On the monadic nature of categories of ordered sets

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Abstract

If $\mathbb{S}$ is an order-adjoint monad, that is, a monad on Set that factors through the category of ordered sets with left adjoint maps, then any monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ makes $\mathbb{T}$ order-adjoint, and the Eilenberg-Moore category of $\mathbb{T}$ is monadic over the category of monoids in the Kleisli category of $\mathbb{S}$.

Keywords: order-adjoint monad, Eilenberg-Moore category, Kleisli monoid, monadic functor

1 Introduction

A monadic functor from A to X determines a unique (up to isomorphism) monad $\mathbb{T}$ on X, and such a monad yields a category of Eilenberg-Moore algebras $\mathsf{X}^{\mathbb{T}}$ that is equivalent to A; however, A can be monadic over a range of different categories. Illustrations of this fact that stem from distributive laws [1] spring to mind, but other examples originate from different contexts. For instance, the category Cnt of continuous lattices that is strictly monadic over Set, as well as over the category Top of topological spaces, the category Sup of complete sup-lattices, and the category CHaus of compact Hausdorff spaces ([4], [17]). The last two examples can be obtained as consequences of the following result (see for example Corollary 4.5.10 in [3]):

> *A morphism* $\tau : \mathbb{S} \to \mathbb{T}$ *between monads on* Set *induces a strictly monadic functor* $\mathsf{Set}^{\tau} : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{S}}$.

Indeed, taking $\mathbb{T}$ to be the filter monad $\mathbb{F}$, and $\mathbb{S}$ the powerset monad $\mathbb{P}$ or the ultrafilter monad $\mathbb{B}$, the principal filter monad morphism $\tau : \mathbb{P} \to \mathbb{F}$ yields the monadicity of $\mathsf{Cnt} \cong \mathsf{Set}^{\mathbb{F}}$ over $\mathsf{Sup} \cong \mathsf{Set}^{\mathbb{P}}$, and the embedding morphism $\mathbb{B} \hookrightarrow \mathbb{F}$ leads to the monadicity of Cnt over $\mathsf{CHaus} \cong \mathsf{Set}^{\mathbb{B}}$ (see [8]). However, the presence of Top in this context remains somewhat idiosyncratic.

The aim of the present work is to describe a setting that leads to a systematic display of monadic functors induced by monad morphisms over categories such as Top or Ord, rather than Eilenberg-Moore categories (and turns out, contrarily to the cited result, not to require recourse to the Axiom of Choice). The main ingredient is an order-adjoint monad, that is, a monad on Set whose extension operation factorizes through the category of ordered sets and left adjoint maps (see 2.3). Another ingredient is inspired by the description of an object in Top as a monoid in the ordered hom-set $\mathsf{Set}(X, FX)$ (where FX is the set of all filters on X, ordered by reverse inclusion): a topological space can be defined as a set X with a neighborhood map $\alpha : X \to FX$ such that

$$\eta_X \leq \alpha \qquad \text{and} \qquad \mu_X \cdot F\alpha \cdot \alpha = \alpha \ ,$$

where η and μ are respectively the unit and multiplication of $\mathbb{F}$ (see [5]). The second identity—idempotency of α in the hom-set of the Kleisli category of $\mathbb{F}$—is central to our construction of a monad $\mathbb{T}'$ from $\mathbb{T}$. In fact, the category $\mathsf{Set}(\mathbb{S})$ of *Kleisli monoids* associated to an order-adjoint monad $\mathbb{S}$ plays the same role as $\mathsf{Set}^{\mathbb{S}}$ previously and leads to an isomorphism

$$\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$$

(Theorem 4.8). After illustrating this result with a number of scattered—and previously unrelated—results occurring in the literature, we show how the intrinsic order-adjoint nature of algebra structures contributes to the study of relevant Eilenberg-Moore categories.

2   Order-adjoint monads

In this section, we recall basic facts and terminology pertaining to order-adjoint monads, and settle a number of notations. Further details can be found in [15].

2.1 Monads. A monad $\mathbb{T}$ on a category X is a triple (T, η, μ) formed by a functor $T : \mathsf{X} \to \mathsf{X}$, and two natural transformations: the *unit* $\eta : \mathrm{Id} \to T$ and *multiplication* $\mu : TT \to T$ of the monad that must satisfy

$$\mu \cdot T\eta = 1 = \mu \cdot \eta T \qquad \text{and} \qquad \mu \cdot T\mu = \mu \cdot \mu T \ .$$

We say that a pair $(R, \sigma) : \mathbb{S} \to \mathbb{T}$ is a *monad morphism* from a monad $\mathbb{S} = (S, \delta, \nu)$ on A to a monad $\mathbb{T} = (T, \eta, \mu)$ on X, if $R : \mathsf{X} \to \mathsf{A}$ is a functor and $\sigma : SR \to RT$ a natural transformation such that

$$R\eta = \sigma \cdot \delta R \qquad \text{and} \qquad R\mu \cdot \sigma T \cdot S\sigma = \sigma \cdot \nu R \ .$$

In the case where $\mathsf{A} = \mathsf{X}$ and R is the identity, we write $\sigma : \mathbb{S} \to \mathbb{T}$ instead of $(1_{\mathsf{X}}, \sigma) : \mathbb{S} \to \mathbb{T}$.

A monad can also be described by way of a *Kleisli triple* $(T, \eta, (-)^{\mathbb{T}})$ on X (Exercise 1.3.12 in [9]), that is,

   (i) a function $T : \mathrm{ob}\,\mathsf{X} \to \mathrm{ob}\,\mathsf{X}$,

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(ii) for every X-object X, an X-morphism $\eta_X : X \to TX$,

(iii) an *extension operation* $(-)^{\mathbb{T}}$ that sends an X-morphism $f : X \to TY$ to an X-morphism $f^{\mathbb{T}} : TX \to TY$,

subject to the conditions

$$\eta_X^{\mathbb{T}} = 1_{TX} \quad , \qquad f^{\mathbb{T}} \cdot \eta_X = f \qquad \text{and} \qquad g^{\mathbb{T}} \cdot f^{\mathbb{T}} = (g^{\mathbb{T}} \cdot f)^{\mathbb{T}} \ . \tag{$*$}$$

Every Kleisli triple $(T, \eta, (-)^{\mathbb{T}})$ yields a monad $\mathbb{T} = (T, \eta, \mu)$ via

$$Tf := (\eta_Y \cdot f)^{\mathbb{T}} \qquad \text{and} \qquad \mu_X := (1_{TX})^{\mathbb{T}} \ ,$$

and every monad $\mathbb{T} = (T, \eta, \mu)$ defines a Kleisli triple thanks to

$$f^{\mathbb{T}} := \mu_Y \cdot Tf \ .$$

These processes are inverse of one another, and we freely switch between the two descriptions: not only is the extension operation $(-)^{\mathbb{T}}$ ubiquitous in our context, but the three Kleisli triple conditions are very economical to verify.

In the case where two Kleisli triples $(S, \delta, (-)^{\mathbb{S}})$ and $(T, \eta, (-)^{\mathbb{T}})$ are defined on the same category X, a family $(\sigma_X : SX \to TX)_{X \in \mathrm{ob}\,\mathsf{X}}$ defines a monad morphism $\sigma : \mathbb{S} \to \mathbb{T}$ if and only if the equalities

$$\eta_X = \sigma_X \cdot \delta_X \qquad \text{and} \qquad (\sigma_Y \cdot f)^{\mathbb{T}} \cdot \sigma_X = \sigma_Y \cdot f^{\mathbb{S}}$$

hold for all X-objects X and X-morphisms $f : X \to SY$.

2.2 Eilenberg-Moore and Kleisli categories. Given a monad $\mathbb{T} = (T, \eta, \mu)$ on a category X, an *Eilenberg-Moore algebra* (or a $\mathbb{T}$-*algebra*) is a pair (X, a), with X an object of X, and $a : TX \to X$ a *structure morphism* that satisfies

$$1_X = a \cdot \eta_X \qquad \text{and} \qquad a \cdot Ta = a \cdot \mu_X \ .$$

In particular, the pair (TX, μ_X) forms an Eilenberg-Moore algebra, the *free $\mathbb{T}$-algebra* on X. A *morphism* of Eilenberg-Moore algebras $f : (X, a) \to (Y, b)$ is an X-morphism $f : X \to Y$ such that

$$f \cdot a = b \cdot Tf \ .$$

Thus, a $\mathbb{T}$-algebra structure a is itself such a morphism $a : (TX, \mu_X) \to (X, a)$. The category of Eilenberg-Moore algebras and their morphisms is denoted by $\mathsf{X}^{\mathbb{T}}$ and is also called the *Eilenberg-Moore category* of $\mathbb{T}$. If $\mathbb{T}$ is given by a Kleisli triple, the conditions for an X-morphism $a : TX \to X$ to form an Eilenberg-Moore structure can be expressed as

$$1_X = a \cdot \eta_X \qquad \text{and} \qquad \forall f, g \in \mathsf{X}(Y, TX) \ (a \cdot f = a \cdot g \implies a \cdot f^{\mathbb{T}} = a \cdot g^{\mathbb{T}}) \ .$$

If $\mathbb{S} = (S, \delta, \nu)$ is a monad on A and $\mathbb{T}$ a monad on X, then a functor $\overline{R} : \mathsf{X}^{\mathbb{T}} \to \mathsf{A}^{\mathbb{S}}$ is said to be *algebraic* over a functor $R : \mathsf{X} \to \mathsf{A}$ if it makes the diagram

$$
\begin{array}{ccc}
\mathsf{X}^{\mathbb{T}} & \xrightarrow{\ \overline{R}\ } & \mathsf{A}^{\mathbb{S}} \\
\downarrow & & \downarrow \\
\mathsf{X} & \xrightarrow{\ R\ } & \mathsf{A}
\end{array}
$$

commute (the vertical arrows represent the respective forgetful functors). Any monad morphism $(R, \sigma) : \mathbb{S} \to \mathbb{T}$ from a monad $\mathbb{S}$ on A to a monad $\mathbb{T}$ on X induces such an algebraic functor; this is defined on objects by

$$\overline{R}(X, a) = (RX, Ra \cdot \sigma_X) \ ,$$

and necessarily sends an X-morphism f to Rf. Conversely, every functor $\overline{R} : \mathsf{X}^{\mathbb{T}} \to \mathsf{A}^{\mathbb{S}}$ that is algebraic over $R : \mathsf{X} \to \mathsf{A}$ is induced by a monad morphism (R, σ): if $\overline{\mu}_X : SRTX \to RTX$ denotes the A-morphism given by $\overline{R}(TX, \mu_X) = (RTX, \overline{\mu}_X)$, then one can define the components of $\sigma : SR \to RT$ by

$$\sigma_X := \overline{\mu}_X \cdot SR\eta_X \ .$$

The objects of the *Kleisli category* $\mathsf{X}_{\mathbb{T}}$ associated to the monad $\mathbb{T}$ are the objects of X, and morphisms $f : X \rightharpoonup Y$ in $\mathsf{X}_{\mathbb{T}}$ are those X-morphisms $f : X \to TY$. The *Kleisli composition* of $f : X \rightharpoonup Y$ and $g : Y \rightharpoonup Z$ in $\mathsf{X}_{\mathbb{T}}$ is defined via the composition in X as
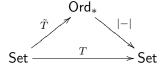
$$g \circ f := \mu_Z \cdot Tg \cdot f = g^{\mathbb{T}} \cdot f \ .$$

The identity $1_X : X \rightharpoonup X$ in this category is just the component $\eta_X : X \to TX$ of the unit.

2.3 Order-adjoint monads. Let Ord denote the category of ordered sets (that is, sets equipped with a reflexive, transitive and antisymmetric relation) with monotone maps, and Ord_* the subcategory of Ord with same objects but whose maps are left adjoint. Explicitly a map $f : X \to Y$ is a morphism of Ord_* if it is monotone and there exists a monotone map, denoted by $f^* : Y \to X$, satisfying

$$1_X \leq f^* \cdot f \qquad \text{and} \qquad f \cdot f^* \leq 1_Y \ .$$

A functor $T : \mathsf{Set} \to \mathsf{Set}$ *factors through* Ord_* if there is a functor $\tilde{T} : \mathsf{Set} \to \mathsf{Ord}_*$ that makes the diagram

$$
\begin{array}{ccc}
 & \mathsf{Ord}_* & \\
{\scriptstyle \tilde{T}} \nearrow & & \searrow {\scriptstyle |-|} \\
\mathsf{Set} & \xrightarrow[\ \ T\ \ ]{} & \mathsf{Set}
\end{array}
$$

commute (where $|-|$ denotes the forgetful functor). For convenience, such a functor T is understood to be *given with* a fixed $\tilde{T}$ that is moreover identified with T; for example, we talk about "the right adjoint $(Tf)^*$ of $Tf : TX \to TY$" to mean "the underlying function of the right adjoint

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$(\tilde{T}f)^*$ of $\tilde{T}f : \tilde{T}X \to \tilde{T}Y$". The hom-sets $\mathsf{Set}(X, TY)$ are then equipped with the pointwise order, so that for $f, f' \in \mathsf{Set}(X, TY)$, one has

$$f \leq f' \iff \forall x \in X \; (f(x) \leq f'(x)) \; .$$

A monad $\mathbb{T} = (T, \eta, \mu)$ on Set is *order-adjoint* if the components of its extension operation $(-)^{\mathbb{T}}$ take values in Ord_*, that is, if and only if T factors through Ord_* and every component μ_X of the monad multiplication is a morphism in Ord_*.

Without any additional assumption on a monad $\mathbb{T}$ on Set whose extension operation takes values in Ord_*, composition in the Kleisli category $\mathsf{Set}_{\mathbb{T}}$ is only monotone in the second variable:

$$f \leq f' \implies h \circ f \leq h \circ f'$$

for all $f, f' \in \mathsf{Set}_{\mathbb{T}}(X, Y)$, $h \in \mathsf{Set}_{\mathbb{T}}(Y, Z)$. We say that an order-adjoint monad $\mathbb{T}$ is *enhanced* if moreover $(-)^{\mathbb{T}}$ preserves the order on the hom-sets $\mathsf{Set}_{\mathbb{T}}(X, Y)$:

$$f \leq f' \implies f^{\mathbb{T}} \leq (f')^{\mathbb{T}}$$

for all $f, f' \in \mathsf{Set}_{\mathbb{T}}(X, Y)$. This condition is equivalent to requiring that composition in $\mathsf{Set}_{\mathbb{T}}$ is monotone in the first variable, so an enhanced order-adjoint monad makes the Kleisli category $\mathsf{Set}_{\mathbb{T}}$ into an ordered category. Note that even if an order-adjoint monad is enhanced, its functor needs not preserve adjoint situations, that is, $T(Tf)^* = (TTf)^*$ does not hold in general.

2.4 Lemma. *An Ord_*-morphism $f : X \to Y$ is split epic in Set if and only if $f \cdot f^* = 1_Y$.*

Proof. If $f \cdot f^* = 1_Y$, then f is split epic by definition. If there is a map $g : Y \to X$ with $f \cdot g = 1_Y$, then $g \leq f^*$. Therefore, $1_Y \leq f \cdot f^* \leq 1_Y$ and equality holds. $\square$

2.5 Proposition. *A monad $\mathbb{T}$ on Set is order-adjoint if and only if the forgetful functor from $\mathsf{Set}^{\mathbb{T}}$ to Set factors through Ord_*.*

Hence, if $\mathbb{T}$ is order-adjoint, then all $\mathbb{T}$-algebras (X, a) are ordered sets, and all $\mathbb{T}$-algebra morphisms $f : (X, a) \to (Y, b)$, in particular $a : (TX, \mu_X) \to (X, a)$, are left adjoint maps.

Proof. We give a brief outline of the proof, while details can be found in [15]. If $\mathbb{T}$ is order-adjoint, the order on the underlying set of a $\mathbb{T}$-algebra (X, a) is inherited from TX via the map $a^\circ := \mu_X \cdot (Ta)^* \cdot \eta_X$:

$$x \leq y \iff a^\circ(x) \leq a^\circ(y)$$

for all $x, y \in X$; this order makes a° into the right adjoint a^* of a (and returns the original order on TX via μ_X°). For a $\mathbb{T}$-algebra morphism $f : (X, a) \to (Y, b)$, one defines the map $f^\circ := a \cdot (Tf)^* \cdot b^*$, which turns out to be the right adjoint f^* of f (relatively to the orders on X and Y induced by a and b respectively, as described above). Conversely, if $\mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}$ factors through Ord_*, one observes that $\mathbb{T}$ is order-adjoint by exploiting that all $Tf : (TX, \mu_X) \to (TY, \mu_Y)$ and $\mu_X : (TTX, \mu_{TX}) \to (TX, \mu_X)$ are $\mathsf{Set}^{\mathbb{T}}$-morphisms. $\square$

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2.6 Proposition. *Let $\tau : \mathbb{S} \to \mathbb{T}$ be a monad morphism from an order-adjoint monad $\mathbb{S} = (S, \delta, \nu)$ to a monad $\mathbb{T} = (T, \eta, \mu)$ on Set. Then $\mathbb{T}$ is order-adjoint, and the components $\tau_X : SX \to TX$ are left adjoints (with respect to the induced order on TX).*

Proof. As in the proof of Proposition 2.5, the $\mathbb{S}$-algebra structure $\mu_X \cdot \tau_{TX} : STX \to TX$ defines an order on TX via

$$x \leq y \iff (\mu_X \cdot \tau_{TX})^\circ(x) \leq (\mu_X \cdot \tau_{TX})^\circ(y) \ ,$$

where $(\mu_X \cdot \tau_{TX})^\circ := \nu_X \cdot (S(\mu_X \cdot \tau_{TX}))^* \cdot \delta_{TX}$, and one then has $(\mu_X \cdot \tau_{TX})^\circ = (\mu_X \cdot \tau_{TX})^*$. For monotonicity of Tf, we use Lemma 2.4 in

$$Tf = Tf \cdot (\mu_X \cdot \tau_{TX}) \cdot (\mu_X \cdot \tau_{TX})^* = (\mu_Y \cdot \tau_{TY}) \cdot STf \cdot (\mu_X \cdot \tau_{TX})^*$$

to observe that Tf is a composite of monotone maps. Monotonicity of μ_X and τ_X are proved similarly. The respective right adjoints are easily seen to be given by

$$(Tf)^* := \mu_X \cdot \tau_{TX} \cdot (STf)^* \cdot (\mu_Y \cdot \tau_{TY})^* \ ,$$
$$\mu_X^* := \mu_{TX} \cdot \tau_{TTX} \cdot (S\mu_X)^* \cdot (\mu_X \cdot \tau_{TX})^* \ ,$$
$$\tau_X^* := \nu_X \cdot (S\tau_X)^* \cdot (\mu_X \cdot \tau_{TX})^* \ .$$

$\square$

For the explicit description of the monads mentioned in the following examples, we refer to [15] or [14]. Further references are given in 4.10.

2.7 Examples. The Eilenberg-Moore algebras of the powerset monad $\mathbb{P}$ on Set are complete lattices, with their morphisms sup-maps:

$$\mathsf{Set}^{\mathbb{P}} \cong \mathsf{Sup} \ .$$

Since $\mathbb{P}$ is order-adjoint, any monad morphism $\tau : \mathbb{P} \to \mathbb{T}$ makes $\mathbb{T}$ order-adjoint (Proposition 2.6). Thus, the filter monad $\mathbb{F}$ on Set becomes order-adjoint via the principal-filter monad morphism $\tau : \mathbb{P} \to \mathbb{F}$; the same statement holds for up-set monad $\mathbb{U}$, and the double-dualization monad $\mathbb{D}$, since there is a chain of monad morphisms

$$\mathbb{P} \to \mathbb{F} \to \mathbb{U} \to \mathbb{D}$$

(the last two simply given by the inclusions $FX \hookrightarrow UX \hookrightarrow DX$ for all sets X). There is also a monad morphism from $\mathbb{P}$ into the monad $\mathbb{U}_{\mathrm{fin}}$ of finitely generated up-sets (obtained by sending an element $A \in PX$ to $\{B \in PX \mid B \cap A \neq \emptyset\}$) that leads to another chain of monad morphisms

$$\mathbb{P} \to \mathbb{U}_{\mathrm{fin}} \to \mathbb{U} \to \mathbb{D} \ .$$

In this case, the order induced by $\mathbb{P}$ on $U_{\mathrm{fin}}X$, UX, or DX, is given by set-inclusion—rather than its opposite as in the previous filter case. There is also a monad morphism

$$\mathbb{P} \to \mathbb{P}_{\mathbf{P}_+}$$

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of the powerset monad, into the $\mathbf{P}_+$-based powerset monad, where $\mathbf{P}_+$ denotes the extended real half-line $[0,\infty]$ equipped with its quantale structure, in which the tensor is given by extended addition, and the order is opposite to the natural order (the components of the morphism are given by the maps $\chi_{(-)} : PX \to P_{\mathbf{P}_+}X$ that send $A \subseteq X$ to its characteristic function $\chi_A : X \to \mathbf{P}_+$ given by $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ otherwise). With the sets $P_{\mathbf{P}_+}X$ ordered pointwise, $\mathbb{P}_{\mathbf{P}_+}$ becomes order-adjoint. For future reference, note that the monad morphism from $\mathbb{P}$ to $\mathbb{P}_{\mathbf{P}_+}$ has a left inverse

$$\mathbb{P}_{\mathbf{P}_+} \to \mathbb{P}$$

(whose component at X is left adjoint to $\chi_{(-)} : PX \to P_{\mathbf{P}_+}X$, and sends a map $\phi : X \to \mathbf{P}_+$ to the set $\{x \in X \mid \phi(x) < \infty\}$).

One readily checks that all of these order-adjoint monads—with the notable exception of the double-dualization monad $\mathbb{D}$—are enhanced.

3   Kleisli monoids

3.1 The category of Kleisli monoids. The definition of a Kleisli monoid is given by way of a category X that is not quite an ordered category, as our main goal is to study the case of order-adjoint monads $\mathbb{T}$ on $\mathsf{X} = \mathsf{Set}$ (see also Proposition 5.3).

Let $\mathbb{T} = (T, e, m)$ be a monad on a category X whose hom-sets $\mathsf{X}(X, TY)$ are equipped with an order that is preserved by composition on the right:

$$f \leq f' \implies f \cdot g \leq f' \cdot g$$

for all $f, f' : X \to TY$, $g : Z \to X$. A *Kleisli monoid* (or $\mathbb{T}$-*monoid*) in X is a pair (X, α) made up of an X-object X and a structure X-morphism $\alpha : X \to TX$ that is *extensive* and *idempotent*:

$$e_X \leq \alpha , \quad \alpha \circ \alpha \leq \alpha$$

(composition is taken in the Kleisli category $\mathsf{X}_{\mathbb{T}}$). In the presence of extensivity, idempotency may be legitimately expressed as an equality $\alpha \circ \alpha = \alpha$; furthermore, $\alpha^{\mathbb{T}} : TX \to TX$ is also idempotent:

$$\alpha^{\mathbb{T}} \cdot \alpha^{\mathbb{T}} = (\alpha^{\mathbb{T}} \cdot \alpha)^{\mathbb{T}} = (\alpha \circ \alpha)^{\mathbb{T}} = \alpha^{\mathbb{T}} .$$

A *Kleisli morphism* (that is, a *morphism of $\mathbb{T}$-monoids*) $f : (X, \alpha) \to (Y, \beta)$ is an X-morphism $f : X \to Y$ such that

$$Tf \cdot \alpha \leq \beta \cdot f$$

and that composes with a Kleisli morphism $g : (Y, \beta) \to (Z, \gamma)$ as in X. The category of Kleisli monoids in X with their morphisms is denoted by $\mathsf{X}(\mathbb{T})$.

In the case where $\mathsf{X} = \mathsf{Set}$, the underlying set of a Kleisli monoid (X, α) can be equipped with the *initial preorder* induced by $\alpha : X \to TX$: for $x, y \in X$

$$x \leq y \iff \alpha(x) \leq \alpha(y) .$$

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This preorder becomes an order exactly when $\alpha : X \to TX$ is a monomorphism; in this case, the Kleisli monoid (X, α) is said to be *separated*. The full subcategory of $\mathsf{Set}(\mathbb{T})$ whose objects are separated Kleisli monoids is denoted by $\mathsf{Set}(\mathbb{T})_0$.

3.2 Examples. The categories of Kleisli monoids of the monads given in 2.7 are the following (see [15] or [14]).

$\mathsf{Set}(\mathbb{P}) \cong \mathsf{PrOrd}$**:** category of preordered sets with monotone maps.

$\mathsf{Set}(\mathbb{P})_0 \cong \mathsf{Ord}$**:** category of ordered sets with monotone maps.

$\mathsf{Set}(\mathbb{F}) \cong \mathsf{Top}$**:** category of topological spaces with continuous maps [5].

$\mathsf{Set}(\mathbb{F})_0 \cong \mathsf{Top}_0$**:** category of T_0 topological spaces with continuous maps.

$\mathsf{Set}(\mathbb{U}) \cong \mathsf{Cls}$**:** category of closure spaces with continuous maps.

$\mathsf{Set}(\mathbb{D}) \cong \mathsf{Cls}_\dagger$**:** category "non-monotone" closure spaces with "continuous" maps.

$\mathsf{Set}(\mathbb{U}_{\mathrm{fin}}) \cong \mathsf{Cls}_{\mathrm{fin}}$**:** category of finitary closure spaces with continuous maps.

$\mathsf{Set}(\mathbb{P}_{\mathbf{P}_+}) \cong \mathsf{Met}$**:** category of generalized metric spaces with contractions.

3.3 $\mathbb{T}$-algebras and $\mathbb{T}$-monoids. If $\mathbb{T}$ is an order-adjoint monad, a $\mathbb{T}$-monoid structure α on X is in particular a Kleisli morphism $\alpha : (X, \alpha) \to (TX, \mu_X^*)$. Moreover, for a $\mathbb{T}$-algebra (X, a), the pair (X, a^*) defines a Kleisli monoid. Indeed, the structure a has a right adjoint a^* by Proposition 2.5, so $a \cdot \eta_X \le 1_X$ and $a \cdot \mu_X \le a \cdot Ta$ imply

$$\eta_X \le a^* \qquad \text{and} \qquad \mu_X \cdot T(a^*) \cdot a^* \le (a^* \cdot a \cdot Ta) \cdot T(a^*) \cdot a^* = a^* \ .$$

Similarly, a morphism $f : (X, a) \to (Y, b)$ of $\mathbb{T}$-algebras yields a morphism $f : (X, a^*) \to (Y, b^*)$. The right adjoint operation on structures therefore defines a functor $L : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{T})$. In fact, since $a \cdot a^* = 1_X$ (Lemma 2.4), the structure a^* is a monomorphism, so L factors through the category of separated Kleisli monoids, and can be seen as having $\mathsf{Set}(\mathbb{T})_0$ as codomain. This functor is both faithful and injective on objects (by unicity of the right adjoint of a structure a), so that $\mathsf{Set}^{\mathbb{T}}$ can be considered as a subcategory of $\mathsf{Set}(\mathbb{T})_0$ or of $\mathsf{Set}(\mathbb{T})$.

3.4 Proposition. *Let $\mathbb{S}$ be an order-adjoint monad. A monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ yields a faithful functor $\mathsf{Set}(\tau) : \mathsf{Set}(\mathbb{S}) \to \mathsf{Set}(\mathbb{T})$ obtained by sending an $\mathbb{S}$-monoid (X, α) to $(X, \tau_X \cdot \alpha)$ and leaving maps untouched.*

Proof. Proposition 2.6 shows that in the given situation, $\mathbb{T}$ factors through Ord, so that $\mathsf{Set}(\mathbb{T})$ can be defined. The claim then follows by straightforward verifications using that τ_X is monotone. $\square$

3.5 Proposition. *Let $\mathbb{S}$ be an order-adjoint monad, and $\tau : \mathbb{S} \to \mathbb{T}$ a monad morphism. There is a functor $Q : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{S})$ that sends a $\mathbb{T}$-algebra (X, a) to $(X, (a \cdot \tau_X)^*)$ and commutes with the underlying-set functors.*

Proof. A $\mathbb{T}$-algebra (X, a) yields an $\mathbb{S}$-algebra $(X, a \cdot \tau_X)$ via $\mathsf{Set}^\tau : \mathsf{Set}^\mathbb{T} \to \mathsf{Set}^\mathbb{S}$, and therefore an $\mathbb{S}$-monoid $(X, (a \cdot \tau_X)^*)$ thanks to the functor $\mathsf{Set}^\mathbb{S} \to \mathsf{Set}(\mathbb{S})$ of 3.3. These operation therefore describe a functor $Q : \mathsf{Set}^\mathbb{T} \to \mathsf{Set}(\mathbb{S})$ that commutes with the underlying-set functors. $\qquad\square$

4 Monads on $\mathsf{Set}(\mathbb{S})$

4.1 The equalizer construction. Consider a morphism $\tau : \mathbb{S} \to \mathbb{T}$ between monads on Set with $\mathbb{S}$ order-adjoint. We proceed to describe the components T', η', and $(-)^{\mathbb{T}'}$ of a Kleisli triple on $\mathsf{Set}(\mathbb{S})$ (Proposition 4.4).

(i) For an object (X, α) of $\mathsf{Set}(\mathbb{S})$, one sets $\beta := \tau_X \cdot \alpha$ and defines $T'X$, the set of $\beta^\mathbb{T}$-*invariants*, as the equalizer in Set of the pair $(\beta^\mathbb{T}, 1_{TX})$:

$$T'X \xrightarrow{\ s_X\ } TX \underset{1_{TX}}{\overset{\beta^\mathbb{T}}{\rightrightarrows}} TX \ .$$

The universal property of s_X yields the existence of a map $r_X : TX \to T'X$ such that

$$s_X \cdot r_X = \beta^\mathbb{T} \qquad \text{and} \qquad r_X \cdot s_X = 1_{T'X} \ .$$

The set $T'X$ can be equipped with the $\mathbb{S}$-monoid structure $\omega_X : T'X \to ST'X$ given by

$$\omega_X := Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \ .$$

Lemma 4.3 below shows that $s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is also an equalizer in $\mathsf{Set}(\mathbb{S})$. This implies that the maps η'_X and $f^{\mathbb{T}'}$ defined in the following points are morphisms of $\mathbb{S}$-monoids.

(ii) Since $\beta^\mathbb{T} \cdot \beta = \beta$ (Proposition 3.4), there exists a map $\eta'_X : X \to T'X$ with $s_X \cdot \eta'_X = \beta$:

$$
\begin{array}{c}
X \\
\eta'_X \big\downarrow \quad\searrow{\scriptstyle \beta} \\
T'X \xrightarrow[s_X]{} TX \underset{1_{TX}}{\overset{\beta^\mathbb{T}}{\rightrightarrows}} TX \ .
\end{array}
$$

This yields a morphism of $\mathbb{S}$-monoids $\eta'_X : (X, \alpha) \to (T'X, \omega_X)$. Let us point out that since $s_X \cdot \eta'_X = \beta$ and $r_X \cdot s_X = 1_{T'X}$, one can equivalently obtain η'_X as either

$$\eta'_X = r_X \cdot \beta \qquad \text{or} \qquad \eta'_X = r_X \cdot \eta_X$$

because $r_X \cdot \beta = r_X \cdot \beta^\mathbb{T} \cdot \eta_X = r_X \cdot \eta_X$.

(iii) If (Y, α_Y) is another $\mathbb{S}$-monoid, and $f : (Y, \alpha_Y) \to (T'X, \omega_X)$ is a $\mathsf{Set}(\mathbb{S})$-morphism, then one observes

$$\beta^{\mathbb{T}} \cdot (s_X \cdot f)^{\mathbb{T}} = (\beta^{\mathbb{T}} \cdot s_X \cdot f)^{\mathbb{T}} = (s_X \cdot f)^{\mathbb{T}} \ .$$

Thus, there exists a unique map $f^{\mathbb{T}'} : T'Y \to T'X$ making the following diagram commute:

$$\begin{array}{c}
T'Y \\
{\scriptstyle f^{\mathbb{T}'}} \Big\downarrow \quad \searrow {\scriptstyle (s_X \cdot f)^{\mathbb{T}} \cdot s_Y} \\
T'X \xrightarrow[s_X]{} TX \underset{1_{TX}}{\overset{\beta^{\mathbb{T}}}{\rightrightarrows}} TX \ .
\end{array}$$

This yields a morphism of $\mathbb{S}$-monoids $f^{\mathbb{T}'} : (T'Y, \omega_Y) \to (T'X, \omega_X)$ that can also be described directly as

$$f^{\mathbb{T}'} = r_X \cdot (s_X \cdot f)^{\mathbb{T}} \cdot s_Y \ .$$

4.2 Remark. The monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ does not need to make $\mathbb{T}$ enhanced (Proposition 2.6), so it is not clear in general whether $r_X : (TX, (\mu_X \cdot \tau_{TX})^*) \to (T'X, \omega_X)$ is a Kleisli morphism or not.

4.3 Lemma. *For a monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ with $\mathbb{S}$ an order-adjoint monad, the map*

$$s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$$

(defined in the previous construction) is an equalizer in $\mathsf{Set}(\mathbb{S})$. As a consequence,

$$\eta'_X : (X, \alpha) \to (T'X, \omega_X) \qquad \text{and} \qquad f^{\mathbb{T}'} : (T'Y, \omega_Y) \to (T'X, \omega_X)$$

are $\mathsf{Set}(\mathbb{S})$-morphisms.

Proof. To verify that $\omega_X : T'X \to ST'X$ is an $\mathbb{S}$-monoid structure, observe that

$$\delta_{T'X} = \delta_{T'X} \cdot r_X \cdot s_X = Sr_X \cdot \delta_{TX} \cdot s_X \le Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X$$

and

$$
\begin{aligned}
\omega_X^{\mathbb{S}} \cdot \omega_X &= Sr_X \cdot \nu_{TX} \cdot S(\mu_X \cdot \tau_{TX})^* \cdot S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X && (s_X \cdot r_X = \beta^{\mathbb{T}}) \\
&\le Sr_X \cdot \nu_{TX} \cdot (S(\mu_X \cdot \tau_{TX}))^* \cdot S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X && \text{(Lemma 2.4)} \\
&\le Sr_X \cdot \nu_{TX} \cdot (S\mu_X \cdot S\tau_{TX})^* \cdot (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X && \text{(Proposition 3.4)} \\
&= Sr_X \cdot \nu_{TX} \cdot (\mu_X \cdot \tau_{TX} \cdot \nu_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X && (\tau \text{ monad morphism}) \\
&= Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X && (\nu_{TX} \cdot \nu_{TX}^* = 1_{STX}) \\
&= \omega_X \ .
\end{aligned}
$$

One can reason similarly to obtain

$$Ss_X \cdot \omega_X = S\beta^{\mathbb{T}} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X \le (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot s_X = (\mu_X \cdot \tau_{TX})^* \cdot s_X \ ,$$

so that $s_X : (T'X, \omega_X) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. Suppose now that $g : (Y, \alpha_Y) \to (TX, (\mu_X \cdot \tau_{TX})^*)$ is a $\mathsf{Set}(\mathbb{S})$-morphism satisfying $\beta^{\mathbb{T}} \cdot g = g$. Since $s_X : T'X \to TX$ is an equalizer of $(\beta^{\mathbb{T}}, 1_{TX})$ in Set, there exists a unique map $h : Y \to T'X$ with $g = s_X \cdot h$; moreover,

$$Sh \cdot \alpha_Y = Sr_X \cdot Sg \cdot \alpha_Y \le Sr_X \cdot (\mu_X \cdot \tau_{TX})^* \cdot g = \omega_X \cdot h \ ,$$

which shows that $h : (Y, \alpha_Y) \to (T'X, \omega_X)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. As a consequence, s_X is an equalizer in $\mathsf{Set}(\mathbb{S})$, and η'_X, $f^{\mathbb{T}'}$ are the underlying maps of the corresponding unique $\mathsf{Set}(\mathbb{S})$-morphisms into $(T'X, \omega_X)$. $\qquad \square$

4.4 Proposition. *If $\tau_X : \mathbb{S} \to \mathbb{T}$ is a monad morphism and $\mathbb{S}$ an order-adjoint monad, then the construction detailed in* (i)–(iii) *above defines a Kleisli triple $(T', \eta', (-)^{\mathbb{T}'})$ on $\mathsf{Set}(\mathbb{S})$.*

Proof. Lemma 4.3 insures that the construction yields components T', η', and $(-)^{\mathbb{T}'}$ of a Kleisli triple on $\mathsf{Set}(\mathbb{S})$. The conditions $(*)$ of 2.1 follow from a straightforward verification, as in [15]. $\quad \square$

4.5 Proposition. *Let (X, α) be an $\mathbb{S}$-monoid. The initial preorder induced by $\omega_X : T'X \to ST'X$ on $T'X$ is an order that moreover makes $s_X : T'X \to TX$ into an order-embedding, and $r_X : TX \to T'X$ into a monotone map. If $\mathbb{T}$ is enhanced, then $r_X : (TX, (\mu_X \cdot \tau_{TX})^*) \to (T'X, \omega_X)$ is a Kleisli morphism and the pair (r_X, s_X) forms an adjoint situation $r_X \dashv s_X$.*

Proof. The cited results follow from straightforward verifications using the fact that

$$\mu_X \cdot \tau_{TX} \cdot Ss_X \cdot \omega_X = s_X \ .$$

See [15] for further details. $\qquad \square$

4.6 Examples. If $\mathbb{S} = \mathbb{T} = \mathbb{P}$, then a $\mathbb{P}$-monoid is a pair (X, α), where $\alpha = {\downarrow}_X : X \to PX$ is the down-set map of X; that is, X is a preordered set (as mentioned in 3.2). An element $A \in PX$ is an $\alpha^{\mathbb{P}}$-invariant precisely when $\alpha^{\mathbb{P}}(A) = A$, that is, when A is down-closed:

$$\bigcup_{x \in A} {\downarrow}_X x = A \ .$$

Hence, the monad $\mathbb{P}'$ yields the *down-set monad* $\mathbb{P}_{\downarrow} = (P_{\downarrow}, \downarrow, \bigcup)$ on PrOrd.

If $\mathbb{S} = \mathbb{T} = \mathbb{F}$ is the filter monad, then an $\mathbb{F}$-monoid is a topological space (X, ν), where $\nu : X \to FX$ is the neighborhood filter map. A filter $f \in FX$ is $\nu^{\mathbb{F}}$-invariant if and only if f is spanned by open sets of X:

$$A \in \nu^{\mathbb{F}}(f) \iff \nu^{-1}(A^{\mathbb{F}}) \in f \iff \{x \in X \mid A \in \nu(x)\} \in f$$

for all $A \in PX$ (and where $A^{\mathbb{F}} = \{\chi \in FX \mid A \in \chi\}$), so $\nu^{\mathbb{F}}(f) = f$ means that if $A \in f$ then its interior must also be in f. The monad $\mathbb{F}'$ is the *open-filter monad* on Top, obtained by considering the neighborhood maps $\nu = e'_X : X \to F'X$ to form its unit, and the restriction of the filtered sum of $\mathbb{F}$ for its multiplication.

Consider now the principal filter natural transformation $\tau : \mathbb{P} \to \mathbb{F}$. The previous examples show that the construction of $\mathbb{F}'$ associates to a preordered set $(X, \downarrow_X)$ the topological space (X, ν) whose neighborhood map is given at each $x \in X$ by the principal filter of $\downarrow_X x \in PX$:

$$\nu(x) = \uparrow_{PX}(\downarrow_X x) \ ,$$

that is, (X, ν) is the Alexandroff space associated to a preordered set X, and open sets are down-closed sets. The set of $\nu^{\mathbb{F}}$-invariant filters can be identified with the set of filters on $P_\downarrow X$, and one obtains the *down-set-filter monad* $\mathbb{F}_\downarrow$ on PrOrd.

4.7 Lemma. *If $R : \mathsf{Set}(\mathbb{S}) \to \mathsf{Set}$ denotes the functor that forgets the structure of objects, then the maps r_X form the components of a natural transformation $r : TR \to RT'$, and the pair $(R, r) : \mathbb{T} \to \mathbb{T}'$ defines a monad morphism.*

Proof. An $\mathbb{S}$-monoid morphism $f : (Y, \alpha_Y) \to (X, \alpha)$ yields a $\mathbb{T}$-monoid morphism $f : (Y, \beta_Y) \to (TX, \beta)$ (where $\beta_Y = \tau_Y \cdot \alpha_Y$ and $\beta = \tau_X \cdot \alpha$), so that

$$\beta \cdot f = (\beta \cdot f)^{\mathbb{T}} \cdot \eta_Y \le (\beta \cdot f)^{\mathbb{T}} \cdot \beta_Y = \beta^{\mathbb{T}} \cdot Tf \cdot \beta_Y \le \beta^{\mathbb{T}} \cdot \beta \cdot f = \beta \cdot f \ .$$

Therefore, one has $(\beta \cdot f)^{\mathbb{T}} \cdot \beta_Y = \beta \cdot f$, and using that $T'f = (\eta'_X \cdot f)^{\mathbb{T}'}$ one obtains

$$T'f \cdot r_Y = r_X \cdot (\beta \cdot f)^{\mathbb{T}} \cdot \beta_Y^{\mathbb{T}} = r_X \cdot ((\beta \cdot f)^{\mathbb{T}} \cdot \beta_Y)^{\mathbb{T}} = r_X \cdot (\beta \cdot f)^{\mathbb{T}} = r_X \cdot \beta^{\mathbb{T}} \cdot Tf = r_X \cdot Tf \ ,$$

which proves that $r : TR \to RT'$ is a natural transformation. Moreover, for $\mu'_X = (1_{T'X})^{\mathbb{T}'} = r_X \cdot (s_X)^{\mathbb{T}} \cdot s_{T'X}$, one has

$$
\begin{aligned}
\mu'_X \cdot r_{T'X} \cdot Tr_X &= r_X \cdot (s_X)^{\mathbb{T}} \cdot (\tau_{T'X} \cdot \omega_X)^{\mathbb{T}} \cdot Tr_X \\
&= r_X \cdot (\mu_{TX} \cdot T\beta^{\mathbb{T}} \cdot \tau_{TX} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X)^{\mathbb{T}} \cdot Tr_X \\
&= r_X \cdot (\beta^{\mathbb{T}} \cdot \mu_X \cdot \tau_{TX} \cdot (\mu_X \cdot \tau_{TX})^* \cdot s_X)^{\mathbb{T}} \cdot Tr_X \\
&= r_X \cdot (\beta^{\mathbb{T}} \cdot s_X)^{\mathbb{T}} \cdot Tr_X \\
&= r_X \cdot (s_X)^{\mathbb{T}} \cdot Tr_X \\
&= r_X \cdot \beta^{\mathbb{T}} \cdot \mu_X \\
&= r_X \cdot \mu_X \ .
\end{aligned}
$$

Since $\eta'_X = r_X \cdot \eta_X$, the pair $(R, r) : \mathbb{T} \to \mathbb{T}'$ forms a monad morphism. $\square$

4.8 Theorem. *If $\tau_X : \mathbb{S} \to \mathbb{T}$ is a monad morphism from an order-adjoint monad $\mathbb{S}$, then there is an isomorphism of Eilenberg-Moore categories that is identical on morphisms:*

$$\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})^{\mathbb{T}'} \ .$$

Proof. Suppose first that (X, a) is a $\mathbb{T}$-algebra. One obtains an $\mathbb{S}$-monoid (X, α), with $\alpha = (a \cdot \tau_X)^*$, that can be equipped with the structure $a' : (T'X, \omega_X) \to (X, \alpha)$ defined by

$$a' := a \cdot s_X \ .$$

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By Proposition 3.5, a' is indeed a morphism of $\mathbb{S}$-monoids. To see that a' satisfies the algebra conditions for the monad $\mathbb{T}'$, we first use the definition of η'_X and Lemma 2.4 to obtain

$$a' \cdot \eta'_X = a \cdot s_X \cdot \eta'_X = a \cdot \beta = a \cdot \tau_X \cdot (a \cdot \tau_X)^* = 1_X \ .$$

Suppose now that $f, g : (Y, \beta) \to (T'X, \omega_X)$ are $\mathsf{Set}(\mathbb{S})$-morphisms satisfying $a' \cdot f = a' \cdot g$, or equivalently, $a \cdot s_X \cdot f = a \cdot s_X \cdot g$; since a is a $\mathbb{T}$-algebra structure, one has $a \cdot (s_X \cdot f)^{\mathbb{T}} = a \cdot (s_X \cdot g)^{\mathbb{T}}$ (see 2.2), so that

$$a' \cdot f^{\mathbb{T}'} = a \cdot s_X \cdot f^{\mathbb{T}'} = a \cdot (s_X \cdot f)^{\mathbb{T}} \cdot s_Y = a \cdot (s_X \cdot g)^{\mathbb{T}} \cdot s_Y = a \cdot s_X \cdot g^{\mathbb{T}'} = a' \cdot g^{\mathbb{T}'} \ .$$

Therefore, $((X, \alpha), a')$ is a $\mathbb{T}'$-algebra. A $\mathbb{T}$-algebra morphism $f : (X, a_X) \to (Y, a_Y)$ yields a $\mathsf{Set}(\mathbb{S})$-morphism $f : (X, (a_X \cdot \tau_X)^*) \to (Y, (a_Y \cdot \tau_Y)^*)$. Since a_Y is a $\mathbb{T}$-algebra structure, one has

$$\begin{aligned}
a_Y \cdot (\tau_Y \cdot (a_Y \cdot \tau_Y)^* \cdot f)^{\mathbb{T}} &= a_Y \cdot \mu_X \cdot T(\tau_Y \cdot (a_Y \cdot \tau_Y)^*) \cdot Tf \\
&= a_Y \cdot Ta_Y \cdot T(\tau_Y \cdot (a_Y \cdot \tau_Y)^*) \cdot Tf \\
&= a_Y \cdot Tf
\end{aligned}$$

by Lemma 2.4. To verify that $a'_Y \cdot (\eta'_Y \cdot f)^{\mathbb{T}'} = f \cdot a'_X$, we use the previous observation in

$$\begin{aligned}
a'_Y \cdot (\eta'_Y \cdot f)^{\mathbb{T}'} &= a_Y \cdot (s_Y \cdot \eta'_Y \cdot f)^{\mathbb{T}} \cdot s_X \\
&= a_Y \cdot (\tau_Y \cdot (a_Y \cdot \tau_Y)^* \cdot f)^{\mathbb{T}} \cdot s_X \\
&= f \cdot a_X \cdot s_X \\
&= f \cdot a'_X \ ,
\end{aligned}$$

which proves that $f : ((X, \alpha_X), a'_X) \to ((Y, \alpha_Y), a'_Y)$ is a morphism of $\mathbb{T}'$-algebras. Thus, the assignment of $((X, (a \cdot \tau_X)^*), a \cdot s_X)$ to a $\mathbb{T}$-algebra (X, a) yields a functor $\overline{Q} : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$ that leaves maps untouched.

The monad morphism $(R, r) : \mathbb{T} \to \mathbb{T}'$ (Lemma 4.7) yields a functor $\overline{R} : \mathsf{Set}(\mathbb{S})^{\mathbb{T}'} \to \mathsf{Set}^{\mathbb{T}}$ by the discussion in 2.2. This functor sends a $\mathbb{T}'$-algebra $((X, \alpha), a')$ to (X, a), where $a : TX \to X$ is defined by

$$a := a' \cdot r_X \ ,$$

and is invariant on maps.

Given a $\mathbb{T}$-algebra (X, a), the structure of $\overline{R}\,\overline{Q}(X, a)$ is described by

$$a \cdot s_X \cdot r_X = a \cdot \mu_X \cdot T(\tau_X \cdot (a \cdot \tau_X)^*) = a \cdot T(a \cdot \tau_X) \cdot T(a \cdot \tau_X)^* = a \ .$$

To study the image of a $\mathbb{T}'$-algebra $((X, \alpha), a')$ via $\overline{Q}\,\overline{R}$, note first that $a' : (T'X, \omega_X) \to (X, \alpha)$ is a $\mathsf{Set}(\mathbb{S})$-morphism. Thus, after setting $\beta = \tau_X \cdot \alpha$ and observing that $(\mu_X \cdot \tau_{TX}) \cdot S(\tau_X \cdot \alpha) = \beta^{\mathbb{T}} \cdot \tau_X$, one obtains

$$\begin{aligned}
1_{SX} = S(a' \cdot r_X) \cdot S(\tau_X \cdot \alpha) \\
\leq S(a' \cdot r_X) \cdot (\mu_X \cdot \tau_{TX})^* \cdot \beta^{\mathbb{T}} \cdot \tau_X \\
= Sa' \cdot \omega_X \cdot r_X \cdot \tau_X \\
\leq \alpha \cdot (a' \cdot r_X \cdot \tau_X) \ .
\end{aligned}$$

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This inequality, combined with $(a' \cdot r_X \cdot \tau_X) \cdot \alpha = 1_X$ and the fact that both α and $(a' \cdot r_X \cdot \tau_X)$ are monotone, yields

$$\alpha = (a' \cdot r_X \cdot \tau_X)^* \ .$$

Hence, the image via $\overline{Q}\,\overline{R}$ of the $\mathbb{T}'$-algebra $((X, \alpha), a')$ returns the $\mathbb{T}'$-algebra whose underlying $\mathbb{S}$-monoid is $(X, (a' \cdot r_X \cdot \tau_X)^*) = (X, \alpha)$; its structure is therefore given by

$$a' \cdot r_X \cdot s_X = a' \ ,$$

so that $\overline{Q}$ and $\overline{R}$ are inverse of one another, and $\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})^{\mathbb{T}'}$. $\qquad\square$

4.9 Corollary. *Given a morphism $\tau : \mathbb{S} \to \mathbb{T}$ with $\mathbb{S}$ an order-adjoint monad, the monad $\mathbb{T}'$ restricts to $\mathsf{Set}(\mathbb{S})_0$, and the isomorphism of Theorem 4.8 becomes*

$$\mathsf{Set}^{\mathbb{T}} \cong \mathsf{Set}(\mathbb{S})_0^{\mathbb{T}'} \ .$$

Proof. The functor $\overline{R}$ restricts to $\mathsf{Set}(\mathbb{S})_0^{\mathbb{T}}$, and $\overline{Q}$ factors through $\mathsf{Set}(\mathbb{S})_0^{\mathbb{T}'}$ since the functor $Q : \mathsf{Set}^{\mathbb{S}} \to \mathsf{Set}(\mathbb{S})$ of 3.1 factors through $\mathsf{Set}(\mathbb{S})_0$. $\qquad\square$

4.10 Examples. The Eilenberg-Moore algebras of the monads mentioned in 2.7 have been described as follows (although most results are classical, we try to give the original printed source in each case and refer to [9] and [6] for further details).

$\mathsf{Set}^{\mathbb{F}} \cong \mathsf{Cnt}$: category of continuous lattices with continuous sup-maps [4]; see also 5.4 below.

$\mathsf{Set}^{\mathbb{U}} \cong \mathsf{Ccd}$: category of constructive completely distributive lattices with maps that preserve all suprema and infima [12].

$\mathsf{Set}^{\mathbb{D}} \cong \mathsf{CaBool}$: category of complete atomistic Boolean algebras with ring homomorphisms that preserve all suprema and infima [9].

$\mathsf{Set}^{\mathbb{U}_{\mathrm{fin}}} \cong \mathsf{Frm}$: category of frames with sup-maps that preserve finite infima, see [2] (in fact, Bénabou describes free frames over meet-semilattices; in conjunction with the free meet-semilattice construction over sets, one obtains monadicity over Set as in [6]).

$\mathsf{Set}^{\mathbb{P}_{\mathbf{P}+}} \cong \mathbf{P}_+\text{-}\mathsf{Mod}$: category of left $\mathbf{P}_+$-modules with sup-maps that commute with the action of $\mathbf{P}_+$ on Sup, see [11].

Hence, one obtains the following table of strict monadicities (of the categories in the entry line over categories displayed in the entry column) using Theorems 4.8 and Corollary 4.9. Previous explicit references are mentioned to the best of our knowledge, though we make absolutely *no* originality claim in their absence. For example, monadicity of Cnt over Ord can hardly be considered novel, but we were not able to find this particular instance in the literature; similarly, the column for CaBool is not surprising in view of [16], even though the results presented therein refer to not-necessarily-strict monadicity.

	Sup	Cnt	Frm	Ccd	CaBool	$\mathbf{P}_+$-Mod
Set	[7]	[4]	[6]	[12]	[9]	[11]
PrOrd	Thm. 4.8	[13]	Thm. 4.8	[13]	Thm. 4.8	Thm. 4.8
Ord	Cor. 4.9	Cor. 4.9	Linton ([10])	[10]	Cor. 4.9	Cor. 4.9
Top		[17]		Thm. 4.8	Thm. 4.8	
Top_0		[4]		Cor. 4.9	Cor. 4.9	
Cls_{fin}			Thm. 4.8	Thm. 4.8	Thm. 4.8	
Cls				Thm. 4.8	Thm. 4.8	
$\text{Cls}_\dagger$					Thm. 4.8	
Met	Thm. 4.8	Thm. 4.8	Thm. 4.8	Thm. 4.8	Thm. 4.8	[11]

5 Algebras of enhanced order-adjoint monads

A morphism $\tau_X : \mathbb{S} \to \mathbb{T}$ between monads on Set induces a functor $\mathsf{Set}^\tau : \mathsf{Set}^{\mathbb{T}} \to \mathsf{Set}^{\mathbb{S}}$, so that a $\mathbb{T}$-algebra (X, a) is an $\mathbb{S}$-algebra $(X, a \cdot \tau_X)$. Theorem 4.8 can be used in the same way to identify categories of $\mathbb{T}$-algebras. We illustrate this further on by giving an original proof of the isomorphism $\mathsf{Set}^{\mathbb{F}} \cong \mathsf{Cnt}$.

5.1 Lemma. *Let $\mathbb{S} = (S, \delta, \nu)$ be an order-adjoint monad and $\tau : \mathbb{S} \to \mathbb{T}$ a monad morphism that makes $\mathbb{T} = (T, \eta, \mu)$ enhanced (via the order described in Proposition 2.6). Given an $\mathbb{S}$-monoid (X, α) and a map $\lambda : X \to T'X$, one has that $((X, \alpha), \lambda)$ is a $\mathbb{T}'$-monoid if and only if $(X, s_X \cdot \lambda)$ is a $\mathbb{T}$-monoid (using the notations of Section 4).*

Proof. Let us first verify that a $\mathbb{T}'$-monoid structure λ on (X, α) yields a $\mathbb{T}$-monoid structure $s_X \cdot \lambda$. From $r_X \cdot \eta_X = \eta'_X \leq \lambda$, one obtains extensivity of $s_X \cdot \lambda$:

$$\eta_X = \tau_X \cdot \delta_X \leq \tau_X \cdot \alpha = (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot \eta_X = s_X \cdot r_X \cdot \eta_X \leq s_X \cdot \lambda .$$

Idempotency is then a consequence of

$$(s_X \cdot \lambda)^{\mathbb{T}} \cdot s_X \cdot \lambda = s_X \cdot \lambda^{\mathbb{T}'} \cdot \lambda \leq s_X \cdot \lambda .$$

Suppose now that $s_X \cdot \lambda$ is a $\mathbb{T}$-monoid structure with $\tau_X \cdot \alpha \leq s_X \cdot \lambda$. As $\mathbb{T}$ is enhanced, one immediately obtains $\eta'_X \leq \lambda$ and $\lambda^{\mathbb{T}'} \cdot \lambda \leq \lambda$ from extensivity and idempotency of $s_X \cdot \lambda$. We are therefore left to verify that $\lambda : (X, \alpha) \to (T'X, \omega)$ is an $\mathbb{S}$-monoid morphism; by composing each side of the extensivity condition with $(\tau_X \cdot \alpha)^{\mathbb{T}} = s_X \cdot r_X$ on the left, we obtain $\tau_X \cdot \alpha = (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot \eta_X \leq s_X \cdot \lambda$, so that

$$\mu_X \cdot \tau_{TX} \cdot S(s_X \cdot \lambda) \cdot \alpha = (s_X \cdot \lambda)^{\mathbb{T}} \cdot \tau_X \cdot \alpha \leq (s_X \cdot \lambda)^{\mathbb{T}} \cdot s_X \cdot \lambda \leq s_X \cdot \lambda .$$

After composing these expressions with $Sr_X \cdot (\mu_X \cdot \tau_{TX})^*$ on the left, we obtain the desired inequality. $\qquad\square$

5.2 Lemma. *Given an enhanced order-adjoint monad $\mathbb{T}$, a $\mathbb{T}$-monoid (X, α) is of the form (X, a^*) for a $\mathbb{T}$-algebra structure $a : TX \to X$ if and only if α has a left adjoint α_* with $\alpha_* \cdot \alpha = 1_X$.*

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Proof. See Corollary 4.11 in [15]. □

5.3 Proposition. *Given an order-adjoint monad $\mathbb{S}$ and a monad morphism $\tau : \mathbb{S} \to \mathbb{T}$ that makes $\mathbb{T}$ enhanced, an $\mathbb{S}$-monoid morphism $a' : (T'X, \omega_X) \to (X, \alpha)$ is a $\mathbb{T}'$-algebra structure if and only if $a' : T'X \to X$ has a right adjoint $(a')^*$ with $a' \cdot (a')^* = 1_X$ that moreover makes $(X, (a')^*)$ into a $\mathbb{T}'$-monoid.*

Proof. Let (X, α) be an $\mathbb{S}$-monoid. If $a' : (T'X, \omega) \to (X, \alpha)$ is a $\mathbb{T}'$-algebra structure, then $a = a' \cdot r_X : TX \to X$ is a $\mathbb{T}$-algebra structure by the correspondence between algebras described in the proof of Theorem 4.8. Proposition 2.5 states that a has a right adjoint $a^* : X \to TX$, and one observes that

$$1_{T'X} \le r_X \cdot a^* \cdot a \cdot s_X = (r_X \cdot a^*) \cdot a' \quad, \qquad a' \cdot (r_X \cdot a^*) = a \cdot a^* = 1_X \;,$$

so $(a')^* = r_X \cdot (a' \cdot r_X)^*$ is the required right adjoint of a'. The proof of Theorem 4.8 shows that $(a \cdot \tau_X)^*$ is the $\mathbb{S}$-monoid structure α; since $\mathbb{T}$ is enhanced, one has $a^* \le (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot a^* \le a^*$, so $s_X \cdot (a')^*$ is also a $\mathbb{T}$-monoid structure:

$$s_X \cdot (a')^* = s_X \cdot r_X \cdot a^* = (\tau_X \cdot \alpha)^{\mathbb{T}} \cdot a^* = a^* \;.$$

It then follows from Lemma 5.1 that $(a')^*$ is a $\mathbb{T}'$-monoid structure on (X, α).

Suppose now that $a' : T'X \to X$ has a right adjoint $\mathbb{T}'$-monoid structure $(a')^* : X \to T'X$ with $a' \cdot (a')^* = 1_X$. Lemma 5.1 yields that $(X, s_X \cdot (a')^*)$ is a $\mathbb{T}$-monoid, and setting $a := a' \cdot r_X$, one has

$$1_{TX} \le s_X \cdot (a')^* \cdot a' \cdot r_X = (s_X \cdot (a')^*) \cdot a \quad, \qquad a \cdot (s_X \cdot (a')^*) = a' \cdot (a')^* = 1_X \;.$$

Hence, we can apply Lemma 5.2 to the right adjoint $\mathbb{T}$-monoid structure $s_X \cdot (a')^*$ to conclude. □

5.4 Continuous lattices. The monad $\mathbb{F}_\downarrow$ on PrOrd can equivalently described using both the down-set monad $\mathbb{P}_\downarrow = (P_\downarrow, \downarrow, \bigcup)$ (see Examples 4.6), and the ordered-filter monad $\mathbb{P}_\Uparrow = (P_\Uparrow, \uparrow, \bigcup)$ on PrOrd, whose functor $P_\Uparrow$ is the restriction of $P_\uparrow$ to filters in X (that is, to up-closed down-directed sets in X). For the up-set map $\uparrow_X : X \to P_\Uparrow X$ to be monotone, the set $P_\Uparrow X$ is ordered by reverse inclusion. It will be convenient to use the following notations for the units and multiplications of the respective monads:

$$d_X(x) = \downarrow_X x \;, \quad \sup_{P_\downarrow X}(\mathcal{A}) = \bigcup \mathcal{A} \;, \quad u_X(x) = \uparrow_X x \;, \quad \inf_{P_\Uparrow X}(\mathcal{B}) = \bigcup \mathcal{B} \;,$$

for all $x \in X$, $\mathcal{A} \in P_\downarrow P_\downarrow X$, and $\mathcal{B} \in P_\Uparrow P_\Uparrow X$. One observes that the down-set-filter monad can be written as

$$\mathbb{F}_\downarrow = (F_\downarrow, \eta', \mu') = (P_\Uparrow P_\downarrow, u P_\downarrow \cdot d, \inf_{P_\Uparrow P_\downarrow} \cdot P_\Uparrow \sup_{P_\Uparrow P_\downarrow}) \;.$$

We say that a complete lattice X is *continuous* if the infimum map $\inf_X : P_\Uparrow X \to X$ has a right adjoint $\Uparrow_X : X \to P_\Uparrow X$ sending x to a filter $\Uparrow_X x = \Uparrow x$; since $P_\Uparrow$ is monotone, we have

$$X \underset{\Uparrow_X}{\overset{\inf_X}{\underset{\perp}{\rightleftarrows}}} P_\Uparrow X \underset{P_\Uparrow d_X}{\overset{P_\Uparrow \sup_X}{\underset{\perp}{\rightleftarrows}}} P_\Uparrow P_\downarrow X = F_\downarrow X \;. \tag{$*$}$$

A sup-map $f : X \to Y$ is *continuous* if it preserves infima of down-directed sets. Recall from 4.10 that the category of continuous lattices and continuous sup-maps is denoted by Cnt.

The diagram $(*)$ suggests that $\inf_X \cdot P_\Uparrow \sup_X$ is the structure of a $\mathbb{F}_\downarrow$-algebra on X, and this is confirmed in the following result. Proposition 5.3 and Lemma 5.1 therefore state that the left adjoint $P_\Uparrow d_X \cdot \Uparrow_X : X \to F_\downarrow X$ is the neighborhood map $P_\Uparrow d_X \cdot \Uparrow_X : X \to FX$ of a topology on the set X: the Scott topology on a continuous lattice.

5.5 Proposition. *There is an isomorphism*

$$\mathsf{Cnt} \cong \mathsf{PrOrd}^{\mathbb{F}_\downarrow}$$

that commutes with the underlying-functors to PrOrd.

Proof. Let us first check that for a continuous lattice X, the map $a' := \inf_X \cdot P_\Uparrow \sup_X$ defines the structure morphism of a $\mathbb{P}_\Uparrow \mathbb{P}_\downarrow$-algebra. We already have $a' \cdot \eta'_X = a' \cdot P_\Uparrow d_X \cdot u_X = 1_X$, so we only need to verify that $a' \cdot P_\Uparrow P_\downarrow a' = a' \cdot \mu'_X$. This follows from

$$
\begin{aligned}
a' \cdot P_\Uparrow P_\downarrow a' &= \inf_X \cdot P_\Uparrow (\sup_X \cdot P_\downarrow a') \\
&= \inf_X \cdot P_\Uparrow (a' \cdot \sup_{P_\Uparrow P_\downarrow X}) \\
&= \inf_X \cdot \inf_{P_\Uparrow X} \cdot P_\Uparrow P_\Uparrow \sup_X \cdot P_\Uparrow \sup_{P_\Uparrow P_\downarrow X} \\
&= \inf_X \cdot P_\Uparrow \sup_X \cdot \inf_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow \sup_{P_\Uparrow P_\downarrow X} \\
&= a' \cdot \mu'_X
\end{aligned}
$$

because a' is left adjoint (see $(*)$) and therefore preserves suprema, $\inf_X$ preserves infima, and $\inf_{P_\Uparrow} : P_\Uparrow P_\Uparrow \to P_\Uparrow$ is a natural transformation.

Consider now an $\mathbb{F}_\downarrow$-algebra $(X, a' : P_\Uparrow P_\downarrow X \to X)$. There is a monad morphism $uP_\downarrow : \mathbb{P}_\downarrow \to \mathbb{F}_\downarrow$, so the preordered set X is a $\mathbb{P}_\downarrow$-algebra, that is, a complete lattice with supremum given by $\sup_X = a' \cdot u_{P_\downarrow X}$. Proposition 5.3 yields that a' has a right adjoint $(a')^* : X \to P_\Uparrow P_\downarrow X$, so we are in the presence of the following adjunctions:

$$
X \underset{(a')^*}{\overset{a'}{\underset{\perp}{\longleftarrow\!\!\!\longrightarrow}}} P_\Uparrow P_\downarrow X \underset{P_\Uparrow (a' \cdot u_{P_\downarrow X})}{\overset{P_\Uparrow d_X}{\underset{\top}{\longleftarrow\!\!\!\longrightarrow}}} P_\Uparrow X \quad .
$$

Since the components of the up-set monad's multiplication are $\inf_{P_\Uparrow X}$, we have in particular $\inf_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow u_{P_\downarrow X} = 1_{P_\Uparrow P_\downarrow X}$. Consequently, by using that $a' \cdot P_\Uparrow P_\downarrow a' = a' \cdot \mu'_X$ we may write

$$
\begin{aligned}
a' \cdot P_\Uparrow d_X \cdot P_\Uparrow (a' \cdot u_{P_\downarrow X}) &= a' \cdot \mu'_X \cdot P_\Uparrow d_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow u_{P_\downarrow X} \\
&= a' \cdot \inf_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow \sup_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow d_{P_\Uparrow P_\downarrow X} \cdot P_\Uparrow u_{P_\downarrow X} = a' \quad .
\end{aligned}
$$

This shows that $a' \cdot P_\Uparrow d_X$ admits $P_\Uparrow (a' \cdot u_{P_\downarrow X}) \cdot (a')^*$ as a right adjoint (and also proves that $a' = \inf_X \cdot P_\Uparrow \sup_X$). But the infimum operation (obtained via the monad morphism $P_\Uparrow d : \mathbb{P}_\Uparrow \to \mathbb{F}_\downarrow$) is precisely $\inf_X = a' \cdot P_\Uparrow d_X$, so X is a continuous lattice.

Finally, a continuous lattice morphism $f : X \to Y$ is also an $\mathbb{F}_\downarrow$-algebra morphism, and a morphism $f : (X, a') \to (Y, b')$ of $\mathbb{F}_\downarrow$-algebras naturally preserves both suprema and down-directed infima because it is both a $\mathbb{P}_\downarrow$-algebra and a $\mathbb{P}_\Uparrow$-algebra morphism. $\qquad\square$

5.6 Corollary. *There is an isomorphism*

$$\mathsf{Cnt} \cong \mathsf{Set}^{\mathbb{F}}$$

that commutes with the underlying-functors to Set.

Proof. Since the monad $\mathbb{F}'$ obtained from $\mathbb{F}$ is the down-set-filter monad $\mathbb{F}_\downarrow$ (Examples 4.6), the results follows from Proposition 5.5 combined with Theorem 4.8. $\qquad\square$

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