COEQUALIZERS IN CATEGORIES OF ALGEBRAS *

by

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Introduction

It is well known [2,§6] that (inverse) limits in a category of algebras over $\mathcal{A}$ - in particular, in the category $\mathcal{A}^{\mathbf{T}}$ of algebras over a triple $\mathbf{T} = (T,\eta,\mu)$ on $\mathcal{A}$ - can be calculated in $\mathcal{A}$. Despite the fact that such a statement is, in general, false for colimits (direct limits), a number of colimit constructions can be carried out in $\mathcal{A}^{\mathbf{T}}$ provided they can be carried out in $\mathcal{A}$ and $\mathcal{A}^{\mathbf{T}}$ has enough coequalizers.

The coequalizers $\mathcal{A}^{\mathbf{T}}$ should have, at a minimum, are, as we shall see in §1, those of <u>reflexive pairs</u>: a pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

of maps f, g in a category $\mathcal{X}$ is <u>reflexive</u> if there is an $\mathcal{X}$-morphism

$$\Delta : Y \longrightarrow X$$

satisfying the identities

$$f\circ\Delta = id_Y = g\circ\Delta .$$

(This terminology arises from the fact that, when $\mathcal{X} = \mathcal{S} = \{\text{sets}\}$, (f,g) is reflexive if and only if the image of the induced function

$$X \xrightarrow[f,g]{} Y \times Y$$

contains the diagonal of $Y \times Y$.)

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In §2 we give two criteria for $\mathcal{A}^{\mathbf{T}}$ to have coequalizers of reflexive pairs, neither of them necessary, of course. In §1, it will turn out, so long as $\mathcal{A}^{\mathbf{T}}$ has such coequalizers, that each functor

$$\mathcal{A}^{\tau} : \mathcal{A}^{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbf{S}} ,$$

induced by a map of triples $\tau : \mathbf{S} \to \mathbf{T}$, has a left adjoint, that $\mathcal{A}^{\mathbf{T}}$ has coproducts if $\mathcal{A}$ does, indeed, has all small colimits if $\mathcal{A}$ has coproducts, and that $\mathcal{A}^{\mathbf{T}}$ has tensor products if $\mathcal{A}$ does. These are, of course, known facts when $\mathcal{A} = \mathcal{S}$ = {sets}; however, at the time of this writing, it is unknown, for example, whether the category of contramodules over an associative coalgebra, presented (in [1]) as $\mathcal{A}^{\mathbf{T}}$ with $\mathcal{A}$ = {ab. groups}, has coequalizers of reflexive pairs.

§ 1. Constructions using coequalizers of reflexive pairs.

We begin with a lemma that will have repeated use. It concerns the following definition, which clarifies what would otherwise be a recurrent conceptual obscurity in the proofs of this section.

Let $U : \mathcal{X} \to \mathcal{A}$ be a functor, let $X \in |\mathcal{X}|$, and let $(f,g) = \{(f_i,g_i)/i \in I\}$ be a family of $\mathcal{A}$-morphisms

$$A_i \overset{f_i}{\underset{g_i}{\rightrightarrows}} UX \quad (i \in I). \tag{1.1}$$

An $\mathcal{X}$-morphism $p : X \to P$ is a <u>coequalizer</u> (<u>rel</u>. U) of the family of pairs (1.1) if

1) $\forall i \in I,\ Up \circ f_i = Up \circ g_i$, and
2) if $q : X \to Y$ satisfies $Uq \circ f_i = Uq \circ g_i$ $(\forall i \in I)$,

then $\exists !\ x : P \to Y$ with $q = x \circ p$.

If $U = id_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$, a coequalizer (rel. U) of the family (1.1) will be called simply a <u>coequalizer</u> of (1.1).

<u>Lemma 1</u>. <u>If</u> U <u>has a left adjoint</u> $F : \mathcal{A} \to \mathcal{X}$ <u>and</u> $\bar{f}_i, \bar{g}_i : FA_i \to X$ <u>are the</u> $\mathcal{X}$-<u>morphisms corresponding to</u> f_i, g_i <u>by adjointness, then</u> $p : X \to P$ <u>is a coequalizer</u> (rel. U) of (f,g) <u>if and only if it is a coequalizer of</u> $(\bar{f},\bar{g})$. <u>If</u> U <u>is faithful</u>

and $\bar{f}_i, \bar{g}_i : X_i \to X$ *are* $\mathcal{X}$*-morphisms with* $U\bar{f}_i = f_i$, $U\bar{g}_i = g_i$, *then* $p : X \to P$ *is a coequalizer* (rel U) *of* (f,g) *if and only if it is a coequalizer of* $(\bar{f},\bar{g})$.

Proof. In the first case, the naturality of the adjunction isomorphisms yields

$$q \cdot \bar{f}_i = q \cdot \bar{g}_i \Longleftrightarrow Uq \cdot f_i = Uq \cdot g_i$$

for every $\mathcal{X}$-morphism q defined on X . In the second case, that relation follows from the faithfulness of U . Clearly, that relation is all the proof required.

Proposition 1. *Let* $\mathbb{S} = (S,\eta',\mu')$ *and* $\mathbb{T} = (T,\eta,\mu)$ *be triples on* $\mathcal{A}$, *suppose the natural transformation* $\tau : S \to T$ *is a map of triples from* $\mathbb{S}$ *to* $\mathbb{T}$, *and let* (A,α) *be an* $\mathbb{S}$*-algebra*, (B,β) *a* $\mathbb{T}$*-algebra*, $p : TA \to B$ *a* $\mathbb{T}$*-homomorphism from* (TA,μ_A) *to* (B,β), *and* $\iota = p \circ \eta_A : A \to B$. *Then the following statements are equivalent.*

1) p *is a coequalizer of the pair*

$$\begin{tikzcd}
(TSA,\mu_{SA}) \arrow[r, "T(\tau_A)"] \arrow[rr, bend right, "T(\alpha)"'] & (TTA,\mu_{TA}) \arrow[r, "\mu_A"] & (TA,\mu_A)
\end{tikzcd} ;$$

2) p *is a coequalizer* (rel $U^{\mathbb{T}}$) *of the pair*

$$\begin{tikzcd}
SA \arrow[rr, "\tau_A"] \arrow[dr, "\alpha"'] & & TA \\
& A \arrow[ur, "\eta_A"'] &
\end{tikzcd} ;$$

3) ι *is an* $\mathbb{S}$*-homomorphism* $(A,\alpha) \to (B,\beta \cdot \tau_B)$ *making the composition*

$$\begin{array}{c} \mathcal{A}^{\mathbf{T}}((B,\beta),X) \\ \downarrow \\ \mathcal{A}^{\mathbf{S}}(\mathcal{A}^{\tau}(B,\beta),\mathcal{A}^{\tau}X) \\ \downarrow = \\ \mathcal{A}^{\mathbf{S}}((B,\beta\cdot\tau_B),\mathcal{A}^{\tau}X) \\ \downarrow \iota \\ \mathcal{A}^{\mathbf{S}}((A,\alpha),\mathcal{A}^{\tau}X) \end{array}$$

<u>a one-one correspondence</u>, $\forall X \in |\mathcal{A}^{\mathbf{T}}|$.

<u>Proof</u>. The equivalence of statements 1) and 2) follows from Lemma 1, since $\mu_A \cdot T(\tau_A)$ is the $\mathcal{A}^{\mathbf{T}}$-morphism corresponding to τ_A by adjointness and $\eta_A \cdot \alpha = T(\alpha) \cdot \eta_{SA}$ is the $\mathcal{A}$-morphism corresponding to $T(\alpha)$ by adjointness.

Next, if $g : TA \to X$ is a $\mathbf{T}$-homomorphism from (TA,μ_A) to a $\mathbf{T}$-algebra (X,ξ), having equal compositions with τ_A and $\eta_A \cdot \alpha$, we show that $g \cdot \eta_A : A \to X$ is an $\mathbf{S}$-morphism from (A,α) to $\mathcal{A}^{\tau}(X,\xi) = (X,\xi\cdot\tau_X)$, i.e., that

$$g \cdot \eta_A \cdot \alpha = \xi \cdot \tau_X \cdot S(g \cdot \eta_A) .$$

Clearly this requires only the proof of

$$g \cdot \tau_A = \xi \cdot \tau_X \cdot Sg \cdot S\eta_A ,$$

for which, consider the diagram

$$\begin{tikzcd}
SA \arrow[r, "S\eta_A"] \arrow[d, "\tau_A"'] & STA \arrow[r, "Sg"] \arrow[d, "\tau_{TA}"] & SX \arrow[d, "\tau_X"] \\
TA \arrow[r, "T\eta_A"'] \arrow[ddr, "g"'] & TTA \arrow[r, "Tg"'] \arrow[d, "\mu_A"] & TX \arrow[ddl, "\xi"] \\
 & TA \arrow[d, "g"] & \\
 & X &
\end{tikzcd}$$

The upper squares commute because τ is natural, the left hand triangle, because $\mu_A \cdot T\eta_A = id_{TA}$, the right hand triangle, because g is a $\mathbb{T}$-homomorphism.

Finally, given an $\mathbb{S}$-homomorphism $f : A \to X$ from (A,α) to $\mathcal{A}^{\tau}(X,\xi) = (X,\xi\cdot\tau_X)$, it turns out that $\xi \cdot Tf$ is a $\mathbb{T}$-homomorphism $(TA,\mu_A) \to (X,\xi)$ having equal compositions with τ_A and $\eta_A \cdot \alpha$. For, the diagram

$$\begin{tikzcd}
& SA \arrow[rr, "\alpha"] \arrow[dl, "\tau_A"'] \arrow[dr, "Sf"] & & A \arrow[r, "\eta_A"] \arrow[d, "f"] & TA \arrow[dd, "Tf"] \\
TA \arrow[dr, "Tf"'] & & SX \arrow[dl, "\tau_X"] & X \arrow[d, "id"] \arrow[dr, "\eta_X"] & \\
& TX \arrow[rr, "\xi"'] & & X & TX \arrow[l, "\xi"]
\end{tikzcd}$$

commutes, since τ is natural, f is an $\mathbb{S}$-homomorphism, η is natural, and $\xi \cdot \eta_X = id_X$.

These arguments form the core of a proof of Proposition 1.

Corollary 1. If $\mathcal{A}^{\mathbb{T}}$ has coequalizers of reflexive pairs, then each functor $\mathcal{A}^{\tau} : \mathcal{A}^{\mathbb{T}} \to \mathcal{A}^{\mathbb{S}}$, induced by a triple map $\tau : \mathbb{S} \to \mathbb{T}$, has a left adjoint $\hat{\tau}$.

Proof. For each $(A,\alpha) \in |\mathcal{A}^{\mathbb{S}}|$, the pair

$$\begin{tikzcd}
F^{\mathbb{T}}SA \arrow[r, "F^{\mathbb{T}}(\tau_A)"] \arrow[rr, bend right, "F^{\mathbb{T}}(\alpha)"'] & F^{\mathbb{T}}TA \arrow[r, "\mu_A"] & F^{\mathbb{T}}A ,
\end{tikzcd}$$

whose coequalizer, if any, is (by Proposition 1) the value $\hat{\tau}(A,\alpha)$ of $\hat{\tau}$ at (A,α), is reflexive by virtue of

$$\Delta = F^{\mathbb{T}}(\eta'_A) .$$

Proposition 2. Let $(A_i,\alpha_i)(i \in I)$ be a family of $\mathbb{T}$-algebras, and assume the coproduct $\bigoplus_{i\in I} A_i$ exists in $\mathcal{A}$, say with injections $j_i : A_i \to \oplus A_i$. Let $p : T(\oplus A_i) \to P$ be a $\mathbb{T}$-homomorphism. Then the following statements are equivalent.

1) p <u>is a coequalizer</u> (rel $U^{\mathbf{T}}$) <u>of the family of pairs</u>

$$\begin{tikzcd} TA_i \arrow[rr, "T(j_i)"] \arrow[d, "\alpha_i"'] & & T(\oplus A_i) \\ A_i \arrow[r, "j_i"'] & \oplus A_i \arrow[ur, "\eta_{\oplus A_i}"'] & \end{tikzcd} \qquad (i \in I)$$

2) <u>each map</u> $h_i = p \cdot \eta_{\oplus A_i} \cdot j_i : A_i \to P$ <u>is a</u> $\mathbf{T}$<u>-homomorphism and the family</u> $(h_i)_{i \in I}$ <u>serves to make</u> P <u>the coproduct in</u> $\mathcal{A}^{\mathbf{T}}$ <u>of</u> $(A_i)_{i \in I}$.

<u>Moreover, if</u> $\oplus TA_i$ <u>is available in</u> $\mathcal{A}$, <u>statements</u> 1) <u>and</u> 2) <u>are equivalent to each of the following statements about</u> p :

3) p <u>is a coequalizer</u> (rel $U^{\mathbf{T}}$) <u>of the pair</u>

$$\begin{tikzcd} \oplus TA_i \arrow[rr, "(\ldots T(j_i)\ldots)"] \arrow[dr, "\oplus \alpha_i"'] & & T(\oplus A_i) \\ & \oplus A_i \arrow[ur, "\eta_{\oplus A_i}"'] & \end{tikzcd}$$

4) p <u>is a coequalizer of the pair</u>

$$\begin{tikzcd} (T(\oplus TA_i),\mu) \arrow[r, "T(\ldots T(j_i)\ldots)"] \arrow[rr, bend right, "T(\oplus \alpha_i)"'] & (TT(\oplus A_i),\mu) \arrow[r, "\mu"] & (T(\oplus A_i),\mu) \end{tikzcd}$$

<u>Proof</u>. The equivalence of statements 1) and 3) is obvious. The equivalence of 3) with 4) is due to Lemma 1, since the top (bottom) maps correspond to each other by adjointness.

Next, let $g : T(\oplus A_i) \to X$ be a $\mathbf{T}$-homomorphism from $F^{\mathbf{T}}(\oplus A_i)$ to (X,ξ), having equal compositions with both components of all the pairs in 1). Then $g \cdot \eta_{\oplus A_i} \cdot j_i : A_i \to X$ is a $\mathbf{T}$-homomorphism $(A_i,\alpha_i) \to (X,\xi)$, for all i , as is shown by the commutativity of the diagrams

$$\begin{tikzcd}
TA_i \arrow[r, "T(j_i)"] \arrow[dd, "\alpha_i"'] & T(\oplus A_i) \arrow[r, "T\eta"] \arrow[dr, "id"'] & TT(\oplus A_i) \arrow[r, "Tg"] \arrow[d, "\mu"] & TX \arrow[dd, "\xi"] \\
 & & T(\oplus A_i) \arrow[dr, "g"] & \\
A_i \arrow[r, "j_i"'] & \oplus A_i \arrow[r, "\eta_{\oplus A_i}"'] & T(\oplus A_i) \arrow[r, "g"'] & X
\end{tikzcd} \qquad (i \in I)$$

Finally, if $f_i : A_i \to X$ is a family of $\mathbb{T}$-homomorphisms $(A_i,\alpha_i) \to (X,\xi)$, then the map

$$g = \xi \cdot T(\ldots f_i \ldots) : T(\oplus A_i) \to TX \xrightarrow{\xi} X$$

is a $\mathbb{T}$-homomorphism (the only one) having $g \cdot \eta_{\oplus A_i} \cdot j_i = f_i$, and, as the diagram below shows, has equal compositions with both members of all the pairs in 1).

$$\begin{tikzcd}
 & & T(\oplus A_i) \arrow[drr, "T(\ldots f_i \ldots)"] & & & \\
TA_i \arrow[urr, "Tj_i"] \arrow[rrrr, "Tf_i"'] \arrow[ddd, "\alpha_i"'] & & & & TX \arrow[r, "\xi"] & X \\
 & & A_i \arrow[urrr, "f_i"] & & & \\
 & & & TA_i \arrow[ul, "\alpha_i"'] \arrow[d, "T(j_i)"] \arrow[drr, "Tf_i"] & & \\
A_i \arrow[uurr, "id"] \arrow[urrr, "\eta A_i"'] \arrow[r, "j_i"'] & \oplus A_i \arrow[rr, "\eta_{\oplus A_i}"'] & & T(\oplus A_i) \arrow[rr, "T(\ldots f_i \ldots)"'] & & TX \arrow[uuu, "\xi"']
\end{tikzcd}$$

This essentially concludes the proof of Proposition 2.

<u>Corollary 2</u>. <u>If $\mathcal{A}^{\mathbb{T}}$ has coequalizers of reflexive pairs, and if $\mathcal{A}$ has all small coproducts, then $\mathcal{A}^{\mathbb{T}}$ has all small colimits</u> (<u>direct limits</u>).

<u>Proof</u>. In the first place, $\mathcal{A}^{\mathbb{T}}$ has small coproducts, because, given $(A_i,\alpha_i) \in |\mathcal{A}^{\mathbb{T}}| (i \in I)$, the pair

$$\begin{tikzcd}
F^{\mathbb{T}}(\oplus TA_i) \arrow[r, "F^{\mathbb{T}}(\ldots T(j_i) \ldots)"] \arrow[rr, bend right, "F^{\mathbb{T}}(\oplus\alpha_i)"'] & F^{\mathbb{T}}\, T(\oplus A_i) \arrow[r, "\mu"] & F^{\mathbb{T}}(\oplus A_i)
\end{tikzcd} ,$$

whose coequalizer, according to Proposition 2, serves as coproduct in $\mathcal{A}^{\mathbf{T}}$ of the family $\{(A_i,\alpha_i) \;/\; i \in I\}$, is reflexive by virtue of

$$\Delta = F^{\mathbf{T}}(\oplus\, \eta_{A_i}) \;.$$

But then, having coproducts and coequalizers of reflexive pairs, $\mathcal{A}^{\mathbf{T}}$ has all small colimits. Indeed, the pair

$$\bigoplus_{\delta\in|\mathcal{D}^{\mathbf{2}}|} D_{\operatorname{dom}\delta} \;\underset{(\ldots j_{\operatorname{dom}\delta}\ \ldots)}{\overset{(\ldots j_{\operatorname{cod}\delta}\cdot D_\delta\ \ldots)_{\delta\in|\mathcal{D}^{\mathbf{2}}|}}{\rightrightarrows}}\; \bigoplus_{i\in|\mathcal{D}|} D_i \quad ,$$

whose coequalizer is well known to serve as colimit of the functor $D : \mathcal{D} \longrightarrow ?$, is reflexive by virtue of

$$\Delta = (\ldots j_{\mathrm{id}_i}\ \ldots)_{i\in|\mathcal{D}|} \;.$$

<u>Remark</u>. If $\mathcal{A}$ is a monoidal category [O] and $\mathbf{T} = (T,\eta,\mu)$ is a suitable triple (meaning at least that $T : \mathcal{A} \to \mathcal{A}$ is a monoidal functor [O], so that there are maps $\tilde{T} : TA \otimes TB \to T(A\otimes B)$ subject to conditions, and η is a monoidal natural transformation, as should probably be μ), then, given $\mathbf{T}$-algebras (A,α), (B,β), a coequalizer (rel $U^{\mathbf{T}}$) of the pair

$$\begin{tikzcd} T(TA\otimes TB) \arrow[r, "T(\tilde{T})"] \arrow[rr, bend right, "T(\alpha\otimes\beta)"'] & TT(A\otimes B) \arrow[r, "\mu"] & T(A\otimes B) \end{tikzcd} \quad ,$$

which is reflexive by virtue of

$$\Delta = T(\eta_A \otimes \eta_B) \quad ,$$

serves equally well as a coequalizer (rel $U^{\mathbf{T}}$) of the pair

$$\begin{tikzcd} TA\otimes TB \arrow[rr, "\tilde{T}"] \arrow[dr, "\alpha\otimes\beta"'] & & T(A\otimes B) \\ & A\otimes B \arrow[ur, "\eta"'] & \end{tikzcd} \quad ,$$

and, if $\mathcal{A}$ is closed monoidal [0] , can be interpreted (in terms of "bilinear maps") as a tensor product, in $\mathcal{A}^{\mathbf{T}}$, of (A,α) and (B,β). Such phenomena hope to be treated in detail elsewhere.

§2. Criteria for the existence of such coequalizers.

In view of §1, it behooves us to find workable sufficient conditions, on $\mathcal{A}$, on $\mathbf{T}$, or on both, that $\mathcal{A}^{\mathbf{T}}$ have coequalizers of reflexive pairs. The first such condition, though rather special, depends on knowing when coequalizers in $\mathcal{A}^{\mathbf{T}}$ can be calculated in $\mathcal{A}$.

Proposition 3. *Let* $\mathbf{T} = (T,\eta,\mu)$ *be a triple in* $\mathcal{A}$, *and let*

$$(A,\alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B,\beta)$$

be a pair of $\mathcal{A}^{\mathbf{T}}$*-morphisms. Assume*

1) *there is an* $\mathcal{A}$*-morphism* $p : B \to C$ *which is a coequalizer (in* $\mathcal{A}$ *) of* $A \underset{g}{\overset{f}{\rightrightarrows}} B$;

2) Tp *is a coequalizer of* (Tf,Tg) ;

3) TTp *is epic*.

Then: there is a map $\gamma : TC \to C$, *uniquely determined by the single requirement that*

$$\begin{array}{ccc} TB & \xrightarrow{Tp} & TC \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow[p]{} & C \end{array}$$

commute; (C,γ) *is a* $\mathbf{T}$*-algebra; and* $p : (B,\beta) \to (C,\gamma)$ *is a coequalizer in* $\mathcal{A}^{\mathbf{T}}$ *of* (f,g).

Proof. The equations $p\circ\beta\circ Tf = p\circ f\circ\alpha = p\circ g\circ\alpha = p\circ\beta\circ Tg$, occurring because of assumption 1, force, because of assumption 2, a unique $\gamma : TC \to C$ with $\gamma Tp = p\beta$.

Then $\gamma \circ \eta_C \circ p = \gamma \circ Tp \circ \eta_B = p \circ \beta \circ \eta_B = p$, but p is epic (by 1) and so $\gamma \circ \eta_C = id_C$. Similarly, using assumption 3, the equation $\gamma \circ T\gamma = \gamma \circ \mu_C$ follows from the calculation

$$\begin{aligned} \gamma \circ T\gamma \circ TTp &= \gamma \circ Tp \circ T\beta = p \circ \beta \circ T\beta \\ &= p \circ \beta \circ \mu_B = \gamma \circ Tp \circ \mu_B \\ &= \gamma \circ \mu_C \circ TTp . \end{aligned}$$

Finally, if $q : B \to X$ is an $\mathcal{A}^{\mathbb{T}}$-morphism from (B,β) to (X,ξ) factoring through C , the factorization must be an $\mathcal{A}^{\mathbb{T}}$-morphism, because the diagram

$$\begin{tikzcd}
TB \arrow[rr] \arrow[dr, "Tp"'] \arrow[dd] & & TX \arrow[dd] \\
& TC \arrow[ur] \arrow[dd] & \\
B \arrow[rr] \arrow[dr] & & X \\
& C \arrow[ur] &
\end{tikzcd}$$

commutes everywhere else, and Tp is epic, by assumption 2.

__Corollary 3__. __If $\mathcal{A}$ has and T preserves coequalizers of reflexive pairs, then $\mathcal{A}^{\mathbb{T}}$ has coequalizers of reflexive pairs, and Corollaries 1 and 2 apply.__

__Examples__. If $\mathbb{T}$ is an adjoint triple, then T preserves all coequalizers (and all colimits, in fact). Samples of such triples:

a) $\mathbb{T} = (- \otimes \Lambda , id \otimes u, id \otimes m)$, where the ground category $\mathcal{A}$ is {k-modules} (k comm.) and Λ is an associative k-algebra with unit $u : k \to \Lambda$ and multiplication $m : \Lambda \otimes \Lambda \to \Lambda (\otimes = \otimes_k)$. $\mathcal{A}^{\mathbb{T}} = \Lambda$-modules.

b) $\mathbb{T} = (- \times \mathbf{2}, id \times (\mathbf{1} \xrightarrow{0} \mathbf{2}), id \times (\mathbf{2} \times \mathbf{2} \xrightarrow{\max} \mathbf{2}))$, where the ground category $\mathcal{A}$ is $\mathcal{C}at$ and $\mathbf{2}$ is the p.o. set $0 \to 1$. $\mathcal{C}at^{\mathbb{T}}$ = {categories with idempotent triples} . Where $\mathbb{S}$ is constructed like $\mathbb{T}$, replacing $\mathbf{2}$ by the category Δ given by

$$|\Delta| = \{0,1,2,\ldots,n,\ldots\}$$

$$\Delta(n,k) = \text{order preserving maps } \{0\ldots\ n-1\} \to \{0\ldots\ k-1\} ,$$

with the obvious composition, $0 : \mathbb{1} \to \Delta$ the inclusion of the object 0, $m : \Delta \times \Delta \to \Delta$ the functor given by

$$
\begin{array}{lcl}
n,\ n' & \longmapsto & n + n' \\
n \xrightarrow{f} k,\ n' \xrightarrow{f'} k' & \longmapsto & \begin{array}{ccc} 0 & \xrightarrow{\quad f \quad} & 0 \\ \vdots & & \vdots \\ n-1 & & k-1 \\ n & & k \\ \vdots & \xrightarrow{k+f'(-\,-n)} & \vdots \\ n+n'-1 & & k+k'-1 \end{array} ,
\end{array}
$$

$\mathcal{C}at^{\mathbf{S}}$ is {categories equipped with a triple} . Define $\tau : \mathbf{S} \to \mathbf{T}$ by crossing with the only functor $\Delta \to \mathbb{2}$ sending $n \neq 0$ to 1 , and 0 to 0 . Then $\mathcal{C}at^{\tau} : \mathcal{C}at^{\mathbf{T}} \to \mathcal{C}at^{\mathbf{S}}$ is the functor interpreting an idempotent triple as a triple on the same category. These constructions and observations are all due to Lawvere. Since $\mathcal{C}at$ has coequalizers and $\mathbf{T}$ is an adjoint triple, $\mathcal{C}at^{\tau}$ has a left adjoint, by Corollary 1 ; roughly speaking, it assigns to a triple in a category, a best idempotent triple on an as closely related other category as possible.

2. Let $\mathcal{A}$ be an additive category, let $m : G \times G \to G$ be an $\mathcal{A}$-morphism satisfying $m(m\times G) = m(G\times m)$, $m(id,0) = id = m(0,id)$. Define a triple $\mathbf{T} = (-\times G,\ (id,0),\ (id\times m))$ on $\mathcal{A}$. Then $\mathbf{T}$ preserves all coequalizers because $A \times G = A \oplus G$.

3. Any functor preserves split coequalizer systems [3]. In particular every triple does, and so Proposition 3 guarantees that coequalizers of $U^{\mathbf{T}}$-split pairs of $\mathcal{A}^{\mathbf{T}}$-morphisms can be computed in $\mathcal{A}$, as was stated in greater generality in [2 ,§6].

The other criterion involves images. We treat images axiomatically, in a manner suggestive of (and perhaps equivalent to) bicategories. Recall that $\mathbb{1}$, $\mathbb{2}$ and $\mathbb{3}$ are the categories depicted as the partially ordered sets

$$
\begin{aligned}
\mathbb{1} &= \{0\} , \\
\mathbb{2} &= \{0 \to 1\} , \\
\mathbb{3} &= \left\{ \begin{tikzcd} 0 \arrow[r, "a"] \arrow[rr, bend right, "c=b\circ a"'] & 1 \arrow[r, "b"] & 2 \end{tikzcd} \right\} .
\end{aligned}
$$

We will need the functors $\mathbf{2} \xrightarrow{c} \mathbf{3}$ and $\mathbf{1} \xrightarrow{1} \mathbf{3}$ (whose values serve as their names). These induce functors $\mathcal{A}^c : \mathcal{A}^{\mathbf{3}} \longrightarrow \mathcal{A}^{\mathbf{2}}$ and $\mathcal{A}^1 : \mathcal{A}^{\mathbf{3}} \longrightarrow \mathcal{A}^{\mathbf{1}} \cong \mathcal{A}$, for any category $\mathcal{A}$.

By an *image factorization functor* for the category $\mathcal{A}$, we mean a functor

$$\mathcal{I} : \mathcal{A}^{\mathbf{2}} \longrightarrow \mathcal{A}^{\mathbf{3}},$$

having the property

1) $\mathcal{A}^{\mathbf{2}} \xrightarrow{\mathcal{I}} \mathcal{A}^{\mathbf{3}} \xrightarrow{\mathcal{A}^c} \mathcal{A}^{\mathbf{2}}$ = identity on $\mathcal{A}^{\mathbf{2}}$, and three more properties which we state using the notations

$$\mathcal{A}^1(\mathcal{I}(f)) = I_f$$
$$\mathcal{I}f = \circ \xrightarrow{f_a} I_f \xrightarrow{f_b} \circ :$$

2) $\qquad f \in |\mathcal{A}^{\mathbf{2}}| \Longrightarrow f_a$ is an epimorphism,

3) $\qquad f \in |\mathcal{A}^{\mathbf{2}}| \Longrightarrow f_b$ is a monomorphism,

4) $\qquad f \in |\mathcal{A}^{\mathbf{2}}| \Longrightarrow (f_b)_a$ and $(f_a)_b$ are isomorphisms.

A functor T *preserves $\mathcal{I}$-images* if there is a natural equivalence, whose composition with $\mathcal{A}^c$ is the identity, between $T^{\mathbf{3}} \circ \mathcal{I}$ and $\mathcal{I} \circ T^{\mathbf{2}}$. This entails, for each $f \in \mathcal{A}(A,B)$, an isomorphism $\iota_f : T(I_f) \to I_{Tf}$ making the triangle

$$\begin{tikzcd}
& I_{TF} \arrow[dr, "(Tf)_b"] & \\
TA \arrow[ur, "(Tf)_a"] \arrow[dr, "T(f_a)"'] & & TB \\
& T(I_f) \arrow[uu] \arrow[ur, "T(f_b)"'] &
\end{tikzcd}$$

commute.

A triple $\mathbf{T} = (T,\eta,\mu)$ on $\mathcal{A}$ preserves $\mathcal{I}$-images if the functor T does.

Lemma 2. If $\mathcal{I} : \mathcal{A}^{\mathbf{2}} \to \mathcal{A}^{\mathbf{3}}$ is an image factorization functor for $\mathcal{A}$ and $\mathbf{T}$ is a triple that preserves $\mathcal{I}$-images, then there is one and only one image factorization functor $\mathcal{I}^{\mathbf{T}}$ for $\mathcal{A}^{\mathbf{T}}$ with the property

$$(U^{\mathbf{T}})^{\mathbf{3}} \circ \mathcal{I}^{\mathbf{T}} = \mathcal{I}.$$

<u>Proof</u>. Given $f : A \to B$, an $\mathscr{A}^{\mathbf{T}}$-morphism from (A,α) to (B,β), the commutativity of the square

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

yields a commutative diagram

$$\begin{tikzcd}
TA \arrow[r, "(Tf)_a"] \arrow[d, "\alpha"'] & I_{Tf} \arrow[r, "(Tf)_b"] \arrow[d, "\mathscr{d}(\alpha,\beta)"'] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f_a"'] & I_f \arrow[r, "f_b"'] & B
\end{tikzcd}$$

Combining this with the commutative diagram arising from the definition of "T preserves $\mathscr{d}$-images", we obtain a map $\gamma = \mathscr{d}(\alpha,\beta) \circ \iota_f : T(I_f) \to I_f$ making the diagram

$$\begin{tikzcd}
TA \arrow[r, "T(f_a)"] \arrow[d, "\alpha"'] & T(I_f) \arrow[r, "T(f_b)"] \arrow[d, "\gamma"] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f_a"'] & I_f \arrow[r, "f_b"'] & B
\end{tikzcd}$$

commute. There is only one such map γ because $T(f_a)$ is epic and f_b is monic. We show now that (I_f,γ) is a $\mathbf{T}$-algebra. Write simply $I = I_f$. $\gamma \circ \eta_I = id_I$ follows from the commutativity of

$$\begin{tikzcd}
I \arrow[r, "f_b"'] \arrow[d, "\eta_I"] & B \arrow[d, "\eta_B"] \arrow[dd, bend left=60, "id_B"] \\
TI \arrow[r, "Tf_b"'] \arrow[d, "\gamma"] & TB \arrow[d, "\beta"] \\
I \arrow[r, "f_b"'] & B
\end{tikzcd}$$

and the fact that f_b is monic. $\gamma \circ T\gamma = \gamma \circ \mu_I$ follows from the commutativity of

$$\begin{tikzcd}
TTI \arrow[r, "TTf_b"] \arrow[d, "T\gamma"] & TTB \arrow[d, "T\beta"'] \arrow[dr, "\mu_B"] & & TTI \arrow[ll, "TTf_b"'] \arrow[d, "\mu_I"] \\
TI \arrow[r, "Tf_b"] \arrow[d, "\gamma"] & TB \arrow[dr, "\beta"] & TB \arrow[d, "\beta"'] & TI \arrow[l, "Tf_b"'] \arrow[d, "\gamma"] \\
I \arrow[rr, "f_b"] & & B & I \arrow[l, "f_b"']
\end{tikzcd}$$

and the fact that f_b is monic. Since $(U^{\mathbf{T}})^{\mathfrak{Z}} \circ \mathcal{L}^{\mathbf{T}} = \mathcal{L}$, the axioms $\mathcal{L}^{\mathbf{T}}$ must satisfy, to be an image factorization functor, are easily verified. The uniqueness assertion is taken care of, essentially, by the obvious uniqueness of $\gamma : TI \to I$, subject to the commutativity relations expressed in the diagram

$$\begin{tikzcd}
TA \arrow[r] \arrow[d] & TI \arrow[r] \arrow[d] & TB \arrow[d] \\
A \arrow[r] & I \arrow[r] & B
\end{tikzcd} .$$

<u>Proposition 4</u>. <u>Let $\mathcal{L}$ be an image factorization functor for $\mathcal{A}$, and let $\mathbf{T}$ be a triple on $\mathcal{A}$ preserving $\mathcal{L}$-images. Assume $\mathcal{A}$ has small products and is co-well-powered (or even just that the isomorphism classes of each class</u>

$$\mathcal{L}\text{-epi }(A) = \{f/f : A \to B,\ f_b : I_f \xrightarrow{\cong} B,\ B \in |\mathcal{A}|\}$$

<u>constitute a set (isomorphisms that are the identity on A, of course)). Then $\mathcal{A}^{\mathbf{T}}$ has all coequalizers</u>.

<u>Proof</u>. Given a pair $(E,\varepsilon) \overset{f}{\underset{g}{\rightrightarrows}} (A,\alpha)$ of $\mathcal{A}^{\mathbf{T}}$-morphisms, let

$$\mathcal{E}_{f,g} = \{h/h \in |\mathcal{A}^{\mathfrak{Z}}|,\ h : (A,\alpha) \to (X,\xi),\ hf = hg,\ h = h_a\} .$$

Observe that an isomorphism class of $\mathcal{E}_{f,g}$ in the sense of $\mathcal{A}^{\mathbf{T}}$ or in the sense of $\mathcal{A}$ is the same thing, because $\mathbf{T}$ preserves $\mathcal{L}$-images, and the maps h, Th are epic. Pick representatives of the isomorphism classes of $\mathcal{E}_{f,g}$, say

$$h_i : (A,\alpha) \longrightarrow (X_i, \xi_i) \qquad (i \in I)$$

and form the induced map (an $\mathcal{A}^{\mathbf{T}}$-morphism by [2, §6]

$$k = \langle \ldots h_i \ldots \rangle : (A,\alpha) \longrightarrow (\Pi_i X_i, \Pi_i \xi_i) .$$

Then k_a is a coequalizer of (f,g) in $\mathcal{A}^{\mathbf{T}}$. Indeed, given $h : (A,\alpha) \to (Z,\zeta)$ with $hf = hg$, we have $h_a \in \mathcal{E}_{f,g}$ and so $\exists i_o \in I$ with $h_a \cong h_{i_o}$. Then the composition

$$(I^{\mathbf{T}})_k \xrightarrow{k_b} (\Pi_i X_i, \Pi_i \xi_i) \xrightarrow{pr_{i_o}} (X_{i_o}, \xi_{i_o}) \xrightarrow{\cong} (I^{\mathbf{T}})_h \xrightarrow{h_b} (Z,\zeta)$$

makes the triangle

$$\begin{tikzcd} (A,\alpha) \arrow[r, "k_a"] \arrow[dr, "h"'] & (I^{\mathbf{T}})_k \arrow[d] \\ & (Z,\zeta) \end{tikzcd}$$

commute, and since k_a is epic, it is the only such map. That $k_a \in \mathcal{E}_{f,g}$ is obvious, and completes the proof of the existence of coequalizers.

<u>Remark</u>. Much the same arguments prove, under the same hypotheses, that $\mathcal{A}^{\mathbf{T}}$ has coequalizers of families of pairs of maps.

<u>Example</u>. $\mathcal{A} = \mathcal{S}$, $\mathcal{U}$ = usual epic-monic factorization. Then any triple preserves $\mathcal{U}$-images (proof below), and consequently $\mathcal{S}^{\mathbf{T}}$ has coequalizers, and, by virtue of Corollary 2, all colimits. That $\mathcal{A}^{\mathbf{T}}$ has all limits if $\mathcal{A}$ has is well known, and this then takes care of the completeness properties of varietal categories.

To see that every triple in $\mathcal{S}$ preserves images, it suffices to see that every triple in $\mathcal{S}$ preserves monomorphisms since the usual epic-monic factorization is determined to within isomorphism by the requirement that it be an epic-monic factorization, and every functor preserves epimorphisms, since they split. So let $\mathbf{T} = (T,\eta,\mu)$ be a triple in $\mathcal{S}$. The only monomorphisms f T has a chance of not preserving are those that are not split, i.e., those with empty domain. Now if $T(\emptyset)$ is $\emptyset$, Tf is surely monic. But if $T\emptyset$ has at least one element, Tf, which may be thought of as a $\mathbf{T}$-morphism from $F^{\mathbf{T}}(\emptyset)$ to $F^{\mathbf{T}}(n)$, admits a retraction, namely the extension to a $\mathbf{T}$-homomorphism of any function

$$n \to U^{\mathbf{T}}F^{\mathbf{T}}\emptyset = T\emptyset \ .$$

That the composition on $F^{\mathbf{T}}(\emptyset)$ is the identity is due to the fact that $F^{\mathbf{T}}(\emptyset)$ is a left zero (is initial, is a copoint) in $\mathcal{S}^{\mathbf{T}}$.

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