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ON LOCALES OF LOCALIZATIONS

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1. Introduction

Our aim is to study the ordered set Loc $\mathscr{A}$ of localizations of a category $\mathscr{A}$, showing it to be a small complete lattice when $\mathscr{A}$ is complete with a (small) strong generator, and further to be the dual of a locale when $\mathscr{A}$ is a locally-presentable category in which finite limits commute with filtered colimits. We also consider the relations between Loc $\mathscr{A}$ and Loc $\mathscr{A}'$ arising from a geometric morphism $\mathscr{A} \to \mathscr{A}'$; and apply our results in particular to categories of modules. We give further details in the rest of this Introduction.

Our results are not sensitive to the view taken of the foundations, so long as it is one appropriate to category theory; we ourselves find the following view most convenient. We suppose the existence of arbitrarily large inaccessible cardinals, our 'inaccessible' being what some have called 'strongly inaccessible'. Such an inaccessible ∞ having been chosen once for all, a set is *small* if its cardinal is less than ∞. The morphisms of any category $\mathscr{A}$ form a set, and $\mathscr{A}$ is *small* if this set is small; while $\mathscr{A}$ is *locally small* if each hom-set $\mathscr{A}(A, B)$ is small. The category $\mathscr{A}$ is *complete* if it admits *small* limits. The reader who so wishes can read 'class' for 'set' and 'set' for 'small set'; but classical Gödel–Bernays set theory will not do for our results, since a localization of $\mathscr{A}$, being in general a large subcategory, is then already a class, so that the collection of all localizations of $\mathscr{A}$ does not exist in GB.

Except where the contrary is made explicit, which only happens twice in the final section, we use 'subcategory' to mean 'full, replete subcategory' – on rare occasions inserting the adjectives for emphasis; and we freely identify a subcategory with the set of its objects. We adopt a standard notation for the data associated with a reflec-

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tive subcategory $\mathscr{C}$ of $\mathscr{A}$: the inclusion is $j : \mathscr{C} \to \mathscr{A}$, and its left adjoint is $\bar{r} : \mathscr{A} \to \mathscr{C}$, always so chosen that $\bar{r}j = 1$; the unit of the adjunction is $\varrho : 1 \to j\bar{r}$; and we write r for the endofunctor $j\bar{r}$ of $\mathscr{A}$, so that $\varrho : 1 \to r$ is the idempotent monad associated to the reflexion. When $\mathscr{C}$ is decorated with subscripts or superscripts, so is each of the associated data. (In general, an *idempotent monad* is a $\varrho : 1 \to r$ for which $r^2 = r$ and $r\varrho = \varrho r = 1 : r \to r^2$. It determines a reflective subcategory $\mathscr{C}$, consisting of the $A \in \mathscr{A}$ for which ϱA is invertible; clearly reflective subcategories are in bijection with isomorphism classes of idempotent monads.)

When $\mathscr{A}$ is finitely complete, we call the reflective subcategory $\mathscr{C}$ a *localization* of $\mathscr{A}$ if $\bar{r}$, or equivalently r, is left exact (that is, preserves finite limits). A functor is said to *have rank* α, where α is a small regular cardinal, if it preserves (small) α-filtered colimits in the sense of [6]; when $\mathscr{A}$ admits such colimits, we call the localization $\mathscr{C}$ of $\mathscr{A}$ an *α-localization*, or a *localization of rank* α, if $\mathscr{C}$ is closed in $\mathscr{A}$ under these colimits – which is to say that j, or equivalently r, has rank α.

The set Loc $\mathscr{A}$ of localizations of $\mathscr{A}$ is ordered by inclusion of subcategories, as is its subset $\mathrm{Loc}_\alpha \mathscr{A}$ of α-localizations, and its subset $\mathrm{Loc}_\mathrm{r} \mathscr{A}$ of those localizations which have *some* (small) rank. After recalling in Section 2 the bijection of [2] between the localizations of a finitely-complete $\mathscr{A}$ and certain factorization systems on $\mathscr{A}$, and analyzing this somewhat further, we apply this bijection in Section 3 to prove that, when $\mathscr{A}$ is complete [resp. finitely complete], any small [resp. finite] family of localizations admits a supremum, given by the closure of their union under small [resp. finite] limits in $\mathscr{A}$. The restriction to *small* families of localizations here is necessary: we give examples of complete and cocomplete $\mathscr{A}$ where Loc $\mathscr{A}$ is a large set and a large family of localizations fails to have a supremum, or has one which is not the closure under small limits of their union. As for $\mathrm{Loc}_\alpha \mathscr{A}$, we show that finite suprema in Loc $\mathscr{A}$ of α-localizations are again α-localizations, *provided that* α-filtered colimits in $\mathscr{A}$ commute with finite limits.

The question of small infima of localizations seems to be much more delicate: we give examples of complete and cocomplete $\mathscr{A}$ where even binary infima fail to exist, or exist but are not given by the intersection of the two localizations. To get positive results here, we have to restrict ourselves to categories $\mathscr{A}$ in which filtered colimits exist and commute with finite limits, and to localizations with rank. After recalling in Section 4 some results of [11] on well-pointed endofunctors and their algebras, and extending these in important ways, we apply them in Section 5 to show that, for a category $\mathscr{A}$ as above, $\mathrm{Loc}_\mathrm{r} \mathscr{A}$ admits small infima given by the intersection of the localizations, and that binary suprema distribute over small infima. Such infima are also the infima in Loc $\mathscr{A}$; and each $\mathrm{Loc}_\alpha \mathscr{A} \subset \mathrm{Loc}_\mathrm{r} \mathscr{A}$ is closed under finite suprema and small infima.

We show in Section 6 that, when the finitely-complete $\mathscr{A}$ is locally small and a small subset of its objects constitutes a *strong generator*, the set Loc $\mathscr{A}$ is small. Then, if $\mathscr{A}$ is complete, Loc $\mathscr{A}$ is by the above a small complete lattice, even without the restrictive conditions of the last paragraph; in the absence of these conditions, however, we have not investigated the nature of infima of localizations, and do not know whether they are the intersections. When these conditions *are* satisfied, it follows from the above that $\mathrm{Loc}_\Gamma \mathscr{A}$ is the dual of a locale, as is each $\mathrm{Loc}_\alpha \mathscr{A}$. (We recall - see [9] - that a complete lattice X is called a *locale*, or equivalently a *frame*, if each $x\wedge -$ preserves arbitrary suprema.) When $\mathscr{A}$ is a locally-presentable category, there is a regular cardinal α such that $\mathrm{Loc}\,\mathscr{A} = \mathrm{Loc}_\alpha \mathscr{A}$; we conclude that $\mathrm{Loc}\,\mathscr{A}$ itself is the dual of a locale whenever $\mathscr{A}$ is a locally-presentable category in which finite limits commute with filtered colimits. Among such $\mathscr{A}$ are locally-finitely-presentable categories, Grothendieck topoi, and more generally the category of models in a Grothendieck topos of a finitary essentially-algebraic theory.

Adopting the language of topos theory, we define a *geometric morphism* $F: \mathscr{A} \to \mathscr{A}'$ between finitely-complete categories to be an adjunction $F^* \dashv F_*: \mathscr{A} \to \mathscr{A}'$ for which the left adjoint $F^*: \mathscr{A}' \to \mathscr{A}$ is left exact. In Section 7 we use the factorization of a geometric morphism into a surjective one followed by an injective one to obtain from such an $F: \mathscr{A} \to \mathscr{A}'$ an adjunction (or Galois connexion) $F^\# \dashv F_\#$: $\mathrm{Loc}\,\mathscr{A}' \to \mathrm{Loc}\,\mathscr{A}$, when each of $\mathscr{A}$ and $\mathscr{A}'$ is complete and locally small and admits a small strong generator.

In Section 8 we turn to the case where $\mathscr{A} = \mathrm{Mod}\text{-}R$ is the locally-finitely-presentable category of right modules for the ring R. As is well known, there is an isomorphism $\mathrm{Loc}(\mathrm{Mod}\text{-}R)^{\mathrm{op}} \cong \mathrm{Top}\,R$ of ordered sets, where $\mathrm{Top}\,R$ is the set of *Gabriel topologies* on R; accordingly, $\mathrm{Top}\,R$ is a locale. A ring-homomorphism ϕ : $R \to S$ induces an algebraic functor $\phi^*: \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$, which has a right adjoint ϕ_* and a left adjoint $\phi_!$. The adjunction $\phi^* \dashv \phi_*$, being a geometric morphism, induces as in the last paragraph an adjunction between $\mathrm{Loc}(\mathrm{Mod}\text{-}R)$ and $\mathrm{Loc}(\mathrm{Mod}\text{-}S)$ which, passing to the duals, we write as $\phi_\# \dashv \phi^\# : \mathrm{Top}\,R \to \mathrm{Top}\,S$. The right adjoint $\phi^\#$ preserves finite infima (in fact *all* infima), and so is a *map of frames*, in the sense of [9], if it also preserves suprema; we give a sufficient condition on ϕ for this to be so, and observe that it is satisfied in several important cases. The other adjunction $\phi_! \dashv \phi^*$ is a geometric morphism if $\phi_!$ is left exact, in which case ϕ is said to be *flat*; for such a ϕ this geometric morphism induces as above an adjunction $\phi_0 \dashv \phi^0 : \mathrm{Top}\,S \to \mathrm{Top}\,R$. We show that $\phi_0 : \mathrm{Top}\,R \to \mathrm{Top}\,S$ is a map of frames if ϕ is a *flat epimorphism*; and that the frame-maps ϕ_0 and $\phi^\#$ in fact coincide when the flat epimorphism ϕ satisfies the 'sufficient condition' referred to above.

$\mathrm{Loc}\,\mathscr{A}$ is contained in the bigger ordered set $\mathrm{Ref}\,\mathscr{A}$ of *all* reflective subcategories of $\mathscr{A}$. The lattice properties of these two are of course not directly comparable: localizations have better properties than general reflections, but it is easier to be an infimum or supremum in $\mathrm{Ref}\,\mathscr{A}$ than in $\mathrm{Loc}\,\mathscr{A}$. Nevertheless, some of the techniques used below may be extended to give results on $\mathrm{Ref}\,\mathscr{A}$. Because localizations and locales have a special interest in themselves, it would be out of place to include such generalizations here; the second author plans to comment upon them in a separate article [14].

2. Localizations and factorization systems

We recall from [4] the notion of a factorization system on a category $\mathscr{A}$. Call a set $\mathscr{M}$ of maps in $\mathscr{A}$ a *skein* if it contains the isomorphisms and is closed under composition. For maps e and m in $\mathscr{A}$ write $e\downarrow m$ if, for every commutative square $ve = mu$, there is a unique 'diagonal' w with $we = u$ and $mw = v$. A *factorization system* $(\mathscr{E}, \mathscr{M})$ on $\mathscr{A}$ consists of two sets $\mathscr{E}$ and $\mathscr{M}$ of maps satisfying:

F1. *$\mathscr{E}$ and $\mathscr{M}$ are skeins.*

F2. *Every map f admits a factorization $f = me$ with $m \in M$ and $e \in \mathscr{E}$.*

F3. *$e\downarrow m$ whenever $e \in \mathscr{E}$ and $m \in \mathscr{M}$.*

Clearly F3 is equivalent, in the presence of F1, to the assertion that, if the exterior of

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & C \arrow[r, "m"] \arrow[d, dashed, "w"] & B \arrow[d, "v"] \\ A' \arrow[r, "e'"'] & C' \arrow[r, "m'"'] & B' \end{tikzcd} \tag{2.1}$$

commutes, with $e, e' \in \mathscr{E}$ and $m, m' \in \mathscr{M}$, then there is a unique w rendering commutative both the inner squares. This shows that the factorization in F2 is unique to within isomorphism, and that the choice of such a factorization for each f gives a functor $\mathscr{A}^2 \to \mathscr{A}^3$ (where the exponents in these functor categories are the ordinals **2** and **3** seen as categories). Note in particular that $\mathscr{E} \cap \mathscr{M}$ consists of the isomorphisms.

For any set $\mathscr{M}$ of maps in $\mathscr{A}$, there are various closure-properties that we have to consider. We first enumerate those that we shall need below, and then explain the meaning of such terms as are not clear.

M1. *$\mathscr{M}$ is closed under pointwise limits in $\mathscr{A}^2$.*

M2. *If $fg \in \mathscr{M}$ and $f \in \mathscr{M}$, then $g \in \mathscr{M}$.*

M3. *If $fg \in \mathscr{M}$ and f is monomorphic, then $g \in \mathscr{M}$.*

M4. *$\mathscr{M}$ is stable under pullbacks.*

M5. *$\mathscr{M}$ is stable under fibred products.*

M6. *$\mathscr{M}$ is stable under rooted limits.*

What we mean by M1 is this: if $f : A \to B : \mathscr{I} \to \mathscr{A}$ is a natural transformation for which $\lim \mathscr{A}$ and $\lim B$ exist, the map $\lim f : \lim A \to \lim B$ is in $\mathscr{M}$ if each $f_i : A_i \to B_i$ is in $\mathscr{M}$; note that the epithet 'pointwise' is otiose if $\mathscr{A}$ admits $\mathscr{I}$-indexed limits. By M4 we mean that the pullback of a map in $\mathscr{M}$ along any map in $\mathscr{A}$ is again in $\mathscr{M}$. The use here of 'stable under pullbacks' rather than 'closed under pullbacks' is to avoid confusion with the special case of M1 where the limits in question are pullbacks, the domain category $\mathscr{I}$ being $(\cdot \to \cdot \leftarrow \cdot)$; M4 is quite a different property from this special case of M1. For the same reasons we also use 'stable' rather than 'closed' in M5 and M6. The property M5 contemplates a family $(f_i : A_i \to B)$ of maps which admits a fibred product $h : C \to B$, and asserts that $h \in \mathscr{M}$ if each $f_i \in \mathscr{M}$; it does not assert that the components $g_i : C \to A_i$ of the limit-cone are in $\mathscr{M}$, but this is of course a consequence if M2 is satisfied. M6 is a generalization of M5: the limit $g_i : \lim A \to A_i$ of a functor $A : \mathscr{I} \to \mathscr{A}$ may be called a *rooted limit* if $\mathscr{I}$ has a terminal object 1, and M6 is the assertion that in these circumstances $g_1 : \lim A \to A_1$ lies in $\mathscr{M}$ if each $A_i \to A_1$ lies in $\mathscr{M}$. When M2 is satisfied, such an A necessarily has each $A_\phi : A_i \to A_j$ in $\mathscr{M}$; and then M6 implies that each $g_i \in \mathscr{M}$. *We use the names* E1–E6 *for the duals of* M1–M6.

It is shown in [4] that, for any factorization system $(\mathscr{E}, \mathscr{M})$, we have

$$\mathscr{M} = \{m \mid e \downarrow m \textit{ for all } e \in \mathscr{E}\}; \tag{2.2}$$

whence it follows easily that $\mathscr{M}$ satisfies M1. Since M2–M6 are, by [8], consequences of M1 for any set $\mathscr{M}$ that contains all the identity maps (and in any case are easily proved directly from (2.2) – see [4]), we have:

Proposition 2.1. *For any factorization system $(\mathscr{E}, \mathscr{M})$, the set $\mathscr{M}$ satisfies* M1–M6 *and $\mathscr{E}$ satisfies their duals* E1–E6.

Observe from (2.2) and its dual that a factorization system $(\mathscr{E}, \mathscr{M})$ is fully determined by the knowledge of $\mathscr{E}$ alone, or of $\mathscr{M}$. We order factorization systems on $\mathscr{A}$ by setting $(\mathscr{E}, \mathscr{M}) \le (\mathscr{E}', \mathscr{M}')$ if $\mathscr{M} \subset \mathscr{M}'$; which by (2.2) and its dual is equivalent to $\mathscr{E}' \subset \mathscr{E}$. There is a largest factorization system $\mathbf{1} =$ (isomorphisms, all maps) and a smallest $\mathbf{0} =$ (all maps, isomorphisms).

We call a factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathscr{A}$ *local* if it satisfies, in addition to F1–F3, the following two further conditions (which should be contrasted with E2 and E4 respectively):

F4. *If $fg \in \mathscr{E}$ and $f \in \mathscr{E}$, then $g \in \mathscr{E}$.*

F5. *$\mathscr{E}$ is stable under pullbacks.*

Clearly the extreme factorization systems $\mathbf{0}$ and $\mathbf{1}$ are local. Our analysis below of the ordered set of localizations of $\mathscr{A}$ is largely based on the following result, which is Corollary 4.8 of [2]. We use the standard notation of the Introduction, and write 1 for the terminal object of $\mathscr{A}$. Recall from [4] that an object C of $\mathscr{A}$ is said to be *orthogonal* to a map $e : A \to B$ if $\mathscr{A}(e, C) : \mathscr{A}(B, C) \to \mathscr{A}(A, C)$ is a bijection.

Theorem 2.2 (Cassidy–Hébert–Kelly). *For a finitely-complete $\mathscr{A}$, there is an order-preserving bijection between the set of localizations $\mathscr{C}$ of $\mathscr{A}$ and the set of local fac-*

torization systems $(\mathscr{E}, \mathscr{M})$ on $\mathscr{A}$. In terms of $\mathscr{C}$, we find $\mathscr{E}$ as the set of maps inverted by r – which are equally those maps to which every $C \in \mathscr{C}$ is orthogonal; and we find $\mathscr{M}$ as the set of maps $m : A \to B$ for which

$$\begin{tikzcd} A \arrow[r, "\varrho A"] \arrow[d, "m"'] & rA \arrow[d, "r(m)"] \\ B \arrow[r, "\varrho B"'] & rB \end{tikzcd} \tag{2.3}$$

is a pullback. We find the $(\mathscr{E}, \mathscr{M})$ factorization $f = me$ of an arbitrary $f : A \to B$ by taking as m the pullback along ϱB of $r(f)$:

$$\begin{tikzcd} A \arrow[dr, "e"] \arrow[drr, "\varrho A"] \arrow[ddr, "f"'] & & \\ & C \arrow[r] \arrow[d, "m"] & rA \arrow[d, "r(f)"] \\ & B \arrow[r, "\varrho B"'] & rB\,. \end{tikzcd} \tag{2.4}$$

In terms of $(\mathscr{E}, \mathscr{M})$, the localization $\mathscr{C}$ consists of those objects C for which $C \to 1$ lies in $\mathscr{M}$, which are equally the objects C orthogonal to every $e \in \mathscr{E}$; and the reflexion $\varrho A : A \to rA$ of A into $\mathscr{C}$ is the $\mathscr{E}$-part of the $(\mathscr{E}, \mathscr{M})$ factorization of $A \to 1$.

For any set $\mathscr{N}$ of maps, we write $\mathscr{N}^{\perp}$ for the set of all objects orthogonal to every map in $\mathscr{N}$, or rather for the corresponding (full, replete) subcategory of $\mathscr{A}$. We shall need the following refinement, which is not in [2], of the description above of $\mathscr{C}$ as the subcategory $\mathscr{E}^{\perp}$. Write $\mathscr{M}on$ for the set of all monomorphisms in $\mathscr{A}$.

Proposition 2.3. *In the circumstances of Theorem* 2.2, *$\mathscr{C}$ is fully determined by the knowledge of $\mathscr{E} \cap \mathscr{M}on$, being in fact $(\mathscr{E} \cap \mathscr{M}on)^{\perp}$.*

Proof. For $C \in (\mathscr{E} \cap \mathscr{M}on)^{\perp}$, form the diagram

$$A \xrightarrow[w]{} B \overset{u}{\underset{v}{\rightrightarrows}} C \xrightarrow[\varrho C]{} rC, \tag{2.5}$$

where u, v is the kernel-pair of ϱC, given by the pullback of ϱC along itself, and w is the equalizer of u, v. Then $r(w)$ is invertible because, r being left exact, it is the equalizer of $r(u)$ and $r(v)$, which are equal because r inverts ϱC; so that $w \in \mathscr{E} \cap \mathscr{M}on$. Thus C is orthogonal to w, so that $uw = vw$ gives $u = v$, showing ϱC

to be monomorphic; whence $\varrho C \in \mathscr{E} \cap \mathscr{M}on$. Therefore C is orthogonal to ϱC, so that $1_C = t \cdot \varrho_C$ for some $t: rC \to C$. Since $rC \in \mathscr{C}$ is certainly orthogonal to $\varrho C \in \mathscr{E}$, it follows from $\varrho C \cdot t \cdot \varrho C = \varrho C$ that $\varrho C \cdot t = 1$; thus ϱC is invertible and $C \in \mathscr{C}$, as desired. □

Following the usual practice when $\mathscr{C}$ is a localization of a *presheaf* category $\mathscr{A}$, we may call $\mathscr{E} \cap \mathscr{M}on$ the set of *dense monomorphisms*. Under an extra hypothesis on $\mathscr{A}$ which (see [10]) is mild in practice, we can describe $\mathscr{E}$ explicitly in terms of $\mathscr{E} \cap \mathscr{M}on$:

Proposition 2.4. *In the circumstances of Theorem 2.2, suppose that every map f in $\mathscr{A}$ factorizes as $f = jp$ where j is a monomorphism and p is a strong epimorphism. Let u, v be the kernel-pair of f (or equivalently of p), and let w be the equalizer of u, v. Then $f \in \mathscr{E}$ if and only if the monomorphisms w and j lie in $\mathscr{E}$.*

Proof. Since $\bar{r}: \mathscr{A} \to \mathscr{C}$ preserves strong epimorphisms (see [10]) because it is a left adjoint, and preserves monomorphisms because it is left exact, $\bar{r}(f)$ is invertible (that is, $f \in \mathscr{E}$) if and only if $\bar{r}(j)$ is invertible (that is, $j \in \mathscr{E}$) and $\bar{r}(p)$ is invertible. Since $\bar{r}(p)$ is a strong epimorphism, it is invertible precisely when it, or equivalently $r(p)$, is monomorphic. Because r is left exact, $r(w)$ is the equalizer of the kernel-pair of $r(p)$; so that $r(p)$ is monomorphic exactly when $r(w)$ is invertible, or $w \in \mathscr{E}$. □

A local factorization system has further closure properties beyond those of Proposition 2.1:

Proposition 2.5. *In the circumstances of Theorem 2.2, we have the following:*

(i) *$\mathscr{E}$ is closed under finite limits in $\mathscr{A}^2$.*

(ii) *If $fm \in \mathscr{E}$ and $m \in \mathscr{M}$, then m is a strong monomorphism.*

(iii) *If $fg \in \mathscr{E}$ and f is monomorphic, then $f \in \mathscr{E}$ and $g \in \mathscr{E}$.*

(iv) *If a monomorphism f has the $(\mathscr{E}, \mathscr{M})$ factorization $f = me$, both m and e are monomorphisms.*

Proof. (i) is immediate since $\mathscr{E}$ consists of the maps inverted by r, and r preserves finite limits. As for (ii), $r(fm) = r(f) \cdot r(m)$ is invertible, so that $r(m)$ is a coretraction and therefore a strong monomorphism, whence its pullback m in (2.3) is a strong monomorphism. (For our purposes below, we need only the weaker result that m is monomorphic.) In (iii), $r(fg) = r(f) \cdot r(g)$ is invertible, so that $r(f)$ is a retraction; but it is also a monomorphism since r is left exact, and is thus invertible, so that f and g lie in $\mathscr{E}$. As for (iv), e is of course monomorphic since me is so, while m is monomorphic as the pullback of the monomorphism $r(f)$ in (2.4). □

We now consider, for $x \in \mathscr{A}$, the category $\mathscr{A}/X$ of objects over X: an object of $\mathscr{A}/X$ is a map $a: A \to X$ in $\mathscr{A}$, and a map in $\mathscr{A}/X$ from $a: A \to X$ to $b: B \to X$ is a map $f: A \to B$ in $\mathscr{A}$ with $bf = a$. The category $\mathscr{A}/X$ admits $\mathscr{I}$-indexed colimits if $\mathscr{A}$ does, the colimit of $(a_i : A_i \to X)$ in $\mathscr{A}/X$ being $\mathrm{colim}_i A_i$ in $\mathscr{A}$ with the obvious augmentation to X. Again, $\mathscr{A}/X$ admits $\mathscr{I}$-indexed limits if $\mathscr{A}$ admits $\mathscr{I}^+$-indexed limits, where $\mathscr{I}^+$ is $\mathscr{I}$ with a new terminal object 1 added; we set $A_1 = X$, and then the limit in $\mathscr{A}/X$ of $(a_i : A_i \to X)_{i \in \mathscr{I}}$ is the limit in $\mathscr{A}$ of $(A_i)_{i \in \mathscr{I}^+}$ with the obvious augmentation to X. When $\mathscr{I}$ is *connected*, this is just the limit in $\mathscr{A}$ of $(A_i)_{i \in \mathscr{I}}$ with an evident augmentation to X.

Any factorization system $(\mathscr{E}, \mathscr{M})$ on $\mathscr{A}$ gives rise to a factorization system $(\mathscr{E}_X, \mathscr{M}_X)$ on $\mathscr{A}/X$; we put $f: a \to b$ as above into $\mathscr{E}_X$ [resp. $\mathscr{M}_X$] precisely when $f: A \to B$ is in $\mathscr{E}$ [resp. $\mathscr{M}$]. The conditions F1 and F2 for $(\mathscr{E}_X, \mathscr{M}_X)$ are trivially satisfied, and F3 almost trivially. Moreover $(\mathscr{E}_X, \mathscr{M}_X)$ satisfies F4 and F5 when $(\mathscr{E}, \mathscr{M})$ does so – the latter because the pullback involved is a connected limit. Supposing now that $\mathscr{A}$ and hence $\mathscr{A}/X$ are finitely complete, we may apply Theorem 2.2 to the local factorization system $(\mathscr{E}_X, \mathscr{M}_X)$ on $\mathscr{A}/X$; the corresponding localization of $\mathscr{A}/X$ is clearly the subcategory $\mathscr{M}/X$ given by those $a: A \to X$ with $a \in \mathscr{M}$. Thus:

Proposition 2.6. *Let $(\mathscr{E}, \mathscr{M})$ be a local factorization system on a finitely-complete $\mathscr{A}$, and let $X \in \mathscr{A}$. Then $\mathscr{M}/X$ is a localization of $\mathscr{A}/X$, with corresponding local factorization system $(\mathscr{E}_X, \mathscr{M}_X)$; the special case $X = 1$ gives $\mathscr{C} = \mathscr{M}/1$ as the localization of $\mathscr{A}$ itself. If $a: A \to X$ has the $(\mathscr{E}, \mathscr{M})$ factorization $a = me$, then m is the reflexion of $a \in \mathscr{A}/X$ into $\mathscr{M}/X$, and e is the unit of this reflexion.*

Suppose now that $\mathscr{A}$ admits $\mathscr{I}$-indexed colimits, and consider a natural transformation $f: A \to B : \mathscr{I} \to \mathscr{A}$. In the diagram

$$\begin{tikzcd}
A_i \arrow[r, "e_i"] \arrow[d, "p_i"'] & C_i \arrow[r, "m_i"] \arrow[d, "s_i"] & B_i \arrow[d, "q_i"] \\
\mathrm{colim}\, A \arrow[r, "\mathrm{colim}\, e"'] & \mathrm{colim}\, C \arrow[r, "\mathrm{colim}\, m"'] & \mathrm{colim}\, B
\end{tikzcd} \tag{2.6}$$

let the top row be the $(\mathscr{E}, \mathscr{M})$ factorization of f_i for some factorization system $(\mathscr{E}, \mathscr{M})$: then by (2.1) we have a functor $C: \mathscr{I} \to \mathscr{A}$ and natural transformations $e: A \to C$ and $m: C \to B$; let the bottom row of (2.6) be the respective colimits, and the vertical maps the colimit-cones. Since $\mathrm{colim}\, e \in \mathscr{E}$ by Proposition 2.1, the bottom row of (2.6) gives the $(\mathscr{E}, \mathscr{M})$ factorization of $\mathrm{colim}\, f$ if and only if $\mathrm{colim}\, m$ lies in $\mathscr{M}$. We can express this by saying that:

Proposition 2.7. *If $\mathscr{A}$ admits $\mathscr{I}$-indexed colimits and $(\mathscr{E}, \mathscr{M})$ is a factorization system, $\mathscr{I}$-indexed colimits preserve $(\mathscr{E}, \mathscr{M})$ factorizations if and only if $\mathscr{M}$ is closed in $\mathscr{A}^2$ under $\mathscr{I}$-indexed colimits.*

The dual of this together with Proposition 2.5(i) gives:

Proposition 2.8. *If $(\mathscr{E}, \mathscr{M})$ is a local factorization system on a finitely-complete $\mathscr{A}$, finite limits preserve $(\mathscr{E}, \mathscr{M})$ factorizations.*

Proposition 2.9. *For some regular cardinal α, let $\mathscr{A}$ admit finite limits and α-filtered colimits, and let these commute with one another. Let $\mathscr{C}$ be a localization of $\mathscr{A}$, and $(\mathscr{E}, \mathscr{M})$ the corresponding local factorization system. Then the following are equivalent:*

(i) *$\mathscr{C}$ is an α-localization of $\mathscr{A}$.*
(ii) *$\mathscr{M}/X$ is, for each X, an α-localization of $\mathscr{A}/X$.*
(iii) *α-filtered colimits preserve $(\mathscr{E}, \mathscr{M})$ factorizations.*
(iv) *$\mathscr{M}$ is closed in $\mathscr{A}^2$ under α-filtered colimits.*

Proof. (iv) is equivalent to (iii) by Proposition 2.7, while (iii) implies (ii) by applying (2.6) to the special case where each e_i is the identity and B is the functor ΔX constant at X, showing that $\mathscr{M}/X$ is closed in $\mathscr{A}/X$ under α-filtered colimits. Since (i) is the special case $X=1$ of (ii), it remains only to show that (i) implies (iv). Consider an α-filtered colimit represented by the right square in (2.6), with each $m_i \in \mathscr{M}$. Since r has rank α, we have the colimit-cone $r(s_i) : rC_i \to r(\operatorname{colim} C)$: and then it follows from the naturality of ϱ that the colimit of $\varrho C_i : C_i \to rC_i$ is $\varrho(\operatorname{colim} C) : \operatorname{colim} C \to r(\operatorname{colim} C)$. Similarly for the colimit of ϱB_i; while the colimit of $r(m_i)$ is of course $r(\operatorname{colim} m)$. For each i we have by (2.3) the pullback on the left in

$$
\begin{tikzcd}
C_i \arrow[r, "\varrho C_i"] \arrow[d, "m_i"'] & rC_i \arrow[d, "r(m_i)"] \\
B_i \arrow[r, "\varrho B_i"'] & rB_i ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{colim} C \arrow[r, "\varrho(\operatorname{colim} C)"] \arrow[d, "\operatorname{colim} m"'] & r(\operatorname{colim} C) \arrow[d, "r(\operatorname{colim} m)"] \\
\operatorname{colim} B \arrow[r, "\varrho(\operatorname{colim} B)"'] & r(\operatorname{colim} B) ,
\end{tikzcd}
$$

whose colimit by the above is the square on the right. The latter is again a pullback since α-filtered colimits commute with finite limits, so that $\operatorname{colim} m$ is in $\mathscr{M}$ by (2.3) of Theorem 2.2. □

3. Small suprema of localizations

In dealing with a family $(\mathscr{C}_k)_{k\in K}$ of localizations, we use the previous notations with a subscript k, and write $(\mathscr{E}_k, \mathscr{M}_k)$ for the local factorization system corresponding to $\mathscr{C}_k$.

Theorem 3.1. *When $\mathscr{A}$ is complete, every small family $(C_k)_{k\in K}$ of localizations admits a supremum $\mathscr{C}$ in* Loc $\mathscr{A}$, *the local factorization system $(\mathscr{E}, \mathscr{M})$ corresponding* *to which has* $\mathscr{E}=\bigcap\mathscr{E}_k$, *while* $\mathscr{M}$ *is the smallest skein containing* $\bigcup\mathscr{M}_k$ *and stable under small fibred products.*

Proof. It suffices by Theorem 2.2 to show that $(\mathscr{E},\mathscr{M})$ as defined above is a local factorization system; for then, since $\mathscr{E}=\bigcap\mathscr{E}_k$, the remarks following Proposition 2.1 show $(\mathscr{E},\mathscr{M})$ to be the supremum of the $(\mathscr{E}_k,\mathscr{M}_k)$ among *all* factorization systems on $\mathscr{A}$, and so *a fortiori* among the local factorization systems. That $(\mathscr{E},\mathscr{M})$ satisfies F1, F4, and F5 is immediate; it also satisfies F3 because clearly $\bigcup\mathscr{M}_k\subset\{m\mid e\downarrow m$ for all $e\in\mathscr{E}\}$ and because the latter set, for any $\mathscr{E}$ whatsoever, is a skein satisfying M1 and hence M5 - see [4] or [8]. Thus it remains only to verify F2.

Given $f:A\to B$, let $f=m_k e_k$ be its $(\mathscr{E}_k,\mathscr{M}_k)$ factorization, and form the fibred product

$$\begin{tikzcd} & C_k \arrow[dr, "m_k"] & \\ B' \arrow[ur, "p_k"] \arrow[rr, "n"'] & & B \end{tikzcd} \tag{3.1}$$

of the m_k for $k\in K$. Then $f=nf'$, where $f':A\to B'$ is the unique map satisfying $p_k f'=e_k$ for $k\in K$. Now repeat the above with f' replacing f; its $(\mathscr{E}_k,\mathscr{M}_k)$ factorization is $f'=m'_k e'_k$, we have the fibred product

$$\begin{tikzcd} & C'_k \arrow[dr, "m'_k"] & \\ B'' \arrow[ur, "p'_k"] \arrow[rr, "n'"'] & & B', \end{tikzcd} \tag{3.2}$$

and $f'=n'f''$ where f'' is determined by $p'_k f''=e'_k$. Since $p_k m'_k e'_k=p_k f'=e_k$, and since $\mathscr{E}_k$ satisfies E2 by Proposition 2.1, we have $p_k m'_k\in\mathscr{E}_k$. By Proposition 2.5(ii), the m'_k are monomorphic; whence the p'_k are monomorphic. Since $p'_k f''=e'_k$, Proposition 2.5(iii) gives $f''\in\mathscr{E}_k$ for each k, so that $f''\in\mathscr{E}$. This establishes F2, since $mn'\in\mathscr{M}$. □

Theorem 3.2. *When the family* $(\mathscr{C}_k)$ *of localizations is finite, Theorem* 3.1 *remains true when* $\mathscr{A}$ *is only finitely complete; and then* $\mathscr{M}$ *is just the smallest skein containing* $\bigcup\mathscr{M}_k$.

Proof. Since the fibred products (3.1) and (3.2) are now finite, we need only finite limits in $\mathscr{A}$, and $\mathscr{M}$ is the smallest skein containing $\bigcup\mathscr{M}_k$ and stable under *finite* fibred products - for which *binary* fibred products suffice. Since, however, the pullback along uv is the pullback along v of the pullback along u, and since each $\mathscr{M}_k$ satisfies M4 by Proposition 2.1, the smallest skein containing $\bigcup\mathscr{M}_k$ is *already* stable under binary fibred products. □

Theorem 3.3. *In the situation of Theorem* 3.1 [*resp. Theorem* 3.2], *the supremum* $\mathscr{C}$ *in* Loc $\mathscr{A}$ *of the* $\mathscr{C}_k$ *is the closure in* $\mathscr{A}$ *under small limits* [*resp. finite limits*] *of the union* $\bigcup \mathscr{C}_k$ *of these* (*full and replete*) *subcategories.*

Proof. Write $\mathscr{D}$ for the closure in question of $\bigcup \mathscr{C}_k$; clearly $\mathscr{D} \subset \mathscr{C}$, and it remains to show the converse.

Define as follows a set $\mathscr{N}$ of maps in $\mathscr{A}$: *the map* $f: A \to B$ *lies in* $\mathscr{N}$ *if, for every pullback*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "u"'] & D \arrow[d, "v"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{3.3}$$

with $D \in \mathscr{D}$, *we have* $C \in \mathscr{D}$.

Since the pullback along fh is the pullback along h of the pullback along f, it is clear that $\mathscr{N}$ is a skein. Since the pullback along wv is the pullback along v of the pullback along w, it is clear that $\mathscr{N}$ is stable under pullbacks. Moreover $\mathscr{N}$ is stable under small fibred products; to see this, consider a diagram

$$\begin{tikzcd} & C_i \arrow[dr, "g_i"] \arrow[dd, "w_i"] & \\ Y \arrow[ur, "t_i"] \arrow[rr, "s"] \arrow[ddd, "u"'] & & D \arrow[ddd, "v"] \\ & A_i \arrow[ddr, "f_i"] & \\ & & \\ X \arrow[uur, "k_i"] \arrow[rr, "h"'] & & B \end{tikzcd},$$

where the base of the prism is a fibred product, and where u, s constitute the pullback of v and h. If we *define* w_i and g_i as the pullback of v and f_i, there are unique t_i that complete the commutative diagram. Since pullback along v is a right-adjoint functor $\mathscr{A}/B \to \mathscr{A}/D$, the top of the prism is again a fibred product. Now if each $f_i \in \mathscr{N}$ and $D \in \mathscr{D}$, we have each $C_i \in \mathscr{D}$, whence $Y \in \mathscr{D}$ because $\mathscr{D}$ is closed under small [resp. finite] limits, so that $h \in \mathscr{N}$.

Since $\mathscr{C}_k \subset \mathscr{D}$ and $\mathscr{D}$ is closed under finite limits, it is clear that mor $\mathscr{C}_k \subset \mathscr{N}$. Since $\mathscr{N}$ is stable under pullbacks, it follows from the pullback (2.3) in Theorem 2.2 that $\mathscr{N}$ contains each $\mathscr{M}_k$; whence $\mathscr{M} \subset \mathscr{N}$ by Theorems 3.1 and 3.2.

If $A \in \mathscr{C}$, the unique map $f: A \to 1$ is in $\mathscr{M}$ by Theorem 2.2, and hence in $\mathscr{N}$. Applying (3.3) with v the identity map of 1, and recalling that $1 \in \mathcal{D}$, we have $A \in \mathcal{D}$. Thus $\mathcal{C} \subset \mathcal{D}$, as required. □

Corollary 3.4. *In the situation of Theorem* 3.1 *or of Theorem* 3.2, *the supremum* $\mathcal{C}$ *of the* $\mathcal{C}_k$ *in* $\mathrm{Loc}\,\mathcal{A}$ *is also their supremum in the larger ordered set of all reflective subcategories of* $\mathcal{A}$.

Remark 3.5. The restriction in Theorems 3.1 and 3.3 to *small* suprema is necessary, as the following examples show. First observe that, when the category $\mathcal{A}$ reduces to a totally-ordered set, every reflexion is a localization; for equalizers are trivial, binary products are just minima, and every reflexion preserves the terminal object. Next, recall the convention that an ordinal is identified with the ordered set of smaller ordinals, so that our inaccessible ∞ denotes the ordered set of all small ordinals β; write ∞^{op} for the dual category, and β^{op} for the small β seen as an object of ∞^{op}. Finally, recall the notion of the *ordinal sum* $A+B$ of ordered sets A and B: it is their disjoint union as sets, with the order given by setting $x \leq y$ if $x \leq y$ in A, or if $x \leq y$ in B, or if $x \in A$ and $y \in B$.

Example 3.6. The category $\mathcal{A} = \infty + \infty^{\mathrm{op}}$ is complete and cocomplete, with a generator and a cogenerator. For each $\beta \in \infty$ the closed interval $\mathcal{C}_\beta = [\beta^{\mathrm{op}}, 0^{\mathrm{op}}]$ of $\mathcal{A}$ is reflective and hence a localization. The union ∞^{op} of the $\mathcal{C}_\beta$ is not reflective, since it lacks an initial object; yet it is the intersection of the reflective subcategories $[\gamma, 0^{\mathrm{op}}]$ for $\gamma \in \infty$. Thus the large family $(\mathcal{C}_\beta)$ has no supremum in $\mathrm{Loc}\,\mathcal{A}$.

Example 3.7. The category $\mathcal{A} = 1 + \infty^{\mathrm{op}}$ (the dual of the ordered set of ordinals $\leq \infty$) admits all limits and colimits, even large ones, and has a generator and a cogenerator. The localizations $\mathcal{C}_\beta = [\beta^{\mathrm{op}}, 0^{\mathrm{op}}]$ for $\beta \in \infty$, whose union is the non-reflective ∞^{op}, do have a supremum: namely $\mathcal{A}$ itself. In contrast to the result for small suprema given by Theorem 3.3, the large supremum $\mathcal{A}$ is not the closure of $\bigcup \mathcal{C}_\beta = \infty^{\mathrm{op}}$ in $\mathcal{A}$ under small limits; for ∞^{op} is already closed in $\mathcal{A}$ under these.

In the situation of Theorem 3.2 (finite suprema in a finitely-complete $\mathcal{A}$) we can describe explicitly the idempotent monad corresponding to the reflexion onto the supremum $\mathcal{C}$; for simplicity we give the result only for a binary supremum.

Proposition 3.8. *For a finitely-complete* $\mathcal{A}$, *the idempotent monad* r *corresponding to the localization* $\mathcal{C}$ *which is the supremum of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *is the limit* r *in the category of endofunctors of* $\mathcal{A}$ *of the diagram*

$$\begin{tikzcd}
r_1 \arrow[dr, "r_1\varrho_2"'] \arrow[drr, "\varrho_2 r_1"] & & \\
& r_1 r_2 & r_2 r_1; \\
r_2 \arrow[ur, "\varrho_1 r_2"] \arrow[urr, "r_2\varrho_1"'] & &
\end{tikzcd} \tag{3.4}$$

and its unit $\varrho : 1 \to r$ *corresponds to the cone over* (3.4) *with vertex* 1 *and generators* $\varrho_1 : 1 \to r_1$ *and* $\varrho_2 : 1 \to r_2$.

Proof. (First note that we are using 1 in several senses: in the statement of the proposition it denotes the identity endofunctor of $\mathcal{A}$, while in the proof it once denotes the terminal object of $\mathcal{A}$ and once the identity map of $r_1 A$. Confusion is unlikely when the contextual clues are reinforced by this warning.) Recalling from Theorem 2.2 that $\varrho A : A \to rA$ is the $\mathscr{E}$-part of the $(\mathscr{E}, \mathcal{M})$ factorization of $f : A \to 1$, we apply to this f the processes used in the proof of Theorem 3.1, with the notation as there. Since e_k is $\varrho_k A$, we find for f' the map $\langle \varrho_1 A, \varrho_2 A\rangle : A \to r_1 A \times r_2 A$. By Theorem 2.2 again, m_1' is the pullback along $\varrho_1(r_1 A \times r_2 A)$ of $r_1\langle \varrho_1 A, \varrho_2 A\rangle$; which, since r_1 preserves finite products, is the pullback along $r_1 A \times \varrho_1 r_2 A$ of $\langle 1, r_1\varrho_2 A\rangle$: $r_1 A \to r_1 A \times r_1 r_2 A$, or equally the pullback along $\varrho_1 r_2 A$ of $r_1 \varrho_2 A$. Interchanging the indices 1 and 2 here gives m_2'. Since $rA = B''$ is the fibred product of m_1' and m_2', it is indeed the A-component of the limit of (3.4). □

Theorem 3.9. *For some small regular cardinal* α, *let* $\mathcal{A}$ *admit finite limits and* α-*filtered colimits, and let these commute with one another. Then the supremum* $\mathscr{C}$ *in* $\operatorname{Loc}\mathcal{A}$ *of a finite family* $(\mathscr{C}_k)$ *of* α-*localizations is again an* α-*localization, and is thus also the supremum in* $\operatorname{Loc}_\alpha \mathcal{A}$.

Proof. Since it suffices to consider binary suprema, we can use Proposition 3.8. When $\mathscr{C}_1$ and $\mathscr{C}_2$ are α-localizations, every endofunctor in (3.4) preserves α-filtered colimits, since r_1 and r_2 do so. Because finite limits commute with α-filtered colimits, the limit r of (3.4) again preserves α-filtered colimits, so that $\mathscr{C}$ is an α-localization. □

Remark 3.10. More generally, if α-filtered colimits commute with β-limits for some regular cardinal β, the supremum of a set of α-localizations of cardinal $<\beta$ is again an α-localization. Although Theorem 3.9 is no longer available when $\beta > \omega$, we can consider directly the effect of passing to α-filtered colimits in the process which, in the proof of Theorem 3.1, constructs $(\mathscr{E}, \mathcal{M})$ factorizations. Since such colimits preserve $(\mathscr{E}_k, \mathscr{M}_k)$ factorizations by Proposition 2.9, and preserve β-limits, they clearly preserve $(\mathscr{E}, \mathscr{M})$ factorizations.

4. Well-pointed endofunctors

Before considering small infima of localizations, we recall from [11], for the reader's convenience, the basic facts about well-pointed endofunctors of $\mathscr{A}$ and their algebras, and augment them with some new results.

By a *pointed endofunctor* (s, σ) we mean an endofunctor s of $\mathscr{A}$ together with a natural transformation $\sigma : 1 \to s$ from the identity endofunctor; we sometimes suppress σ and write s for (s, σ). An *action* of (s, σ) on $A \in \mathscr{A}$ is a map $a : sA \to A$ satisfying $a \cdot \sigma A = 1_A$; and an *s-algebra* (A, a) is an $A \in \mathscr{A}$ together with such an action a. If we define an *s-algebra-map* $f : (A, a) \to (B, b)$ to be a map $f : A \to B$ satisfying $f \cdot a = b \cdot s(f)$, the s-algebras form a category s-Alg with a faithful forgetful functor $U : s\text{-Alg} \to \mathscr{A}$.

We say that (s, σ) is *well pointed* if $s\sigma = \sigma s : s \to s^2$. An example of a well-pointed endofunctor is the idempotent monad (r, ϱ) associated as in the Introduction to a reflective subcategory $\mathscr{C}$ of $\mathscr{A}$. The following lemma, whose proof is easy, is very useful:

Lemma 4.1 (= [11, Lemma 5.1]). *If (s, σ) is well-pointed, $g : sB \to A$ is any map, and $f : B \to A$ is the composite $g \cdot \sigma B$, then $s(f) = \sigma A \cdot g$.*

Recall from the sentence before Theorem 2.2 the meaning of *orthogonal*. The following is easily verified, using Lemma 4.1 in the proofs that (i) implies (ii) and that (ii) implies (iii):

Proposition 4.2 (= [11, Proposition 5.2]). *For a well-pointed (s, σ) the following properties of $A \in \mathscr{A}$ are equivalent:*

(i) *A admits some s-action $a : sA \to A$.*

(ii) *σA is invertible, so that A admits the unique s-action $a = (\sigma A)^{-1}$.*

(iii) *A is orthogonal to σB for each $B \in \mathscr{A}$.*

*Thus s-*Alg *is isomorphic to, and will henceforth be identified with, the (full and replete) subcategory of $\mathscr{A}$ determined by such objects A; whereupon $U : s\text{-Alg} \to \mathscr{A}$ is identified with the inclusion.*

Note that, when (s, σ) is the idempotent monad (r, ϱ) associated to a reflective subcategory $\mathscr{C}$ of $\mathscr{A}$, we have $r\text{-Alg} = \mathscr{C}$. The following observation is new:

Proposition 4.3. *If $\sigma : 1 \to s$ and $\tau : 1 \to t$ are well pointed, so is $\sigma\tau : 1 \to st$; and (st)-*Alg $= (s\text{-Alg}) \cap (t\text{-Alg})$. *Consequently, there is a similar result for any finite composite $\sigma_1 \cdots \sigma_n : 1 \to s_1 \cdots s_n$ of well-pointed s_i.*

Proof. Recall that $\sigma\tau$ denotes the diagonal of the commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] \arrow[d, "\tau"'] \arrow[dr, "\sigma\tau"] & s \arrow[d, "s\tau"] \\ t \arrow[r, "\sigma t"'] & st. \end{tikzcd} \tag{4.1}$$

The corresponding diagram for $\tau\sigma$ gives $t\sigma\cdot\tau=\tau s\cdot\sigma$, and operating on this with s before and t behind gives $st\sigma t\cdot s\tau t=s\tau st\cdot s\sigma t$, which (using $\tau t=t\tau$ and $s\sigma=\sigma s$) may be written as $st\sigma t\cdot st\sigma=s\tau st\cdot\sigma st$; this last is, by (4.1), the statement $st\sigma t=\sigma\tau st$ of the well-pointedness of st.

If $A\in(s\text{-Alg})\cap(t\text{-Alg})$, both σA and τA are invertible by Proposition 4.2(ii); so that $(\sigma\tau)A$ is invertible by the top triangle of (4.1), and $A\in(st)\text{-Alg}$. Conversely, if $A\in(st)\text{-Alg}$, both σA and τA are coretractions by (4.1) since $(\sigma\tau)A$ is invertible, so that $A\in(s\text{-Alg})\cap(t\text{-Alg})$ by Proposition 4.2(i). □

We write End $\mathscr{A}$ for the category of endofunctors of $\mathscr{A}$ and natural transformation between them. The following generalizes Proposition 9.1 of [11], which dealt with the special case of fibred coproducts, to certain *rooted colimits*; recall that we mentioned *rooted limits* in Section 2, in connexion with M6.

Proposition 4.4. *Let $t:\mathscr{I}\to\mathrm{End}\,\mathscr{A}$ be a functor with a pointwise colimit s, the colimit-cone being $\sigma_i:t_i\to s$. Suppose that $\mathscr{I}$ has an initial object* 0 *and that t_0 is the identity endofunctor* 1; *write $\tau_i:1\to t_i$ for the value of t on $(0\to i)$, and $\sigma:1\to s$ for σ_0. If each (t_i,τ_i) is well pointed, so is (s,σ); and* $s\text{-Alg}=\bigcap t_i\text{-Alg}$.

Proof. Composing the equality $t_i\tau_i=\tau_i t_i:t_i\to t_i^2$ with $\sigma_i\sigma_i:t_i^2\to s^2$ gives, since $\sigma_i\tau_i=\sigma_0=\sigma$, the equality $\sigma_i\sigma=\sigma\sigma_i:t_i\to s^2$, which can equally be written as $s\sigma\cdot\sigma_i=\sigma s\cdot\sigma_i$; because the σ_i are jointly epimorphic, we have $s\sigma=\sigma s$.

If $A\in\bigcap t_i\text{-Alg}$, each τ_iA is invertible by Proposition 4.2(ii); clearly σA is then invertible, so that $A\in s\text{-Alg}$. Conversely, if $A\in s\text{-Alg}$, each τ_iA is a coretraction since $\sigma A=\sigma_iA\cdot\tau_iA$ is invertible, so that $A\in\bigcap t_i\text{-Alg}$ by Proposition 4.2(i). □

Remark 4.5. It follows at once from Proposition 4.2(iii) that, for a well-pointed (s,σ), the subcategory s-Alg is closed under limits in $\mathscr{A}$; we are about to establish conditions sufficient for it to be reflective. First, however, note that it is certainly reflective if $s\sigma=\sigma s$ is invertible, the reflexion then being given by $\sigma:1\to s$; such an (s,σ) may be called a *weakly idempotent monad*, to distinguish it from what we called an *idempotent monad* in the Introduction. Observe that, in the notation of the Introduction, the adjunction $\bar r\dashv j$ associated to a reflective subcategory $\mathscr{C}$ gives only a *weakly* idempotent monad $\varrho:1\to r=j\bar r$ if we do not take care so to choose $\bar r$ that $\bar rj=1$.

Construction 4.6. By a *sequence* we mean a functor whose domain is the category ∞ of all small ordinals. Given a well-pointed (s, σ) on an $\mathscr{A}$ that admits small filtered colimits, we define inductively a sequence $S: \infty \to \mathrm{End}\,\mathscr{A}$. We set $S_0 = 1$, and $S_{\beta+1} = sS_\beta$, with connecting-map $S_\beta^{\beta+1} : S_\beta \to S_{\beta+1}$ given by $\sigma S_\beta : S_\beta \to sS_\beta$; for a limit-ordinal β we set $S_\beta = \mathrm{colim}_{\gamma<\beta} S_\gamma$, with the generators of the colimit-cone as the connecting-maps $S_\gamma^\beta : S_\gamma \to S_\beta$. Since $S_n = s^n$ for finite n, we shall henceforth write s^β for S_β; and we write $\sigma_\beta : 1 \to s^\beta$ for S_0^β. By Propositions 4.3 and 4.4, *each* (s^β, σ_β) *for* $\beta > 0$ *is a well-pointed endofunctor with the same algebras as* (s, σ).

Lemma 4.7 (= [11, Lemma 5.5]). *In the situation above, the connecting-map* $S_{\gamma+1}^{\beta+1} : s^{\gamma+1} \to s^{\beta+1}$ *for* $\gamma < \beta$ *coincides with* $sS_\gamma^\beta : ss^\gamma \to ss^\beta$.

Proof. Since $S_\gamma^\beta = S_{\gamma+1}^\beta S_\gamma^{\gamma+1} = S_{\gamma+1}^\beta \cdot \sigma s^\gamma$, Lemma 4.1 gives $sS_\gamma^\beta = \sigma s^\beta \cdot S_{\gamma+1}^\beta$; which is $S_\beta^{\beta+1} S_{\gamma+1}^\beta = S_{\gamma+1}^{\beta+1}$. □

The following special case of Theorem 6.2 of [11] is the only one that we shall use:

Theorem 4.8. *Let* (s, σ) *be a well-pointed endofunctor on a category* $\mathscr{A}$ *that admits small filtered colimits, and form the sequence S as above. If, for some small regular cardinal* α, *the endofunctor s has rank* α, *the map* $\sigma s^\alpha : s^\alpha \to s^{\alpha+1}$ *is invertible; and* s-Alg *is a reflective subcategory of* $\mathscr{A}$, *the reflexion of A into* s-Alg *being* $\sigma_\alpha A : A \to s^\alpha A$.

Proof. Since the $S_\beta^\alpha : s^\beta \to s^\alpha$ for $\beta < \alpha$ constitute a colimit-cone, and since s preserves α-filtered colimits, we have a colimit-cone $sS_\beta^\alpha : s^{\beta+1} \to s^{\alpha+1}$, which by Lemma 4.7 is $S_{\beta+1}^{\alpha+1}$, or $\sigma s^\alpha \cdot S_{\beta+1}^\alpha$. The ordinals of the form $\beta+1$ for $\beta<\alpha$ being final in α, the colimit of $(s^{\beta+1})_{\beta<\alpha}$ is also $S_{\beta+1}^\alpha : s^{\beta+1} \to s^\alpha$. Thus σs^α is invertible.

For any $A \in \mathscr{A}$, therefore, $\sigma s^\alpha A$ is invertible, so that $s^\alpha A \in s$-Alg by Proposition 4.2. For any B in s-Alg, which by the last remark in Construction 4.6 is s^α-Alg, and any A, Proposition 4.2(iii) applied to s^α shows that B is orthogonal to $\sigma_\alpha A : A \to s^\alpha A$; which is to say that any $f: A \to B$ is $g \cdot \sigma_\alpha A$ for a unique g. So $\sigma_\alpha A$ is indeed the reflexion of A into s-Alg. □

Remark 4.9. In the circumstances of Theorem 4.8, $\sigma_\alpha : 1 \to s^\alpha$ is in general only a *weakly* idempotent monad in the sense of Remark 4.5. It is, however, an honest idempotent monad if, as is always the case in our applications, σA is the identity for $A \in s$-Alg: provided that we make the obvious choice of A itself as the colimit of a filtered diagram constant at A. See Remark 5.4 below.

5. Small infima of localizations with rank

Although completeness of $\mathscr{A}$ guarantees by Theorem 3.1 the existence of small suprema in Loc $\mathscr{A}$, it does not guarantee the existence of even binary infima.

Example 5.1. Let $\mathscr{A}$ be the complete and cocomplete category $\infty + \infty^{\mathrm{op}}$ of Example 3.6, and write ∞_0 and ∞_1 for the (full) subcategories of ∞ given respectively by the even ordinals and by the odd ones. Then $\mathscr{C}_0 = \infty_0 + \infty^{\mathrm{op}}$ and $\mathscr{C}_1 = \infty_1 + \infty^{\mathrm{op}}$ are reflective subcategories of $\mathscr{A}$, and hence localizations. Their intersection $\mathscr{C}_0 \cap \mathscr{C}_1 = \infty^{\mathrm{op}}$ is not reflective, but is the union of the reflective subcategories $[\beta^{\mathrm{op}}, 0^{\mathrm{op}}]$ for $\beta \in \infty$; hence $\mathscr{C}_0$ and $\mathscr{C}_1$ admit no infimum in $\mathrm{Loc}\,\mathscr{A}$.

Example 5.2. The infimum of two localizations $\mathscr{C}_0$ and $\mathscr{C}_1$ of $\mathscr{A}$ may exist without being $\mathscr{C}_0 \cap \mathscr{C}_1$, even when $\mathscr{A}$ is locally small, is complete and cocomplete, admits arbitrary (even large) intersections of subobjects, is weakly wellpowered (that is, each object has but a small set of *strong* subobjects), and is weakly cowellpowered. For an example, let $\mathscr{A}$ be the ordinal sum $\infty + \mathscr{G}$ (defined by an obvious extension of Remark 3.5) where $\mathscr{G}$ is the category G/Grp of small groups under G, and G is the large group which is the coproduct of all small simple groups. The localizations $\mathscr{C}_0$ and $\mathscr{C}_1$ are $\infty_0 + \mathscr{G}$ and $\infty_1 + \mathscr{G}$ in the language of Example 5.1; in contrast to that example, the only localization of $\mathscr{A}$ contained in $\mathscr{C}_0 \cap \mathscr{C}_1 = \mathscr{G}$ is $\{1\}$, where 1 is the terminal object of $\mathscr{G}$; so that $\{1\}$ is the infimum in $\mathrm{Loc}\,\mathscr{A}$ of $\mathscr{C}_0$ and $\mathscr{C}_1$.

We establish the existence of small infima of localizations only under restrictive conditions on $\mathscr{A}$ - filtered colimits are to commute with finite limits - and restrictive conditions on the localizations - each is to have some rank. Since we deal only with *small* families $(\mathscr{C}_k)$ of localizations, the latter condition is equivalent to the requirement that, for some small regular cardinal α, each $\mathscr{C}_k$ is an α-localization. In these circumstances, the infimum in $\mathrm{Loc}\,\mathscr{A}$ of the small family $(\mathscr{C}_k)$ is in fact $\bigcap \mathscr{C}_k$, in contrast to the more general situation of Example 5.2.

We use of course our standard notation: the reflexion onto $\mathscr{C}_k$ is given by $\varrho_k : 1 \to r_k$. If the $\mathscr{C}_k$ were merely reflective subcategories, not necessarily localizations, and we wished only to prove the reflectivity of $\bigcap \mathscr{C}_k$, under the hypotheses that $\mathscr{A}$ is cocomplete and each r_k has rank α, it would be easy: by Proposition 4.4 the fibred coproduct $\sigma : 1 \to s$ of the $\varrho_k : 1 \to r_k$ is a well-pointed endofunctor with $s\text{-Alg} = \bigcap r_k\text{-Alg} = \bigcap \mathscr{C}_k$; and s has rank α since 1 and each r_k have rank α, and colimits commute with colimits; so that $\bigcap \mathscr{C}_k$ is reflective by Theorem 4.8. It is in showing that $\bigcap \mathscr{C}_k$ is a localization when each $\mathscr{C}_k$ is so that we need the more delicate argument below, as well as the restrictive condition on $\mathscr{A}$ referred to above.

Theorem 5.3. *Let $\mathscr{A}$ admit finite limits and small filtered colimits, and let these commute with one another. Let $(\mathscr{C}_k)_{k \in K}$ be a small family of α-localizations of $\mathscr{A}$. Then $\mathscr{C} = \bigcap \mathscr{C}_k$ is an α-localization of $\mathscr{A}$, and thus the infimum of the $\mathscr{C}_k$ both in* $\mathrm{Loc}\,\mathscr{A}$ *and in* $\mathrm{Loc}_\alpha \mathscr{A}$.

Proof. Write $\mathscr{I}$ for the set of finite subsets of the indexing set K, ordered by inclusion. The small set $\mathscr{I}$ is filtered, and has the empty subset of K as its initial object 0. Choose a total ordering of K; then every $i \in \mathscr{I}$ inherits a total order, and has the form $\{i(1), i(2), \dots, i(n)\}$ for some n and some $i(1) < i(2) < \cdots < i(n)$ in K. We define a functor $t : \mathscr{I} \to \operatorname{End} \mathscr{A}$, whose value at the object i shall be written as t_i, with $t_0 = 1$.

We set $t_i = r_{i(1)} r_{i(2)} \cdots r_{i(n)}$, and for $j \subset i$ we take as the connecting map $t_j \to t_i$ the natural transformation

$$\varrho_{i(1)} \cdots \varrho_{j(1)-1}\, r_{j(1)}\, \varrho_{j(1)+1} \cdots \varrho_{j(2)-1}\, r_{j(2)}\, \varrho_{j(2)+1} \cdots \varrho_{i(n)};$$

that is to say, the m-th factor here is $r_{i(m)}$ if $i(m) \in j$ and $\varrho_{i(m)}$ otherwise. The functoriality of t is clear.

As in Proposition 4.4, we write $\tau_i : 1 \to t_i$ for the connecting map $t_0 \to t_i$, which is of course $\varrho_{i(1)} \varrho_{i(2)} \cdots \varrho_{i(n)} : 1 \to r_{i(1)} r_{i(2)} \cdots r_{i(n)}$. By Proposition 4.3, (t_i, τ_i) is a well-pointed endofunctor with $t_i\text{-Alg} = \bigcap_{k \in i} r_k\text{-Alg} = \bigcap_{k \in i} \mathscr{C}_k$. Moreover t_i is left exact and has rank α, since the same is true of each of its factors $r_{i(m)}$.

Write $\sigma_i : t_i \to s$ for the colimit of t, and as in Proposition 4.4 write $\sigma : 1 \to s$ for σ_0. By Proposition 4.4, (s, σ) is a well-pointed endofunctor with $s\text{-Alg} = \bigcap_{k \in K} \mathscr{C}_k = \mathscr{C}$. Moreover s is left exact and has rank α, since filtered colimits commute with finite limits and with all colimits.

We now apply Construction 4.6 to (s, σ), and conclude by Theorem 4.8 that $s\text{-Alg} = \mathscr{C}$ is reflective in $\mathscr{A}$, with reflexion $\sigma_\alpha : 1 \to s^\alpha$. It is clear from the inductive definition of S in Construction 4.6 that each s^β, and thus in particular s^α, is left exact and has rank α; so that $\mathscr{C}$ is an α-localization of $\mathscr{A}$. $\square$

Remark 5.4. For $A \in \mathscr{C}$, each $\varrho_k A$ and hence each $\tau_i A$ is the identity, by our choice of ϱ_k and r_k in the Introduction; so that σA too is the identity, if we make the obvious choice of A itself as the filtered colimit of the functor constant at A. Thus, as in Remark 4.9, $\sigma_\alpha : 1 \to s^\alpha$ is the 'strict' idempotent monad $\varrho : 1 \to r$ associated to $\mathscr{C}$.

Theorem 5.5. *In the situation of Theorem 5.3, let $(\mathscr{E}, \mathscr{M})$ be the local factorization system corresponding to the α-localization $\mathscr{C} = \bigcap \mathscr{C}_k$. Then $\mathscr{M} = \bigcap \mathscr{M}_k$, while $\mathscr{E}$ is the smallest skein containing $\bigcup \mathscr{E}_k$ and stable (in the sense of the dual* E6 *of* M6) *under rooted filtered small colimits.*

Proof. $\mathscr{C} \subset \mathscr{C}_k$ gives $\mathscr{M} \subset \mathscr{M}_k$ by Theorem 2.2, so that $\mathscr{M} \subset \bigcap \mathscr{M}_k$. Suppose therefore that $m : A \to B$ lies in $\bigcap \mathscr{M}_k$; we are to show that $m \in \mathscr{M}$. We use the language of the proof of Theorem 5.3.

Since $m \in \mathscr{M}_k$, it follows from Theorem 2.2 that

$$\begin{tikzcd} A \arrow[r, "\varrho_k A"] \arrow[d, "m"'] & r_k A \arrow[d, "r_k(m)"] \\ B \arrow[r, "\varrho_k B"'] & r_k B \end{tikzcd} \tag{5.1}$$

is a pullback for each k. We prove by induction on the cardinal of $i \in \mathscr{I}$ that each

$$\begin{tikzcd} A \arrow[r, "\tau_i A"] \arrow[d, "m"'] & t_i A \arrow[d, "t_i(m)"] \\ B \arrow[r, "\tau_i B"'] & t_i B \end{tikzcd} \tag{5.2}$$

is a pullback, which is trivial when this cardinal is 0. When the cardinal is $n>0$, observe from the proof of Theorem 5.3 that (5.2) is the exterior of

$$\begin{tikzcd} A \arrow[r, "\varrho_{i(1)} A"] \arrow[d, "m"'] & r_{i(1)} A \arrow[r, "r_{i(1)} \tau_j A"] \arrow[d, "r_{i(1)}(m)"] & r_{i(1)} t_j A \arrow[d, "r_{i(1)} t_j(m)"] \\ B \arrow[r, "\varrho_{i(1)} B"'] & r_{i(1)} B \arrow[r, "r_{i(1)} \tau_j B"'] & r_{i(1)} t_j B \end{tikzcd}, \tag{5.3}$$

where $j=\{i(2), \dots, i(n)\}$ (which is empty if $n=1$). The right square of (5.3) is a pullback by the inductive hypothesis (replacing i by j in (5.2)) and the left-exactness of $r_{i(1)}$, while the left square is a pullback by (5.1); so that the exterior is a pullback, as desired.

Since $\mathscr{I}$ is filtered and filtered colimits commute with finite limits, it follows from (5.2) that

$$\begin{tikzcd} A \arrow[r, "\sigma A"] \arrow[d, "m"'] & sA \arrow[d, "s(m)"] \\ B \arrow[r, "\sigma B"'] & sB \end{tikzcd} \tag{5.4}$$

is a pullback. We use transfinite induction to prove that each

$$\begin{tikzcd} A \arrow[r, "\sigma_\beta A"] \arrow[d, "m"'] & s^\beta A \arrow[d, "s^\beta(m)"] \\ B \arrow[r, "\sigma_\beta B"'] & s^\beta B \end{tikzcd} \qquad (5.5)_\beta$$

is a pullback. This is trivially so for $\beta=0$. The diagram $(5.5)_{\beta+1}$ is the exterior of

$$\begin{tikzcd} A \arrow[r, "\sigma A"] \arrow[d, "m"'] & sA \arrow[r, "s\sigma_\beta A"] \arrow[d, "s(m)"] & ss^\beta A \arrow[d, "ss^\beta(m)"] \\ B \arrow[r, "\sigma B"'] & sB \arrow[r, "s\sigma_\beta B"'] & ss^\beta B, \end{tikzcd} \tag{5.6}$$

since the connecting-map $S_1^{\beta+1}: s \to s^{\beta+1}$ is, by Lemma 4.7, $sS_0^\beta : s \to ss^\beta$, which is $s\sigma_\beta$; and (5.6) is a pullback since the left square is one by (5.4) and the right square is one by the inductive hypothesis $(5.5)_\beta$ and the left-exactness of s. For a limit-ordinal γ, the diagram $(5.5)_\gamma$ is the filtered colimit of $(5.5)_\beta$ for $\beta<\gamma$, and is thus a pullback since filtered colimits commute with finite limits. Since $\sigma_\alpha : 1 \to s^\alpha$ is the idempotent monad $\varrho : 1 \to r$ for the localization $\mathscr{C}$, it follows from Theorem 2.2 that $m \in \mathscr{M}$.

Turning now to the consideration of $\mathscr{E}$, write $\mathscr{E}^*$ for the smallest skein containing $\bigcup \mathscr{E}_k$ and stable under small rooted filtered colimits. Since $\mathscr{E}$ is a skein containing each $\mathscr{E}_k$ and stable, by Proposition 2.1, under rooted colimits, we have $\mathscr{E}^* \subset \mathscr{E}$; so that it remains to prove the converse.

Using again the language of the proof of Theorem 5.3, we shall show that $\varrho : 1 \to r$ lies in $\mathscr{E}^*$, in the sense that each $\varrho A : A \to rA$ does so. First, each $\varrho_k A : A \to r_k A$, being inverted by r_k, is in $\mathscr{E}_k$ by Theorem 2.2; so that $\varrho_k : 1 \to r_k$ is in $\mathscr{E}^*$. It follows inductively that each $\tau_i : 1 \to t_i$ is in $\mathscr{E}^*$, since τ_i is the composite

$$1 \xrightarrow[\varrho_{i(n)}]{} r_{i(n)} \xrightarrow[\tau_j r_{i(n)}]{} t_j r_{i(n)},$$

where $j = \{(i(1), i(2), \ldots, i(n-1)\}$. Thus the rooted filtered colimit $\sigma : 1 \to s$ of t is in $\mathscr{E}^*$. It now follows inductively that each $\sigma_\beta : 1 \to s^\beta$ is in $\mathscr{E}^*$, since $\sigma_{\beta+1} = \sigma s^\beta \cdot \sigma_\beta$ and since, when β is a limit ordinal, σ_β is the rooted filtered colimit of $(s^\gamma)_{\gamma<\beta}$. Thus $\varrho : 1 \to r$, being σ_α, is in $\mathscr{E}^*$.

Now we do not use this directly, but apply it instead to the localization $\mathscr{M}/X$ of $\mathscr{A}/X$ described in Proposition 2.6. Since $\mathscr{M} = \bigcap \mathscr{M}_k$, the localization $\mathscr{M}/X$ is the intersection $\bigcap(\mathscr{M}_k/X)$ of the localizations $\mathscr{M}_k/X$, which by Proposition 2.7 are α-localizations. Since it is easy to see that finite limits commute with filtered colimits in $\mathscr{A}/X$, the constructions in the proof of Theorem 5.3 apply with $\mathscr{A}/X$, $\mathscr{M}/X$, and $\mathscr{M}_k/X$ replacing $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{C}_k$.

Because the local factorization system $(\mathscr{E}_{kX}, \mathscr{M}_{kX})$ on $\mathscr{A}/X$ corresponding to the localization $\mathscr{M}_k/X$ has $f \in \mathscr{E}_{kX}$ precisely when f, seen as a map in $\mathscr{A}$, lies in $\mathscr{E}_k$, and because colimits in $\mathscr{A}/X$ are formed as in $\mathscr{A}$, it follows from the penultimate paragraph that the unit of the reflexion of $\mathscr{A}/X$ into $\mathscr{M}/X$, seen as a map in $\mathscr{A}$, again lies in $\mathscr{E}^*$. Now let $f : A \to X$ be in $\mathscr{E}$; by Proposition 2.6 its reflexion in $\mathscr{M}/X$ is 1_X, and the unit is f itself. So $f \in \mathscr{E}^*$; and we have $\mathscr{E} \subset \mathscr{E}^*$, as required. □

Theorem 5.6. *Let $\mathscr{A}$ admit finite limits and small filtered colimits, and let these* *commute with one another. Then, for each small regular cardinal α, the ordered set* $\mathrm{Loc}_\alpha \mathscr{A}$ *admits finite suprema and small infima, and the former distribute over the latter.*

Proof. We have the finite suprema by Theorems 3.2 and 3.9, and the small infima by Theorem 5.3; so that it remains to prove the distributivity – namely that $\mathscr{C}_0 \vee \bigcap \mathscr{C}_k = \bigcap (\mathscr{C}_0 \vee \mathscr{C}_k)$, where $\mathscr{C}_0$ is a localization and $(\mathscr{C}_k)$ is a small set of localizations, and $\vee$ denotes the supremum in $\mathrm{Loc}_\alpha \mathscr{A}$. In spite of the information we now have from Theorems 3.1–3.3 and 3.8, and Theorems 5.3 and 5.5, we have been unable to find a really direct proof, and have instead to rely on Proposition 2.3.

Write $\mathscr{C}$ for $\bigcap \mathscr{C}_k$, write $\mathscr{C}'_k$ for $\mathscr{C}_0 \vee \mathscr{C}_k$, and write $\mathscr{C}'$ for $\bigcap \mathscr{C}'_k$. We use the standard notation of the Introduction, we systematically use $(\mathscr{E}'_k, \mathscr{M}'_k)$ for the local factorization system corresponding to $\mathscr{C}'_k$ and so on, and we use the language of the proof of Theorem 5.3. The inclusion $\mathscr{C}_0 \vee \mathscr{C} \subset \mathscr{C}'$ is trivial, and we need only its converse, which by Theorems 3.1 and 3.9 is equivalent to $\mathscr{E}_0 \cap \mathscr{E} \subset \mathscr{E}'$. By Proposition 2.3, it suffices to show that $\mathscr{E}_0 \cap \mathscr{E} \cap \mathscr{M}\!on \subset \mathscr{E}'$. Accordingly we take a monomorphism $f: A \to X$ in $\mathscr{E}_0 \cap \mathscr{E}$, and show that it lies in $\mathscr{E}'$.

Since a map in $\mathscr{A}/X$ is monomorphic precisely when it is so as a map in $\mathscr{A}$, the argument in the last two paragraphs of the proof of Theorem 5.5 allows us to work in $\mathscr{A}/X$, or equivalently to work in $\mathscr{A}$ under the simplifying hypothesis that $X = 1$.

So we have an object A of $\mathscr{A}$ such that the unique map $f: A \to 1$ is monomorphic, and lies in $\mathscr{E}_0 \cap \mathscr{E}$; we are to show that it lies in $\mathscr{E}'$. To say that $f \in \mathscr{E}$ is, of course, to say that $r(A) \cong 1$; and similarly for $\mathscr{E}'$ and $\mathscr{E}_0$. We use the explicit construction of r given in the proof of Theorem 5.3.

Since $\mathscr{C}_k \subset \mathscr{C}'_k$ we have maps $\xi_k : r'_k \to r_k$ with $\xi_k \varrho'_k = \varrho_k$, which induce maps $\eta_i : t'_i \to t_i$ with $\eta_i \tau'_i = \tau_i$, which in turn induce a map $\zeta : s' \to s$ with $\zeta\sigma' = \sigma$, and finally $\theta : r' \to r$ with $\theta\varrho' = \varrho$; if we prove each $\xi_k A$ to be invertible, it will follow that θA is invertible, so that $r'A \cong 1$ and $f \in \mathscr{E}'$.

Consider the diagram

$$\begin{tikzcd} & r_k A \arrow[dr, "g"] & \\ A \arrow[ur, "\varrho_k A"] \arrow[rr, "f"'] & & 1 \end{tikzcd};$$

by Theorem 2.2, $\varrho_k A$ (being inverted by r_k) is in $\mathscr{E}_k$, while g (being a map in $\mathscr{C}_k$) is in $\mathscr{M}_k$. By Proposition 2.5(iv), g is monomorphic. Since $f \in \mathscr{E}_0$, Proposition 2.5(iii) gives $\varrho_k A \in \mathscr{E}_0$. So $\varrho_k A \in \mathscr{E}_0 \cap \mathscr{E}_k = \mathscr{E}'_k$; while $g \in \mathscr{M}_k \subset \mathscr{M}'_k$. It follows from Theorem 2.2 that $\varrho_k A$ is the reflexion of A into $\mathscr{C}'_k$, so that $\xi_k A$ is invertible. □

Let us write $\mathrm{Loc}_\mathrm{r} \mathscr{A}$ for the ordered set of those localizations of $\mathscr{A}$ that have rank α for *some* small regular cardinal α. Since any small set $(\mathscr{C}_k)$ of localizations in $\mathrm{Loc}_\mathrm{r} \mathscr{A}$ all have rank α for some α, and since finite suprema and small infima in $\mathrm{Loc}_\alpha \mathscr{A}$ are also, by Theorems 3.9 and 5.3, the suprema and infima in $\mathrm{Loc}\, \mathscr{A}$, we have:

Theorem 5.7. *The results of Theorem* 5.6 *remain true if we replace* $\mathrm{Loc}_\alpha \mathscr{A}$ *by* $\mathrm{Loc}_\Gamma \mathscr{A}$; *and each* $\mathrm{Loc}_\alpha \mathscr{A}$ *is closed in* $\mathrm{Loc}_\Gamma \mathscr{A}$ *under finite suprema and small infima.*

6. The consequences of the existence of a small strong generator

The definition in [10] of *strong epimorphism* generalizes at once to *families* $(p_i : G_i \to A)$ of maps in $\mathscr{A}$; such a family is *strongly epimorphic* if it is jointly epimorphic and satisfies the evident extension from a map p to a family (p_i) of the condition: $p \downarrow j$ *for all monomorphic* j. When, as is always the case in this article, $\mathscr{A}$ is finitely complete, the family (p_i) is strongly epimorphic precisely when there is no proper subobject of A through which every p_i factorizes. When the coproduct $\sum G_i$ exists, the family (p_i) is strongly epimorphic exactly when the corresponding map $\sum G_i \to A$ is so.

A set $\mathscr{G}$ of objects of $\mathscr{A}$ is said to be *strongly generating*, or to be a *strong generator*, if for each $A \in \mathscr{A}$ the family of all maps $g : G_g \to A$ with domain $G_g \in \mathscr{G}$ is strongly epimorphic. When such a $\mathscr{G}$ is small we call it a *small strong generator.* Of course a strong generator is a *generator*, in the sense that the $g : G_g \to A$ are jointly epimorphic.

Proposition 6.1. *A locally-small and finitely-complete* $\mathscr{A}$ *with a small strong generator is wellpowered; that is, each* $A \in \mathscr{A}$ *has but a small set of subobjects.*

Proof. The function from the set of subobjects of A to the set of subsets of the small set $\mathscr{A}(\mathscr{G}, A)$ of maps with codomain A and domain in $\mathscr{G}$, which sends the subobject B of A to the set of those g which factorize through B, is injective; for if it takes the same value at the subjects B and C, it also takes this value at $B \cap C$, giving $B = B \cap C = C$ because $\mathscr{G}$ is a strong generator. □

The following refines Proposition 2.3:

Proposition 6.2. *Let* $\mathscr{G}$ *be a strong generator for the finitely-complete* $\mathscr{A}$, *and* $\mathscr{C}$ *a localization of* $\mathscr{A}$ *with corresponding local factorization system* $(\mathscr{E}, \mathscr{M})$. *Then* $\mathscr{C} = \mathscr{T}^\perp$ *where* $\mathscr{T} = \{ e \in \mathscr{E} \cap \mathscr{M}\!on \mid \text{codomain}\; e \in \mathscr{G} \}$.

Proof. Given $C \in \mathscr{T}^\perp$ we are to prove that $C \in \mathscr{C}$. Form the diagram (2.5) as in the proof of Proposition 2.3, recalling from that proof that $w \in \mathscr{E} \cap \mathscr{M}\!on$. For each $g : G_g \to B$ with domain in $\mathscr{G}$, form the pullback

$$\begin{tikzcd} H_g \arrow[r, "t_g"] \arrow[d, "s_g"'] & G_g \arrow[d, "g"] \\ A \arrow[r, "w"'] & B, \end{tikzcd} \tag{6.1}$$

observing that $t_g \in \mathscr{T}$ since $\mathscr{E}$ and $\mathscr{Mon}$ are stable under pullbacks. Since $uw = vw$ gives $ugt_g = vgt_g$ and since $C \in \mathscr{T}^\perp$, we have $ug = vg$; which now gives $u = v$ since $\mathscr{G}$ is a strong generator and *a fortiori* a generator; so that ϱC is monomorphic, and hence lies in $\mathscr{E} \cap \mathscr{Mon}$.

Now for each $h : Q_h \to rC$ with domain in $\mathscr{G}$, form the pullback

$$\begin{tikzcd} P_h \arrow[r, "x_h"] \arrow[d, "y_h"'] & Q_h \arrow[d, "h"] \\ C \arrow[r, "\varrho C"'] & rC, \end{tikzcd}$$

observing that $x_h \in \mathscr{T}$. Since $C \in \mathscr{T}^\perp$ there is a unique $z_h : Q_h \to C$ with $z_h x_h = y_h$. Thus $\varrho C \cdot z_h \cdot x_h = \varrho C \cdot y_h = hx_h$, giving $\varrho C \cdot z_h = h$ since $x_h \in \mathscr{E}$ and $rC \in \mathscr{C} = \mathscr{E}^\perp$. Since $\mathscr{G}$ is a strong generator, and every h as above factorizes through the monomorphism ϱC, the latter is invertible; so that $C \in \mathscr{C}$. □

We can in fact describe $\mathscr{E} \cap \mathscr{Mon}$ (and thus $\mathscr{E}$ itself, by Proposition 2.4) explicitly in terms of $\mathscr{T}$:

Proposition 6.3. *In the circumstances of Proposition 6.2, let $w : A \to B$ be any monomorphism in $\mathscr{A}$. Then $w \in \mathscr{E}$ if and only if, for every $g : G_g \to B$ with domain in $\mathscr{G}$, the pullback t_g of w in (6.1) lies in $\mathscr{T}$.*

Proof. The 'only if' part is clear. For the 'if' part, apply $\bar{r} : \mathscr{A} \to \mathscr{C}$ to (6.1). Since each $\bar{r}(t_g)$ is invertible, each $\bar{r}(g)$ factorizes through $\bar{r}(w)$. Now left adjoints clearly preserve strongly epimorphic families, so that the family $(\bar{r}(g))$ is strongly epimorphic in $\mathscr{C}$; thus $\bar{r}(w)$, which is monomorphic in $\mathscr{C}$ because $\bar{r}$ is left exact, is invertible, and $w \in \mathscr{E}$. □

The strong generator $\mathscr{G}$ in Proposition 6.2 need not be small. The totally of objects of $\mathscr{A}$ is always a strong generator; and if we take $\mathscr{G}$ to be this totality, Proposition 6.2 just reduces to Proposition 2.3. Our main interest, however, is of course in the case where $\mathscr{G}$ is small. Then, because $\mathscr{A}$ is wellpowered by Proposition 6.1, the number of possibilities for $\mathscr{T}$ in Proposition 6.2 is small – since if one representative of a subobject of $G \in \mathscr{G}$ is in $\mathscr{T}$, so is any other representative of that subobject. Thus:

Theorem 6.4. *A finitely-complete and locally-small $\mathscr{A}$ with a small strong generator has only a small set of localizations.*

Now Theorem 3.1 gives:

Proposition 6.5. *If the complete and locally-small $\mathscr{A}$ has a small strong generator,* $\operatorname{Loc} \mathscr{A}$ *is a small complete lattice.*

Similarly, Theorems 5.6 and 5.7 give:

Proposition 6.6. *Let $\mathscr{A}$ be locally small with a small strong generator, let it admit finite limits and small filtered colimits, and let these commute with one another. Then* $\operatorname{Loc}_{\mathrm{r}} \mathscr{A}$ *is the dual of a locale, as is* $\operatorname{Loc}_\alpha \mathscr{A}$ *for each small regular cardinal α.*

To get $\operatorname{Loc} \mathscr{A} = \operatorname{Loc}_{\mathrm{r}} \mathscr{A}$ in the conditions of Proposition 6.6, we impose the extra hypothesis that each $A \in \mathscr{A}$ is *α-presentable* for some small regular cardinal α depending on A; by which is meant that $\mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ preserves α-filtered colimits. A locally-small cocomplete $\mathscr{A}$, with a small strong generator $\mathscr{G}$ each object of which is α-presentable for some small α, is what Gabriel and Ulmer [6] (see also Kelly [13]) call a *locally-presentable category*; so we restrict ourselves now to such categories, recalling from [6] or [13] that they are also complete.

Proposition 6.7. *If $\mathscr{A}$ is a locally-presentable category, there is a small regular cardinal α such that every localization of $\mathscr{A}$ has rank α.*

Proof. By [6] or [13], every $A \in \mathscr{A}$ is β_A-presentable for some small β_A. Let $\mathscr{G}$ be a small strong generator of $\mathscr{A}$; since the subobjects of the $G \in \mathscr{G}$ form a small set by Proposition 6.1, there is a small regular cardinal α such that each of these subobjects is α-presentable. If $\mathscr{C}$ is a localization of $\mathscr{A}$, Proposition 6.2 gives $\mathscr{C} = \mathscr{T}^{\perp}$ where each $e : A \to B$ in $\mathscr{T}$ has $B \in \mathscr{G}$ and A a subobject of B, so that both A and B are α-presentable. The subcategory $\mathscr{C}$, consisting of those C such that $\mathscr{A}(e, C) : \mathscr{A}(B, C) \to \mathscr{A}(A, C)$ is invertible for each $e \in \mathscr{T}$, is accordingly closed in $\mathscr{A}$ under α-filtered colimits, as desired. $\square$

In view of this, Proposition 6.6 and the detailed results of Sections 3 and 5 give:

Theorem 6.8. *Let $\mathscr{A}$ be locally-presentable category in which finite limits commute with filtered colimits. Then* $\operatorname{Loc} \mathscr{A}$ *is the dual of a locale, infima in* $\operatorname{Loc} \mathscr{A}$ *being intersections $\bigcap \mathscr{C}_k$ and a binary supremum $\mathscr{C}_1 \vee \mathscr{C}_2$ being the closure in $\mathscr{A}$ under finite limits of the union $\mathscr{C}_1 \cup \mathscr{C}_2$; while the local factorization systems correspon-*

ding to infima or to finite suprema are given by the results of Theorems 5.5 *and* 3.1.

Examples 6.9. We give some examples of locally-presentable categories in which finite limits commute with filtered colimits.

(i) It is classical that Set is such a category.

(ii) If $\mathscr{A}$ is such a category, so too of course is any functor category $[\mathscr{K}^{\mathrm{op}}, \mathscr{A}]$ with small $\mathscr{K}$. In particular, any presheaf category $[\mathscr{K}^{\mathrm{op}}, \mathrm{Set}]$ is such.

(iii) Every localization $\mathscr{C}$ of such a category $\mathscr{A}$ is again such. For by Proposition 6.7, the idempotent monad r on $\mathscr{A}$ corresponding to $\mathscr{C}$ has a rank, whence the category $\mathscr{C}$ of algebras for this monad is locally presentable by Corollary 10.8 of [6]. Since both limits and colimits in $\mathscr{C}$ may be formed by applying $j : \mathscr{C} \to \mathscr{A}$ to the appropriate diagram, taking the limit or the colimit in $\mathscr{A}$, and then applying $\bar{r} : \mathscr{A} \to \mathscr{C}$, and since $\bar{r}$ preserves finite limits and all colimits, the result follows.

(iv) Grothendieck topoi are such categories - as is of course well known - since they are precisely the localizations of presheaf categories.

(v) If $\mathscr{A}$ is such a category and $\mathscr{C}$ is a reflective subcategory for which the inclusion $j : \mathscr{C} \to \mathscr{A}$ is finitary (that is, preserves filtered colimits), then $\mathscr{C}$ is such a category; for $\mathscr{C}$ is locally presentable since the idempotent monad r is finitary, and $\mathscr{C}$ is closed in $\mathscr{A}$ under finite limits and filtered colimits. Since the locally-finitely-presentable categories are precisely the finitary reflexions of presheaf categories, they are such categories; as is, of course, also well known.

(vi) If $\mathscr{A}$ is such a category and if $\mathscr{L}$ is a small finitely-complete category, the subcategory $\mathrm{Lex}[\mathscr{L}, \mathscr{A}]$ of $[\mathscr{L}, \mathscr{A}]$ determined by the left-exact functors is again such a category; for it is locally presentable by (an obvious extension of) Proposition 10.4 of [13], and it is clearly closed in $[\mathscr{L}, \mathscr{A}]$ under finite limits and filtered colimits. In particular, $\mathrm{Lex}[\mathscr{L}, \mathscr{F}]$ is such a category whenever $\mathscr{F}$ is a (Grothendieck) topos.

Remark 6.10. Recall from Section 6.3 of [12] that a small finitely-complete $\mathscr{L}$ as above is what is usefully called a *finitary essentially-algebraic theory*, and that $\mathrm{Lex}[\mathscr{L}, \mathscr{A}]$ is *the category of models* of $\mathscr{L}$ *in* $\mathscr{A}$. The example $\mathrm{Lex}[\mathscr{L}, \mathscr{F}]$ where $\mathscr{F}$ is a topos has as special cases the example of a topos $\mathscr{F}$ - for, when we take $\mathscr{L}$ to be the dual of (a skeleton of) the category of finite sets, so that $\mathscr{L}$ is *the theory of an object*, we have $\mathrm{Lex}[\mathscr{L}, \mathscr{A}] \simeq \mathscr{A}$ - and also the example of a locally-finitely-presentable category; for such a category (by Theorem 9.8 of [13]) is equivalent to $\mathrm{Lex}[\mathscr{L}, \mathrm{Set}]$ for some small finitely-complete $\mathscr{L}$. We observe, using Section 10 of [13], that Theorem 6.8 applied to $\mathscr{A} = \mathrm{Lex}[\mathscr{L}, \mathscr{F}]$ generalizes the result of Borceux and Van den Bossche [1], who prove that the localizations of $\mathscr{A}$ form the dual of a locale when $\mathscr{A}$ is the category of models in a topos $\mathscr{F}$ of a finitary, one-sorted, purely-algebraic theory (that is, a *Lawvere theory* as described in [15]), which is *commutative* in the sense of Linton [16] (this last meaning that every operation $\omega : A^n \to A$ is a homomorphism of algebras).

7. The effect of a geometric morphism

Making more precise what we said in the Introduction, define a *geometric morphism* $F:\mathscr{A}\to\mathscr{A}'$ between finitely-complete categories to be a left-exact functor $F^*:\mathscr{A}'\to\mathscr{A}$ which admits a right adjoint $F_*:\mathscr{A}\to\mathscr{A}'$; and define a 2-cell $F\to G$ between geometric morphisms to be a natural transformation $F^*\to G^*$. Then finitely-complete categories and geometric morphisms form a 2-category; but it suffices for our purposes to consider the *mere* category Geom of finitely-complete categories and *isomorphism classes* of geometric morphisms. (Since we make only a very innocuous use of Geom, we ignore the fact that it is not a legitimate category in the sense of our foundations, its morphisms not forming a set unless we impose size-restrictions on $\mathscr{A}$ and $\mathscr{A}'$.)

Such a geometric morphism F is called an *injection* if F_* is fully faithful (so that $\mathscr{A}$ is, to within equivalence, a localization of $\mathscr{A}'$), and is called a *surjection* if F^* is conservative (that is, isomorphism-reflecting). We extend our standard notation for localizations $\mathscr{C}$ of $\mathscr{A}$ by writing $i:\mathscr{C}\to\mathscr{A}$ for the injective geometric morphism given by $\bar{r}\dashv j$.

The surjections and the injections form a factorization system on Geom. F1 is obvious, and F3 is proved in Proposition 5.1 of Day [3]. Day also proves F2 here; but for general left adjoints, and under stronger hypotheses on $\mathscr{A}$ and $\mathscr{A}'$. The alternative proof of this result given in Proposition 3.5 of [2] restricts, as in Remark 3.6 of [2], to give the desired F2 for Geom. If $F=i'G$ is the factorization of F into a surjection $G:\mathscr{A}\to\mathscr{C}'$ and an injection $i':\mathscr{C}'\to\mathscr{A}'$, the latter can be taken to be an actual localization of $\mathscr{A}'$; and if $(\mathscr{E}',\mathscr{M}')$ is the corresponding local factorization system, $\mathscr{E}'$consists of the maps inverted by F^* - since G^* is conservative and $\mathscr{E}'$ consists (by Theorem 2.2) of the maps inverted by $\bar{r}'$. Note that an injective geometric morphism is clearly a monomorphism in Geom.

Given a localization $\mathscr{C}$ of $\mathscr{A}$ and a geometric morphism $F:\mathscr{A}\to\mathscr{A}'$, let the geometric morphism $Fi:\mathscr{C}\to\mathscr{A}'$ factorize as the surjection $G:\mathscr{C}\to\mathscr{C}^{\#}$ followed by the injection $i^{\#}:\mathscr{C}^{\#}\to\mathscr{A}'$. Then $\mathscr{C}^{\#}$ is a localization of $\mathscr{A}'$; and $\mathscr{C}\mapsto\mathscr{C}^{\#}$ defines a function $F^{\#}:\mathrm{Loc}\,\mathscr{A}\to\mathrm{Loc}\,\mathscr{A}'$. In terms of the corresponding local factorization systems, since $\mathscr{E}^{\#}$ consists of the maps inverted by $\bar{r}F^*$, we have

$$\mathscr{E}^{\#}=(F^*)^{-1}\mathscr{E}; \tag{7.1}$$

which shows that $F^{\#}$ is an increasing map, and further shows, by Theorem 3.1, that $F^{\#}$ preserves small suprema of localizations when $\mathscr{A}$ and $\mathscr{A}'$ are complete. If $\mathrm{Loc}\,\mathscr{A}$ and $\mathrm{Loc}\,\mathscr{A}'$ are small complete lattices, this implies of course that $F^{\#}$ admits a right adjoint $F_{\#}$. So Proposition 6.5 gives:

Theorem 7.1. *Let each of $\mathscr{A}$ and $\mathscr{A}'$ be a locally-small complete category with a small strong generator. Then a geometric morphism $F:\mathscr{A}\to\mathscr{A}'$ induces as above an adjunction $F^{\#}\dashv F_{\#}:\mathrm{Loc}\,\mathscr{A}'\to\mathrm{Loc}\,\mathscr{A}$.*

We can give a somewhat more explicit description of the right adjoint $F_{\#}$. We denote injective and surjective geometric morphisms by arrows of the respective forms $\rightarrowtail$ and $\twoheadrightarrow$. Let $i' : \mathscr{C}' \to \mathscr{A}'$ be any localization of $\mathscr{A}'$, and use $i_{\#}$ for the geometric morphism $F_{\#}\ \mathscr{C}' \to \mathscr{A}$. Since the counit of the adjunction $F^{\#} \dashv F_{\#}$ is the inequality $F^{\#}\ F_{\#} \leq 1$, we have from the construction of $F^{\#}$ the following commutative diagram of geometric morphisms:

$$
\begin{tikzcd}
F_{\#}\,\mathscr{C}' \arrow[r, two heads] \arrow[d, tail, "i_{\#}"'] & F^{\#}F_{\#}\,\mathscr{C}' \arrow[r, tail] \arrow[dr, tail] & \mathscr{C}' \arrow[d, tail, "i'"] \\
\mathscr{A} \arrow[rr, "F"'] & & \mathscr{A}'
\end{tikzcd}
\tag{7.2}
$$

Proposition 7.2. *In the situation of Theorem* 7.1, *the exterior of* (7.2) *is a pullback in* Geom.

Proof. Let the exterior of the following diagram be a commutative square in Geom:

$$
\begin{tikzcd}
\mathscr{B} \arrow[rrr, "H"] \arrow[dr, "L"] \arrow[dd, "K"'] & & & \mathscr{C}' \arrow[dd, "i'"] \\
& \mathscr{C} \arrow[r, two heads, "G"] \arrow[dl, tail, "i"] & F^{\#}\,\mathscr{C} \arrow[ur, tail, dashed, "t"] \arrow[dr, tail, "i^{\#}"] & \\
\mathscr{A} \arrow[rrr, "F"'] & & & \mathscr{A}'.
\end{tikzcd}
\tag{7.3}
$$

Let $K = iL$ be the factorization of K, and form $F^{\#}\ \mathscr{C}$ as above by factorizing Fi as $i^{\#}\ G$. Since $i'H = i^{\#}\ GL$, with i' injective and GL surjective, there is a unique t (clearly injective) rendering (7.3) commutative. Thus $F^{\#}\ \mathscr{C} \subset \mathscr{C}'$, so that $\mathscr{C} \subset F_{\#}\ \mathscr{C}'$. Therefore i, and hence K, factorizes through $i_{\#}$; uniquely so, since $i_{\#}$ is monomorphic in Geom. Because i' too is monomorphic, this suffices to show that (7.2) is a pullback. $\square$

Remark 7.3. It would be of interest to find conditions under which, when F_* has rank α, the induced $F^{\#}$ takes α-localizations to α-localizations.

The assignment $F \mapsto F^{\#}$ of Theorem 7.1 is functorial:

Theorem 7.4. *Let each of $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{A}''$ be a locally-small complete category with a strong generator, and let $F : \mathscr{A} \to \mathscr{A}'$ and $H : \mathscr{A}' \to \mathscr{A}''$ be geometric morphisms. Then $(HF)^{\#} = H^{\#}\ F^{\#}$. Moreover, if $F : \mathscr{A} \to \mathscr{A}$ is the identity, so is $F^{\#}$* : Loc $\mathscr{A} \to$ Loc $\mathscr{A}$.

Proof. The latter assertion is trivial, while the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, two heads] \arrow[d, tail] & \mathscr{C}^{\#} \arrow[r, two heads] \arrow[d, tail] & \mathscr{C}^{\#\#} \arrow[d, tail] \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{A}' \arrow[r, "H"'] & \mathscr{A}''
\end{tikzcd}$$

establishes the former assertion. □

8. Applications to rings and modules

We now specialize to the case where $\mathscr{A}$ is the category Mod-R of right modules for a ring R. A localization $\mathscr{C}$ of Mod-R, with corresponding local factorization system $(\mathscr{E}, \mathscr{M})$, determines the set $\mathscr{T}$ of those monomorphisms $I \to R$ that lie in $\mathscr{E}$, which we also interpret as a set $\mathscr{T}$ of right ideals of R. Since R alone constitutes a strong generator of Mod-R, it follows by Proposition 6.2 that $\mathscr{C}$ is given in terms of $\mathscr{T}$ by $\mathscr{C} = \mathscr{T}^{\perp}$. The sets $\mathscr{T}$ of right ideals which occur in this way were determined by Gabriel [5] (see also Section 5 of Chapter VI of [17]), and are now called *Gabriel topologies* on R; they are those which satisfy the three conditions

T1. $R \in \mathscr{T}$.

T2. *If* $I \in \mathscr{T}$ *and* $x \in R$, *then* $[I : x] \in \mathscr{T}$.

T3. *The right ideal* J *is in* $\mathscr{T}$ *if, for some* $I \in \mathscr{T}$ *and every* $y \in I$, *we have* $[J : y] \in \mathscr{T}$.

Here, as usual, $[I : x]$ denotes $\{z \in R \mid xz \in I\}$. These conditions also imply the following two, where we use $\leq$ for inclusion of right ideals: see [17, *loc. cit.*]

T4. *If* $I \in \mathscr{T}$ *and* $I \leq J$, *then* $J \in \mathscr{T}$.

T5. *If* $I, J \in \mathscr{T}$, *then* $I \cap J \in \mathscr{T}$.

If we write Top R for the set of Gabriel topologies on R, ordered by the inclusion $\mathscr{T} \subset \mathscr{T}'$, Proposition 6.2 provides an isomorphism $\mathrm{Top}\, R \cong \mathrm{Loc}(\text{Mod-}R)^{\mathrm{op}}$ of ordered sets. Since Mod-R is locally finitely presentable, it follows from Theorem 6.8 that Top R is a locale: that it is a distributive lattice was shown (in the very different language of *hereditary torsion theories*) in Proposition 8.11 of Golan [7].

It is immediate from the conditions T1–T3 that the infimum in Top R of a family $(\mathscr{T}_k)_{k \in K}$ is just their intersection: observe that this agrees with Theorem 3.1 through the connexion in Proposition 6.2. The supremum $\mathscr{T}$ of this family is of course the smallest topology containing $\bigcup \mathscr{T}_k$; it reduces to $\{R\}$ if K is empty, and otherwise can be constructed by transfinite induction as follows.

Construction 8.1. For each small ordinal β we define a set $\mathcal{S}_\beta$ of right ideals of R, starting with $\mathcal{S}_0 = \bigcup \mathcal{T}_k$; this satisfies T1 and T2 (as well as T4), but not in general T3. We put I in $\mathcal{S}_{\beta+1}$ if, for some $L \in \mathcal{S}_\beta$ and some $M \leq R$ such that $[M:y] \in \mathcal{S}_\beta$ for all $y \in L$, we have $[M:x] \leq I$ for some $x \in R$; and we set $\mathcal{S}_\alpha = \bigcup_{\beta<\alpha} \mathcal{S}_\beta$ for a limit-ordinal α. Clearly each $\mathcal{S}_\beta$ satisfies T4 and hence T1; it further satisfies T2 since, for any $z \in R$, the inequality $[M:x] \leq I$ gives $[M:xz] = [[M:x]:z] \leq [I:z]$. Moreover the $\mathcal{S}_\beta$ increase with β, since if $L \in \mathcal{S}_\beta$, we have $[L:y] = R \in \mathcal{S}_\beta$ for all $y \in L$, while $[L:1] = L$. It follows from T1–T4 that each $\mathcal{S}_\beta$ is contained in the supremum $\mathcal{T}$ of the $\mathcal{T}_k$. The set of sets of ideals of R being small, we have $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha$ for some small α; since $\mathcal{S}_\alpha$ then satisfies T3, we have $\mathcal{S}_\alpha = \mathcal{T}$. $\square$

We now consider the effect of a homomorphism $\phi : R \to S$ of rings. Write $\phi^* : \text{Mod-}S \to \text{Mod-}R$ for the algebraic functor induced by ϕ, sending the S-module A to the R-module with the same underlying group and with the R-action given by $ax = a(\phi x)$ for $x \in R$; on maps ϕ^* is the identity. Since ϕ similarly sends left S-modules to left R-modules, S itself becomes an R-bimodule, in terms of which we can express the left and right adjoints of ϕ^*:

$$\phi_! = - \otimes_R S \dashv \phi^* \dashv \mathrm{Hom}_R(S, -) = \phi_*. \tag{8.1}$$

Since ϕ^*, having a left adjoint, is left exact, and since it is clearly conservative, the adjunction $\phi^* \dashv \phi_*$ constitutes a surjective geometric morphism $F : \text{Mod-}R \to \text{Mod-}S$. By Theorem 7.1, this induces an adjunction $F^\# \dashv F_\# : \mathrm{Loc}(\text{Mod-}S) \to \mathrm{Loc}(\text{Mod-}R)$; on passing to the duals this becomes an adjunction $F_\# \dashv F^\#$: $\mathrm{Top}\, R \to \mathrm{Top}\, S$, which we henceforth denote by

$$\phi_\# \dashv \phi^\# : \mathrm{Top}\, R \to \mathrm{Top}\, S. \tag{8.2}$$

Note that, because $(\psi\phi)^* = \phi^* \psi^*$ and $1^* = 1$, Theorem 7.4 gives

$$(\psi\phi)^\# = \psi^\# \phi^\# \quad \textit{and} \quad 1^\# = 1. \tag{8.3}$$

Proposition 8.2. *For $\phi : R \to S$ and $\mathcal{T} \in \mathrm{Top}\, R$ we have*

$$\phi^\# \mathcal{T} = \{ J \leq S \mid \sigma^{-1}[J:y] \in \mathcal{T} \textit{ for all } y \in S \}.$$

Proof. If $\mathcal{T}$ corresponds to the localization $\mathcal{C}$ of Mod-R with associated local factorization system $(\mathcal{E}, \mathcal{M})$, its image $\phi^\# \mathcal{T}$ corresponds, in the language of Section 7, to the localization $\mathcal{C}^\# = F^\# \mathcal{C}$ of Mod-S; so that a right ideal J of S lies in $\phi^\# \mathcal{T}$ precisely when the monomorphism $J \to S$ lies in $\mathcal{E}^\#$, which by (7.1) is to say that the monomorphism $\phi^* J \to \phi^* S$ lies in $\mathcal{E}$. By Proposition 6.3, this is in turn to say that the pullback of $\phi^* J \to \phi^* S$ along any map $R \to \phi^* S$ lies in $\mathcal{T}$; and since a map $R \to \phi^* S$ is just $x \mapsto y(\phi x)$ for some $y \in S$, the pullback in question is the right ideal $\{ x \in R \mid y(\phi x) \in J \} = \phi^{-1}[J:y]$. $\square$

The $\phi^{\#}$ of (8.2), being a right adjoint, preserves all infima, and in particular finite ones. It is therefore a *map of frames*, in the sense of Section 1.1 of Chapter II of Johnstone [9], precisely when it preserves suprema - or equivalently has a right adjoint. (Since it then has *both* adjoints, it is not only a map of frames, but a map of complete lattices.) We now give conditions on ϕ sufficient for this to be so. First note that, for $\mathcal{T} \in \operatorname{Top} R$, it follows from Proposition 8.2 (since $[J:1]=J$) that

$$\phi^{\#} \mathcal{T} \subset \{J \leq S \mid \phi^{-1} J \in \mathcal{T}\}. \tag{8.4}$$

Theorem 8.3. *Consider the following conditions on the ring-homomorphism* $\phi: R \to S$:

(i) *S is a reduced regular ring; that is, a von Neumann regular ring without nilpotent elements.*

(ii) *S is commutative.*

(iii) *ϕ is surjective.*

(iv) *S, as a module over its centre, is generated by $\phi(R)$.*

(v) *For any right ideal J of S and any $y \in S$, there is a finite sequence $x_1, \ldots, x_n$ of elements of R such that $\phi^{-1}[J:y] \geq \bigcap_i [\phi^{-1} J : x_i]$.*

(vi) *For every $\mathcal{T} \in \operatorname{Top} R$ the inclusion* (8.4) *is an equality.*

(vii) *$\phi^{\#}: \operatorname{Top} R \to \operatorname{Top} S$ is a map of frames, and hence a map of complete lattices.*

Then each of (i)–(iv) *implies* (v), *while* (v) *implies* (vi) *and* (vi) *implies* (vii).

Proof. Since every right ideal in a reduced regular ring S is two-sided (see Proposition 12.3 of Chapter I of [17]), (i) implies (v) trivially on taking $n=1$ and $x_1=1$. To see that (iv), which contains (ii) and (iii) as special cases, implies (v), let $y = w_1 \phi(x_1) + \cdots + w_n \phi(x_n)$ with each w_i in the centre of S. Then

$$[J:y] \geq \bigcap_i [J: w_i \phi(x_i)] \geq \bigcap_i [J: \phi(x_i)],$$

so that

$$\phi^{-1}[J:y] \geq \bigcap_i \phi^{-1}[J:\phi(x_i)] = \bigcap_i [\phi^{-1} J : x_i].$$

It is clear from Proposition 8.2 and the properties T2, T5 (or T1 when $n=0$), and T4 of $\mathcal{T}$ that (v) implies (vi); and it remains only to show that (vi) implies (vii).

Given (vi) then, we are to show that $\phi^{\#}$ preserves suprema. First, it preserves the empty supremum: for if $\mathcal{T}$ is the smallest topology $\{R\}$, any $J \in \phi^{\#} \mathcal{T}$ has $\phi^{-1} J = R$ by (vi), so that J contains ϕR; in particular $1 \in J$, whence $J = S$.

Now let $(\mathcal{T}_k)$ be a non-empty family in $\operatorname{Top} R$, with supremum $\mathcal{T}$. To show that $\phi^{\#} \mathcal{T}$ is the supremum of the $\phi^{\#} \mathcal{T}_k$ is to show that it is contained in this supremum, since the converse is trivial. What we have to show, then, is that if $\mathcal{T}' \in \operatorname{Top} S$ contains each $\phi^{\#} \mathcal{T}_k$, it contains $\phi^{\#} \mathcal{T}$. By (vi), to say that $\mathcal{T}'$ contains each $\phi^{\#}\mathscr{T}_k$ is to say that, for any right ideal J of S, we have

$$\phi^{-1}J\in\bigcup\mathscr{T}_k \quad \textit{implies} \quad J\in\mathscr{T}'; \tag{8.5}$$

and we are to deduce from this that

$$\phi^{-1}J\in\mathscr{T} \quad \textit{implies} \quad J\in\mathscr{T}'. \tag{8.6}$$

We turn to the transfinite construction of $\mathscr{T}$ given in Construction 8.1, retaining the notation there, and prove inductively that

$$\phi^{-1}J\in\mathscr{S}_\beta \quad \textit{implies} \quad J\in\mathscr{T}', \tag*{$(8.7)_\beta$}$$

which gives (8.6) since $\mathscr{T}=\mathscr{S}_\alpha$ form some α.

The case $\beta=0$ of $(8.7)_\beta$ is (8.5); and the inductive step at a limit-ordinal is trivial. So it remains to deduce $(8.7)_{\beta+1}$ from $(8.7)_\beta$. Let $\phi^{-1}J\in\mathscr{S}_{\beta+1}$. Then by Construction 8.1 there is an $L\in\mathscr{S}_\beta$, an $M\le R$ such that $[M:y]\in\mathscr{S}_\beta$ for all $y\in L$, and an $x\in R$ such that $[M:x]\le\phi^{-1}J$.

Since $L\in\mathscr{S}_\beta$ and $\mathscr{S}_\beta$ satisfies T2 we have $[L:x]\in\mathscr{S}_\beta$. Consider the right ideal $\phi[L:x]\cdot S$ of S generated by the image of $[L:x]$ under ϕ; since $[L:x]\le\phi^{-1}(\phi[L:x]\cdot S)$ and $\mathscr{S}_\beta$ satisfies T4, we have $\phi^{-1}(\phi[L:x]\cdot S)\in\mathscr{S}_\beta$; so that $\phi[L:x]\cdot S\in\mathscr{T}'$ by the inductive hypothesis $(8.7)_\beta$. Because the topology $\mathscr{T}'$ satisfies T3, the desired conclusion $J\in\mathscr{T}'$ will follow if we show that

$$[J:y]\in\mathscr{T}' \quad \textit{for all } y\in\phi[L:x]\cdot S.$$

Let $y\in\phi[L:x]\cdot S$, so that y has the form $y=(\phi x_1)y_1+\cdots+(\phi x_n)y_n$ where each $y_i\in S$ and each $x_i\in R$ with $xx_i\in L$. Since $[J:y]\ge\bigcap[J:(\phi x_i)y_i]=\bigcap[[J:\phi x_i]:y_i]$, and since $\mathscr{T}'$ satisfies T2, T5 (or T1 when $n=0$), and T4, it suffices to show that $[J:\phi x_i]\in\mathscr{T}'$ for each i.

We have $\phi^{-1}[J:\phi x_i]=[\phi^{-1}J:x_i]\ge[[M:x]:x_i]=[M:xx_i]$. Since $xx_i\in L$ we have $[M:xx_i]\in\mathscr{S}_\beta$, whence $\phi^{-1}[J:\phi x_i]\in\mathscr{S}_\beta$ since $\mathscr{S}_\beta$ satisfies T4; now the inductive hypothesis $(8.7)_\beta$ gives $[J:\phi x_i]\in\mathscr{T}'$, as desired. □

Remark 8.4. The reduced regular rings, and the commutative rings, form full subcategories of the category Rng of rings; the homomorphisms that are surjective, and those that satisfy (v) or (vi) of Theorem 8.3 (although not those that satisfy (iv)), form non-full subcategories of Rng with all rings as objects - skeins, in fact. If Rng' denotes any one of these subcategories, it follows from (8.3) that $R\mapsto \operatorname{Top}R$ and $\phi\mapsto\phi^{\#}$ constitute a functor Top from Rng' to the category Frm of frames, and hence (in the terminology of [9]) to the dual $\mathsf{Loc}^{\mathrm{op}}=\mathsf{Frm}$ of the category of locales.

For certain ring-homomorphisms $\phi:R\to S$, there is another way of producing an adjunction between $\operatorname{Top}R$ and $\operatorname{Top}S$. We obtained (8.2) from the adjunction $\phi^*\dashv\phi_*$ in (8.1), which always constitutes a geometric morphism $F:\mathsf{Mod}\text{-}R\to\mathsf{Mod}\text{-}S$; however the adjunction $\phi_!\dashv\phi^*$ in (8.1) constitutes a geometric morphism $H:\mathsf{Mod}\text{-}S\to\mathsf{Mod}\text{-}R$ when $\phi_!=-\otimes_R S$ is left exact - which is to say that S *is flat as* *a left R-module.* When this is so we call ϕ a *flat homomorphism of rings*; clearly rings and flat homomorphisms form a non-full subcategory $\mathsf{Rng}^{\mathrm{f}}$ of Rng. For such a flat ϕ the geometric morphism H above induces, by Theorem 7.1, an adjunction $H^{\#} \dashv H_{\#} : \mathrm{Loc}(\mathsf{Mod}\text{-}R) \to \mathrm{Loc}(\mathsf{Mod}\text{-}S)$; on passing to the duals this becomes an adjunction $H_{\#} \dashv H^{\#} : \mathrm{Top}\, S \to \mathrm{Top}\, R$, which we henceforth denote by

$$\phi_0 \dashv \phi^0 : \mathrm{Top}\, S \to \mathrm{Top}\, R. \tag{8.8}$$

Once again, because $(\psi\phi)^* = \phi^* \psi^*$ and $1^* = 1$, Theorem 7.4 gives functoriality: for flat $\phi : R \to S$ and $\psi : S \to T$, we have

$$(\psi\phi)_0 = \psi_0 \phi_0 \quad \textit{and} \quad 1_0 = 1. \tag{8.9}$$

Proposition 8.5. *For a flat $\phi : R \to S$ and $\mathscr{T}' \in \mathrm{Top}\, S$ we have*

$$\phi^0 \mathscr{T}' = \{ I \le R \mid \phi I \cdot S \in \mathscr{T}' \}.$$

Proof. As in the proof of Proposition 8.2, I lies in $\phi^0 \mathscr{T}'$ precisely when the monomorphism $\phi_! I \to \phi_! R$, or $I \otimes_R S \to R \otimes_R S \cong S$, lies in $\mathscr{T}'$. The isomorphism $R \otimes_R S \cong S$ sending $x \otimes y$ to xy, which by definition is $\phi(x)y$, the image in S of $I \otimes_R S$ is $\phi I \cdot S$. □

For any $\phi : R \to S$, the algebraic functor ϕ^* is fully faithful precisely when ϕ is an epimorphism of rings (in the categorical sense – we do not mean a surjection): see Proposition 1.2 of Chapter XI of [17]. When ϕ is both flat and epimorphic, we call it a *flat epimorphism*; for such a ϕ, the geometric morphism $H : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$ above is an injection. Then $\mathsf{Mod}\text{-}S$ is itself, to within equivalence, a localization of $\mathsf{Mod}\text{-}R$, corresponding say to the topology $\mathscr{S} \in \mathrm{Top}\, R$. (It is easy to give $\mathscr{S}$ explicitly: the right ideal I of R lies in $\mathscr{S}$ precisely when the morphism $\phi_! I \to \phi_! R$ is invertible, which is clearly to say that $\phi I \cdot S = S$, or equivalently that $1 \in \phi I \cdot S$.)

Accordingly we may, for a flat epimorphism $\phi : R \to S$, identify $\mathrm{Loc}(\mathsf{Mod}\text{-}S)$ with $\{ \mathscr{C} \in \mathrm{Loc}(\mathsf{Mod}\text{-}R) \mid \mathscr{C} \subset \mathsf{Mod}\text{-}S \}$. Then $H^{\#} : \mathrm{Loc}(\mathsf{Mod}\text{-}S) \to \mathrm{Loc}(\mathsf{Mod}\text{-}R)$ is just the inclusion, so that its right adjoint $H_{\#}$ sends $\mathscr{C} \in \mathrm{Loc}(\mathsf{Mod}\text{-}R)$ to $\mathscr{C} \cap \mathsf{Mod}\text{-}S$ (which agrees with Proposition 7.2). Passing now to the duals, we have identified $\mathrm{Top}\, S$ with $\{ \mathscr{T} \in \mathrm{Top}\, R \mid \mathscr{T} \supset \mathscr{S} \}$, the functor $\phi^0 : \mathrm{Top}\, S \to \mathrm{Top}\, R$ with the inclusion, and its left adjoint $\phi_0 : \mathrm{Top}\, R \to \mathrm{Top}\, S$ with the map sending $\mathscr{T}$ to the supremum $\mathscr{S} \vee \mathscr{T}$ in $\mathrm{Top}\, R$. Since $\mathrm{Top}\, R$ is a locale and hence a distributive lattice, $\phi_0 = \mathscr{S} \vee -$ preserves finite infima. Using the functoriality given by (8.9), and writing $\mathsf{Rng}^{\mathrm{fe}}$ for the category of rings and flat epimorphisms, we therefore have:

Theorem 8.6. *When $\phi : R \to S$ is a flat epimorphism, $\phi_0 : \mathrm{Top}\, R \to \mathrm{Top}\, S$ is a map of frames; so that $R \mapsto \mathrm{Top}\, R$ and $\phi \mapsto \phi_0$ constitute a functor $\mathsf{Rng}^{\mathrm{fe}} \to \mathsf{Frm}$.*

When Theorems 8.3 and 8.6 both apply, we do not get two different maps of frames:

Theorem 8.7. *When the flat epimorphism* $\phi : R \to S$ *satisfies* (vi) *of Theorem* 8.3 – *that is, when* (8.4) *is an equality – the frame-maps* $\phi^{\#}, \phi_0 : \mathrm{Top}\, R \to \mathrm{Top}\, S$ *coincide.*

Proof. We give an indirect proof by showing that $\phi^{\#}$ is the left adjoint of ϕ^0. We first observe that, for $I \leq R$ and $J \leq S$, we have

$$I \leq \phi^{-1}(\phi I \cdot S) \quad \textit{and} \quad \phi(\phi^{-1} J) \cdot S \leq J. \tag{8.10}$$

We are to prove that, for $\mathscr{T} \in \mathrm{Top}\, R$ and $\mathscr{T}' \in \mathrm{Top}\, S$, we have

$$\phi^{\#}\, \mathscr{T} \subset \mathscr{T}' \quad \textit{if and only if} \quad \mathscr{T} \subset \phi^0\, \mathscr{T}'. \tag{8.11}$$

Given the left assertion in (8.11), let $I \in \mathscr{T}$. Then $\phi^{-1}(\phi I \cdot S) \in \mathscr{T}$ by (8.10) and T4, so that $\phi I \cdot S \in \phi^{\#}\, \mathscr{T}$ since (8.4) is an equality; thus $\phi I \cdot S \in \mathscr{T}'$ by the left assertion of (8.11), and now Proposition 8.5 gives $I \in \phi^0\, \mathscr{T}'$, as desired.

Given the right assertion of (8.11), let $J \in \phi^{\#}\, \mathscr{T}$; then $\phi^{-1} J \in \mathscr{T}$ by (8.4), so that $\phi^{-1} J \in \phi^0\, \mathscr{T}'$ by the right assertion of (8.11); by Proposition 8.5, therefore, $\phi(\phi^{-1} J) \cdot S \in \mathscr{T}'$; and now (8.10) and T4 give $J \in \mathscr{T}'$, as desired. □

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