Quaestiones Mathematicae

ISSN: 1607-3606 (Print) 1727-933X (Online) Journal homepage: http://www.tandfonline.com/loi/tqma20

On Categories of Monoids, Comonoids, and Bimonoids

Hans-E. Porst

Quaestiones Mathematicae **31**(2008), 127–139.

ON CATEGORIES OF MONOIDS, COMONOIDS, AND BIMONOIDS

Hans-E. Porst*†

Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria.
E-Mail porst@math.uni-bremen.de

For the sixtieth birthday of my friend and colleague Jiří Adámek

Abstract. The categories of monoids, comonoids and bimonoids over a symmetric monoidal category $\mathbb{C}$ are investigated. It is shown that all of them are locally presentable provided $\mathbb{C}$'s underlying category is. As a consequence numerous functors on and between these categories are shown to be part of an adjoint situation; in particular, the category of comonoids is monoidally closed.

Mathematics Subject Classification (2000): Primary 18D10; Secondary 18A40.

Key words: Monoids, comonoids, bimonoids, free and cofree constructions, monoidally closed categories.

1. Introduction. The interest in coalgebraic structures has risen quite impressively over the last two decades (see e.g. [4]). While this was on the one hand due to certain aspects of computer science and therefore restricted to the set based case, more recently the methods developed there and the module based case, originating from H. Hopf's observation of the existence of a "comultiplication" on certain homology groups, are starting to converge (see e.g. [10]).

It is in this spirit that in the present note we focus explicitely on the notion of monoid over a symmetric monoidal category $\mathbb{C}$ (a concept classical to category theory) and its dual notion of comonoid. Restricting ourselves to those $\mathbb{C}$ whose underlying category is locally presentable as, e.g., the category **Set** of sets or $\mathbf{Mod}_R$ of modules over a commutative ring R, we show that the categories of monoids and comonoids respectively over $\mathbb{C}$ (and even the category of bimonoids) share this property. This not only yields a lot of completeness and cocompleteness results but also, by use of the Special Adjoint Functor Theorem (SAFT), a wealth of adjoint situations in and between the categories occuring here naturally.

In a sequel to this paper we shall – based on the results presented here – investigate corresponding properties of categories of Hopf monoids and Hopf algebras in particular.

*Granting of sabbatical leave from my permanent affiliation, University of Bremen, Germany, is gratefully acknowledged, as is the hospitality of Tshwane University of Technology, Pretoria.

†Permanent address: Department of Mathematics, University of Bremen, 28359 Bremen, Germany.

DOI 10.2989/QM.2008.31.2.2.474

We would like to make the following comment concerning monoids, in particular the existence of free monoids. For the results presented here it would have been sufficient to rely on the classical criterion as given by MacLane (see [8]). Our preference to the functor-algebra approach is mainly due to the attempt to provide a uniform treatment of monoids and comonoids; note however, that this approach also guarantees the existence of free monoids, where the classical one might fail.

2. Monoids and comonoids.

2.1. Admissible monoidal categories. In the sequel we will assume that $\mathbb{C} = (\mathbf{C}, -\otimes-, I, a, l, r, s)$ is a symmetric monoidal category, where $a, (l, r, s)$ denote the natural isomorphisms expressing associativity (left and right unit law, symmetry) and which – except for the symmetry – we will suppress occasionally. We moreover assume that $\mathbf{C}$ is a locally presentable category (though some of the results hold more generally).

Definition. A symmetric monoidal category $\mathbb{C}$ with $\mathbf{C}$ locally presentable will be called *admissible*, provided for each C in $\mathbf{C}$ the functor $C\otimes -$ is finitary (i.e., if $C\otimes -$ preserves directed colimits).

Our main, but not sole, example of an admissible monoidal category will be the category $\mathbf{Mod}_R$ of R-modules with respect to a commutative ring R. Also every locally presentable category is admissible with respect to to binary product $-\times-$.

Remarks.

1. If $\mathbb{C}$ is monoidally closed it is clearly admissible (but not conversely – see 3.3).
2. If $\mathbb{C}$ is admissible, for each $n \in \mathbb{N}$, the "n^{th} tensor power functor"
$$T^n\colon \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & \otimes^n C \end{array}$$
is finitary (see [9]).
3. If $\mathbb{C}$ is admissible we can form (by our (co)completeness assumptions) the functors
$$T_+\colon \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & (C\otimes C)+I \end{array} \quad\text{and}\quad T_\times\colon \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & (C\otimes C)\times I \end{array}$$
and these are finitary, too; for T_+ this is obvious, for $T_\times$ it follows from 2. and local presentability of $\mathbf{C}$.
4. A morphism $T_+C \to C$ is usually denoted by $[m, e]$ with $m\colon C\otimes C \to C$ and $e\colon I \to C$ its components; analogously, a morphism $C \to T_\times C$ with coordinates $\mu\colon C \to C\otimes C$ and $\epsilon\colon C \to I$ will be denoted by $\langle \mu, \epsilon\rangle$.

2.2. The categories Mon$\mathbb{C}$ and Comon$\mathbb{C}$. The category **Mon**$\mathbb{C}$ of monoids over $\mathbb{C}$ then is defined as usual: its objects are triples $(C, C\otimes C \xrightarrow{m} C, I \xrightarrow{e} C)$ such that the diagrams

$$\begin{tikzcd} C\otimes C\otimes C \arrow[r, "m\otimes 1_C"] \arrow[d, "1_C\otimes m"'] & C\otimes C \arrow[d, "m"] \\ C\otimes C \arrow[r, "m"'] & C \end{tikzcd} \qquad \begin{tikzcd} C\otimes I \arrow[r, "1_C\otimes e"] \arrow[dr, Rightarrow, "r_C"'] & C\otimes C \arrow[d, "m"] & I\otimes C \arrow[l, "e\otimes 1_C"'] \arrow[dl, Rightarrow, "l_C"] \\ & C & \end{tikzcd}$$

commute. A monoid homomorphism $(C, m, e) \longrightarrow (C', m', e')$ then is any $f\colon C \to C'$ making the diagrams

$$\begin{tikzcd} C\otimes C \arrow[r, "m"] \arrow[d, "f\otimes f"'] & C \arrow[d, "f"] \\ C'\otimes C' \arrow[r, "m'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "e'"'] & C \arrow[d, "f"] \\ & C' \end{tikzcd}$$

commutative.

The category **Comon**$\mathbb{C}$ of comonoids over $\mathbb{C}$ is defined to be $(\mathbf{Mon}\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$, i.e., its objects are triples $(C, C \xrightarrow{\mu} C\otimes C,\ C \xrightarrow{\epsilon} I)$ such that the diagrams

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\mu"'] & C\otimes C \arrow[d, "1_C\otimes\mu"] \\ C\otimes C \arrow[r, "\mu\otimes 1_C"'] & C\otimes C\otimes C \end{tikzcd} \qquad \begin{tikzcd} C\otimes I & C\otimes C \arrow[l, "1_C\otimes\epsilon"'] \arrow[r, "\epsilon\otimes 1_C"] & I\otimes C \\ & C \arrow[ul, Rightarrow, "r_C^{-1}"] \arrow[u, "\mu"] \arrow[ur, Rightarrow, "l_C^{-1}"'] & \end{tikzcd}$$

commute, while a comonoid homomorphism $(C, \mu, \epsilon) \longrightarrow (C', \mu', \epsilon')$ is any $f\colon C \to C'$ making the diagrams

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "f"'] & C\otimes C \arrow[d, "f\otimes f"] \\ C' \arrow[r, "\mu'"'] & C'\otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "f"'] & I \\ C' \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

commute.

In particular, $\mathbf{MonMod}_R$ equals $\mathbf{Alg}_R$, the categories of R-algebras, while $\mathbf{ComonMod}_R =: \mathbf{Coalg}_R$ is called the *category of R-coalgebras.*

A monoid (C, m, e) is called *commutative* iff $m = m\circ s_C$ with $s_C\colon C\otimes C \to C\otimes C$ the symmetry; dually, a comonoid (C, μ, ϵ) is called *cocommutative*, provided that $\mu = s_C \circ \mu$. By ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$, respectively, we denote the categories of commutative monoids (cocommutative comonoids) with all (co)monoid homomorphisms. One has ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C} = ({}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}^{\mathsf{op}})^{\mathsf{op}}$.

For later use we note

Lemma. *The symmetry s of $\mathbb{C}$ induces functorial isomorphisms*

$$\begin{array}{rccc} (-)^{\mathsf{op}}\colon & \mathbf{Mon}\mathbb{C} & \longrightarrow & \mathbf{Mon}\mathbb{C} \\ & (C,m,e) & \longmapsto & (C, m\circ s_C, e) \end{array}$$

and

$$\begin{array}{rccc} (-)^{\mathsf{op}}\colon & \mathbf{Comon}\mathbb{C} & \longrightarrow & \mathbf{Comon}\mathbb{C} \\ & (C,\mu,\epsilon) & \longmapsto & (C, s_C\circ \mu, \epsilon) \end{array}$$

respectively.

2.3. Lifting adjunctions. It is well known (see e.g. [12]) that every monoidal functor $F\colon \mathbb{C} \to \mathbb{C}'$ induces a functor $\hat{F}\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$ by $\hat{F}(C,m,e) = (FC, Fm, Fe)$.

Lemma. *Let the monoidal functor $F\colon \mathbb{C} \to \mathbb{C}'$ have a right adjoint G with counit $\varepsilon\colon FG \to 1_{\mathbb{C}'}$. Then $\hat{F}$ has a right adjoint $\tilde{G}$ with counit $\tilde{\varepsilon}$ such that*

- *the following diagram commutes*

$$\begin{tikzcd} \mathbf{Mon}\mathbb{C}' \arrow[r, "\tilde{G}"] \arrow[d, "|-|'"'] & \mathbf{Mon}\mathbb{C} \arrow[d, "|-|"] \\ \mathbb{C}' \arrow[r, "G"'] & \mathbb{C} \end{tikzcd}$$

  and

- *$|\tilde{\varepsilon}_{(C,m,e)}|' = \varepsilon_C$ holds for each (C,m,e) in $\mathbf{Mon}\mathbb{C}'$.*

Proof. Given a monoid (C,m,e) over $\mathbb{C}'$ define a monoid $\tilde{G}(C,m,e) = (GC, \bar{m}, \bar{e})$ by commutativity of the following diagrams:

$$\begin{tikzcd} FGC\otimes FGC \arrow[r, "\sim"] \arrow[d, "\varepsilon_C\otimes\varepsilon_C"'] & F(GC\otimes GC) \arrow[r, "F\bar{m}"] & FGC \arrow[d, "\varepsilon_C"] \\ C\otimes C \arrow[rr, "m"'] & & C \end{tikzcd} \qquad \begin{tikzcd} J \arrow[r, "\sim"] \arrow[drr, "e"'] & FI \arrow[r, "F\bar{e}"] & FGC \arrow[d, "\varepsilon_C"] \\ & & C \end{tikzcd}$$

Then $\varepsilon_C\colon \hat{F}\tilde{G}(C,m,e) \to (C,m,e)$ is a monoid homomorphism. If now $f\colon \hat{F}(C',m',e') = (FC', Fm', Fe') \to (C,m,e)$ is any monoid homomorphism then the $\mathbb{C}'$-morphism $f^\sharp\colon C' \to GC$ corresponding to f by adjunction is easily seen to be a monoid homomorphism $(C',m',e') \to \tilde{G}(C,m,e)$ which proves that ε_C indeed is $\hat{F}$-couniversal for (C,m,e). $\square$

2.4. The cartesian case. Clearly, **MonSet** is just the category **Monoids** of ordinary monoids (when **Set** is considered a monoidal category by binary products), and, somewhat more generally: if $\mathbb{C}$ is monoidal with $- \otimes - = - \times -$, the binary product, and $I = 1$, the terminal object, then $\mathbf{Mon}\mathbb{C}$ is the category of monoid objects in $\mathbf{C}$. We will call this the *cartesian case.* Forming comonoids in the cartesian case seems to be uninteresting at this stage: each $\mathbf{C}$-object C carries precisely one cartesian comonoid structure: $(C, \Delta, !)$ with $\Delta \colon C \longrightarrow C \times C$ the diagonal and $! \colon C \longrightarrow 1$ the unique morphism. Thus, in this case, $\mathbf{C} \simeq \mathbf{Comon}(\mathbf{C}, - \times -, 1)$.

2.5. Functor algebras and -coalgebras. Recall that, given an endofunctor $F \colon \mathbf{K} \longrightarrow \mathbf{K}$ on some category $\mathbf{K}$ the category $\mathbf{Alg}F$ of F-algebras has objects $(K, FK \xrightarrow{\alpha} K)$ with K, α in $\mathbf{K}$ and morphisms $(K, \alpha) \longrightarrow (K', \alpha')$ those $\mathbf{K}$-morphisms $f \colon K \longrightarrow K'$ making the diagram

$$\begin{tikzcd}
FK \arrow[r, "\alpha"] \arrow[d, "Ff"] & K \arrow[d, "f"] \\
FK' \arrow[r, "\alpha'"'] & K'
\end{tikzcd}$$

commute. The category $\mathbf{Coalg}F$ of F-coalgebras is $(\mathbf{Alg}F^{\mathsf{op}})^{\mathsf{op}}$, i.e., it has objects $(K, K \xrightarrow{\alpha} FK)$ with α, K in $\mathbf{K}$ and morphisms $f \colon (K, \alpha) \longrightarrow (K', \alpha')$ those $f \colon K \longrightarrow K'$ such that

$$\begin{tikzcd}
K \arrow[r, "\alpha"] \arrow[d, "f"'] & FK \arrow[d, "Ff"] \\
K' \arrow[r, "\alpha'"'] & FK'
\end{tikzcd}$$

commutes. $\mathbf{Alg}F$ and $\mathbf{Coalg}F$ thus are concrete categories over $\mathbf{K}$ with obvious forgetful functors.

We recall the following facts about these categories.

Facts. (See [1] and [2])

1. The underlying functor $\mathbf{Alg}F \to \mathbf{K}$ creates limits and those colimits which are preserved by F; consequently, it is monadic as soon as it has a left adjoint.

2. If $\mathbf{K}$ is cocomplete and F preserves directed colimits, then $\mathbf{Alg}F \to \mathbf{K}$ has a left adjoint.

3. The underlying functor $\mathbf{Coalg}F \to \mathbf{K}$ creates colimits and those limits which are preserved by F; it is comonadic as soon as it has a right adjoint. (This is just the dual of 1. above.)

4. If $\mathbf{K}$ is a locally presentable category and F preserves directed colimits, then $\mathbf{Coalg}F \longrightarrow \mathbf{K}$ has a right adjoint.

5. If $\mathbf{K}$ is locally presentable and F preserves directed colimits the categories $\mathbf{Alg}F$ and $\mathbf{Coalg}F$ are accessible.

REMARKS. Let $\mathbb{C}$ be an admissible monoidal category. By the previous facts (and the identifications of (C, m, e) and $(C, [m, e])$, and, respectively (C, μ, ϵ) and $(C, \langle \mu, \epsilon \rangle)$) we obtain:

1. The category $\mathbf{Alg}T_+$ is finitary monadic over $\mathbf{C}$ and contains $\mathbf{Mon}\mathbb{C}$ as a full subcategory.
2. The category $\mathbf{Coalg}T_\times$ is comonadic over $\mathbf{C}$ and contains $\mathbf{Comon}\mathbb{C}$ as a full subcategory.

2.6. Closure properties. Investigating $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ as subcategories of $\mathbf{Alg}T_+$ and $\mathbf{Coalg}T_\times$, respectively, it is important to get further information on these embeddings.

PROPOSITION.

1. $\mathbf{Mon}\mathbb{C}$ *is closed in* $\mathbf{Alg}T_+$ *with respect to the formation of limits and of directed and absolute colimits.*
2. $\mathbf{Comon}\mathbb{C}$ *is closed in* $\mathbf{Coalg}T_\times$ *with respect to colimits and absolute limits.*

Proof. Assume $\big((C, m, e) \xrightarrow{\pi_i} (C_i, m_i, e_i)\big)_I$ is a limit in $\mathbf{Alg}T_+$ with all (C_i, m_i, e_i) satisfying the associativity axioms. By a simple diagram chase one obtains (in $\mathbf{C}$)

$$\forall i \in I \quad \pi_i \circ m \circ (m \otimes 1_C) = \pi_i \circ m \circ (1_C \otimes m).$$

Since $\mathbf{Alg}T_+ \longrightarrow \mathbf{C}$ preserves limits, the π_i are jointly cancellable. Thus (C, m, e) is associative.

Let now $\big((C_i, m_i, e_i) \xrightarrow{d_i} (C, m, e)\big)_i$ be a directed colimit in $\mathbf{Alg}T_+$ with all (C_i, m_i, e_i) satisfying the associativity axiom. Now a simple diagram chase gives (again in $\mathbf{C}$)

$$\forall i \in I \quad m \circ (m \otimes 1_C) \circ (d_i \otimes d_i \otimes d_i) = m \circ (1_C \otimes m) \circ (d_i \otimes d_i \otimes d_i).$$

Since T^3 is finitary and $\mathbf{Alg}T_+ \longrightarrow \mathbf{K}$ preserves directed colimits we can jointly cancel the $d_i \otimes d_i \otimes d_i$. The argument concerning absolute colimits is the same.

Analogous arguments show satisfaction of the unitary laws for (C, m, e).

Statement 2. is dual to the first part of 1. $\square$

COROLLARY. *Let* $\mathbb{C}$ *be an admissible monoidal category. Then*

1. $\mathbf{Mon}\mathbb{C}$ *is finitary monadic over* $\mathbf{C}$.
2. $\mathbf{Mon}\mathbb{C}$ *is a locally* λ*-presentable category provided* $\mathbf{C}$ *is.*

Proof. Since the Eilenberg-Moore category of a finitary monad on a locally λ-presentable category is locally λ-presentable again, 2. follows from 1.

Since $\mathbf{Alg}T_+$ is a locally finitely presentable category (see 2.5) it follows from the previous proposition (by the reflection theorem for accessible categories [2, 2.48]) that $\mathbf{Mon}\mathbb{C}$ is reflective in $\mathbf{Alg}T_+$. Consequently, the forgetful functor $\mathbf{Mon}\mathbb{C} \longrightarrow \mathbf{C}$ is finitary and has a left adjoint. Monadicity of $\mathbf{Alg}T_+$ and closure of $\mathbf{Mon}\mathbb{C}$ under absolute coequalizers now proves monadicity (using the Beck-Paré-Theorem). $\square$

Remarks. Using essentially the same arguments one obtains

1. ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Mon}\mathbb{C}$ under limits, directed colimits and absolute colimits.

   ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ is reflective in $\mathbf{Mon}\mathbb{C}$ and finitary monadic over $\mathbf{C}$. ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ is a locally presentable category.

2. ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ is closed in $\mathbf{Comon}\mathbb{C}$ under colimits and absolute limits.

2.7. Comon$\mathbb{C}$ as an equifier. Since the arguments used above with respect to $\mathbf{Mon}\mathbb{C}$ cannot be dualized (the dual of a locally presentable category is – nearly always – not locally presentable) we need to use a different approach.

Definition. ([2, 2.76]) Let $F_1^t, F_2^t\colon \mathbf{K} \longrightarrow \mathbf{L}_t$ $(t \in T)$ be a family of functors and, for each $t \in T$,

$$\varphi^t, \psi^t\colon F_1^t \longrightarrow F_2^t$$

be a pair of natural transformations. Then the full subcategory of $\mathbf{K}$ spanned by those objects K which satisfy $\varphi_K^t = \psi_K^t$ for all $t \in T$ is called the *equifier* $\mathbf{Eq}\big((\varphi^t, \psi^t)_T\big)$ of the above family of pairs of natural transformations.

Fact. (see [2]) With notation as above the category $\mathbf{Eq}\big((\varphi^t, \psi^t)_T\big)$ is an accessible category provided that all functors F_1^t, F_2^t are accessible functors.

Now consider the category of $T_\times$-coalgebras with its underlying functor $|-|\colon \mathbf{Coalg}T_\times \longrightarrow \mathbf{C}$. Define natural transformations as follows:

1. $\varphi^1, \psi^1\colon |-| \longrightarrow T^3 \circ |-|$

   $$\varphi^1_{(C,[\mu,\epsilon])} := (\mu \otimes 1_C) \circ \mu, \qquad \psi^1_{(C,[\mu,\epsilon])} := (1_C \otimes \mu) \circ \mu$$

2. $\varphi^2, \psi^2\colon |-| \longrightarrow |-| \otimes I$

   $$\varphi^2_{(C,[\mu,\epsilon])} := (1_C \otimes \epsilon) \circ \mu, \qquad \psi^2_{(C,[\mu,\epsilon])} := r_C^{-1}$$

3. $\varphi^3, \psi^3\colon |-| \longrightarrow I \otimes |-|$

   $$\varphi^3_{(C,[\mu,\epsilon])} := (\epsilon \otimes 1_C) \circ \mu, \qquad \psi^3_{(C,[\mu,\epsilon])} := l_C^{-1}$$

Then, obviously,

$$\mathbf{Comon}\mathbb{C} = \mathbf{Eq}\big((\varphi^t, \psi^t)_{t=1,2,3}\big).$$

Since the domain and the codomains of the above natural transformations clearly are accessible we obtain

PROPOSITION. *Let $\mathbb{C}$ be an admissible monoidal category. Then*

1. $\mathbf{Comon}\mathbb{C}$ *is a locally presentable category.*
2. $\mathbf{Comon}\mathbb{C}$ *is comonadic over C.*

Proof. $\mathbf{Comon}\mathbb{C}$ is accessible by the above. $\mathbf{Coalg}T_\times$ is cocomplete, since $\mathbf{C}$ is, and $\mathbf{Comon}\mathbb{C}$ is closed in $\mathbf{Coalg}T_\times$ under colimits. But a cocomplete accessible category is locally presentable ([2, 2.47]).

To prove 2. we only need to show that the underlying functor $\mathbf{Coalg}T_\times \longrightarrow \mathbf{C}$ has a right adjoint. Since this functor preserves all colimits the special Adjoint Functor Theorem applies (its assumptions are fulfilled since $\mathbf{Comon}\mathbb{C}$ is locally presentable). □

REMARKS.

1. The same arguments show that ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ is locally presentable and comonadic over $\mathbf{C}$. Since its embedding into $\mathbf{Comon}\mathbb{C}$ preserves colimits, it is a coreflective subcategory (again by the SAFT).
2. Clearly, by duality, also $\mathbf{Mon}\mathbb{C}$ can be described as an equifier. Our approach used for $\mathbf{Mon}\mathbb{C}$, however, allowed for more: we got an additional result concerning the degree of local presentability.

3. Lifting admissibility.

3.1. Monoidal structures for monoids and comonoids. It is well known how to lift the tensor product from $\mathbb{C}$ to $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$, respectively: one simply defines

- $(C_1, m_1, e_1) \otimes (C_2, m_2, e_2) \colon = (C_1 \otimes C_2, m, e)$ with $m =$

  $$(C_1 \otimes C_2) \otimes (C_1 \otimes C_2) \xrightarrow{C_1 \otimes s \otimes C_2} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow{m_1 \otimes m_2} C_1 \otimes C_2$$

  and $e = I \simeq I \otimes I \xrightarrow{e_1 \otimes e_2} C_1 \otimes C_2$,

- $(C_1, \mu_1, \epsilon_1) \otimes (C_2, \mu_2, \epsilon_2) \colon = (C_1 \otimes C_2, \mu, \epsilon)$ with $\mu =$

  $$C_1 \otimes C_2 \xrightarrow{\mu_1 \otimes \mu_2} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow{C_1 \otimes s \otimes C_2} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2)$$

  and $\epsilon = C_1 \otimes C_2 \xrightarrow{\epsilon_1 \otimes \epsilon_1} I \otimes I \simeq I$.

By these constructions and I made into a monoid and comonoid respectively in the obvious way $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ become symmetric monoidal categories again (see [7]). We will always consider $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ as endowed with these monoidal structures. Note that the tensor product of two (co)commutative (co)monoids is a (co)commutative (co)monoid again and that, moreover, the following holds:

Fact. With the monoidal structures defined above ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ are symmetric monoidal categories and, moreover,

1. $(C_1, m_1, e_1) \xrightarrow{\iota_1} (C_1, m_1, e_1) \otimes (C_2, m_2, e_2) \xleftarrow{\iota_2} (C_2, m_2, e_2)$
   with $\iota_2 = C_1 \simeq C_1 \otimes I \xrightarrow{1_{C_1} \otimes e_2} C_1 \otimes C_2$
   and $\iota_2 = C_2 \simeq I \otimes C_2 \xrightarrow{e_1 \otimes 1_{C_2}} C_1 \otimes C_2$ is a coproduct in ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$.

2. $(C_1, \mu_1, \epsilon_1) \xleftarrow{\pi_1} (C_1, \mu_1, \epsilon_1) \otimes (C_2, \mu_2, \epsilon_2) \xrightarrow{\pi_2} (C_2, \mu_2, \epsilon_2)$
   with $\pi_1 = C_1 \otimes C_2 \xrightarrow{1_{C_1} \otimes \epsilon_2} C_1 \otimes I \simeq C_1$
   and $\pi_2 = C_1 \otimes C_2 \xrightarrow{\epsilon_1 \otimes 1_{C_2}} I \otimes C_2 \simeq C_2$ is a product in ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$.

3.2. The case of comonoids. It has been observed by Barr [3] that ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ is cartesian closed. Using the idea of Barr's proof together with the results of 2.7 we obtain

Proposition. *If $\mathbb{C}$ is an admissible monoidal category, then $\mathbf{Comon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ are admissible. If $\mathbb{C}$ is even monoidally closed, then so are $\mathbf{Comon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$, where ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$ is then in fact cartesian closed.*

Proof. Each functor $(C, \mu, \epsilon) \otimes -$ on $\mathbf{Comon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$, respectively, preserves those colimits which are preserved by $C \otimes -$ on $\mathbf{C}$, since the underlying functors of the categories of comonoids create colimits. Since these categories are locally presentable, the Special Adjoint Functor Theorem applies. □

3.3. The case of monoids. It is clear that for monoids we cannot expect the same to hold; for example, the category of commutative R-algebras, i.e., ${}_{\mathbf{c}}\mathbf{MonMod}_R$ is not a cocartesian closed category. We do get, however, the following result:

Proposition. *Let $\mathbb{C}$ be an admissible monoidal category, then $\mathbf{Mon}\mathbb{C}$ and ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ are admissible.*

Proof. Since the underlying functors of the categories of (commutative) monoids create directed colimits the arguments of the proof above apply here as well. □

4. Bimonoids. Having seen that both of the categories, $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$, are again suitable to build categories of monoids and comonoids over them one should have the following fact in mind:

FACT. (ECKMANN-HILTON PRINCIPLE). The category of monoids over $\mathbf{Mon}\mathbb{C}$ is isomorphic to $_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ (see e.g. [6]). By duality, the category of comonoids over $\mathbf{Comon}\mathbb{C}$ is isomorphic to $_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$.

Thus only the constructions $\mathbf{MonComon}\mathbb{C}$ and $\mathbf{ComonMon}\mathbb{C}$ can be of interest.

4.1. The category $\mathbf{Bimon}\mathbb{C}$. The following is well known (see e.g.[12]):

FACT. For any symmetric monoidal category $\mathbb{C}$ and data

- $C \in \mathbf{C}^{obj}$
- $m\colon C \otimes C,\ e\colon I \longrightarrow C \in \mathbf{C}^{mor}$
- $\mu\colon C \longrightarrow C \otimes C,\ \epsilon\colon C \longrightarrow I \in \mathbf{C}^{mor}$

such that

a. $(C, m, e) \in \mathbf{Mon}\mathbb{C}^{obj}$

b. $(C, \mu, \epsilon) \in \mathbf{Comon}\mathbb{C}^{obj}$

the following conditions are equivalent:

i) $\mu\colon (C, m, e) \longrightarrow (C, m, e) \otimes (C, m, e)$ and
$\epsilon\colon (C, m, e) \longrightarrow I$ are monoid homomorphisms.

ii) $m\colon (C, \mu, \epsilon) \otimes (C, \mu, \epsilon) \longrightarrow (C, \mu, \epsilon)$ and
$e\colon I \longrightarrow (C, \mu, \epsilon)$ are comonoid homomorphisms.

Quintuples $\mathcal{C} = (C, m, e, \mu, \epsilon)$ satisfying the above conditions then are called *bimonoids* in $\mathbb{C}$. These form a category $\mathbf{Bimon}\mathbb{C}$ when using as morphisms $f\colon \mathcal{C} = (C, m, e, \mu, \epsilon) \to (C', m', e', \mu', \epsilon') = \mathcal{C}'$ those $\mathbf{C}$-morphisms $f\colon C \to C$ which simultaneously are both, a monoid homomorphism $(C, m, e) \to (C', m', e')$ and a comonoid homomorphism $(C, \mu, \epsilon) \to (C', \mu', \epsilon')$.

By the definition of the monoidal structures on $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$, respectively, condition ii) above means that (C, m, e) is not only a monoid in $\mathbb{C}$ as required by a., but even a monoid in $\mathbf{Comon}\mathbb{C}$, while i) says that (C, μ, ϵ) is even a comonoid in $\mathbf{Mon}\mathbb{C}$. We thus observe

LEMMA. *Up to concrete isomorphism of categories one has*

$$\mathbf{MonComon}\mathbb{C} = \mathbf{Bimon}\mathbb{C} = \mathbf{ComonMon}\mathbb{C}.$$

From this and the results of 2.6, 2.7 and 3.2 we obtain immediately

Proposition. *For any admissible monoidal category* $\mathbb{C}$ *the category* $\mathbf{Bimon}\mathbb{C}$ *is locally presentable. It is finitary monadic over* $\mathbf{Comon}\mathbb{C}$ *and comonadic over* $\mathbf{Mon}\mathbb{C}$.

Remark. Note that the adjoints of $\mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ and $\mathbf{Bimon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ can, alternatively, be constructed as lifts of the adjoints of $\mathbf{Mon}\mathbb{C} \to \mathbb{C}$ and $\mathbf{Comon}\mathbb{C} \to \mathbb{C}$ respectively using Lemma 2.3.

In the sequel we occasionally will use the following notation concerning bimonoids: If $\mathcal{C} = (C, m, e, \mu, \epsilon)$ is bimonoid, $\mathcal{C}^m \colon = (C, m, e)$ will denote its monoid part and ${}^c\mathcal{C} = (C, \mu, \epsilon)$ its comonoid part. We then will write $\mathcal{C} = ({}^c\mathcal{C}, \mathcal{C}^m)$. In case of $\mathbb{C} = \mathbf{Mod}_R$ we might use $\mathcal{C}^a$ instead of $\mathcal{C}^m$. Also $\mathbf{Bialg}_R \colon = \mathbf{BimonMod}_R$. Given a bimonoid $\mathcal{C} = ({}^c\mathcal{C}, \mathcal{C}^m)$ then so are

- $\mathcal{C}^{\mathsf{cop}} \colon = \big(({}^c\mathcal{C})^{\mathsf{op}}, \mathcal{C}^m\big)$.
- $\mathcal{C}^{\mathsf{mop}} \colon = \big({}^c\mathcal{C}), (\mathcal{C}^m)^{\mathsf{op}}\big)$.
- $\mathcal{C}^{\mathsf{op}} \colon = \big(({}^c\mathcal{C})^{\mathsf{op}}, (\mathcal{C}^m)^{\mathsf{op}}\big)$.

This defines functorial isomorphisms

$$(-)^{\mathsf{cop}}, (-)^{\mathsf{mop}}, (-)^{\mathsf{op}} \colon \mathbf{Bimon}\mathbb{C} \longrightarrow \mathbf{Bimon}\mathbb{C}.$$

4.2. Commutative and cocommutative bimonoids. A bimonoid is called *commutative* and *cocommutative* respectively, provided its underlying monoid is commutative or its underlying comonoid is cocommutative. By ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C}$ respectively we denote the full subcategories of $\mathbf{Bimon}\mathbb{C}$ spanned by the commutative or cocommutative bimonoids. Then ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C} = \mathbf{Comon}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C} = \mathbf{Mon}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$; the latter is even the category of cartesian monoids in ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$.

Again from previous results we obtain most of the following

Proposition.

1. ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ *is a locally presentable category. It is comonadic over* ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C}$. ${}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C}$ *is a reflexive subcategory of* $\mathbf{Bimon}\mathbb{C}$ *closed with respect to directed colimits, and, consequently monadic over* $\mathbf{Comon}\mathbb{C}$.
2. ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C}$ *is a locally presentable category. It is finitary monadic over* ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C}$. ${}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C}$ *is a coreflexive subcategory of* $\mathbf{Bimon}\mathbb{C}$ *and, consequently, comonadic over* $\mathbf{Mon}\mathbb{C}$.

Proof. Only the (co)reflexivity statements still need proofs. These however follow from (the dual of) Lemma 2.3 applied to the embeddings ${}_{\mathbf{c}}\mathbf{Mon}\mathbb{C} \hookrightarrow \mathbf{Mon}\mathbb{C}$ and ${}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C} \hookrightarrow \mathbf{Comon}\mathbb{C}$ (use Remark 1. in 2.6 and 2.7). $\square$

Note that the proof above in fact even shows the following

Facts.

1. The cofree bimonoid over a commutative monoid $\mathcal{C}$ is commutative and, thus, the cofree commutative bimonoid over $\mathcal{C}$.

2. The free bimonoid over a cocommutative comonoid $\mathcal{C}$ is cocommutative and, thus, the free cocommutative bimonoid over $\mathcal{C}$.

which, however, could also have been obtained without resorting to the lifting argument of 2.3 by rather applying the following simple observation to the appropriate functorial isomorphisms taking opposites.

Lemma. *Let*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "I"] \arrow[d, "U"'] & \mathbf{C} \arrow[d, "U"] \\ \mathbf{K} \arrow[r, "J"'] & \mathbf{K} \end{tikzcd}$$

be a commutative diagram with equivalences I and J. Let $\mathbf{Fix}I$ and $\mathbf{Fix}J$ denote the full subcategories of $\mathbf{C}$ and $\mathbf{K}$ spanned by the objects fixed by I and J respectively.

If U maps $\mathbf{Fix}I$ into $\mathbf{Fix}J$, then so does any adjoint of U.

4.3. Summary. We might summarize our results thus far by the following diagram in which

- all categories are locally presentable,
- all arrows labelled fm are finitary monadic functors,
- all arrows labelled cm are comonadic functors,
- all hooked arrows are accessible embeddings with r denoting a reflective and c a coreflective one.

$$\begin{tikzcd}
 & & \mathbf{Bimon}\mathbb{C} \arrow[dll, bend right, "fm"'] \arrow[drr, bend left, "cm"] & & \\
 & {}_{\mathbf{coc}}\mathbf{Bimon}\mathbb{C} \arrow[ur, hook, "c"'] \arrow[d, "fm"] & & {}_{\mathbf{c}}\mathbf{Bimon}\mathbb{C} \arrow[ul, hook', "r"] \arrow[d, "cm"'] & \\
\mathbf{Comon}\mathbb{C} \arrow[drr, "cm"'] & {}_{\mathbf{coc}}\mathbf{Comon}\mathbb{C} \arrow[l, hook', "c"'] \arrow[dr, "cm"] & & {}_{\mathbf{c}}\mathbf{Mon}\mathbb{C} \arrow[r, hook, "r"] \arrow[dl, "fm"'] & \mathbf{Mon}\mathbb{C} \arrow[dll, "fm"] \\
 & & \mathbf{C} & &
\end{tikzcd}$$

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Received 17 January, 2008.