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Some Remarks on Total Categories

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Introduction

Let *set* denote the category of small sets. A category $\mathscr{B}$ is said to have *small hom sets* if it has a hom functor $(-,-)\colon \mathscr{B}^{\mathrm{op}} \times \mathscr{B} \to set$. For such $\mathscr{B}$ we will write

$$Y\colon \mathscr{B} \to \hat{\mathscr{B}} = set^{\mathscr{B}^{\mathrm{op}}} \qquad \text{and} \qquad Z\colon \mathscr{B} \to \check{\mathscr{B}} = (set^{\mathscr{B}})^{\mathrm{op}}$$

for the associated Yoneda embeddings. (In general, $\hat{\mathscr{B}}$ and $\check{\mathscr{B}}$ do not have small hom sets.) Following Street and Walters [3], a category $\mathscr{B}$ is said to be *total* (respectively *cototal*) if $\mathscr{B}$ has small hom sets and Y (respectively Z) has a left (respectively right) adjoint. We will reserve L (respectively R) for such adjoints. Of course $\mathscr{B}$ is cototal if and only if $\mathscr{B}^{\mathrm{op}}$ is total.

Totality is a strong form of cocompleteness. In the interests of completeness (in some sense or other) we recall a few basic results from [3, 6]. For $\Phi \in \hat{\mathscr{B}}$, $\Phi L \simeq \varinjlim(\Phi\mathrm{el} \to \mathscr{B})$, where Φel denotes the usual category of elements of Φ. Any functor $D\colon \mathscr{G} \to \mathscr{B}$ admits a factorization

$$\begin{tikzcd} \mathscr{G} \arrow[r, "E"] \arrow[dr, "D"'] & \mathscr{E} \arrow[d, "M"] \\ & \mathscr{B} \end{tikzcd}$$

with E final and M a discrete fibration (see [4] or [2]). It follows that for $\mathscr{B}$ with small hom sets: $\mathscr{B}$ is total if and only if every diagram in $\mathscr{B}$, whose associated discrete fibration has small fibres, has a colimit. Furthermore, for any $\mathscr{B}$, $\hat{\mathscr{B}}$ is small bicomplete. So any total category, being a full reflective subcategory of a small bicomplete category, is small bicomplete. A category with small hom sets which is small bicomplete need not be total though. (A counterexample which shows somewhat more is given in Section 1.)

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An indication of the strength of totality is provided by Theorem 1 below. Recall that a functor $F: \mathscr{B} \to \mathscr{X}$ is *admissible* if $\mathscr{X}(BF, X)$ is small for all $B \in \mathscr{B}$ and $X \in \mathscr{X}$. If $\mathscr{X}$ has small hom sets any such F is admissible.

THEOREM 1 (Street and Walters). *If $\mathscr{B}$ is total and $F: \mathscr{B} \to \mathscr{X}$ is admissible, then F has a right adjoint if and only if F preserves all colimits.*

A wealth of examples is provided by the next result.

THEOREM 2 (Street and Walters). (i) set *is total.*

(ii) *If $\mathscr{B}$ is total and $\mathscr{A}$ is small, then $\mathscr{B}^{\mathscr{A}}$ is total.*

(iii) *If $\mathscr{C}$ is a full reflective subcategory of $\mathscr{B}$ and $\mathscr{B}$ is total, then $\mathscr{C}$ is total.*

Totality is a notion which makes sense in a large class of 2-categories. Indeed, it was at an abstract level that the definition was given in [3]. "Total object relative to a Yoneda structure" promises to unify many cocompleteness notions. At this time, however, it seems reasonable to pursue the concept for categories. Our remarks in Section 4 can be construed as attempts to identify the total objects in some closely related 2-categories.

1. DUALITY

For a small category $\mathscr{A}$ one has the Isbell conjugation functors

$$\begin{tikzcd} \hat{\mathscr{A}} \arrow[r, shift left=2, "(\ )^{+}"] \arrow[r, phantom, "\perp"] & \check{\mathscr{A}}, \arrow[l, shift left=2, "(\ )^{-}"] \end{tikzcd}$$

defined by $A\Phi^{+} = set^{\mathscr{A}^{\mathrm{op}}}(\Phi, (-, A))$ and $A\Psi^{-} = set^{\mathscr{A}}(\Psi, (A, -))$ for $\Phi \in \hat{\mathscr{A}}$, $\Psi \in \check{\mathscr{A}}$ and $A \in \mathscr{A}$. Clearly $(\ )^{+}$ and $(\ )^{-}$ commute with Y and Z. For a general $\mathscr{B}$ they do not exist, but if $\mathscr{B}$ is total we can define $(\ )^{+}$ by the above formula, since in that case $\hat{\mathscr{B}}(\Phi, (-, B)) \simeq \mathscr{B}(\Phi L, B)$ is small. Similarly, if $\mathscr{B}$ is cototal $(\ )^{-}$ exists. In conjunction with the adjointness relation $\check{\mathscr{B}}(\Phi^{+}, \Psi) \simeq \hat{\mathscr{B}}(\Phi, \Psi^{-})$, when $(\ )^{+}$ and $(\ )^{-}$ exist, the following formulas are useful:

PROPOSITION 3. *For $\Phi \in \hat{\mathscr{B}}$, $\Psi \in \check{\mathscr{B}}$.*

(i) *If $\mathscr{B}$ is total, $\check{\mathscr{B}}(\Psi, \Phi^{+}) \simeq (\Phi L)\Psi$.*

(ii) *If $\mathscr{B}$ is cototal, $\hat{\mathscr{B}}(\Psi^{-}, \Phi) \simeq (\Psi R)\Phi$.*

Proof. (i) $\check{\mathscr{B}}(\Psi, \Phi^+) \simeq \int_X set(X\Phi^+, X\Psi) \simeq \int_X set(\hat{\mathscr{B}}(\Phi(-,X)), X\Psi) \simeq \int_X set((\Phi L, X), X\Psi) \simeq (\Phi L)\Psi$.

(ii) Similarly.

PROPOSITION 4. *For $\Phi \in \hat{\mathscr{B}}$, $\Psi \in \check{\mathscr{B}}$. If $\mathscr{B}$ is total and cototal, then*

$$\begin{array}{ccc} \check{\mathscr{B}}(\Phi^+, \Psi) & \simeq & \hat{\mathscr{B}}(\Phi, \Psi^-) \\ \wr\| & & \wr\| \\ (\Phi L)\,\Psi^- & & (\Psi R)\,\Phi^+ \\ & \searrow\!\!\simeq \quad \simeq\!\!\swarrow & \\ & \mathscr{B}(\Phi L, \Psi R). & \end{array}$$

Proof. $\hat{\mathscr{B}}(\Phi, \Psi^-) \simeq \int_X set(X\Phi, X\Psi^-) \simeq \int_X set(X\Phi, \check{\mathscr{B}}((X,-), \Psi)) \simeq \int_X set(X\Phi, (X, \Psi R)) \simeq \hat{\mathscr{B}}(\Phi, (-, \Psi R)) \simeq (\Psi R)\,\Phi^+ \simeq \mathscr{B}(\Phi L, \Psi R)$.

If $\mathscr{B}$ is an ordered set, that is, if $(-,-)\colon \mathscr{B}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{2}$, it makes sense to adjust the Yoneda arrows accordingly. Y (respectively Z) becomes the embedding of $\mathscr{B}$ in the set of left (respectively right) order ideals of $\mathscr{B}$. Now $\mathscr{B}$ is sup complete if and only if Y has a left adjoint. For a right ideal J, J^- is the left ideal of lower bounds of J. A familiar calculation shows that $JR \simeq J^-L$. In other words, if $\mathscr{B}$ is sup complete, it is also inf complete. For general $\mathscr{B}$ the analogue of this statement is false, but if $\mathscr{B}$ is total and cototal the formula above is correct.

THEOREM 5. *If $\mathscr{B}$ is total and cototal,*

(i) $R \simeq (\ )^- L$ *and*

(ii) $L \simeq (\ )^+ R$.

Proof of (i). For $\Psi \in \check{\mathscr{B}}$, $B \in \mathscr{B}$. $\mathscr{B}(\Psi R, B) \simeq \hat{\mathscr{B}}((-, \Psi R), (-, B)) \simeq \int_X set((X, \Psi R), (X, B)) \simeq \int_X set(\check{\mathscr{B}}((X,-), \Psi), (X, B)) \simeq \int_X set(X\Psi^-, (X, B)) \simeq \hat{\mathscr{B}}(\Psi^-, (-, B)) \simeq \mathscr{B}(\Psi^- L, B)$.

The above proof also shows that $(\ )^{-+} \simeq RZ$. Z, being fully faithful, is cotripleable, so in the context of bitotality L is the canonical comparison functor.

The following "natural" example of a total category which is not cototal was shown to the author by Bob Paré: The category of small groups, *grp*, is easily seen to be total by Theorem 2. For each infinite cardinal α let S_α denote a simple group of cardinality α and consider the diagram suggested by

$$\begin{tikzcd} & (S_\alpha,-) \\ & \cdot \\ & \cdot \\ (1,-) \arrow[uuur, "{(!,-)}"] & \cdot \end{tikzcd}$$

in set^{grp}. Evaluating the diagram at an arbitrary group G produces a diagram of sets almost all of which are 1. Thus the diagram has a limit in $\check{grp}$. Were R to exist for this example so also would $\prod_\alpha S_\alpha$ in grp, which is not the case since the forgetful functor $grp \to set$ preserves all limits.

2. Existence

The next result provides a reasonable number of examples of bitotal categories. Further examples of such will be given in Section 3.

THEOREM 6. *If $\mathscr{B}$ is total and has a small set of cogenerators, then $\mathscr{B}$ is cototal.*

Proof. Z preserves all colimits so, by Theorem 1, we have only to show that it is admissible.

Let $\mathscr{C}$ denote the small full subcategory of $\mathscr{B}$ determined by a small set of cogenerators. Then for all $X \in \mathscr{B}$ the canonical morphism

$$X \to \int_C \{(X, C), C\},$$

where the integral is over $C \in \mathscr{C}$ and $\{\ ,\ \}$'s denote powers, is mono. For all $B \in \mathscr{B}$ and $\Psi \in \check{\mathscr{B}}$ we have

$$\begin{aligned}\check{\mathscr{B}}(BZ, \Psi) &\simeq set^{\mathscr{B}}(\Psi, (B, -)) \rightarrowtail set^{\mathscr{B}}(\Psi, (B, \int_C \{(-, C), C\})) \\ &\simeq \int_C set^{\mathscr{B}}(\Psi, ((-, C), (B, C))) \simeq \int_C \hat{\mathscr{B}}((-, C), (\Psi, (B, C))) \\ &\simeq \int_C (C\Psi, (B, C)) \in set.\end{aligned}$$

Any Grothendieck topos has a small set of cogenerators (if $(G_i)_{i\in I}$ generate, $(\Omega^{G_i})_{i\in I}$ cogenerate), so Grothendieck topoi are bitotal categories. The next result is, in a sense, similar to Theorem 6.

THEOREM 7. *If $\mathscr{A}$ is small, $D: \mathscr{A} \to \mathscr{B}$ is dense and $\mathscr{B}$ is small cocomplete, then $\mathscr{B}$ is total.*

Proof. D being dense, $(D, -): \mathscr{B} \to \hat{\mathscr{A}}: B \mapsto \mathscr{B}(-D, B)$ is fully faithful. It is a simple matter to check that $\Gamma \mapsto \int^A A\Gamma \cdot AD$, where the integral is over $A \in \mathscr{A}$ and $-\cdot-$'s denote multiples, is left adjoint to it.

3. Topological Examples

It is instructive to begin by reviewing the construction of colimits in *top*, the category of small topological spaces, and related examples. To form $\varinjlim M$ for $M: \mathscr{E} \to top$, one composes with $U: top \to set$, the underlying set functor; forms the colimit $\iota: MU \to \varinjlim MU$ in *set*; and finally equips $\varinjlim MU$ with the finest topology for which all components of ι are continuous. The definition which follows abstracts this last mentioned lifting of ι.

A functor $U: \mathscr{T} \to \mathscr{B}$ is said to be a *total opfibration* if $\mathscr{T}$ has small hom sets; $\hat{U}$ has a left adjoint, $U_!$; and U satisfies the following *lifting condition*: The functor $D: \mathscr{T} \to U_!/Y$ defined by

$$\begin{tikzcd}
 & T \arrow[ddl, "Y"'] \arrow[d, dashed, "D"] \arrow[ddr, "U"] & \\
 & U_!/Y \arrow[dl] \arrow[dr] & \\
\hat{T} \arrow[rr, phantom, "\longrightarrow"] \arrow[dr, "U_!"'] & & B \arrow[dl, "Y"] \\
 & \hat{B} \arrow[loop below, "\sim"] &
\end{tikzcd}$$

has a left adjoint over $\mathscr{B}$.

To understand the condition recall first that an object $\Phi \in \hat{\mathscr{T}}$ is coextensive with a diagram $M: \mathscr{E} \to \mathscr{T}$ in $\mathscr{T}$, where M is a discrete fibration with small fibres. Then $\Phi U_!$ corresponds to the diagram obtained by factoring MU, and an object $(\Phi, \gamma: \Phi U_! \to (-, B), B)$ of $U_!/Y$ corresponds to a cone from this diagram to B. The hypothesized left adjoint for D, together with its unit, provides, for such an object of $U_!/Y$, an object $T \in \mathscr{T}$ and a "cone" $\Phi \to (-, T)$ which is a best lifting of γ. The paradigmatic example of a total opfibration is of course $U: top \to set$.

THEOREM 8. *If $U: \mathscr{T} \to \mathscr{B}$ is a total opfibration and $\mathscr{B}$ is total, then $\mathscr{T}$ is total and U is cocontinuous.*

Proof. Each of the factors of Y in the diagram above has a left adjoint. That of $P: U_!/Y \to \hat{\mathscr{T}}$ is obtained by general comma nonsense from L, the left adjoint of $\mathscr{B}$'s Y. Explicitly it sends $\Phi \in \hat{\mathscr{T}}$ to $(\Phi, \eta: \Phi U_! \to (-, \Phi U_! L), \Phi U_! L) \in U_!/Y$. This description, together with the fact that the left adjoint of D is over $\mathscr{B}$, shows that U is cocontinuous.

It is clear that the two step construction of L for $\mathscr{T}$ is only a generalization of the familiar procedure recalled at the beginning of this section.

A somewhat weaker definition of total opfibration could have been given. Recall from [3] that a functor $U: \mathscr{T} \to \mathscr{B}$ is *total* if it is admissible and $(U, -)$ (which thus exists) has an admissible left adjoint, which we will denote by $-\cdot U$. For total U and $\mathscr{E}$ with small hom sets, the lifting condition above can be expressed by requiring that D, defined by

$$\begin{tikzcd}
 & \mathscr{T} \arrow[ddl, "Y"'] \arrow[d, dashed, "D"] \arrow[ddr, "U"] & \\
 & -\cdot U/\mathscr{B} \arrow[dl] \arrow[dr] & \\
\hat{\mathscr{T}} \arrow[dr, "-\cdot U"'] \arrow[rr, phantom, "\longrightarrow"] & & \mathscr{B} \arrow[dl, "\mathscr{B}"] \\
 & \mathscr{B} \arrow[loop below, "\tilde{U}"] &
\end{tikzcd}$$

where $\tilde{U}$ corresponds, via

$$\frac{Y\cdot U\to U}{Y\to(U,U)},$$

to the strength of U, has a left adjoint over $\mathscr{B}$. $-\cdot U/\mathscr{B}\to\hat{\mathscr{E}}$ has a left adjoint and it follows that if $\mathscr{E}$ has small hom sets, U is total and U satisfies the lifting condition, then $\mathscr{E}$ is total and U is cocontinuous. From

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "Y"] \arrow[d, "{(U,-)}"'] \arrow[dr, phantom, "\simeq"] & \hat{\mathscr{B}} \arrow[dl, "\hat{U}"] \\
\hat{\mathscr{E}} &
\end{tikzcd}$$

Theorem 8 now follows, but the added generality does not yet seem useful and the earlier discussion is closer to present experience.

It is a simple exercise in dualization to define a *cototal fibration* and conclude that *top*, etc., are also cototal. Note that we can now conclude that *ct2*, the category of small compact Hausdorff spaces, is bitotal. This theorem (together with other results of this paper) was presented at Oberwolfach in 1979. Subsequently, Tholen [5] showed that totality lifts through semitopological functors. Recall that $U:\mathscr{E}\to\mathscr{B}$ is semitopological if and only if every U-cone has a rigid U-semi-initial lifting. In this terminology U is a cototal fibration if and only if every U-cone, obtained from a discrete opfibration with small fibres over $\mathscr{E}$, has a U-initial lifting. The two notions are incomparable; each is a generalization of "topological functor."

4. Classification

It would seem desirable to express additional properties that a total category may have in terms of L. The following result is illustrative of what we have in mind. It is reminiscent of Day's reflection theorem [1], but note that $\hat{\mathscr{B}}$ is not cartesian closed.

THEOREM 9. *If $\mathscr{B}$ is total, then $\mathscr{B}$ is cartesian closed if and only if L preserves binary products.*

Proof. $\mathscr{B}$ is cartesian closed if and only if for each $A \in \mathscr{B}$, $A \times -$ is cocontinuous. Since $(A \times -)_!$ exists and is given by $(-, A) \times -$, the latter is equivalent to $((-, A) \times \Phi)L \simeq A \times \Phi L$, for all $\Phi \in \mathscr{B}$. $(-, A)L \simeq A$, so it only remains to be shown that such isomorphisms for all $A \in \mathscr{B}$ and $\Phi \in \hat{\mathscr{B}}$ imply that L preserves binary products.

$$\begin{aligned}(\Psi \times \Phi)L &\simeq \left(\left(\int^A A\Psi \cdot (-, A)\right) \times \Phi\right)L \simeq \left(\int^A A\Psi \cdot ((-, A) \times \Phi)\right) L\\ &\simeq \int^A A\Psi \cdot ((-, A) \times \Phi)L \simeq \int^A A\Psi \cdot (A \times \Phi L)\\ &\simeq \left(\int^A A\Psi \cdot A\right) \times \Phi L \simeq \Psi L \times \Phi L.\end{aligned}$$

A total category is said to be *lex total* if L preserves finite limits. Walters [6] has shown that "$\mathscr{B}$ is lex total and has a small set of generators" is equivalent to "$\mathscr{B}$ is a Grothendieck topos."

For small $\mathscr{A}$, $\mathscr{A}Y: \mathscr{A} \to \hat{\mathscr{A}}$ is such that $\widehat{\mathscr{A}Y}$ has both right and left adjoints. $(\mathscr{A}Y)_! \dashv \widehat{\mathscr{A}Y} \dashv \check{\mathscr{A}}Y$ expresses the fact that small powers of *set* are considerably more than lex total. The exponential transpose of $(\mathscr{A}Y)_!$, $\hat{\mathscr{A}}^{\mathrm{op}} \times \hat{\mathscr{A}} \to set$ can be shown to be given by $\langle F, G\rangle \mapsto F^+ \cdot G$, ($F^+ \cdot G \simeq \int^A AG \cdot AF^+$). That of $\hat{\mathscr{A}}Y$ is $\langle F, G\rangle \mapsto \hat{\mathscr{A}}(F, G)$. If idempotents split in $\mathscr{A}$, a familiar result says that $F \in \hat{\mathscr{A}}$ is representable if and only if $F^+ \cdot F \simeq \hat{\mathscr{A}}(F, F)$. This suggests a candidate for $\mathscr{A}$ with $\hat{\mathscr{A}}$ equivalent to $\mathscr{B}$, if $\mathscr{B}$ is total with $T \dashv L \dashv Y$.

Finally, we note that for $\mathscr{B} = set$ we have

$$\begin{tikzcd}[column sep=huge, row sep=huge]
set \arrow[r, shift left=12, "K"] \arrow[r, shift left=6, leftarrow, "(\phi)-"] \arrow[r, "\Delta"] \arrow[r, shift right=6, leftarrow, "(1)-"'] \arrow[r, shift right=12, "Y"'] \arrow[dr, shift left=2, "Z"] \arrow[dr, leftarrow, shift right=2, "R"'] \arrow[dr, phantom, "\perp"] & \hat{set} \arrow[d, shift right=2, "(\ )^+"'] \arrow[d, phantom, "\dashv"] \\
& \check{set} \arrow[u, shift right=2, "(\ )^-"']
\end{tikzcd}$$

where $XK \simeq X \cdot (-, \phi)$ and Δ is the usual diagonal. K does not have a left adjoint and by Theorem 9 it follows that R does not have a right adjoint. Other examples of total categories with such a configuration of adjoints do not seem to be forthcoming.

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