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ON THE ORDERED SET OF REFLECTIVE SUBCATEGORIES

G.M. KELLY

Given a category A, we consider the (often large) set $Ref\, A$ of its reflective (full, replete) subcategories, ordered by inclusion. It is known that, even when A is complete and cocomplete, wellpowered and cowellpowered, the intersection of two reflective subcategories need not be reflective. Supposing that A admits (i) small limits and (ii) arbitrary (even large) intersections of strong subobjects, we prove that an infimum $\bigwedge_i C_i$ in $Ref\, A$ must necessarily be the intersection $\bigcap_i C_i$. Accordingly $Ref\, A$ is not in general, even for good A, a complete lattice. We show, however, under the same conditions on A, that $Ref\, A$ does admit *small* suprema $\bigvee_i C_i$, given by the closure in A of the union $\bigcup_i C_i$ under the limits of types (i) and (ii) above.

1. Strongly complete categories

We suppose an inaccessible cardinal ∞ chosen once for all, and call a set *small* if its cardinal is less than ∞. The morphisms of any category A form a set, and A is *small* if this set is small; while A

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is *locally small* if each hom-set $A(A,B)$ is small. We use *large* to mean *not small*. Although a category is said to be *complete* when it admits all *small* limits, an ordered set is called a *complete lattice* only when it admits *all* suprema and infima - even large ones if it is a large set.

We refer to [7] for the definition and properties of *strong monomorphisms*. We call a category *strongly complete* if, besides admitting small limits, it admits the fibred product (or *intersection*) of any family $(f_\alpha : A_\alpha \to B)$ of strong monomorphisms; in other words, it admits *all intersections of strong subobjects*. Of course a category is strongly complete if it is complete and wellpowered, or even *weakly wellpowered* - in the sense that each object has but a small set of *strong* subobjects.

In a strongly-complete category, the epimorphisms and the strong monomorphisms form a factorization system: given $f: A \to B$ we take for $i: C \to B$ the intersection of those strong subobjects of B through which f factorizes then $f = ip$, and p is epimorphic by Lemma 4.4 of [7].

We use *subcategory* to mean *full replete* subcategory. A subcategory $\mathcal{B}$ of the strongly-complete $\mathcal{A}$ is said to be *closed in* $\mathcal{A}$ *under strong limits* if the limit in $\mathcal{A}$ of a diagram in $\mathcal{B}$ lies in $\mathcal{B}$, whenever the diagram is either small or else of the form $(f_\alpha : A_\alpha \to B)$ where the maps f_α in $\mathcal{B}$ are strong monomorphisms in $\mathcal{A}$. Clearly the intersection of subcategories closed under strong limits is itself so closed. Given any subcategory $\mathcal{D}$ of $\mathcal{A}$, there is accordingly a smallest subcategory $\mathcal{D}^*$ containing $\mathcal{D}$ which is closed in $\mathcal{A}$ under strong limits; we call $\mathcal{D}^*$ the *strong-limit closure of* $\mathcal{D}$ *in* $\mathcal{A}$.

PROPOSITION 1. *If* $\mathcal{B}$ *is closed in the strongly-complete* $\mathcal{A}$ *under strong limits, every strong monomorphism* $f: C \to B$ *in* $\mathcal{B}$ *is a strong monomorphism in* $\mathcal{A}$. *Accordingly* $\mathcal{B}$ *is itself strongly complete, and the inclusion* $\mathcal{B} \to \mathcal{A}$ *preserves small limits and arbitrary intersections of strong monomorphisms.*

Proof. Consider the totality of those maps $j: E_j \to B$ through which f factorizes and which are strong monomorphisms in $\mathcal{A}$ with $E_j \in \mathcal{B}$. Then their intersection $k: E \to B$ in $\mathcal{A}$ is a strong monomorphism in A, and $E \in B$ since B is closed in A under strong limits. Let the factorization of f through k be $f = kp$; we show that p is invertible, so that f like k is a strong monomorphism in A. Since f is by hypothesis a strong monomorphism in B, so is p; thus it suffices to show that p is epimorphic in B. Let $x,y \colon E \to F$ in B satisfy $xp = yp$, and let $z \colon G \to E$ be the equalizer in A of x and y. Since B is closed in A under small limits, we have $G \in B$; and f factorizes through kz, which is a strong monomorphism in A. By the definition of k, therefore, z is invertible and $x = y$. □

2. Strong cogenerators

The definition in [7] of strong monomorphism admits an evident generalization to a definition of a *strongly monomorphic family* $(f_\alpha \colon A \to B_\alpha)$, in such a way that, if the product ΠB_α exists, the family is strongly monomorphic if and only if the corresponding map $A \to \Pi B_\alpha$ is so.

A *small* set G of objects of A is said to be a *strong cogenerator* if, for each $A \in A$, the family $(h \colon A \to G_h)$ of all maps with codomain in G is strongly monomorphic. When A is complete and locally small, this is to say that the canonical map $\eta_A \colon A \to \Pi_{G \in G}\{A(A,G) \pitchfork G\}$ (where $X \pitchfork G$ denotes the product of X copies of G) is a strong monomorphism. From the form of the Special Adjoint Functor Theorem given by Börger *et al.* in Corollary 1.10 of [3], it follows that:

PROPOSITION 2. *Let the strongly-complete and locally-small A have a strong cogenerator, and let B be locally small. Then a functor $U \colon A \to B$ has a left adjoint if and only if it preserves small limits (and hence monomorphisms), and sends an intersection of strong monomorphisms to an intersection of monomorphisms.* □

If G is a strong cogenerator, the set-valued functors $A(-,G)$ for $G \in G$ jointly reflect isomorphisms. Taking for simplicity the case of a locally-small complete A, suppose $f \colon A \to B$ to be such that each $A(f,G)$ is invertible. The naturality of η gives $u\eta_A = \eta_B f$ where u

is the invertible map $\Pi_G\{A(f,G) \pitchfork G\}$; since $u\eta_A$ is a strong monomorphism, so is f . On the other hand, G being *a fortiori* a cogenerator since the η_A are *a fortiori* monomorphic, the $A(-,G)$ are jointly faithful and hence jointly reflect epimorphisms, so that f is epimorphic and hence invertible.

Suppose conversely that the small set G is such that the $A(-,G)$ jointly reflect isomorphims. Then if η_A factorizes through an epimorphism $f: A \to B$, this f is invertible. For to say that η_A factorizes through f is to say that every $h: A \to G$ with $G \in G$ factorizes through f , or that each $A(f,G)$ is surjective; but $A(f,G)$ is injective since f is epimorphic.

If A is such that $g: A \to C$ is a strong monomorphism whenever g factorizes through no non-trivial epimorphism $f: A \to B$ (which is the case whenever A admits finite colimits, or whenever epimorphisms and strong monomorphisms constitute a factorization system on A), a small set G is a strong cogenerator exactly when the $A(-,G)$ for $G \in G$ jointly reflect isomorphisms; this is really a special case of Proposition 4.3 of Im and Kelly [6]. Since we have seen that epimorphisms and strong monomorphisms do form a factorization system when A is strongly complete, we have:

PROPOSITION 3. *A small set G of objects of a locally-small and strongly-complete A form a strong cogenerator if and only if a map f is invertible whenever $A(f,G)$ is so for each $G \in G$.* □

PROPOSITION 4. *Let $\mathcal{D}$ be a small subcategory of the locally-small and strongly-complete A , and let $\mathcal{D}^*$ be the strong-limit closure of $\mathcal{D}$ in A . Then $\mathcal{D}$ is a strong cogenerator for $\mathcal{D}^*$ (which is itself strongly complete by Proposition 1.)*

Proof. By Proposition 3, we have to show that a map f in $\mathcal{D}$ is invertible if $A(f,D)$ is so for each $D \in \mathcal{D}$. Consider the subcategory $\mathcal{B}$ of A given by all those objects B of A for which $A(f,B)$ is invertible. If any diagram in $\mathcal{B}$ admits a limit in A , this limit lies in B ; for $A(f, \lim B_i) \cong \lim A(f, B_i)$. So B is closed in A under strong limits; therefore, since it contains $\mathcal{D}$, it contains $\mathcal{D}^*$. Thus f is invertible since it is a map in $\mathcal{D}^*$ with $A(f,B)$ invertible for each $B \in \mathcal{D}^*$. □

Propositions 1, 2, and 4 now give:

PROPOSITION 5. *If* $\mathcal{D}$ *is a small subcategory of the locally-small and strongly-complete* A *and* $\mathcal{D}^*$ *is the strong-limit closure of* $\mathcal{D}$ *in* A *, then* $\mathcal{D}^*$ *is reflective in* A . □

3. Infima of reflective subcategories

We write *Ref* A for the set of reflective subcategories of A , ordered by inclusion. We use $\bigvee$ and $\bigwedge$ to denote suprema and infima in *Ref* A , so far as they exist, retaining $\cup$ and $\cap$ for union and intersection.

Remark 6. Even when A is locally small, complete and cocomplete, wellpowered and cowellpowered, and has a generator and a strong cogenerator, the intersection of two reflective subcategories need not be reflective. The following simple counter-example is contained in a forthcoming article [1] by Adámek and Rosický. An object of A is a set with two topologies, a first and a second; and a map $f: A \to B$ is a function continuous both for the first topology and for the second. The subcategory C_1 [respectively C_2] consists of those objects for which the first [respectively second] topology is compact Hausdorff. A reflexion of $A \in A$ into C_1 is given by taking its Stone-Čech compactification $r: A \to B$ with respect to the first topology, and giving to B as its second topology the final one with respect to r ; similarly for the reflexion into C_2 . Yet $C_1 \cap C_2$ is not reflective; the formal proof is in [1], but the intuition is clear enough - if we take the reflexion of A into C_1 , and then the reflexion of this into C_2 , and then the reflexion of this last into C_1 , and so on alternately, the cardinal of the successive reflexions increases unboundedly. {The author knows of no such counter-example for a well-behaved $\mathcal{A}$ with a *strong* generator.}

This counter-example does not of itself show that *Ref* $\mathcal{A}$ for such an $\mathcal{A}$ may lack binary infima; but it does so when combined with the following:

THEOREM 7. *Let* $\mathcal{A}$ *be locally small and strongly complete. If a subset* $\{C_i\}$ *of Ref* $\mathcal{A}$ *admits an infimum* $\wedge C_i$ *, this must be the intersection* $\cap C_i$.

Proof. If *any* diagram in a reflective C_i admits a limit in $\mathcal{A}$, this limit lies in C_i ; hence each C_i is closed in $\mathcal{A}$ under strong limits. Thus if $D \in \cap C_i$ and if $\{D\}$ denotes the subcategory of $\mathcal{A}$ consisting of D alone, its strong-limit closure $\{D\}^*$ is contained in $\cap C_i$. Since $\{D\}^*$ is reflective in $\mathcal{A}$ by Proposition 5, we have $\{D\}^* \subset \wedge C_i$. Since this is true for every $D \in \cap C_i$ and since $D \in \{D\}^*$, we have $\cap C_i \subset \wedge C_i$; whence $\cap C_i = \wedge C_i$. □

Remark 8. Since *Ref* $\mathcal{A}$ for a strongly-complete $\mathcal{A}$, or even for an $\mathcal{A}$ so good as that of Remark 6, may lack binary infima, it is not in general a complete lattice. Yet we show in Theorem 14 below that *Ref* $\mathcal{A}$ admits *small* suprema when $\mathcal{A}$ is strongly complete. It follows that *Ref* $\mathcal{A}$ is a *large* set for the $\mathcal{A}$ of Remark 6. Since this $\mathcal{A}$ has a strong cogenerator $G = \{G_1, G_2, G_3\}$, where each G_i is a two-element set and the respective pairs of topologies are (chaotic, chaotic), (chaotic, Sierpinski), and (Sierpinski, chaotic), *Ref* $\mathcal{A}$ can be large even when the locally-small strongly-complete $\mathcal{A}$ has a strong cogenerator. The following result, therefore, has no converse:

PROPOSITION 9. *Let* $\mathcal{A}$ *be locally small and strongly complete. If Ref* $\mathcal{A}$ *is a small set,* $\mathcal{A}$ *has a strong cogenerator.*

Proof. Since $\{D\}^*$ is reflective in $\mathcal{A}$ for each $D \in \mathcal{A}$ by Proposition 5, the set of *distinct* $\{D\}^*$ is small; let it consist of the $\{G\}^*$ where G runs through the small set $\mathcal{G}$. Given $A \in \mathcal{A}$, let

$\{A\}^* = \{H\}^*$ where $H \in G$. By Proposition 4, H is a strong cogenerator for $\{A\}^*$, so that the canonical $\zeta\colon A \to A(A,H) \pitchfork H$ is a strong monomorphism in $\{A\}^*$. Here $A(A,H) \pitchfork H$ is in the first instance the product in $\{A\}^*$; but it is equally the product in A. Moreover ζ is a strong monomorphism in A by Proposition 1. Since ζ factorizes through $\eta_A\colon A \to \Pi_{G \in G}\{A(A,G) \pitchfork G\}$, the latter too is a strong monomorphism in A; so that G is a strong cogenerator for A. □

4. Reflective factorization systems

We call a set M of maps in a category A a *skein* if it contains the isomorphisms and is closed under composition. A skein M is *stable under pullbacks* if the pullback of a map in M along any map in A is again in M. The skein M is *stable under fibred products* if, whenever a family $(f_\alpha\colon A_\alpha \to B)$ of maps in M admits a fibred product $h\colon C \to B$, we have $h \in M$; similarly for stability under *small* or *finite* fibred products. (Note the distinction between stability under pullbacks and stability under binary fibred products.) *Stability under the intersections of strong monomorphisms* means stability under those fibred products, possibly large, in which each f_α is a strong monomorphism.

For maps e and m in A we write $e \downarrow m$ if, for every commutative square $ve = mu$, there is a unique "diagonal" w with $we = u$ and $mw = v$. If N is any set of maps in A we write $N^\downarrow = \{m \mid n \downarrow m$ for all $n \in N\}$ and $N^\uparrow = \{e \mid e \downarrow n$ for all $n \in N\}$. By Proposition 2.1.1 of Freyd and Kelly [5], $N^\downarrow$ is a skein, stable under pullbacks and fibred products, such that

(1) if fg and f are in $N^\downarrow$, so is g.

We recall from [5] that a *factorization system* (E,M) on A consists of two skeins E and M such that every map f in A admits a factorization $f = me$ with $m \in M$ and $e \in E$, and such that $e \downarrow m$ for each $e \in E$ and $m \in M$. This last requirement may be expressed equivalently as $M \subset E^\downarrow$, or as $E \subset M^\uparrow$; in fact, by Proposition 2.2.1

of [5], every factorization system satisfies $M = E^{\downarrow}$ and $E = M^{\uparrow}$, and hence is fully determined by the knowledge of either E or M. In any factorization system $E \cap M$ consists of the isomorphisms, since $e \downarrow e$ only for invertible e. We order factorization systems by setting $(E,M) \leq (E',M')$ when $M \subset M'$; or equivalently when $E' \subset E$.

By the dual of (1), any factorization system (E,M) satisfies

(2) if fg and g are in E, so is f;

we call (E,M) a *reflective* factorization system if it also satisfies

(3) if fg and f are in E, so is g.

It was proved by Cassidy, Hébert, and Kelly in Section 2 of [4] that, when A is *strongly finitely complete* (that is, when A admits finite limits and arbitrary intersections of strong monomorphisms), there is an order-preserving bijection between reflective factorization systems on A and reflective subcategories of A. We now give a modification of their proof of that result; the new proof provides extra information on M needed in our applications below.

PROPOSITION 10. *Let C be a reflective subcategory of the strongly-finitely-complete A, the reflexion of A into C being $\rho_A : A \to rA$. Let $\Phi C = (E,M)$ where E is the set of maps inverted by $r : A \to A$ and where M is the smallest skein in A containing* mor C *and stable under pullbacks and arbitrary intersections of strong monomorphisms. Then*

$$(4) \qquad C = E^{\perp} =_{Df} \{A \in A \mid A(e,A) \text{ is invertible for all } e \in E\},$$

and (E,M) is a reflective factorization system with factorizations constructed as in (5) and (6) below.

Proof. If $C \in C$ and $e \in E$, we have $A(e,C)$ invertible since it is conjugate to the invertible $C(r(e),C)$: thus $C \subset E^{\perp}$. We may always so choose the reflexion that $\rho_C : C \to rC$ for $C \in C$ is $1 : C \to C$; then $r(\rho_A) = 1$ for any $A \in A$. If $A \in E^{\perp}$ we have $A(\rho_A, A)$ invertible, since $\rho_A \in \mathcal{E}$; so there is an $h\colon rA \to A$ with $h\rho_A = 1$, and now $\rho_A h\rho_A = \rho_A$ gives $\rho_A h = 1\colon rA \to rA$; thus ρ_A is invertible and $A \in \mathcal{C}$. This proves (4).

$\mathcal{E}$ is clearly a skein, and $\mathcal{M}$ is a skein by definition. For $C \in \mathcal{C}$, we have $C \in \mathcal{E}^{\perp}$ by (4); this is equally the assertion that $C \to 1$, where 1 is the terminal object of $\mathcal{A}$, lies in $\mathcal{E}^{\downarrow}$. It follows from (1) that mor $\mathcal{C} \subset \mathcal{E}^{\downarrow}$. Since the skein $\mathcal{E}^{\downarrow}$, by the remarks preceding (1), is stable under pullbacks and fibred products, it follows from the definition of $\mathcal{M}$ that $\mathcal{M} \subset \mathcal{E}^{\downarrow}$.

Since $\mathcal{E}$ clearly satisfies (3), it remains to show that any $f\colon A \to B$ in $\mathcal{A}$ has an $(\mathcal{E},\mathcal{M})$ factorization. In the diagram

(5)
$$
\begin{tikzcd}
A \arrow[dr, "w"] \arrow[drr, "\rho_A"] \arrow[ddr, "f"'] & & \\
& D \arrow[r, "v"'] \arrow[d, "u"] & rA \arrow[d, "r(f)"] \\
& B \arrow[r, "\rho_B"'] & rB
\end{tikzcd}
,
$$

let the square be a pullback. Since $r(f) \in$ mor $\mathcal{C} \subset \mathcal{M}$, we have $u \in \mathcal{M}$ by the definition of $\mathcal{M}$. Now let $k\colon E \to D$ be the intersection of all those strong subobjects of D , lying in $\mathcal{M}$, through which w factorizes; then we have a factorization

(6) $$A \xrightarrow[w]{} D \quad = \quad A \xrightarrow[e]{} E \xrightarrow[k]{} D \qquad ,$$

where the strong monomorphism k lies in $\mathcal{M}$ by the definition of $\mathcal{M}$. We now show that $e \in \mathcal{E}$, so that f has $(uk)e$ as its $(\mathcal{E},\mathcal{M})$ factorization.

Since $r(\rho_A) = 1$, applying r to the top triangle of (5) shows that $r(w)$ is a coretraction; whence $r(e)$ is a coretraction by (6). Let the analogue of (5) with e in place of f be

(7)
$$\begin{tikzcd}
A \arrow[dr, "w'"] \arrow[drr, "\rho_A"] \arrow[ddr, "e"'] & & \\
& D' \arrow[r, "v'"'] \arrow[d, "u'"] & rA \arrow[d, "r(e)"] \\
& E \arrow[r, "\rho_E"'] & rE
\end{tikzcd}$$

Since $r(e)$, being a coretraction, is a strong monomorphism, so is its pullback u'. Being a strong monomorphism in M through which e factorizes, u' is invertible by the definition of k; so we may as well take $D' = E$, $u' = 1$, and $w' = e$. Applying r to the top triangle of (7) gives $r(v')r(e) = 1$, while applying r to the square in (7) gives $r(e)r(v') = 1$. So $r(e)$ is invertible and $e \in E$. □

PROPOSITION 11. *For any factorization system* (E, M) *on an* A *with a terminal object* 1, *define a subcategory* $C = \Psi(E, M)$ *by*

(8) $$C = M/1 =_{Df} \{A \in A \mid A \to 1 \text{ lies in } M\} .$$

Then C *is reflective in* A, *the reflexion* $\rho_A : A \to rA$ *being the* E*-part of the* (E, M) *factorization of* $A \to 1$. □

THEOREM 12. *For a strongly-finitely-complete* A, *the (clearly order-preserving) functions* Φ *and* Ψ *constitute a bijection between reflective factorization systems on* A *and reflective subcategories of* A.

Proof. $\Psi\Phi C = M/1$, where M is as in Proposition 10. Since M is $E^{\downarrow}$, we have $M/1 = E^{\perp}$, which is C by (4); hence $\Psi\Phi = 1$.

Let $\Psi(E,M)$, where (E,M) is now reflective, be C as in Proposition 11, and let ΦC be (E',M') . Then $r(f)$ for $f\colon A \to B$ in A is the unique map rendering commutative

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\rho_A"'] & & B \arrow[dl, "\rho_B"] \\ rA \arrow[r, "r(f)"] \arrow[dr] & rB \arrow[d] & \\ & 1 & \end{tikzcd}$$

Since $rA \to 1$ and $rB \to 1$ lie in M , we have $r(f) \in M$ by (1). Hence $r(f)$ is invertible, so that $f \in E'$, if and only if $r(f) \in E$. Since $\rho_A \in E$, this is equivalent by (2) to $r(f)\rho_A \in E$, and hence to $\rho_B f \in E$. Since $\rho_B \in E$, this last is equivalent by (3) to $f \in E$. So $E' = E$, and $\Phi\Psi = 1$. □

The following is contained in Borceux and Kelly [2] in the special case in which C is a *localization*, in the sense that the reflexion r is left exact:

PROPOSITION 13. *Let the reflective subcategory C of the strongly-finitely-complete A correspond as in Theorem 12 to the reflective factorization system (E,M) . Then if $gf \in E$ and $f \in M$, the map f is a strong monomorphism.*

Proof. Since $r(gf) = r(g)r(f)$ is invertible, $r(f)$ is a coretraction and hence a strong monomorphism, whence its pullback u in (5) is a strong monomorphism. Since k is a strong monomorphism in (6) and e in (6) is invertible because $f \in M$, the composite $f = uke$ is a strong monomorphism. □

5. Small suprema of reflective subcategories

Consider reflective subcategories C_i of a strongly-finitely-complete A and the reflective factorization systems (E_i,M_i) that correspond to them by Theorem 12. If there is a factorization system (E,M) where $E = \cap E_i$, it is clearly reflective by (3), and it is obviously the supremum of the (E_i, M_i) in the set of all factorization systems, and *a fortiori* in the set of reflective factorization systems. Accordingly the corresponding $C = M/1$ is the supremum of the C_i in $Ref\ A$. The following is an analogue of Theorem 3.1 of Borceux and Kelly [2], who dealt with the simpler case of small suprema in the ordered set $Loc\ A$ of localizations of A:

THEOREM 14. *Let* A *be strongly complete [respectively strongly finitely complete] and let* $\{C_i\}_{i \in I}$ *be a set of reflective subcategories with* I *small [respectively finite]. Set* $E = \cap E_i$, *and let* M *be the smallest skein containing* $\cup M_i$ *which is stable under small [respectively finite] fibred products and all intersections of strong monomorphisms. Then* (E, M) *is a reflective factorization system, so that the corresponding* $C = M/1$ *is the supremum of the* C_i *in* $Ref\ A$.

Proof. E is a skein since each E_i is so, and M is a skein by definition. Since $M_i = E_i^{\downarrow} \subset E^{\downarrow}$, and since $E^{\downarrow}$ is a skein stable under fibred products by the remarks preceding (1), we have $M \subset E^{\downarrow}$. It remains to show that any $f: A \to B$ has an (E, M) factorization. Let $f = m_i e_i$ be its (E_i, M_i) factorization, and form the fibred product

$$\begin{tikzcd} & D_i \arrow[dr, "m_i"] & \\ B' \arrow[ur, "p_i"] \arrow[rr, "n"'] & & B \end{tikzcd} ;$$

then $f = nf'$, where $f': A \to B'$ is the unique map satisfying $p_i f' = e_i$ for all i. Factorize f' as

$$A \xrightarrow[f']{} B' \quad = \quad A \xrightarrow[e]{} E \xrightarrow[k]{} B' ,$$

where k is the intersection of all those strong subobjects of B', lying in M, through which f' factorizes. By the definition of M, we have $n \in M$ and $k \in M$; it remains to show that $e \in E$, or equivalently that $e \in \mathcal{E}_i$ for each i .

Let $e = m_i'e_i'$ be its $(\mathcal{E}_i, \mathcal{M}_i)$ factorization. Since

$$e_i = p_i f' = p_i ke = p_i km_i'e_i'$$

and since e_i and e_i' belong to $\mathcal{E}_i$, we have $p_i km_i' \in \mathcal{E}_i$ by (2). Because $m_i' \in \mathcal{M}_i$, it follows from Proposition 13 that m_i' is a strong monomorphism. Since e factorizes through the strong monomorphism m_i' lying in $\mathcal{M}$, it follows from the definition of k that m_i' is invertible. Thus $e \in \mathcal{E}_i$ for each i , and hence $e \in \mathcal{E}$. □

The following, in its statement and in its proof, is a modification of the result given for localizations in Theorem 3.3 of Borceux and Kelly [2]:

THEOREM 15. *In the situation of Theorem 14,* $\mathcal{C}$ *is the strong-limit closure [respectively strong-finite-limit closure] in* $\mathcal{A}$ *of* $\cup\, \mathcal{C}_i$.

Proof. Write $\mathcal{D}$ for the closure in question of $\cup\, \mathcal{C}_i$; clearly $\mathcal{D} \subset \mathcal{C}$ since the reflective $\mathcal{C}$ is closed in $\mathcal{A}$ under *all* limits that exist; and it remains to show the converse.

Define as follows a set $\mathcal{N}$ of maps in $\mathcal{A}$: the map $f\colon A \to B$ lies in $\mathcal{N}$ if, for every pullback

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "u"'] & D \arrow[d, "v"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{9}$$

with $D \in \mathcal{D}$, we have $C \in \mathcal{D}$.

Since the pullback along fh is the pullback along h of the pullback along f , it is clear that $\mathcal{N}$ is a skein. Since the pullback along wv is the pullback along v of the pullback along w , it is clear that $\mathcal{N}$ is stable under pullbacks. Moreover $\mathcal{N}$ is stable under such fibred products as exist, and hence under small [respectively finite] ones and all intersections of strong monomorphisms. To see this, consider a diagram

$$\begin{tikzcd}
& C_i \arrow[dr, "g_i"] \arrow[dd, "w_i"] & \\
Y \arrow[ur, "t_i"] \arrow[rr, "s"'] \arrow[dd, "u"'] & & D \arrow[dd, "v"] \\
& A_i \arrow[dr, "f_i"] & \\
X \arrow[ur, "k_i"] \arrow[rr, "h"'] & & B
\end{tikzcd},$$

where the base of the prism is a fibred product, and where u, s constitute the pullback of v and h. If we *define* w_i and g_i as the pullback of v and f_i, there are unique t_i that complete the commutative diagram. Since pullback along v is a right-adjoint functor $\mathcal{A}/B \to \mathcal{A}/D$, the top of the prism is again a fibred product. Now if each $f_i \in \mathcal{N}$ and $D \in \mathcal{D}$, we have each $C_i \in \mathcal{D}$, whence $Y \in \mathcal{D}$ because $\mathcal{D}$ is closed under small [respectively finite] limits, so that $h \in \mathcal{N}$.

Since $\mathcal{C}_i \subset \mathcal{D}$ and $\mathcal{D}$ is closed under finite limits, it is clear that mor $\mathcal{C}_i \subset \mathcal{N}$. Because $\mathcal{N}$ is stable under pullbacks and fibred products, it follows from Proposition 10 that each $\mathcal{M}_i \subset \mathcal{N}$; and now Theorem 14 gives $\mathcal{M} \subset \mathcal{N}$.

If $A \in \mathcal{C}$, the unique map $f: A \to 1$ is in $\mathcal{M}$ by Proposition 10 (since of course $1 \in \mathcal{C}$) and is hence in $\mathcal{N}$. Applying (9) with v the identity map of 1, and recalling that $1 \in \mathcal{D}$, we have $A \in \mathcal{D}$. Thus $\mathcal{C} \subset \mathcal{D}$, as required. □

Remark 16. We have seen in Remark 8 above that large suprema need not exist in $Ref\ A$ for a strongly-complete A, even under extra hypotheses on A. We now observe that, even when a large supremum $C = \bigvee C_i$ does exist, it need not be the strong-limit closure of $\cup C_i$. The counter-example is contained in Example 3.7 of [2]. We take for A the ordinal sum $1 + \infty^{op}$, the dual of the ordered set $\infty + 1$ consisting of all ordinals $\leq \infty$. This A is locally small and admits all limits and colimits, even large ones; it is trivially weakly wellpowered and weakly cowellpowered, although neither wellpowered nor cowellpowered; and it trivially admits a generator and a cogenerator. The reflective subcategories $C_\beta = [\beta^{op}, 0^{op}]$ for $\beta \in \infty$, where β^{op} denotes the ordinal $\beta < \infty$ as an element of ∞^{op}, have as their supremum A itself; but $\cup C_\beta = \infty^{op} \subset A$ is already closed in A under small limits and all intersections of strong monomorphisms.

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Pure Mathematics Department,
University of Sydney,
New South Wales, 2006
Australia.