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A note on complete semilattices

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Let $\mathscr{S}$ be the category whose objects are complete (semi)lattices (i.e. partially ordered sets in which every subset has a supremum), and whose morphisms are sup-preserving maps. In an earlier note in this column [4], Dana May Latch observed that $\mathscr{S}$ has arbitrary (set-indexed) products and coproducts, and that these coincide – i.e. $\prod_{i\in I} L_i \cong \coprod_{i\in I} L_i$ for any set $\{L_i \mid i \in I\}$ of objects of $\mathscr{S}$. (She contrasted this with the situation in various finitary algebraic categories such as abelian groups and semilattices, where finite products and coproducts coincide but infinite ones do not.) However, she failed to observe a more fundamental reason for this coincidence, namely that the category $\mathscr{S}$ is isomorphic to its dual.

The proof of this fact rests on two simple and well-known lemmas:

LEMMA 1. *Let P be a poset. If P is complete, so is P^{op}.*

Proof. To construct the inf of a subset $P' \subseteq P$, we simply take the sup of the set of lower bounds for P' (cf. [3], p. 29).

LEMMA 2. *Let $P \xrightarrow{f} Q$ be a sup-preserving map between complete posets. Regard P and Q as small categories and f as a functor; then f has a right adjoint $Q \xrightarrow{R(f)} P$, and $R(f)$ preserves infs.*

Proof. This is a special case of the Adjoint Functor Theorem ([2], p. 84). (The "solution-set condition" is automatic since P and Q are small categories.)

In general, the right adjoint of a functor is determined only up to natural isomorphism; but we are dealing with categories in which every isomorphism is an identity map, so $R(f)$ is uniquely determined by f. Thus we have a functor $\mathscr{S} \to \mathscr{S}^{op}$, which sends P to P^{op} and f to $R(f)$; and this functor is an isomorphism, since its square is the identity.

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Now products in $\mathcal{S}$ are simply Cartesian products, and so the operation of taking the opposite of a poset commutes with forming products; thus we recover the result of Latch.

Latch also observes that the category $\mathcal{S}$ is "algebraic" in the sense of being monadic over Sets. In fact, it is isomorphic to the category of algebras for the monad structure induced by union on the covariant power-set functor; I believe this result is originally due to E. G. Manes [5]. This raises the question of whether there are any other nontrivial examples of self-dual categories which are monadic over Sets; I'm not quite sure what "nontrivial" means in this sentence, but it should certainly exclude the ordinals 1 and 2, both of which are monadic over Sets.

In conclusion, it should be remarked that all the observations of this note are still valid if we replace Sets by a topos $\mathcal{E}$, and $\mathcal{S}$ by the category of internally complete lattices in $\mathcal{E}$. A detailed proof of the monadicity in this case will be found in [1].

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