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A CHARACTERIZATION OF RING CONGRUENCES ON SEMIRINGS

BY

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Abstract. The main result in this paper is a complete characterization of ring congruences on semi-rings in terms of certain special class of k-ideals. As an application, field congruences on a special class of semirings are characterized.

1. Introduction

A semiring S is an algebraic structure $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups, satisfying $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in S$. Henceforth S will denote a semiring and we shall write simply ab instead of $a \cdot b$ for all $a, b \in S$. A subsemiring H of S is called a semi-ideal of S if SH and HS are contained in H. A k-ideal I {Henriksen [1]} is a semi-ideal of S such that whenever $x + y \in I$, $x, y \in S$, then $x \in I$ iff $y \in I$.

By a congruence on a semirings, we mean a congruence on both $(S, +)$ and $(S, \cdot)$. A congruence ρ on S is called a ring congruence if its quotient semiring $(S/\rho, +, \cdot)$ is a ring. Bourne [5] defined a binary relation σ_H (say) on an additively commutative semiring S relative to a semi-ideal H of S as $(a, b) \in \sigma_H$ iff there exist $h_1, h_2 \in H$ such that $a + h_1 = b + h_2$. Bugenhagen

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[6] investigated the congruence nature of this relation and some properties of the quotient semiring S/σ_H in case of an additively commutative semiring S. Iizuka [2] also defined another type of congruence which Latorre [7] showed to be a ring congruence in a very special class of semirings.

The main purpose of this paper is to completely characterize ring congruences in terms of certain special class of k-ideals of semirings (Theorem 3.5). This result includes the results of Bugenhagen and Latorre as special cases.

2. Basic Concepts and Preliminaries

Definition 2.1. A non-empty subset H of S is called a c-subset of S if for any $a, b \in S$, there exist $h_1, h_2 \in H$ such that $a + b + h_1 = h_2 + b + a$.

Definition 2.2. A c-semi-ideal [c-k-ideal] of S is a semi-ideal [k-ideal] of S which is also a c-subset of S.

Definition 2.3. Let $H \subseteq S$, $H \neq \Phi$ and $a \in S$. We define $a(+)H = \{x \in S : a + x \in H\}$ and $H_w = \{a \in S : a(+)H = \Phi\}$. $H(+)a$ and ${}_wH$ are defined dually.

Definition 2.4. Let $H \subseteq S$, $H \neq \Phi$. ρ_H is a binary relation on S defined as $\rho_H = \{(a,b) \in S \times S : a + h_1 = h_2 + b$ for some $h_1, h_2 \in H\}$.

Remark 2.5. ρ_H coincides with σ_H for each semi-ideal H of S when $(S,+)$ is commutative.

Theorem 2.6. *If H is a c-semi-ideal of S, then ρ_H is a congruence on S.*

Proof. Let $a \in S, h \in H$. Since H is a c-subset of S, there are $h_1, h_2 \in H$ such that $a + h + h_1 = h_2 + h + a$. This implies $(a,a) \in \rho_H$ as H is a semi-ideal. Let $(a,b) \in \rho_H$ for some $a, b \in S$. Then there are $h_1, h_2 \in H$ such that $a + h_1 = h_2 + b$. Since H is a c-subset of S, there exist $h_i \in H$, $i = 3, 4, 5, 6$

such that $a + h_1 + h_3 = h_4 + h_1 + a$ and $b + h_2 + h_5 = h_6 + h_2 + b$. Thus $h_6 + h_4 + h_1 + a = b + h_2 + h_5 + h_3$ which implies $(b, a) \in \rho_H$. The transitive property of ρ_H is clear. Therefore ρ_H is an equivalence relation on S.

Again since H is a semi-ideal of S, $(a, b) \in \rho_H$ for some $a, b \in S$ implies (ac, bc) and (ca, cb) both belong to ρ_H. Finally let $a, b, c \in S$ such that $(a, b) \in \rho_H$. This implies there exist $h_1, h_2 \in H$ such that $a + h_1 = h_2 + b$. Therefore $c + (a + h_1) + h_3 = c + (h_2 + b) + h_3 = h_4 + (h_2 + b) + c$ for some $h_3, h_4 \in H$ as H is a c-subset of S. Thus $(c + a, b + c) \in \rho_H$. Noting that the c-subset property of H implies $(x + y, y + x) \in \rho_H$ for all $x, y \in S$, we get $(c + a, c + b)$ and $(a + c, b + c)$ belong to ρ_H.

We close this section with an example which shows that there exist k-ideals of a semiring which are not c-k-ideals.

Example 2.7. Let S be the free monoid generated by the set $\{a, b\}$. In this free monoid the operation is denoted by "+" and the identity element of it is denoted by 0. We define a new binary operation "$\cdot$" on S by $u \cdot v = 0$ for all $u, v \in S$. Thus we obtain a semiring $(S, +, \cdot)$. In this semiring the subsemiring generated by "a" is a k-ideal but not a c-k-ideal of S.

3. Ring Congruences

Throughout this section we consider H as a c-semi-ideal of a semiring S.

Lemma 3.1.

(i) *H is contained in a ρ_H-class, say U.*

(ii) $\rho_H = \rho_U$.

(iii) *U is a c-k-ideal of S.*

(iv) *$H = U$ iff H is a c-k-ideal of S.*

Proof. (i) Immediate from the fact that $h_1 + h_2 = h_1 + h_2$ for all $h_1, h_2 \in H$.

(ii) $\rho_H \subseteq \rho_U$ is clear from $H \subseteq U$. Let $a, b \in S$ such that $(a, b) \in \rho_U$. Then $a + u_1 = u_2 + b$ for some $u_1, u_2 \in U$. Let $h \in H$. Then $(u_i, h) \in \rho_H$

by (i), $i = 1,2$. This implies $u_1 + h_1 = h_1' + h$ and $h + h_2' = h_2 + u_2$ for some $h_i, h_i' \in H$, $i = 1,2$. Therefore $h_2 + a + h_1' + h = h_2 + a + u_1 + h_1 = h_2 + u_2 + b + h_1 = h + h_2' + b + h_1$. Applying c-semi-ideal property of H on both sides we get $(a,b) \in \rho_H$.

(iii) Let $u_1, u_2 \in U$. By (i) $(h, u_2) \in \rho_H$, for any $h \in H$. Then $(u_1 + h, u_1 + u_2) \in \rho_H$ as ρ_H is a congruence on S. Thus there exist $h_1, h_2 \in H$ such that $u_1 + h + h_1 = h_2 + u_1 + u_2$, which shows $(u_1, u_1 + u_2) \in \rho_H$ for H is a semi-ideal of S. Therefore $u_1 + u_2 \in U$. Again since ρ_H is a congruence and H is a semi-ideal, SU, $US \subseteq U$ follows from (i). Thus U is a semi-ideal and hence a c-semi-ideal of S for $H \subseteq U$.

Finally to show U is a k-ideal also, let $x \in S$ and $u \in U$ such that $x + u \in U$. By (i), $(x + u, u) \in \rho_H = \rho_U$. This implies $(x, u) \in \rho_U$ as U is a semi-ideal. Thus $x \in U$ as desired. Similarly $x \in S$ and $u, u + x \in U$ implies $x \in U$.

(iv) If $H = U$, then H is a c-k-ideal is clear from (iii). Conversely, let H be a c-k-ideal of S. Let $x \in S$, $h \in H$ such that $(x,h) \in \rho_H$. Then by definition of ρ_H, $x + h_1 \in H$ for some $h_1 \in H$. Since H is a k-ideal, $x \in H$. Thus $H = U$.

Bugenhagen [6] showed that if H is a semi-ideal of an additively commutative semiring S, then H is contained in a σ_H-class, say U, which is a k-ideal of S and $S/\sigma_H = S/\sigma_U$. Also $H = U$ iff H is a k-ideal. These results are special cases of the above lemma, since here $\rho_H = \sigma_H$ and every semi-ideal is a c-semi-ideal.

Let [a] denote the ρ_H-class of $a \in S$.

Lemma 3.2. *Let U be the ρ_H-class containing H with $H_w = \Phi$. Then*

(i) $U_w = \Phi$.

(ii) $x + a$, $b + x$, $x + b \in U$, *when* $a, b, x \in S$ *such that* $a + x \in H$ *and* $(a,b) \in \rho_H$

(iii) ${}_wU = \Phi$.

Proof. (i) Trivial for $H \subseteq U$.

(ii) Since H is a c-subset of S, $(a+x, x+a) \in \rho_H$. Thus $x+a \in U$ as $a+x \in H$. Again $[a]=[b]$. Then $[b+x]=[b]+[x]=[a]+[x]=[a+x]=U=[x+a]=[x]+[a]=[x]+[b]=[x+b]$. Hence $b+x$, $x+b \in U$.

(iii) Immediate from (ii).

Lemma 3.3. *For each c-semi-ideal H with $H_w = \Phi$, the quotient semiring $(S/\rho_H, +, \cdot)$ is a ring.*

Proof. Since ρ_H is a congruence on S, $(S/\rho_H, +, \cdot)$ is already a semiring. We show the semigroup $(S/\rho_H, +)$ is a commutative group. For this let $a \in S$ and $u \in U$. As H is a c-subset of S, $a+u+h_1 = h_2+u+a$ for some $h_1, h_2 \in H$. Then $(a+u, a) \in \rho_U$. But $\rho_U = \rho_H$ by Lemma 3.1 (ii). This implies $[a]+U=[a]+[u]=[a+u]=[a]$. Similarly $U+[a]=[a]$. Thus U is the identity of $(S/\rho_H, +)$.

Again let $a \in S$. Since $H_w = \Phi$, there exists $x \in S$ such that $a+x \in H$. Also by Lemma 3.2 (ii), $x+a \in U$. Therefore $[a]+[x]=[a+x]=U=[x+a]=[x]+[a]$. Thus $[x]$ is the additive inverse of $[a]$. Hence $(S/\rho_H, +)$ is a group. Its commutative property follows directly from the fact that H is a c-subset of S.

A zero element, denoted by 0, of a semiring S is defined by $0 \cdot x = x \cdot 0 = 0$ and $0+x=x+0=x$ for all $x \in S$. An h-ideal J {Iizuka [2], Latorre [7]} is a semi-ideal of a hemiring T (additively commutative semiring with zero) such that if $x, z \in T$, $j_1, j_2 \in J$ and $x+j_1+z=j_2+z$, then $x \in J$. Iizuka [2] defined a congruence Γ_J (say) on a hemiring T, relative to a semi-ideal J as $(a,b) \in \Gamma_J$ iff there are $j_1, j_2 \in J$ and $z \in T$ such that $a+j_1+z=b+j_2+z$. Latorre [7] showed that if J is an h-ideal of an additively regular (for any $a \in T$, there is $x \in T$ such that $a+x+a=a$) hemiring T, then T/σ_J is a ring and $T/\sigma_J = T/\Gamma_J$ {[7], Corollary 2.8}. Now it is obvious that any h-ideal is a k-ideal and for an additively commutative semi-ring $\sigma_J = \rho_J$ (Remark 2.5). It is easy to verify that any h-ideal J in an additively regular hemiring satisfies $J_w = \Phi$. Also as before the c-subset property is redundant here. Thus the above result is a very special case of the Lemma 3.3.

Lemma 3.4. *Let ρ be a ring congruence on a semiring S and U be the identity of $(S/\rho, +)$. Then*

(i) $U_w = {}_wU = \Phi$

(ii) $\rho = \rho_U$

(iii) U *is a c-k-ideal of* S.

Proof. Let $[a]$ denote the ρ-class of $a \in S$.

(i) Let $a \in S$. Since $(S/\rho, +)$ is a group, there exists $x \in S$ such that $[a] + [x] = U = [x] + [a]$ i.e. $[a + x] = U = [x + a]$. Thus $a + x$, $x + a \in U$.

(ii) Since U is the additive identity of S/ρ, $\rho_U \subseteq \rho$ is clear. Conversely, let $a, b \in S$ such that $(a, b) \in \rho$ i.e. $[a] = [b]$. Since $(S/\rho, +)$ is a group, there exists $x \in S$ such that $U = [a] + [x] = [x] + [b]$. This implies $a + x$, $x + b \in U$. Now $a + (x + b) = (a + x) + b$ and so $(a, b) \in \rho_U$.

(iii) Since U is the identity of $(S/\rho,+)$ i.e. the "zero" of the ring $(S/\rho,+,\cdot)$, $U + U = U$ and for any $a \in S$, $[a]\ U = U = U[a]$. These imply U is a semi-ideal of S. Also let $a \in S$ and $u \in U$ such that $a + u \in U$ (or $u + a \in U$). Then $[a] = [a] + U = [a] + [u] = [a + u] = U$ (or $[a] = U + [a] = U$, similarly). Thus $a \in U$ and so U is a k-ideal of S.

Finally since $(S/\rho, +)$ is commutative $[a+b] = [a]+[b] = [b]+[a] = [b+a]$. Therefore $(a + b, b + a) \in \rho_U$ by (ii), whence it follows the result.

We are now in position to state the main result of this paper. It is analogous to a theorem {[4], p. 200}, which characterizes group congruences on semigroups.

Theorem 3.5. *Let S be a semiring [an additively commutative semiring] and H be a c-semi-ideal [semi-ideal] of S with $H_w = \Phi$. Then H is contained in a ρ_H-class, say U. U is a c-k-ideal [k-ideal] of S with $U_w = {}_wU = \Phi$. $H = U$ iff H is a c-k-ideal [k-ideal] of S. Moreover $\rho_H = \rho_U$ and $(S/\rho_H, +, \cdot)$ is a ring.*

Conversely, let ρ be a ring congruence on a semiring [an additively commutative semiring] S and U be the additive identity of S/ρ. Then U is a c-k-ideal [k-ideal] of S with $U_w = {}_wU = \Phi$. Moreover $\rho = \rho_U$.

The correspondence described above between c-k-ideals [k-ideals] U of S with $U_w = \Phi$ and ring congruences on S is one-to-one.

Proof. The unbracketed assertion is immediate from the above lemmas and the other one is merely a corollary of it, since every semi-ideal is a c-semi-ideal there.

Remark 3.6. In view of the remark 2.5, we see from the above theorem that any ring congruence on an additively commutative semiring S is a Bourne congruence σ_H relative to a semi-ideal H of S with $H_w = \Phi$ and conversely.

A congruence ρ on S is called a field congruence if its quotient semiring $(S/\rho, +, \cdot)$ is a field. From the theorem 3.5 one can easily deduce the following:

Corollary 3.7. *Let S be a multiplicatively commutative semiring with multiplicative identity such that S has a proper c-k-ideal U with $U_w = \Phi$. Then there exists a maximal (proper) c-k-ideal $\bar{U}$ of S with $\bar{U}_w = \Phi$ such that $S/\rho_{\bar{U}}$ is a field.*

Conversely, if ρ is a field congruence on a semiring S and U is the additive identity of S/ρ, then U is a maximal (proper) c-k-ideal with $U_w = \Phi$ and $\rho = \rho_U$.

We wish to prove here another interesting result which is very much relevant to our discussion.

The principal congruence P_H (Shyr [3], Clifford and Preston [4]) on a semigroup $(X, +)$ relative to a subset H of X is defined as $P_H = \{(a,b) \in X \times X : x + a + y \in H \text{ iff } x + b + y \in H \text{ for all } x, y \in X\}$.

Theorem 3.8. *Let U be a c-k-ideal [k-ideal] of a semiring [an additively commutative semiring] S with $U_w = \Phi$. Then $\rho_U = P_U$.*

Proof. As before we prove only the unbracketed assertion. $\rho_U \subseteq P_U$ follows from the lemma 3.1 (iv) and the fact that ρ_U is a congruence on S.

Conversely, let $a, b \in S$ such that $(a,b) \in P_U$. Since $U_w = \Phi$, there exists $x \in S$ such that $b + x \in U$. Now U is a c-subset of S. Then there are

$u_1, u_2 \in U$ such that $x + b + u_1 = u_2 + b + x$. But $u_2 + (b + x) \in U$ as U is a semi-ideal of S. Thus $x + b \in U$, since U is also a k-ideal. Also $u_2 + b + x \in U$ implies $u_2 + a + x \in U$ for $(a, b) \in P_U$. So again since U is a k-ideal, $a + x \in U$. Now $a + (x + b) = (a + x) + b$. This shows $(a, b) \in \rho_U$.

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