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Constructing locales from quantales

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Introduction

We recall that a *locale* is a complete lattice L satisfying $a \wedge (\bigvee b_\alpha) = \bigvee(a \wedge b_\alpha)$, for all $a \in L$, and $\{b_\alpha\} \subseteq L$. Examples of locales include the lattices $\mathcal{O}(X)$ of open subsets of topological spaces X. Following Joyal and Tierney [**7**], a *morphism* $f: L \to M$ *of locales* is a $\bigvee$-, $\wedge$-, and τ-preserving map. Such functions are sometimes called 'frame homomorphisms', in which case the right adjoint $f_*: M \to L$ (which exists since f preserves $\bigvee$) is then called a 'morphism of locales'.

In many areas of ring theory, it is desirable to construct a space or, more generally, a locale of ideals of a ring. Indeed, the first step of any sheaf representation of a ring can be viewed as the introduction of a locale of ideals on which to define the requisite sheaf of rings.

This article is concerned with two general ways in which the lattice of ideals of a ring has been modified to obtain a locale of ideals. In the constructions of the Zariski spectrum of a commutative ring (see [**6**]), the spectrum of a C^*-algebra (see [**4**]), as well as the Levitski, Brown-McCoy and the Jacobson radical ideals of a non-commutative ring (see [**1**]), one considers a complete lattice of ideals in which the meet is given by intersection, and the usual multiplicative structure is altered to agree with the binary meet. For the Pierce spectrum (see [**6**]) and the pure spectrum (see [**2**]) it is the join (i.e. sum of ideals) that remains unchanged, while the existing product becomes the binary meet.

The general setting in which we shall work is that of quantales, which are a generalization of both locales and ideal lattices. Each locale of ideals discussed above will arise as a quotient or a subquantale of the quantale of ideals of the ring in question.

We begin with a brief introduction to quantales. In §2, we study quotients of a quantale Q via certain closure operators on Q which are called quantic nuclei. These operators generalize the 'quasi-radical' operations on the lattice of ideals of a commutative ring with identity, as introduced by D. Kirby [**8**]. As in the ideal case, we establish the completeness of the lattice of all quantic nuclei on Q. We then apply this completeness in §3 to obtain a largest localic quotient of Q, as well as a left adjoint to the inclusion of the category of locales in the category of quantales. Along the way we also consider commutative, right-sided and idempotent quantales separately, obtaining analogous results in each of these cases. Finally, after presenting some general examples in §4 we conclude the paper with a brief study in §5 of quantic conuclei and subquantales of Q.

1. *Quantales*

Quantales were first introduced by Mulvey [9] in order to provide a possible setting for constructive foundations for quantum mechanics, as well as a non-commutative analogue of the maximal spectrum of a C^*-algebra. Borceux and van den Bossche [3] have also studied quantales, but the quantales in [3] are not general enough to include all of our ring-theoretic examples. In particular, they are assumed to be right-sided and idempotent (as defined below).

A *quantale* is a complete lattice Q together with an associative binary operation & satisfying

$$a \,\&\, (\bigvee b_\alpha) = \bigvee (a \,\&\, b_\alpha) \quad \text{and} \quad (\bigvee b_\alpha) \,\&\, a = \bigvee (b_\alpha \,\&\, a) \quad \text{for all} \quad a \in Q \quad \text{and} \quad \{b_\alpha\} \subseteq Q.$$

As $\bigvee$-preserving maps, the functions $a \,\&\, -$ and $- \,\&\, a$ have right adjoints which we shall denote by $a \underset{r}{\to} -$ and $a \underset{l}{\to} -$, respectively. A straightforward calculation shows that

$$(a \,\&\, b) \underset{r}{\to} c = b \underset{r}{\to} (a \underset{r}{\to} c),\ (a \,\&\, b) \underset{l}{\to} c = a \underset{l}{\to} (b \underset{l}{\to} c),$$

and

$$a \underset{r}{\to} (b \underset{l}{\to} c) = b \underset{l}{\to} (a \underset{r}{\to} c), \quad \text{for all} \quad a, b, c \in Q.$$

Let Q be a quantale. Then Q is *commutative* if & is commutative. Note that Q is commutative if and only if $a \underset{r}{\to} b = a \underset{l}{\to} b$, for all $a, b \in Q$. An element a of Q is called *left-sided*, if $\tau \,\&\, a \leqslant a$, where τ denotes the largest element of Q. It is not difficult to show that a is left-sided if and only if $a \underset{l}{\to} a = \tau$. Now, if every element of Q is left-sided, then Q is called a *left-sided quantale*. The notions of *right-sided* and *two-sided* are defined similarly. Next, an element 1 of Q is a *left unit* if $1 \,\&\, a = a$, for all $a \in Q$. An easy calculation shows that 1 is a left unit if and only if $1 \underset{l}{\to} a = a$, for all $a \in Q$. If Q has a left unit, then Q is called a *left unital quantale*. The notions of *right unit* and (two-sided) *unit* are defined similarly. Finally, an element e of Q is *idempotent*, if $e^2 = e \,\&\, e = e$, and Q is an *idempotent quantale*, if every element of Q is idempotent.

Examples of quantales are numerous. Any locale is a quantale (with $\& = \wedge$) which is commutative, two-sided, unital, and idempotent. In fact, it is well known that a quantale is a locale if and only if it is idempotent and two-sided.

Let R be any ring. Then it is easy to see that the set Sub (R) of additive subgroups of R is a quantale with

$$\bigvee = \Sigma \quad \text{and} \quad A \,\&\, B = AB = \{a_1 b_1 + \ldots + a_n b_n \mid a_i \in A,\, b_i \in B\}.$$

Moreover $A \underset{r}{\to} B$ and $A \underset{l}{\to} B$ are the right and left residuations $B \underset{r}{:} A$ and $B \underset{l}{:} A$, respectively. Similarly, the sets LIdl (R), RIdl (R) and Idl (R) of left, right and two-sided ideals of R are quantales. Note that $B \underset{l}{:} A$ is a left ideal if B is a left ideal, but $B \underset{r}{:} A$ need not be a left ideal unless A is a right ideal. Thus in LIdl (R) we have $A \underset{l}{\to} B = B \underset{l}{:} A$, but $A \underset{r}{\to} B = \Sigma \{C \in \text{LIdl}\,(R) \mid AC \subseteq B\}$. Analogous results hold for RIdl (R). However if A and B are both two-sided ideals then so are $B \underset{r}{:} A$ and $B \underset{l}{:} A$. Therefore $A \underset{r}{\to} B = B \underset{r}{:} A$ and $A \underset{l}{\to} B = B \underset{l}{:} A$ in Idl (R). Clearly LIdl (R) is left-sided, RIdl (R) is right-sided and Idl (R) is two-sided. Moreover, if R is a ring with identity, then these three quantales are left unital, right unital and unital, respectively. If R is commutative, then all of the above associated quantales are commutative.

Next, suppose R is a C^*-*algebra*. Then (as a consequence of our general construction in §3) we shall see that the set of closed right (respectively, left and two-sided) ideals forms an idempotent quantale with $\vee = \overline{\Sigma}$ and $A \& B = \overline{AB}$. In particular, the closed two-sided ideals form a locale.

If R is a sheaf of rings on a locale L, then the set of ideals of R is also a quantale. This example will be studied in detail in [**12**].

Other examples include the set End (X) of endomorphisms of a $\vee$-lattice X with $\vee$ computed point-wise and & given by composition, and the set $\mathrm{Mat}_N(L)$ of $N \times N$ matrices with coefficients in a locale L, i.e. functions $N \times N \to L$, with point-wise $\vee$ and $(a \& b)(m, n) = \vee_x a(m, x) \& b(x, n)$, where N is any set. Moreover, with the notion of homomorphism presented below, one can show that $\mathrm{Mat}_N(L)$ is isomorphic to the quantale $\mathrm{End}_L L^N$ of L-endomorphisms of the L-module L^N, in the sense of Joyal and Tierney [**7**].

Suppose Q and Q' are quantales. A *homomorphism* $f: Q \to Q'$ of quantales is a $\vee$, τ and & preserving function. The category of quantales and homomorphisms is denoted by **Quant**.

Note that if $f: Q \to Q'$ is a homomorphism, then since f preserves $\vee$, it has a right adjoint f_*. Since f also preserves &, one checks that $f_*(f(a) \underset{r}{\to} a') = a \underset{r}{\to} f_*(a')$ and $f_*(f(a) \underset{l}{\to} a') = a \underset{l}{\to} f_*(a')$ for all $a \in Q$ and $a' \in Q'$.

Any morphism of locales is a quantale homomorphism. Also, if $g: R \to R'$ is a homomorphism of commutative rings with identity, then $f: \mathrm{Idl}\,(R) \to \mathrm{Idl}\,(R')$ defined by $f(A) = R'g(A)$ is a homomorphism. Note that, if R' is not commutative, then $A \mapsto R'g(A)R'$ need not preserve &, although it does preserve $\vee$ (being left adjoint to $A' \mapsto g^{-1}(A')$), as well as τ.

2. *Quantic Nuclei*

In [**8**], D. Kirby studies closure operations on Idl (R), where R is a commutative ring with identity. In particular, 'quasi-radical' operations are considered, i.e. those satisfying $j(AB) = j(A) \cap j(B)$. These operations include the 'radical' operations, which are those defined by $j(A) = \bigcap\{P \mid P \in \mathscr{P}, A \subseteq P\}$, for some set $\mathscr{P}$ of prime ideals of R. The main theorem of [**8**] states that every quasi-radical operation on Idl (R) is radical if and only if every proper ideal of R has an essential prime divisor. Using the observations that quasi-radical operations on Idl (R) correspond to sublocales of the locale Rad (R) of radical ideals of R, and radical operations correspond to spatial quotients of Rad (R), Niefield and Rosenthal [**11**] translated and generalized Kirby's theorem to characterize those locales such that every sublocale is spatial. Here morphisms of locales are going in the geometric direction.

Closure operations (i.e. nuclei) on a locale have been extensively studied (see Simmons [**14**] or Johnstone [**6**], for example) as a tool for understanding sublocales. Certain operations (called 'multiplicative nuclei') on Idl (R), for a non-commutative ring R with identity, have also been studied as a tool for constructing compact locales of ideals (see Banaschewski and Harting [**1**]).

In this section, we generalize Kirby's definition of a closure operation to obtain the notion of a 'quantic nucleus' which then applies to all of our examples of quantales. As in the case of locales, these nuclei will be used to study quotients of a quantale, in particular, those which are locales.

Recall that a *closure operator* on a partially ordered set Q is an order-preserving map $j: Q \to Q$ which is increasing (i.e. satisfies $a \leqslant j(a)$, for all $a \in Q$) and idempotent (i.e. satisfies $j(j(a)) = j(a)$, for all $a \in Q$). Note that, if j is a closure operator, then $a \leqslant j(b)$ if and only if $j(a) \leqslant j(b)$.

Let Q be a quantale. A *quantic nucleus* on Q is a closure operator j such that $j(a) \& j(b) \leqslant j(a \& b)$, for all $a, b \in Q$. Note that if j is a quantic nucleus, then

$$j(a \& b) = j(a \& j(b)) = j(j(a) \& b) = j(j(a) \& j(b)),$$

for all $a, b \in Q$. The set of quantic nuclei on Q will be denoted by $N(Q)$.

If L is a locale, then the quantic nuclei on L are precisely the nuclei. For $Q = \mathrm{Idl}(R)$ where R is a commutative ring with identity, it is not difficult to show that a quantic nucleus on $\mathrm{Idl}(R)$ is precisely what Kirby calls a 'closure operation'. Kirby's examples include $j(A) = \sqrt{A} = \{r \in R \mid r^n \in A\}$, $j(A) = C:(C:A)$, where $C \in \mathrm{Idl}(R)$, and $j(A) = S^{-1}A \cap R$, the S-component of A with respect to a multiplicatively closed set S. The first two of these examples will be generalized to an arbitrary quantale in §4. The third example will be considered in detail in [**13**]. In the case where R is not necessarily commutative, the 'multiplicative nuclei' of [**1**] are also quantic nuclei. Examples of such quantic nuclei on $\mathrm{Idl}(R)$ include the Levitsky, Brown-McCoy and Jacobson radicals on an ideal. Finally, if R is a C^*-algebra (or more generally, a Banach algebra or just a normed algebra), then the map $A \mapsto \bar{A}$ (where $\bar{A}$ denotes the closure of A) is easily seen to be a quantic nucleus on $\mathrm{RIdl}(R)$, $\mathrm{LIdl}(R)$, and $\mathrm{Idl}(R)$. We shall return to these examples after we present the connection between quantic nuclei and quotients.

Recall that if j is a closure operator on a complete lattice Q, then

$$Q_j = \{a \in Q \mid j(a) = a\}$$

is closed under $\bigwedge$ and hence is complete. In particular, $\top \in Q_j$. Note that the join in Q_j is given by $\bigvee_j(a_\alpha) = j(\bigvee a_\alpha)$, and the map $j: Q \to Q_j$ is a $\bigvee$-preserving surjection. Moreover every surjective $\bigvee$-lattice homomorphism $f: Q \to Q'$ arises (up to isomorphism) in this manner via $j = f_* \circ f$, where f_* is the right adjoint to f.

Theorem 2·1. *If $j: Q \to Q$ is a quantic nucleus, then Q_j is a quantale via $a \&_j b = j(a \& b)$; and $j: Q \to Q_j$ is a quantale homomorphism. Moreover every surjective quantale homomorphism arises in this manner.*

Proof. First, $\&_j$ is clearly associative. Also, if $a, b_\alpha \in Q_j$, then

$$a \&_j (\textstyle\bigvee_j b_\alpha) = j(a \& j(\bigvee b_\alpha)) = j(a \& \bigvee b_\alpha) = j(\bigvee(a \& b_\alpha)) \leqslant j(\bigvee(j(a \& b_\alpha))) = \bigvee_j (a \&_j b_\alpha),$$

and hence $a \&_j (\bigvee_j b_\alpha) = \bigvee_j(a \&_j b_\alpha)$. Similarly $(\bigvee_j b_\alpha) \&_j a = \bigvee_j(b_\alpha \&_j a)$. Thus Q_j is a quantale via $\&_j$, and $j: Q \to Q_j$ is clearly a quantale homomorphism.

Suppose that $f: Q \to Q'$ is a surjective quantale homomorphism and $j = f_* \circ f$. It remains to show that $j(a) \& j(b) \leqslant j(a \& b)$. Since f is &-preserving, we know that

$$f(f_*(f(a)) \& f_*(f(b))) = f(f_*(f(a))) \& f(f_*(f(b))) = f(a) \& f(b) = f(a \& b).$$

Hence, using the adjointness of f and f_*, we obtain

$$j(a) \& j(b) = f_*(f(a)) \& f_*(f(b)) \leqslant f_*(f(a \& b)) = j(a \& b),$$

as desired. ∎

Now, if j is a quantic nucleus on Q, since $j(a \& b) = j(j(a) \& j(b))$, it follows that Q_j is a locale with $\wedge = \&_j$ if and only if $j(a \& b) = j(a) \wedge j(b)$, for all $a, b \in Q$. Such an operator is called a *localic nucleus* on Q. The corresponding quotient is called a *localic quotient* of Q.

We now return to some of our examples. First, consider $\surd : \mathrm{Idl}\,(R) \to \mathrm{Idl}\,(R)$, where R is a commutative ring with identity. It is not difficult to show that $\mathrm{Idl}\,(R)_{\surd} = \mathrm{Rad}\,(R)$, the set of radical ideals of R. Moreover, since $\surd AB = \surd A \cap \surd B$, we obtain the well-known result that $\mathrm{Rad}\,(R)$ is a locale. Of course, this is just the locale associated with the Zariski spectrum. Note that a localic nucleus on $\mathrm{Idl}\,(R)$ is precisely what Kirby [**8**] calls a 'quasi-radical operation'.

Next, if R is a C^*-algebra, then since $\overline{(\;)}$ is a quantic nucleus on $Q = \mathrm{RIdl}\,(R)$, $\mathrm{LIdl}\,(R)$ and $\mathrm{Idl}\,(R)$, we see that $Q_{\overline{(\;)}}$ is a quantale. But this is just the lattice of closed ideals in question. Thus we obtain (as promised in §1) the result that the closed left (respectively, right and two-sided) ideals form a quantale with $\bigvee = \overline{\Sigma}$ and $A \& B = \overline{AB}$. Recall that, for any left ideal A, there exists a directed family $\{u_\alpha\} \subseteq A$ of positive elements such that $\|u_\alpha\| \leqslant 1$, such that $\alpha \leqslant \beta$ implies $u_\alpha \leqslant u_\beta$, and such that if $x \in \bar{A}$ then $\|u_\alpha x - x\| \to 0$ (see 1·7·3 in [**4**]). Using this, one shows that $\bar{A} \cap \bar{B} \subseteq \overline{AB}$ for all $A, B \in \mathrm{LIdl}\,(R)$. Note that the reverse containment holds if A and B are two-sided. Thus the closed left (respectively, right) ideals form an idempotent quantale, and the closed two-sided ideals form a locale. This locale is precisely the one which corresponds to the spectrum of R: see [**4**].

It is sometimes useful to have criteria to determine if a subset of Q is of the form Q_j, for some quantic nucleus j. For example, in the case of the ideals of a ring, it will sometimes be easier to work with a collection of ideals without explicit reference to a nucleus. The following proposition generalizes a well-known result about locales.

Proposition 2·2. *If $S \subseteq Q$, then $S = Q_j$, for some quantic nucleus j, if and only if S is closed under $\bigwedge$, and $a \underset{r}{\to} s$, $a \underset{l}{\to} s \in S$, whenever $a \in Q$ and $s \in S$.*

Proof. Suppose $S = Q_j$, for some quantic nucleus j. If $\{s_\alpha\} \subseteq S$, then

$$j(\bigwedge s_\alpha) \leqslant \bigwedge j(s_\alpha) = \bigwedge s_\alpha,$$

and hence $\bigwedge s_\alpha \in S$. If $a \in Q$ and $s \in S$ then $j(a \underset{r}{\to} s) \leqslant a \underset{r}{\to} s$, since

$$a \& j(a \underset{r}{\to} s) \leqslant j(a) \& j(a \underset{r}{\to} s) \leqslant j(a \& (a \underset{r}{\to} s)) \leqslant j(s) = s.$$

Hence $a \underset{r}{\to} s \in S$. Similarly $a \underset{l}{\to} s \in S$.

Conversely, suppose that S is closed under $\bigwedge$ and that $a \underset{r}{\to} s$, $a \underset{l}{\to} s \in S$, whenever $a \in Q$ and $s \in S$. Define $j : Q \to Q$ by $j(a) = \bigwedge\{s \in S \mid a \leqslant s\}$. Clearly j is order-preserving, increasing and idempotent. To see that $j(a) \& j(b) \leqslant j(a \& b)$, it suffices to show that if $a \& b \leqslant s$ and $s \in S$, then $j(a) \& j(b) \leqslant s$. However $a \& b \leqslant s$ implies $a \leqslant b \underset{l}{\to} s$. Since $b \underset{l}{\to} s \in S$, we know that $j(a) \leqslant b \underset{l}{\to} s$. Thus $j(a) \& b \leqslant s$, and so $b \leqslant j(a) \underset{r}{\to} s$. Hence $j(b) \leqslant j(a) \underset{r}{\to} s$. Therefore $j(a) \& j(b) \leqslant s$. ∎

A subset S of Q satisfying the conditions of Proposition 2·2 is called a *quantic quotient* of Q. Note that the '&' of S is given by $a \&_S b = \bigwedge\{c \in S \mid a \& b \leqslant c\}$.

Corollary 2·3. *If Q is a right-sided (respectively, left-sided) quantale, then the set $T(Q)$ of two-sided elements is a quantic quotient of Q.*

Proof. Suppose that Q is right-sided. The left-sided case is similar. If $\{t_\alpha\} \subseteq T(Q)$, then $\bigwedge t_\alpha$ is right-sided since Q is, and left-sided since $\tau \,\&\, (\bigwedge t_\alpha) \leqslant \bigwedge(\tau \,\&\, t_\alpha) \leqslant \bigwedge t_\alpha$. If $a \in Q$ and $t \in T(Q)$, then $a \underset{r}{\rightarrow} t$ and $a \underset{\ell}{\rightarrow} t$ are right-sided since Q is right-sided. Now $a \underset{r}{\rightarrow} t$ is left-sided since a is right-sided, i.e. $a \,\&\, \tau \,\&\, (a \underset{r}{\rightarrow} t) \leqslant a \,\&\, (a \underset{r}{\rightarrow} t) \leqslant t$ implies $\tau \,\&\, (a \underset{r}{\rightarrow} t) \leqslant a \underset{r}{\rightarrow} t$, and $a \underset{\ell}{\rightarrow} t$ is left-sided since t is left-sided, i.e.

$$\tau \,\&\, (a \underset{\ell}{\rightarrow} t) \,\&\, a \leqslant \tau \,\&\, t \leqslant t$$

implies $\tau \,\&\, (a \underset{\ell}{\rightarrow} t) \leqslant a \underset{\ell}{\rightarrow} t$. Thus $a \underset{r}{\rightarrow} t,\ a \underset{\ell}{\rightarrow} t \in T(Q)$, as desired. ∎

If Q is not left-sided or right-sided, we shall see in §4 that $T(Q)$ is still a quantale, but it is a subquantale rather than a quotient of Q.

Note that if S and T are quantic quotients of Q corresponding to j_S and j_T in $N(Q)$, then $S \subseteq T$ if and only if $j_S \geqslant j_T$, where $N(Q)$ has the point-wise order. If $\{S_\alpha\}$ is a family of quantic quotients of Q, it is easy to see that $\bigcap S_\alpha$ is a quantic quotient. Also, if $\{j_\alpha\}$ is a family of quantic nuclei, consider $j(a) = \bigwedge j_\alpha(a)$. Clearly j is increasing; and

$$j(j(a)) \leqslant j_\alpha(j(a)) \leqslant j_\alpha(j_\alpha(a)) = j_\alpha(a)$$

for all α, and so $j(j(a)) \leqslant \bigwedge j_\alpha(a) = j(a)$. Since

$$j(a) \,\&\, j(b) \leqslant j_\alpha(a) \,\&\, j_\alpha(b) \leqslant j_\alpha(a \,\&\, b)$$

for all α, we have
$$j(a) \,\&\, j(b) \leqslant \bigwedge j_\alpha(a \,\&\, b) = j(a \,\&\, b).$$

Hence j is a quantic nucleus. Thus we obtain the following proposition.

PROPOSITION 2·4. *$N(Q)$ is a complete lattice with $\bigwedge$ computed point-wise, and $\bigvee j_\alpha$ given by the nucleus corresponding to $\bigcap Q_{j_\alpha}$.*

In the case where L is a locale, it is well known that $N(L)$ is a locale (cf. [**6**]). One might ask whether $N(Q)$ is a quantale in general. First we note that if j and k are quantic nuclei, then $a \mapsto j(a) \,\&\, k(a)$ need not be a quantic nucleus. For example, if R is a commutative ring, $Q = \mathrm{Idl}\,(R)$, and $j = k = \surd$, then the above map is not increasing. It seems natural to define

$$j \,\&\, k = \bigwedge\{m \in N(Q) \mid j(a) \,\&\, k(a) \leqslant m(a) \quad \text{for all} \quad a \in Q\}.$$

Note that $j \,\&\, k \leqslant m$ if and only if $j(a) \,\&\, k(a) \leqslant m(a)$ for all $a \in Q$. Although we do not know if $\&$ is associative, we do have the following result:

COROLLARY 2·5. *If $j \in N(Q)$, then $j\&-$ and $-\&j$ have right adjoints.*

Proof. Define $j \underset{r}{\rightarrow} m = \bigvee\{n \in N(Q) \mid n(a) \leqslant j(a) \underset{r}{\rightarrow} m(a)$ for all $a \in Q\}$. Then $n \leqslant j \underset{r}{\rightarrow} m$ if and only if $n(a) \leqslant j(a) \underset{r}{\rightarrow} m(a)$ for all $a \in Q$. Hence

$$\begin{aligned} j \,\&\, k \leqslant m &\Leftrightarrow j(a) \,\&\, k(a) \leqslant m(a) \quad \text{for all} \quad a \in Q \\ &\Leftrightarrow k(a) \leqslant j(a) \underset{r}{\rightarrow} m(a) \quad \text{for all} \quad a \in Q \\ &\Leftrightarrow k \leqslant j \underset{r}{\rightarrow} m. \end{aligned}$$

Similarly $-\,\&j$ has a right adjoint, namely $j \underset{\ell}{\rightarrow} -$. ∎

Finally, we provide a characterization of quantic nuclei in terms of $\rightarrow_r$ and $\rightarrow_l$, generalizing proposition 2 of [**8**]. If $j:Q\rightarrow Q$ is a quantic nucleus, then one easily shows that $j(a)\rightarrow_r j(b) = a\rightarrow_r j(b)$ and $j(a)\rightarrow_l j(b) = a\rightarrow_l j(b)$, for all $a, b\in Q$. For the proof of the converse, we need some assumption on Q.

PROPOSITION 2·6. *If Q is left or right unital, then a function $j:Q\rightarrow Q$ is a quantic nucleus if and only if $j(a)\rightarrow_r j(b) = a\rightarrow_r j(b)$ and $j(a)\rightarrow_l j(b) = a\rightarrow_l j(b)$, for all $a, b\in Q$.*

Proof. Let Q be a left unital quantale. The right unital case is analogous.

Suppose $j:Q\rightarrow Q$ satisfies $j(a)\rightarrow_r j(b) = a\rightarrow_r j(b)$ and $j(a)\rightarrow_l j(b) = a\rightarrow_l j(b)$ for all $a, b\in Q$. Since $1\,\&\,j(a) = j(a)$ we have $1\leqslant j(a)\rightarrow_l j(a) = a\rightarrow_l j(a)$, and so $a = 1\,\&\,a\leqslant j(a)$. Hence j is increasing. To see that j is order-preserving, suppose that $a\leqslant b$. Since $1\,\&\,a = a\leqslant b\leqslant j(b)$, we know that $1\leqslant a\rightarrow_l j(b) = j(a)\rightarrow_l j(b)$. Thus we have $j(a) = 1\,\&\,j(a)\leqslant j(b)$, as desired. Next, since $1\leqslant j(a)\rightarrow_r j(a) = j(j(a))\rightarrow_l j(a)$, it follows that $j(j(a))\leqslant j(a)$, and so j is idempotent. Finally, $a\,\&\,b\leqslant j(a\,\&\,b)$ implies $a\leqslant b\rightarrow_l j(a\,\&\,b) = j(b)\rightarrow_l j(a\,\&\,b)$, and so $a\,\&\,j(b)\leqslant j(a\,\&\,b)$. Thus $j(b)\leqslant a\rightarrow_r j(a\,\&\,b) = j(a)\rightarrow_r j(a\,\&\,b)$, and hence $j(a)\,\&\,j(b)\leqslant j(a\,\&\,b)$. ▌

COROLLARY 2·7. *If L is a locale, then a function $j:L\rightarrow L$ is a nucleus if and only if $j(a)\rightarrow j(b) = a\rightarrow j(b)$, for all $a, b\in L$.* ▌

We conclude this section by noting that as in the case of locales (cf. [**5**]), quotients can be studied via congruences, i.e. equivalence relations θ on Q satisfying $(a\,\&\,c, b\,\&\,d)\in\theta$ for all $(a, b), (c, d)\in\theta$ and $(\bigvee a_\alpha, \bigvee b_\alpha)\in\theta$ for all $\{(a_\alpha, b_\alpha)\}\subseteq\theta$. In particular one can show that $N(Q)$ is isomorphic to the lattice of congruences on Q.

3. *Applications of the completeness of* $N(Q)$

Let R be a commutative ring with identity. Now, although Kirby does not discuss quotients of $\mathrm{Idl}(R)$ in [**8**], he does show that the set of localic nuclei is closed under $\bigwedge$ and that $j(A)$ is a radical for every ideal A, whenever j is a localic nucleus. These two results tell us that $\mathrm{Rad}(R)$ is the largest localic quotient of $\mathrm{Idl}(R)$, or equivalently that $\surd$ is the least localic nucleus on $\mathrm{Idl}(R)$. Thus, to study localic quotients of $\mathrm{Idl}(R)$, it suffices to consider nuclei on the locale $\mathrm{Rad}(R)$.

The goal of this section is to generalize the above to an arbitrary quantale. In particular, we would like to have an analogue of $\surd$ when R is non-commutative. Although the completeness of the lattice of localic nuclei is straightforward, in order to obtain a description of the largest localic quotient, it is helpful to approach the problem in stages via the following lemma.

LEMMA 3·1. *If j is a quantic nucleus on Q, then j is localic if and only if*

(i) $j(a\,\&\,b) = j(b\,\&\,a)$ *for all* $a, b\in Q$,

(ii) $j(a\,\&\,\top)\leqslant j(a)$ *for all* $a\in Q$, *and*

(iii) $j(a) = j(a^2)$ *for all* $a\in Q$.

Proof. If j is localic, then (i)–(iii) clearly hold. Conversely, suppose j satisfies (i)–(iii) and $a, b\in Q$. Since

$$j(a\,\&\,b)\leqslant j(a\,\&\,\top)\leqslant j(a)\quad\text{and}\quad j(a\,\&\,b)\leqslant j(\top\,\&\,b) = j(b\,\&\,\top)\leqslant j(b),$$

it follows that $j(a \& b) \leqslant j(a) \wedge j(b)$. Also

$$j(a) \wedge j(b) \leqslant j(j(a) \wedge j(b)) = j((j(a) \wedge j(b))^2) = j((j(a) \wedge j(b)) \& (j(a) \wedge j(b))) \leqslant j(j(a) \& j(b)) \leqslant j(j(a \& b)) = j(a \& b).$$

Hence $j(a \& b) = j(a) \wedge j(b)$, as desired. ∎

Let j be a quantic nucleus on Q. Then j is called *commutative* if $j(a \& b) = j(b \& a)$ for all $a, b \in Q$. Since $j(a \& b) = j(j(a) \& j(b))$, j is commutative if and only if Q_j is commutative. Consider

$$j_c = \bigwedge\{j \in N(Q) \mid j \text{ is commutative}\}.$$

Since

$$j_c(a \& b) = \bigwedge\{j(a \& b) \mid j \text{ is commutative}\} = \bigwedge\{j(b \& a) \mid j \text{ is commutative}\} = j_c(b \& a),$$

it follows that j_c is commutative. Clearly any meet of commutative nuclei is commutative. Now if j is commutative then $j_c \leqslant j$. Conversely, if $j_c \leqslant j$, then

$$j \circ j_c \leqslant j \circ j = j \quad \text{and} \quad j = j \circ \mathrm{id} \leqslant j \circ j_c,$$

and so $j \circ j_c = j$. Hence

$$j(a \& b) = j(j_c(a \& b)) = j(j_c(b \& a)) = j(b \& a),$$

i.e. j is commutative. Thus we obtain the following lemma.

LEMMA 3·2. *The following are equivalent for* $j \in N(Q)$:

(1) Q_j *is commutative*;

(2) j *is commutative*;

(3) $j_c \leqslant j$;

(4) $Q_j \subseteq Q_{j_c}$.

Next we present a description of the elements of the quantic quotient Q_{j_c}. An element a of Q is *symmetric* if $\sigma \in \Sigma_n$ (the permutation group on n letters) and $a_1 \& \ldots \& a_n \leqslant a$ implies $a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant a$. The set of symmetric elements of Q is denoted by $\Sigma(Q)$.

PROPOSITION 3·3. $\Sigma(Q)$ *is the largest commutative quantic quotient of* Q.

Proof. First we show that $\Sigma(Q)$ is a commutative quantic quotient of Q. Suppose $\{a_\alpha\} \subseteq \Sigma(Q)$, $a_1 \& \ldots \& a_n \leqslant \bigwedge a_\alpha$ and $\sigma \in \Sigma_n$. For every α we have $a_1 \& \ldots \& a_n \leqslant a_\alpha$ and a_α is symmetric. Thus $a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant a_\alpha$. Hence $a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant \bigwedge a_\alpha$, and so $\bigwedge a_\alpha \in \Sigma(Q)$. Next suppose $a \in \Sigma(Q)$, $b \in Q$, $a_1 \& \ldots \& a_n \leqslant b \to_r a$, and $\sigma \in \Sigma_n$. Since $b \& a_1 \& \ldots \& a_n \leqslant a$ and a is symmetric, it follows that $b \& a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant a$, and so $a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant b \to_r a$. Hence $b \to_r a$ is symmetric. Similarly $b \to_l a \in \Sigma(Q)$. To see that $\Sigma(Q)$ is commutative, it suffices to show that $b \to_r a = b \to_l a$, for all $a \in \Sigma(Q)$. Since $(b \to_l a) \& b \leqslant a$ and a is symmetric, it follows that $b \& (b \to_l a) \leqslant a$. Hence $b \to_l a \leqslant b \to_r a$. Similarly $b \to_r a \leqslant b \to_l a$. Therefore $\Sigma(Q)$ is a commutative quotient of Q, and so $\Sigma(Q) \subseteq Q_{j_c}$, by Lemma 3·2.

Next we show that $Q_{j_c} \subseteq \Sigma(Q)$. Suppose $a \in Q_{j_c}$, $a_1 \& \ldots \& a_n \leqslant a$ and $\sigma \in \Sigma_n$. Since $a \in Q_{j_c}$, it follows that $j_c(a_1 \& \ldots \& a_n) \leqslant a$. However

$$j_c(a_1 \& \ldots \& a_n) = j_c(a_{\sigma(1)} \& \ldots \& a_{\sigma(n)}),$$

since j_c is commutative. Thus $j_c(a_{\sigma(1)} \& \ldots \& a_{\sigma(n)}) \leqslant a$, and so $a_{\sigma(1)} \& \ldots \& a_{\sigma(n)} \leqslant a$. Therefore $\Sigma(Q) = Q_{j_c}$, and the desired result follows. ▮

Let **CQuant** denote the full subcategory of **Quant** consisting of commutative quantales.

PROPOSITION 3·4. Σ: **Quant** $\to$ **CQuant** *is a functor which is left adjoint to the inclusion* I_c: **CQuant** $\to$ **Quant**.

Proof. Suppose that $f: Q \to Q'$ is a homomorphism. Let j_c and j'_c denote the smallest commutative elements of $N(Q)$ and $N(Q')$, respectively. Define $\Sigma(f): \Sigma(Q) \to \Sigma(Q')$ by $a \mapsto j'_c(f(a))$. Note that, if $f_*: Q' \to Q$ denotes the right adjoint to f, then f_* preserves symmetric elements since f preserves &. Thus, given $a \in \Sigma(Q)$ and $a' \in \Sigma(Q')$, we have $j'_c(f(a)) \leqslant a' \Leftrightarrow f(a) \leqslant a' \Leftrightarrow a \leqslant f_*(a')$, and so $\Sigma(f) \dashv f_*$. Hence $\Sigma(f)$ preserves $\bigvee$. Clearly $\Sigma(f)$ preserves $\top$. If $a, b \in \Sigma(Q)$, then

$$j'_c(f(a)) \&_{j'_c} j'_c(f(b)) = j'_c(f(a) \& f(b)) = j'_c(f(a \& b)) \leqslant j'_c(f(j_c(a \& b))) = j'_c(f(a \&_{j_c} b)).$$

Since $f(a \& b) \leqslant j'_c(f(a \& b))$, we know that $a \& b \leqslant f_*(j'_c(f(a \& b)))$. However f_* preserves symmetric elements, and so

$$a \&_{j_c} b = j_c(a \& b) \leqslant f_*(j'_c(f(a \& b))).$$

Thus $f(a \&_{j_c} b) \leqslant j'_c(f(a \& b))$. Hence

$$j'_c(f(a \&_{j_c} b)) \leqslant j'_c(f(a \& b)) = j'_c(f(a)) \&_{j'_c} j'_c(f(b)),$$

and so $\Sigma(f)$ preserves &. Therefore Σ is a functor.

Taking $\epsilon: \Sigma \circ I_c \to \mathrm{id}$ given by the identity transformation, and $\eta: \mathrm{id} \to I_c \circ \Sigma$ given by $\eta_Q: Q \to \Sigma(Q)$, $\eta_Q(a) = j_c(a)$, we see that $\Sigma \dashv I_c$. ▮

A quantic nucleus $j: Q \to Q$ is called *right-sided* if $j(a \& \top) \leqslant j(a)$ for all $a \in Q$, or, equivalently, if Q_j is right-sided. Consider $j_r = \bigwedge\{j \in N(Q) \mid j \text{ is right-sided}\}$. It is not difficult to show, as in the commutative case, that any meet of right-sided nuclei is right-sided. Thus j_r is right-sided, and $j_r \leqslant j$ if and only if j is right-sided. Therefore we have

LEMMA 3·5. *The following are equivalent for* $j \in N(Q)$:
(1) Q_j *is right-sided*;
(2) j *is right-sided*;
(3) $j_r \leqslant j$;
(4) $Q_j \subseteq Q_{j_r}$. ▮

Let $\mathscr{R}(Q) = \{a \in Q \mid a \text{ is right-sided and } b \underset{\ell}{\rightarrow} a \text{ is right-sided for all } b \in Q\}$. Note that if Q is commutative or left-sided, then $\mathscr{R}(Q) = T(Q)$.

PROPOSITION 3·6. $\mathscr{R}(Q)$ *is the largest right-sided quantic quotient of* Q.

Proof. If $\{a_\alpha\}$ is a family of right-sided elements, then $(\bigwedge a_\alpha) \& \top \leqslant \bigwedge(a_\alpha \& \top) \leqslant \bigwedge a_\alpha$. Since $b \underset{\ell}{\rightarrow} \bigwedge a_\alpha = \bigwedge(b \underset{\ell}{\rightarrow} a_\alpha)$, it follows that $\mathscr{R}(Q)$ is closed under $\bigwedge$. Next let $a \in \mathscr{R}(Q)$ and $c \in Q$. Then $c \underset{\ell}{\rightarrow} a$ is right-sided, and $b \underset{\ell}{\rightarrow} (c \underset{\ell}{\rightarrow} a) = (b \& c) \underset{\ell}{\rightarrow} a$ is right-sided for all $b \in Q$. Also $c \underset{r}{\rightarrow} a$ is right-sided since a is, and $b \underset{\ell}{\rightarrow} (c \underset{r}{\rightarrow} a) = c \underset{r}{\rightarrow} (b \underset{\ell}{\rightarrow} a)$ is right-sided for all b, since $b \underset{\ell}{\rightarrow} a$ is right-sided. Thus $c \underset{\ell}{\rightarrow} a$, $c \underset{r}{\rightarrow} a \in \mathscr{R}(Q)$. Hence $\mathscr{R}(Q)$ is a quantic quotient. Since $a \mathbin{\&} \tau \leqslant a$ for all $a \in \mathscr{R}(Q)$, it follows that $\mathscr{R}(Q)$ is right-sided since $a \mathbin{\&}_{R(Q)} \tau = \bigwedge\{b \in \mathscr{R}(Q) \mid a \mathbin{\&} \tau \leqslant b\} \leqslant a$. Therefore $\mathscr{R}(Q) \subseteq Q_{j_r}$, by Proposition 3·5.

Now if $a \in Q_{j_r}$, then a is right-sided since $a \mathbin{\&} \tau \leqslant j_r(a \mathbin{\&} \tau) \leqslant j_r(a) = a$. Since $b \underset{\ell}{\rightarrow} a \in Q_{j_r}$ for all $b \in Q$, it follows that $b \underset{\ell}{\rightarrow} a$ is right-sided. Hence $a \in \mathscr{R}(Q)$. Therefore $Q_{j_r} \subseteq \mathscr{R}(Q)$, and this completes the proof. ∎

Let **RQuant** denote the full subcategory of **Quant** consisting of right-sided quantales. If $f: Q \to Q'$ is in **Quant,** then its right adjoint f_* preserves right-sided elements since $f(f_*(a') \mathbin{\&} \tau) = f(f_*(a')) \mathbin{\&} f(\tau) \leqslant a' \mathbin{\&} \tau \leqslant a'$ implies that $f_*(a') \mathbin{\&} \tau \leqslant f_*(a')$. Using the fact that $b \underset{\ell}{\rightarrow} f_*(a') = f_*(f(b) \underset{\ell}{\rightarrow} a')$, we see that $f_*(\mathscr{R}(Q')) \subseteq \mathscr{R}(Q)$. Thus, replacing j_c by j_r in the proof of Proposition 3·4, we obtain the following

PROPOSITION 3·7. *$\mathscr{R}$*: **Quant** $\to$ **RQuant** *is a functor which is left adjoint to the inclusion I_r*: **RQuant** $\to$ **Quant**. ∎

Similarly one obtains a left adjoint $\mathscr{L}$ to the inclusion I_ℓ: **LQuant** $\to$ **Quant**, where **LQuant** denotes the category of left-sided quantales.

A quantic nucleus j is called *idempotent* if $j(a^2) = j(a)$ for all $a \in Q$, or, equivalently, if Q_j is an idempotent quantale. Consider $j_e = \bigwedge\{j \in N(Q) \mid j$ is idempotent$\}$. As in the two previous cases, a meet of idempotent nuclei is idempotent and we obtain the following result.

LEMMA 3·8. *The following are equivalent for $j \in N(Q)$*:

(1) *Q_j is idempotent*;

(2) *j is idempotent*;

(3) $j_e \leqslant j$;

(4) $Q_j \leqslant Q_{j_e}$. ∎

An element $a \in Q$ is *semiprime* if $b \leqslant a$ whenever $b^2 \leqslant a$. Since $(b \wedge c)^2 \leqslant b \mathbin{\&} c$, it follows that a is semiprime if and only if $b \wedge c \leqslant a$ whenever $b \mathbin{\&} c \leqslant a$. If Q is right-sided or left-sided, then it is not difficult to show that a is semiprime if and only if $b \leqslant a$ whenever $b^n \leqslant a$ for some positive integer n. Let $S(Q)$ denote the set of semiprime elements of Q.

Note that, if Q is LIdl(R), RIdl(R) or Idl(R), then the above agrees with the usual definition of semiprime ideal. Of course, in the commutative case, these ideals are just the radical ideals.

PROPOSITION 3·9. *If Q is right-sided or left-sided, then $S(Q)$ is the largest idempotent quotient of Q.*

Proof. Assume that Q is right-sided. The left-sided case is analogous. One easily shows that $S(Q)$ is closed under $\bigwedge$. Suppose that $a \in S(Q)$ and $c \in Q$. If $b^2 \leqslant c \underset{r}{\rightarrow} a$, then $(c \mathbin{\&} b)^2 \leqslant c \mathbin{\&} (b \mathbin{\&} \tau) \mathbin{\&} b \leqslant c \mathbin{\&} b^2 \leqslant a$. Since a is semiprime it follows that $c \mathbin{\&} b \leqslant a$, and so $b \leqslant c \underset{r}{\rightarrow} a$. Thus $c \underset{r}{\rightarrow} a \in S(Q)$. Similarly $c \underset{\ell}{\rightarrow} a \in S(Q)$. Hence $S(Q)$ is a quantic quotient. To see that $S(Q)$ is idempotent, let $a \in S(Q)$. Since $a \mathbin{\&} a \leqslant a \mathbin{\&} \tau \leqslant a$, we know that $a \leqslant b$ implies $a \mathbin{\&} a \leqslant b$ for all $b \in Q$. Of course the reverse implication holds when $b \in S(Q)$. Thus

$$a \mathbin{\&}_{S(Q)} a = \bigwedge\{b \in S(Q) \mid a \mathbin{\&} a \leqslant b\} = \bigwedge\{b \in S(Q) \mid a \leqslant b\} = a.$$

Therefore $S(Q)$ is an idempotent quotient of Q, and so $S(Q) \subseteq Q_{j_e}$ by Lemma 3·8.

It remains to show that $Q_{j_e} \subseteq S(Q)$. If $a \in Q_{j_e}$ and $b^2 \leqslant a$, then $b \leqslant j_e(b) = j_e(b^2) \leqslant j_e(a) = a$. Hence $a \in S(Q)$, and this completes the proof. ∎

Let **ERQuant** denote the full subcategory of **Quant** consisting of idempotent right-sided quantales. If $f: Q \to Q'$ is in **RQuant**, then its right adjoint preserves semiprime elements, for if $a' \in S(Q')$ and $a^2 \leqslant f_*(a')$, then $f(a)^2 = f(a^2) \leqslant a'$ which implies $f(a) \leqslant a'$ and so $a \leqslant f_*(a')$. Thus, replacing j_c by j_e in the proof of Proposition 3·4, we obtain the following:

PROPOSITION 3·10. S: **RQuant** → **ERQuant** *is a functor which is left adjoint to the inclusion.* ∎

Similarly one obtains a left adjoint in the left-sided and two-sided cases. In fact, the latter actually goes into the (algebraic) category of locales. To generalize the above to **Quant**, one considers the set $\mathscr{E}(Q)$ of $a \in S(Q)$ such that $a^2 \leqslant a$ and for $x = b \underset{r}{\rightarrow} a$, $b \underset{\ell}{\rightarrow} a$ or $c \underset{r}{\rightarrow} (b \underset{\ell}{\rightarrow} a)$, we have $x \in S(Q)$ and $x^2 \leqslant x$. As in the above cases one proves that $\mathscr{E}(Q) = Q_{j_e}$ and one proves

PROPOSITION 3·11. $\mathscr{E}$: **Quant** → **EQuant** *is a functor which is left adjoint to the inclusion.*

Returning to localic nuclei, consider $j_{\mathrm{loc}} = \bigwedge\{j \in N(Q) \mid j \text{ is localic}\}$. By Proposition 3·1, j is localic if and only if j is commutative, right-sided and idempotent. Thus j_{loc} is commutative, right-sided and idempotent, being a meet of such nuclei, and so j_{loc} is localic.

LEMMA 3·12. *The following are equivalent for* $j \in N(Q)$:

(1) Q_j *is a locale*;

(2) j *is localic*;

(3) $j_{\mathrm{loc}} \leqslant j$;

(4) $Q_j \subseteq Q_{j_{\mathrm{loc}}}$.

Proof. We know that the implications (1) ⇔ (2) and (3) ⇔ (4) hold, and (2) ⇒ (3) follows from the definition of j_{loc}. For the proof that (3) implies (2), assume $j_{\mathrm{loc}} \leqslant j$. Since j_{loc} is commutative, right-sided and idempotent, it follows that $j_c \leqslant j_{\mathrm{loc}}$, $j_r \leqslant j_{\mathrm{loc}}$ and $j_e \leqslant j_{\mathrm{loc}}$, by Lemmas 3·2, 3·5 and 3·8. Thus $j_c \leqslant j$, $j_r \leqslant j$ and $j_e \leqslant j$. Applying 3·2, 3·5 and 3·8 again, we see that j is commutative, right-sided and idempotent. Therefore j is localic, as desired. ∎

COROLLARY 3·13. $j_{\mathrm{loc}} = j_c \vee j_r \vee j_e$.

Proof. This follows from (2) ⇔ (3) of Lemmas 3·2, 3·5, 3·8 and 3·12. ∎

An element of Q is called *localic* if it is symmetric, right-sided and semiprime. Note that a symmetric, right-sided element is necessarily two-sided. The set of localic elements of Q is denoted by Loc(Q).

THEOREM 3·14. Loc(Q) *is the largest localic quotient of* Q.

Proof. By Corollary 3.13, $Q_{j_{\mathrm{loc}}} = Q_{j_c} \cap Q_{j_r} \cap Q_{j_c} \subseteq \mathrm{Loc}(Q)$. It remains to show that $\mathrm{Loc}(Q) \subseteq Q_{j_{\mathrm{loc}}}$ or equivalently, Loc(Q) is a localic quotient of Q.

We know that Loc(Q) is closed under $\bigwedge$. If $a \in \mathrm{Loc}(Q)$ and $b \in Q$, then $b \underset{r}{\rightarrow} a = b \underset{\ell}{\rightarrow} a$, since a is symmetric. Thus, to see that Loc(Q) is a quantic quotient, it suffices to show that $b \underset{r}{\rightarrow} a \in \mathrm{Loc}(Q)$. As in the proof of Proposition 3·3, we know $b \underset{r}{\rightarrow} a$ is symmetric, since a is. Also, as in 3·6, $b \underset{r}{\rightarrow} a$ is right-sided since a is. To see that $b \underset{r}{\rightarrow} a$ is semiprime, suppose that $c^2 \leqslant b \underset{r}{\rightarrow} a$. Then $b \,\&\, c^2 \leqslant a$, and so

$$b \,\&\, c^2 \,\&\, b \leqslant a \,\&\, b \leqslant a \,\&\, \top \leqslant a.$$

Since a is symmetric, it follows that $(b \,\&\, c)^2 \leqslant a$. Hence $b \,\&\, c \leqslant a$, since a is semiprime, and so $c \leqslant b \underset{r}{\rightarrow} a$.

Finally we show that $\mathrm{Loc}(Q)$ is localic. Let $\hat{\&}$ denote the '&' for $\mathrm{Loc}(Q)$, i.e.

$$a \,\hat{\&}\, b = \bigwedge\{c \in \mathrm{Loc}(Q) \mid a \,\&\, b \leqslant c\}.$$

Since every element of $\mathrm{Loc}(Q)$ is two-sided, we have $a \,\&\, b \leqslant a \wedge b$. However $a \wedge b \in \mathrm{Loc}(Q)$, and so $a \,\hat{\&}\, b \leqslant a \wedge b$. Also $(a \wedge b) \,\&\, (a \wedge b) \leqslant a \,\&\, b \leqslant a \,\hat{\&}\, b$ implies $a \wedge b \leqslant a \,\hat{\&}\, b$, since the latter is semiprime. ∎

COROLLARY 3·15. *If Q is two-sided, then $S(Q)$ is the largest localic quotient of Q.*

Proof. To see that $\mathrm{Loc}(Q) = S(Q)$, we shall show that every $a \in S(Q)$ is symmetric by showing that $a_1 \,\&\, \ldots \,\&\, a_n \leqslant a$ if and only if $a_1 \wedge \ldots \wedge a_n \leqslant a$. If $a_1 \,\&\, \ldots \,\&\, a_n \leqslant a$ then $(a_1 \wedge \ldots \wedge a_n)^n \leqslant a_1 \,\&\, \ldots \,\&\, a_n \leqslant a$, and so $a_1 \wedge \ldots \wedge a_n \leqslant a$, since a is semiprime and two-sided. Since every element of Q is two-sided, we know that

$$a_1 \,\&\, \ldots \,\&\, a_n \leqslant a_1 \wedge \ldots \wedge a_n.$$

Hence $a_1 \,\&\, \ldots \,\&\, a_n \leqslant a$ whenever $a_1 \wedge \ldots \wedge a_n \leqslant a$. ∎

COROLLARY 3·16. *If R is a ring, then the lattice $S(\mathrm{Idl}(R))$ of semiprime ideals of R is the largest localic quotient of $\mathrm{Idl}(R)$.* ∎

Note that Corollary 3·16 provides a non-commutative version of the result that the Zariski spectrum (i.e the locale of radical ideals of a commutative ring) is the largest localic quotient of $\mathrm{Idl}(R)$. Moreover it follows that other ring theoretic examples (e.g. the Levitski, Brown-McCoy and Jacobson radicals of [**1**]) arise via nuclei on the locale of semiprime ideals of R. The reader should also note that Simmons [**15**] has a similar result in a slightly different context. He works with 'idioms' rather than quantales.

COROLLARY 3·17. *If Q is a right-sided idempotent quantale, then the set $T(Q)$ of two-sided elements is the largest localic quotient of Q.*

Proof. It suffices to show that every two-sided element of Q is symmetric. To see this we shall show that $b \,\&\, a_1 \,\&\, \ldots \,\&\, a_n = b \,\&\, a_{\sigma(1)} \,\&\, \ldots \,\&\, a_{\sigma(n)}$ for all $\sigma \in \Sigma_n$ and all $b, a_1, \ldots, a_n \in Q$, and $b \,\&\, c \leqslant a$ implies $c \,\&\, b \leqslant a$ for all two-sided a.

Suppose $\sigma \in \Sigma_n$ and $b, a_1, \ldots, a_n \in Q$. Since Q is idempotent and right-sided, we see that

$$\begin{aligned} b \,\&\, a_1 \,\&\, \ldots \,\&\, a_n &= (b \,\&\, a_1 \,\&\, \ldots \,\&\, a_n)^n \\ &\leqslant (b \,\&\, \top \,\&\, a_{\sigma(1)} \,\&\, \top) \,\&\, (\top \,\&\, a_{\sigma(2)} \,\&\, \top) \,\&\, \ldots \,\&\, (\top \,\&\, a_{\sigma(n)} \,\&\, \top) \\ &\leqslant b \,\&\, a_{\sigma(1)} \,\&\, \ldots \,\&\, a_{\sigma(n)}. \end{aligned}$$

The reverse inequality follows on applying σ^{-1}.

If a is two-sided and $b \& c \leqslant a$, then $c \& b \& c = c \& c \& b$, by the above. Thus $c \& b = c \& c \& b = c \& b \& c \leqslant c \& a \leqslant \top \& a \leqslant a$. ▮

Applying this corollary in the case where Q is the quantale of closed right (respectively, left) ideals of a C^*-algebra R, we obtain a result about the locale of two-sided ideals of R, i.e. the spectrum (in the sense of [**4**]).

COROLLARY 3·18. *If R is a C^*-algebra, then the locale of closed two-sided ideals is the largest localic quotient of the quantale of closed right (respectively, left) ideals.*

Let **Loc** denote the full subcategory of **Quant** consisting of locales. We have seen that if $f: Q \to Q'$ is a morphism of quantales, then f_* preserves symmetric, right-sided and semiprime elements, i.e. f_* preserves localic elements. Replacing j_c by j_{loc} in the proof of Proposition 3·4, we obtain

THEOREM 3·19. Loc: **Quant** $\to$ **Loc** *defines a functor which is left adjoint to the inclusion.* ▮

We conclude this section by noting that it is also possible to generalize the result that every semiprime ideal is an intersection of primes (i.e. $S(\mathrm{Idl}(R))$ is spatial) to arbitrary algebraic quantales. However, as we are not aware of any examples of algebraic quantales other than algebraic locales and those of the form $\mathrm{Idl}(R)$, we have chosen not to present the generalization here. The latter does include some new examples, for example the case where R is a sheaf of commutative rings on an algebraic locale. However such quantales are covered by a stronger theorem which states that $S(\mathrm{Idl}(R))$ is spatial for all commutative rings R in $\mathrm{Sh}(L)$ if and only if L is spatial (cf. [**12**]).

4. *General examples*

In this section, we generalize some examples of Kirby [**8**], as well as several examples from locale theory (see [**6**]). First, we note that $j_{\text{loc}}: Q \to Q$ (see §3) can be thought of as a generalization of $\surd : \mathrm{Idl}(R) \to \mathrm{Idl}(R)$. It is the smallest localic nucleus on Q, and so every localic quotient of Q factors through $\mathrm{Loc}(Q)$. Thus localic quotients of Q are completely determined by nuclei on the locale $\mathrm{Loc}(Q)$.

Let s be a two-sided element of a quantale Q. Consider $s \vee - : Q \to Q$. Clearly $s \vee -$ is order-preserving, increasing and idempotent. Since s is two-sided, it follows that

$$(s \vee a) \& (s \vee b) = (s \& s) \vee (s \& b) \vee (a \& s) \vee (a \& b) \leqslant s \vee (a \& b).$$

Hence $s \vee -$ is a quantic nucleus, and $Q_{s\vee -} = \uparrow(s) = \{a \mid s \leqslant a\}$ is a quotient of Q. Note that if Q is the locale $\mathcal{O}(X)$ of open subsets of a space X and $s = U$, then $Q_{s\vee -} \cong \mathcal{O}(X \backslash U)$.

Suppose s is an idempotent element of a commutative right-sided quantale Q. Consider $s \to - : Q \to Q$. Clearly $s \to -$ is order-preserving and increasing. Also $s \to -$ is idempotent since s is. Finally,

$$(s \to a) \& (s \to b) \leqslant s \to (a \& b),$$

since

$$s \& (s \to a) \& (s \to b) = s^2 \& (s \to a) \& (s \to b) = s \& (s \to a) \& s \& (s \to b) \leqslant a \& b.$$

Thus $s \to -$ is a quantic nucleus. Now consider

$$s \,\&\, Q = \{s \,\&\, a \mid a \in Q\} = \{a \in Q \mid s \,\&\, a = a\}.$$

It is not difficult to show that $s \,\&\, Q$ is a quantale, and $a \mapsto s \to a$ defines an isomorphism $s \,\&\, Q \to Q_{s\to -}$. Note that, if $Q = \mathcal{O}(X)$ for some space X and $s = U$, then $s \,\&\, Q = \mathcal{O}(U)$.

Next we generalize the preceding example to an idempotent right-sided quantale. In such a quantale, one can show that $s \,\&\, a \,\&\, b = s \,\&\, b \,\&\, a$ for all $a, b \in Q$ (cf. [10] or see the proof of 3·17 for a more general result).

First consider $s \underset{l}{\to} -$, which is clearly order-preserving and idempotent. Since Q is right-sided, it follows that $s \underset{l}{\to} -$ is increasing. Also

$$(s \underset{l}{\to} a) \,\&\, (s \underset{l}{\to} b) \leqslant s \underset{l}{\to} (a \,\&\, b),$$

since

$$(s \underset{l}{\to} a) \,\&\, (s \underset{l}{\to} b) \,\&\, s = (s \underset{l}{\to} a) \,\&\, (s \underset{l}{\to} b) \,\&\, s \,\&\, s = (s \underset{l}{\to} a) \,\&\, s \,\&\, (s \underset{l}{\to} b) \,\&\, s \leqslant a \,\&\, b.$$

Therefore $s \underset{l}{\to} -$ is a quantic nucleus and, as in the commutative case, one can show that $Q \,\&\, s \cong Q_{s \underset{l}{\to} -}$.

Now, although $s \underset{r}{\to} -$ is not necessarily increasing, we can consider

$$s \underset{r}{\to} (s \,\&\, -) = s \underset{r}{\to} (\top \,\&\, -).$$

Clearly $s \underset{r}{\to} (s \,\&\, -)$ is order-preserving and increasing. Since $s \,\&\, (s \underset{r}{\to} (s \,\&\, a)) \leqslant s \,\&\, a$, it follows that $s \underset{r}{\to} (s \,\&\, (s \underset{r}{\to} (s \,\&\, a))) \leqslant s \underset{r}{\to} (s \,\&\, a)$, and so $s \underset{r}{\to} (s \,\&\, -)$ is idempotent. Finally,

$$[s \underset{r}{\to} (s \,\&\, a)] \,\&\, [s \underset{r}{\to} (s \,\&\, b)] \leqslant s \underset{r}{\to} (s \,\&\, a \,\&\, b)$$

since

$$\begin{aligned} s \,\&\, [s \underset{r}{\to} (s \,\&\, a)] \,\&\, [s \underset{r}{\to} (s \,\&\, b)] &= s^2 \,\&\, [s \underset{r}{\to} (s \,\&\, a)] \,\&\, [s \underset{r}{\to} (s \,\&\, b)] \\ &= s \,\&\, [s \underset{r}{\to} (s \,\&\, a)] \,\&\, s \,\&\, [s \underset{r}{\to} (s \,\&\, b)] \leqslant (s \,\&\, a) \,\&\, (s \,\&\, b) = s^2 \,\&\, a \,\&\, b \\ &= s \,\&\, a \,\&\, b. \end{aligned}$$

Thus $s \underset{r}{\to} (s \,\&\, -)$ is a quantic nucleus and, as above, one can show that

$$s \,\&\, Q \cong Q_{s \underset{r}{\to} (s \,\&\, -)}.$$

However, in this case, we can prove more. First, $s \underset{r}{\to} (s \,\&\, -)$ is localic, since

$$s \underset{r}{\to} (s \,\&\, a \,\&\, b) \leqslant [s \underset{r}{\to} (s \,\&\, a)] \wedge [s \underset{r}{\to} (s \,\&\, b)],$$

and

$$[s \underset{r}{\to} (s \,\&\, a)] \wedge [s \underset{r}{\to} (s \,\&\, b)] =$$

$$([s \underset{r}{\to} (s \,\&\, a)] \wedge [s \underset{r}{\to} (s \,\&\, b)])^2 \leqslant [s \underset{r}{\to} (s \,\&\, a)] \,\&\, [s \underset{r}{\to} (s \,\&\, b)] \leqslant s \underset{r}{\to} (s \,\&\, a \,\&\, b).$$

Thus $s \,\&\, Q$ is a locale for all s. Moreover $\top \,\&\, Q$ is the locale of two-sided elements of Q. This is an extension of the known result that the closed two-sided ideals of a C^*-algebra form a locale.

We conclude this section with a generalization of Kirby's example

$$C:(C:-):\mathrm{Idl}\,(R) \to \mathrm{Idl}\,(R),$$

where R is a commutative ring with identity. Such a nucleus is localic precisely when C is radical. As a corollary of our generalization, we show that this result holds in the non-commutative case if we replace 'radical' by 'semiprime'.

LEMMA 4·1. *If s is an element of a quantale Q, then $(- \to_l s) \to_r s$ and $(- \to_r s) \to_l s$ are closure operators.*

Proof. First $(- \to_l s) \to_r s$ is clearly order-preserving, and $a \leqslant (a \to_l s) \to_r s$ since $(a \to_l s) \& a \leqslant s$. To prove idempotency, we show that $a \to_r s = ((a \to_r s) \to_l s) \to_r s$. Since $a \leqslant (a \to_l s) \to_r s$, it follows that $((a \to_r s) \to_l s) \to_r s \leqslant a \to_r s$. Also $((a \to_r s) \to_l s) \& (a \to_r s) \leqslant s$ implies $a \to_r s \leqslant ((a \to_r s) \to_l s) \to_r s$. Thus $(- \to_l s) \to_r s$ is a closure operator. The proof for $(- \to_r s) \to_l s$ is similar. ∎

To show that the above operators are quantic nuclei, we shall assume $(a \& b) \to_l s = (b \& a) \to_l s$ (respectively, $(a \& b) \to_r s = (b \& a) \to_r s$). This will be the case if s is symmetric (in particular, if Q is commutative), or if Q is an idempotent right-sided (respectively, left-sided) quantale. In the latter case, the desired condition follows from the fact that $t \& a \& b = t \& b \& a$ (respectively, $a \& b \& t = b \& a \& t$) for all $a, b, t \in Q$. Note that when L is a locale the operators $(- \to s) \to s$ generalize the nucleus $\neg\neg$, and $L_{(-\to s)\to s} \cong (L_{s \vee -})_{\neg\neg}$ (cf. [**11**]).

LEMMA 4·2. *If $(a \& b) \to_l s = (b \& a) \to_l s$ (respectively, $(a \& b) \to_r s = (b \& a) \to_r s$) for all $a, b \in Q$, then $(- \to_l s) \to_r s$ (respectively, $(- \to_r s) \to_l s$) is a quantic nucleus.*

Proof. Suppose that $(a \& b) \to_l s = (b \& a) \to_l s$. Then it easily follows that

$$t \& a \& b \leqslant s \quad \text{if and only if} \quad t \& b \& a \leqslant s. \tag{*}$$

First we show that $[(a \& b) \to_l s] \& [(a \to_l s) \to_r s] \leqslant b \to_l s$. Since

$$\begin{aligned}[(a \& b) \to_l s] \& b \& [(a \to_l s) \to_r s] &= [(b \& a) \to_l s] \& b \& [(a \to_l s) \to_r s] \\ &= [b \to_l (a \to_l s)] \& b \& [(a \to_l s) \to_r s] \leqslant (a \to_l s) \& [(a \to_l s) \to_r s] \leqslant s,\end{aligned}$$

applying (*), we obtain $[(a \& b) \to_l s] \& [(a \to_l s) \to_r s] \& b \leqslant s$, and so the desired result follows.

Using the above we have

$$[(a \& b) \to_l s] \& [(a \to_l s) \to_r s] \& [(b \to_l s) \to_r s] \leqslant (b \to_l s) \& [(b \to_l s) \to_r s] \leqslant s.$$

Hence
$$[(a \to_l s) \to_r s] \& [(b \to_l s) \to_r s] \leqslant ((a \& b) \to_l s) \to_r s.$$

Therefore $(- \to_l s) \to_r s$ is a quantic nucleus. The other case is similar. ∎

THEOREM 4·3. *If $s \in \mathrm{Loc}(Q)$, then $(- \to_l s) \to_r s$ is localic and equal to $(- \to_r s) \to_l s$. Moreover, if Q is right (respectively, left) unital, then $s \in \mathrm{Loc}(Q)$ if and only if $(- \to_r s) \to_l s$ (respectively, $(- \to_l s) \to_r s$) is localic.*

Proof. Suppose that s is localic. Then $a \to_r s = a \to_l s$ for all $a \in Q$, since s is symmetric. Hence $(- \to_l s) \to_r s = (- \to_r s) \to_l s$.

Since $(a \to_l s) \& a \leqslant s$ and s is right-sided, we have

$$(a \to_l s) \& a \& \top \leqslant s \& \top \leqslant s,$$

and so $a \rightarrow_{\ell} s \leqslant (a \,\&\, \tau) \rightarrow_{\ell} s$. Hence

$$((a \,\&\, b) \rightarrow_{\ell} s) \rightarrow_{r} s \leqslant ((a \,\&\, \tau) \rightarrow_{\ell} s) \rightarrow_{r} s \leqslant (a \rightarrow_{\ell} s) \rightarrow_{r} s.$$

Similarly
$$((a \,\&\, b) \rightarrow_{\ell} s) \rightarrow_{r} s \leqslant (b \rightarrow_{\ell} s) \rightarrow_{r} s,$$

since s is left-sided. Therefore

$$((a \,\&\, b) \rightarrow_{\ell} s) \rightarrow_{r} s \leqslant [(a \rightarrow_{\ell} s) \rightarrow_{r} s] \wedge [(b \rightarrow_{\ell} s) \rightarrow_{r} s].$$

For the proof of the reverse inequality, we shall show that if

$$x \leqslant [(a \rightarrow_{\ell} s) \rightarrow_{r} s] \wedge [(b \rightarrow_{\ell} s) \rightarrow_{r} s],$$

then

$$x \leqslant ((a \,\&\, b) \rightarrow_{\ell} s) \rightarrow_{r} s.$$

Suppose that $x \leqslant ((a \rightarrow_{\ell} s) \rightarrow_{r} s)$ and $x \leqslant ((b \rightarrow_{\ell} s) \rightarrow_{r} s) = ((b \rightarrow_{\ell} s) \rightarrow_{\ell} s)$. Then $(a \rightarrow_{\ell} s) \,\&\, x \leqslant s$ and $x \,\&\, (b \rightarrow_{\ell} s) \leqslant s$. Now

$$x \,\&\, ((a \,\&\, b) \rightarrow_{\ell} s) = x \,\&\, (a \rightarrow_{\ell} (b \rightarrow_{\ell} s)) \leqslant a \rightarrow_{\ell} (x \,\&\, (b \rightarrow_{\ell} s)) \leqslant a \rightarrow_{\ell} s.$$

Hence

$$\begin{aligned}[((a \,\&\, b) \rightarrow_{\ell} s) \,\&\, x]^2 &= ((a \,\&\, b) \rightarrow_{\ell} s) \,\&\, x \,\&\, ((a \,\&\, b) \rightarrow_{\ell} s) \,\&\, x \\ &\leqslant \tau \,\&\, [x \,\&\, ((a \,\&\, b) \rightarrow_{\ell} s)] \,\&\, x \leqslant \tau \,\&\, (a \rightarrow_{\ell} s) \,\&\, x \leqslant \tau \,\&\, s \leqslant s.\end{aligned}$$

Since s is semiprime, it follows that $((a \,\&\, b) \rightarrow_{\ell} s) \,\&\, x \leqslant s$, and so $x \leqslant ((a \,\&\, b) \rightarrow_{\ell} s) \rightarrow_{r} s$, as desired. Therefore $(- \rightarrow_{\ell} s) \rightarrow_{r} s$ is localic.

Assume that Q is right unital and that $j = (- \rightarrow_{r} s) \rightarrow_{\ell} s$ is localic. The left unital case is similar. Since $s \,\&\, 1 \leqslant s$, we know that $1 \leqslant s \rightarrow_{r} s$, and so

$$(s \rightarrow_{r} s) \rightarrow_{\ell} s \leqslant 1 \rightarrow_{\ell} s = (1 \rightarrow_{\ell} s) \,\&\, 1 \leqslant s.$$

Hence $j(s) = (s \rightarrow_{r} s) \rightarrow_{\ell} s \leqslant s$. Since j is localic and $j(s) = s$, it follows that s is localic, and this completes the proof. ∎

COROLLARY 4·4. *If R is a ring with identity and $C \in \mathrm{Idl}(R)$, then C is semiprime if and only if $C :_{r} (C :_{\ell} -)$ is a localic nucleus on $\mathrm{Idl}(R)$ if and only if $C :_{\ell} (C :_{r} -)$ is a localic nucleus on $\mathrm{Idl}(R)$.* ∎

Note that the above corollary generalizes the result of Kirby to non-commutative rings.

5. *Subquantales*

A subset S of Q is called a *subquantale* if S is closed under $\bigvee$, & and τ. Clearly S is a subquantale of Q if and only if S is a quantale and the inclusion $S \rightarrow Q$ is a homomorphism.

If R is any ring, then $\mathrm{Idl}(R)$ is a subquantale of $\mathrm{RIdl}(R)$ and $\mathrm{LIdl}(R)$. If R is commutative with identity, then the set $\mathrm{EIdl}(R)$ of idempotent ideals of R and the set $\mathrm{PIdl}(R)$ of pure ideals (i.e. ideals A such that $AB = A \cap B$ for all ideals B) of R are subquantales of $\mathrm{Idl}(R)$. It is not difficult to show that these two subquantales are in fact locales. The pure spectrum is discussed in [**2**].

Let Q be any quantale. Then the sets $R(Q)$, $L(Q)$ and $T(Q)$ of right-, left- and two-sided elements of Q are clearly closed under $\bigvee$, & and τ, and thus are subquantales of Q.

Recall that a *coclosure operator* on a partially ordered set Q is an order-preserving map $g : Q \to Q$ which is decreasing and idempotent. Such a map is easily seen to satisfy $g(a) \leqslant b$ if and only if $g(a) \leqslant g(b)$. Moreover, if Q is complete, then

$$Q_g = \{a \in Q \mid g(a) = a\}$$

is closed under $\bigvee$ and hence is a sub-$\bigvee$-lattice of Q. In fact every sub-$\bigvee$-lattice S of Q arises in this manner, where $g(a) = \bigvee\{s \in S \mid s \leqslant a\}$. Note that the meet in Q_g is given by $\bigwedge_g a_\alpha = g(\bigwedge a_\alpha)$.

Let Q be a quantale. A *quantic conucleus* on Q is a coclosure operator g such that $g(\tau) = \tau$ and $g(a) \,\&\, g(b) \leqslant g(a \,\&\, b)$, for all $a, b \in Q$. The set of all quantic conuclei on Q is denoted by $CN(Q)$.

PROPOSITION 5·1. *If $g : Q \to Q$ is a quantic conucleus, then Q_g is a subquantale of Q. Moreover, if S is any subquantale of Q, then $S = Q_g$, where $g(a) = \bigvee\{s \in S \mid s \leqslant a\}$.*

Proof. We know that Q_g is closed under $\bigvee$ and τ, since g is a coclosure operator and since $g(\tau) = \tau$. If $a, b \in Q_g$, then $a \,\&\, b = g(a) \,\&\, g(b) \leqslant g(a \,\&\, b)$, and so $a \,\&\, b \in Q_g$. Hence Q_g is a subquantale of Q.

Suppose that S is a subquantale of Q, and $g(a) = \bigvee\{s \in S \mid s \leqslant a\}$. Then $S = Q_g$ and g is a coclosure operator on Q. Now $g(\tau) = \tau$, since $\tau \in S$. If $s \leqslant a$ and $t \leqslant b$, where $s, t \in S$, then $s \,\&\, t \leqslant a \,\&\, b$. Since S is closed under &, it follows that $s \,\&\, t \leqslant g(a \,\&\, b)$. Hence

$$g(a) \,\&\, g(b) = \bigvee\{s \mid s \leqslant a\} \,\&\, \bigvee\{t \mid t \leqslant b\} = \bigvee\{s \,\&\, t \mid s \leqslant a,\, t \leqslant b\} \leqslant g(a \,\&\, b).$$

Therefore g is a quantic conucleus. ∎

As in the case of $N(Q)$, we can give $CN(Q)$ the pointwise order. Note that $g \leqslant h$ in $CN(Q)$ if and only if $Q_g \subseteq Q_h$. Also, if $\{g_\alpha\} \subseteq CN(Q)$, then $\bigcap Q_{g_\alpha}$ is a subquantale. Thus we obtain

PROPOSITION 5·2. *$CN(Q)$ is a complete lattice.* ∎

We now turn our attention to those subquantales of Q which are, in fact, locales. If g is a quantic conucleus on Q, then $g(a) \wedge_g g(b) = g(g(a) \wedge g(b)) = g(a \wedge b)$.

PROPOSITION 5·3. *The following are equivalent for $g \in CN(Q)$:*

(1) *Q_g is a locale with $\wedge = \&$;*

(2) *$g(a) \,\&\, g(b) = g(a \wedge b)$, for all $a, b \in Q$;*

(3) *$g(a)^2 = g(a)$, $g(a) \,\&\, \tau \leqslant g(a)$, and $\tau \,\&\, g(a) \leqslant g(a)$, for all $a \in Q$;*

(4) *every element of Q_g is idempotent and two-sided.*

Proof. Clearly the implications (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) and (2) $\Rightarrow$ (3) hold. For the proof of the implication (3) $\Rightarrow$ (2) let $a, b \in Q$ and assume that (3) holds. Then

$$g(a \wedge b) = g(a \wedge b)^2 = g(a \wedge b) \,\&\, g(a \wedge b) \leqslant g(a) \,\&\, g(b).$$

Also $$g(a)\,\&\,g(b) \leqslant (g(a)\,\&\,\top) \wedge (\top\,\&\,g(b)) = g(a) \wedge g(b)$$

implies that $$g(a)\,\&\,g(b) = g(g(a)\,\&\,g(b)) \leqslant g(g(a) \wedge g(b)) = g(a \wedge b).$$

Therefore $g(a)\,\&\,g(b) = g(a \wedge b)$. ∎

A *localic conucleus* is a quantic conucleus satisfying the above properties. If g is a localic conucleus on Q, then Q_g is called a *localic subquantale* of Q.

Let R be a commutative ring with identity. If $A \in \mathrm{Idl}\,(R)$, let $E(A) = \{e \in A \mid e^2 = e\}$. Note that $E(R)$ is a Boolean algebra with respect to the operations $e \wedge f = ef$, $e \vee f = e+f-ef$, and $\neg e = 1-e$, and $E(A)$ is an ideal of $E(R)$ for all $A \in \mathrm{Idl}\,(R)$. Define $g : \mathrm{Idl}\,(R) \to \mathrm{Idl}\,(R)$ by $g(A) = \langle E(A) \rangle$. Then g is clearly a coclosure operator, $g(R) = R$, and

$$g(A)\,g(B) = \langle E(A)\,E(B) \rangle = \langle ef \mid e \in E(A), f \in E(B) \rangle \subseteq \langle E(AB) \rangle = g(AB).$$

Thus g is a quantic conucleus which is easily seen to be localic, and $(\mathrm{Idl}\,(R))_g$ is the set of ideals of R which are generated by idempotents. Moreover, using the following lemma, one can show that $(\mathrm{Idl}\,(R))_g$ is isomorphic to the locale of ideals of $E(R)$, i.e. the locale associated with the Pierce spectrum of R (see [**6**]).

LEMMA 5·4. *If I is an ideal of $E(R)$ and A is the ideal of R generated by I, then $A = \{re \mid r \in R,\ e \in I\}$ and $E(A) = I$.*

Proof. If $r_1, r_2 \in R$ and $e_1, e_2 \in I$, consider $r = r_1 e_1 + r_2 e_2$ and $e = e_1 \vee e_2$. Since

$$re = (r_1 e_1 + r_2 e_2)(e_1 + e_2 - e_1 e_2) = r_1 e_1 + r_2 e_2,$$

it follows that $r_1 e_1 + r_2 e_2 \in A$. Hence $A = \{re \mid r \in R, e \in I\}$. Clearly $I \subseteq E(A)$. If $e \in E(A)$ and $e = rf$ for some $f \in I$, then $ef = rf^2 = rf = e$, and so $e \leqslant f$. Hence $e \in I$, since I is an ideal of $E(R)$. Therefore $E(A) = I$ as desired.

In contrast to the nucleus case, there is not necessarily a largest localic element in $CN(Q)$. However we have the following partial results.

PROPOSITION 5·5. *If $\{S_\alpha\}$ is a non-empty family of localic subquantales of Q, then $\bigcap S_\alpha$ is a localic subquantale.*

Proof. By Proposition 5·3, it suffices to show that every element s of $\bigcap S_\alpha$ is two-sided and idempotent. Since $\{S_\alpha\}$ is non-empty, we know that s is in some localic subquantale S_α of Q. Hence s is two-sided and idempotent, by Proposition 5·3. ∎

Let $E(Q)$ denote the set of two-sided idempotents of Q. Note that $E(Q)$ contains every localic subquantale of Q.

PROPOSITION 5·6. *The following are equivalent for a quantale Q:*

(1) *Q has a largest localic subquantale;*

(2) *$E(Q)$ is a locale with $\wedge = \&$;*

(3) *$\top \in E(Q)$ and $e\,\&\,f = f\,\&\,e$, for all $e, f \in E(Q)$.*

Proof. (1) ⇒ (2). Suppose that Q has a largest localic subquantale L. Then $L \subseteq E(Q)$ by Proposition 5·3. If e is any two-sided idempotent, then $\{\bot, e, \top\}$ is a localic subquantale of Q, and so $\{\bot, e, \top\} \subseteq L$. Hence $E(Q) \subseteq L$. Therefore $E(Q)$ is a locale with $\wedge = \&$.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Suppose that $\top \in E(Q)$ and $e \& f = f \& e$, for all $e, f \in Q$. Then $E(Q)$ is closed under &. Now $E(Q)$ is closed under $\bigvee$ since

$$(\bigvee e_\alpha) \& (\bigvee e_\alpha) = \bigvee(e_\alpha \& e_\beta) \leqslant \bigvee(e_\alpha \& \top) = \bigvee e_\alpha$$

and

$$\bigvee e_\alpha = \bigvee(e_\alpha \& e_\alpha) \leqslant \bigvee(e_\alpha \& e_\beta) = (\bigvee e_\alpha) \& (\bigvee e_\alpha).$$

Hence $E(Q)$ is a subquantale of Q, and it is localic since every element is two-sided and idempotent. Moreover $E(Q)$ contains every localic subquantale of Q by Proposition 5·3.

COROLLARY 5·7. *If Q is a commutative unital quantale and $1 = \top$, then $CN(Q)$ is complete and $E(Q)$ is the largest subquantale of Q.*

As we noted earlier, if R is a commutative ring with identity, then EIdl(R) is a localic subquantale of Idl(R). Corollary 5·7 asserts that it is the largest localic subquantale of Idl(R).

We conclude with an example a ring R such that Idl(R) does not have a largest localic subquantale, i.e. EIdl(R) is not closed under products. Let R be the ring of 2×2 upper triangular matrices with coefficients in $\mathbb{Z}$. Consider the two-sided idempotent ideals

$$E = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$$

and

$$F = \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}.$$

Then $FE = 0 \neq EF$. Therefore Idl(R) does not have a largest localic subquantale.

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