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Spectra of monoidal-lattices

SHU-HAO SUN*

0 Introduction

For well over 50 years now (starting with Ward and Dilworth [20]) there has been a steady development of the theory of 'ideal lattices'. This is the theory of certain lattices L which carry a multiplication. The motivating example is the ideal lattice Id R of a ring R. The idea is to prove various ideal theoretic results in this multiplicative lattice context, i.e., without reference to ring elements. In this direction, a very general notion, called m-lattice, was introduced (see Birkhoff [3]). This is nothing but a join-semilattice together with a multiplication which distributes over finite joins. It was expected that the ideal lattice of a ring might be described by the ideal lattice of an m-lattice but the latter does not seem to be well developed yet.

In this paper, we shall study the theory of ideals of m-lattices. Clearly, this theory extends those of the bound distributive lattices and of the commutative rings (more generally, of the neo-commutative rings in the sense of Kaplansky [9] or m-rings in the sense of [19]).

The main results contained in this paper are:

(a) For those m-lattices L whose top element is also a multiplication identity, (such m-lattices are also called monoidal-lattices since they are monoidal categories), we establish a separation lemma which is a common generalization of the classical separation lemmas of Krull for rings and of Stone for distributive lattices. Furthermore, we show that the properties of Id L are similar to the ideal lattices for both the distributive lattices and rings. Under this circumstance, we discuss extensively their spectra, maximal spectra and minimal spectra. Note that our results not only generalize the corresponding one for distributive lattices, but also have some ring-interpretations.

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(b) For general m-lattices, one can't get a similar kind of separation lemma which guarantees the existence of (m-)prime ideals. We will illustrate with an example of an m-lattice with an identity, which does not have any (m-)prime ideals. In other words, our setting to establish spectra theory is, to some extent, the most general. However, a full discussion of it from the categorical point of view as well as the relationship with module theory over locales can be referred to the author's forthcoming paper.

Because both the spectra theories of commutative rings and of distributive lattices depend only on the Prime Ideal Theorem (PIT) rather than the full axiom of Choice (AC) – it is known that PIT is strictly weaker than AC (e.g., see [8] and [2]), we do not assume the full AC unless otherwise is indicated explicitly. However, we shall be free to use PIT.

Throughout the paper, we assume all the rings to have identity. For the relationship between the spectra of distributive lattices and that of commutative rings, we refer to Simmons [14].

First, we shall give a precise definition of an unital m-lattice or a monoidal-lattice.

DEFINITION. A join-semilattice Q with the bottom 0 and the top 1 is called a *monoidal-lattice* if there is an associative multiplication $(a, b) \mapsto a \cdot b$ satisfying the following properties:

(i) $a \cdot (b_1 \vee b_2) = (a \cdot b_1) \vee (a \cdot b_2)$ and $(b_1 \vee b_2) \cdot a = (b_1 \cdot a) \vee (b_2 \cdot a)$ and $0 \cdot a = a \cdot 0 = 0$.

(ii) The top element 1 is also the multiplicative identity

The class of monoidal-lattices includes:

(1) all 2-sided ideal lattices of rings;
(2) bounded distributive lattices;
(3) the sub-semilattice consisting of all finitely generated 2-sided ideals of an m-ring;
(4) multiplicative ideal lattices in the sense of Martinez [12];
(5) integral *cl*-groupoids (see Birkhoff [3]); in particular, multiplicative ideal structures in the sense of Georgescu and Voiculescu [6];
(6) unital quantales in the sense of Niefield and Rosenthal [13].

Let Q be a monoidal-lattice. In §1, we shall discuss the prime spectrum, consisting of all m-prime ideals, of Q and the dual prime spectrum of Q consisting of all prime m-filters and we shall prove that both of them are spectral spaces (that is, it is a not necessarily Hausdorff compact space and has a basis consisting of compact open subsets which is closed under binary intersections; and is a sober space, i.e., each irreducible closed subset is generated by a unique point). These results extend some classical results of rings and of distributive lattices. In particular, our Theorem 1.6 generalizes some results of Keimel [10] and of [13].

In §2, we shall prove that the category of distributive lattices and hence, the dual of the category of coherent locale and coherent maps (for the definitions see below), is reflective full subcategory of the category of monoidal-lattices.

In §3, we shall study the maximal spectrum of Q and prove the analogue of that of the distributive lattices and that of the commutative rings and we also find some applications to noncommutative rings.

In §4, we investigate the minimal spectrum of Q and establish the analogue of that of the distributive lattices.

Finally in §5, we shall illustrate with an example which is an m-lattice with a multiplication identity but without any (m-)prime ideals. This means that the analogue of ideal theory of monoidal-lattices could not be generalized to general m-lattices.

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§1 Prime spectra

Let Q be a monoidal-lattice. Then a subset F of Q is called an *m-filter* if $a, b \in F$ implies $a \cdot b \in F$ and $a \leq b$, $a \in F \Rightarrow b \in F$; and called a *prime m-filter* if it is a proper m-filter and $a \vee b \in F$ implies $a \in F$ or $b \in F$. A proper ideal P of Q is said to be *m-prime* if $ab \in P$ implies either $a \in P$ or $b \in P$.

In [16], we showed that the Prime Ideal Theorem (PIT) implies that the following separation lemma holds:

SEPARATION LEMMA 1.0 ([16, Theorem 1.1]). *Let Q be a monoidal-lattice. Let F be an m-filter of Q and I an ideal of Q disjoint from F. Then there exists an m-prime ideal P of Q containing I and disjoint from F.*

We write Spec Q for the set of all m-prime ideals of Q with the hull-kernel topology $\{O_I \mid I \in \text{Id } Q\}$, where $O_I = \{P \in \text{Spec } Q \mid P \not\supseteq I\}$.

Our aim is to show that Spec Q is always spectral. Recall that an order-preserving map $j : Q \to Q$ on Q is called a *closure operator* if j is *inflationary* (i.e., satisfies $a \leq j(a)$ for all $a \in Q$) and *idempotent* (i.e., satisfies $j(j(a)) = j(a)$ for all $a \in Q$).

Now we describe the open-set lattice of Spec Q by defining $S : \text{Id } Q \to \text{Id } Q$ as follows: $S(I) = \bigcap \{P \in \text{Spec } Q \mid P \supseteq I\}$ for each $I \in \text{Id } Q$. It is easy to check that the image $S(\text{Id } Q)$ is isomorphic to the open-set lattice of Spec Q and that S is a closure operator which satisfies $S(IJ) = S(I) \cap S(J)$.

Following ring-theoretic terminologies, an ideal I of Q is called a semiprime-ideal if $b^2 \in I$ implies $b \in I$. Then we have the following:

LEMMA 1.1. *For any monoidal-lattice Q, $S(\mathrm{Id}\,Q)$ is isomorphic to the open-set lattice of* Spec Q *whose prime elements are precisely m-prime ideals of Q; and precisely consists of all semiprime-ideals of Q.*

Let Q be a monoidal-lattice. Then we say that an element $a \in Q$ is *compact* if for every family of elements of Q whose join is $\geq a$, there is a finite subfamily whose join is $\geq a$. Q is called *compact* if 1 is compact. Q is called *algebraic* if each element of Q can be expressed as a join of compact elements, and *coherent* if it is algebraic and the multiplication of two compact elements is also compact.

For any monoidal-lattice Q, Id Q is also a monoidal-lattice in which the join of ideals I_i is the ideal generated by their union and the multiplication JK of ideals J and K is the ideal generated by the set of all jk, where $j \in J$ and $k \in K$.

It is easy to check that Id Q is coherent for each monoidal-lattice Q.

LEMMA 1.2. *The compact elements in $S(\mathrm{Id}\,Q)$ are exactly of the form $S(\downarrow a)$ for some $a \in Q$.*

Proof. Let I be a compact element in $S(\mathrm{Id}\,Q)$. Then $\bigvee_s \{S(\downarrow a) \mid a \in I\} = I$; and we can find $a_0 \in I$ such that $S(\downarrow a_0) = I$, since I is a compact ideal.

Conversely, to show that each $S(\downarrow a)$ is compact, it suffices to show that for each ideal J with $S(J) = S(\downarrow a)$, there exists an element b in J such that $S(\downarrow b) = S(\downarrow a)$. In fact, if there is no such $b \in J$, then for each n, $a^n \notin J$, since $S(\downarrow a) = S(\downarrow a^n)$. So $F_a = \bigcup \{\downarrow a^n \mid n \in N\}$ is an m-filter disjoint from J. Using the Separation Lemma, we obtain an m-prime ideal P containing J and disjoint from F; in particular, $a \notin P$ which is impossible, since $S(\downarrow a) = S(J) \subseteq P$.

Recall in [8] that a locale is nothing but a complete Heyting algebra (a Brouwerian lattice) and hence is a monoidal-lattice.

For a sober space X (in particular, a Hausdorff space) X is a spectral iff its open-set locale is coherent.

Thus we have:

THEOREM 1.3. *For each monoidal-lattice Q, $S(\mathrm{Id}\,Q)$ is a coherent locale and hence* Spec Q *is spectral.*

Slightly generalizing the notion of semiprime-ideal, an element $a \in Q$ is called *semiprime* if $b^2 \leq a$ implies that $b \leq a$. The set of all semiprime elements of Q will be denoted by $S(Q)$.

LEMMA 1.4. For any monoidal-lattice Q, the following conditions are equivalent:

(1) $I \in \operatorname{Id} Q$ is a semiprime element.

(2) For each $b \in Q$, $b^2 \in I$ implies $b \in I$.

(3) I is an intersection of m-prime ideals of Q.

Proof. The non-trivial part is (1) $\Rightarrow$ (3). Let $I \in \operatorname{Id} Q$ be semiprime and let J be the intersection of m-prime ideals containing I. It remains to show that $J = I$. For this, suppose that there is $a \in J$ but $a \notin I$. Then $a^n \notin I$ for each natural number n, since I is semiprime and $a^{2n+1} \le a^{2n}$. Now, let $F = \bigcup \{\uparrow a^n \mid n \in \mathcal{N}\}$. Then F is an m-filter which is disjoint from I and so, by the Separation Lemma, there exists an m-prime ideal P containing I and disjoint from F. But $a \in F$, so that $J \nsubseteq P$, a contradiction. This completes the proof.

We shall show below that each coherent monoidal-lattice in which the multiplication distributes over infinite joins is, up to isomorphism, exactly the $\operatorname{Id} Q$ for some monoidal-lattice Q (see Theorem 2.2). Then it follows from Theorem 1.3 and Lemma 1.4 that $S(Q)$ is spatial for any coherent monoidal-lattice Q in which the multiplication distributes over infinite joins.

However, we shall strengthen this result by showing that $S(Q)$ is spatial for each algebraic complete monoidal-lattice Q (that is, Q does not need to be a quantale), which also generalizes some results of [10].

To show the following result, we need another form of the separation lemma. Recall that an m-filter F is called compactly generated if for each $a \in F$ there is a compact element $c \in F$ with $c \le a$.

LEMMA 1.5 ([16, Theorem 1.4]). Let A be a complete monoidal-lattice; F a compactly generated m-filter and I an ideal of A disjoint from F. Then AC implies that there exists a principal m-prime ideal P containing I which is disjoint from F.

Remark. Very recently, Banaschewski [0] proved that in the above result PIT suffices.

Recall that a locale L is called *spatial* if it is isomorphic to a lattice of open subsets of a space, or equivalently, each pair of distinct elements of L can be separated by a prime element and that a coherent locale is always spatial.

By Lemma 1.5, we have the following result which extends some of those in [7] and in [10].

THEOREM 1.6. Let Q be an algebraic complete monoidal-lattice. Then AC implies that $S(Q)$ is a spatial locale.

Proof. First we show that each semiprime is a meet of m-primes. Let $s \in Q$ be a semiprime. If $a \nleq s$, we need to find an m-prime p with $p \geq s$ and $p \ngeq a$. Since Q is algebraic, there exists a compact element $a_1 \leq a$ but $a_1 \nleq s$ so that $a_1^2 \nleq s$, since s is semiprime. Then there is a compact element $a_2 \leq a_1^2$ but $a_2^2 \nleq s$. In this way, we obtain a sequence $\{a_1, a_2, \ldots, a_n \ldots\}$ such that $a_i \nleq s$ and $a_{i+1} \leq a_i^2 \leq a_i$ and all a_i are compact. Now let F be the m-filter generated by this sequence. Then F is disjoint from the principal ideal generated by s, and is compactly generated since $a_i a_j \geq a_{\min\{i,j\}+1}$. By Lemma 1.5, there is an m-prime p such that $p \geq s$ and $\downarrow p$ is disjoint from F so that $a \nleq p$.

Now it remains to show that $S(Q)$ is a locale. Let a, s_i be in Q; clearly we have $a \wedge (\bigvee s_i) \geq \bigvee (a \wedge s_i)$. For the converse, let p be an m-prime with $\bigvee (a \wedge s_i) \leq p$ (note that each m-prime is prime in the usual lattice sense). Then for each i, $a \wedge s_i \leq p$. If $a \leq p$, then $a \wedge (\bigvee s_i) \leq p$; if $a \nleq p$, then $s_i \leq p$ for all i, so $\bigvee s_i \leq p$ and therefore $a \wedge (\bigvee s_i) \leq p$. This implies that $a \wedge (\bigvee s_i) \leq \bigwedge \{p \mid p \geq \bigvee (a \wedge s_i)\} = \bigvee (a \wedge s_i)$. Thus $a \wedge (\bigvee s_i) = \bigvee (a \wedge s_i)$ and hence $S(Q)$ is a locale (in fact, a spatial locale) by the first paragraph. This completes the proof.

Remark 1. In fact, we, in the proof of Theorem 1.6, used the Countable Dependent Choice – a weak form of AC – rather than the full AC.

Remark 2. Note that we do not assume that in Q, the multiplication is distributive over infinite joins; that is, Q is not necessarily a quantale. In [13], Niefield and Rosenthal claimed that each semiprime is a meet of primes in an algebraic quantale (without proof) which now becomes a special case of our Theorem 1.6.

Remark 3. I thank the referee for him letting me know the following history: In [10], Keimel proves (Theorem A), for algebraic quantales that semi-primes are the intersections of primes and also shows that the following are equivalent for such an L:

(1) each element is a meet of primes;
(2) each element is semi-prime;
(3) each element is idempotent;
(4) in L, $ab = a \wedge b$, and the lattice is distributive.

So the result in [13] mentioned in Remark 2 is in fact one of the results of Keimel. However, our result 1.6 also generalizes the result of Keimel in the following way:

(1) As mentioned in Remark 2, we only assume that the multiplication is distributive over finite joins rather than over infinite joins. The simple example 1.7(2) shows that our assumption is strictly weaker than that of Keimel.

(2) We do not assume that $Q = S(Q)$; in particular, we do not assume that $S(Q)$ is algebraic. The following example 1.7(1) shows the difference.

EXAMPLE 1.7. (1) Let R be a general ring. Then Id R is a algebraic complete monoidal-lattice (in fact, it is a quantale) and SId R is the lattice of prime radical ideals of R but it is not necessarily algebraic. For example, let R be the free ring having two variables. Then SID R is not algebraic (for the details see the author [19]).

In particular, this example also shows that the conclusion of Theorem 1.6 can't be strengthened to be a coherent locale or an algebraic locale.

(2) Let L be the set of ordinal numbers $\omega_0 + 1 = \{0, 1, \ldots, n, \ldots, \omega_0\}$ with the usual order. Then L is an algebraic completely distributive complete lattice. Now we define a multiplication on L such that L be a monoidal-lattice but not a quantale.

When $n = 1$, define $n \cdot m = 0$ if $m \neq \omega_0$ and $n \cdot \omega_0 = 1$; when $n \neq 1$, define $n \cdot m = \min\{n, m\}$ for all $m \in L$. Then it is easy to check that the multiplication is associative and it distributes over finite joins. But it does not distribute over infinite joins.

Next, we shall consider the dual prime spectrum DSpec Q consisting of all prime m-filters of a monoidal-lattice Q.

Let

$$\mathrm{qFil}\, Q = \{F \subseteq Q \mid F \text{ is a } m\text{-filter}\}.$$

Then qFil Q is a complete lattice ordered by inclusion.

By the dual prime spectrum DSpec Q of Q, we mean the set of all prime m-filters of Q with the usual hull-kernel topology, i.e., the typical open subset of it has the form: $O_F = \{P \in \mathrm{DSpec}\, Q \mid F \nsubseteq P\}$ for some $F \in \mathrm{qFil}\, Q$. We shall show that it is a spatial locale (see Theorem 1.9); that is, the lattice of open sets of DSpec Q is isomorphic to qFil Q; or equivalently, the mapping sending F to O_F is injective.

For $a \in Q$, we denote by

$$F_a = \bigcup \{\uparrow a^n \mid n \in N\}.$$

Then we have:

LEMMA 1.8. For any $a, b \in Q$, $F_a \cap F_b = F_{a \vee b}$.

Proof. Let $a, b \in Q$. Then $(a \vee b)^{2n} \leq (a^n \vee b^n)$, for each $n \geq 1$. Now suppose $z \in F_a \cap F_b$. Then there are $n_1, n_2 \in N$ such that $a^{n_1} \leq z$ and $b^{n_2} \leq z$. Take $n = \max\{n_1, n_2\}$. Then $a^n \vee b^n \leq z$ so that $(a \vee b)^{2n} \leq z$, and hence $z \in F_{a \vee b}$. The converse is clear.

THEOREM 1.9. Let Q be a monoidal-lattice. Then qFil Q is a spatial locale whose prime elements are precisely the prime m-filters of Q.

Proof. First we will show that qFil Q is a locale. For each pair $F_1, F_2 \in$ qFil Q, we define $(F_1 \to F_2) = \{a \in Q \mid a \vee F_1 \subseteq F_2\}$, where $a \vee F = \{a \vee x \mid x \in F\}$. Then $(F_1 \to F_2)$ is an m-filter. In fact, take any $a_1, a_2 \in (F_1 \to F_2)$. Then for each $b \in F$, we have $(a_1 \vee b)$ and $(a_2 \vee b) \in F_2$, so that $(a_1 \vee b)(a_2 \vee b) \in F_2$. On the other hand, since $(a_1 \vee b)(a_2 \vee b) \leq (a_1 a_2) \vee b$, the latter must be in F_2. Thus $a_1 a_2 \in (F_1 \to F_2)$. The other condition of m-filter is obvious. Hence $(F_1 \to F_2)$ is an m-filter.

Now we want to show that

$$F_1 \cap F_2 \subseteq F_3 \quad \Leftrightarrow \quad F_1 \subseteq (F_2 \to F_3).$$

Suppose that $F_1 \cap F_2 \subseteq F_3$. If $a \in F_1$, then $a \vee b \in F_1 \cap F_2$, for each $b \in F_2$ so that $a \vee b \in F_3$ and hence $a \in (F_2 \to F_3)$. Next suppose $F_1 \subseteq (F_2 \to F_3)$. If $a \in F_1 \cap F_2$, then $a = a \vee a \in F_3$ and hence $F_1 \cap F_2 \subseteq F_3$. Hence qFil Q is a locale.

Secondly, we will show that the prime elements of qFil Q are precisely the prime m-filters of Q. Suppose that P is a prime m-filter. If P is not a prime element in qFil Q, then there are $F_1, F_2 \in$ qFil Q with $F_1 \cap F_2 \subseteq P$ and $F_i \nsubseteq P$, for $i = 1, 2$. Thus there are $a_i \in F_i \backslash P$; and so $a_1 \vee a_2 \in F_1 \cap F_2 \subseteq P$ which is impossible, since P is a prime m-filter. Next, let P be a prime element in qFil Q; and so in particular, P is an m-filter. We want to show that p is a prime m-filter. If not, there are elements $a_1, a_2 \notin P$ with $a_1 \vee a_2 \in P$. But then $F_{a_1 \vee a_2} \subseteq P$ and so $F_{a_1} \cap F_{a_2} \subseteq P$, by Lemma 1.8. Therefore either $F_{a_1} \subseteq P$ or $F_{a_2} \subseteq P$ since both F_{a_1} and F_{a_2} are in qFil Q and P is a prime element in qFil Q, a contradiction.

Finally, we show that qFil Q is spatial. Suppose $F_1, F_2 \in$ qFil Q with $F_1 \nsubseteq F_2$, i.e., there is $a \in F_1 \backslash F_2$. Then $\downarrow a$ is disjoint from F_2. By the Separation Lemma, there is an m-prime ideal P containing $\downarrow a$ and disjoint from F_2. Let $F = Q \backslash P$. Then F is a prime m-filter; or equivalently, F is a prime element in qFil Q, by the above result. Moreover, we have $F_2 \subseteq F$ but $F_1 \nsubseteq F$. Hence qFil Q is spatial and this completes the proof.

Furthermore, we shall prove that qFil Q is further a coherent locale and hence the dual prime spectrum DSpec Q of Q is a spectral space.

LEMMA 1.10. *Let Q be a monoidal-lattice. Then*

(a) $F_1 \vee F_2 = \{\downarrow a_1 b_1 a_2 b_2 \cdots a_n b_n \mid a_i \in F_1, b_i \in F_2, i \leq n, n \in N\}$, *for each pair* $F_1, F_2 \in \mathrm{qFil}\, Q$. *In particular,* $F_a \vee F_b = F_{ab}$.

(b) *The compact elements in* qFil Q *are exactly of the form* F_a, *for some* $a \in Q$.

Proof. (a) Clear.

(b) Let $F \in \mathrm{qFil}\, Q$ be a compact element. Then $\bigvee \{F_a \mid a \in F\} = F$, and there is some $a_0 \in F$ such that $F_{a_0} = F$.

Conversely, suppose $F_a = \bigvee F_i$. Then $a \in \bigvee F_i$ and so there are n, m such that

$$a \geq (b_{11} b_{12} \cdots b_{1n})(b_{21} \cdots b_{2n}) \cdots (b_{m1} \cdots b_{mn}),$$

where $b_{ji} \in F_j, i \leq n, j \leq m$; i.e., $a \in \bigvee \{F_j \mid j \leq m\}$. Thus $F_a \subseteq \bigvee \{F_j \mid j \leq m\}$ and hence F_a is compact. This completes the proof.

Now, combining Lemma 1.8 and Lemma 1.10, we see that the set of all compact elements generates qFil Q (and hence qFil Q is algebraic) and is closed under finite joins and finite meets (and hence it is further coherent). Thus:

THEOREM 1.11. *Let Q be a monoidal-lattice. Then* qFil Q *is a coherent locale and hence the dual prime spectrum* DSpec Q *of Q is a spectral space.*

The relationship between the prime spectrum and the dual prime spectrum of Q is given by the following proposition.

PROPOSITION 1.12. *For any $a, b \in Q$,*

$$F_a \subseteq F_b \quad \Leftrightarrow \quad S(\downarrow b) \subseteq S(\downarrow a).$$

Proof. Suppose $F_a \subseteq F_b$. Then $a \in F_b$ and so there is n such that $b^n \leq a$ which implies that $S(\downarrow b^n) \subseteq S(\downarrow a)$. Then $S(\downarrow b) \subseteq S(\downarrow a)$ follows from the fact that $S(\downarrow b) = S(\downarrow b^n)$.

Conversely, suppose $S(\downarrow b) \subseteq S(\downarrow a)$ and $F_a \nsubseteq F_b$. Then there is an n such that $a^n \notin F_b$ which implies that $(\downarrow a^n)$ is disjoint from F_b. Now using the Separation Lemma, there is an m-prime ideal P containing a and disjoint from F_b. Thus $S(\downarrow a^n)$ is disjoint from F_b and so in particular, $b \notin S(\downarrow a)$, a contradiction.

§2 Reflective subcategory

In this section, we shall prove that the category *DLA* of distributive lattices is a reflective full subcategory of the category *MLA* of monoidal-lattices and monoidal-morphisms, where a map $f : Q_1 \to Q_2$ is called a monoidal-morphism if it preserves finite joins and multiplication.

First we observe from Theorem 1.3 that there is a map $\mathrm{SId} : MLA \to DLA$ given by

$$\mathrm{SId}(Q) = S(\mathrm{Id}\, Q).$$

Now for a monoidal-morphism $f : Q_1 \to Q_2$, we define

$$\mathrm{SId}(f)(S(\downarrow a)) = S(\downarrow f(a)).$$

Then SId is a functor. To see this, we first check that $\mathrm{SId}(f)$ is a lattice morphism. In fact, for any $a, b \in Q_1$, we have

$$\begin{aligned}\mathrm{SId}(f)(S(\downarrow a) \cap S(\downarrow b)) &= \mathrm{SId}(f)(S(\downarrow ab)) = S(\downarrow f(ab)) = S(\downarrow f(a)f(b))\\ &= S(\downarrow f(a)) \cap S(\downarrow f(b)) = \mathrm{SId}(f)(S(\downarrow a)) \cap \mathrm{SId}(f)(S(\downarrow b)),\end{aligned}$$

and similarly $\mathrm{SId}(f)$ is finite join-preserving. Other axioms follow easily from the definition of $\mathrm{SId}(f)$.

THEOREM 2.1. *Let Q be a monoidal lattice. Then the functor* SId *is a left adjoint to the inclusion functor. That is, DLA is a reflective full subcategory of MLA.*

Proof. It suffices to show that the monoidal-morphism $\mathrm{SId}_Q : Q \to \mathrm{SId}\, Q$ is the unit. For each monoidal-morphism $g : Q \to L$, where L is a distributive lattice, we see that $g(ab) = g(a) \wedge g(b)$, for all $a, b \in Q$. Hence there exists a unique lattice-morphism $f : \mathrm{SId}\, Q \to L$ defined by sending each $S(\downarrow a)$ to $g(a)$. The rest of the proof is straightforward.

It is known that DLA is dual to the category of coherent locales and coherent maps (see [8, Proposition 3.2 and Corollary 3.3]) and so we have the following:

COROLLARY. *The dual of the category of coherent locales and coherent maps is a reflective full subcategory of MLA.*

Next we shall establish the analogue of the relationship between coherent locales and bounded distributive lattices.

Clearly, for each monoidal-lattice Q, $\mathrm{Id}\, Q$ is coherent. Now we want to show the converse under some conditions.

THEOREM 2.2. *Let Q be a complete monoidal-lattice (resp. m-lattice) with the multiplication distributes over infinite joins. Then Q is coherent if and only if there is an isomorphism between Q and* Id L, *for some monoidal-lattice (resp. m-lattice) L.*

Proof. Let Q be a complete monoidal-lattice with multiplication distributes over infinite joins. Suppose that Q is coherent. Let $L = L(Q)$ be the set of all compact elements of Q. Then L is a sub-monoidal-lattice of Q. For $a \in Q$, we let $I_a = \{k \in L \mid k \leq a\}$ and define $\phi : Q \to \mathrm{Id}\, L$ by $\phi(a) = I_a$. We claim that ϕ is an isomorphism. First, we show that ϕ is surjective. If I is any ideal of L, then it is a directed set of Q, and so $k \in L$, $k \leq \bigvee_Q I$ implies $k \leq i$ for some $i \in I$. Thus $k \in I$ and hence $\phi(\bigvee_Q I) = I$. Next, since Q is algebraic, it follows that $\bigvee_Q (\phi(a)) = a$ for all $a \in Q$ and therefore ϕ is injective. Clearly ϕ is order-preserving. It remains to show that ϕ preserves multiplication. Clearly $\phi(ab) \geq \phi(a)\phi(b)$. Now let $k \in L$ and $k \leq ab$. Then

$$k \leq (\bigvee \{x_i \in L \mid x_i \leq a\}) \cdot (\bigvee \{y_i \in L \mid y_j \leq b\})$$

$$= \bigvee \{x_i y_j \mid x_i \in L, y_j \in L, x_i \leq a, y_j \leq b\};$$

so that $k \in \phi(a) \cdot \phi(b)$. Hence $\phi(ab) \leq \phi(a) \cdot \phi(b)$. This completes the proof.

Let Q_1 and Q_2 be coherent complete monoidal-lattices. A monoidal-morphism $f : Q_1 \to Q_2$ is called coherent if f preserves arbitrary joins and sends compact elements of Q_1 to compact elements of Q_2. Therefore, we have the following:

COROLLARY. *The category MLA is dual to the category of coherent complete monoidal-lattices and coherent monoidal-morphisms.*

§3 Maximal ideal spaces

Let Q be a monoidal-lattice. Then the separation lemma 1.5 guarantees the existence of m-primes of Q. Now let Max Q be the maximal ideals space of Q. We shall show that Max Q is a subspace of Spec Q, and with some additional conditions on Q, we then show that Max Q is a retract of Spec Q.

The following Lemma is obvious.

LEMMA 3.1. *Let Q be a monoidal-lattice. Then each maximal ideal of Q is an m-prime ideal.*

Remark. For distributive lattices D, it is known that its maximal ideals space $\operatorname{Max} D$ is always compact T_1 and is T_2 if D is normal (i.e., for each pair $a, b \in D$ with $a \vee b = 1$, there exist $c, d \in D$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$, the details see [8]).

Now, we shall extend these results to monoidal-lattices and as an application, we extend some ring-theoretic results for the commutative case (see [5]) to the non-commutative case.

Similar to the distributive case, we have the following:

LEMMA 3.2. *Let Q be a monoidal-lattice. Then* $\operatorname{Max} Q$ *is a compact T_1 space.*

DEFINITION. A monoidal-lattice Q is called *almost-normal* (resp. *normal*) if for each pair b_1, b_2 in Q, with $b_1 \vee b_2 = 1$ there exist $c_1, c_2 \in Q$ such that $(c_1 c_2)^2 = 0$ (resp. $c_1 c_2 = 0$), $c_1 \vee b_2 = 1$ and $c_2 \vee b_1 = 1$. For the commutative monoidal-lattices; in particular, for distributive lattices, the normality and the almost-normality coincide.

PROPOSITION 3.3. *Let Q be an almost-normal monoidal-lattice. Then* $\operatorname{Max} Q$ *is T_2.*

Proof. Let $M_1, M_2 \in \operatorname{Max} Q$ with $M_1 \nsubseteq M_2$. Let $b_1 \in M_1 \backslash M_2$. Since M_2 is maximal, there is a $b_2 \in M_2$ with $b_1 \vee b_2 = 1$. Then by the almost-normality of Q, there exist $c_1, c_2 \in Q$ such that $(c_1 c_2)^2 = 0$, $c_1 \vee b_1 = 1$ and $c_2 \vee b_2 = 1$. Hence $c_1 \notin M_1$ and $c_2 \notin M_2$; that is, $M_i \in O_{c_i}$, $i = 1, 2$. Since $(c_1 c_2)^2 = 0$, it follows from Lemma 3.1 that O_{c_1} is disjoint from O_{c_2}.

We note that the $S : \operatorname{Id} Q \to \operatorname{Id} Q$ defined in §1 has the following two properties: $S(I^n) = S(I)$ for any n, and $1 \in S(I)$ implies $1 \in I$. Then we have the following lemma.

LEMMA 3.4. *Let Q be a monoidal-lattice. Then Q is almost-normal (resp. normal) if and only if* $\operatorname{Id} Q$ *is almost-normal (resp. normal).*

In fact, we have the following:

THEOREM 3.5. *Let Q be a monoidal-lattice. Then the following conditions are equivalent:*

(1) *Q is almost-normal.*
(2) id *Q is almost-normal.*
(3) Spec *Q is a normal space.*
(4) *Each m-prime ideal is contained in a unique maximal ideal.*
(5) Max *R is a non-empty retract of* Spec *R.*

Proof. (1) $\Leftrightarrow$ (2): This is just Lemma 3.4.

Now we will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (5): It suffices to define a continuous mapping $m : \operatorname{Spec} Q \to \operatorname{Max} Q$ such that $m(M) = M$ for each $M \in \operatorname{Max} Q$. For each $P \in \operatorname{Spec} Q$, define

$$F_P = \{b \in Q \mid (\exists c \in P)(b \vee c = 1)\}.$$

We claim that F_P is a prime m-filter. The fact that F_P is an upper set is clear. Now suppose $b_1, b_2 \in F_P$. Then there exist $c_1, c_2 \in P$ such that $b_1 \vee c_1 = 1 = b_2 \vee c_2$, and so

$$(b_1 b_2) \vee c_1 \vee c_2 \geq (b_1 \vee c_1)(b_2 \vee c_2) = 1.$$

Hence $b_1 b_2 \in F_P$. Next suppose $b_1 \vee b_2 \in F_P$. Then there is $c \in P$ such that $b_1 \vee (b_2 \vee c) = 1$. Since Q is almost-normal, there exist $d_1, d_2 \in Q$ such that $(d_1 d_2)^2 = 0$, $d_1 \vee (b_2 \vee c) = 1$ and $d_2 \vee b_1 = 1$. Moreover, by the m-primeness of P, we have either $d_1 \in P$ or $d_2 \in P$ so that either $(d_1 \vee c) \in P$ since $c \in P$ or $d_2 \in P$, which imply $b_1 \in F_P$ or $b_2 \in F_P$. Hence F_P is a prime m-filter.

Now let $m(P) = Q \backslash F_P$. Then it is clear that $m(P)$ is a maximal ideal and $m(M) = M$ if M is maximal. Hence the mapping m is well defined. It remains to show that m is continuous. Consider a basic open set $O_b = \{M \in \operatorname{Max} Q \mid b \notin M\}$ for some $b \in Q$. Then $m(P) \in O_b$ if and only if there exists a $c \in P$ with $b \vee c = 1$. Let $d_1, d_2 \in Q$ be such that $(d_1 d_2)^2 = 0$, $d_1 \vee c = 1$ and $d_2 \vee b = 1$. Then $d_1 \notin P$. On the other hand, if P' is any m-prime ideal not containing d_1, then $d_2 \in P'$ which implies $b \notin m(P')$ and so that

$$\{P \in \operatorname{Spec} Q \mid d_1 \notin P\} \subseteq m^{-1}(O_b).$$

(5) $\Rightarrow$ (4): Let $m : \operatorname{Spec} Q \to \operatorname{Max} Q$ be continuous mapping with $m(M) = M$ for each $M \in \operatorname{Max} Q$. Then for each $P \in \operatorname{Spec} Q$ and any maximal $M \supseteq P$, we have $M = m(M) = m(P)$. In other words, $m(P)$ must be the unique maximal ideal containing P.

(4) $\Rightarrow$ (1): Let $b_1, b_2 \in Q$ be such that $b_1 \vee b_2 = 1$. Consider

$$F = \{\downarrow c_1 c_2 \cdots c_{2n} \mid c_{2i+1} \vee b_1 = 1 = c_{2i} \vee b_2,\ i = 0, 1, \ldots n, n \in N\}.$$

Then F is an m-filter. If $0 \notin F$, then by the Separation Lemma, there is an m-prime ideal P which is disjoint from F. Thus for each $p \in P$, we have $p \vee b_i \neq 1$, $i = 1, 2$. If not, say $p \vee b_1 = 1$, then $p \geq p1 \in F$, a contradiction. Now let J_i be the ideal generated by P and b_i, $i = 1, 2$. Then $J_1 \neq J_2$. Since $\{1\}$ is an m-filter and $b_1 \vee b_2 = 1$, there are distinct m-prime ideals P_i, $i = 1, 2$ with $P_i \supseteq J_i$. Hence there are distinct maximal ideals M_i containing P_i. Then both M_i contain P, a contradiction. Hence F must contain 0 and so there are n and c_i, $i \leq 2n$ such that

$$c_1 c_2 \cdots c_{2n} = 0, \qquad c_{2i+1} \vee b_1 = 1 = c_{2i} \vee b_2, \quad i = 0, 1, \ldots n.$$

Finally, by letting $d_1 = c_1 c_3 \cdots c_{2n-1}$ and $d_2 = c_2 c_4 \cdots c_{2n}$, we see that $d_1 \vee b_1 = 1 = d_2 \vee b_2$ and $(d_1 d_2)^2 \leq c_1 c_2 \cdots c_{2n} = 0$. Hence Q is almost-normal.

This completes the proof.

Recall that a ring is called an m-ring (or neo-commutative ring in the sense of Kaplansky [9]) if for each pair of finitely generated ideals, their product is also finitely generated. We showed in [18] that if R is an m-ring, then Id R is isomorphic to Id $L(R)$, where $L(R)$ is the sub-monoidal-lattice consisting of all finitely generated ideals of R. As an application of Theorem 3.5, we have:

THEOREM 3.6. *Let R be an m-ring. The the following conditions are equivalent*:
(1) Id R *is almost-normal.*
(2) Spec R *is a normal space.*
(3) *Each prime* 2*-sided ideal is contained in a unique maximal* 2*-sided ideal of R.*
(4) Max R *is a non-empty retract of* Spec R.

Since a commutative ring is clearly an m-ring and the two notions of normality and almost-normality coincide, we have extended the following results of DeMarco–Orsatti [5] and Simmons [14] (see also [8]).

COROLLARY 1 [5]. *Let R be a commutative ring. Then the following conditions are equivalent*:
(1) Id R *is normal*
(2) Spec R *is normal.*
(3) *Each prime ideal is contained in a unique maximal ideal of R.*
(4) Max R *is a retract of* Spec R.

COROLLARY 2 [14]. *Let L be a bounded distributive lattice. Then the following conditions are equivalent*:
(1) *L is normal.*
(2) Spec L *is normal.*
(3) *Each prime ideal is contained in a unique maximal ideal of l.*
(4) Max L *is a retract of* Spec L.

Remark 1. We have shown that the above corollary does not hold for general non-commutative rings (see [17, Example 2.5)]. Another related result which was proved in [17] is the following: (2), (3) and (4) remain equivalent for weakly symmetric rings but not for general rings.

Remark 2. This Remark is only for those readers who are interested in 'constructive' proof (i.e., do not assume any non-constructive principals such as AC, PIT, etc).

Just like the distributive case (see [8]), it may be possible to describe the open sets lattice of Max Q as a localic quotient of Id Q.

Define $j : \mathrm{Id}\, Q \to \mathrm{Id}\, Q$ by

$$j(I) = \{a \in Q \mid (\forall x \in Q)(a \vee x = 1) \Rightarrow (\exists c \in I)(c \vee x = 1)\}.$$

Then $j(I)$ is an ideal and j is a localic nucleus and moreover, if we assume AC (it is known that the Maximal Ideal Theorem (MIT) is logical equivalent to AC), $j(I)$ is the intersection of all maximal ideals containing I.

We claim that the lattice $j(\mathrm{Id}\, Q)$ is isomorphic to the open-set lattice of Max Q. This description allows us to study the properties of the open-set lattice of Max Q without MIT (or equivalently AC), if one is only interested in the properties of the open-set locale rather than the space itself.

EXAMPLE 3.7. Let $C(X)$ be the ring of all continuous mapping from a topological space X to the space $\mathbf{R}$ of real numbers in the usual metric and $Q = \mathrm{Id}(C(X))$. Then Q is a unital quantale and in particular, a monoidal-lattice. Then Q need not be distributive but it is always normal, using Theorem 3.6.

§4 Minimal spectra

In this section, we will be free to use AC. A monoidal-lattice Q is called *semiprime* if Q has a semiprime bottom 0 (that is, $a^2 = 0$ implies $a = 0$) and the underlying semi-lattice of Q is a lattice. Clearly, if R is a semiprime ring, then Id R of all 2-sided ideals of R is a semiprime monoidal-lattice. For any monoidal-lattice Q, let Min Q be the minimal prime ideal space of Q. Then we shall show that each minimal prime ideal is m-prime and so Min Q is actually a subspace of Spec Q. Next we show that Min Q is compact if and only if Q is quasi-complemented, which extends a main result of [15, Theorem 2.6].

LEMMA 4.1. *Let Q be a prime monoidal-lattice. Then, for any $a, b \in Q$, $ab = 0 \Leftrightarrow a \wedge b = 0$. If, in addition, the multiplication of Q distributes over infinite* *joins, then the right polar* $\bigvee \{x \in Q \mid ax = 0\}$ *and the left polar* $\bigvee \{x \in Q \mid ax = 0\}$ *of a coincide (which is denoted by* a^**). Moreover, for each* $a \in Q$, a^* *is semiprime and* $(a \vee a^*)$ *is dense. (An element z is called dense if* $zy = 0$ *implies* $y = 0$*).*

Proof. The proof of the first two results is similar to those in [12]. Now suppose that $b^2 \leq a^*$ and $b \nleq a^*$. Then $ba \neq 0$. Since 0 is semiprime, it follows that $baba \neq 0$ and hence $bba \neq 0$. Thus $a^*a \neq 0$ since $b^2 \leq a^*$, a contradiction. Thus we have shown that a^* is semiprime. To show that $(a \vee a^*)$ is dense, let $b \in Q$ with $b(a \vee a^*) = 0$. Then $ba = 0$ so that $b \leq a^*$. On the other hand, we have $ba^* = 0$ and so $b \wedge a^* = 0$. Hence $b = b \wedge a^* = 0$ and therefore $a \vee a^*$ is dense.

LEMMA 4.2. *Let Q be a prime monoidal-lattice. Then each maximal filter of Q is a maximal m-filter and each minimal prime ideal is a minimal m-prime ideal.*

Proof. Let M be a maximal filter of Q. Then for each $a, b \in M$, $a \wedge b \neq 0$ and so $ab \neq 0$, by Lemma 4.1. Let $\tilde{M}$ be the m-filter generated by M. Then $\tilde{M}$ is a filter, since $ab \leq a \wedge b$ for all $a, b \in Q$. Thus $\tilde{M} = M$, since M is maximal. Hence M is an m-filter. The second result follows from the fact that P is a minimal m-prime ideal if and only if the complement $(Q \backslash P)$ is a maximal m-filter.

The following result follows immediately from Lemma 4.2.

PROPOSITION 4.3. *Let Q be a prime monoidal-lattice. Then each prime ideal of Q contains a minimal m-prime ideal.*

Remark. The result in Proposition 4.3 extends the corresponding classical results for rings and for distributive lattices with 0.

For any distributive lattice L, we know that $\mathrm{Min}\, L$ is 0-dimensional (that is, each element is a join of complemented elements), but not necessarily compact. In fact, we have the following extension:

LEMMA 4.4. *Let Q be a semiprime monoidal-lattice. Then* $\mathrm{Min}\, Q$ *is 0-dimensional.*

Proof. It suffices to show that each $O_{(\downarrow a)} \cap \mathrm{Min}\, Q$ is closed. Suppose $P \in \mathrm{Min}\, Q \backslash O_{(\downarrow a)}$. Then $a \in P$. Since $Q \backslash P$ is a maximal filter, there is $m \notin P$ with $m \wedge a = 0$. Thus $O_{(\downarrow m)}$ is an open neighbourhood of P which is disjoint from $O_{(\downarrow a)}$, and the proof is completed.

DEFINITION. A monoidal-lattice Q is called quasi-complemented if for each $a \in Q$, there is $a' \in Q$ such that $a \wedge a' = 0$ and $(a \vee a')$ is dense.

Remark. The referee pointed out that this notion also appeared in [11, Theorem 37.4], [7] and [4, Theorem 2.2] for lattice-ordered groups.

EXAMPLE 4.5. Clearly, every quasi-complemented distributive lattice is a quasi-complemented monoidal-lattice. Every quantale Q, satisfying that $a \cdot b \leq a \wedge b$ for any $a, b \in Q$ and that 0 is semiprime, is a quasi-complemented monoidal-lattice (see Lemma 4.2). Thus the $\mathrm{Id}(C(X))$ in Example 3.8 is a quasi-complemented prime monoidal-lattice but not necessarily distributive.

We have shown in [15] that if L is a bounded distributive lattice, then $\mathrm{Min}\, L$ is compact if and only if L is quasi-complemented. Now we generalize it to prime monoidal-lattices. First, we need the following lemma, the proof of which is similar to [15, Lemma 2.5].

LEMMA 4.6. *Let Q be a prime monoidal-lattice. Then* $\mathrm{Min}\, Q$ *is compact if and only if Q satisfies the following condition: If $I \in \mathrm{Id}\, Q$ and $I \nsubseteq P$, for any $P \in \mathrm{Min}\, Q$, then I must contain some dense element in Q.*

THEOREM 4.7. *Let Q be a prime monoidal-lattice. Then* $\mathrm{Min}\, Q$ *is compact if and only if Q is quasi-complemented.*

Proof. Suppose that $\mathrm{Min}\, Q$ is compact. For each $a \in Q$, let a^0 denote the ideal $\{b \in Q \mid a \wedge b = 0\}$ (it is an ideal by Lemma 4.1). Then the ideal generated by $\{a\} \cup a^0$ is not contained in any minimal ideal P since each maximal filter not containing a^0 must contain a by the maximality. So by Lemma 4.6, this ideal must contain some dense element; i.e., there exists $a' \in a^0$ such that $(a \vee a')$ is dense. Hence Q is quasi-complemented.

Now, suppose that Q is quasi-complemented and $\mathrm{Min}\, Q$ is not compact. By Lemma 4.6, there is an ideal I such that $I \nsubseteq P$ for each $P \in \mathrm{Min}\, Q$ and I does not contain any dense element in Q. Then for each $a \in I$, we see that $a' \notin I$, since $(a \vee a')$ is dense; in particular, $a' \neq 0$. Moreover, if $a_1 \ldots a_n \in I$, then

$$a_1 \vee \cdots \vee a_n \vee (a_1' \ldots a_n') \geq (a_1 \vee a_1') \ldots (a_n \vee a_n')$$

is dense, since dense elements form an m-filter. So $\{a' \mid a \in I\}$ can be extended to a maximal filter M. Clearly, M is disjoint from I so that the minimal prime ideal $Q \backslash M$ contains I, which is a contradiction.

§5 Example

We illustrate with an example of an m-lattice with an identity which does not have any m-prime ideals.

First we note that the following counterpart of Theorem 2.2 holds for general m-lattices.

PROPOSITION 5.1. *A complete m-lattice Q, in which the multiplication distributes over infinite joins, is coherent if and only if it is isomorphic to the* Id L, *where L is the sub-m-lattice consisting of all compact elements of Q.*

Now let R be any ring with an identity. Consider the set Sub R of all subgroups of R. For $J, K \in \mathrm{Sub}\, R$, the multiplication JK is defined to be the subgroup of R generated by all elements of the form jk, where $j \in J$ and $k \in K$. Then Sub R is a complete coherent m-lattice in which the multiplication distributes over infinite joins. Thus there is an m-lattice isomorphism between Sub R and Id L, where L is the sub-m-lattice of Sub R, consisting of all finitely generated subgroups of R. Therefore, to give an example of an m-lattice which does not have any m-prime ideal, it suffices to construct a ring R with the property that Sub R does not have any m-prime element; or equivalently, R does not have any prime subgroup.

EXAMPLE. Let R be the ring of all 2×2 real matrices. Then Sub R does not have any m-prime. First we note that the bottom of Sub R is not m-prime since

$$\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now let I be an m-prime of Sub R.
Since $I \neq R$, we can suppose

$$\begin{pmatrix} r_0 & 0 \\ 0 & 0 \end{pmatrix} \notin I.$$

Then we have

$$\begin{pmatrix} r_0 & r_0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_0 & -r_0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_0 & 0 \\ -r_0 & 0 \end{pmatrix} \notin I,$$

since

$$\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \in I$$

for any real number r.

Thus we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r_0 & r_0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} r_0 & 0 \\ -r_0 & 0 \end{pmatrix} \times \begin{pmatrix} r_0 & -r_0 \\ 0 & 0 \end{pmatrix} \notin I,$$

since I is prime.

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Department of Pure Mathematics
University of Sydney
Australia