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Introduction to extensive and distributive categories

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Abstract

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In recent years, there has been considerable discussion as to the appropriate definition of distributive categories. Three definitions which have had some support are:

(1) A category with finite sums and products such that the canonical map $\delta : A \times B + A \times C \to A \times (B + C)$ is an isomorphism (Walters).

(2) A category with finite sums and products such that the canonical functor $+ : \mathbf{A}/A \times \mathbf{A}/B \to \mathbf{A}/(A + B)$ is an equivalence (Monro).

(3) A category with finite sums and finite limits such that the canonical functor $+$ of (2) is an equivalence (Lawvere and Schanuel).

There has been some confusion as to which of these was the natural notion to consider. This resulted from the fact that there are actually two elementary notions being combined in the above three definitions. The first, to which we give the name *distributivity*, is exactly that of (1). The second notion, which we shall call *extensivity*, is that of a category with finite sums for which the canonical functor $+$ of definitions (2) and (3) is an equivalence. Extensivity, although it implies the existence of certain pullbacks, is essentially a property of having well-behaved sums. It is the existence of these pullbacks which has caused the confusion. The connections between definition (1) and definitions (2) and (3) are that any extensive category with products is distributive in the first sense, and that any category satisfying (3) satisfies (1) locally.

The purpose of this paper is to present some basic facts about extensive and distributive categories, and to discuss the relationships between the two notions.

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1. Introduction

Many of the results in this paper concerning distributive categories are well known. What is not well known is that the natural context for many of these results is a category with finite sums. Typically in the past the content of the major theorems and proofs has been obscured by working in the presence of various limits. In particular, the notion of extensivity, defined at the beginning of the following section, implies the existence of pullbacks along the injection of a sum. It is probably the existence of these limits that has caused the confusion.

The Burnside rig of a distributive category is well known [6]. It has as elements isomorphism classes of objects of the category, and its addition and multiplication are given by sums and products in the category. Just as a distributive category can be thought of as a category with a rig-like structure, so should an extensive category be thought of as a category with an Abelian-group-like structure. As many results as possible will be proved using only this additive structure. In a later paper, the 2-category of extensive categories will be considered as analogous to the category **Ab** of Abelian groups, and, in particular, the tensor product defined.

Considerable work on distributive categories has been done by Cockett [1], Lawvere [3], Monro [5], Schanuel [6] and Walters [7], and this paper depends on the work of all of them. The isolation of extensivity, and the realization that it is an essentially additive notion was made over a period of time and is due to the authors, Lawvere and Schanuel. Lawvere has independently reported this discovery in [4].

2. Extensive categories

2.1. The notion of extensivity

Definition 2.1. A category $\mathbf{A}$ with finite sums is called *extensive* if for each pair X_1, X_2 of objects in $\mathbf{A}$, the canonical functor

$$+ : \mathbf{A}/X_1 \times \mathbf{A}/X_2 \to \mathbf{A}/(X_1 + X_2)$$

is an equivalence.

It should be noted here that all limits and colimits discussed in this paper are finite, and mention of limits, colimits, products and so on means finite such, even if this is not explicitly stated. We also note that the terms 'sum' and 'coproduct' are synonymous, and we shall feel free to use either, from time to time.

The following result is the key technical lemma concerning extensive categories. The proof is long and not particularly enlightening, but it provides an equivalent definition of extensivity which is both more intuitively accessible, and allows simpler proofs of later theorems.

Proposition 2.2. *A category with sums is extensive iff it has pullbacks along coprojections of coproducts and every commutative diagram*

$$\begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A \arrow[d, "f"] & A_2 \arrow[l, "a_2"'] \arrow[d, "f_2"] \\ X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"'] \end{tikzcd}$$

comprises a pair of pullback squares in **A** *just when the top row is a coproduct diagram in* **A**.

Proof. We can state this more simply by noting that provided the diagram commutes, the top row will be a coproduct diagram in **A** iff the whole thing is a coproduct diagram in $\mathbf{A}/X_1 + X_2$. Thus the proposition becomes: A category with sums is extensive iff it has pullbacks along injections and every diagram of the above form is a pair of pullback squares iff it is a coproduct diagram in $\mathbf{A}/X_1 + X_2$. Throughout the proof we shall abbreviate 'is a pair of pullback squares' to 'is a pullback', and 'is a coproduct in $\mathbf{A}/X_1 + X_2$' to 'is a coproduct'. We shall also say that a category 'has the extensivity condition for X_1 and X_2' if the canonical functor + corresponding to X_1,X_2 is an equivalence.

Assume the extensivity condition for X_1 and X_2. Suppose that the above diagram, which we call D, is a coproduct. Given a commutative diagram

$$\begin{tikzcd} B_1 \arrow[r, "\beta_1"] \arrow[d, "g_1"'] & A \arrow[d, "f"] & B_2 \arrow[l, "\beta_2"'] \arrow[d, "g_2"] \\ X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"'] \end{tikzcd}$$

we can form the coproduct $g_1 + g_2 : B_1 + B_2 \to X_1 + X_2$ in $\mathbf{A}/X_1 + X_2$, with coprojections b_1 and b_2. Then β_1 and β_2 induce a unique arrow $(\beta_1, \beta_2) : g_1 + g_2 \to f$. By assumption, however, f is actually $f_1 + f_2$. Then by full-fidelity of the functor + we get a unique pair of arrows $\alpha_i : g_i \to f_i$ satisfying $(\beta_1, \beta_2) \circ b_i = a_i \circ \alpha_i = \beta_i$. But this, along with the fact that the α_i are in the comma category over $X_1 + X_2$, makes them the unique arrows required to make diagram D a pullback. Thus D is a pullback if it is a coproduct. Now we must prove the converse.

Suppose that D is a pullback. We shall show that it is a coproduct in $\mathbf{A}/X_1 + X_2$. Essential surjectivity of + means that we can find a coproduct

$$\begin{tikzcd} A_1' \arrow[r, "a_1'"] \arrow[d, "f_1'"'] & A \arrow[d, "f"] & A_2' \arrow[l, "a_2'"'] \arrow[d, "f_2'"] \\ X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"'] \end{tikzcd}$$

But these are a pair of commuting squares and so, as D is a pair of pullback squares, they induce a unique pair of arrows $\theta_i : A'_i \to A_i$ which satisfy $f_i \circ \theta_i = f'_i$ (making them arrows $\theta_i : f'_i \to f_i$ in $\mathbf{A}/X_i$) and $a_i \circ \theta_i = a'_i$.

We are still trying to show that D is a coproduct, but as sums exist we can form the actual coproduct

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_1 + A_2 \arrow[d, "f_1+f_2"'] & A_2 \arrow[l, "\alpha_2"'] \arrow[d, "f_2"'] \\ X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"'] \end{tikzcd}$$

Then the arrows $a_i : x_i \circ f_i \circ f_i \to f$ induce an arrow $(a_1, a_2) : f_1 + f_2 \to f$ satisfying $(a_1, a_2) \circ \alpha_i = a_i$. But (a_1, a_2) is from $f_1 + f_2$ to $f'_1 + f'_2$ and so, by full fidelity of $+$ we can uniquely write it as $\phi_1 + \phi_2$ for arrows ϕ_i from f_i to f'_i satisfying $a'_i \circ \phi_i = (a_1, a_2) \circ \alpha_i = a_i$.

Now if we can show that ϕ_i and θ_i are mutually inverse we shall have shown that diagram D is a coproduct and thus have completed half the proof: given the extensivity condition for X_1 and X_2, diagram D is a pullback iff it is a coproduct. But

$$a_i \circ \theta_i \circ \phi_i = a'_i \circ \phi_i = a_i ,$$

$$f_i \circ \theta_i \circ \phi_i = f'_i \circ \phi_i = f_i ,$$

and then as D was assumed a pullback it follows that $\theta_i \circ \phi_i = 1$. On the other hand, by the definition of the sum of two maps, we have

$$\begin{aligned}(\phi_1 \circ \theta_1 + \phi_2 \circ \theta_2) \circ a'_i &= a'_i \circ \phi_i \circ \theta_i = (a_1, a_2) \circ \alpha_i \circ \theta_i \\ &= a_i \circ \theta_i = a'_i = (1_{A'_1} + 1_{A'_2}) \circ a'_i\end{aligned}$$

and so

$$\phi_1 \circ \theta_1 + \phi_2 \circ \theta_2 = 1_{A'_1} + 1_{A'_2}$$

giving $\theta_i \circ \phi_i = 1$ by fidelity of $+$.

In the second part of the proof, we assume that a diagram D is a pullback iff it is a coproduct, and that these pullbacks exist. We now must prove the extensivity condition for X_1 and X_2. The functor $+$ induces a function

$$+ : \mathrm{Hom}(A_1, B_1) \times \mathrm{Hom}(A_2, B_2) \to \mathrm{Hom}(A_1 + A_2, B_1 + B_2) .$$

The function $+$ can be inverted by pulling back a given arrow in the codomain over the injections. This guarantees full fidelity. To show essential surjectivity, we must, given an arrow in the codomain pull it back along the injections; the result we know will have top row a coproduct. This completes the proof. $\square$

Henceforth we shall use 'extensivity condition' interchangably to mean these two equivalent conditions.

2.2. Extensive categories

We note that being extensive is genuinely stronger than just having finite sums. The category of vector spaces over a field k has finite sums but is not extensive.

One of the major ideas of this paper is contained in the following slogan.

Slogan 2.3. An extensive category is one in which sums exist and are well-behaved.

This manifests itself in several ways. The first is the content of the following theorem.

Proposition 2.4. *The free category with sums on a category* **A** *is extensive.*

Proof. We use the original form of the extensivity condition. An arrow from $(A_i)_{i\in I}$ to $X_1 + X_2$ amounts to a function $f : I \to [2]$ and a collection of arrows $\alpha_i : A_i \to X_{f(i)}$. Then we just split up the family $(A_i)_{i\in I}$ as $((A_i)_{i\in f^{-1}(1)}, (A_i)_{i\in f^{-1}(2)})$. There is an obvious pair of arrows to (X_1, X_2) which gives us essential surjectivity. Full fidelity similarly follows from the fact that we are really only dealing with disjoint unions of sets. □

By well-behaved we mean behaving similarly to those in **Sets**. Of course **Sets** is extensive: it is this fact which is central to the standard combinatorial proof of the binomial theorem. We count the functions from a set N to a disjoint union $X + Y$, and then pull back each function, using extensivity to establish a bijection with pairs of functions, one into X and one into Y. Also **Sets** is distributive; if an extensive category has products, then these automatically distribute over the sums. We defer the proof of this fact, but shall prove several other results which support our newly adopted slogan. The first two exhibit further similarities between sums in extensive categories and sums in **Sets**: one treating binary sums, the second being the corresponding result for initial objects.

Definition 2.5. In a category with sums and pullbacks along injections, sums are said to be *disjoint* if the pullback of the injections of a binary sum is the initial object, and all injections are monic.

Proposition 2.6. *In an extensive category, sums are disjoint.*

Proof. As certainly

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & A_2 \arrow[d, "a_2"'] & A_2 \arrow[l, "1"'] \arrow[d, "1"] \\ A_1 \arrow[r, "a_1"] & A_1 + A_2 & A_2 \arrow[l, "a_2"'] \end{tikzcd}$$

is a coproduct, also it is a pullback. By the right-hand pullback, a_2 is monic. □

Definition 2.7. An initial object in a category is said to be *strict* if any arrow into it is invertible.

Proposition 2.8. *In an extensive category, initials are strict.*

Proof. Given an arrow $\alpha : A \to 0$ the diagram

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "\alpha"'] & A \arrow[d, "\alpha"'] & A \arrow[l, "1"'] \arrow[d, "\alpha"'] \\ 0 \arrow[r, "1"] & 0 & 0 \arrow[l, "1"'] \end{tikzcd}$$

is a pullback and so a coproduct. Any sum with injections equal can have at most one arrow to another object; thus $! \circ \alpha = 1_A$. □

Corollary 2.9. *If* $\mathbf{A}$ *is an extensive category, then the functor* $0 : 1 \to \mathbf{A}/0$ *is an equivalence of categories.* □

The next two results give us an alternative description of extensivity. The following definition is normally given in the context of a category with all pullbacks [2, p. 16]. It does, however, make perfect sense given only pullbacks along injections, as we have in an extensive category. The reason for introducing a new name for what is exactly half of the extensivity condition is that it appears as part of the conditions for Giraud's theorem, as will be mentioned below.

Definition 2.10. In a category with finite sums and pullbacks along their injections, a coproduct diagram

$$X_1 \xrightarrow{x_1} X_1 + X_2 \xleftarrow{x_2} X_2$$

is said to be *universal* if pulling it back along any morphism into $X_1 + X_2$ gives a coproduct diagram.

Lemma 2.11. *In an extensive category, sums are universal.* □

Lemma 2.12. *In a category with universal sums, initials are strict.*

Proof. The proof that initials in an extensive category are strict carries over precisely. □

Lemma 2.13. *In any category, if sums are universal then they will be disjoint provided only the pullback of the injections of binary sums is the initial object.*

Proof. Form the pullback

$$\begin{array}{ccccc} X_1 & \xrightarrow{x_1} & X & \xleftarrow{x_2} & X_2 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle x}\downarrow & & {\scriptstyle h_2}\downarrow \\ X & \xrightarrow{x} & X+Y & \xleftarrow{y} & Y \end{array}$$

Then X_2 is initial and so by universality of sums, x_1 is invertible, and so x monic. □

Proposition 2.14. *A category with finite sums and pullbacks along their injections is extensive iff the sums are universal and disjoint.*

Proof. We know that in extensive categories, sums are universal and disjoint. In fact, universality of sums is exactly half of the condition of extensivity. Thus it remains to show that if sums are universal and disjoint then

$$\begin{array}{ccccc} A_1 & \xrightarrow{a_1} & A_1+A_2 & \xleftarrow{a_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_1+h_2}\downarrow & & {\scriptstyle h_2}\downarrow \\ X_1 & \xrightarrow{x_1} & X_1+X_2 & \xleftarrow{x_2} & X_2 \end{array}$$

is a pullback. Suppose now we are given a commuting square

$$\begin{array}{ccc} B & \xrightarrow{f} & A_1+A_2 \\ {\scriptstyle g}\downarrow & & \\ X_1 & \xrightarrow{x_1} & X_1+X_2 \end{array}$$

First form the pullback

$$\begin{tikzcd}
B_1 \arrow[r, "b_1"] \arrow[d, "f_1"'] & B \arrow[d, "f"'] & B_2 \arrow[l, "b_2"'] \arrow[d, "f_2"'] \\
A_1 \arrow[r, "a_1"] \arrow[d, "h_1"'] & A_1 + A_2 \arrow[d, "h_1+h_2"'] & A_2 \arrow[l, "a_2"'] \arrow[d, "h_2"'] \\
X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"']
\end{tikzcd}$$

Now

$$x_1 \circ g \circ b_2 = (h_1 + h_2) \circ f \circ b_2 = x_2 \circ h_2 \circ f_2$$

and so by strictness of initials, B_2 is initial. But then by universality of sums, b_1 is invertible, and so $f_1 \circ b_1^{-1}$ is the required unique arrow of the pullback. Similarly, A_2 is a pullback. $\square$

This result, which gives us another description of extensivity, is the last we prove about mere extensive categories. In the next section we look at distributive categories, after which we return to extensive categories, considering the effect of adding various limits.

3. Distributive categories

Definition 3.1. A category with finite products and sums is said to be *distributive* if the canonical arrow

$$\delta : A \times B + A \times C \to A \times (B + C)$$

is an isomorphism.

Sometimes this definition includes a condition concerning the initial object. This is redundant, as the following proposition, due to Cockett, shows.

Proposition 3.2. *In a distributive category the projection $p : A \times 0 \to 0$ is invertible.*

Proof. There is only one possible inverse, the unique arrow $! : 0 \to A$. Certainly we have $p \circ ! = 1$. On the other hand, the distributivity axiom establishes $A \times (0 + 0)$ as the coproduct of $A \times 0$ with itself, the coprojections being equal. But any sum with coprojections equal can have at most one arrow to any other object and so $! \circ p = 1$. $\square$

This is analogous to the result that in an extensive category, the canonical functor $0 : \mathbf{1} \to \mathbf{A}/0$ is an equivalence. We can prove a few simple consequences of distributivity, already known to be true in an arbitrary extensive category.

Proposition 3.3. *In a distributive category injections are monic.*

Proof. Given an injection $a_1 : A_1 \to A_1 + A_2$ and arrows $f, g : B \to A_1$ such that $a_1 \circ f = a_1 \circ g$, we must prove that $f = g$. Consider the arrow

$$B \times (A_1 + A_2) \xrightarrow{\delta^{-1}} B \times A_1 + B \times A_2 \xrightarrow{(1_B \times 1_{A_1}, \binom{1_B}{f} \circ p_B)} B \times A_1 .$$

If we compose it with $1_B \times a_1$ then we get

$$\begin{aligned}&\left(1_B \times 1_{A_1}, \binom{1_B}{f} \circ p_B\right) \circ \delta^{-1} \circ (1_B \times a_1)\\ &\quad = \left(1_B \times 1_{A_1}, \binom{1_B}{f} \circ p_B\right) \circ i_{B \times A_1} = 1_B \times 1_{A_1}\end{aligned}$$

and so $1_B \times a_1$ has a retract, hence is monic. Now

$$(1_B \times a_1) \circ \binom{1_B}{f} = \binom{1_B}{a_1 \circ f} = \binom{1_B}{a_1 \circ g} = (1_B \times a_1) \circ \binom{1_B}{g} .$$

Then as $1_B \times i_1$ is monic we have $\binom{1_B}{f} = \binom{1_B}{g}$ and so $f = g$. $\square$

Proposition 3.4. *In a distributive category initials are strict.*

Proof. In any distributive category we have $A \times 0 \cong 0$, and so given an arrow $\alpha : A \to 0$ there is an induced arrow $\binom{1_A}{\alpha} : A \to 0$ which satisfies $p \circ \binom{1_A}{\alpha} = 1_A$, where p is the projection $A \times 0 \to A$. Also, as $A \times 0$ is initial, $\binom{1_A}{\alpha} \circ p = 1_{A \times 0}$ and so A is isomorphic to $A \times 0$, and hence to 0. But as 0, and so A is initial, the isomorphism must be α. $\square$

In the case of extensive categories, the companion result to this last gave us disjointness of sums. It does not necessarily hold in a distributive category. An example of a category that is distributive but does not have disjoint sums is $\mathcal{P}X$: the power set of a finite non-empty set X, ordered by inclusion.

4. Extensive categories with limits

4.1. Extensive categories that have a terminal object

We now examine the effect of adding further limits, beginning with only a terminal object. First we should note that this is a genuine condition: there are extensive categories that do not have a terminal object. A simple example is obtained by taking the free category with sums on the category comprising only two parallel arrows. We have seen that freely adding sums to any category gives an extensive one, and it is clear that this category has neither terminal object nor products. Another example is the category of manifolds of dimension 5 or $-\infty$.

If a category with sums does have a terminal object, we can simplify the extensivity condition.

Proposition 4.1. *If a category with sums has a terminal object then it is extensive iff it satisfies the extensivity condition for* 1 *and* 1.

Proof. Suppose that the extensivity condition for 1 and 1 holds. Given that

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A \arrow[d, "f"'] & A_2 \arrow[l, "a_2"'] \arrow[d, "f_2"'] \\
X_1 \arrow[r, "x_1"] & X_1 + X_2 & X_2 \arrow[l, "x_2"']
\end{tikzcd}$$

is a coproduct then

$$\begin{tikzcd}
A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & A \arrow[d, "f"'] & A_2 \arrow[l, "a_2"'] \arrow[d, "f_2"'] \\
X_1 \arrow[r, "x_1"] \arrow[d, "!"'] & X_1 + X_2 \arrow[d, "!+!"'] & X_2 \arrow[l, "x_2"'] \arrow[d, "!"'] \\
1 \arrow[r, "i_1"] & 1 + 1 & 1 \arrow[l, "i_2"']
\end{tikzcd}$$

is a coproduct and hence a pullback. Also

$$\begin{tikzcd}
X_1 \arrow[r, "x_1"] \arrow[d, "!"'] & X_1 + X_2 \arrow[d, "!+!"'] & X_2 \arrow[l, "x_2"'] \arrow[d, "!"'] \\
1 \arrow[r, "i_1"] & 1 + 1 & 1 \arrow[l, "i_2"']
\end{tikzcd}$$

is a coproduct and hence a pullback, and so by a general theorem about pullbacks, the first diagram is a pullback too. If on the other hand the first diagram is a pullback, as the last is a coproduct and so a pullback, by a general theorem about pullbacks the second is a pullback too. But then the second is a coproduct, and so the first also is one. $\square$

4.2. *Boolean categories*

Definition 4.2. An extensive category is said to be *Boolean* if it has a terminal object and the first injection $T : 1 \to 1 + 1$ is a subobject classifier.

A Boolean category has even nicer sums that an extensive category. An example of a category that is extensive and has a terminal object, but is not Boolean is the category of manifolds of dimension less than five.

We saw that in extensive categories, injections are monic. If the category is moreover Boolean, then the converse is also true.

Proposition 4.3. *In a Boolean category, all monics are injections.*

Proof. Suppose **A** is Boolean and $i : A_1 \to A$ is monic. Then as T is a subobject classifier we get, uniquely, a pullback diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "!"'] & A \arrow[d, "\chi_i"] \\ 1 \arrow[r, "T"] & 1+1 \end{tikzcd}$$

But then we form the pullback of χ_i along F to give a pair of pullbacks, and hence, by extensivity, a coproduct, with i as coprojection. □

Proposition 4.4. *An extensive category that has a terminal object is Boolean iff every monic is the injection of a coproduct, with complement unique up to a unique isomorphism commuting with the injection.*

Proof. One direction is essentially the content of the preceding proposition. Conversely if every monic is an injection as described then given i there is a unique map $\chi_i : A \to 1+1$ such that $x_i = !+!$ and so the diagram is a pullback. □

4.3. Extensive categories that have products

We now add the condition of not just a terminal object, but binary products, so guaranteeing all finite products. Again there is an example to show that this is a genuinely stronger notion. We take manifolds of dimension less than 5. This has singletons as terminal objects. In the category of manifolds, the product of two manifolds of dimension 4 has dimension 8. In manifolds of dimension less than 5, the product of two manifolds of dimension 4 does not exist. Alternatively, we could take the free category with sums on the category with three objects and two non-identity arrows: $\cdot \to \cdot \leftarrow \cdot$.

Proposition 4.5. *An extensive category with products is distributive.*

Proof. Given objects A, B_1, B_2, we know that

$$\begin{tikzcd} B_1 \arrow[r, "b_1"] \arrow[d, "!"'] & B_1 + B_2 \arrow[d, "!+!"] & B_2 \arrow[l, "b_2"'] \arrow[d, "!"] \\ 1 \arrow[r, "i_1"] & 1+1 & 1 \arrow[l, "i_2"'] \end{tikzcd}$$

is a coproduct and hence a pullback. Given this pullback, it is the case in any category that

$$\begin{tikzcd} A \times B_1 \arrow[r, "1_A \times b_1"] \arrow[d, "!"'] & A \times (B_1 + B_2) \arrow[d, "(!+!)\circ p_2"] & A \times B_2 \arrow[l, "1_A \times b_2"'] \arrow[d, "!"] \\ 1 \arrow[r, "i_1"] & 1 + 1 & 1 \arrow[l, "i_2"'] \end{tikzcd}$$

is a pullback, and so, in this case, a coproduct. But this is exactly the requirement of distributivity. □

The converse is not true. A counterexample is again $\mathscr{P}X$. This is distributive but not extensive.

There is a related fact about extensive categories with products worth noting at this stage. If we freely add sums to a category with all finite products we get a category that is extensive and has finite products. In particular, if we freely add products and then freely add sums, we get an extensive category. But freely adding products and then sums gives the free distributive category on a category. Thus we have the following important result:

Proposition 4.6. *The free distributive category on a category is extensive.* □

4.4. Lextensive categories

We now move on to the last stopping point in our survey of extensive categories with various limits: extensive categories with all finite limits. These rejoice in the somewhat unwieldy name of lextensive categories.

Definition 4.7. An extensive category with all finite limits is called *lextensive*.

Again the strengthening is genuine: this time the example is **Hty**, the category of topological spaces and homotopy classes of maps. This is extensive and has finite products but not all finite limits. We now look at some properties of lextensive categories

Proposition 4.8. *A lextensive category is locally lextensive.*

Proof. This is immediate once we know how to form limits and coproducts in the comma category $\mathbf{A}/X$. In particular, sums and pullbacks are formed as in $\mathbf{A}$. Products are given by pullbacks in $\mathbf{A}$. The terminal object is the identity arrow on X. □

Corollary 4.9. *A category is lextensive iff it is locally distributive and has disjoint sums.*

Proof. The previous result tells us that lextensive categories are locally lextensive, and so a fortiori locally distributive. An earlier one tells us that they have disjoint sums. Suppose then that a category **A** is locally distributive and has disjoint sums. Local distributivity is clearly just the condition that sums are universal and finite limits exist. Thus we have universal and disjoint sums, finite limits, and therefore the category is lextensive. □

Disjoint sums are necessary here. The example $\mathscr{P}X$ once more illustrates this: in fact it is not only locally distributive, but locally Cartesian closed, and is still not extensive.

Remark 4.10. A category is lextensive iff it has finite limits, and finite sums which are disjoint and universal. Lextensivity therefore forms the first part of the conditions given in Giraud's characterization of Grothendieck toposes [2, pp. 16–17].

We have seen that the notion of lextensive category is genuinely stronger than that of extensive category with products. It turns out, however, that adding the Boolean property erodes this difference.

Proposition 4.11. *Boolean categories with products have all finite limits.*

Proof. It will suffice to show that equalizers exist. Given then a pair of arrows $f,g : A \to B$ in a Boolean category with products, we form the diagonal arrow $B \to B \times B$. This is certainly monic, and hence by the preceding result, an injection. Then, as we are in an extensive category, we can pull back along it.

$$\begin{array}{ccc} E & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle \binom{f}{g}} \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

Then it follows in any category that $e : E \to A$ is the equalizer of f and g. □

On the other hand, being Boolean with products is strictly stronger than being lextensive.

Also, adding products to a Boolean category is itself a genuine strengthening. Let C_6 be the cyclic group of order 6. Then the category of C_6-sets with orbits of length 1, 2, or 3 (i.e. the non-transitive C_6-sets) is Boolean but does not have finite products.

This is the end of our results on extensive and distributive categories. The main results are shown in the following diagram. The objects are categorical properties. The arrows are to be read as implications. The labels give examples to show that the implications are one-way.

$$
\begin{tikzcd}
\text{Boolean Topos} \arrow[r, "\mathbf{Sets}^{\cdot\rightrightarrows\cdot}"] \arrow[dddd, "\mathbf{Sets}_{\mathrm{denum.}}"'] & \text{Topos} \arrow[dr, "\mathcal{P}X"] \arrow[dd, "\mathbf{CRng}^{\mathrm{op}}"'] & \\
 & & \text{Sum} + \text{loc.C.Cl.} \arrow[d, "(\mathcal{P}\mathbb{N})_{\mathrm{fin}}"] \\
 & \text{Lext.} \arrow[r, "\mathcal{P}X"] \arrow[dd, "\mathbf{Hty}"'] & \text{loc.Dist.} \arrow[d, "\mathbf{Hty}"] \\
 & & \text{Dist.} \arrow[d, "\mathbf{V.Sp}"] \\
\text{Boolean} + \text{Pr.} \arrow[uur] \arrow[d, "\mathbf{C_6\text{-}sets}_{\mathrm{non\text{-}tr}}"'] & \text{Ext.} + \text{Pr.} \arrow[ur, "\mathcal{P}X"] \arrow[d, "\mathbf{Mnf}_{<5}"'] & \text{Sum} + \text{Pr.} \arrow[d, "\mathbf{Mnf}_{<5}"] \\
\text{Boolean} \arrow[r, "\mathbf{Mnf}_{<5}"] & \text{Ext.} + 1 \arrow[r, "\mathbf{V.Sp}"] \arrow[d, "\mathbf{Mnf}_5"'] & \text{Sum} + 1 \arrow[d, "\mathbf{Mnf}_5"] \\
 & \text{Ext.} \arrow[r, "\mathbf{V.Sp}"] & \text{Sum}
\end{tikzcd}
$$

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