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Bireflectivity

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Dedicated to John C. Reynolds, in honour of his 60th birthday

Abstract

Motivated by a model for syntactic control of interference, we introduce a general categorical concept of bireflectivity. Bireflective subcategories of a category $\mathscr{A}$ are subcategories with left and right adjoint equal, subject to a coherence condition. We characterise them in terms of split-idempotent natural transformations on $\mathrm{id}_{\mathscr{A}}$. In the special case that $\mathscr{A}$ is a presheaf category, we characterise them in terms of the domain, and prove that any bireflective subcategory of $\mathscr{A}$ is itself a presheaf category. We define diagonal structure on a symmetric monoidal category which is still more general than asking the tensor product to be the categorical product. We then obtain a bireflective subcategory of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and deduce results relating its finite product structure with the monoidal structure of $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ determined by that of $\mathscr{C}$. We also investigate the closed structure. Finally, for completeness, we give results on bireflective subcategories in $\mathrm{Rel}(\mathscr{A})$, the category of relations in a topos $\mathscr{A}$, and a characterisation of bireflection functors in terms of modules they define.

Keywords: Reflective subcategories; Split-idempotents; Presheaves; Monoidal structure; Categories of relations; Modules

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1. Introduction

This paper is a companion paper to *Syntactic Control of Interference Revisited* [15] in this volume. In this paper, we introduce a general categorical concept, *bireflectivity*, to analyse the properties of the model of the SCIR type system given in [15]. This paper is purely categorical: it can be read independently of [15] as a category theoretic paper. The bireflectivity concept has much wider applicability, but this paper concentrates on our leading example taken from [15]; although we will describe it again here, we will not explain its significance.

The central surprising category theoretic feature of the model of SCIR given in [15] is the concept of a "bireflective" subcategory, by which we mean a subcategory with inclusion having both left and right adjoint, with those adjoints equal, and satisfying an evident coherence condition relating the unit and counit. In [15], the one and only nontrivial bireflective subcategory of the semantic category is the subcategory of *passive* objects. For many categories, such as **Set**, **Poset**, and the category of ω-cpo's, there is no nontrivial such subcategory, and in fact, we prove that any well pointed category has no nontrivial such subcategory.

In this paper, we characterise bireflective subcategories of a category $\mathscr{A}$ as equivalent to split-idempotent natural transformations from the identity functor on $\mathscr{A}$ to itself. The construction implicit in this result uses a limit in the 2-category **Cat**, called an *identifier*. So we describe the notion of identifier, give the construction, and prove our result. In the particular case that $\mathscr{A}$ is a presheaf category $[\mathscr{A}'^{\mathrm{op}}, \mathbf{Set}]$, we prove more: that any bireflective subcategory $\mathscr{B}$ of $\mathscr{A}$ must itself be a presheaf category; and we give an explicit description of a $\mathscr{B}'$ for which $\mathscr{B} = [\mathscr{B}'^{\mathrm{op}}, \mathbf{Set}]$.

The semantic category of [15] is the functor category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ for a certain small category $\mathbf{X}$ of worlds and the category $\mathbf{D}$ of domains. This is a mild variant of a presheaf category, and for our purposes, satisfies the same properties. So although we study presheaf categories in this paper, it is routine to verify that our results all extend to $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$; we may replace in our analysis **Set** by **D**, ordinary categories by **D**-enriched categories, and apply it to the trivially **D**-enriched version of **X**. In fact, our analysis all extends to the $\mathscr{V}$-enriched case for any $\mathscr{V}$ that is cartesian closed, complete and cocomplete. For the ease of exposition, however, we express our results in terms of **Set** and ordinary categories.

A specific property of presheaf categories we use, which also stands in appropriately enriched contexts, is that, if $\mathscr{A}'$ is a small monoidal category, then $[\mathscr{A}'^{\mathrm{op}}, \mathbf{Set}]$ is the free monoidal cocompletion of $\mathscr{A}'$ [5, 9, 15]. With a little more structure on $\mathscr{A}'$, which we call *diagonal* structure, we can construct an idempotent natural transformation from $\mathrm{id}_{\mathscr{A}'}$ to itself, and hence a split one from $\mathrm{id}_{[\mathscr{A}'^{\mathrm{op}}, \mathbf{Set}]}$ to itself, thus yielding a bireflective subcategory of $[\mathscr{A}'^{\mathrm{op}}, \mathbf{Set}]$. For an example, in [15], the category of worlds is a small monoidal category with diagonal structure, and generalising mildly from **Set** to the category of domains, our construction yields the monoidal structure on the semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ of [15] and its restriction to the passive objects. Here, we use diagonal structure to deduce several results about the interaction of the bireflective subcategory

$\mathscr{B}$ with $\mathscr{A}$: both adjunctions between them are monoidal adjunctions; $\mathscr{B}$ has finite products given by the restriction of the tensor product on $\mathscr{A}$; $\mathscr{B}$ is contained in the category of commutative comonoids on $\mathscr{A}$; and $\mathscr{B}$ is an exponential ideal of $\mathscr{A}$. These results are central to the analysis of [15].

As an extended example beyond the presheaf case, we study the bireflective subcategories of $\mathrm{Rel}(\mathscr{A})$, the category of relations in a topos $\mathscr{A}$. They are shown to correspond to the reflective full subcategories of $\mathscr{A}$ closed under power object formation. As the latter structure has been studied in the literature [7] with an application in logic, this exemplifies the generality and usefulness of the bireflectivity concept. We also give a characterisation of bireflection functors without specified inclusions; this is given in terms of modules those bireflections define.

For this paper, we do not make heavy use of 2-categories beyond their definition; a standard reference to an analysis of the definition is [12] by Kelly and Street.

2. The semantic category

In this section, we recall the semantic definitions of [15].

Definition 1 (*The category of worlds*). The category $\mathbf{X}$ has as objects countable sets; a morphism (f,R) from X to Y is a function $f: X \to Y$ with an equivalence relation R on X such that $xRy \wedge fx = fy \Rightarrow x = y$; and the composite $(g,S)\circ(f,R)$ is the function $g\circ f$ with the relation T, where $xTy \Leftrightarrow xRy \wedge fxSfy$. The identities are the identity functions with the total relations.

Proposition 2. *Finite product of sets gives a symmetric monoidal structure on* $\mathbf{X}$ *with unit the terminal object.*

Proof. For $(f,R): X \to Z$ and $(g,S): Y \to W$ in $\mathbf{X}$, the tensor $(f,R)\otimes(g,S)$ is given by $(f\times g, R\times S): X\times Y \to Z\times W$, where $(x,y)(R\times S)(x',y') \Leftrightarrow xRx' \wedge ySy'$. The canonical isomorphisms are given by those of finite products with total relations. The singleton set 1 is terminal in $\mathbf{X}$ with the unique morphism t_W from W given by the unique function and the equality relation on W. $\square$

Definition 3. Given $W \in \mathbf{X}$, define the *state change constraint* morphism $\alpha_W : W \to W$ by the identity function and the identity on W.

Note that α is an idempotent natural transformation on $\mathrm{id}_{\mathbf{X}}$. We write $\pi_0 : X\otimes Y \to X$ for $X\otimes t_Y$ and similarly for $\pi_1 : X\otimes Y \to X$.

In [15], the type theory SCIR is modelled in the *semantic category* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, where $\mathbf{D}$ is the category of possibly bottomless ω-complete posets and continuous functions.

Definition 4 (*Passive objects*). For $f \in [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, f is called *passive* if $f\alpha$ is the identity natural transformation on f. The full subcategory $\mathbf{P}$ of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ is given by the passive objects. The full inclusion is written $J : \mathbf{P} \to [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$.

Definition 5. Define a monoidal structure on $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ as follows. For f and g in $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, the tensor product $f \otimes g$ is the functor that sends W in $\mathbf{X}$ to $\{(a,b) \mid a \in fW, b \in gW, a \# b\}$ with the componentwise order, where

$$a \# b \Leftrightarrow \exists u : W \to X \otimes Y \text{in } \mathbf{X}.$$

$$\exists a' \in fX . \exists b' \in gY.$$

$$a = f(\pi_0 u)a' \wedge b = g(\pi_1 u)b',$$

and $u : W \to Z$ in $\mathbf{X}$ to $(f \otimes g)(u)(a,b) = (f(u)a, g(u)b)$. The unit is the terminal object of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$. The associativity and symmetry isomorphisms are given pointwise as for the binary product structure on $\mathbf{D}$.

Through the course of this paper, we prove the following properties of the category of passive objects used in [15].

Proposition 6. (1) *The full subcategory* $\mathbf{P}$ *is both reflective and coreflective in the semantic category* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$; *moreover, the reflector and coreflector coincide.*

(2) *the category* $\mathbf{P}$ *has finite products.*

(3) *the symmetric monoidal structure on* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ *restricts to the cartesian structure on* $\mathbf{P}$.

(4) *both the inclusion and the reflector (=coreflector) are strong symmetric monoidal functors, i.e., they preserve the monoidal structure up to coherent isomorphism.*

(5) $\mathbf{P}$ *is an exponential ideal of* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, *i.e., given* $P \in \mathbf{P}$ *and* $A \in [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, *the exponential object* $[A, P]$ *lies in* $\mathbf{P}$.

For the precise definition of strong monoidal functors, see [9].

Of course, one could prove these results directly, rather than by appeal to the abstract theory we develop here. However, it seems likely that other models of syntactic control of interference will be developed in future, so rather than having to prove such results every time one discovers a new model, it seems useful to have a general result from which one can deduce them automatically. Moreover, our general results provide necessary and sufficient conditions for the natural level of generality of the arguments, so they set parameters to the search for models that satisfy the properties we study.

Remark 7. Robin Cockett and Robert Seely have pointed out (personal communication) that a second tensor can be defined on the category $\mathbf{X}$ of worlds: on objects it yields the disjoint union of the sets, and on morphisms, yields the sum of the function parts and the “join” of the equivalence-relation parts. The second tensor also lifts to the semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ which, together with the bireflective subcategory $\mathbf{P}$, provides an example of a weakly distributive model of negation-free linear logic [3], with ! and ? both given by the bireflector. This construction cannot be non-trivially generalised to model *full* linear logic, for if the semantic category were $*$-autonomous, the bireflective subcategory (which is both the category of algebras for ? and the category of co-algebras for !) would be both cartesian closed and co-cartesian closed, and hence degenerate.

3. Bireflectivity

In this section, we define the notion of bireflective subcategory and characterise bireflective subcategories in a given category. After giving a few examples, we use this characterisation to show that any bireflective subcategory of a presheaf category is itself a presheaf category. So in particular, the category of passive objects of the previous section is a presheaf category. In fact, it follows from our analysis that it is the *only* nontrivial bireflective subcategory of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$. We also use the characterisation to simplify the condition for bireflections to be monoidal adjunctions.

Definition 8. A *bireflective* subcategory of a category $\mathscr{A}$ is a subcategory $\mathscr{B}$ of $\mathscr{A}$ with inclusion $J:\mathscr{B}\to\mathscr{A}$ that has left and right adjoints equal, say $S:\mathscr{A}\to\mathscr{B}$, with

$$\begin{tikzcd} JSA \arrow[r, "\varepsilon'_A"] \arrow[dr, "\mathrm{id}"'] & A \arrow[d, "\eta_A"] \\ & JSA \end{tikzcd}$$

commuting, where η is the unit of adjunction $S\dashv J$ and ε' is the counit of $J\dashv S$.

A full subcategory $\mathscr{B}$ of $\mathscr{A}$ is said to be closed under subobject formation if, for any monomorphism $f:A\to B$ in $\mathscr{A}$ with $B\in\mathscr{B}$, there is an object $A'\in\mathscr{B}$ with $A'\cong A$. It is closed under quotient formation if $\mathscr{B}^{\mathrm{op}}$ is closed under subobject formation in $\mathscr{A}^{\mathrm{op}}$.

Proposition 9. *Any bireflective subcategory $\mathscr{B}$ of a category $\mathscr{A}$ is full, closed under subobject formation, and closed under quotient formation.*

Proof. With notation as in Definition 8, for $B\in\mathscr{B}$, $\eta_{JB} = \eta_{JB}\varepsilon'_{JB}J\eta'_B = J\eta'_B$. So, for any $f:JB\to JC$ in $\mathscr{A}$, $f = J(\overline{f}\eta'_B)$ with $\overline{f}$ the transposition of f under $S\dashv J$. For subobject formation, let $m:A\to JB$ be a monomorphism with $A\in\mathscr{A}$ and $B\in\mathscr{B}$. We show that $\varepsilon'_A\eta_A = \mathrm{id}_A$. This is equivalent to $m\varepsilon'_A\eta_A = m$; using the coherence condition,

$m = J(\overline{m})\eta_A = J(\overline{m})\eta_A\varepsilon'_A\eta_A = m\varepsilon'_A\eta_A$, with $\overline{m}$ the transposition of m. Closure under quotient formation is proved dually. □

The following can be checked by routine calculations.

Proposition 10. *The coherence condition* $\eta\varepsilon' = \mathrm{id}$ *in Definition* 8 *is equivalent to any of the following.*
(1) $\varepsilon S = S\varepsilon' : SJS \to S$.
(2) $S\eta = \eta' S : S \to SJS$.
(3) *For* $A \in \mathscr{A}$ *and* $B \in \mathscr{B}$,

$$\begin{tikzcd}
\mathscr{B}(B,SA) \arrow[r, phantom, "\overset{(J\dashv S)}{\cong}"] \arrow[d, "J"'] & \mathscr{A}(JB,A) \arrow[d, "S"] \\
\mathscr{A}(JB,JSA) \arrow[r, phantom, "\underset{(S\dashv J)}{\cong}"] & \mathscr{B}(SJB,SA)
\end{tikzcd}$$

Proposition 11. *The bireflective-subcategory relation is transitive.*

Remark 12. The adjointness $S \dashv J \dashv S$ does not imply the coherence condition. For a natural counter example, consider $\mathscr{B}$ = the category of commutative monoids and $\mathscr{A} = \mathscr{B} \times \mathscr{B}$. The diagonal functor $\Delta : \mathscr{B} \to \mathscr{B} \times \mathscr{B} = \mathscr{A}$ is faithful, giving $\mathscr{B}$ as a subcategory of $\mathscr{A}$. Given a commutative comonoid M and N, the binary direct sum $M \oplus N$ has the base set $M \times N$ and multiplication $(m,n)(m',n') = (mm',nn')$. It is both the product and coproduct of M and N in $\mathscr{B}$ (the projections are obvious, and the coprojection are $i : m \mapsto (m, e_N)$ and $j : n \mapsto (e_M, n)$ where e_M and e_N are the unit of M and N, respectively). So, the functor $\oplus : \mathscr{A} \to \mathscr{B}$ is a left and right adjoint to Δ. However, the composition $\Delta(M \oplus N) = (M \oplus N, M \oplus N) \xrightarrow{(\pi,\pi')} (M,N) \xrightarrow{(i,j)} (M \oplus N, M \oplus N)$ is not the identity, sending $((m,n),(m',n'))$ to $((m,e_N),(e_M,n'))$.

An idempotent $f : A \to A$ in a category is called *split* if there exist $g : A \to B$ and $h : B \to A$ with $hg = f$ and $gh = \mathrm{id}_B$. The splitting is unique up to isomorphism in that, for any other splitting $(B', g' : A \to B', h' : B' \to A)$ of f, there is an isomorphism $c : B \to B'$ with $cg = g'$ and $h'c = h$.

In this and subsequent sections, endo-natural transformations whose components are all split idempotents play a central role. We call such a natural transformation a *split-idempotent natural transformation.*

Theorem 13. *Given a category* $\mathscr{A}$, *a bireflective subcategory of* $\mathscr{A}$ *with specified adjunction corresponds bijectively to a split-idempotent natural transformation on* $\mathrm{id}_{\mathscr{A}}$ *with specified splitting.*

In order to prove this, we need the construction of a bireflective subcategory from a split-idempotent natural transformation on $\mathrm{id}_{\mathscr{A}}$. This is given by a limit in the 2-category **Cat**, called an *identifier*.

Definition 14. Let $\mathscr{K}$ be a 2-category and

$$
\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}
$$

be a 2-cell in it. The *identifier* of α is the universal 1-cell $h : Z \to X$ such that $fh = gh$ and $\alpha h = \mathrm{id} : fh \Rightarrow gh$.

Spelling this out, h has two properties:

(1) given $k : W \to X$ such that αk is an identity 2-cell, there exists a unique 1-cell $k^\alpha : W \to Z$ such that

$$
\begin{tikzcd} Z \arrow[r, "h"] & X \\ W \arrow[u, dashed, "k^\alpha"] \arrow[ur, "k"'] & \end{tikzcd}
$$

commutes.

(2) given $k, k' : W \to X$ with αk and $\alpha k'$ both identities, and $\beta : k \Rightarrow k'$, there exists a unique 2-cell $\beta^\alpha : k^\alpha \Rightarrow k'^\alpha$ such that $h\beta^\alpha = \beta$.

Identifiers are limits in 2-categories, as explained in Kelly's article [11].

Proof of Theorem 13. Let $\alpha : \mathrm{id}_{\mathscr{A}} \Rightarrow \mathrm{id}_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ be a split-idempotent natural transformation, $J : \mathscr{A}^\alpha \to \mathscr{A}$ the identifier of α, and $\mathrm{id}_{\mathscr{A}} \overset{r}{\Longrightarrow} R \overset{j}{\Longrightarrow} \mathrm{id}_{\mathscr{A}}$ the splitting of α. Then $\alpha R = \mathrm{id}_R$. By the universality of the identifier J, one has a unique functor $S : \mathscr{A} \to \mathscr{A}^\alpha$ with $R = JS$. The adjunction $S \dashv J$ is given by

$$
\begin{aligned}
&\mathscr{A}^\alpha(SA, B) \cong \mathscr{A}(A, JB)\\
&f \mapsto Jf \circ r_A\\
&J^{-1}_{SA,B}(g \circ j_A) \leftarrow\!\shortmid g
\end{aligned}
$$

with unit r. Applying the same argument to $\mathscr{A}^{\mathrm{op}}$, one obtains $J \dashv S$ with counit j. For the reverse direction, $\varepsilon' \cdot \eta$ gives the desired split idempotent. It is easy to verify these constructions are mutually inverse. $\square$

Theorem 13 allows one to replace an analysis of bireflective subcategories by that of split-idempotent natural transformations, which is often easier.

Example 15. The category of finite semilattices is bireflective in the category of finite commutative semigroups. First note that any one-generator finite commutative semigroup G has exactly one idempotent. With the additive notation, let G be generated by x with relation $ix=(i+k)x$ $(i,k>0)$. There is a unique h with $0\leqslant h<k$ and $k|(i+h)$, say $nk=i+h$. Since $(i+h)x+kx=(i+k)x+hx=(i+h)x$, one has $2(i+h)x=(i+h)x+nkx=(i+h)x$, i.e., $(i+h)x$ is an idempotent. For unicity, if jx is also an idempotent, $jx=(i+h)(jx)=j((i+h)x)=(i+h)x$. Given a finite commutative semigroup G and $x\in G$, let x' be the unique idempotent in $\langle x\rangle\subset G$, the finite subsemigroup of G generated by x. The function $\alpha_G:x\mapsto x'$ is an endomorphism on G since given $x,y\in G$, $x'+y'$ is an idempotent in $\langle x+y\rangle\subset G$, and hence $(x+y)'=x'+y'$ by the uniqueness. The uniqueness also implies that α_G is natural in G. Finally, α_G splits with the retract $\{x\in G\,|\,x+x=x\}$, which is a semilattice with the order $x\leqslant y\Leftrightarrow x+y=y$. Similarly, semilattices in the category of torsion commutative semigroups form a bireflective subcategory. One may also replace semigroups by monoids.

Example 16. The category **Rel** of sets and relations is bireflective in **SProc**, the interaction category of synchronous processes [1, Proposition 3.0.4]. In short, an object A of **SProc** is a pair (Σ_A,S_A) of sets with S_A a nonempty prefix closed subset of Σ_A^*; a morphism from A to B is a strong bisimilar class of $\Sigma_A\times\Sigma_B$-labelled transition systems whose traces are contained in $S_A\times S_B$ in the obvious sense; the composite $(P;Q):A\xrightarrow{P}B\xrightarrow{Q}C$ is given by "synchronisation" at B, i.e., for $(a,c)\in\Sigma_A\times\Sigma_C$, there is a $(P;Q)$-transition $(p;q)\xrightarrow{(a,c)}(p';q')$ if and only if there exists $b\in\Sigma_B$ with a P-transition $p\xrightarrow{(a,b)}p'$ and a Q-transition $q\xrightarrow{(b,c)}q'$; and finally, the identity on A is given by the $\Sigma_A\times\Sigma_A$-labelled transition system whose traces are $\{$"$(a_0,a_0)(a_1,a_1)\cdots$"$\,|\,$"$a_0a_1\cdots$"$\in S_A\}$. Given an object A, let $S_A|_n$ be the subset of S_A given by the strings of length at most n. There is a trivial, one-step $\Sigma_A\times\Sigma_A$-labelled transition system α_A with 'start' $\xrightarrow{(a,b)}$ 'end' $\Leftrightarrow a=b\wedge a\in S_A|_1$. For $P:A\to B$, both $\alpha_A;P$ and $P;\alpha_B$ are bisimilar to the transition system P "truncated" to at most one-step. So $\alpha_A:A\to A$ is natural in $A\in$ **SProc**. This also splits, giving the retract $((S_A|_1)',S_A|_1)$, where $(S_A|_1)'$ is $S_A|_1$ minus the empty string. The statement at the beginning holds since the full subcategory of **SProc** given by those A with α_A the identity transition system is precisely **Rel**.

Given a 2-category $\mathscr{K}$, a coidentifier in $\mathscr{K}$ is an identifier in $\mathscr{K}^{\mathrm{op}}$, reversing the 1-cells in the definition. For our main example of a coidentifier,

Example 17. Let $\mathscr{C}$ be a category, and let $\alpha:\mathrm{id}\Rightarrow\mathrm{id}:\mathscr{C}\to\mathscr{C}$ be an idempotent natural transformation. Then, the coidentifier $\mathscr{C}_\alpha$ is given by factoring $\mathscr{C}$ by the congruence $\sim$, where for $f,g:A\to B$,

$$f\sim g\;\Leftrightarrow\;\alpha_B\cdot f=\alpha_B\cdot g.$$

To see this, first observe that $\sim$ is a congruence on $\mathscr{C}$: it is obviously an equivalence on each hom-set $\mathscr{C}(A,B)$; it respects composition in $\mathscr{C}$ because α is natural. Now if α is identified with the identity and $f \sim g$, then f is identified with $\alpha_B \cdot f = \alpha_B \cdot g$, which is identified with g. Conversely, $\alpha_B \cdot \mathrm{id}_B = \alpha_B = \alpha_B \cdot \alpha_B$, so $\mathrm{id}_B \sim \alpha_B$.

So, we may describe the coidentifier $\mathscr{C}_\alpha$ of α by

$$\mathrm{Ob}(\mathscr{C}_\alpha) = \mathrm{Ob}(\mathscr{C})$$
$$\mathscr{C}_\alpha(A,B) = \mathscr{C}(A,B)/\sim \quad \text{where } f \sim g \text{ if and only if } \alpha_B \cdot f = \alpha_B \cdot g.$$

Given any functor $h : \mathscr{A} \to \mathscr{B}$ between small categories, one has a functor $[h,\mathscr{D}] : [\mathscr{B},\mathscr{D}] \to [\mathscr{A},\mathscr{D}]$. If $\mathscr{D}$ is complete and cocomplete, $[h,\mathscr{D}]$ has left and right adjoints given by left and right Kan extension (see, e.g., [14]). If $h : \mathscr{A} \to \mathscr{B}$ is the coidentifier of a natural transformation between functors whose value is equal on objects, it follows from the universal property that $[h,\mathscr{D}]$ is fully faithful, exhibiting $[\mathscr{B},\mathscr{D}]$ as equivalent to a full subcategory of $[\mathscr{A},\mathscr{D}]$. That subcategory is given by those $f : \mathscr{A} \to \mathscr{D}$ such that $f\alpha = \mathrm{id}$, where α is the natural transformation.

Proposition 18. *Given a category $\mathscr{D}$ where every idempotent splits and a natural idempotent $\alpha : \mathrm{id} \Rightarrow \mathrm{id} : \mathscr{C} \to \mathscr{C}$, the full inclusion $[h,\mathscr{D}] : [\mathscr{C}_\alpha,\mathscr{D}] \to [\mathscr{C},\mathscr{D}]$ exhibits $[\mathscr{C}_\alpha,\mathscr{D}]$ as a bireflective subcategory of $[\mathscr{C},\mathscr{D}]$. The adjoint of $[h,\mathscr{D}]$ takes f to the splitting of $f\alpha : f \Rightarrow f$.*

Proof. This follows by using $[-,\mathscr{D}] : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$ to send colimits in **Cat** to limits, hence coidentifiers to identifiers, and by applying the construction of Theorem 13. Note that $[\alpha,\mathscr{D}]$ always splits. $\square$

Replacing bireflective subcategories by idempotent natural transformations also makes clearer the constraints in finding nontrivial examples. First we recall

Definition 19. A fully faithful functor $Z : \mathscr{G} \to \mathscr{C}$ is *generating* if the functor $\tilde{Z} : \mathscr{C} \to [\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$, $C \mapsto \mathscr{C}(Z-, C)$, is faithful.

Spelling this out, a full subcategory $\mathscr{G}$ of $\mathscr{C}$ generates $\mathscr{C}$ if for any parallel pair of distinct maps $f, g : A \to B$ in $\mathscr{C}$, there exists an object X of $\mathscr{C}$ and a map $h : X \to A$ such that fh and gh are distinct. For example, the unit category $\{1\}$ is generating in **Set** and in **Poset**, and the arrow category is generating in **Cat**.

A category $\mathscr{C}$ with a terminal object 1 is *well-pointed* if the inclusion $\{1\} \to \mathscr{C}$ is generating.

Proposition 20. *Given a generating functor $Z : \mathscr{G} \to \mathscr{C}$, any endo-natural transformation on $\mathrm{id}_{\mathscr{C}}$ is uniquely determined by its restriction to Z.*

Proof. Given $s, t : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$, $sZ = tZ$ implies for each $C \in \mathscr{C}$, $G \in \mathscr{G}$, and $f : ZG \to C$, by naturality, $s_C f = f s_{ZG} = f t_{ZG} = t_C f$. $\square$

So, there are no more endo-natural transformations on $\mathrm{id}_{\mathscr{C}}$ than there are on Z.

Corollary 21. *For a well-pointed category $\mathscr{C}$, there is no nontrivial idempotent natural transformation on* $\mathrm{id}_{\mathscr{C}}$.

Proof. The only natural transformation on the inclusion of $\{1\}$ is the identity on 1. □

Remark 22. In our category $\mathbf{X}$ of worlds (Definition 1), $\{1\}$ is not a generator (as one cannot distinguish two morphisms which differ only in their equivalence relation parts), but $\{2\}$, the one object subcategory of $\mathbf{X}$ given by the two element set 2, is. Applying the above proposition, there are at most six natural transformations on $\mathrm{id}_{\mathbf{X}}$, of which four can be idempotent. By examining each, one can conclude $\alpha_X = (\mathrm{id}_X, \Delta_X)$ is the only idempotent natural transformation on $\mathrm{id}_{\mathbf{X}}$ other than the identity.

We may use Remark 22 to deduce that our semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ has only one nontrivial bireflective subcategory, which is of course the subcategory of passive objects. In fact, we show a stronger result: to give an (split-)idempotent natural transformation on $\mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}$ is to give an idempotent natural transformation on $\mathrm{id}_{\mathscr{C}}$. This gives a converse to Proposition 18 in case the base category is **Set**. The lifting of this result from **Set** to **D** is routine: we give it for **Set** for ease of exposition.

Proposition 23. *For a small category $\mathscr{C}$, the idempotent natural transformations on* $\mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}$ *are in bijection with those on* $\mathrm{id}_{\mathscr{C}}$.

Proof. Given an idempotent natural transformation $\alpha : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$, it extends to $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ by homming α^{op} into **Set**. Now given any idempotent natural transformation $\beta : \mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]} \Rightarrow \mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}$, by the fact that every $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is a colimit of representables, β is fully determined by its behaviour on representables. Thus every such β arises from a unique $\alpha : \mathrm{id}_{\mathscr{C}} \Rightarrow \mathrm{id}_{\mathscr{C}}$. □

Example 24. Given a monoid M with zero element 0 ($x0 = 0x = 0$ for all $x \in M$), **Set** is bireflective in the category of M-sets, corresponding to the idempotent 0 on id_M with M regarded as a one object category. The inclusion sends a set A to the M-set $(A, (x \in M, a \in A) \mapsto a)$.

Remark 25. Bireflectivity seems to be the distinctive categorical property that differentiates the model based on the category $\mathbf{X}$ from other extant examples of functor category semantics. For example, Oles originally used a subcategory $\mathbf{X}'$ of $\mathbf{X}$ with the same objects, the maps (f, R) being those where the restriction of f to any R-equivalence class is bijective. Clearly, this rules out the state change constraint endomorphisms α_W, and so there is only one natural idempotent on $\mathrm{id}_{\mathbf{X}'}$, namely the identity. As a result, the functor category used by Oles possesses no nontrivial bireflective subcategories.

In our application for models of SCIR [15], we need both reflection and coreflection to be monoidal adjunctions. Theorem 13 can be used to replace this requirement by a simpler condition on the corresponding split-idempotent natural transformation.

Proposition 26. *Let $\mathscr{A}$ be a monoidal category with unit being the terminal object* 1 *and $\alpha : \mathrm{id}_{\mathscr{A}} \Rightarrow \mathrm{id}_{\mathscr{A}}$ a split-idempotent natural transformation with $\alpha_{A\otimes B} = \alpha_A \otimes \alpha_B$ ($A, B \in \mathscr{A}$). Then, the bireflective subcategory $\mathscr{B}$ of $\mathscr{A}$ corresponding to α has a monoidal structure with unit the terminal object, and the reflection and the coreflection are monoidal adjunctions with respect to this monoidal structure.*

Proof. Let J and S be the inclusion and reflection (=coreflection), η and ε the unit and counit of $S \dashv J$, η' and ε' those of $J \dashv S$ (so α splits as $\varepsilon'\eta$). For $B, C \in \mathscr{B}$, define $B \otimes' C \in \mathscr{B}$ by $B \otimes' C = S(JB \otimes JC)$. It is easy to check that, together with the terminal object $S1 \in \mathscr{B}$ as unit, this gives a monoidal structure on $\mathscr{B}$.

First we check that J is strong monoidal with respect to the tensor $\otimes'$, defining the comparison morphisms by

$$JB \otimes JC \xrightarrow{\eta_{JB\otimes JC}} JS(JB \otimes JC) = J(B \otimes' C),$$
$$1 \xrightarrow{\eta_1} J(S1).$$

Since JS is a right adjoint, η_1 is clearly an isomorphism. To see $\eta_{JB\otimes JC}$ is an isomorphism, note that η_{JB}, η_{JC}, ε'_{JB}, and ε'_{JC} are all isomorphisms because of the fullyfaithfulness of J and the coherence condition. So, $\varepsilon'_{JB\otimes JC}\eta_{JB\otimes JC} = \alpha_{JB\otimes JC} = \alpha_{JB} \otimes \alpha_{JC} = \varepsilon'_{JB}\eta_{JB} \otimes \varepsilon'_{JC}\eta_{JC}$ is an isomorphism, hence $\eta_{JB\otimes JC}$ is a split monomorphism. By the coherence condition, it is also an epimorphism, therefore an isomorphism. The coherence axioms for these comparison morphisms are checked by routine calculations.

Next, we define the comparison morphisms for S in the only way that makes $J \dashv S$ a monoidal adjunction. The comparison for the unit is the identity id_{S1}. For $A \otimes B \in \mathscr{A}$, the comparison morphism is the transposition of

$$J(SA \otimes' SB) \xrightarrow{\cong} JSA \otimes JSB \xrightarrow{\varepsilon'_A \otimes \varepsilon'_B} A \otimes B,$$

i.e.,

$$SA \otimes' SB \xrightarrow{\eta'} SJ(SA \otimes' SB) \xrightarrow{\cong} S(JSA \otimes JSB) \xrightarrow{S(\varepsilon'_A \otimes \varepsilon'_B)} S(A \otimes B).$$

This is an isomorphism since η' and $S(\varepsilon'_A \otimes \varepsilon'_B)$ are; the latter is checked similarly to $\eta_{JB\otimes JC}$.

Finally, one can check that the above data make $S \dashv J$ a monoidal adjunction by routine calculations. □

4. Diagonal categories

In this section, we define *diagonal* structure on a symmetric monoidal category. A diagonal structure consists of the data and some of the axioms required to force the monoidal structure to be finite product structure. Of course, the category of worlds **X** has diagonal structure, as does any category with finite products. From diagonal structure, one can obtain an idempotent natural transformation that, in a precise sense, measures the extent to which the diagonal structure fails to be finite product structure. This idempotent allows us to define a bireflective subcategory of the presheaf category as in the previous section, and the diagonal structure further allows us to deduce results such as that the monoidal structure on the presheaf category restricts to finite product structure on the bireflective subcategory, and that the adjunction becomes a monoidal adjunction.

Definition 27. A *diagonal* category is a symmetric monoidal category $\mathscr{C}$ whose unit is the terminal object of $\mathscr{C}$, together with a natural transformation with components $\delta_A : A \to A \otimes A$, called the *diagonal* morphism on A, such that

$$\begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & A \otimes A \arrow[d, "A \otimes \delta_A"] \\ A \otimes A \arrow[r, "\delta_A \otimes A"'] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\delta_A"'] \arrow[dr, "\delta_A"] & \\ A \otimes A \arrow[rr, "c"'] & & A \otimes A \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes B \arrow[dl, "\delta_{A \otimes B}"'] \arrow[dr, "\delta_A \otimes \delta_B"] & \\ A \otimes B \otimes A \otimes B \arrow[rr, "A \otimes c \otimes B"'] & & A \otimes A \otimes B \otimes B \end{tikzcd}$$

commute.

It is routine to verify that in a diagonal category $\mathscr{C}$, the maps $(t \otimes A) \cdot \delta_A$ form an idempotent natural transformation from $\mathrm{id}_{\mathscr{C}}$ to $\mathrm{id}_{\mathscr{C}}$.

Our leading example of diagonal structure is as follows.

Example 28. On **X**, define $\delta_A : A \to A \otimes A$ by the diagonal together with the total relation.

Example 29. Consider the category $\mathbf{Set}_*$ of pointed sets, together with the symmetric monoidal closed structure given by smash product. A $\mathbf{Set}_*$-category is a category with zero morphisms. Consider any $\mathbf{Set}_*$-category with finite products. Then, the finite products define the symmetric monoidal structure and we may define $\delta_A : A \to A \times A$ to be the zero morphism. Specific examples of such categories are the categories of monoids, of pointed sets, and of ω-cpo's with bottom and bottom preserving maps.

Example 30. Any category with finite products.

It is easy to see that in Examples 28 and 29, the structure is not that of finite products since the diagram

$$\begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[dr, "\mathrm{id}"'] & A \otimes A \arrow[d, "t \otimes A"] \\ & A \end{tikzcd}$$

does not commute. Observe that in Example 28, $(t \otimes A) \cdot \delta_A$ is the state change constraint idempotent α_A. Also in Example 28 and the three specific examples in 29, $(t \otimes A) \cdot \delta_A$ is the only nontrivial idempotent natural transformation on the identity functor by Proposition 20.

Proposition 31. *The data for a diagonal category form the finite product structure if and only if*

$$\begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[dr, "\mathrm{id}"'] & A \otimes A \arrow[d, "t \otimes A"] \\ & A \end{tikzcd}$$

commutes.

Proof. ($\Leftarrow$) In any category with finite products, the composite of the diagonal with the projection must be the identity.

($\Rightarrow$) Given $(f : C \to A, g : C \to B)$, define $h : C \to A \times B$ to be

$$C \xrightarrow{\delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes B.$$

It is routine to verify, using the equation and the terminal object condition, that the appropriate two diagrams commute. Unicity is similar, using the third of the three diagonal commutativities. $\square$

Proposition 32. *Let $\mathscr{C}$ be a diagonal category. Then the free category on $\mathscr{C}$ that forces the diagonal data of $\mathscr{C}$ to be finite products, is given by the coidentifier of the natural transformation determined by $(t\otimes A)\cdot\delta_A : A\to A$.*

Proof. This follows from Proposition 31 because sending the diagonal data to finite product structure necessitates the identification of $(t\otimes A)\cdot\delta_A : A\to A$ with id_A, and such identification with the addition of no further objects or arrows yields a finite product structure. □

Example 33. The construction of Proposition 32 applied to Example 28 yields the category of countable sets. For Example 29, the construction of 32 yields a category equivalent to the unit category. The construction of the category **X** of worlds from the category of countable sets generalizes easily to a construction on any small category with finite limits. One still acquires a diagonal category, and following that construction by that of Proposition 32 returns the original category.

To end this section, we digress briefly to observe that, for general reasons, *any* monoidal structure on **X** giving rise to an idempotent natural transformation on $\mathrm{id}_{\mathbf{X}}$ necessarily has as the unit either the terminal object 1 or the initial object 0. The argument goes as follows.

Proposition 34 (Foltz et al. [6]). *Given a monoidal category $\mathscr{C}$ with unit I, any idempotent f on I extends to an idempotent natural transformation f^* on* $\mathrm{id}_{\mathscr{C}}$; *moreover, $(-)^*$ is injective.*

Proof. Writing r_C for the right identity, one has a monoid homomorphism $(-)^*$: $\mathscr{C}(I,I)\to[\mathscr{C},\mathscr{C}](\mathrm{id}_{\mathscr{C}},\mathrm{id}_{\mathscr{C}})$, $f\mapsto(f^*_C : C\xrightarrow{r_C^{-1}} C\otimes I\xrightarrow{\mathrm{id}_C\otimes f} C\otimes I\xrightarrow{r_C} C)_{C\in\mathscr{C}}$ with right inverse $(-)_I : [\mathscr{C},\mathscr{C}](\mathrm{id},\mathrm{id})\to\mathscr{C}(I,I)$, $t\mapsto t_I$. So, $(-)^*$ is a monomorphism. This easily restricts to idempotents. □

So, natural transformations on $\mathrm{id}_{\mathscr{C}}$ limit the possible monoidal structures on $\mathscr{C}$. In particular,

Remark 35. For the category **X**, there are exactly two idempotent natural transformations $\mathrm{id},\alpha : \mathrm{id}_{\mathbf{X}}\to\mathrm{id}_{\mathbf{X}}$ (Remark 22). So, for any monoidal structure on **X** whose unit has an idempotent on it, the unit is either the terminal object 1 or the initial object 0, since otherwise there would be more than two endomorphisms on the unit.

5. Presheaves

In this section, we take a small diagonal category $\mathscr{C}$, construct the presheaf category on it, and apply the construction of Proposition 18 to the idempotent $(t\otimes A)\cdot\delta_A$ to obtain a bireflective subcategory of $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$. The presheaf category is the free monoidal cocompletion of $\mathscr{C}$. We use this fact, together with the diagonal structure on $\mathscr{C}$, to deduce the relationship between the induced monoidal structure on $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ and finite products in the bireflective subcategory. It follows that the latter is a full subcategory of the category of commutative monoids on $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$.

As explained in the Introduction, we only refer to **Set** as our base category in this section merely for ease of exposition. All our results here extend to the $\mathscr{V}$-enriched case for any cartesian closed, complete and cocomplete category $\mathscr{V}$, if we start with a small diagonal $\mathscr{V}$-category $\mathscr{C}$. Note that every idempotent splits in such $\mathscr{V}$. The category **D** of domains satisfies the conditions, and **X**, in fact any small diagonal category, can be seen trivially as a small diagonal **D**-category, so we can deduce results for our leading example immediately.

Theorem 36 (Im and Kelly [9]). *Let $\mathscr{C}$ be a small* (*symmetric*) *monoidal category. Then, the free* (*symmetric*) *monoidal cocompletion of $\mathscr{C}$ is $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ with* (*symmetric*) *monoidal structure given by left Kan extension*

$$\begin{tikzcd}
\mathscr{C}\times\mathscr{C} \arrow[r, "\mathscr{Y}\times\mathscr{Y}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\cong"] & {[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]\times[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]} \arrow[d, "\mathrm{Lan}_{\mathscr{Y}\times\mathscr{Y}}(\mathscr{Y}\otimes)=\widehat{\otimes}"] \\
\mathscr{C} \arrow[r, "\mathscr{Y}"'] & {[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}
\end{tikzcd}$$

Spelling this out, $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ is cocomplete and $f\widehat{\otimes}-$ and $-\widehat{\otimes}f$ preserve colimits. An explicit formula for $\widehat{\otimes}$ is $(f\widehat{\otimes}g)C=\int^{X,Y\in\mathscr{C}}\mathscr{C}(C,X\otimes Y)\times fX\times gY$, i.e., the coequalizer in **Set**

$$\coprod_{\substack{u\in\mathscr{C}(X_0,X_1)\\ v\in\mathscr{C}(Y_0,Y_1)}}\mathscr{C}(C,X_0\otimes Y_0)\times fX_1\times gY_1 \rightrightarrows \coprod_{X,Y\in\mathscr{C}}\mathscr{C}(C,X\otimes Y)\times fX\times gY\to(f\widehat{\otimes}g)C$$

of the evident two maps $(X_0\xrightarrow{u}X_1,Y_0\xrightarrow{v}Y_1,w,x,y)\mapsto(w,f(u)x,g(v)y)$ and $(u,v,w,x,y)\mapsto((u\otimes v)w,x,y)$. See [14] for a background for coends $\int^{A\in\mathscr{A}}S(A,A)\in\mathscr{B}$ with $S:\mathscr{A}^{\mathrm{op}}\times\mathscr{A}\to\mathscr{B}$ (colimits of a particular type) and their relationship with left Kan extensions.

Remark 37. The monoidal structure on the semantic category $[\mathbf{X}^{\mathrm{op}},\mathbf{D}]$ given in Definition 5 agrees with the monoidal structure determined by Proposition 2 and the extension of Theorem 36 to **D** rather than **Set**.

Now assume $\mathscr{C}$ is a diagonal category. We write α_C for the idempotent $(t\otimes C)\cdot\delta_C$. Let $\mathscr{C}/{\sim}$ be the coidentifier determined by α (Example 17). Consider the following diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "y"] \arrow[d, "h"'] & {[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, shift right, "S"'] \\
\mathscr{C}/{\sim} \arrow[r, "y"'] & {[\mathscr{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]} \arrow[u, hook, shift right, "{[h, \mathbf{Set}]}"']
\end{tikzcd}$$

We have seen by Proposition 18 that $[h,\mathbf{Set}]$ is fully faithful with left and right adjoint equal, given by sending $f : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ to the splitting of $f\alpha$. Moreover, h sends the monoidal structure of $\mathscr{C}$ to finite product structure on $\mathscr{C}/{\sim}$ (Proposition 32). The category $[\mathscr{C}/{\sim}^{\mathrm{op}},\mathbf{Set}]$ is cartesian closed and cocomplete. So, by the universal property of $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$, we have

Proposition 38. *Given the diagram above, S sends $\widehat{\otimes}$ on $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ to finite products on $[\mathscr{C}/{\sim}^{\mathrm{op}},\mathbf{Set}]$.*

Proposition 39. *For any $f,g\in[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$, $(f\widehat{\otimes}g)\alpha = f\alpha\widehat{\otimes}g\alpha$.*

Proof. The families $((f\widehat{\otimes}g)\alpha)_{f,g\in[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}$ and $(f\alpha\widehat{\otimes}g\alpha)_{f,g\in[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]}$ both form natural transformations from $\widehat{\otimes} : [\mathscr{C}^{\mathrm{op}},\mathbf{Set}]\times[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]\to[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ to itself. These two natural transformations are equal if and only if their restrictions under $\mathscr{Y}\times\mathscr{Y}$ to $\mathscr{C}\times\mathscr{C}$ are equal; this is immediate from the definition of $\widehat{\otimes}$ as a left Kan extension. So, it suffices to prove that for each Z in $\mathscr{C}$, and for each $X,Y\in\mathscr{C}$, the maps $\mathscr{C}(Z,\alpha_{X\otimes Y})$: $\mathscr{C}(Z,X\otimes Y)\to\mathscr{C}(Z,X\otimes Y)$ and $\mathscr{C}(Z,\alpha_X\otimes\alpha_Y)$ are equal; but that holds by a routine calculation using the third commutativity in the definition of diagonal category. $\square$

So, the split-idempotent natural transformation $[\alpha^{\mathrm{op}},\mathbf{Set}]$ on $\mathrm{id}_{[\mathscr{C}^{\mathrm{op}},\mathrm{Set}]}$ satisfies the condition of Proposition 26. Putting this together,

Theorem 40. *The full inclusion J sends finite products in $[\mathscr{C}/{\sim}^{\mathrm{op}},\mathbf{Set}]$ to the monoidal structure of $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$, and has left and right adjoint sending f to the splitting of $f\alpha$, sending the symmetric monoidal structure to finite products. So, both $S\dashv J$ and $J\dashv S$ are monoidal adjunctions.*

Corollary 41. *$[\mathscr{C}/{\sim}^{\mathrm{op}},\mathbf{Set}]$ is a full subcategory of the category of commutative comonoids in $([\mathscr{C}^{\mathrm{op}},\mathbf{Set}],\widehat{\otimes})$.*

Proof. Since $\widehat{\otimes}$ restricts to finite products on $[\mathscr{C}/\sim^{\mathrm{op}}, \mathbf{Set}]$, each object f of $[\mathscr{C}/\sim^{\mathrm{op}}, \mathbf{Set}]$ possesses a unique commutative comonoid structure. Fully faithfulness is obvious. $\square$

In the particular case of Example 28, calculation of the formula for $\widehat{\otimes}$ reveals that $\mathbf{P}$ is precisely the category of commutative comonoids.

6. Closedness

In this section, we address closed structure. None of our results here strictly requires the fact that we have a bireflective subcategory; in fact, they do not require we have presheaves either. However, the leading example is, as through the course of the paper, the inclusion of $\mathbf{P}$ into $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$. Recall from the previous section, that given a small symmetric monoidal category $\mathscr{C}$, the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is the free symmetric monoidal cocompletion of $\mathscr{C}$. In fact, more is true: $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is symmetric monoidal closed. That result, together with all our analysis of the previous section, extends to small symmetric monoidal $\mathscr{V}$-categories, provided $\mathscr{V}$ is *locally presentable as a Cartesian closed category* (see [10]). The category of domains is such a category, so for general reasons, $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ is symmetric monoidal closed.

To prove the results of this section, we consider a more general situation. A full subcategory $\mathscr{B}$ of a symmetric monoidal closed $\mathscr{A}$ is called an exponential ideal if, for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$, (an isomorphic copy of) $[A, B]$ lies in $\mathscr{B}$ ([8]).

Proposition 42. *Let $\mathscr{A}$ and $\mathscr{B}$ be symmetric monoidal closed, with $J : \mathscr{B} \to \mathscr{A}$ a full inclusion with left adjoint F preserving symmetric monoidal structure up to coherent isomorphism. Then $\mathscr{B}$ is an exponential ideal of $\mathscr{A}$.*

Proof. It suffices to show that for any X in $\mathscr{B}$ and A in $\mathscr{A}$, $[A, JX]_{\mathscr{A}}$ lies in the image of J. To see that, apply Yoneda to the following sequence of natural isomorphisms, for any C in $\mathscr{A}$:

$$\begin{aligned} \mathscr{A}(C, [A, JX]_{\mathscr{A}}) &\cong \mathscr{A}(C \otimes A, JX) \cong \mathscr{B}(F(C \otimes A), X) \\ &\cong \mathscr{B}(FC \otimes FA, X) \cong \mathscr{B}(FC, [FA, X]_{\mathscr{B}}) \\ &\cong \mathscr{A}(C, J[FA, X]_{\mathscr{B}}). \qquad \square \end{aligned}$$

One can also show that given any full coreflective subcategory $\mathscr{B}$ of a symmetric monoidal closed category $\mathscr{A}$ such that $\mathscr{B}$ is closed under the monoidal structure of $\mathscr{A}$, then $\mathscr{B}$ is symmetric monoidal closed. This allows us to deduce

Theorem 43. *Let $\mathscr{B}$ be a full reflective and coreflective subcategory of symmetric monoidal closed $\mathscr{A}$, and assume $\mathscr{B}$ is closed under the monoidal structure of $\mathscr{A}$ and* *the left adjoint preserves the symmetric monoidal structure. Then, $\mathscr{B}$ is symmetric monoidal closed and is in fact an exponential ideal of $\mathscr{A}$.*

Putting this together with earlier results, we may conclude

Corollary 44. *Any small diagonal category $\mathscr{C}$ induces a bireflective subcategory $\mathscr{B}$ of* $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, *such that $\mathscr{B}$ is a presheaf category, hence cartesian closed, and an exponential ideal in* $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, *with both the inclusion and adjoint preserving the symmetric monoidal structure.*

7. Bireflective subcategories of Rel($\mathscr{A}$)

This section is devoted to an extended example of bireflectivity in Rel($\mathscr{A}$), where $\mathscr{A}$ is a regular category and Rel($\mathscr{A}$) is the category of relations in $\mathscr{A}$.

First we recall some definitions (see, e.g., [8]). A morphism $f : A \to B$ in a category $\mathscr{A}$ is a *cover* if it does not factor through any proper subobject of B. Given $g : A \to C$, the *image* of g, if it exists, is a monomorphism $m : B \to C$ together with a cover $f : A \to B$ such that $g = mf$; such factorisation is unique up to isomorphism. A category $\mathscr{A}$ is *regular* if it has finite limits, if all morphisms in $\mathscr{A}$ has images, and if the pullback of a cover along an arbitrary morphism is a cover. Given a regular category $\mathscr{A}$, the category Rel($\mathscr{A}$) of *relations* in $\mathscr{A}$ has the same objects as $\mathscr{A}$; a morphism from A to B in Rel($\mathscr{A}$) is a subobject of $A \times B$ in $\mathscr{A}$, i.e., a monic pair (f, g) with $f : R \to A$ and $g : R \to B$ in $\mathscr{A}$ for some R and two pairs (f, g) and (k, l) are identified if $k = fh$ and $l = gh$ for some isomorphism h. We write $A \nrightarrow B$ for a morphism in Rel($\mathscr{A}$). Given $(f, g) : A \nrightarrow B$ and $(k, l) : B \nrightarrow C$, the composition $(f, g); (k, l) : A \nrightarrow C$, written in the diagrammatic order, is given by first taking the pullback of g and k in $\mathscr{A}$, say with projections k' and g', and then taking the image of $\langle fk', lg' \rangle$; the result of the composition is not dependent on the choice of representing monic pairs. The *reciprocation* $(-)^{\circ} : \mathrm{Rel}(\mathscr{A})^{\mathrm{op}} \to \mathrm{Rel}(\mathscr{A})$ sends $(f, g) : A \nrightarrow B$ to $(g, f) : B \nrightarrow A$, exhibiting Rel($\mathscr{A}$) as a self-dual category. A morphism $(\mathrm{id}_A, f) : A \nrightarrow B$ with $f : A \to B$ in $\mathscr{A}$ is called a *map*. For a subcategory $\mathscr{B}$ of Rel($\mathscr{A}$), Map($\mathscr{B}$) is the subcategory of $\mathscr{A}$ given by maps in $\mathscr{B}$; $\mathscr{A}$ may be identified with the subcategory Map(Rel($\mathscr{A}$)) of Rel($\mathscr{A}$).

Given a regular category $\mathscr{A}$, let $U \in \mathscr{A}$ be a subobject of the terminal object 1. Observe

- $U \xleftarrow{\mathrm{id}} U \xrightarrow{\mathrm{id}} U$ is a product cone in $\mathscr{A}$.
- for any $A \in \mathscr{A}$, there is at most one morphism from A to U.
- $\mathscr{A}/U$ may be identified with a full subcategory of $\mathscr{A}$ given by those A with a morphism from A to U.
- $\mathscr{A}/U$ is regular. Rel($\mathscr{A}/U$) and the full subcategory of Rel($\mathscr{A}$) given by the objects of $\mathscr{A}/U$ agree.
- the projection $\pi_0 : A \times U \to A$ is an isomorphism if and only if $A \in \mathscr{A}/U$.

Proposition 45. *Given a regular category $\mathscr{A}$ and a subobject U of the terminal object 1 of $\mathscr{A}$,* $\mathrm{Rel}(\mathscr{A}/U)$ *is bireflective in* $\mathrm{Rel}(\mathscr{A})$.

Proof. $\mathrm{Rel}(\mathscr{A})$ has a natural idempotent α on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$ given by $\alpha_A = (\pi_0, \pi_0)$ with $\pi_0 : A \times U \to A$. This splits as $(\pi_0, \mathrm{id}_{A\times U}); (\mathrm{id}, \pi_0) : A \nrightarrow A \times U \nrightarrow A$. So, by Theorem 13, the objects of the form $A \times U$ with $A \in \mathscr{A}$ give a bireflective full subcategory of $\mathrm{Rel}(\mathscr{A})$. But by the above observation, it is $\mathrm{Rel}(\mathscr{A}/U)$. $\square$

Definition 46. Given a regular category $\mathscr{A}$, a bireflective subcategory $\mathscr{B}$ of $\mathrm{Rel}(\mathscr{A})$ is *well-supported* if the bireflection of the terminal object 1 of $\mathscr{A}$ is isomorphic to 1 in $\mathscr{A}$.

Lemma 47. *Suppose $\mathscr{B}$ is bireflective in* $\mathrm{Rel}(\mathscr{A})$ *with $\mathscr{A}$ regular. The reflection U of terminal object 1 of $\mathscr{A}$ is a subobject of 1 in $\mathscr{A}$ and $\mathscr{B}$ is a well-supported bireflective subcategory of* $\mathrm{Rel}(\mathscr{A}/U)$.

Proof. By Theorem 13, we have a split natural idempotent α on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$ corresponding to $\mathscr{B}$. Let $(!_V, !_V)$ with $!_V : V \to 1$ be a monic pair representing $\alpha_1 : 1 \nrightarrow 1$. Then, $!_V$ is monic in $\mathscr{A}$ and α_1 splits as $(!_V, \mathrm{id}_V); (\mathrm{id}_V, !_V) : 1 \nrightarrow V \nrightarrow 1$; i.e., $V \cong U$ and $(!_U, \mathrm{id}_U)$ is the reflector of $1 \in \mathrm{Rel}(\mathscr{A})$ into $\mathscr{B}$, with $!_U$ being monic. The restriction of α to $\mathscr{A}/U$ gives a bireflective subcategory $\mathscr{B}'$ of $\mathrm{Rel}(\mathscr{A}/U)$. Obviously $\mathscr{B}'$ is well-supported and contained in $\mathscr{B}$. For the converse containment, let $J : \mathscr{B} \to \mathrm{Rel}(\mathscr{A})$ be the inclusion and consider, for $B \in \mathscr{B}$, the relation $(!_{JB}, \mathrm{id}_{JB}) : 1 \nrightarrow JB$. There is a unique $(f, g) : U \nrightarrow JB$ with $(!_U, \mathrm{id}_U); (f, g) = (!, \mathrm{id}_{JB})$. Then, g is both a monomorphism and a cover, i.e., it is an isomorphism. So, there is $fg \in v : JB \to U$ in $\mathscr{A}$, hence $\mathscr{B}$ is contained in $\mathscr{B}'$. $\square$

Since one can recover, up to isomorphisms, the reflection on $\mathrm{Rel}(\mathscr{A})$ from that on $\mathrm{Rel}(\mathscr{A}/U)$ by Propositions 11 and 45, we have

Corollary 48. *To give a bireflective subcategory of* $\mathrm{Rel}(\mathscr{A})$ *is to give a well-supported bireflective subcategory of* $\mathrm{Rel}(\mathscr{A}/U)$ *for some subobject U of* 1.

So, we concentrate on well-supported bireflective subcategories of $\mathrm{Rel}(\mathscr{A})$.

Proposition 49. *Let $\mathscr{B}$ be a well-supported bireflective subcategory of* $\mathrm{Rel}(\mathscr{A})$ *with $\mathscr{A}$ regular. Then,* $\mathrm{Map}(\mathscr{B})$ *is reflective in $\mathscr{A}$ and each component of the reflector is a cover in $\mathscr{A}$.*

The proof rests on the following lemmas.

Lemma 50. *Let $\mathscr{A}$ be a regular category. Suppose that $\alpha = (\alpha^0, \alpha^1) : A \nrightarrow A$ is a split idempotent in* $\mathrm{Rel}(\mathscr{A})$, *giving a retract R of A in* $\mathrm{Rel}(\mathscr{A})$. *Further suppose that*

α^0 and α^1 are covers in $\mathscr{A}$. Then, (α^0,α^1) has a coequalizer $r : A \to R$ in $\mathscr{A}$ with codomain R; the kernel pair of r in $\mathscr{A}$ is (α^0,α^1); and the splitting of α is given by $(\mathrm{id}_A, r);(r,\mathrm{id}_A) : A \nrightarrow R \nrightarrow A$. Conversely, any cover $r : A \to R$ in $\mathscr{A}$ gives rise to a split idempotent $(\mathrm{id}, r);(r,\mathrm{id}) : A \nrightarrow A$ in $\mathrm{Rel}(\mathscr{A})$.

Proof. Let $A \overset{(f,g)}{\nrightarrow} R \overset{(k,l)}{\nrightarrow} A$ be the splitting of α. The equation $(\alpha^0,\alpha^1) = (f,g);(k,l)$ in $\mathrm{Rel}(\mathscr{A})$ is equivalent to the existence of a cover h from D, the vertex of the pullback of g and k, to A that makes the left diagram bellow commute in $\mathscr{A}$. Similarly, by $(\mathrm{id}_R,\mathrm{id}_R) = (k,l);(f,g)$, there is a cover i for which the right diagram commutes.

$$
\begin{tikzcd}
A \arrow[drrr, "\alpha^1"] \arrow[dddr, "\alpha^0"'] & & & \\
& D \arrow[ul, "h"'] \arrow[r, "g'"] \arrow[d, "k'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "l"'] \arrow[d, "k"] & A \\
& B \arrow[r, "g"'] \arrow[d, "f"] & R & \\
& A & &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[drrr, "\mathrm{id}_R"] \arrow[dddr, "\mathrm{id}_R"'] & & & \\
& E \arrow[ul, dashed, "i"'] \arrow[r, "l'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "g"'] \arrow[d, "f"] & R \\
& C \arrow[r, "l"'] \arrow[d, "k"] & A & \\
& R & &
\end{tikzcd}
$$

Since $\alpha^0 h$, $\alpha^1 h$, and i are covers, so are f, l, k, and g. A pullback of covers is also a pushout (see e.g., [8]), so, by the second diagram, we have a unique map $r : A \to R$ with $k = rl$ and $g = rf$. This r is a cover because k is. Since $\langle f,g\rangle$ is monic, $\langle \mathrm{id}, r\rangle f = \langle f,g\rangle$ implies that the cover f is monic, so it is an isomorphism. Similarly l is an isomorphism. Substituting rf for g and rl for k, and using again the fact that a pullback of covers is a pushout, one can check that r is a coequalizer of α^0 and α^1 by the first diagram. Since (k',g') is a monic pair, so is (fk', lg'); this implies that the cover h is monic, hence an isomorphism. Similarly, i is an isomorphism. So, α splits as $(\mathrm{id}, r);(r,\mathrm{id})$. The converse direction is immediate. □

Lemma 51. *Suppose $\mathscr{B}$ is a reflective full subcategory of $\mathrm{Rel}(\mathscr{A})$ with $\mathscr{A}$ regular. Further suppose that the reflector of $A \in \mathrm{Rel}(\mathscr{A})$ is of the form $(\mathrm{id}_A, r) : A \nrightarrow R$ with some cover $r : A \to R$ in $\mathscr{A}$. Then r is a reflector of $A \in \mathscr{A}$ into the full subcategory $\mathrm{Map}(\mathscr{B})$ of $\mathscr{A}$.*

Proof. Let $J : \mathscr{B} \to \mathrm{Rel}(\mathscr{A})$ be the inclusion. We show the reflection $\mathrm{Rel}(\mathscr{A})(A, JB) \cong \mathrm{Rel}(\mathscr{A})(R, JB)$ with $B \in \mathscr{B}$ restricts to a natural bijection between maps. From the right hand side to the left, the composition with (id, r) sends maps to maps. In the other direction, let $f : A \to JB$ be a morphism in $\mathscr{A}$, and let $(k,l) : R \nrightarrow JB$ be the unique morphism with $(\mathrm{id}, r); (k, l) = (\mathrm{id}, f)$, i.e., there is a cover h for which

$$\begin{tikzcd}
A \arrow[drrr, "f"] \arrow[dddr, "\mathrm{id}"'] & & & \\
& D \arrow[ul, "h"'] \arrow[r, "r'"] \arrow[d, "k'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[r, "l"'] \arrow[d, "k"] & JB \\
& A \arrow[r, "r"'] \arrow[d, "\mathrm{id}"] & R & \\
& A & &
\end{tikzcd}$$

commutes. We show that (k, l) is a map. Since $rk' = rh$ is a cover, k is a cover. So it suffices to show that k is also a monomorphism. Given $m, n : X \to C$ with $km = kn$, let r'' be the pullback of r along km, and take m' and n' with $k'm' = k'n'$, $r'm' = mr''$, and $r'n' = nr''$.

$$\begin{tikzcd}
A'' \arrow[r, "r''"] \arrow[d, shift right, "m'"'] \arrow[d, shift left, "n'"] & X \arrow[d, shift right, "m"'] \arrow[d, shift left, "n"] & \\
A' \arrow[r, "r'"] \arrow[d, "k'"'] & C \arrow[r, "l"] \arrow[d, "k"] & JB \\
A \arrow[r, "r"] & R &
\end{tikzcd}$$

Since r'' is a cover, and hence epic, the equation $lmr'' = lr'm' = fhm' = fk'm' = fk'n' = lnr''$ implies $lm = ln$. Because (k, l) is a monic pair, this, together with $km = kn$, implies $m = n$. $\square$

Proof of Proposition 49. Let $J : \mathscr{B} \to \mathrm{Rel}(\mathscr{A})$ be the inclusion, S the bireflection. By Theorem 14, one has a split natural idempotent α on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$ that corresponds to $\mathscr{B}$.

Since $1 \in \mathscr{B}$, $\alpha_1 = (\mathrm{id}_1, \mathrm{id}_1)$ in $\mathrm{Rel}(\mathscr{A})$. Write (α_A^0, α_A^1) for $\alpha_A : A \nrightarrow A$. By naturality with respect to $(\mathrm{id}_A, !_A) : A \nrightarrow 1$ and $(!_A, \mathrm{id}_A) : 1 \nrightarrow A$, α_A^0 and α_A^1 are covers in $\mathscr{A}$.

By definition, α_A splits as $\eta_A; \varepsilon'_A : A \nrightarrow JSA \nrightarrow A$ with η_A the unit of $S \dashv J$ and ε'_A the counit of $J \dashv S$. So, by Lemma 50, (α^0, α^1) has a coequalizer $r_A : A \to JSA$ in $\mathscr{A}$, and without loss of generality, one may assume $\eta_A = (\mathrm{id}, r_A)$.

Since a coequalizer is a cover, by Lemma 51, r_A reflects $A \in \mathscr{A}$ into $\mathrm{Map}(\mathscr{B})$. The reflection of $\mathscr{A}$ into $\mathrm{Map}(\mathscr{B})$ is the restriction of $S : \mathrm{Rel}(\mathscr{A}) \to \mathscr{B}$ to $\mathscr{A}$. $\square$

For the converse direction of Proposition 49, one needs a stronger condition on the reflector of $\mathscr{A}$ into Map($\mathscr{B}$). Let $\mathscr{B}$ be a collection of objects of a regular category $\mathscr{A}$, and write Rel($\mathscr{B}$) for the full subcategory of Rel($\mathscr{A}$) given by the objects in $\mathscr{B}$. Suppose that $\mathscr{B}$, as a full subcategory of $\mathscr{A}$, is reflective in $\mathscr{A}$ with the reflector $r_A : A \to R_A$ of each $A \in \mathscr{A}$ being a cover. Further, without loss of generality, one may assume $r_A = \mathrm{id}_A$ for $A \in \mathscr{B}$. Then, $(\mathrm{id}, r_A); (r_A, \mathrm{id}) : A \nrightarrow A$ is a split idempotent in Rel($\mathscr{A}$). *If* the family $\alpha = ((\mathrm{id}, r_A); (r_A, \mathrm{id}))_{A \in \mathscr{A}}$ forms a natural transformation on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$, Theorem 13 gives Rel($\mathscr{B}$) as a bireflective full subcategory of Rel($\mathscr{A}$). The above lemmas show that this construction is the inverse of the one given by Proposition 49.

One can analyze the naturality condition on the family $\alpha = (\mathrm{id}, r); (r, \mathrm{id})$ a little further. For $f : A \to B$ in $\mathscr{A}$, let $R_f : R_A \to R_B$ be the unique morphism with $R_f r_A = r_B f$. The naturality of α is equivalent to that of $((r_A, \mathrm{id}_A))_{A \in \mathscr{A}}$ with respect to maps, i.e., for all $f : A \to B$ in $\mathscr{A}$, the following commutes.

$$
\begin{tikzcd}
R_A \arrow[r, "{(r_A,\mathrm{id})}", "|" marking] \arrow[d, "{(\mathrm{id},R_f)}"', "-" marking] & A \arrow[d, "{(\mathrm{id},f)}", "-" marking] \\
R_B \arrow[r, "{(r_B,\mathrm{id})}"', "|" marking] & B
\end{tikzcd}
$$

To wit, suppose that α is natural. For all maps $(\mathrm{id}, f) : A \nrightarrow B$ in particular, one has $(\mathrm{id}, f); (\mathrm{id}, r_B); (r_B, \mathrm{id}) = (\mathrm{id}, r_A); (r_A, \mathrm{id}); (\mathrm{id}, f)$. By definition, $(\mathrm{id}, f); (\mathrm{id}, r_B) = (\mathrm{id}, r_A); (\mathrm{id}, R_f)$. Because (id, r_A) is a split epimorphism, $(r_A, \mathrm{id}); (\mathrm{id}, f) = (\mathrm{id}, R_f); (r_B, \mathrm{id})$ as desired. Conversely, if $(r_A, \mathrm{id}_A)_{A \in \mathscr{A}}$ is natural with respect to maps, for all morphism $(f, g) : A \nrightarrow B$, each square in the diagram

$$
\begin{tikzcd}
A \arrow[r, "{(\mathrm{id},r_A)}", "|" marking] \arrow[d, "{(f,\mathrm{id})}"', "-" marking] \arrow[dr, phantom, "\boxed{1}"] & R_A \arrow[r, "{(r_A,\mathrm{id})}", "|" marking] \arrow[d, "{(R_f,\mathrm{id})}" description] \arrow[dr, phantom, "\boxed{2}"] & A \arrow[d, "{(f,\mathrm{id})}", "-" marking] \\
C \arrow[r, "{(\mathrm{id},r_C)}", "|" marking] \arrow[d, "{(\mathrm{id},g)}"', "-" marking] \arrow[dr, phantom, "\boxed{3}"] & R_C \arrow[r, "{(r_C,\mathrm{id})}", "|" marking] \arrow[d, "{(\mathrm{id},R_g)}" description] \arrow[dr, phantom, "\boxed{4}"] & C \arrow[d, "{(\mathrm{id},g)}", "-" marking] \\
B \arrow[r, "{(\mathrm{id},r_B)}"', "|" marking] & R_B \arrow[r, "{(r_B,\mathrm{id})}"', "|" marking] & B
\end{tikzcd}
$$

commutes ($\boxed{2}$ and $\boxed{3}$ by definition, $\boxed{4}$ by the assumption, and $\boxed{1}$ by the assumption and the reciprocation). We record

Proposition 52. *Let $\mathscr{B}$ be a collection of objects of a regular category $\mathscr{A}$ that includes the terminal object* 1. *Then* $\mathrm{Rel}(\mathscr{B})$, *the full subcategory of* $\mathrm{Rel}(\mathscr{A})$ *given by the objects in* $\mathscr{B}$, *is bireflective in* $\mathrm{Rel}(\mathscr{A})$ *if and only if the following are satisfied*:
(1) *$\mathscr{B}$, as a full subcategory of $\mathscr{A}$, is reflective in $\mathscr{A}$,*
(2) *the reflector $r_A : A \to R_A$ of each $A \in \mathscr{A}$ is a cover, and*
(3) *for each $f : A \to B$ in $\mathscr{A}$, $(r_A, \mathrm{id}_A);(\mathrm{id}_A, f) = (\mathrm{id}_{R_A}, R_f);(r_B, \mathrm{id}_B)$ in* $\mathrm{Rel}(\mathscr{A})$ *where R_f is the unique morphism with $R_f r_A = r_B f$.*

In $\mathscr{A}$, condition (iii) translates to the one requiring that for each $f : A \to B$, the unique morphism h from A to the vertex of the pullback of η_B and R_f determined by (f, η_A) is a cover. This is a special case of 'cover pullback' defined by Cockett and Spooner [4] though the word 'cover' here has a different meaning from ours.

An idempotent $(f, g) : A \nrightarrow A$ in $\mathrm{Rel}(\mathscr{A})$ is a *partial equivalence relation* if $\langle f, g\rangle = \langle g, f\rangle h$ for some h in $\mathscr{A}$; if further $\langle id_A, id_A\rangle = \langle f, g\rangle k$ holds for some k, it is called an *equivalence relation*. Every kernel pair of a morphism in $\mathscr{A}$ gives an equivalence relation. A regular category $\mathscr{A}$ is *effective* if the converse holds, i.e., every equivalence relation is a kernel pair of some morphism in $\mathscr{A}$.

Corollary 53. *For an effective regular category $\mathscr{A}$, a bireflective subcategory of* $\mathrm{Rel}(\mathscr{A})$ *bijectively corresponds to a natural partial equivalence relation on* $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$; *the subcategory is well-supported if the corresponding natural partial equivalence is an equivalence relation.*

Proof. The well-supported case is immediate by the proof of Proposition 52. For the general case, suppose $\mathscr{B}$ is bireflective in $\mathrm{Rel}(\mathscr{A})$. By Corollary 48, $\mathscr{B}$ is well-supported in $\mathrm{Rel}(\mathscr{A}/U)$ with U the bireflection of 1. Let $\alpha = (\alpha^0, \alpha^1)$ be the natural split idempotent on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A}/U)}$ corresponding to $\mathscr{B}$. The natural split idempotent β on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$ is given by $\beta_A = (\pi_0 \alpha^0_{A\times U}, \pi_0 \alpha^1_{A\times U})$ with $\pi_0 : A \times U \to A$; it is immediate to check this is a partial equivalence relation.

Conversely, let $\beta = (\beta^0, \beta^1)$ be a natural partial equivalence relation on $\mathrm{id}_{\mathrm{Rel}(\mathscr{A})}$. Let $(!_U, !_U) : 1 \nrightarrow 1$ for some $U \in \mathscr{A}$ be a monic pair representing β_1; U is a subobject of 1. For $A \in \mathscr{A}$, let $\pi_0 : A \times U \to A$ be the projection. The naturality of β with respect to $(\mathrm{id}, !_U) : U \nrightarrow 1$ shows that $\beta^0_U = \mathrm{id}_U$. Similarly, $\beta^1_U = \mathrm{id}_U$. Since U is the terminal object of $\mathscr{A}/U$, repeating the first part of the proof of Proposition 49, one may conclude that β^0_A and β^1_A with $A \in \mathscr{A}/U$ are covers. By the assumption, $(\beta^0_A, \beta^1_A) : A \nrightarrow A$ is a kernel pair of some morphism $r : A \to R$ in $\mathscr{A}$; r is necessarily a cover when β^0_A and β^1_A are covers. So, the restriction of β to $\mathscr{A}/U$ splits, giving a well-supported bireflective subcategory of $\mathrm{Rel}(\mathscr{A}/U)$, and hence giving a bireflective subcategory of $\mathrm{Rel}(\mathscr{A})$ by Corollary 48. To see this construction and the one given in the previous paragraph are mutually inverse, note that the naturality of β with respect to $(\pi_0, \mathrm{id}_{A\times U}) : A \nrightarrow A \times U$ with $A \in \mathscr{A}$ shows that $\beta^0_A = \pi_0 \beta^0_{A\times U}$. Similarly, $\beta^1_A = \pi_0 \beta^1_{A\times U}$. Finally note that $A \in \mathscr{A}/U$ if and only if π_0 is an isomorphism. $\square$

When $\mathscr{A}$ is a topos, Proposition 52 can be replaced by an elegant characterisation with a simpler proof for the converse direction.

Theorem 54. *Let $\mathscr{A}$ be a topos, $\mathscr{B}$ a collection of objects of $\mathscr{A}$ that includes the terminal object* 1. *Then,* $\mathrm{Rel}(\mathscr{B})$ *is bireflective in* $\mathrm{Rel}(\mathscr{A})$ *if and only if $\mathscr{B}$ is reflective in $\mathscr{A}$ and closed under power object formation.*

Proof. ($\Rightarrow$) By Proposition 49, it suffices to show that given $B \in \mathscr{B}$, the power object Ω^{JB} is in $\mathscr{B}$. Let $\varepsilon : \Omega^{JB} \nrightarrow JB$ be the relation corresponding to the evaluation $\mathrm{ev} : \Omega^{JB} \times JB \to \Omega$ in $\mathscr{A}$, let $\bar{\varepsilon} : S(\Omega^{JB}) \nrightarrow B$ in $\mathrm{Rel}(\mathscr{B})$ be its transposition under $S \dashv J$, and let $(\mathrm{id}, r) : \Omega^{JB} \nrightarrow JS(\Omega^{JB})$ be the reflector. By the universality of Ω^{JB}, i.e., $\mathscr{A}(X, \Omega^{JB}) \cong \mathrm{Rel}(\mathscr{A})(X, JB)$, one has a map $(\mathrm{id}, f) : JS(\Omega^{JB}) \nrightarrow \Omega^{JB}$ with $(\mathrm{id}, f); \varepsilon = J\bar{\varepsilon} : JS(\Omega^{JB}) \nrightarrow JB$; moreover, the universality implies $r; f = \mathrm{id}_{\Omega^{JB}}$, i.e., r is a monomorphism. Since r is also a cover, it is an isomorphism.

($\Leftarrow$) Let $J : \mathscr{B} \to \mathscr{A}$ be the inclusion with the reflection S. For $A \in \mathscr{A}$, the reflector $\eta : A \to JSA$ in $\mathscr{A}$ gives a reflector $(\mathrm{id}_A, \eta) : A \nrightarrow JSA$ of $A \in \mathrm{Rel}(\mathscr{A})$ into $\mathrm{Rel}(\mathscr{B})$. To wit, for any $B \in \mathrm{Rel}(\mathscr{B})$, one has

$$
\begin{aligned}
\mathrm{Rel}(\mathscr{A})(A, JB) &\cong \mathscr{A}(A, \Omega^{JB}) = \mathscr{A}(A, JC) && \text{for some } C \in \mathscr{B} \\
&\cong \mathscr{B}(SA, C) && S \dashv J \\
&\cong \mathscr{A}(JSA, JC) = \mathscr{A}(JSA, \Omega^{JB}) && J \text{ is fully faithful} \\
&\cong \mathrm{Rel}(\mathscr{A})(JSA, JB).
\end{aligned}
$$

By the self-duality of $\mathrm{Rel}(\mathscr{A}), (\mathrm{id}, \eta)^\circ = (\eta, \mathrm{id}) : JSA \nrightarrow A$ is automatically a coreflector, so it remains to check the coherence condition $(\eta, \mathrm{id}); (\mathrm{id}, \eta) = \mathrm{id}_{JSA}$ in $\mathrm{Rel}(\mathscr{A})$, i.e., η is a cover. Since the generic monomorphism $\top : 1 \to \Omega$ is in $\mathscr{B}$, and since a reflective full subcategory is closed under the limit construction, in particular pullbacks, it follows that $\mathscr{B}$ is closed under subobject formation. So, the universality of η implies that η is a cover. $\square$

Remark 55. A full subcategory of a complete topos is called an *exponential* variety if it is closed under the formation of subobjects, cartesian products, and power-objects. Exponential varieties are studied in detail in [7] with regard to the interpretation of the axiom of choice in a topos. There, given a Grothendieck topology j on a small category $\mathscr{C}$, exponential varieties of j-sheaves are shown to be in natural correspondence with the so-called exponential congruence relation on morphisms in $\mathscr{C}$.

8. Bireflectivity and modules

In this section, we characterise bireflectivity solely in terms of the reflection (=coreflection) functor, without taking the corresponding inclusion as a given. We use some definitions from the theory of the bicategory of modules. For the basic definition of bicategory, see [2]. Given two small categories $\mathscr{A}$ and $\mathscr{B}$, a *module* φ from

$\mathscr{A}$ to $\mathscr{B}$, written $\varphi : \mathscr{A} \nrightarrow \mathscr{B}$, is a functor $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$ (note: in some literature, the direction of a module is the other way round). For the basic theory of modules, see [13].

Definition 56. The bicategory **Mod** of modules has as objects small categories. A 1-cell $\varphi : \mathscr{A} \nrightarrow \mathscr{B}$ is a module from $\mathscr{A}$ to $\mathscr{B}$. A 2-cell $\lambda : \varphi \Rightarrow \psi : \mathscr{A} \nrightarrow \mathscr{B}$ is a natural transformation from φ to ψ as functors. Given two 1-cells $\varphi : \mathscr{A} \to \mathscr{B}$ and $\psi : \mathscr{B} \to \mathscr{C}$, the composition $\psi\varphi : \mathscr{A} \to \mathscr{C}$ is given by, for objects $A \in \mathscr{A}$ and $C \in \mathscr{C}$, $\psi\varphi(C,A) = \int^{B \in \mathscr{B}} \psi(C, B) \times \varphi(B,A)$, i.e., the coequalizer

$$\coprod_{f \in \mathscr{B}(B_0,B_1)} \psi(C,B_0) \times \varphi(B_1,A) \rightrightarrows \coprod_{B \in \mathscr{B}} \psi(C,B) \times \varphi(B,A) \to \psi\varphi(C,A),$$

where the two parallel arrows are given by the morphism parts of ψ and φ, respectively. The morphism part of $\psi\varphi$ and the vertical and horizontal compositions of 2-cells are given by the universality of the coequalizer. The identity 1-cell $\mathscr{Y}_{\mathscr{A}} : \mathscr{A} \nrightarrow \mathscr{A}$ is $\mathscr{Y}_{\mathscr{A}}(A',A) = \mathscr{A}(A',A)$.

Given a functor $F : \mathscr{A} \to \mathscr{B}$ with $\mathscr{A}$ and $\mathscr{B}$ small, one can define two modules $F_* : \mathscr{A} \nrightarrow \mathscr{B}$ and $F^* : \mathscr{B} \nrightarrow \mathscr{A}$ by $F_*(B,A) = \mathscr{B}(B,FA)$ and $F^*(A,B) = \mathscr{B}(FA,B)$. By the Yoneda lemma, $\mathbf{Cat}(F,G) \cong \mathbf{Mod}(F_*,G_*) \cong \mathbf{Mod}(F^*,G^*)$, thus $(-)_*$ and $(-)^*$ give identity-on-objects embeddings of **Cat** and $\mathbf{Cat}^{\mathrm{op}}$ in **Mod**, respectively.

A module $\varphi : \mathscr{A} \to \mathscr{B}$ is called *convergent* if φ is isomorphic to F_* for some functor $F : \mathscr{A} \to \mathscr{B}$. An essential property of a convergent module $\varphi \cong F_*$ is that it has a right adjoint F^*. The unit $\lambda : \mathscr{Y}_{\mathscr{A}} \Rightarrow F^*F_*$ of the adjunction is given by the morphism part of F

$$\lambda_{A',A} : \mathscr{A}(A',A) \xrightarrow{F} \mathscr{B}(FA',FA) \cong \int^{B \in \mathscr{B}} \mathscr{B}(FA',B) \times \mathscr{B}(B,FA),$$

and the counit $\mu : F_*F^* \Rightarrow \mathscr{Y}_{\mathscr{B}}$ by

$$\mu_{B',B} : \int^{A \in \mathscr{A}} \mathscr{B}(B',FA) \times \mathscr{B}(FA,B) \xrightarrow{\bar{\circ}} \mathscr{B}(B',B)$$

where $\bar{\circ}$ corresponds to the family $(\mathscr{B}(B',FA) \times \mathscr{B}(FA,B) \xrightarrow{\circ} \mathscr{B}(B',B))_{A \in \mathscr{A}}$ of composition maps via the universality of $\int^{A \in \mathscr{A}} \mathscr{B}(B',FA) \times \mathscr{B}(FA,B)$.

Proposition 57. *Given a small category $\mathscr{B}$, the following are equivalent.*
(1) *any module into $\mathscr{B}$ with a right adjoint is convergent.*
(2) *every idempotent splits in $\mathscr{B}$.*

Proof. See [13].

Idempotent splitting is a relatively mild condition; for example, any category with equalizers or coequalizers has idempotent splitting.

Now we characterise bireflective subcategories in an idempotent-splitting category in terms of the modules defined by bireflection functors.

Proposition 58. *Let $\mathscr{A}$ be an idempotent-splitting small category, and $S : \mathscr{A} \to \mathscr{B}$ a functor with $\mathscr{B}$ small. Then $\mathscr{B}$ is a bireflective subcategory of $\mathscr{A}$ with bireflection S if and only if*

(1) $S^* \dashv S_*$ *in* **Mod**,

(2) $S^* S_* \overset{\mu'}{\Longrightarrow} \mathscr{Y}_{\mathscr{A}} \overset{\lambda}{\Longrightarrow} S^* S_*$ *is the identity, where λ is the unit of $S_* \dashv S^*$ and μ' is the counit of $S^* \dashv S_*$, and*

(3) *any $f : B' \to B$ in $\mathscr{B}$ can be factorised as $B' \to SA \to B$ for some $A \in \mathscr{A}$.*

Proof. The only if part is clear. For the if part, by Proposition 57, $S^* \dashv S_*$ implies that there exists a functor $J : \mathscr{B} \to \mathscr{A}$ with isomorphism $\alpha : S^* \cong J_*$, i.e., $S \dashv J$ in **Cat**. Let λ and μ be the unit and counit of $S_* \dashv S^*$, λ' and μ' those of $S^* \dashv S_*$, and ι and κ those of $J_* \dashv J^*$, respectively.

Since both S_* and J^* are right adjoint to J_*, $S_* \cong J^*$, i.e. $J \dashv S$ in **Cat**. The isomorphism $\beta : S_* \cong J^*$ is the mate of α under the adjunctions $S^* \dashv S_*$ and $J_* \dashv J^*$, i.e., suppressing the surrounding invertible 2-cells,

$$\beta \cong \begin{tikzcd}
\mathscr{A} \arrow[rr, "y"] \arrow[dr, "S_*"'] \arrow[dr, phantom, shift left=6, "\mu' \Uparrow"] & & \mathscr{A} \arrow[dr, "J^*"] \arrow[dr, phantom, shift right=6, "\Uparrow \iota"] & \\
& \mathscr{B} \arrow[ur, bend left, "S^*"] \arrow[ur, bend right, "J_*"'] \arrow[ur, phantom, "\overset{\alpha}{\cong}"] \arrow[rr, "y"'] & & \mathscr{B}
\end{tikzcd}$$

We show that α and β satisfy the condition (3) of Proposition 10. By the condition (2), we have

$$\begin{tikzcd}
\mathscr{A} \arrow[dr, "S_*"'] & & \mathscr{A} \arrow[dr, "J^*"] \arrow[dr, phantom, shift right=6, "\Uparrow \iota"] & \\
& \mathscr{B} \arrow[ur, bend left, "S^*"] \arrow[ur, bend right, "J_*"'] \arrow[ur, phantom, "\overset{\alpha}{\cong}"] \arrow[rr, "y"'] & & \mathscr{B}
\end{tikzcd}$$

$$= \begin{tikzcd}
& \mathscr{B} \arrow[dr, "S^*"] & & \\
\mathscr{A} \arrow[ur, "S_*"] \arrow[rr, "y"] \arrow[rr, phantom, shift left=6, "\lambda \Uparrow"] \arrow[dr, "S_*"'] \arrow[dr, phantom, shift left=6, "\mu' \Uparrow"] & & \mathscr{A} \arrow[dr, "J_*"] \arrow[dr, phantom, shift right=6, "\Uparrow \iota"] & \\
& \mathscr{B} \arrow[ur, bend left, "S^*"] \arrow[ur, bend right, "J_*"'] \arrow[ur, phantom, "\overset{\alpha}{\cong}"] \arrow[rr, "y"'] & & \mathscr{B}
\end{tikzcd}$$

$$
= \begin{tikzcd}
& \mathscr{B} \arrow[dr, "S^*"] \arrow[d, phantom, "\Uparrow\lambda"] & & \\
\mathscr{A} \arrow[ur, "S_*"] \arrow[rr, "y"] \arrow[dr, "S_*"'] & {} \arrow[r, phantom, "\cong"] & \mathscr{A} \arrow[dr, bend left, "J^*"] \arrow[dr, bend right, "S_*"'] \arrow[dr, phantom, "\beta\,\cong"] & \\
& \mathscr{B} \arrow[rr, "y"'] & & \mathscr{B}.
\end{tikzcd}
$$

This translates to the condition (3) of Proposition 10, hence the adjointness $S \dashv J \dashv S$ satisfies the coherence condition of bireflectivity.

It remains to show that J is faithful. This is equivalent to ι being a monomorphism, which in turn is equivalent to λ' being a monomorphism. By the condition (3), each $\mu_{B',B} : \int^A \mathscr{B}(B', SA) \times \mathscr{B}(SA, B) \to \mathscr{B}(B', B)$ is surjective. By the adjointness $S^* \dashv S_* \dashv S^*$ and the condition (2), one has

$$
\mathscr{A} \xrightarrow{S^*} \mathscr{B} \xrightarrow{S_*} \mathscr{A}
$$

$$
\cong \begin{tikzcd}
\mathscr{B} \arrow[rr, "y"] \arrow[dr, "S^*"'] & {} \arrow[d, phantom, "\Uparrow\mu"] & \mathscr{B} \arrow[dr, "S^*"] \arrow[d, phantom, "\Uparrow\lambda"] & & \\
& \mathscr{A} \arrow[ur, "S_*"] \arrow[rr, "y"] \arrow[dr, "S_*"'] & {} \arrow[d, phantom, "\Uparrow\mu'"] & \mathscr{A} \arrow[dr, "S_*"] \arrow[d, phantom, "\Uparrow\lambda'"] & \\
& & \mathscr{B} \arrow[ur, "S^*"] \arrow[rr, "y"'] & {} & \mathscr{B}
\end{tikzcd}
$$

$$
\cong \begin{tikzcd}
& \mathscr{A} \arrow[dr, "S_*"] \arrow[d, phantom, "\Uparrow\lambda'"] & \\
\mathscr{B} \arrow[ur, "S^*"] \arrow[rr, "y"] \arrow[dr, "S^*"'] & {} \arrow[d, phantom, "\Uparrow\mu"] & \mathscr{B} \\
& \mathscr{A}. \arrow[ur, "S_*"'] &
\end{tikzcd}
$$

So, each $\mu_{B',B}$ is also injective, hence μ is an isomorphism. Therefore λ' is also an isomorphism, hence J is (fully) faithful. □

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