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Are Colimits of Algebras Simple to Construct?

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INTRODUCTION

The answer to the question in the title is: yes, for finitary algebraic theories. In general, colimits can be constructed naturally, if not precisely simply. In more detail, we assume that an epis-preserving monad (triple) $\mathscr{C}$ in a cocomplete, co-well-powered category $\mathscr{K}$ is given. Then we present a construction of colimits of $\mathscr{C}$-algebras, which is based on iteration of pushouts in the base category $\mathscr{K}$. If the monad $\mathscr{C}$ is finitary then we show that no iteration is necessary (under mild additional assumptions). Thus, colimits of finitary algebras can be directly constructed in $\mathscr{K}$—we call such colimits simple. For example, colimits of lattices are simple (since lattices are finitary); colimits of σ-complete semi-lattices are not simple, though coproducts are.

If, on the other hand, $\mathscr{C}$ does not preserve epis then (1) the category $\mathscr{K}^{\mathscr{C}}$ of $\mathscr{C}$-algebras need not be cocomplete and (2) even if $\mathscr{C}$ is finitary, colimits need not be simple. An example of (1) in the category $\mathscr{K} = \text{Pos}$ of posets is exhibited, based on the first example (with $\mathscr{K} = \text{graphs}$) of this kind, due to the first author [2].

The present paper is a combination of several results:

(a) Linton's observation that coproducts in $\mathscr{K}^{\mathscr{C}}$ can be computed as reflexive coequalizers [8];

(b) A construction of coequalizers in categories $\mathscr{K}(T)$ of "algebras

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without axioms" [3], plus the fact that the category $\mathscr{K}^{\mathscr{T}}$ is a subcategory of $\mathscr{K}(T)$ closed with respect to coequalizers [2];

(c) Barr's result that finitary, right exact algebraic theories have reflexive coequalizers, preserved by the forgetful functor $\mathscr{U}^{\mathscr{T}}: \mathscr{K}^{\mathscr{T}} \to \mathscr{K}$ [5]. Barr assumes $\mathscr{K}$ to be $EX\,5$, which is rather special; we generalize his result, weakening extremely the hypothesis on $\mathscr{K}$.

In a subsequent paper we shall investigate more closely simple colimits of universal algebras (in the category of sets). Inter alia, we shall present an infinitary variety which has simple colimits.

I. Colimit Construction

I.1. Throughout this section we assume that a cocomplete category $\mathscr{K}$ is given together with a monad [9] (algebraic theory [10]) $\mathscr{T} = (T, \mu, \eta)$. Linton [8] has shown that the category $\mathscr{K}^{\mathscr{T}}$ of $\mathscr{T}$-algebras is cocomplete whenever it has (reflexive) coequalizers. In more detail, given a collection of $\mathscr{T}$-algebras, say (A_i, δ_i), $i \in I$, denote the coproducts

$$A = \coprod_{i \in I} A_i \qquad \text{and} \qquad B = \coprod_{i \in I} TA_i \qquad \text{in } \mathscr{K}$$

(with injections $v_i: A_i \to A$ and $w_i: TA_i \to B$). We have natural morphisms

$$\delta = \coprod_{i \in I} \delta_i : B \to A;\ k: B \to TA \qquad \text{with} \quad k \cdot w_i = Tv_i \ (i \in I).$$

I.2 Theorem (Linton [8]). *The coproduct of algebras (A_i, δ_i) is obtained as the coequalizer (in $\mathscr{K}^{\mathscr{T}}$) of $T\delta$ and $\mu \cdot Tk$. More precisely, let*

$$\begin{tikzcd}
(TB, \mu) \arrow[r, "T\delta"] \arrow[d, "Tk"'] & (TA, \mu) \arrow[r, "c"] & (C, \gamma) \\
(T^2A, \mu) \arrow[ur, "\mu"'] & &
\end{tikzcd}$$

be a coequalizer, then $\coprod (A_i, \delta_i) = (C, \gamma)$ (with injections $c \cdot \eta_A \cdot v_i : A_i \to C$).

The pair $T\delta$ and $\mu \cdot Tk$ is reflexive, i.e., it has a common coretraction: put $h = \coprod_{i \in I} \eta_{A_i}: A \to B$ then $T\delta \cdot Th = T(\coprod \delta_i \cdot \eta_{A_i}) = 1$; $(\mu \cdot Tk) \cdot Th = \mu \cdot T\eta_A = 1$. Hence, if $\mathscr{K}^{\mathscr{T}}$ has coequalizers of reflexive pairs then it is cocomplete (for, as is well known, the colimit of each diagram can be computed via coproducts and a reflexive coequalizer).

I.3. We can proceed analogously with general colimits, not only coproducts. Let $D: \mathscr{D} \to \mathscr{K}^{\mathscr{T}}$ be a diagram of $\mathscr{T}$-algebras ($\mathscr{D}$ is a small category; for each $i \in \mathscr{D}^{\mathrm{obj}}$ we have a $\mathscr{T}$-algebra $Di = (A_i, \delta_i)$). Let $D_0 : \mathscr{D} \to \mathscr{K}$ be the underlying diagram ($D_0 i = A_i$). Put

$$A = \operatorname{colim} D_0 \qquad \text{and} \qquad B = \operatorname{colim} TD_0 \qquad \text{in} \quad \mathscr{K}$$

(with colimit injections $v_i : A_i \to A$ and $w_i : TA_i \to B$). Again, we have natural morphisms

$$\delta = \operatorname{colim} \delta_i : B \to A;\ k : B \to TA \qquad \text{with} \quad k \cdot w_i = Tv_i.$$

The following theorem is proved precisely as the above one in [8].

I.4. THEOREM. *Let*

$$\begin{tikzcd}
(TB, \mu) \arrow[r, "T\delta"] \arrow[dr, "Tk"'] & (TA, \mu) \arrow[r, "c"] & (C, \gamma) \\
& (T^2A, \mu) \arrow[u, "\mu"'] &
\end{tikzcd}$$

be a coequalizer in $\mathscr{K}^{\mathscr{T}}$. Then

$$(C, \gamma) = \operatorname{colim} D \qquad \text{in} \quad \mathscr{K}^{\mathscr{T}}$$

with injections $c \cdot \eta_A \cdot v_i : A_i \to C$.

Starting from $T\delta$ and $\mu \cdot Tk$ we shall now exhibit a construction of $\operatorname{colim} D$ in the category $\mathscr{K}^{\mathscr{T}}$, by iterating pushouts in $\mathscr{K}$.

I.5. *Colimit Construction*

Given a diagram D in $\mathscr{K}^{\mathscr{T}}$, let $\delta : B \to A$ and $k : B \to TA$ be as above. We shall define chains U, V in $\mathscr{K}$, i.e., functors from Ord (the ordered category of ordinals) into $\mathscr{K}$, and a transformation $\rho : U \to V$. (Given $U_{i,j}$ and $U_{j,k}$ we always assume that $U_{i,k}$ is defined as $U_{j,k} \cdot U_{i,j}$, analogously with $V_{i,k}$.)

First step. Let $p = \operatorname{coeq}(T\delta, \mu \cdot Tk)$ and $q = \operatorname{coeq}(T^2\delta, T\mu \cdot T^2k)$ be coequalizers in $\mathscr{K}$:

$$\begin{tikzcd}
T^2B \arrow[r, shift left, "T^2\delta"] \arrow[r, shift right, "T\mu \cdot T^2k"'] \arrow[d, dashed, "\mu"'] & T^2A \arrow[r, "q"] \arrow[d, dashed, "\mu"'] \arrow[rr, dashed, bend left, "Tp"] & U_0 \arrow[r, dashed, "U_{0,1}"'] \arrow[d, dashed, "\rho_0"'] & TV_0 = U_1 \\
TB \arrow[r, shift left, "T\delta"] \arrow[r, shift right, "\mu \cdot Tk"'] & TA \arrow[r, "p"] & V_0 &
\end{tikzcd}$$

This defines U_0 and V_0; put $U_1 = TV_0$. Since Tp merges $T^2\delta$ and $T\mu \cdot T^2k$, there exists $U_{0,1} : U_0 \to U_1$ with $Tp = U_{0,1} \cdot q$. Also $p \cdot \mu$ merges $T^2\delta$ and

$T\mu \cdot T^2k$ because $p \cdot \mu \cdot T^2\delta = p \cdot T\delta \cdot \mu = p \cdot \mu \cdot Tk \cdot \mu = p \cdot \mu \cdot (T\mu \cdot T^2k)$. Hence there exists $\rho_0 : U_0 \to V_0$ with $p \cdot \mu = \rho_0 \cdot q$.

Isolated step. Put $U_{j+1} = TV_j$.

Given $\rho_j : U_j \to V_j$ and $U_{j,j+1} : U_j \to U_{j+1} = TV_j$, consider their pushout $V_{j,j+1} \cdot \rho_j = \rho_{j+1} \cdot U_{j,j+1}$:

$$\begin{tikzcd}
U_j \arrow[r, "U_{j,j+1}"] \arrow[d, "\rho_j"'] & U_{j+1} = TV_j \arrow[r, dashed, "TV_{j,j+1}"] \arrow[d, "\rho_{j+1}"] & TV_{j+1} \\
V_j \arrow[r, "V_{j,j+1}"] & V_{j+1} &
\end{tikzcd}$$

This defines V_{j+1}, ρ_{j+1} and $V_{j,j+1}$. Put $U_{j+1,j+2} = TV_{j,j+1}$.

Limit step.

$$\begin{tikzcd}
T^2B \arrow[r, shift left, "T^2\delta"] \arrow[r, shift right, "T\mu \cdot T^2k"'] \arrow[d, "\mu"'] & T^2A \arrow[r, "q"] \arrow[d, "\mu"] & U_0 \arrow[r, "U_{01}"] \arrow[d, "\rho_0"] \arrow[dr, phantom, "\text{push out}"] & U_1 = TV_0 \arrow[r, "U_{12} = TV_{01}"] \arrow[d, "\rho_1"] \arrow[dr, phantom, "\text{push out}"] & U_2 = TV_1 \arrow[r, "U_{23} = TV_{12}"] \arrow[d, "\rho_2"] & \cdots \\
TB \arrow[r, shift left, "T\delta"] \arrow[r, shift right, "\mu \cdot Tk"'] & TA \arrow[r, "p"'] & V_0 \arrow[r, "V_{01}"'] & V_1 \arrow[r, "V_{12}"'] & V_2 \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
U_\omega \arrow[r, "U_{\omega,\omega+1}"] \arrow[d, "\operatorname*{colim}_{j<\omega} \rho_j = \rho_\omega"'] \arrow[dr, phantom, "\text{push out}"] & U_{\omega+1} \arrow[r, "U_{\omega+1,\omega+2} = TV_{\omega,\omega+1}"] \arrow[d, "\rho_{\omega+1}"] & \cdots \\
V_\omega \arrow[r, "V_{\omega,\omega+1}"'] & V_{\omega+1} \arrow[r] & \cdots
\end{tikzcd}$$

Given a limit ordinal α and given the restriction of U, V and ρ to all ordinals smaller than α we define

$$U_\alpha = \operatorname*{colim}_{j<\alpha} U_j \quad \text{with canonical} \quad U_{j,\alpha} : U_j \to U_\alpha,$$

$$V_\alpha = \operatorname*{colim}_{j<\alpha} V_j \quad \text{with canonical} \quad V_{j,\alpha} : V_j \to V_\alpha,$$

$$\rho_\alpha = \operatorname*{colim}_{j<\alpha} \rho_j.$$

Finally, $U_{\alpha,\alpha+1} : U_\alpha \to U_{\alpha+1} = TV_\alpha$ arises naturally from $U_{j+1} = TV_j$ $(j < \alpha)$. More precisely, $U_{\alpha,\alpha+1}$ is the unique morphism for which the following squares commute:

$$\begin{tikzcd} U_j \arrow[r, "U_{j,j+1}"] \arrow[d, "U_{j,\alpha}"'] & TV_j \arrow[d, "TV_{j,\alpha}"] \\ U_\alpha \arrow[r, "U_{\alpha,\alpha+1}"] & TV_\alpha \end{tikzcd} \qquad (j < \alpha).$$

DEFINITION. We say that *colimit construction stops* after n steps ($n \in \mathrm{Ord}$) provided that $U_{n,n+1}$ is an isomorphism (equivalently, all $U_{n,n+j}$ and all $V_{n,n+j}$ are isomorphisms for $j \in \mathrm{Ord}$). In that case we define $\gamma: TV_n \to V_n$ by $\gamma = \rho_n \cdot (U_{n,n+1})^{-1}: TV_n = U_{n+1} \to U_n \to V_n$.

In the following theorem we assume that $\mathscr{K}$ has a factorization system $(\mathscr{E}, \mathscr{M})$. This means that (1) $\mathscr{E}$ is a class of epis, $\mathscr{M}$ is a class of monos and $\mathscr{E} \cap \mathscr{M}$ is the class of all isomorphisms and (2) each morphism factorizes, uniquely up to isomorphism, as an $\mathscr{E}$-epi, followed by an $\mathscr{M}$-mono.

I.6. COLIMIT THEOREM. *Let $\mathscr{K}$ be a cocomplete, $\mathscr{E}$-co-well-powered category with a factorization system $(\mathscr{E}, \mathscr{M})$. Let $\mathfrak{T}$ be a monad in $\mathscr{K}$, preserving $\mathscr{E}$-epis (i.e., $e \in \mathscr{E}$ implies $Te \in \mathscr{E}$). Then for every diagram D in $\mathscr{K}^{\mathfrak{T}}$*

(i) *the colimit construction stops (say, after n steps); (V_n, γ) is a $\mathfrak{T}$-algebra;*

(ii) $\operatorname{colim} D = (V_n, \gamma)$ *with respect to injections* $V_{0,n} \cdot p \cdot \eta_A \cdot v_i: A_i \to V_n$.

Particularly, the category $\mathscr{K}^{\mathfrak{T}}$ is cocomplete.

Proof. (a) We shall prove (by transfinite induction) that U and V are $\mathscr{E}$-chains, i.e., that $U_{i,j} \in \mathscr{E}$ and $V_{i,j} \in \mathscr{E}$. For this we shall need the following properties of (an arbitrary) factorization system:

a1. $\mathscr{E}$ is right cancellative, i.e., $e_1 \cdot e_2 \in \mathscr{E}$ implies $e_1 \in \mathscr{E}$;

a2. $\mathscr{E}$ is closed to regular epis;

a3. pushouts carry $\mathscr{E}$-epis (if $e' \cdot e = f' \cdot f$ is a pushout of e, f then $e \in \mathscr{E}$ implies $f' \in \mathscr{E}$);

a4. colimits of chains carry $\mathscr{E}$-epis (if $S_{i,j}: S_i \to S_j$ is an $\mathscr{E}$-chain for $i \leqslant j < \alpha$ and if $S_{i,\alpha}: S_i \to S_\alpha$ is its colimit then

(1) $S_{i,\alpha} \in \mathscr{E}$ for each i;

(2) given $f: S_\alpha \to T$ such that each $f \cdot S_{i,\alpha}$ is in $\mathscr{E}$ then so is f).

All of these properties are easy to verify; see, e.g., [2] for (a3), [3] for (a4).

Since p is a coequalizer, we have $p \in \mathscr{E}$; by hypothesis, $Tp \in \mathscr{E}$ follows. Now $Tp = U_{0,1} \cdot q \in \mathscr{E}$ implies $U_{0,1} \in \mathscr{E}$. Also $V_{0,1} \in \mathscr{E}$ because it is opposite to $U_{0,1}$ in a pushout. The inductive step follows: if $U_{j,j+1} \in \mathscr{E}$ then also $V_{j,j+1} \in \mathscr{E}$ (opposite in a pushout) and $U_{j+1,j+2} = TV_{j,j+1} \in \mathscr{E}$. The limit step follows from (a4).

(b) The colimit construction stops: indeed, U_0 has only a set of non-equivalent $\mathscr{E}$-quotients (because $\mathscr{K}$ is $\mathscr{E}$-co-well-powered) and each

$$U_{0,j} : U_0 \to U_j \qquad (j \in \mathrm{Ord})$$

is an $\mathscr{E}$-quotient. Thus, there exist ordinals $n < m$ with $U_{0,n}$ equivalent to $U_{0,m}$; it easily follows that $U_{n,m}$ is an isomorphism and then so is $U_{n,n+1}$.

(c) $(V_n, \gamma) = \operatorname{colim} D$. To prove this it suffices to show that

$$(TB, \mu) \underset{\mu \cdot Tk}{\overset{T\delta}{\rightrightarrows}} (TA, \mu) \xrightarrow{V_{0,n} \cdot p} (V_n, \gamma)$$

is a coequalizer in $\mathscr{K}^{\mathscr{T}}$ (see Theorem I.4). Let us introduce here the category $\mathscr{K}(T)$ of "$\mathscr{T}$-algebras without axioms":

Objects are pairs (Q, δ) with $Q \in \mathscr{K}^{\mathrm{obj}}$ and $\delta : TQ \to Q$ a morphism; morphisms $f : (Q, \delta) \to (Q', \delta')$ are $\mathscr{K}$-morphisms $f : Q \to Q'$ with $f \cdot \delta = \delta' \cdot Tf$.

Then $\mathscr{K}^{\mathscr{T}}$ is a full subcategory of $\mathscr{K}(T)$; it is closed under the formation of coequalizers, see [2]. And $V_{0,n} \cdot p$ is a coequalizer of $T\delta$ and $\mu \cdot Tk$ in $\mathscr{K}(T)$: this is proved in [3], where a colimit construction is exhibited, coinciding (for coequalizers) with the above and proved to yield the colimits whenever it stops.

I.7. Corollary. *Let $\mathscr{K}$ be a cocomplete, co-well-powered category. Then the category $\mathscr{K}^{\mathscr{T}}$ is cocomplete for any monad $\mathscr{T}$ preserving epis.* (*Apply Theorem* I.6 *to* (*epis, extreme monos*)—*see* [6].)

I.8. *Remarks.* (a) The preceding corollary is proved in [2] and [7]—each time with a somewhat distinct proof. See [13] for a general result.

(b) Another iterative construction of colimits of $\mathscr{T}$-algebras has been presented by Schubert [11]. His construction differs basically from the one above by using the transformations μ, η on each isolated step (whereas they play no role in the above construction except, of course, to define $\mu \cdot Tk$). Starting with p, q, ρ_0 and $U_{0,1}$ as in I.5, Schubert's construction continues by letting ρ_1 be the coequalizer of $T\rho_0$ and $\mu_{V_0} \cdot TU_{0,1}$ and ρ_2 the coequalizer of $T\rho_1$ and $\mu_{V_1} \cdot TU_{1,2}$, etc.:

$$
\begin{tikzcd}
& & & & & & TU_0 \arrow[d, shift right, "T\rho_0"'] \arrow[d, shift left, "\mu\cdot TU_{01}"] & & TU_1 \arrow[d, shift right, "T\rho_1"'] \arrow[d, shift left, "\mu\cdot TU_{12}"] & \\
T^2B \arrow[rr, shift left, "T^2\delta"] \arrow[rr, shift right, "T\mu\cdot T^2\delta"'] \arrow[dr, "\mu"'] & & T^2A \arrow[rr, "q"] \arrow[dr, "\mu"] & & U_0 \arrow[rr, "U_{01}"] \arrow[dr, "\rho_0"] & & U_1 = TV_0 \arrow[rr, "U_{12} = TV_{01}"] \arrow[dr, "\rho_1"] & & U_2 = TV_1 \arrow[r] \arrow[d, "\rho_2"] & \cdots \\
& TB \arrow[rr, shift left, "T\delta"] \arrow[rr, shift right, "\mu\cdot Tk"'] & & TA \arrow[rr, "p"'] & & V_0 \arrow[rr, "V_{01} = \rho_1\cdot\eta"'] & & V_1 \arrow[r, "V_{12} = \rho_2\cdot\eta"'] & V_2 \arrow[r] & \cdots
\end{tikzcd}
$$

Construction I.5 is considerably simpler; also, it is not clear whether Schubert's construction stops whenever $\mathscr{F}$ preserves epis. It does stop if $\mathscr{F}$ preserves colimits of n_0-chains for some limit ordinal n_0—and the same is true about Construction I.5:

$$U_{n_0} = \operatorname*{colim}_{i<n_0} U_{i+1} = \operatorname*{colim}_{i<n_0} TV_i = T(\operatorname*{colim}_{i<n_0} V_i) = TV_{n_0} = U_{n_0+1}.$$

(c) Construction I.5 can be generalized to situations where morphisms

$$\delta : B \to A \qquad \text{and} \qquad k : B \to TA$$

are given, not necessarily via any diagram. Thus, let us denote by $\mathscr{K}^*(T)$ the category of "relational algebras," i.e., quadruples (A, B, δ, k) as above, where morphisms are pairs

$$(f, g) : (A, B, \delta, k) \to (A', B', \delta', k')$$

of $\mathscr{K}$-morphisms making the diagram

$$
\begin{tikzcd}
TA \arrow[d, "Tf"'] & B \arrow[l, "\delta"'] \arrow[r, "k"] \arrow[d, "g"] & A \arrow[d, "f"] \\
TA' & B' \arrow[l, "\delta'"] \arrow[r, "k'"'] & A'
\end{tikzcd}
$$

commutative. Then $\mathscr{K}^{\mathscr{F}}$ can be obviously considered as a full subcategory of $\mathscr{K}^*(T)$. And Construction I.5 yields, if it stops, reflections of relational algebras in the category $\mathscr{K}^{\mathscr{F}}$.

Thus, under the hypotheses of Theorem I.6, the category $\mathscr{K}^{\mathscr{F}}$ is ("constructively") reflective in $\mathscr{K}^*(T)$. This is proved in [7].

Related categories have already been considered by Kelly [12]. Generalizing Schubert's result above, he proves the reflexivity of the category $\mathscr{K}^{\mathscr{F}}$ in case $\mathscr{F}$ preserves colimits of n_0-chains (or weaker colimits of this type).

Let us observe that, regardless of whether $\mathscr{F}$ preservers epis, $\mathscr{K}^{\mathscr{F}}$ is reflexive in $\mathscr{K}^*(T)$ iff $\mathscr{K}^{\mathscr{T}}$ is cocomplete. The reflexion is a coequalizer of $T\delta$, $\mu \cdot Tk$ in $\mathscr{K}^{\mathscr{T}}$. Conversely, for a diagram $D: \mathscr{D} \to \mathscr{K}^{\mathscr{T}}$, its colimit is the coequalizer of $T\delta$ and $\mu \cdot Tk$ (see I.3) which is obtained as the reflection of (A, B, δ, k).

The categories $\mathscr{K}^*(T)$ are discussed in [7].

II. Finitary Monads and Relations

II.1. The aim of the present section is to prepare ground for the claim, made in the next section that finitary algebras have "simple colimits." We shall prove that right exact monads $\mathscr{T}$, preserving directed unions (both notions are explained below) have the property that the forgetful functor

$$\mathscr{U}^{\mathscr{T}}: \mathscr{K}^{\mathscr{T}} \to \mathscr{K}$$

preserves reflexive coequalizers. Now, this is a theorem of Barr [5], who supposed the category $\mathscr{K}$ to be so-called $EX5$, which is a very restrictive set of conditions. (All varieties of finitary algebras and the category of sets are $EX5$. But topological spaces, posets, σ-lattices or complete lattices, etc., fail to be $EX5$.) Barr's result was generalized by the first author [1] who weakened the hypothesis on the theory $\mathscr{T}$: in place of preservation of filtered colimits he proved that it suffices that $\mathscr{T}$ preserves unions of ω-sequences of split monos.

We shall now generalize Barr's result in another direction, weakening the hypothesis on $\mathscr{K}$. All we need is that $\mathscr{K}$ be well powered and have finite limits, directed unions and regular factorizations, i.e., a factorization system (regular epis, monos). These are very mild conditions, of course. The method for proving the mentioned theorem is a modification of that exhibited by Barr. We work with relations in a category . Since we do not assume that pullbacks preserve regular epis, relations must be treated carefully; e.g., composition of relations need not be associative!

II.2. In what follows we assume that $\mathscr{K}$ is a finitely complete category with a factorization system $(\mathscr{E}, \mathscr{M})$. Then a *relation* $r: A \rightharpoonup B$ is a subobject of $A \times B$. This can be represented either by an $\mathscr{M}$-mono $r: R \to A \times B$ or by a collectively $\mathscr{M}$-mono pair $r_1: R \to A$ and $r_2: R \to B$ (i.e., a pair such that the induced morphism $R \to A \times B$ is an $\mathscr{M}$-mono). Relations are naturally ordered—as subobjects.

The *composition* of relations $r: A \rightharpoonup B$ and $s: B \rightharpoonup C$ is defined as follows. Let $r: R \to A \times B$ and $s: S \to B \times C$ be representations of r and s by $\mathscr{M}$-monos. Then also $r \times 1: R \times C \to A \times B \times C$ and $1 \times s: A \times S \to A \times B \times C$ are $\mathscr{M}$-monos, representing two subobjects of $A \times B \times C$. Now $s \cdot r: A \rightharpoonup C$ is the image (in the sense of the factorization system) of $\pi \cdot [(r \times 1) \cap$

$(1 \times s)]$, where $\pi: A \times B \times C \to A \times C$ is the projection. In more detail, let $(r \times 1) \cdot \tilde{s} = (1 \times s) \cdot \tilde{r}$

$$\begin{tikzcd}
 & & \tilde{T} \arrow[dl, "\tilde{s}"'] \arrow[dr, "\tilde{r}"] \arrow[dddll, dashed, "e"'] & \\
 & R \times C \arrow[dr, "r \times 1"'] & & A \times S \arrow[dl, "1 \times s"] \\
 & & A \times B \times C \arrow[d, "\pi"] & \\
T \arrow[rr, "m"] & & A \times C &
\end{tikzcd}$$

be a pullback and let

$$m \cdot e = \pi \cdot (r \times 1) \cdot \tilde{s} = \pi \cdot (s \times 1) \cdot \tilde{r}$$

be an $(\mathscr{E}, \mathscr{M})$-factorization. Then $s \cdot r$ is the subobject of $A \times C$ represented by m.

It is easy to see that composition of relations is isotonic:

$$r \subset r' \quad \text{and} \quad s \subset s' \quad \text{imply} \quad r \cdot s \subset r' \cdot s'.$$

The *inverse relation* $r^{-1}: B \rightharpoonup A$ is defined as follows for a relation $r: A \rightharpoonup B$: if r is represented by

$$r: R \to A \times B$$

then r^{-1} is represented by

$$\xi \cdot r: R \to B \times A,$$

where $\xi: A \times B \to B \times A$ is the natural isomorphism. It is easy to verify that "inverse" reverses composition: $(r \cdot s)^{-1} = s^{-1} \cdot r^{-1}$. Also $(r^{-1})^{-1} = r$.

II.3. A relation $r: A \rightharpoonup A$ is an *equivalence* if it is

(i) reflexive, i.e., $\Delta_A \subset r$, where $\Delta_A: A \to A \times A$ is the diagonal;

(ii) symmetric, i.e., $r = r^{-1}$;

(iii) transitive, i.e., $r \cdot r \subset r$.

An example of equivalence is the kernel pair of a morphism $f: A \to B$, i.e., the pair $r_1, r_2: R \to A$, forming the pullback of f and f. In a number of categories (e.g., all varieties of algebras) kernel pairs are the only equivalences. On the other hand, in the category of topological spaces (or posets) there exist equivalences which are not even strict subobjects of $A \times A$.

II.4. DEFINITION [5]. A functor is *right exact* if it preserves coequalizers of pairs, representing equivalences.

EXAMPLE. Every functor $F: \mathrm{Set} \to \mathscr{K}$ and $F: R\text{-Vect} \to \mathscr{K}$ (where R-Vect is the category of vector spaces over a field R) is right exact. Indeed, coequalizers of equivalences in Set and R-Vect are split [9] hence absolute.

Generally, to be right exact is a slightly stronger condition than to preserve regular epis. (If F is right exact, it preserves regular epis, because each regular epi is the coequalizer of its kernel pair.)

II.5. *Construction.* Let $\mathscr{K}$ be a well-powered, finitely complete category with directed unions (of $\mathscr{M}$-monos). Let $r: A \rightharpoonup A$ be a reflexive relation. Define a chain of relations r_i, i an arbitrary ordinal, by transfinite induction:

$$r_0 = r \cdot r^{-1},$$

$$r_{i+1} = \bigcup_{s \in \overline{\{r_i\}}} s,$$

$$r_\alpha = \bigcup_{i<\alpha} r_i \qquad \text{for a limit ordinal } \alpha.$$

Here $\overline{\{r_i\}}$ denotes the least set of relations on A which contains r_i and is closed under composition; its elements are:

$$r_i;\ r_i \cdot r_i;\ r_i \cdot (r_i \cdot r_i);\ (r_i \cdot r_i) \cdot r_i;\ (r_i \cdot r_i) \cdot (r_i \cdot r_i);\ \text{etc.}$$

Then

(i) $i \leqslant j$ implies $r_i \subset r_j$ and each set $\overline{\{r_i\}}$ is directed;

(ii) there exists an ordinal α such that

$$r^* = r_\alpha$$

is an equivalence;

(iii) r^* is the least equivalence, containing r;

(iv) pairs, representing r and r^*, say,

$$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} A \qquad \text{and} \qquad R^* \overset{r_1^*}{\underset{r_2^*}{\rightrightarrows}} A,$$

have the same coequalizer (if any).

Proof. Since r is reflexive, clearly so are all r_i because a composition of reflexive relations is reflexive. Now, for each i, $r_i \in \overline{\{r_i\}}$ implies $r_i \subset r_{i+1}$. It easily follows that $r_i \subset r_j$ whenever $i \leqslant j$. Also each $\overline{\{r_i\}}$ is directed: given s', $s'' \in \overline{\{r_i\}}$ then also $s = s' \cdot s'' \in \overline{\{r_i\}}$ and we have

$$s' = s' \cdot \Delta \subset s' \cdot s'' = s \qquad \text{and} \qquad s'' = \Delta \cdot s'' \subset s' \cdot s'' = s.$$

Since $\mathscr{K}$ is well powered, there exist ordinals $\alpha < \beta$ with $r_\alpha = r_\beta$ (or else, $A \times A$ would have a proper class of distinct subobjects). It follows that

$$r_\alpha \cdot r_\alpha \subset r_{\alpha+1} \subset r_\beta = r_\alpha;$$

hence r_α is transitive. Each r_i is reflexive and symmetric (r_0 is symmetric since $r_0^{-1} = (r \cdot r^{-1})^{-1} = r \cdot r^{-1} = r_0$; by transfinite induction each r_i is symmetric). Therefore, r_α is an equivalence. Put $r^* = r_\alpha$.

If t is any equivalence, containing r, then t contains each r_i. Indeed (1) $r_0 = r \cdot r^{-1} \subset t \cdot t^{-1} = t$; (2) if $r_i \subset t$ then also $s \subset t$ for each $s \in \overline{\{r_i\}}$, hence $r_{i+1} \subset t$; and (3) $r_\beta = \bigcup_{i<\beta} r_i \subset t$ provided that $r_i \subset t$ for each $i < \beta$. Thus, t contains r^*.

Finally, given representations as in (iv) and given a morphism $f: A \to B$ we shall verify that

$$f \cdot r_1 = f \cdot r_2 \qquad \text{iff} \quad f \cdot r_1^* = f \cdot r_2^*.$$

Since $r \subset r^*$, clearly $f \cdot r_1^* = f \cdot r_2^*$ implies $f \cdot r_1 = f \cdot r_2$. The converse is also proved easily: let t be the equivalence, represented by the kernel pair of f. If $f \cdot r_1 = f \cdot r_2$ then $r \subset t$. There follows $r^* \subset t$, hence $f \cdot r_1^* = f \cdot r_2^*$. Thus, r and r^* have the same coequalizer.

II.6. Convention. A functor, not necessarily preserving subobjects, is said to *preserve a union* $m = \bigcup_{i \in I} m_i$ if

$$\operatorname{im} Fm = \bigcup_{i \in I} \operatorname{im} Fm_i$$

where im denotes the image with respect to the factorization system $(\mathscr{E}, \mathscr{M})$.

II.7. Lemma. *Let $\mathscr{K}$ be a category with pullbacks and directed unions. Then for every monad $\mathbb{F}$, preserving directed unions, the category $\mathscr{K}^{\mathbb{F}}$ has directed unions, preserved by $\mathscr{U}^{\mathbb{F}}: \mathscr{K}^{\mathbb{F}} \to \mathscr{K}$.*

Proof. Let $m_i: (A_i, \delta_i) \to (C, \gamma)$ be a directed family of subalgebras (i.e., $m_i \in \mathscr{M}$). Let

$$m = \bigcup_{i \in I} m_i: A \to C$$

and let $Tm = n \cdot e$; $Tm_i = n_i \cdot e_i$ be image factorizations.

$$\begin{tikzcd}
TA \arrow[d, "e"'] & & D \arrow[dll, dashed, "\tilde{m}"'] \arrow[rr, dashed, "h"] & & A \arrow[ddd, bend left, "m"] \\
A^* \arrow[dd, "n"'] & & TA_i \arrow[u, dashed, "u_i"'] \arrow[rr, "\delta_i"] \arrow[dl, "e_i"] & & A_i \arrow[u, dashed, "v_i"'] \arrow[dd, "m_i"] \\
& A_i^* \arrow[uur, dotted, "\tilde{u}_i"] \arrow[ul, dotted, "w_i"] \arrow[dl, "n_i"] & & & \\
TC \arrow[rrrr, "\gamma"'] & & & & C
\end{tikzcd}$$

By hypothesis, $n = \bigcup_{i \in I} n_i$. Consider the pullback $(n \cdot \gamma) \cdot \tilde{m} = m \cdot h$ of $n \cdot \gamma$ and m. For each i we have the embeddings $v_i : A_i \to A$ and $w_i : A_i^* \to A^*$ and

$$m \cdot (v_i \cdot \delta_i) = m_i \cdot \delta_i = \gamma \cdot Tm_i = (\gamma \cdot n) \cdot (w_i \cdot e_i).$$

Therefore, there exists u_i: $TA_i \to D$ with $\tilde{m} \cdot u_i = w_i \cdot e_i$ (and $h \cdot u_i = v_i \cdot \delta_i$). Now, $m \in \mathscr{M}$ implies $\tilde{m} \in \mathscr{M}$; since $e_i \in \mathscr{E}$ we can use the diagonal fillin (see [10]) to obtain

$$\tilde{u}_i : A_i^* \to D \qquad \text{with} \quad \tilde{m} \cdot \tilde{u}_i = w_i$$

hence

$$n_i = n \cdot w_i = (n \cdot \tilde{m}) \cdot \tilde{u}_i.$$

Now, each n_i factorizes through $n \cdot \tilde{m}$ and $n = \bigcup n_i$—therefore $\tilde{m}$ is an isomorphism.

Define δ: $TA \to A$ by

$$\delta = h \cdot \tilde{m}^{-1} \cdot e$$

and note that

$$m \cdot \delta = \gamma \cdot n \cdot e = \gamma \cdot Tm. \tag{$*$}$$

It suffices to verify that (A, δ) is a $\mathscr{T}$-algebra; then clearly (A, δ) is the union of (A_i, δ_i) (more precisely, $m = \bigcup m_i$ in $\mathscr{K}^{\mathscr{T}}$ with respect to m: $(A, \delta) \to (C, \gamma)$, which is a homomorphism by $(*)$). This will prove the lemma.

To prove $\delta \cdot \eta_A = 1$ we use the fact that m is a mono:

$$\begin{aligned}
m \cdot \delta \cdot \eta_A &= \gamma \cdot Tm \cdot \eta_A && \text{by } (*) \\
&= \gamma \cdot \eta_C \cdot m && \eta \text{ is a transformation} \\
&= m && \gamma \cdot \eta_C = 1.
\end{aligned}$$

Analogously, to prove $\delta \cdot T\delta = \delta \cdot \mu_A$ we use the same mono:

$$\begin{aligned} m \cdot \delta \cdot \mu_A &= \gamma \cdot Tm \cdot \mu_A && \text{by } (*) \\ &= \gamma \cdot \mu_C \cdot T^2 m && \mu \text{ is a transformation} \\ &= \gamma \cdot T\gamma \cdot T^2 m && \gamma \cdot \mu_C = \gamma \cdot T\gamma \\ &= \gamma \cdot T(m \cdot \delta) && \text{by } (*) \\ &= m \cdot \delta \cdot T\delta && \text{by } (*) \text{ again.} \end{aligned}$$

II.8. THEOREM. *Let $\mathscr{K}$ be a finitely complete, well-powered category with directed unions and regular factorizations. Let $\mathscr{T}$ be a right exact monad, preserving directed unions. Then $\mathscr{U}^{\mathscr{T}}: \mathscr{K}^{\mathscr{T}} \to \mathscr{K}$ preserves reflexive coequalizers.*

Remark. In the above theorem we have abandoned an arbitrary factorization system and we work with (regular epi, mono)-factorizations. The only reason for this is that we need a transfer to a factorization system $(\mathscr{E}^{\mathscr{T}}, \mathscr{M}^{\mathscr{T}})$ in $\mathscr{K}^{\mathscr{T}}$; hence we need $\mathscr{T}$ to preserve $\mathscr{E}$-epis, see [10]. Since $\mathscr{T}$ is right exact, it preserves regular epis. If $\mathscr{T}$, moreover, preserves $\mathscr{E}$-epis, then it suffices that $\mathscr{K}$ have $(\mathscr{E}, \mathscr{M})$-factorizations for an arbitrary $\mathscr{M}$.

In III.12 we shall see that it does not suffice in the above theorem that $\mathscr{T}$ preserve regular epis (rather then be right exact) even for "nearly unary" theories $\mathscr{T}$ in the category of Hausdorff spaces.

Proof of theorem. The category $\mathscr{K}^{\mathscr{T}}$ has finite limits, is well powered and has directed unions (II.7) and regular factorizations—indeed, $\mathscr{U}^{\mathscr{T}}$ creates all limit and all those colimits, preserved by $\mathscr{T}$ (hence, creates coequalizers of equivalences). Thus, Construction II.5 applies to $\mathscr{K}^{\mathscr{T}}$.

Let $f, g: P \to Q$ be a reflexive pair in $\mathscr{K}^{\mathscr{T}}$. The induced morphism $P \to Q \times Q$ has a regular factorization, say

$$P \xrightarrow{e} R \xrightarrow{r} Q \times Q.$$

Moreover, $\mathscr{U}^{\mathscr{T}} r \cdot \mathscr{U}^{\mathscr{T}} e$ is the regular factorization of the morphism, induced by $\mathscr{U}^{\mathscr{T}} f$, $\mathscr{U}^{\mathscr{T}} g$. Since e is epi, the coequalizer of f, g coincides with that of $\pi_1 \cdot r$, $\pi_2 \cdot r$ (where $\pi_i: Q \times Q \to Q$ are projections). The same holds for the coequalizer of $\mathscr{U}^{\mathscr{T}} f$, $\mathscr{U}^{\mathscr{T}} g$. Thus, it suffices to show that $\mathscr{U}^{\mathscr{T}}$ preserves coequalizers of pairs representing reflexive relations (r is reflexive since f, g have a common coretraction).

Since $\mathscr{U}^{\mathscr{T}}$ creates finite limits, it preserves relations, their composition and inverse relations. It also preserves directed unions (II.7). Hence, it "preserves" Construction II.5; more precisely:

$$\mathscr{U}^{\mathscr{T}} r^* = (\mathscr{U}^{\mathscr{T}} r)^*.$$

Since r and r^* have the same coequalizer, and so do $\mathscr{U}^{\mathscr{T}} r$ and $(\mathscr{U}^{\mathscr{T}} r)^*$, and since $\mathscr{U}^{\mathscr{T}}$ preserves the coequalizer of the equivalence r^*, it follows that $\mathscr{U}^{\mathscr{T}}$ preserves the coequalizer of r.

II.9. If $\mathscr{K}$ and $\mathscr{T}$ are as in the above theorem then the functor $\mathscr{U}^{\mathscr{T}}$ preserves finite cointersections (pushouts) of regular quotients, see [5]. If the theory $\mathscr{T}$ not only preserves directed unions but also directed colimits (is *finitary*) then $\mathscr{U}^{\mathscr{T}}$ preserves cointersections. Indeed, $\mathscr{U}^{\mathscr{T}}$ preserves both finite and directed cointersections, thus all cointersections.

II.10. COROLLARY. *Let $\mathscr{K}$ be a complete and cocomplete, well-powered category with regular factorizations. Then for each finitary, right exact monad $\mathscr{T}$ in $\mathscr{K}$*

(a) *$\mathscr{U}^{\mathscr{T}}$ preserves cointersections;*

(b) *quotient algebras of any $\mathscr{T}$-algebra (A, δ) form a complete lattice, which is a complete sublattice of the lattice of quotients of A.*

Proof. Since $\mathscr{K}$ is well powered, it is also regularly co-well powered (proof: for distinct regular quotients, represented by $f_1: A \to B_1$ and $f_2: A \to B_2$, their kernel pairs p_1, q_1 and p_2, q_2 are distinct subobjects of $A \times A$, because f_i is the coequalizer of p_i, q_i for $i = 1, 2$). Since $\mathscr{K}$ is complete, the quotient posets $Q(A)$ of any object A form a complete lattice: the infimum of any set of regular epis $f_i: A \to B_i$ is obtained as regular coimage of the induced morphism $A \to \prod B_i$.

$\mathscr{U}^{\mathscr{T}}$ preserves (indeed, creates, in the sense of [9]) regular factorizations and limits. Hence $\mathscr{K}^{\mathscr{T}}$ is well powered (hence, regularly co-well powered) and complete. Moreover, the embedding

$$Q^{\mathscr{T}}(A, \delta) \mapsto Q(A)$$

of the quotient lattice of (A, δ) in $\mathscr{K}^{\mathscr{T}}$, which is induced by $\mathscr{U}^{\mathscr{T}}$, preserves infima. Now, by (a), which was proved above, $\mathscr{U}^{\mathscr{T}}$ also preserves suprema (=cointersections). Hence, $Q^{\mathscr{T}}(A, \delta)$ is a complete sublattice of $Q(A)$.

III. SIMPLE COLIMITS

III.1. Given a diagram D of algebras, its colimit is the coequalizer of $T\delta$ and $\mu \cdot Tk$, see I.4. The situation with $\operatorname{colim} D$ is extremely simplified provided that this coequalizer can be computed in the base category $\mathscr{K}$. If this is the case, we say that D has a simple colimit:

III.2. DEFINITION. A diagram D in the category $\mathscr{K}^{\mathscr{T}}$ is said to have a *simple colimit* if the forgetful functor $\mathscr{U}^{\mathscr{T}}$ creates the coequalizer of $T\delta$ and $\mu \cdot Tk$. In more detail let

$$TB \underset{\mu \cdot Tk}{\overset{T\delta}{\rightrightarrows}} TA \xrightarrow{c} C$$

be a coequalizer in $\mathscr{K}$. Then colim D is simple if there exists γ: $TC \to C$ such that c: $(TA, \mu) \to (C, \gamma)$ is the coequalizer of $T\delta$ and $\mu \cdot Tk$ in $\mathscr{K}^{\mathscr{T}}$.

For example, if the monad $\mathscr{T}$ preserves reflexive coequalizers then $\mathscr{U}^{\mathscr{T}}$ creates them, hence $\mathscr{K}^{\mathscr{T}}$ has simple colimits. This is the case, e.g., of β-compactification of completely regular spaces; thus, compact spaces (which are the corresponding algebras) have simple colimits in the category of completely regular spaces.

III.3. EXAMPLE. Coproducts of complete semi-lattices are simple. Denote by $\mathscr{P} = (P, \mu, \eta)$ the power-set monad of complete semi-lattices (in $\mathscr{K} = \mathrm{Set}$). Given complete semi-lattices $(A_i, \sup_i)$ for $i \in I$, put

$$\begin{array}{llll} A = \coprod_{i \in I} A_i & \text{and} & B = \coprod_{i \in I} PA_i & \text{in Set;} \\ & \delta: B \to A; & \delta(X) = \sup_i X & \text{for } X \in PA_i; \\ & k: B \to PA; & k(X) = X & (PA_i \subset PA \text{ for each } i). \end{array}$$

Then $P\delta$, $\mu \cdot Pk$: $PB \to PA$ are as follows. Given $\mathfrak{X} \in PB$ we have $\mathfrak{X} = \{X_t; t \in T\}$, where $X_t \in PA_{i_t}$ (with $i_t \in I$ for $t \in T$). Then

$$P\delta(\mathfrak{X}) = \{\sup_{i_t} X_t\}_{t \in T},$$
$$Pk(\mathfrak{X}) = \bigcup_{t \in T} X_t.$$

It is easy to find the coequalizer of $P\delta$ and $\mu \cdot Pk$ in Set: it is

$$c: P(\coprod A_i) \to \prod A_i,$$
$$c(Y) = \{x_i\}_{i \in I}, \qquad \text{where } x_i = \sup_i(Y \cap A_i) \text{ for } Y \subset \coprod A_i.$$

Moreover, endowing the set $\prod A_i$ with the semi-lattice structure of a product, the map c becomes a homomorphism. It follows that $(\prod A_i, \sup) = \coprod(A_i, \sup_i)$ is a simple coproduct, via the observation below.

III.4. *Observation.* Let $\mathscr{K}$ have a factorization system $(\mathscr{E}, \mathscr{M})$ and let the monad $\mathscr{T}$ preserve $\mathscr{E}$-epis. Then colim D is simple iff for the coequalizer (in $\mathscr{K}$) $c = \mathrm{coeq}(T\delta, \mu \cdot Tk)$

$$\begin{tikzcd} TB \arrow[r, shift left, "T\delta"] \arrow[r, shift right, "\mu \cdot Tk"'] & TA \arrow[r, "c"] & C \\ & T^2A \arrow[u, dashed, "\mu"] \arrow[r, "Tc"] & TC \arrow[u, dashed, "?"'] \end{tikzcd}$$

there exists a morphism $\gamma\colon TC \to C$ with $\gamma \cdot Tc = c \cdot \mu$. For $\mathscr{K} = \mathrm{Set}$, colim D is simple iff the equivalence $\ker(c)$ is a congruence on the free algebra (TA, μ).

Proof. This follows from the result on categories $\mathscr{K}(T)$, mentioned already in I.6 (part (c) of the proof): it suffices to show that c is a coequalizer in $\mathscr{K}(T)$, and this is evident. For $\mathscr{K} = \mathrm{Set}$, each monad preserves epis.

III.5. COROLLARY. *Let $\mathscr{K}$ be a cocomplete, $\mathscr{E}$-co-well-powered category with a factorization system $(\mathscr{E}, \mathscr{M})$; let $\mathscr{T}$ preserve $\mathscr{E}$-epis. If the forgetful functor $\mathscr{U}^{\mathscr{T}}\colon \mathscr{K}^{\mathscr{T}} \to \mathscr{K}$ preserves reflexive coequalizers, then $\mathscr{K}^{\mathscr{T}}$ has simple colimits.*

Proof. The pair $T\delta$ and $\mu \cdot Tk$ is reflexive (see Theorem I.2). Under the present hypothesis, $\mathscr{K}^{\mathscr{T}}$ is cocomplete (see Theorem I.6); hence there exists a coequalizer

$$(TB, \mu) \underset{\mu \cdot Tk}{\overset{T\delta}{\rightrightarrows}} (TA, \mu) \xrightarrow{c} (C, \gamma)$$

in $\mathscr{K}^{\mathscr{T}}$. And $\mathscr{U}^{\mathscr{T}}$ preserves this coequalizer; thus

$$TB \underset{\mu \cdot Tk}{\overset{T\delta}{\rightrightarrows}} TA \xrightarrow{c} C$$

is a coequalizer in $\mathscr{K}$. The operation γ is what is required in the preceding observation.

III.6. THEOREM. *Let $\mathscr{K}$ be a cocomplete, finitely complete and well-powered category with regular factorizations. Let $\mathscr{T}$ be a right exact monad.*

If $\mathscr{T}$ preserves directed unions (particularly, if $\mathscr{T}$ is finitary), then $\mathscr{K}^{\mathscr{T}}$ has simple colimits.

Proof. Combine Theorems III.5 and II.8. Since $\mathscr{K}$ is well powered, it is also regularly-co-well powered, for distinct regular quotients have distinct kernel pairs.

III.7. *Remark.* In particular, all finitary monads in Set (groups, lattices, vector spaces, etc.) have simple colimits.

III.8. EXAMPLE. Coproducts of semigroups.

Let $\mathscr{S} = ((-)^+, \mu, \eta)$ be the (string) monad of semigroups in Set: X^+ is the set of all non-void strings $(x_1, ..., x_n)$ in X; f^+ sends $(x_1, ..., x_n)$ to $(f(x_1), ..., f(x_n))$; η sends $x \in X$ to $(x) \in X^+$ and $\mu: X^{++} \to X^+$ is the concatenation:

$$\mu((x_{1,1}, ..., x_{1,n_1}), (x_{2,1}, ..., x_{2,n_2}), ..., (x_{r,1}, ..., x_{r,n_r}))$$
$$= (x_{1,1}, ..., x_{1,n_1}, x_{2,1}, ..., x_{2,n_2}, ..., x_{r,1}, ..., x_{r,n_r}).$$

Given semigroups $(A_i, \cdot_i)$ for $i \in I$, we put

$$A = \coprod_{i \in I} A_i \qquad \text{and} \qquad B = \coprod_{i \in I} A_i^+ \qquad \text{in Set}$$

(or, simply, $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{i \in I} A_i^+$ assuming $A_i^+ \cap A_j^+ = \emptyset$ for $i \neq j$). Then $\delta: B \to A$ is defined by $\delta(x_1, ..., x_n) = x_1 \cdot_i x_2 \cdot_i \cdots \cdot_i x_n$, if $(x_1, ..., x_n) \in A_i^+$, $i \in I$, while $k: B \to A^+$ is the inclusion map ($B \subset A^+$). Thus, the maps δ^+ and $\mu \cdot k^+$ are defined as follows. Given $\xi = (\xi_1, ..., \xi_n)$ in B^+ with $\xi_t = (x_{t,1}, ..., x_{t,m_t}) \in A_{i_t}^+$ then

$$\delta^+(\xi) = (x_{1,1} \cdot_{i_1} x_{1,2} \cdot_{i_1} \cdots \cdot_{i_1} x_{1,m_1}, ..., x_{n,1} \cdot_{i_n} x_{n,2} \cdot_{i_n} \cdots \cdot_{i_n} x_{n,m_n}),$$

$\mu(\mathscr{U} \cdot k^+(\xi) = (x_{1,1}, x_{1,2}, ..., x_{1,n_1}, ..., x_{n,1}, x_{n,2}, ..., x_{n,m_n})$. The coequalizer of δ^+, $\mu \cdot k^+: B^+ \to A^+ = (\bigcup A_i)^+$ is c: $(\bigcup A_i)^+ \to C$ with $C = \{(x_1, ..., x_r) \in (\bigcup A_i)^+;\ x_t \in A_i$ implies $x_{t+1} \notin A_i$ for $1 \leqslant t < r\}$. The map c is the obvious reduction of an arbitrary string $(y_1, ..., y_m)$ in $\bigcup A_i$ to a string belonging to C (in which $y_t, y_{t+1} \in A_i$ are reduced to $y_t \cdot_i y_{t+1}$).

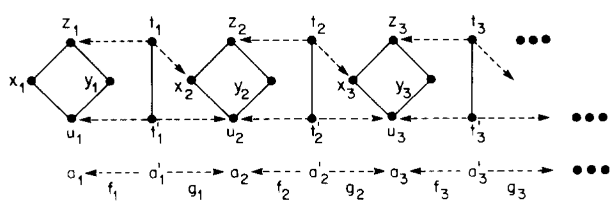
Thus, we know that C can be equipped with a semigroup multiplication $*$ such that $(C, *) = \coprod_{i \in I} (A_i, \cdot_i)$. Indeed, this is

$$(x_1, ..., x_n) * (y_1, ..., y_m) = (x_1, ..., x_n, y_1, ..., y_m) \text{ if } x_n \in A_i,\ y_1 \in A_j \text{ and } i \neq j,$$

$$(x_1, ..., x_n) * (y_1, ..., y_m) = (x_1, ..., x_{n-1}, x_n \cdot_i y_1, y_2, ..., y_m) \quad \text{if } x_n, y_1 \in A_i.$$

III.9. EXAMPLE. Colimits of complete (or σ-complete) semi-lattices are not simple (though coproducts are, by III.3!).

Let us consider the following diagram D of complete semi-lattices (objects $A_1, A_1', A_2, A_2', ...$ and morphisms $f_1, g_1, f_2, g_2, ...$):



The colimit of the underlying diagram D_0 in Set is

$$A = \{y_i\}_{i=1}^{\infty} \cup \{z_i\}_{i=1}^{\infty} \cup \{x_1, u_1\}$$

with

$v_1 : A_1 \to A$ the inclusion map and

$v_i' : A_i' \to A; \qquad t_i \mapsto z_i, t_i' \mapsto u_1,$

$v_i : A_i \to A; \qquad y_i \mapsto y_i, z_i \mapsto z_i, x_i \mapsto z_{i-1}, u_i \mapsto u_1$ (for $i \neq 1$).

The colimit of PD_0 is

$$B = \left[\bigcup_{i=2}^{\infty} (PA_i - \{\varnothing, \{x_i\}, \{u_i\}, \{x_i, u_i\}\}) \right] \cup PA_1$$

with

$w_1 : PA_1 \to B$ the inclusion map and

$w_i' : PA_i' \to B; \qquad \varnothing \mapsto \varnothing, \{t_i\} \mapsto \{z_i\}, \{t_i'\} \mapsto \{u_1\}, \{t_i, t_i'\} \mapsto \{z_i, u_i\};$

$w_i : PA_i \to B; \qquad \{x_i\} \mapsto \{z_{i-1}\}, \{u_i\} \mapsto \{u_1\}, \{x_i, u_i\} \mapsto \{z_{i-1}, u_1\},$

$T \mapsto T$ for all other $T \subset A_i$ (for $i \neq 1$).

We shall denote $\bar{A}_i = v_i(A_i) - \{u_1\}$ for $i = 1, 2, 3, \ldots$; $\bar{A}_i \subset A$. The maps $\delta: B \to A$ and $k: B \to PA$ are as follows. Given $T \subset A_i$, $T \in B$:

$$\delta(T) = z_i \qquad \text{if card } T \geqslant 2,\ T \neq \{u_i, y_i\} \text{ and } T \neq \{u_1, x_1\};$$

$$\delta(\varnothing) = u_1;\ \delta(\{y_i\}) = y_i,\ \delta(\{z_i\}) = z_i,\ \delta(\{x_1\}) = x_1;$$

$$\delta(\{u_1\}) = u_1;\ \delta(\{x_1, u_1\}) = x_1;\ \delta(\{u_i, y_i\}) = y_i;$$

while

$k(T) = T \qquad$ if $u_i, x_i \notin T$ or $i = 1$;

$k(T) = (T - \{x_i\}) \cup \{z_{i-1}\} \qquad$ if $x_i \in T$, $u_i \notin T$ and $i \neq 1$;

$k(T) = (T - \{u_i\}) \cup \{u_1\} \qquad$ if $u_i \in T$, $x_i \notin T$ and $i \neq 1$;

$k(T) = (T - \{u_i, x_i\}) \cup \{u_1, z_{i-1}\} \qquad$ if $x_i, u_i \in T$ and $i \neq 1$.

Let

$$PB \underset{\mu \cdot Pk}{\overset{P\delta}{\rightrightarrows}} PA \xrightarrow{c} C$$

be a coequalizer in Set. We shall find sets $X_1, X_2 \in PA$ such that

$$c(X_1) \neq c(X_2)$$

but, in the colimit of D in Set$^{\mathscr{T}}$, i.e., in the coequalizer

$$(PB, \mu) \underset{\mu \cdot Pk}{\overset{P\delta}{\rightrightarrows}} (PA, \mu) \xrightarrow{c_0} (C_0, \gamma)$$

we have $c_0(X_1) = c_0(X_2)$.

This will show that the colimit of D is not simple. To this end, let $X_1 = A$ and $X_2 = \{z_{2^n}; n \in N\}$. It is evident that $c_0(X_1) = c_0(X_2)$ is the largest element of C_0. We shall now verify that $c(X_1) \neq c(X_2)$.

Given $\Sigma \in PB$ and $i = 1, 2, 3, \ldots$ then

(1) $P\delta(\Sigma) \cap \bar{A}_i \neq \varnothing$ implies $\mu \cdot Pk(\Sigma) \cap (\bar{A}_{i-1} \cap \bar{A}_i) \neq \varnothing$ (proof: we have $T_0 \in \Sigma$ with $\delta(T_0) \in \bar{A}_i$, then $T_0 - \{u_i\} \subset \bar{A}_i$ or $T_0 - \{u_{i-1}\} \subset \bar{A}_{i-1}$; hence $k(T_0) \subset \bar{A}_i \cup \bar{A}_{i-1}$; we have $\mu \cdot Pk(\Sigma) = \bigcup_{T \in \Sigma} k(T)$);

(2) $\mu \cdot Pk(\Sigma) \cap \bar{A}_i \neq \varnothing$ implies $P\delta(\Sigma) \cap (\bar{A}_i \cup \bar{A}_{i+1}) \neq \varnothing$ (proof: we have $T_0 \in \Sigma$ with $k(T_0) \cap \bar{A}_i \neq \varnothing$, then $T_0 \subset A_i$ or $z_i \in T_0$; hence $\delta(T_0) \subset \bar{A}_i \cup \bar{A}_{i+1}$).

It easily follows that $c(A) \neq c(\{z_{2^n}\}_{n=0}^{\infty})$. Indeed, if the coequalizer merges A and $\{z_{2^n}\}$ then there exist $\Sigma_0, \ldots, \Sigma_{p-1} \in PB$

$$\begin{tikzcd}[column sep=small]
& \Sigma_0 \arrow[dl, "P\delta"'] \arrow[dr, "\mu.Pk"] & & \Sigma_1 \arrow[dl, "P\delta"'] \arrow[dr, "\mu.Pk"] & & \Sigma_2 \arrow[dl, "P\delta"'] \; \cdots & & \Sigma_{p-1} \arrow[dl, "P\delta"'] \arrow[dr, "\mu.Pk"] & \\
A & & Y_1 & & Y_2 & & Y_{p-1} & & \{z_{2^n}\}
\end{tikzcd}$$

such that $P\delta(\Sigma_j) = \mu \cdot Pk(\Sigma_{j-1}) = Y_j$ and $A = P\delta(\Sigma_0)$; $\{z_{2^n}\} = \mu \cdot Pk(\Sigma_{p-1})$ (or, conversely, $A = \mu \cdot Pk(\Sigma_{p-1})$; $\{z_{2^n}\} = P\delta(\Sigma_0)$, which is analogous). For every $i = 1, 2, 3, \ldots$ we have $A \cap \bar{A}_i \neq \varnothing$; hence, by (1)

$$\begin{aligned}
&Y_1 \cap (\bar{A}_{i-1} \cup \bar{A}_i) \neq \varnothing, \\
&Y_2 \cap (\bar{A}_{i-2} \cup \bar{A}_{i-1} \cup \bar{A}_i) \neq \varnothing, \\
&\vdots \\
&\{z_{2^n}\} \cap (\bar{A}_{i-p} \cup \cdots \cup \bar{A}_i) \neq \varnothing.
\end{aligned}$$

This is a contradiction: we can certainly find i such that no power of 2 lies in the interval $\langle i - p, i \rangle$ and then $\{z_{2^n}\} \cap (\bar{A}_{i-p} \cup \cdots \cup \bar{A}_i) = \varnothing$.

Since all semi-lattices considered above were countable, the same example serves to show that σ-complete semi-lattices do not have simple colimits.

III.10. EXAMPLE. Ordered algebras need not have coproducts.

Let us define a monad $\mathscr{E}$ in the category Pos of posets and isotonic maps, such that Pos$^{\mathscr{E}}$ does not have coproducts ($\mathscr{E}$ will not preserve epis, of course). An analogous example in the category of graphs has been exhibited in [2].

For each poset A denote

$$A^{(3)} = \{(x, y, z) \in A \times A \times A; x < y < z\} + \{\xi\}$$

and for each isotonic map $f: A \to B$ define $f^{(3)}: A^{(3)} \to B^{(3)}$ by

$$\begin{aligned} f^{(3)}(x, y, z) &= f^{(3)}(\xi) = \xi & &\text{if } f(x) = f(y) \text{ or } f(y) = f(z); \\ f^{(3)}(x, y, z) &= (f(x), f(y), f(z)) & &\text{if } f(x) < f(y) < f(z). \end{aligned}$$

Denote by $PA^{(3)}$ the following poset of all subsets of $A^{(3)}$:

$$X \leqslant Y \text{ iff} \qquad X = \varnothing \text{ or } X = Y \; (X, Y \subset A^{(3)}).$$

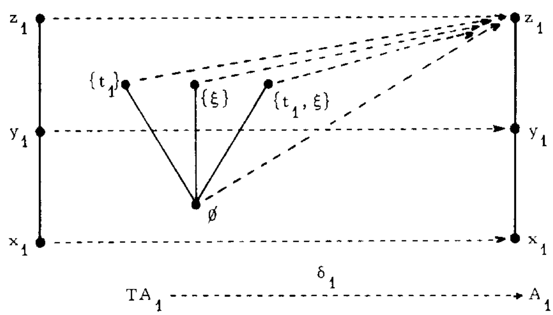
Now we define $\mathscr{T} = (T, \mu, \eta)$.

$TA = A + PA^{(3)}$ (a disjoint union of two posets);
$\eta_A; A \to A + PA^{(3)}$ is the first injection;
$\mu_A: A + PA^{(3)} + P(A + PA^{(3)})^{(3)} = A + PA^{(3)} + PA^{(3)} \to A + PA^{(3)}$ is 1_A on A, $1_{PA^{(3)}}$ on both copies of $PA^{(3)}$.

Define $\mathscr{T}$-algebras (A_1, δ_1) and (A_2, δ_2) as follows:

$$A_1 = \{x_1, y_1, z_1\} \qquad \text{with} \quad x_1 < y_1 < z_1;$$

then $A_1^{(3)} = \{t_1, \xi\}$, where $t_1 = (x_1, y_1, z_1)$, and we define $\delta_1: A_1 + P\{t_1, \xi\} \to A_1$: $\delta_1 = 1_{A_1}$ on A_1, $\delta_1 = \text{const } z_1$ on $P\{t_1, \xi\}$.



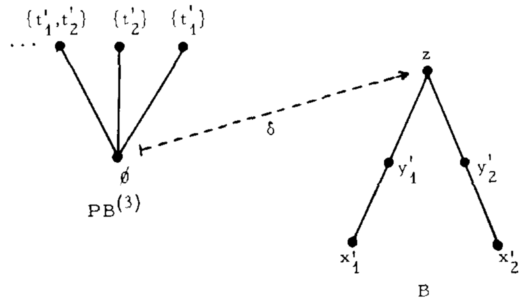
Analogously, $A_2 = \{x_2, y_2, z_2\}$, $A_2^{(3)} = \{t_2, \xi\}$, $\delta_2 = \text{const } z_2$ on $P\{t_2, \xi\}$. These $\mathscr{T}$-algebras (A_1, δ_1) and (A_2, δ_2) have no coproduct in $\text{Pos}^{\mathscr{T}}$.

Proof. Assume that, to the contrary, (B, δ) is the coproduct with injections $v_1: A_1 \to B$, $v_2: A_2 \to B$. Since (A_1, δ_1) and (A_2, δ_2) are isomorphic, v_1 and v_2 are clearly split monos (hence, one-to-one maps). Put $z = \delta(\varnothing)$; then

$$v_1(z_1) = v_1(\delta_1(\varnothing)) = \delta(Tv_1(\varnothing)) = z,$$

$$v_2(z_2) = z;$$

put $x_i' = v_i(x_i)$ and $y_i' = v_i(y_i)$ for $i = 1, 2$; then $x_i' < y_i' < z$ in B. Put $t_i' = (x_i', y_i', z) \in B^{(3)}$ for $i = 1, 2$. It is easy to verify that $y_1' \neq y_2'$, hence $t_1' \neq t_2'$.



Since v_1 is a $\mathscr{E}$-homomorphism, we see that

$$\delta(Q) = z \qquad \text{for any } Q \subset \{t_1', \xi\}$$

and, analogously,

$$\delta(Q) = z \qquad \text{for any } Q \subset \{t_2', \xi\}.$$

On the other hand, it follows from the properties of coproducts, that δ is one-to-one on $PB^{(3)} - (\exp\{t_1', \xi\} \cup \exp\{t_2', \xi\})$. Put $K = \{z' \in B;\ z' > z\}$. We have $\varnothing \preccurlyeq \{t_1', t_2'\}$ in $B^{(3)}$, thus $z = \delta(\varnothing) \leqslant \delta(\{t_1', t_2'\})$; put $z_0 = \delta(\{t_1', t_2'\})$: as mentioned above, $z_0 \neq z$, hence $z_0 \in K$. Analogously, define $t_3 = (y_1', z, z_0)$, then $\varnothing \preccurlyeq t_3$ implies $z = \delta(\varnothing) < \delta\{t_3\}$—we see that $\operatorname{card} K \geqslant 2$. Now, for every non-void subset $R \subset K$ put

$$R^* = \{(y_1', z, r);\ r \in R\} \subset B^{(3)}.$$

Then $\delta(R^*) \in K$ (because $\varnothing \preccurlyeq R^*$ and $\delta(\varnothing) \neq \delta(R^*)$). Therefore we get a one-to-one map $R \mapsto \delta(R^*)$ from $PK - \{\varnothing\}$ into K—a contradiction, because $\operatorname{card} K \geqslant 2$ implies $\operatorname{card}(PK - \{\varnothing\}) > \operatorname{card} K$.

III.11. Example. Finitary ordered algebras need not have simple coproducts.

Proceed analogously as above, defining a submonad $\mathscr{E}_\omega = (T_\omega, \mu_\omega, \eta_\omega)$ of $\mathscr{E}$ by

$$T_\omega A = A + P_\omega A^{(3)}, \qquad \text{where } P_\omega A^{(3)} \text{ is the sub-poset of } PA^{(3)} \text{ over } \textit{finite} \text{ subsets of } A^{(3)}.$$

Clearly, $\mathscr{E}_\omega$ is a finitary monad which, moreover, preserves finite algebras. The algebras (A_1, δ_1) and (A_2, δ_2) above are $\mathscr{E}_\omega$-algebras which do not have a simple coproduct (but they have a coproduct, by I.8c). Indeed, if the coproduct were simple, it would be a quotient of $T_\omega(A_1 + A_2)$, which is finite. But the coproduct $(A_1, \delta_1) + (A_2, \delta_2)$ is infinite. The proof is as above: we define K, show that card $K \geqslant 2$ and we have a one-to-one map

$$R \mapsto \delta(R^*)$$

from $P_\omega K - \{\varnothing\}$ into K. Then K is infinite, else card $(P_\omega K - \{\varnothing\}) >$ card K.

III.12. EXAMPLE. Unary algebras on Hausdorff spaces do not have simple colimits, though their monad is finitary and preserves regular epis.

Consider pretopological unary algebras, i.e., topological spaces X equipped with a transformation $\varepsilon: X \to X$, not necessarily continuous. Homomorphisms between them are continuous maps, respecting the unary operation. This is a monad in the category Top, which is easily seen to have simple colimits (its coequalizers, indeed all colimits, are preserved by the forgetful functor). Yet, its restriction to Hausdorff spaces fails to have simple colimits.

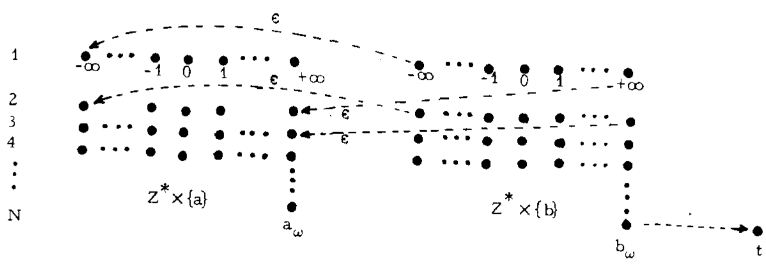
For simplicity, we consider the monad $\mathscr{T} = (T, \mu, \eta)$ of idempotent unary pretopological algebras in Haus (the category of Hausdorff spaces). This monad is a sum of the identity and the discrete identity. (We denote by X_0 the discrete space on the same set as the given space X; points $x \in X$ are denoted by x_0 when considered in X_0). More precisely:

$TX = X + X_0$; $Tf = f + f$ for a continuous map $f: X \to Y$;

$\mu_X: (X + X_0) + (X + X_0)_0 \to X + X_0$ embeds X onto X and all three copies of X_0 onto X_0; thus $\mu_X(x) = x$ and $\mu_X(x_0) = x_0$ $(x \in X)$;

$\eta_X: X \to X + X_0$ embeds X onto X; thus $\eta_X(x) = x$.

It is clear that T preserves filtered colimits, coproducts, and regular epis (=Hausdorff quotient spaces). Yet, $\mathrm{Haus}^{\mathscr{T}}$ does not have simple colimits, as we illustrate.



Denote by $Z^* = Z + \{-\infty, +\infty\}$ the usual two-point completion of integers (with the finest topology such that $\lim_{n\to\infty} n = +\infty$ and

$\lim_{n\to\infty}(-n) = -\infty$). Define a space $X = Z^* \times N \times \{a, b\} + \{a_\omega, b_\omega, t\}$ to have the finest topology such that

$$a_\omega = \lim_{n\to\infty}(+\infty, n, a) \qquad \text{and} \qquad b_\omega = \lim_{n\to\infty}(+\infty, n, b)$$

and that each subspace $Z^* \times \{n, x\}$ with $n \in N$, $x = a, b$ is homeomorphic to Z^*. Define an idempotent operation $\varepsilon: X_0 \to X$ by

$$\begin{aligned}
\varepsilon(-\infty, n, b) &= (-\infty, n, a),\\
\varepsilon(+\infty, n, b) &= (+\infty, n+1, a),\\
\varepsilon(b_\omega) &= t,\\
\varepsilon(x) &= x \qquad \text{for all remaining } x \in X.
\end{aligned}$$

Finally, define a continuous map $f: X \to X$ by

$$\begin{aligned}
f(z, n, b) &= (z+1, n, b),\\
f(x) &= x \qquad \text{for all} \quad x \in X - Z \times N \times \{b\}.
\end{aligned}$$

We claim that the coequalizer of $\mathscr{E}$-homomorphisms

$$f, 1_X: (X, \varepsilon) \to (X, \varepsilon)$$

in $\mathrm{Haus}^{\mathscr{E}}$ is not simple.

The coequalizer of $f, 1_X: X \to X$ in Haus is clearly the following quotient space $h: X \to A$ of X:

$$\begin{aligned}
A &= (Z^* \times N \times \{a\}) + \{r_n\}_{n\in N} + \{a_\omega, b_\omega, t\},\\
h(z, n, b) &= r_n \qquad \text{for } z \in Z^*, n \in N,\\
h(x) &= x \qquad \text{otherwise.}
\end{aligned}$$

Further, the coequalizer of $f, 1_X: X_0 \to X_0$ in Haus is the following discrete space $h': X_0 \to A_0'$:

$$\begin{aligned}
A_0' &= A \cup \{(+\infty, n, b), (-\infty, n, b)\}_{n\in N}\\
h'(\pm\infty, n, b) &= (\pm\infty, n, b), \qquad \text{else } h' = h.
\end{aligned}$$

Finally the coequalizer of $Tf, T1_X$, i.e., of $f + f, 1_X + 1_X: X + X_0 \to X + X_0$ is

$$h + h': X + X_0 \to A + A_0' = B.$$

Now, the maps δ: $A + A_0' \to A$ and k: $A + A_0' \to A + A_0$ are defined as follows:

$\delta = 1_A \qquad \text{on } A;$

$\delta(-\infty, n, b)_0 = (-\infty, n, a), \quad \delta(+\infty, n, b)_0 = (+\infty, n+1, a), \quad \delta(b_\omega)_0 = t;$

$\delta(x_0) = x \qquad \text{otherwise};$

$k = 1_A + \rho, \qquad \text{where } \rho\colon A_0' \to A_0 \text{ is the natural factorization.}$

Denote by $c\colon A + A_0 \to C$ the coequalizer of $T\delta$ and $\mu \cdot Tk$.

$$\begin{tikzcd}
B + B_0 \arrow[r, shift left, "\delta+\delta"] \arrow[r, shift right, "\mu\cdot(k+k)"'] & A + A_0 \arrow[r, "c"] & C \\
& X + X_0 \arrow[u, dashed, "h+h"'] & \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "1_X"'] & X \arrow[u, dashed, "\eta"'] &
\end{tikzcd}$$

With the above description of δ and k, it is easy to see that (since b_ω is not in the image of δ):

$$c(t) \neq c(a_\omega) \qquad \text{with} \quad t, a \in A \subset A + A_0. \tag{$\dagger$}$$

Now we can readily verify that the coequalizer of f, 1_X is not simple. If, conversely, it is simple, then there exists an operation $\gamma\colon C_0 \to C$ such that $\tilde{c} = c \cdot (h+h) \cdot \eta\colon (X, \varepsilon) \to (C, \gamma)$ is the coequalizer of f, 1_X: $(X, \varepsilon) \to (X, \varepsilon)$. Since $\varepsilon(z, n, b) = (z, n, b)$ for $z \in Z$, $n \in N$, clearly $\gamma(c(r_n)) = c(r_n)$. Further, $\tilde{c}(+\infty, n, b) = c(r_n)$ while $\varepsilon(+\infty, n, b) = (+\infty, n+1, a)$. By $\tilde{c} \cdot \varepsilon = \gamma \cdot \tilde{c}$ it follows that $\tilde{c}(+\infty, n, b) = \tilde{c}(+\infty, n+1, a)$, $n \in N$. Since c is continuous, we get

$$\tilde{c}(b_\omega) = \tilde{c}(a_\omega).$$

Finally, $\varepsilon(b_\omega) = t$ and $\varepsilon(a_\omega) = a_\omega$ yields the contradiction with ($\dagger$): $\tilde{c}(t) = \tilde{c}(a_\omega)$.

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