A Graph Theoretic Perspective on CPM(Rel)

Daniel Marsden

Mixed states are of interest in quantum mechanics for modelling partial information. More recently categorical approaches to linguistics have also exploited the idea of mixed states to describe ambiguity and hyponym / hypernym relationships. In both these application areas the category **Rel** of sets and binary relations is often used as an alternative model. Selinger's CPM construction provides the setting for mixed states in Hilbert space based categorical quantum mechanics. By analogy, applying the CPM construction to **Rel** is seen as introducing mixing into a relational setting.

We investigate the category **CPM**(**Rel**) of completely positive maps in **Rel**. We show that the states of an object in **CPM**(**Rel**) are in bijective correspondence with certain graphs. Via map-state duality this then allows us provide a graph theoretic characterization of the morphisms in **CPM**(**Rel**). By identifying an appropriate composition operation on graphs, we then show that **CPM**(**Rel**) is isomorphic to a category of sets and graphs between them. This isomorphism then leads to a graph based description of the complete join semilattice enriched †-compact structure of **CPM**(**Rel**). These results allow us to reason about **CPM**(**Rel**) entirely in terms of graphs.

We exploit these techniques in several examples. We give a closed form expression for the number of states of a finite set in **CPM**(**Rel**). The pure states are characterized in graph theoretic terms. We also demonstrate the possibly surprising phenomenon of a pure state that can be given as a mixture of two mixed states.

1 Introduction

We study aspects of Selinger's CPM construction [9], and particularly its application to the category **Rel** of sets and binary relations between them. The CPM construction was originally introduced to extend the categorical approach to quantum mechanics [2] to support mixed states. The construction takes a †-compact closed category $\mathscr{C}$ and produces a new †-compact closed category **CPM**($\mathscr{C}$). In the case of quantum mechanics this construction takes us from the setting of [2], modelling pure state quantum mechanics, to a setting suitable for mixed state quantum mechanics.

The distributional compositional semantics approach to linguistics [4] uses similar mathematical methods to categorical quantum mechanics. Density operators have been proposed as candidates for modelling linguistic ambiguity [7, 8] and hyponym / hypernym relationships [3]. In this way the CPM construction becomes important in categorical models of linguistics.

In both linguistics and quantum mechanics it is common to consider **Rel** as an alternative model. By analogy with the situation in quantum mechanics, the category **CPM**(**Rel**) is then seen as introducing mixing in the relational setting. For example, a toy model of linguistic ambiguity in **CPM**(**Rel**) is discussed in [8], with the set:

$$\{\text{true}, \text{false}\}$$

considered as representing truth values. In **Rel** this set has 4 states, corresponding to its subsets. In **CPM**(**Rel**) there are 5 states, 4 corresponding to the pure states, and a new mixed state. This raises several questions. Why 5 states? How many states would we expect for an arbitrary finite set in **CPM**(**Rel**)? Can we easily identify the pure states? Direct computation gives the following state counts for small sets in **Rel** and **CPM**(**Rel**):

Elements	Rel States	CPM(Rel) States
0	1	1
1	2	2
2	4	5
3	8	18
4	16	113
5	32	1450

The numbers of states in **Rel** is simply the size of the corresponding power set. The pattern for the **CPM**(**Rel**) states is more complex, understanding it properly leads to a better understanding of the structure of **CPM**(**Rel**). It turns out that the states in **CPM**(**Rel**) correspond to certain graphs on the elements of the underlying sets. In this paper we:

- Give a graph based classification of the states in **CPM**(**Rel**).

- Via map-state duality give a classification of the morphisms in **CPM**(**Rel**).

- By defining an appropriate graph composition operation, we exhibit a category of graphs isomorphic to **CPM**(**Rel**). In this way we give a combinatorial description of **CPM**(**Rel**).

- Via this isomorphism, we transfer the complete join semilattice enriched †-compact monoidal structure of **Rel** to our category of graphs, and give explicit graph based descriptions of this structure.

Along the way, we also characterize the pure states in **CPM**(**Rel**) in terms of graphs. We then use this characterization to exhibit a pure state in **CPM**(**Rel**) that can be constructed as a mixture of two mixed states. We also provide a closed form for the number of states of a finite set in **CPM**(**Rel**).

We remark that although we must formalize our observations mathematically in later sections, the spirit of our intention is that a simple observation improves our understanding of **CPM**(**Rel**). We provide explicit graphical examples throughout, and it is in these simple finite examples that we anticipate the graphical reasoning to be of practical use. We also consider our approach to be complementary to the usual string diagrammatic reasoning, rather than a competitor to it. An application of the characterization of **CPM**(**Rel**) in terms of graphs, in a quantum mechanical setting, can be found in [5].

2   Preliminaries

We assume some familiarity with †-compact closed categories and their graphical calculus [1, 11, 6]. This section provides some mathematical background on the key CPM construction, mainly from [9].

Definition 2.1 (Positive Morphism)**.** An endomorphism $f : A \to A$ in a †-compact monoidal category is **positive** if there exists an object B and a morphism $g : A \to B$ with:

$$f = g^{\dagger} \circ g$$

Definition 2.2 (Completely Positive Morphisms)**.** For a †-compact monoidal category $\mathscr{C}$, the category of **completely positive morphisms** , **CPM**($\mathscr{C}$), has:

- **Objects**: The same objects as $\mathscr{C}$

- **Morphisms**: Let X^* denote the dual of an object X. A morphism $f : A \to B$ is a $\mathscr{C}$ morphism $f : A \otimes A^* \to B \otimes B^*$ such that the following composite is positive:

$$A^* \quad B$$

$$A^* \quad B$$
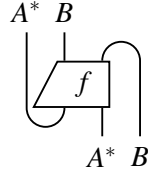
Composition and identities are inherited from $\mathscr{C}$.

The underlying category embeds into the result of the CPM construction [9]:

Lemma 2.3. *If $\mathscr{C}$ is a compact closed category there is a surjective and identity on objects functor:*

$$\mathscr{C} \to \mathbf{CPM}(\mathscr{C})$$

Definition 2.4. We say that a compact closed category is **self-dual** if each object is equal to its dual.

Remark 2.5. Definition 2.4 is a naive definition of self-duality, but sufficient for the purposes of this paper. More careful analysis of self-duality, including a broader definition and suitable coherence conditions is given in [10].

The following theorem is a small variation on [9, theorem 4.2]. By slightly changing the definition of the tensor product used in Selinger's work, we show that *self-dual* compact structure on $\mathscr{C}$ lifts to *self-dual* compact structure on $\mathbf{CPM}(\mathscr{C})$. We provide this variation as the symmetrical form of the tensor product is slightly more convenient to work with for self-dual compact closed categories such as **Rel**.

Theorem 2.6. *If $\mathscr{C}$ is a †-compact closed category with self-dual †-compact structure then $\mathbf{CPM}(\mathscr{C})$ is also a †-compact closed category with self-dual †-compact structure.*

Proof. We must define the tensor product to account for self-duality in $\mathscr{C}$. The action on objects is inherited from the tensor in $\mathscr{C}$. For arbitrary $\mathbf{CPM}(\mathscr{C})$-morphisms $f : A \to B$ and $g : B \to D$, the action of the tensor product bifunctor on morphisms is given by the following composite in $\mathscr{C}$:
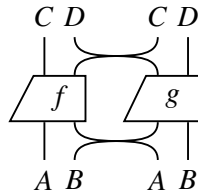
$\mathbf{CPM}(\mathscr{C})$ morphisms are closed under this operation, and bifunctoriality is easy to see. The monoidal unit and associators and unitors, along with the †-compact structure are inherited from $\mathscr{C}$. $\qquad\square$

As **Rel** has self-dual †-compact structure, we shall assume the monoidal structure given in theorem 2.6 in later sections. We will require the following observation from [9]:

Lemma 2.7. *A relation $R : X \to X$ is a positive morphism in* **Rel** *if and only if it is symmetric and:*

$$R(x, x') \Rightarrow R(x, x)$$

This can be generalized to **CPM**(**Rel**) morphisms via map-state duality:

Lemma 2.8. *The morphisms in* **CPM**(**Rel**) *are exactly the relations $R : A \otimes A \to B \otimes B$ satisfying the following axioms:*

$$R(a_1, a_2, b_1, b_2) \quad \Rightarrow \quad R(a_2, a_1, b_2, b_1)$$
$$R(a_1, a_2, b_1, b_2) \quad \Rightarrow \quad R(a_1, a_1, b_1, b_1)$$

Proof. This follows from map-state duality and lemma 2.7. For an arbitrary relation $R : A \otimes A \to B \otimes B$, we define relation:

$$\overline{R} : A \otimes B \to A \otimes B$$
$$\overline{R}(a_1, b_1, a_2, b_2) \Leftrightarrow R(a_2, a_1, b_2, b_1)$$

$\overline{R}$ is symmetric iff:

$$R(a_1, a_2, b_1, b_2) \Rightarrow R(a_2, a_1, b_2, b_1)$$

$\overline{R}$ satisfies:

$$\overline{R}(a_1, b_1, a_2, b_2) \Rightarrow \overline{R}(a_1, b_1, a_1, b_1)$$

iff:

$$R(a_2, a_1, b_2, b_1) \Rightarrow R(a_1, a_1, b_1, b_1)$$

$\square$

3 CPM(Rel) States as Graphs

We now start to relate **CPM**(**Rel**) to suitable types of graphs.

Definition 3.1. An undirected graph γ is:

- **Labelled** if the vertices can be distinguished.
- **Simple** if every vertex has a self-loop and there are no duplicate edges.

A **complete graph** is a graph in which there is an edge between each pair of vertices. We will write:

- V_γ for the vertex set of γ.
- E_γ for the edge set of γ.

In the sequel, we will refer to simple undirected labelled graphs simply as graphs.

Remark 3.2. In the literature, simple graphs are normally required to have *no* self-loops. Such graphs clearly bijectively correspond to those of our definition. The presence of self-loops will render some later definitions and reasoning more uniform. When drawing diagrams of graphs we will leave self-loops implicit to avoid clutter.

The following observation is at the heart of everything that follows:

Proposition 3.3. *The states of a set X in* **CPM**(**Rel**) *bijectively correspond to the graphs on subsets of elements of X.*

Proof. Given a point of X in **CPM**(**Rel**), we construct a graph with vertices:

$$\{x \mid R(x,x)\}$$

We have an edge from x to x' iff:

$$R(x,x')$$

For a graph on a subset of elements of X, we define a positive morphism R in **Rel** with $R(x,x')$ iff there is an edge from x to x'. This is a positive morphism as our graphs are undirected and have self-loops on all vertices.

These two constructions are inverse to each other, and therefore witness the required bijection. $\square$

We are now able to provide a closed form for the number of states of a finite set in **CPM**(**Rel**).

Corollary 3.4. *An n-element finite set in* **CPM**(**Rel**) *has:*

$$\sum_{0 \leq i \leq n} \binom{n}{i} 2^{i(i-1)/2}$$

states.

Proof. There are $2^{i(i-1)/2}$ graphs with i vertices. There are $\binom{n}{i}$ ways of choosing $0 \leq i \leq n$ vertices from X. We then simply count all these possible graphs to get our total. $\square$

We now characterize pure states in a manner that is literally easy to see:

Proposition 3.5. *A state in* **CPM**(**Rel**) *is pure iff the corresponding graph is complete.*

*Proof.* We identify subsets $U \subseteq X$ with relations $U : \{*\} \to X$ in the obvious way.

For $U \subseteq X$, the positive morphism corresponding to this pure state is given by $U \circ U^\circ$. Now assume distinct $u,v \in U$. We then have $U^\circ(u,*) \wedge U(*,v)$ and so $U \circ U^\circ(u,v)$. The corresponding graph of U therefore has an edge between each pair of distinct elements of U.

In the opposite direction, a connected graph on vertex set V corresponds to the positive morphism given by the relation $V \circ V^\circ$. $\square$

We note the following observation that runs counter to some of our intuition about how mixing should behave:

Lemma 3.6. *There exist pure states in* **CPM**(**Rel**) *that can be given as a "convex combination" (union) of mixed states.*

Proof. We consider the following two states of the set $\{x,y,z\}$:
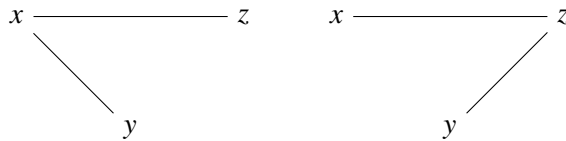
We note that these are not pure states by proposition 3.5. If we form the union of the underlying relations, we get the state:

$$x \rule{2cm}{0.4pt} z$$
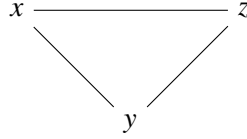
As this graph is complete, the corresponding state is a pure state by proposition 3.5. $\square$

4   A Category of Graphs

We now pursue a description of the category **CPM**(**Rel**) entirely in terms graphs and operations upon graphs. Via map-state duality, as a corollary of proposition 3.3, there is a bijection between **CPM**(**Rel**)(A,B) and the set of graphs with vertices a subset of $A \times B$. We will therefore be interested in graphs of this form when considering the morphisms in **CPM**(**Rel**).

Definition 4.1. For sets A, B:

- If γ is a graph with vertex labels a subset of $A \times B$, we will write $\gamma : A \to B$.

- For a set A we define the graph $1_A : A \to A$ as the complete graph with vertices the diagonal of $A \times A$.

Definition 4.2 (Graph Composition)**.** For graphs $\gamma : A \to B$ and $\gamma' : B \to C$, we define the graph $\gamma' \circ \gamma : A \to C$ as follows:

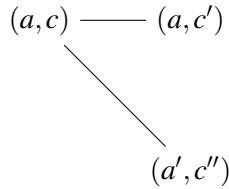$$V_{\gamma' \circ \gamma} := \{(a,c) \mid \exists b \in B.\ (a,b) \in V_\gamma \wedge (b,c) \in V_{\gamma'}\}$$
$$E_{\gamma' \circ \gamma} := \{\{(a,c),(a',c')\} \mid \exists b,b' \in B.\ \{(a,b),(a',b')\} \in E_\gamma \wedge \{(b,c),(b',c')\} \in E_{\gamma'}\}$$

Remark 4.3. Informally, we glue together vertices and edges where their B components agree. Notice that in order to calculate the vertices of the composite, we only need consider vertices in the components. Similarly, when calculating the edges, we need only consider edges in the components.

Example 4.4. The composite of the following two graphs:

$$(b,c) \rule{1cm}{0.4pt} (b,c')$$

$$(a,b) \rule{1cm}{0.4pt} (a',b')$$

$$(b'',c) \qquad (b',c'')$$

is given by the graph:

$$(a,c) \rule{1cm}{0.4pt} (a,c')$$

$$(a',c'')$$

Lemma 4.5. *For graphs* $\gamma : A \to B$, $\gamma' : B \to C$, $\gamma'' : C \to D$, *and* $1_A, 1_B$ *as in definition 4.1:*

- $\gamma'' \circ (\gamma' \circ \gamma) = (\gamma'' \circ \gamma') \circ \gamma$

- $1_B \circ \gamma = \gamma$

- $\gamma \circ 1_A = \gamma$

Definition 4.6. We define the category $\mathscr{G}$ of sets and graphs between them as having:

- **Objects**: Sets

- **Morphisms**: A morphism $A \to B$ is a graph $\gamma : A \to B$ as in definition 4.1

Composition is as described in definition 4.2. The identities and associativity of composition then follow from lemma 4.5

We will require the following technical lemma:

Lemma 4.7. *Let $R : A \to B$ and $S : B \to C$ be morphisms in* **CPM**(**Rel**). *Then for all $a \in A$ and $c \in C$:*

$$(S \circ R)(a,a,c,c) \quad \Leftrightarrow \quad \exists b \in B.\, R(a,a,b,b) \wedge S(b,b,c,c)$$

Proof. The right to left direction is immediate from the definition of relational composition. For the other direction, if:

$$(S \circ R)(a,a,c,c)$$

there then exist $b, b' \in B$ such that:

$$R(a,a,b,b') \quad \wedge \quad S(b,b',c,c)$$

By lemma 2.8 we then have:

$$R(a,a,b,b) \quad \wedge \quad S(b,b,c,c)$$

$\square$

Definition 4.8. For a **CPM**(**Rel**) morphism $R : A \to B$, we define the graph $G(R) : A \to B$:

$$V_{G(R)} := \{(a,b) \mid R(a,a,b,b)\}$$
$$E_{G(R)} := \{\{(a,b),(a',b')\} \mid R(a,a',b,b')\}$$

Proposition 4.9. *The mapping of definition 4.8 extends to an identity on objects functor:*

$$G : \textbf{CPM}(\textbf{Rel}) \to \mathscr{G}$$

Proof. We first check identities are preserved:

$$
\begin{aligned}
&V_{G(1_A)} && &&E_{G(1_A)} \\
=\ &\{ \text{ definition (4.8) } \} && =\ &&\{ \text{ definition (4.8) } \} \\
&\{(a_1,a_2) \mid R(a_1,a_1,a_2,a_2)\} && &&\{\{(a_1,a_2),(a_3,a_4)\} \mid 1_A(a_1,a_3,a_2,a_4)\} \\
=\ &\{ \text{ identity relations } \} && =\ &&\{ \text{ identity relations } \} \\
&\{(a_1,a_2)\} && &&\{\{(a_1,a_1),(a_3,a_3)\} \\
=\ &\{ \text{ definition } \} && =\ &&\{ \text{ definition } \} \\
&V_{1_{G(A)}} && &&E_{1_{G(A)}}
\end{aligned}
$$

We must also check composition is preserved. For vertices:

$$
\begin{aligned}
&V_{G(S \circ R)} \\
=\ & \{\ \text{definition 4.8}\ \} \\
&\{(a,c) \mid (S \circ R)(a,a,c,c)\} \\
=\ & \{\ \text{lemma 4.7}\ \} \\
&\{(a,c) \mid \exists b.\ R(a,a,b,b) \wedge S(b,b,c,c)\} \\
=\ & \{\ \text{definition 4.8}\ \} \\
&\{(a,c) \mid \exists b.\ (a,b) \in V_{G(R)} \wedge (b,c) \in V_{G(S)}\} \\
=\ & \{\ \text{definition 4.2}\ \} \\
&V_{G(S) \circ G(R)}
\end{aligned}
$$

For edges:

$$
\begin{aligned}
&E_{G(S \circ R)} \\
=\ & \{\ \text{definition 4.8}\ \} \\
&\{\{(a,c),(a',c')\} \mid (S \circ R)(a,a',c,c')\} \\
=\ & \{\ \text{relation composition}\ \} \\
&\{\{(a,c),(a',c')\} \mid \exists b,b' \in B.\ R(a,a',b,b') \wedge S(b,b',c,c')\} \\
=\ & \{\ \text{definition 4.8}\ \} \\
&\{\{(a,c),(a',c')\} \mid \exists b,b' \in B.\ \{(a,b),(a',b')\} \in E_{G(R)} \wedge \{(b,c),(b',c')\} \in E_{G(S)}\} \\
=\ & \{\ \text{definition 4.2}\ \} \\
&E_{G(S) \circ G(R)}
\end{aligned}
$$

<div align="right">□</div>

Definition 4.10. For a graph $\gamma : A \to B$ we define the **CPM(Rel)** morphism $C(\gamma) : A \to B$ as follows:

$$
C(\gamma)(a,a',b,b') \Leftrightarrow \{(a,b),(a',b')\} \in E_\gamma
$$

Proposition 4.11. *The mapping of definition 4.10 extends to an identity on objects functor:*

$$
C : \mathscr{G} \to \mathbf{CPM}(\mathbf{Rel})
$$

Proof. That the mapping is identity on objects is trivial. To confirm identities are preserved, we have:

$$
\begin{aligned}
&C(1_A)(a_1,a_2,a_3,a_4) \\
\Leftrightarrow\ & \{\ \text{definition (4.10)}\ \} \\
&\{(a_1,a_3),(a_2,a_4)\} \in E_{1_A} \\
\Leftrightarrow\ & \{\ \text{definition}\ \} \\
&a_1 = a_3 \wedge a_2 = a_4 \\
\Leftrightarrow\ & \{\ \text{identity relations}\ \} \\
&1_{C(A)}(a_1,a_2,a_3,a_4)
\end{aligned}
$$

For associativity of composition, we reason as follows:

$$C(\gamma' \circ \gamma)(a, a', c, c')$$
$\Leftrightarrow$ { definition 4.10 }
$$\{(a, c), (a', c')\} \in E_{\gamma' \circ \gamma}$$
$\Leftrightarrow$ { definition 4.2 }
$$\exists b, b' \in B. \{(a, b), (a', b')\} \in E_\gamma \wedge \{(b, c), (b', c')\} \in E_{\gamma'}$$
$\Leftrightarrow$ { definition 4.10 }
$$\exists b, b' \in B. C(\gamma)(a, a', b, b') \wedge C(\gamma')(b, b', c, c')$$
$\Leftrightarrow$ { relation composition }
$$(C(\gamma') \circ C(\gamma))(a, a', c, c')$$

$\square$

Theorem 4.12. *The functors $G : \mathbf{CPM}(\mathbf{Rel}) \to \mathscr{G}$ of definition 4.8 and $C : \mathscr{G} \to \mathbf{CPM}(\mathbf{Rel})$ of definition 4.10 are inverse to each other.*

Proof. In the first direction:

$$(C \circ G)(R)(a, a', b, b')$$
$\Leftrightarrow$ { definition 4.10 }
$$\{(a, b), (a', b')\} \in E_{G(R)}$$
$\Leftrightarrow$ { definition 4.8 }
$$R(a, a', b, b')$$

For the other direction, for vertices:

$$V_{(G \circ C)(\gamma)}$$
$=$ { definition 4.8 }
$$\{(a, b) \mid C(\gamma)(a, a, b, b)\}$$
$=$ { definition 4.10 }
$$\{(a, b) \mid \{(a, b)\} \in E_\gamma\}$$
$=$ { self-loops correspond to vertices }
$$\{(a, b) \mid (a, b) \in V_\gamma\}$$
$=$ { set theory }
$$V_\gamma$$

Finally, for edges:

$$E_{(G \circ G)(\gamma)}$$
$=$ { definition 4.8 }
$$\{\{(a, b), (a', b')\} \mid C(\gamma)(a, a', b, b')\}$$

$$= \quad \{ \text{ definition 4.10 } \}$$
$$\{\{(a,b),(a',b')\} \mid \{(a,b),(a',b')\} \in E_\gamma\}$$
$$= \quad \{ \text{ set theory } \}$$
$$E_\gamma$$

$\square$

5   A Structured Category of Graphs

As the categories $\mathscr{G}$ and **CPM**(**Rel**) are isomorphic, we can transfer structure from **CPM**(**Rel**) to $\mathscr{G}$. Firstly, the monoidal structure on **CPM**(**Rel**) induces a monoidal structure on $\mathscr{G}$. As the functors witnessing the isomorphism are identity on objects, the tensor of objects in $\mathscr{G}$ is also given by the cartesian product of sets. The tensor product of morphisms in $\mathscr{G}$ is given explicitly as follows:

Definition 5.1 (Graph Tensor). For graphs $\gamma : A \to C$, $\gamma' : B \to D$ we define the graph $\gamma \otimes \gamma' : A \otimes B \to C \otimes D$ as follows:

$$V_{\gamma \otimes \gamma'} := \{(a,b,c,d) \mid (a,c) \in V_\gamma \wedge (b,d) \in V_{\gamma'}\}$$
$$E_{\gamma \otimes \gamma'} := \{\{(a,b,c,d),(a',b',c',d')\} \mid \{(a,c),(a',c')\} \in E_\gamma \wedge \{(b,d),(b',d')\} \in E_{\gamma'}\}$$

Remark 5.2. Informally, the vertices of the composite graph are all combinations of vertices from the two component graphs. There is then an edge between a pair of vertices if there are edges between both pairs of component vertices. Again, when calculating the vertices, we need only consider the vertices of the components, and when calculating edges we need only consider component edges.

Example 5.3. The tensor of the following pair of graphs:

$$(b,d) - (b',d')$$
$$\searrow$$
$$(b'',d'')$$

$$(a,c) - (a',c')$$

is given by the graph:
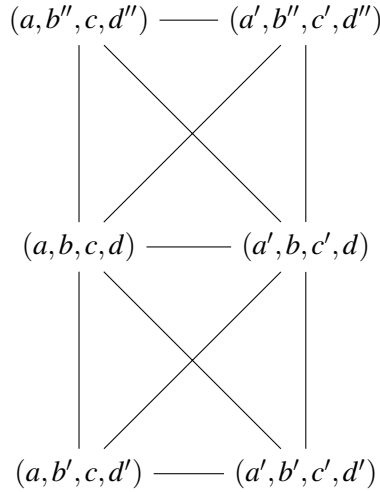
Definition 5.4. **cJSLat** is the category of complete join semilattices and homomorphisms.

The category **CPM**(**Rel**) is **cJSLat**-enriched, via the usual inclusion order on relations. Again, as **CPM**(**Rel**) is isomorphic to $\mathscr{G}$, we can transfer this enrichment to the homsets of $\mathscr{G}$. Concretely this ordering is given as follows:

Definition 5.5. We define a partial order on graphs:

$$\gamma \leq \gamma' \quad \Leftrightarrow \quad E_\gamma \subseteq E_{\gamma'}$$

These graphs are closed under unions in the following sense:

$$V_{\bigcup_i \gamma_i} := \bigcup_i V_{\gamma_i} \qquad\qquad E_{\bigcup_i \gamma_i} := \bigcup_i E_{\gamma_i}$$

We can also recover a dagger operation on $\mathscr{G}$, induced from the isomorphism with **CPM**(**Rel**). The dagger is given by "reversing the order of the vertex pairs", explicitly:

Definition 5.6 (Graph Dagger). We define identity on objects involution $(-)^\dagger : \mathscr{G}^{op} \to \mathscr{G}$. The action on morphisms for graph $\gamma : A \to B$ is given by:

$$V_{\gamma^\dagger} := \{(b,a) \mid (a,b) \in V_\gamma\}$$
$$E_{\gamma^\dagger} := \{\{(b,a),(b',a')\} \mid \{(a,b),(a',b')\} \in E_\gamma\}$$

Via our isomorphism, we describe a direct, graph based version of the embedding of lemma 2.3, generalizing proposition 3.5:

Proposition 5.7. *There is a faithful, surjective and identity on objects functor:*

$$\mathbf{Rel} \to \mathscr{G}$$

The action on morphisms is to send a relation $R : A \to B$ to the complete graph on $R \subseteq A \times B$.

Using proposition 5.7, we can easily transfer symmetric and compact structure to $\mathscr{G}$. Concretely, this is given as follows:

Definition 5.8. The symmetry $A \otimes B \to B \otimes A$ is given by the complete graph on:

$$\{(a,b,b,a) \mid a \in A, b \in B\}$$

The cup of the compact structure corresponds to the complete graph on:

$$\{(*,a,a) \mid a \in A\}$$

The cap is then given by applying the † graph operation. Clearly the vertex sets described above are isomorphic to simpler sets. The definitions above will interact correctly with our definitions of composition and tensor product.

We can now extend theorem 4.12 to account for the order-enriched †-compact structure:

Theorem 5.9. *Consider $\mathscr{G}$ as a* **cJSLat***-enriched †-compact closed category, with the required structure as in definitions 5.1, 5.5, 5.6 and 5.8. The categories* **CPM**(**Rel**) *and $\mathscr{G}$ are isomorphic as* **cJSLat***-enriched †-compact categories.*

6   Conclusion

From the observations in this paper, we can view **CPM**(**Rel**) as a †-compact monoidal category with a highly combinatorial flavour.

We already encounter some strange behaviour in **CPM**(**Rel**), such as the unusual mixing behaviour noted in lemma 3.6. One direction for further investigation is to explore the CPM construction for other more general categories of relations. Every regular category has a corresponding category of relations, giving a broad class of examples. Another generalized class of categories of relations is given by matrix algebras over commutative quantales. Study of these examples may clarify the intuitive interpretation of the CPM construction as introducing mixing, and furnish a broad class of models for string diagrammatic calculi.

A second area of interest for further investigation is iteration of the CPM construction. This is of practical interest in linguistics where multiple aspects of mixing occur, such as ambiguity and hyponym / hypernym relations. Here again, categories of relations might prove useful test cases for our intuitions as to how "multiple levels of mixing" should behave.

Acknowledgements

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