Trees in Distributive Categories

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Dedicated to Max Kelly in the occasion of his 60th birthday.

1 Introduction

This note presents a brief account of one aspect of the talk given at the conference, and its aim is to relate work done by the authors on a labelled version of Bénabou's *motors* with *distributive categories*, viewed as a conceptual framework (as well as a practical tool) for the foundations of Computer Science (see [Law90, Coc90, Wal89, Wal]).

Motors, introduced by Jean Bénabou in his talk at this conference (and also in lectures held in Rome and Milan [Bén90]) have been shown to provide a deep algebraic insight on the inductive nature of *forests* and also of *labelled trees* [KV91]. Hence they represent a highly valuable tool for manifold applications, specially in Computer Science. We would like to mention, for instance, the algebraic characterization of Milner's *observational equivalence* shown in [DKV].

The relevance and ubiquity of *distributive categories* in many fields of Mathematics has been stressed by Lawvere and Schanuel (see e.g. [Law90]). They promise to become the conceptual, unifying framework for several aspects of Computer Science, as shown, e.g., in [Wal] and [Wal89]. In particular the Theory of Concurrency seems to benefit from the contrast between the "petit" and the "gros" aspects in distributive categories.

In analogy with the classical formula for the free monoid [Mac71] we give a characterization of the initial X-motor in a distributive category $\mathcal{D}$ (where X is an object of $\mathcal{D}$) as a power series in X with Catalan numbers as coefficients. Further, we extend Bénabou's *unique decomposition theorem* [Bén90, KV91] to distributive categories.

We would like to thank Jean Bénabou, who taught us motors and gave us advice and support, and Bob Walters, since the idea behind this work arose in conversations with him, as witnessed (at least) by the flavour of his "apparently illegitimate calculations" [Wal89]. We would like also to thank the referee for his useful suggestions.

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2 X-Motors

Before introducing the general setting in distributive categories, we want to recall briefly the definition of X-motor as given in terms of set with operators.

Definition 2.1 *Given a set X (to be thought of as an alphabet*[1]*), an X-motor is a tuple $\langle M, \oplus, 0, f \rangle$, where $\langle M, \oplus, 0 \rangle$ is a monoid, and*

$$f : X \times M \longrightarrow M$$

is a map.

The definition above generalizes Bénabou's notion of *motor*, which we recover when $X = \{*\}$, so that f becomes an endoarrow of M. If $\alpha, \beta, \gamma, \ldots$ are elements of X, and t is an element of M, we will write $\alpha(x)$ (or even αx) for $f(\alpha, x)$, according to the tradition of Ω-group and R-module theory.

By cartesian closedness, an alternative definition could be given using an X-indexed family of endoarrows of M. The resulting structure (which is basically a one-sorted algebra given by a monoid and by $|X|$ unary operators) is soundly established in Computer Science as the basic tool for the study of *synchronization trees* (see, for instance, [Mil80, Mil89]). Indeed, this line of thought leads to a powerful calculus of trees which is basically shaped as the classical one, but is far more general (see [KV91]). Here, however, we are concerned with distributive categories, so we prefer Definition 2.1.

As for the morphisms between X-motors, let us give the following

Definition 2.2 *A morphism between two X-motors $\langle M, \oplus, 0, f \rangle$ and $\langle M', \oplus', 0', f' \rangle$, is an arrow $\phi : M \longrightarrow M'$ which is a monoid morphism and satisfies $\phi(\sigma t) = \sigma\phi(t)$, for all $t \in M$ and $\sigma \in X$. Denote by $\mathbb{M}_X$ the initial object of X-**Mot**, the category of X-motors.*

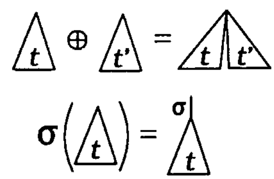
In terms of the alternative definition (see above), the condition $\phi(\sigma t) = \sigma\phi(t)$ simply means that ϕ is *equivariant* with respect to all the endofunctions. This viewpoint is certainly more intuitive, but Definition 2.2 generalizes to distributive categories (see Definition 3.2). Further, the axioms for an X-motor as given in Definition 2.1 are, after all, a weakening of the axioms for a left R-module.

New we want to sketch some of the reasons justifying the use of X-motors to manage trees. Indeed, at first glance one could think that $\mathbb{M}_X$ is "too simple": after all, the initial object of **Mon** (the category of monoids) is the trivial monoid. But recall that $0 \in \mathbb{M}_X$, and that necessarily $\alpha(0) \neq 0$ for all $\alpha \in X$, by the initiality of $\mathbb{M}_X$. Further, still by initiality, we have that $\alpha(0) \oplus \beta(0)$ is a "new" element. Repeating these considerations one sees that $\mathbb{M}_X$ has actually many elements, but perhaps one does not have an intuitive insight of its structure. So we start with some heuristic remarks, which are essentially an adaptation of Bénabou's account of unlabelled forests.

Let us consider an "object" $\mathbb{T}_X$ whose members are to be thought of as trees ordered and labelled on their arcs. This object could have been defined in many different ways, and should also have been equipped with some operations. However, all we need to know

[1] Elsewhere we used Σ to denote the alphabet, according to the tradition of Computer Science. Here, however, Σ will be widely used to denote sums.

is that we can generate a new tree either by *joining the roots* of a pair of trees (preserving the order), or by adding a new labelled arc at the top of a tree. This second operation (left-prefix) can of course be performed with each label. The following picture illustrates these basic operations:



Of course, the "join" operation should be associative, and the "root-only" tree should play the rôle of the identity. So we would end up with a monoid. On the other hand, the "arc-creating" operation would be an X-indexed set of endofunctions, i.e., a map from $X \times M$ to M, so we would get a X-motor. Thus, we would have a unique arrow in X-**Mot**

$$\mathbb{M}_X \xrightarrow{\mathsf{Pict}} \mathbb{T}_X$$

since $\mathbb{M}_X$ is initial.

Now, let us show some of the basic properties of labelled trees. For instance, we might wish to count how many *arcs* of a tree $t \in \mathbb{T}_X$ are labelled by a given label α. This means a map $n_\alpha : \mathbb{T}_X \longrightarrow \mathbb{N}$. But if $t = t_1 \oplus t_2$ then $n_\alpha(t) = n_\alpha(t_1) + n_\alpha(t_2)$, and when one creates a new tree by adding a labelled arc at the top of a tree, $n_\alpha(\alpha(t)) = n_\alpha(t) + 1$, while $n_\alpha(\beta(t)) = n_\alpha(t)$ for every $\beta \neq \alpha$. This means that

$$n : \mathbb{T}_X \longrightarrow < \mathbb{N}, +, 0, s_\alpha >,$$

where $s_\alpha(\alpha, n) = n + 1$ and $s_\alpha(\beta, n) = n$ for all $\beta \neq \alpha$, is an X-motor map. But then, the *unique* map

$$\nu_\alpha : \mathbb{M}_X \longrightarrow < \mathbb{N}, +, 0, s_\alpha >$$

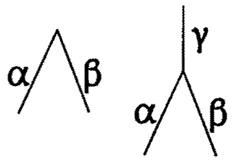
must factor through Pict, since by the initiality of $\mathbb{M}_X$ the diagram

$$\begin{tikzcd}
\mathbb{M}_X \arrow[r, "\mathsf{Pict}"] \arrow[dr, "\nu_\alpha"'] & \mathbb{T}_X \arrow[d, "n_\alpha"] \\
 & \mathbb{N}
\end{tikzcd}$$

necessarily commutes. Hence the slogan: "count the labels in $\mathbb{M}_X$ rather than in $\mathbb{T}_X$".

This process can be carried on again with entirely different X-motors: many other counting maps can be created this way (see [KV91]).

It is now obvious in which sense $\mathbb{M}_X$ is, indeed, the object of X-labelled trees. The operation f creates new labelled arcs, $\oplus$ joins the roots and 0 is the root-only tree. Thus, $\alpha(0) \oplus \beta(0)$ and $\gamma(\alpha(0) \oplus \beta(0))$ represent just:



It is worth observing that we disregard the real nature of $\mathbb{T}_X$. We suggest that most (if not all) of the interesting properties of trees can be studied in $\mathbb{M}_X$, since most (if not all) of them should factor through Pict.

Many interesting theorems can be proved about $\mathbb{M}_X$. The fundamental one is that every element $t \in \mathbb{M}_X$ has a unique decomposition $\sigma_1(x_1) \oplus \sigma_2(x_2) \oplus \cdots \oplus \sigma_n(x_n)$: this property is the link between the completely invariant definition and a more descriptive calculus which can be easily developed (see [KV91]).

3 Distributive categories

There is no established terminology about distributive categories. We give here a few definitions to encompass the cases which are of interest for us.

Definition 3.1 *A category $\mathcal{D}$ is called* distributive *(in the sense of Walters) iff it has finite products and coproducts, and the finite products distribute with respect to the finite coproducts, i.e., the arrow*

$$A \times B + A \times C \longrightarrow A \times (B + C),$$

arising from the two arrows $1_A \times \mu_B$ and $1_A \times \mu_C$, is a natural isomorphism (μ_B and μ_C denote the injections).

Since we want to be able to write power series, we will consider *countably* (resp. *infinitely*) distributive categories, i.e., distributive categories which have countable (resp. infinite) coproducts, and finite products which distribute over them. Further, a (countably, infinitely) distributive category will be said to be *monoidal* if it satisfies Definition 3.1 when the product is weakened to a tensor product.

Extremely important are also distributive categories in the sense of Lawvere-Schanuel. Recall that in their definition every slice category is required to be distributive: the relevance to Computer Science is due to the fact that in order to model labelled transition systems, one needs a category distributive in this sense.

From now on we will often omit the product symbol. Integer indices without bounds are assumed to range over $\mathbb{N}$. When only natural isomorphisms are involved, we will denote them by equality.

3.1 X-Motors in a distributive category

Let us fix a countably distributive category $\mathcal{D}$ (actually, our results hold even in a *monoidal* countably distributive category). We are interested in the most proper lifting of Definition 2.1 to this setting. It turns out that the appropriate definition is the following one:

Definition 3.2 *A motor on the object (data type) X, or simply an X-motor, in a distributive category $\mathcal{D}$ is a tuple $\langle M, \mu, \eta, f\rangle$, where $\langle M, \mu, \eta\rangle$ is a monoid and*

$$f : XM \longrightarrow M$$

is an arrow of $\mathcal{D}$. A morphism between two X-motors $\langle M, \mu, \eta, f\rangle$ and $\langle M', \mu', \eta', f'\rangle$ is an arrow $\psi : M \longrightarrow M'$ in $\mathcal{D}$ such that the following diagrams commute:

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M \arrow[d, "\psi"] \\ & M' \end{tikzcd}$$

$$\begin{tikzcd} M \times M \arrow[r, "\mu"] \arrow[d, "\psi \times \psi"'] & M \arrow[d, "\psi"] \\ M' \times M' \arrow[r, "\mu'"'] & M' \end{tikzcd}$$

$$\begin{tikzcd} XM \arrow[r, "f"] \arrow[d, "\mathbf{1}_X \times \psi"'] & M \arrow[d, "\psi"] \\ XM' \arrow[r, "f'"'] & M' \end{tikzcd}$$

The reason for using in this context the term *data type* can be found in [Wal89]: an immediate naive motivation, however, is that in practical applications trees are "labelled" exactly in the sense that they carry data on each arc.

This definition is completely internal, and allows us to speak of X-motors in $\mathcal{D}$: the main result we will prove is the *existence of the initial X-motor:*

Theorem 3.1 *Given an object X of a distributive category $\mathcal{D}$, the initial X-motor in $\mathcal{D}$ is*

$$M_X = 1 + X + 2X^2 + 5X^3 + 14X^4 + \cdots = \sum_{n\geq 0} C_n X^n$$

together with suitable μ, η and f; C_n is here the n-th Catalan number, i.e.,

$$C_n = \binom{2n}{n}\frac{1}{n+1}$$

Before proving the theorem, we would like to give an insight. Catalan numbers are well-known in combinatorics, because C_n is the number of binary trees with n nodes. However, we began with a heuristic discussion of trees, which are different from binary trees, even if the two concepts are known to be equivalent: indeed, in practical programming trees are always managed as binary trees[2]. This correspondence is important, because (as we will see in the proof) the (apparently) complicated monoid operation of M_X can be really understood only keeping binary trees in mind.

To define μ, η and f we need a simple property of Catalan numbers, namely that if $n > 0$ then

$$C_n = \sum_{0\leq k\leq n-1} C_k C_{n-1-k} = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0.$$

We start by defining $f : XM \longrightarrow M$. Take the nth term of XM after a distribution, XC_nX^n, and consider the $(n+1)$th term of M:

$$C_{n+1}X^{n+1} = C_0C_nX^{n+1} + \cdots + C_nC_0X^{n+1}$$

Since $C_0 = 1$, we just inject XC_nX^n in the last summand, and then in M_X:

$$XC_nX^n = C_nX^{n+1} \xrightarrow{i_n} C_0C_nX^{n+1} + \cdots + C_nC_0X^{n+1} = C_{n+1}X^{n+1} \xrightarrow{i_{n+1}} M_X.$$

The morphism $\mu : M_X \times M_X \longrightarrow M_X$ is defined recursively: the distribution of the product $M_X \times M_X$ gives

$$M_X \times M_X = 1\times 1 + 1\times X + X\times 1 + X\times X + 1\times 2X^2 + \cdots = \sum_{n,m\geq 0} C_nX^n \times C_mX^m$$

and we will denote with $\mu_{n,m}$ the (n,m)th component of μ. Since we want to use induction, we note that $\mu_{n,m}$ will land in $C_{n+m}X^{n+m}$ and eventually in M_X via the suitable injection, that will be understood with a slight abuse of notation.

The map $\mu_{0,0}$ is just the canonical isomorphism $1\times 1 = 1$. Now, let us define $\mu_{k,n-k}$ for each integer $0 \leq k \leq n$, and a fixed n.

If $k = 0$, let $\mu_{0,n} : C_0X^0 \times C_nX^n \longrightarrow C_nX^n$ be the canonical isomorphism. If $k > 1$, given the term $C_kX^k \times C_{n-k}X^{n-k}$, we know that

$$\begin{aligned} &C_kX^k \times C_{n-k}X^{n-k} = \\ &\quad C_0C_{k-1}X^k \times C_{n-k}X^{n-k} + C_1C_{k-2}X^k \times C_{n-k}X^{n-k} + \cdots + C_{k-1}C_0X^k \times C_{n-k}X^{n-k} \end{aligned}$$

so we will break again $\mu_{k,n-k}$ in k maps $\mu^p_{k,n-k}$, $0 \leq p \leq k-1$, from the last sum, and since

$$C_nX^n = \sum_{0\leq r\leq n-1} C_rC_{n-r-1}X^n$$

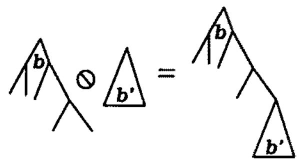
we can simply define $\mu^p_{k,n-k}$ as

$$\begin{aligned} C_pC_{k-p-1}X^k \times C_{n-k}X^{n-k} &= C_pX^{p+1} \times C_{k-p-1}X^{k-p-1} \times C_{n-k}X^{n-k} \\ &\xrightarrow{C_p1_X^{p+1}\times\mu_{k-p-1,n-k}} C_pX^{p+1} \times C_{n-p-1}X^{n-p-1} \\ &= C_pC_{n-p-1}X^n \end{aligned}$$

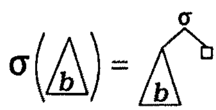
[2]The correspondence between trees and binary trees is described in [Knu73], and has an algebraic formalization which can be found in [KV91].

and being $C_pC_{n-p-1}X^n$ just the pth summand of C_nX^n, we can inject in it.

The intuition behind this definition arises from the identification of trees with labelled arcs with binary trees with labelled nodes, and the corresponding identification of the sum of trees with the ***bottom-right*** product of binary trees [KV91]. The bottom-right product of binary trees means appending the second tree to the extreme right node of the first, as illustrated in the following picture, where it is denoted by $\oslash$:



Now think of C_kX^k as the set of binary trees with k labelled nodes. Our description of μ is a description of the bottom-right product. The "labels" in X^k are "divided" in the first one (the top node), the following p ones (the nodes of the left subtree) and the remaining ones (the nodes of the right subtree). We use identities on the top node and on the left subtree (which, in a sense, remain "unchanged"), and then we take recursively the product of the right subtree with $C_{n-k}X^{n-k}$. Note that binary trees equipped with the bottom-right product and with the map illustrated in the following picture, where $\square$ denotes the empty binary tree:



form an X-motor isomorphic to $\mathbb{M}_X$, as has been shown in [KV91]. This is the real reason why Catalan numbers work. Further, these pictures give an insight on the definition of μ and f.

A very special situation appears if we take $p = k - 1$. The general definition then yields

$$\begin{array}{rcl} C_{k-1}C_0X^k \times C_{n-k}X^{n-k} & = & C_{k-1}X^k \times C_0X^0 \times C_{n-k}X^{n-k} \\ & \xrightarrow{C_{k-1}1_X^k \times \mu_{0,n-k}} & C_{k-1}X^k \times C_{n-k}X^{n-k} \\ & = & C_{k-1}C_{n-k}X^n \end{array}$$

and this is just an isomorphism. Now, $\mu_{k,n-k}$ is the map defined by the $\mu^p_{k,n-k}$'s. Taking all the $\mu_{k,n-k}$'s together (composed with the injections i_n in M_X) we get the desired map $\mu : M_X \times M_X \longrightarrow M_X$. Of course, $\eta : 1 \longrightarrow M_X$, the identity, is simply the injection i_0. It is straightforward to show that μ and η satisfy the monoid axioms.

Proof of Theorem 3.1. All we need to prove is the initiality of $\langle M_X, \mu, \eta, f\rangle$, i.e., that given any X-motor $\langle M', \mu', \eta', f'\rangle$ in the category, there is a unique X-motor morphism $\psi : M_X \longrightarrow M'$. Suppose such a map exists: we will write ψ_k, $k \geq 0$, for the kth component of ψ, and we start by observing that the first condition

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & M_X \arrow[d, "\psi"] \\ & M' \end{tikzcd}$$

defines $\psi_0 : 1 \longrightarrow M'$. We will use f and the monoid structure to extend the map to M_X. We can rewrite the equivariance condition using only maps from the first component of M_X:

$$\begin{tikzcd} X \times 1 = X \arrow[r, "f_0"] \arrow[d, "\mathbf{1}_X \times \eta'"'] & M_X \arrow[d, "\psi"] \\ XM' \arrow[r, "f'"'] & M' \end{tikzcd}$$

and since f_0 is just the injection in the coproduct, this means $\psi_1 = f' \circ (\mathbf{1}_X \times \eta)$.

Consider now the general ψ_n. We decompose again the map in the parts corresponding to the decomposition of $C_n X^n$, and we write ψ_n^k, with $0 \leq k \leq n-1$. Suppose all ψ_m, $m < n$, have been shown to be derived from the first two maps ψ_0, ψ_1: then the equivariance condition, read on the $(n-1)$th term is

$$\begin{tikzcd} X \times C_{n-1}X^{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\mathbf{1}_X \times \psi_{n-1}"'] & M_X \arrow[d, "\psi"] \\ XM' \arrow[r, "f'"'] & M' \end{tikzcd}$$

and since f_{n-1} is the injection in the coproduct in the last term of the decomposition of C_n the diagram can be rewritten as

$$\begin{tikzcd} X \times C_{n-1}X^{n-1} \arrow[r, "="] \arrow[d, "\mathbf{1}_X \times \psi_{n-1}"'] & C_{n-1}C_0X^n \arrow[d, "\psi_n^{n-1}"] \\ XM' \arrow[r, "f'"'] & M' \end{tikzcd}$$

so $\psi_n^{n-1} = f' \circ (\mathbf{1}_X \times \psi_{n-1})$.

On the other hand, take a ψ_n^k, with $0 \leq k \leq n-2$. The preservation of the monoid product diagram, when restricted to the pair $C_{k+1}X^{k+1} \times C_{n-k-1}X^{n-k-1}$ becomes

$$\begin{tikzcd}
C_{k+1}X^{k+1} \times C_{n-k-1}X^{n-k-1} \arrow[r, "\mu_{k+1,n-k-1}"] \arrow[d, "\psi_{k+1} \times \psi_{n-k-1}"'] & M_X \arrow[d, "\psi"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

but we observed that if we decompose C_{k+1} and take just the last component, the map μ becomes an isomorphism (plus an injection), so

$$\begin{tikzcd}
C_kC_0X^{k+1} \times C_{n-k-1}X^{n-k-1} \arrow[r, "\mu^k_{k+1,n-k-1}"] \arrow[d, "\psi^k_{k+1} \times \psi_{n-k-1}"'] & C_kC_{n-k-1}X^n \arrow[d, "\psi^k_n"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

implies that ψ^k_n is defined by ψ^k_{k+1} and ψ_{n-k-1}. Thus, if ψ exists it is unique.

Now, as usual, we define ψ as we did above, using equivariance and preservation of the product, and we show it is an X-motor map. This will be proved by decomposing ψ on the sum.

We start observing that there is a class of diagrams which commute trivially, namely the diagrams with which we defined ψ. The class includes all equivariance diagrams and the product diagrams

$$\begin{tikzcd}
C_pC_{k-p-1}X^k \times C_{n-k}X^{n-k} \arrow[r, "\mu^p_{k,n-k}"] \arrow[d, "\psi^p_k \times \psi_{n-k}"'] & M_X \arrow[d, "\psi"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

with $p = k - 1$. All we have to prove is that these diagrams commute for $0 \leq p \leq k-2$. We will use induction on the sum of the exponents of X in the top left corner of the diagram above, so that the base step is trivial, recalling that $\psi_0 = \eta'$.

Looking at the definition of μ, we see that the previous diagram is actually

$$\begin{tikzcd}
C_pC_{k-p-1}X^k \times C_{n-k}X^{n-k} \arrow[r, "\mu^p_{k,n-k}"] \arrow[d, "\psi^p_k \times \psi_{n-k}"'] & C_pC_{n-p-1}X^n \arrow[d, "\psi^p_n"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

and using again the definitions of the maps involved in the diagram we get

$$\begin{tikzcd}
(C_pX^{p+1} \times C_{k-p-1}X^{k-p-1}) \times C_{n-k}X^{n-k} \arrow[rr, "\equiv"] \arrow[d, "(\psi^p_{p+1} \times \psi_{k-p-1}) \times \psi_{n-k}"'] & & C_pX^{p+1} \times (C_{k-p-1}X^{k-p-1} \times C_{n-k}X^{n-k}) \arrow[d, "C_p\mathbf{1}_X^{p+1} \times \mu_{k-p-1,n-k}"] \\
(M' \times M') \times M' \arrow[d, "\mu' \times \mathbf{1}_M"'] & & C_pX^{p+1} \times C_{n-p-1}X^{n-p-1} \arrow[d, "\psi^p_{p+1} \times \psi_{n-p-1}"] \\
M' \times M' \arrow[r, "\mu'"'] & M' & M' \times M' \arrow[l, "\mu'"]
\end{tikzcd}$$

Now, by induction the following diagram commutes:

$$\begin{tikzcd}
C_{k-p-1}X^{k-p-1} \times C_{n-k}X^{n-k} \arrow[r, "\mu_{k-p-1,n-k}"] \arrow[d, "\psi_{k-p-1} \times \psi_{n-k}"'] & C_{n-p-1}X^{n-p-1} \arrow[d, "\psi_{n-p-1}"] \\
M' \times M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}$$

so we end up with

$$\begin{tikzcd}
(C_pX^{p+1} \times C_{k-p-1}X^{k-p-1}) \times C_{n-k}X^{n-k} \arrow[rr, "\equiv"] \arrow[d, "(\psi^p_{p+1} \times \psi_{k-p-1}) \times \psi_{n-k}"'] & & C_pX^{p+1} \times (C_{k-p-1}X^{k-p-1} \times C_{n-k}X^{n-k}) \arrow[d, "\psi^p_{p+1} \times (\psi_{k-p} \times \psi_{n-k})"] \\
(M' \times M') \times M' \arrow[d, "\mu' \times \mathbf{1}_{M'}"'] & & M' \times (M' \times M') \arrow[d, "\mathbf{1}_{M'} \times \mu'"] \\
M' \times M' \arrow[r, "\mu'"'] & M' & M' \times M' \arrow[l, "\mu'"]
\end{tikzcd}$$

which obviously commutes. □

4 The unique decomposition property

As we pointed out in Section 2, the main property of $\mathbf{M}_X$ was a unique decomposition theorem. This theorem can be also proved in the present setting, in a more general form.

In our hypotheses, there is a left adjoint $(-)^*$ to the forgetful functor $U : \mathbf{Mon}(\mathcal{D}) \longrightarrow \mathcal{D}$, and if ψ is an arrow of $\mathcal{D}$ ending at $U(M)$ for some $M \in \mathbf{Obj}(\mathbf{Mon}(\mathcal{D}))$, we will write $h(\psi)$ for the map in $\mathbf{Mon}(\mathcal{D})$ corresponding to ψ in the adjunction. X-$\mathbf{Mot}(\mathcal{D})$ will denote the category of X-motors in $\mathcal{D}$.

Theorem 4.1 *Let $\mathbb{M}_X = \langle M_X, \mu, \eta, f\rangle$ be the initial X-motor, i.e., the initial object of X-$\mathbf{Mot}(\mathcal{D})$. Consider the functor $\Lambda_X : X$-$\mathbf{Mot}(\mathcal{D}) \longrightarrow X$-$\mathbf{Mot}(\mathcal{D})$, which to each X-motor $\langle M, \mu, \eta, f\rangle$ associates the X-motor given by the monoid $(XM)^* = \sum_n (XM)^n$ endowed with the arrow $f' = j \circ (\mathbf{1}_X \times U(h(f)))$, where j is the injection of generators*

$$XM \xrightarrow{j} (XM)^*,$$

and to each arrow ψ in X-$\mathbf{Mot}(\mathcal{D})$ associates $(\mathbf{1}_X \times U(\psi))^* = \sum_n (\mathbf{1}_X \times U(\psi))^n$. Then,

$$\mathbb{M}_X \cong \Lambda_X(\mathbb{M}_X).$$

Proof. First of all, we have to prove that Λ_X takes effectively morphisms of X-motors to morphisms of X-motors. Given an X-motor morphism

$$\psi : \langle M, \mu, \eta, f\rangle \longrightarrow \langle M', \mu', \eta', g\rangle,$$

$\Lambda_X(\psi)$ is trivially a monoid morphism. Moreover (dropping the U's with an abuse of notation),

$$\begin{aligned}
\Lambda_X(\psi) \circ f' &= \Lambda_X(\psi) \circ j \circ (\mathbf{1}_X \times h(f)) \\
&= j \circ (\mathbf{1}_X \times \psi) \circ (\mathbf{1}_X \times h(f)) \\
&= j \circ (\mathbf{1}_X \times (\psi \circ h(f))) \\
&= j \circ (\mathbf{1}_X \times (h(\psi \circ f))) \\
&= j \circ (\mathbf{1}_X \times (h(g \circ (\mathbf{1}_X \times \psi)))) \\
&= j \circ (\mathbf{1}_X \times (h(g) \circ (\mathbf{1}_X \times \psi)^*)) \\
&= j \circ (\mathbf{1}_X \times h(g)) \circ (\mathbf{1}_X \times \Lambda_X(\psi)) \\
&= g' \circ (\mathbf{1}_X \times \Lambda_X(\psi))
\end{aligned}$$

So that $\Lambda_X(\psi)$ is also an X-motor map. We have of course a unique morphism

$$\mathbb{M}_X \xrightarrow{!} \Lambda_X(\mathbb{M}_X)$$

To build an inverse, take the map $(XM_X)^* \longrightarrow M_X$ which corresponds to f in the adjunction $(-)^* \dashv U$, i.e., $h(f)$. It is an X-motor map, because

$$h(f) \circ f' = h(f) \circ j \circ (\mathbf{1}_X \times h(f)) = f \circ (\mathbf{1}_X \times h(f)).$$

Now, $h(f)\circ ! = \mathbf{1}_{\mathbb{M}_X}$ trivially because $\mathbb{M}_X$ is initial, and we have

$$\begin{aligned}
! \circ h(f) &= h(! \circ f) \\
&= h(f' \circ (\mathbf{1}_X \times !)) \\
&= h(j \circ (\mathbf{1}_X \times h(f)) \circ (\mathbf{1}_X \times !)) \\
&= h(((\mathbf{1}_X \times h(f)) \circ (\mathbf{1}_X \times !))^* \circ j) \\
&= (\mathbf{1}_X \times (h(f) \circ !))^* \\
&= (\mathbf{1}_X \times \mathbf{1}_{\mathbb{M}_X})^* \\
&= \mathbf{1}_{\Lambda_X(\mathbb{M}_X)}
\end{aligned}$$

and this proves $\mathbb{M}_X \cong \Lambda_X(\mathbb{M}_X)$. □

As a final remark, we want to stress that Theorem 4.1 shows in which sense trees are to be thought of as a "recursive closure" of the "labelled list of" operator. $(XM)^*$ can be built in **Sets** as the monoid of lists $\langle x_1, m_1\rangle, \langle x_2, m_2\rangle, \ldots, \langle x_k, m_k\rangle$ where $x_j \in X$ and $m_j \in M$, and thus represents the obvious generalization to distributive categories of the construction "labelled list of". Hence, to say that $\mathbb{M}_X \cong \Lambda_X(\mathbb{M}_X)$ is exactly to say that every X-labelled tree is a finite list of X-labelled trees, each one with an associated label in X (see also [Wal89]).

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