], [23], compare with the definition of a topological sum [2]). If the (X_α, d_α) are ultrametric, then so is their metric sum (see [10], [23]). Given lattices $\mathbf{L}(X_\alpha)$, to get $\mathbf{L}(X_\Sigma)$ we take their disjoint sum, add the point 1_Σ (= the greatest element of $\mathbf{L}(X_\Sigma)$), glue it with all 1_α such that $r_\alpha(1_\alpha) = \operatorname{diam} X_\alpha = c$, and connect it with all 1_α such that $r_\alpha(1_\alpha) = \operatorname{diam} X_\alpha < c$ (see Figure 3).

Problem 3. Describe behavior of the lattice $\mathbf{L}(X)$ under the action of various functors.

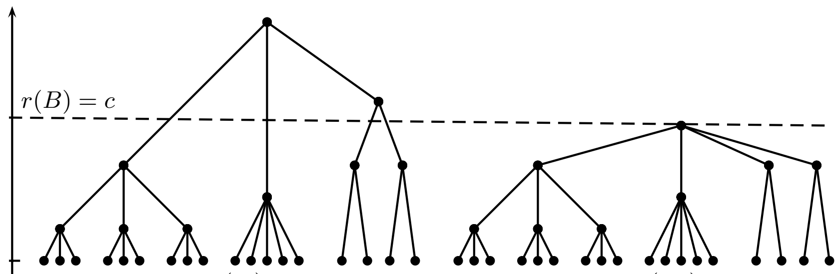

FIGURE 4

• Let $|_c\colon$ $\mathbf{METR} \to \mathbf{METR}_c$ be a trimming functor assigning a space (X_c, d_c) to any metric space (X, d). Here $X_c = X$ and $d_c(x, y) = d(x, y)$ for $d(x, y) \leq c$, $d_c(x, y) = c$ otherwise. It is known that $|_c$ is a reflective functor from **METR** to $\mathbf{METR}_c$ and **ULTRAMETR** is closed in **METR** under its action (see [10]). Given a graph of $\mathbf{L}(X)$, to get the graph of $\mathbf{L}(X_c)$ we cut off the upper part of $\mathbf{L}(X)$ along the horizontal $r(B) = c$ and replace it with the vertex 1_c, i.e., the greatest element of $\mathbf{L}(X_c)$ (see Figure 4).

• Let $(\operatorname{Hexp} X, d_H)$ be the set of all closed subsets of $(X, d) \in \operatorname{Ob}(\mathbf{METR}_c)$ with the Hausdorff metric $d_H(A, C) = \inf\{\varepsilon \mid A \subseteq O_\varepsilon(C), C \subseteq O_\varepsilon(A)\}$, see [2]. It is known that $\operatorname{Hexp} X$ is a functor in $\mathbf{METR}_c$ and that $(\operatorname{Hexp} X, d_H)$ is ultrametric iff so is (X, d), [10]. The metric Δ can be extended from $X \otimes X$ onto $\mathbf{L}(X) \otimes \mathbf{L}(X)$ by a formula similar to $(***)$:

$$\Delta(B, B') = r(B \vee B') \text{ for } B \neq B' \text{ and } \Delta(B, B) = 0.$$

It is easily verified that Δ is an ultrametric on $\mathbf{L}(X)$ and that any non-extensive map of lattices $\varphi\colon (L, r) \to (L', r')$ turns out to be a non-expanding map of spaces $\varphi\colon (L, \Delta) \to (L', \Delta')$. Balls $B \in \mathbf{L}(X)$ are closed subsets of X; hence, the Hausdorff distance $d_H(B, B')$ is well-defined. Moreover, for any two balls B and B' we have, $d_H(B, B') = \Delta(B, B')$. Therefore, there exists an isometric imbedding of $\mathbf{L}(X)$ into $(\operatorname{Hexp} X, d_H)$, which extends the Hausdorff imbedding $i_H\colon (X, d) \to (\operatorname{Hexp} X, d_H)$ defined by $i_H(x) = \{x\}$.

Finally we want to remind the reader that **ULTRAMETR** is a reflective subcategory in **METR**, [9], [3]. For any metric space (X, d), there are an ultrametric space (uX, d_u) (called an *ultrametrization* of X) and a non-expanding surjection $u\colon (X, d) \to (uX, d_u)$ such that for any non-expanding map $f\colon (X, d) \to (Y, \delta)$, where (Y, δ) is ultrametric, there is a unique non-expanding map $uf\colon (uX, d_u) \to (Y, \delta)$ such that $f = uf \cdot u$. This correspondence is a reflective functor $\mathbf{u}\colon \mathbf{METR} \to$ **ULTRAMETR**. Two metric spaces are called *u-equivalent* if their ultrametrizations are isometric. It is easy to prove that any n-point metric space is u-equivalent to an n-point subset of the real line [23], [25]. Using this Theorem, E. Schepin wrote an effective scanning algorithm and realized it on IBM-compatible computer [25]. Combining the functors **L** and **u** we get a functor $\mathbf{L} \cdot \mathbf{u}$: **METR** $\to$ **LAT*** that provides us with a lattice classification of general metric spaces. The latter is none other than the classification up to u-equivalence. These algebra-topology relations will be the subject of another paper.

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