A SURVEY OF TENSOR PRODUCTS
AND RELATED CONSTRUCTIONS
IN TWO LECTURES

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ABSTRACT. We survey tensor products of lattices with zero and related constructions focused on two topics: amenable lattices and box products.

PART I. FIRST LECTURE:
AMENABLE LATTICES

ABSTRACT. Let A be a finite lattice. Then A is *amenable* ($A \otimes B$ is a lattice, for *every* lattice B with zero) iff A (as a join-semilattice) is *sharply transferable* (whenever A has an embedding φ into $\operatorname{Id} L$, the ideal lattice of a lattice L, then A has an embedding ψ into L satisfying $\psi(x) \in \varphi(x)$ and $\psi(x) \notin \varphi(y)$, if $y < x$).

In Section 1, we survey tensor products. In Section 2, we introduce transferability. These two topics are brought together in Section 3 in the characterization theorem of amenable lattices.

1. TENSOR PRODUCT

For a $\{\vee, 0\}$-semilattice A, we use the notation $A^- = A - \{0\}$.

Tensor products were introduced in J. Anderson and N. Kimura [1] and G. A. Fraser [8]. Let A and B be $\{\vee, 0\}$-semilattices. We denote by $A \otimes B$ the *tensor product* of A and B, defined as the free $\{\vee, 0\}$-semilattice generated by the set $A^- \times B^-$ and subject to the relations

$$\langle a, b_0 \rangle \vee \langle a, b_1 \rangle = \langle a, b_0 \vee b_1 \rangle, \qquad \text{for } a \in A^-, \ b_0, b_1 \in B^-;$$
$$\langle a_0, b \rangle \vee \langle a_1, b \rangle = \langle a_0 \vee a_1, b \rangle, \qquad \text{for } a_0, a_1 \in A^-, \ b \in B^-.$$

1.1. **The set representation.** Let A and B be $\{\vee, 0\}$-semilattices. We introduce a partial binary operation, the *lateral join*, on $A \times B$: let $\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle \in A \times B$; the *lateral join* $\langle a_0, b_0 \rangle \vee \langle a_1, b_1 \rangle$ is defined if $a_0 = a_1$ or $b_0 = b_1$, in which case, it is the join, $\langle a_0 \vee a_1, b_0 \vee b_1 \rangle$; that is,

$$\langle a, b_0 \rangle \vee \langle a, b_1 \rangle = \langle a, b_0 \vee b_1 \rangle, \qquad \text{for } a \in A, \ b_0, b_1 \in B;$$
$$\langle a_0, b \rangle \vee \langle a_1, b \rangle = \langle a_0 \vee a_1, b \rangle, \qquad \text{for } a_0, a_1 \in A, \ b \in B.$$

A nonempty subset I of $A \times B$ is a *bi-ideal* of $A \times B$, if it is hereditary, it contains

$$\perp_{A,B} = (A \times \{0\}) \cup (\{0\} \times B),$$

and it is closed under lateral joins.

The *extended tensor product* of A and B, denoted by $A \overline{\otimes} B$, is the lattice of all bi-ideals of $A \times B$. It is easy to see that it is an algebraic lattice. For $a \in A$ and $b \in B$, we define $a \otimes b \in A \overline{\otimes} B$ by

$$a \otimes b = \perp_{A,B} \cup \{ \langle x, y \rangle \in A \times B \mid \langle x, y \rangle \le \langle a, b \rangle \}$$

and call $a \otimes b$ a *pure tensor*. A pure tensor is a principal (that is, one-generated) bi-ideal.

Now we can state the representation:

Theorem 1. *The tensor product $A \otimes B$ can be represented as the $\{\vee, 0\}$-subsemilattice of compact elements of $A \overline{\otimes} B$.*

Let $a_0 \le a_1$ in A and $b_0 \ge b_1$ in B. Then

$$(a_0 \otimes b_0) \vee (a_1 \otimes b_1) = (a_0 \otimes b_0) \cup (a_1 \otimes b_1).$$

Such an element is called a *mixed tensor*.

A bi-ideal I is *capped*, if it a *finite union* of pure tensors; pure tensors and mixed tensors are the simplest examples. A tensor product $A \otimes B$ is *capped*, if (in the set representation) all its elements are capped bi-ideals. It is easy to see that a capped tensor product is always a lattice. (It is an open problem whether the converse holds; we do not think so.)

1.2. **Representation by homomorphisms.** Let A and B be $\{\vee, 0\}$-semilattices. Note that $\operatorname{Id} B$, the set of all ideals of $\langle B; \vee \rangle$, is a semilattice under intersection. So we can consider the set of all semilattice homomorphisms from the semilattice $\langle A^-; \vee \rangle$ into the semilattice $\langle \operatorname{Id} B; \cap \rangle$,

$$A \vec{\otimes} B = \operatorname{Hom}(\langle A^-; \vee \rangle, \langle \operatorname{Id} B; \cap \rangle),$$

ordered componentwise, that is, $f \le g$ iff $f(a) \le g(a)$ (that is, $f(a) \subseteq g(a)$), for all $a \in A^-$. The arrow indicates which way the homomorphisms go. Note that the elements of $A \vec{\otimes} B$ are *antitone* functions from A^- to $\operatorname{Id} B$.

With any element φ of $A \vec{\otimes} B$, we associate the subset $\varepsilon(\varphi)$ of $A \times B$:

$$\varepsilon(\varphi) = \{ \langle x, y \rangle \in A \times B \mid y \in \varphi(x) \} \cup \perp_{A,B}.$$

Theorem 2. *The map ε is an isomorphism between $A \vec{\otimes} B$ and $A \overline{\otimes} B$.*

If A is *finite*, then a homomorphism from $\langle A^-; \vee \rangle$ to $\langle \operatorname{Id} B; \cap \rangle$ is determined by its restriction to $\operatorname{J}(A)$, the set of all join-irreducible elements of A.

For an interesting application of the representation of tensor products by homomorphisms, see G. Grätzer and F. Wehrung [27].

1.3. **Examples.** Let B_n denote the Boolean lattice with 2^n elements.

Let L be a lattice with zero. Then

   (i) $L \otimes B_1 \cong L$;
   (ii) $L \otimes B_n \cong L^n$;
   (iii) for a finite distributive lattice D and $P = \operatorname{J}(D)$, $M_3 \otimes D$ can be represented as the set $M_3[D]$ of all balanced triples of D (a triple $\langle x, y, z \rangle$ is *balanced* iff $x \wedge y = x \wedge z = y \wedge z$) or as M_3^P.
   (iv) $N_5 \otimes L$ can be represented as the set of all triples $\langle x, y, z \rangle$ of L satisfying $y \wedge z \le x \le z$.

The representations in (iii) and (iv) utilize the representation by homomorphisms of Section 1.2.

The four examples share the property that the tensor product is a lattice. R. W. Quackenbush [35] raised the question whether this is true, in general. We answered this in [29]. In $M_3 \otimes \mathrm{F}(3)$, let a, b, and c be the atoms of M_3, let x, y, and z be the free generators of $\mathrm{F}(3)$, and form the elements

$$\alpha = (a \otimes x) \vee (b \otimes y) \vee (c \otimes z),$$
$$\beta = a \otimes 1,$$

where 1 is the unit of $\mathrm{F}(3)$. We proved that $\alpha \wedge \beta$ does not exist in $M_3 \otimes \mathrm{F}(3)$.

1.4. **Congruences.** The main result of G. Grätzer, H. Lakser, and R. W. Quackenbush [17] is the statement that

$$\operatorname{Con} A \otimes \operatorname{Con} B \cong \operatorname{Con}(A \otimes B)$$

holds for *finite* lattices A and B. For infinite lattices with zero, this cannot hold, in general, because

- the tensor product of two algebraic distributive lattices is not necessarily algebraic;
- the tensor product of lattices with zero is not necessarily a lattice.

We compensate for the first by switching to the semilattice with zero of compact congruences and for the second by assuming that the tensor product is capped:

The Isomorphism Theorem for Capped Tensor Products. *Let A and B be lattices with zero. If $A \otimes B$ is capped, then the following isomorphism holds:*

$$\operatorname{Con_c} A \otimes \operatorname{Con_c} B \cong \operatorname{Con_c}(A \otimes B).$$

To describe this isomorphism, we need some notation. Let α be a congruence of A and let β be a congruence of B. Define a binary relation $\alpha \, \overline{\square} \, \beta$ on $A \, \overline{\otimes} \, B$ as follows: for H, $K \in A \, \overline{\otimes} \, B$, let $H \equiv K \; (\alpha \, \overline{\square} \, \beta)$ iff, for all $\langle x, y \rangle \in H$, there exists an $\langle x', y' \rangle \in K$ such that $x \equiv x' \; (\alpha)$ and $y \equiv y' \; (\beta)$, and symmetrically. Let $\alpha \, \square \, \beta$ be the restriction of $\alpha \, \overline{\square} \, \beta$ to $A \otimes B$. If $A \otimes B$ is a lattice, then $\alpha \, \square \, \beta$ is a lattice congruence on $A \otimes B$.

For $\alpha \in \operatorname{Con} A$ and $\beta \in \operatorname{Con} B$, we define $\alpha \odot \beta$, the *tensor product* of α and β, by the formula

$$\alpha \odot \beta = (\alpha \, \square \, \omega_B) \wedge (\omega_A \, \square \, \beta).$$

Theorem 3. *Let A and B be lattices with zero such that $A \otimes B$ is a lattice. The map $\alpha \otimes \beta \mapsto \alpha \odot \beta$ extends to a $\{\vee, 0\}$-embedding*

$$\varepsilon \colon \operatorname{Con_c} A \otimes \operatorname{Con_c} B \to \operatorname{Con_c}(A \otimes B).$$

If $A \otimes B$ is capped, then ε establishes the Isomorphism Theorem.

The Isomorphism Theorem can be proved in a more general setup.

Let A and B be lattices with zero. A *sub-tensor product* of A and B is a subset C of $A \otimes B$ satisfying the following conditions:

(i) C contains all the mixed tensors in $A \otimes B$;
(ii) C is closed under finite intersection;
(iii) C is a lattice with respect to containment.

If every element of C (as a bi-ideal) is capped, then C is a *capped sub-tensor product*.

The Isomorphism Theorem for Capped Sub-Tensor Products. *Let A and B be lattices with zero. If C is a capped sub-tensor product of A and B, then the following isomorphism holds:*

$$\mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B \cong \mathrm{Con_c}\, C.$$

The lattice tensor product of Lecture Two is a sub-tensor product.

For some earlier results on congruence lattices of lattices of the type $L \otimes D$, where D is distributive, see B. A. Davey, D. Duffus, R. W. Quackenbush, and I. Rival [5], D. Duffus, B. Jónsson, and I. Rival [6], J. D. Farley [7], G. Grätzer and E. T. Schmidt [20], G. Grätzer and F. Wehrung [28], and E. T. Schmidt [37].

2. Transferable lattices

Transferable lattices were introduced in [14] in order to provide a nice class of first-order sentences that hold for the ideal lattice of a lattice iff they hold for the lattice.

A finite lattice T is *transferable*, if for every embedding φ of T into $\mathrm{Id}\, L$, the ideal lattice of a lattice L, there exists an embedding ξ of T into L.

However, from a structural point of view, the following stronger form is of more interest.

A finite lattice T is *sharply transferable*, if for every embedding φ of T into $\mathrm{Id}\, L$, there exists an embedding ξ of T into L satisfying $\xi(x) \in \varphi(y)$ iff $x \leq y$.

The motivation for these definitions comes from the fact that the well-known result: *a lattice L is modular iff* $\mathrm{Id}\, L$ *is modular*, can be recast: N_5 *is a (sharply) transferable lattice.*

It is easy to verify that N_5 *is a sharply transferable lattice.* It is somewhat more difficult to see the negative result: M_3 *is not a (sharply) transferable lattice.*

To give the characterization theorem of (sharply) transferable lattices, we need the following definitions, see H. Gaskill [10].

Let P be a poset and let X and Y be subsets of P. Then X *is dominated by* Y, in notation, $X \ll Y$, if for all $x \in X$, there exists $y \in Y$ such that $x \leq y$.

Let A be a finite join-semilattice. A *minimal pair* of A is a pair $\langle p, I \rangle$ such that $p \in \mathrm{J}(A)$, $I \subseteq \mathrm{J}(A)$, $|I| \geq 2$, $p \notin I$, and $p \leq \bigvee I$; moreover, for all $J \subseteq \mathrm{J}(A)$, if $J \ll I$ and $p \leq \bigvee J$, then $I \subseteq J$.

A finite join-semilattice A satisfies condition (T), if $\mathrm{J}(A)$ has a linear ordering $\trianglelefteq$ such that for every minimal pair $\langle p, J \rangle$ of A and $j \in J$, the relation $p \trianglelefteq j$ holds. A lattice A satisfies condition $(\mathrm{T}_\vee)$ (respectively, $(\mathrm{T}_\wedge)$), if the semilattice $\langle A; \vee \rangle$ (respectively, $\langle A; \wedge \rangle$) satisfies (T).

Finally, we need the *Whitman condition*:

(W) $x \wedge y \leq u \vee v$ implies that $[x \wedge y, u \vee v] \cap \{x, y, u, v\} \neq \varnothing.$

Now we can state the result from H. S. Gaskill, G. Grätzer, and C. R. Platt [11]:

The Characterization Theorem for Sharply Transferable Lattices. *Let A be a finite lattice. Then A is sharply transferable iff it satisfies the three conditions $(\mathrm{T}_\vee)$, $(\mathrm{T}_\wedge)$, and (W).*

As discussed in Appendix A and R. Freese's Appendix G of [16], this result shows that sharply transferable lattices are the same as finite sublattices of a free lattice (see J. B. Nation [32]).

Sharply transferable semilattices are defined analogously. H. Gaskill [10] proved the following result:

The Characterization Theorem for Sharply Transferable Semilattices.
Let S be a finite semilattice. Then S is sharply transferable iff it satisfies (T).

See R. Freese, J. Ježek, and J. B. Nation [9] for a discussion on how $(T_\vee)$ is the same as D-*cycle free* and on the structure of this class of lattices.

3. Amenable lattices

Of course, the tensor product of two finite lattices is always a lattice. In Section 1.3, we noted that $M_3 \otimes F(3)$ is not a lattice. Now we introduce the class of finite lattices A for which $A \otimes L$ is always a lattice.

Let us call the finite lattice A *amenable*, if $A \otimes L$ is a lattice, for any lattice L with zero. So M_3 is not amenable. Every finite distributive lattice is amenable. It is easy to see using the representation in Example (iv) of Section 1.3 that N_5 is amenable.

Now we state the characterization theorem of finite amenable lattices [25]:

Theorem 4. *For a finite lattice A, the following conditions are equivalent:*
  (i) *A is amenable.*
  (ii) *A is* transferable *as a join-semilattice.*
  (iii) *$A \otimes F(3)$ is a lattice.*
  (iv) *A satisfies* $(T_\vee)$.

The equivalence of (i) and (iii) states that $F(3)$ is a "test lattice"; the equivalence of (ii) and (iv) is a restatement of the result of H. Gaskill [10] stated above.

The proof of this result is fairly long. Curiously, the crucial step is based on a construction in H. S. Gaskill, G. Grätzer, and C. R. Platt [11] for lattice (not semilattice) transferability; while we are unable to apply this result directly, the idea is clearly borrowed.

It follows that the class of finite amenable lattices and the class of finite *lower bounded* lattices coincide, see R. Freese, J. Ježek, and J. B. Nation [9]. By Theorem 2.43 of [9], a finite lattice is lower bounded iff it can be obtained from a one-element lattice by a sequence of doubling constructions with respect to lower pseudo-intervals.

Recently, we have succeeded in generalizing Theorem 4 to arbitrary lattices with zero:

Theorem 5. *For a lattice A with zero, the following conditions are equivalent:*
  (i) *A is amenable.*
  (ii) *A is locally finite and $A \otimes B$ is a lattice, for every lattice B with zero.*
  (iii) *A is locally finite and $A \otimes F(3)$ is a lattice.*
  (iv) *A is locally finite and every finite sublattice of A satisfies* $(T_\vee)$.

For a finite amenable lattice A, there is a close connection between $J(A)$ and $J(\operatorname{Con} A)$. Let $a \in J(A)$; let a_* be the unique element of A covered by a. Then $a \mapsto \Theta(a, a_*)$ is a bijection between $J(A)$ and $J(\operatorname{Con} A)$. (In fact, the converse is also true, showing that amenability is the same as *fermentability* in the sense of P. Pudlák and J. Tůma [34].) This suggests that the congruence lattice of a finite amenable lattice is very special.

A *spike* in a finite poset P is a pair $a < b$ of elements of P such that b is maximal in P, b covers a in P, and b is the *only* maximal element of P containing a. A poset P is *spike-free*, if it has no spikes.

Theorem 6. *A finite distributive lattice D can be represented as the congruence lattice of an amenable lattice iff* $\mathrm{J}(D)$ *is spike-free.*

This result is a special case of a more general theorem in [31].

PART II. SECOND LECTURE:
BOX PRODUCTS

ABSTRACT. We have seen in Part I that the tensor product of two lattices with zero is not necessarily a lattice. We survey a new lattice construction, the *box product* that always yields a lattice. If A and B are lattices and either both A and B have a zero or one of them is bounded, then the box product $A \,\square\, B$ of A and B has an ideal, $A \boxtimes B$, for which an analogue of the Isomorphism Theorem for capped sub-tensor products holds, without any further restriction on A or B. In general, $A \boxtimes B$ is a subset of $A \otimes B$; equality holds, if A or B is distributive.

4. THE $M_3\langle L \rangle$ CONSTRUCTION AND THE $N_5\langle L \rangle$ CONSTRUCTION

Let L be a lattice. A lattice K is a *congruence-preserving extension* of L, if K is an extension of L and every congruence of L extends to exactly one congruence of K. The extension is *proper*, if $K \neq L$. Similarly, we can define a congruence-preserving embedding of lattices. In [21], the first author and E. T. Schmidt asked whether every lattice L with more than one element has a proper congruence-preserving extension. If L is a modular lattice, the answer is already provided by Schmidt's $M_3[L]$ construction, see E. T. Schmidt [36], R. W. Quackenbush [35], and Section 1.3. By definition, $M_3[L]$ is the set of all *balanced triples* of L, ordered componentwise, see Section 1.3:

$$M_3[L] = \{\, \langle x, y, z \rangle \in L^3 \mid x \wedge y = x \wedge z = y \wedge z \,\}.$$

Unfortunately, $M_3[L]$ is not always a lattice, see G. Grätzer and F. Wehrung [28] for a planar example L. The answer to the problem mentioned in the previous paragraph was finally provided by a simple trick that we describe now, see [24]. For every lattice L, define $M_3\langle L \rangle$, a subset of L^3, as follows:

$$(1) \qquad M_3\langle L \rangle = \{\, \langle v \wedge w, u \wedge w, u \wedge v \rangle \mid u,\, v,\, w \in L \,\}.$$

We call an element of $M_3\langle L \rangle$ a *Boolean triple* of L. In particular, $M_3\langle L \rangle$ is a subset of $M_3[L]$. Endow $M_3\langle L \rangle$ with the componentwise ordering.

Theorem 7. *Let L be a lattice. Then $M_3\langle L \rangle$ is a lattice, and the diagonal map,*

$$x \mapsto \langle x, x, x \rangle,$$

defines a congruence-preserving embedding from L into $M_3\langle L \rangle$.

In particular, if L has more than one element, then $M_3\langle L \rangle$ properly contains L, thus solving the above problem.

It appears desirable to generalize the $M_3\langle L \rangle$ construction to any pair of lattices with zero, thus creating an analogue of the tensor product that never fails to be a lattice. One (heuristic) way to proceed is the following. We note that the Boolean triples of L are exactly those triples of L that are balanced "for a good reason". Of course, one has to define precisely what a "good reason" is. Formula (1) suggests to look for "meet-parametrizations" of the solutions of the equational system defining balanced triples, that is, $x \wedge y = x \wedge z = y \wedge z$.

Now let us do the same with the pentagon, N_5, instead of M_3. By using the representation by homomorphisms of the elements of the tensor product $N_5 \otimes L$,

see Section 1.2, we define a certain object that we denote by $N_5[L]$, see Section 1.3:

$$(2) \qquad N_5[L] = \{\, \langle x, y, z \rangle \in L^3 \mid y \wedge z \leq x \leq z \,\}.$$

The situation here is quite different from the situation with $M_3[L]$: indeed, since N_5 is amenable, $N_5[L]$ is always a lattice; furthermore, if L has a zero, then $N_5[L]$ is isomorphic to $N_5 \otimes L$.

However, we may still look for those triples of elements of L that belong to $N_5[L]$ "for a good reason" (say, a meet-parametrization of the solutions of the equational system defining $N_5[L]$). An easy computation gives us the definition of a new object that we denote, of course, by $N_5\langle L \rangle$:

$$(3) \qquad N_5\langle L \rangle = \{\, \langle v \wedge w, u \wedge w, v \rangle \mid u,\ v,\ w \in L \,\}.$$

Again, it is not hard to prove that $N_5\langle L \rangle$, endowed with componentwise ordering, is a lattice. It is strange that even though $N_5[L]$ is a lattice, for every lattice L, $N_5\langle L \rangle$ is, as a rule, a *proper* subset of $N_5[L]$; for example, for $L = N_5$.

A similar method to the one outlined above gives a definition of $A\langle L \rangle$, for a finite lattice A and a lattice L. A precise description of this method would be lengthy, and it would involve the study of the structure of solution sets of systems of equations in distributive semilattices. Furthermore, it may not be very useful at this point, because we found a general, short definition that encompasses all these constructions and more. The starting point is the construction of the *box product* defined in the next section.

5. The box product $A \,\square\, B$

We refer to [26], for more detail and for proofs.

Let A and B be lattices. For $\langle a, b \rangle \in A \times B$, define

$$a \,\square\, b = \{\, \langle x, y \rangle \in A \times B \mid x \leq a \text{ or } y \leq b \,\}.$$

We define the *box product* of A and B, denoted by $A \,\square\, B$, as the set of all finite *intersections* of the form

$$H = \bigcap (\, a_i \,\square\, b_i \mid i < n \,),$$

where n is a positive integer and $\langle a_i, b_i \rangle \in A \times B$, for all $i < n$.

It is clear that $A \square B$ is a meet-subsemilattice of the powerset lattice $\mathrm{Pow}(A \times B)$ of $A \times B$. To obtain that $A \,\square\, B$ is also a join-semilattice, we prove that it is a closure system in a sublattice, denoted by $A \,\boxdot\, B$, of $\mathrm{Pow}(A \times B)$. The definition of $A \,\boxdot\, B$ is the following. For $\langle c, d \rangle \in A \times B$, put

$$c \circ d = \{\, \langle x, y \rangle \in A \times B \mid x \leq c \text{ and } y \leq d \,\},$$

and define $A \,\boxdot\, B$ as the set of all finite *unions* of the form

$$(4) \qquad H = \bigcup (\, a_i \,\square\, b_i \mid i < m \,) \cup \bigcup (\, c_j \circ d_j \mid j < n \,),$$

where $m > 0$, $n \geq 0$, and all pairs $\langle a_i, b_i \rangle$ and $\langle c_j, d_j \rangle$ belong to $A \times B$.

Theorem 8. *Let A and B lattices. Then $A \,\boxdot\, B$ is a sublattice of $\mathrm{Pow}(A \times B)$ and $A \,\square\, B$ is a closure system in $A \,\boxdot\, B$. In particular, $A \,\square\, B$ is a lattice.*

The statement that $A \,\square\, B$ is a closure system in $A \,\boxdot\, B$ means that, for every element H of $A \,\boxdot\, B$, there exists a least element K of $A \,\square\, B$ such that $H \subseteq K$; we denote this element by $\overline{H}$. It is important to note that $\overline{H}$ is given by a *formula*, as follows. If H is written as in (4), then $\overline{H}$ is given by

$$\overline{H} = \bigcap (\, a^{(X)} \,\square\, b^{(n-X)} \mid X \subseteq n \,),$$

where

$$a^{(X)} = \bigvee (\, a_i \mid i < m \,) \vee \bigvee (\, c_j \mid j \in X \,),$$
$$b^{(X)} = \bigvee (\, b_i \mid i < m \,) \vee \bigvee (\, d_j \mid j \in X \,),$$

for all $X \subseteq n$.

6. The lattice tensor product $A \boxtimes B$

For lattices A and B, the box product $A \,\square\, B$ has a unit element iff either A or B has a unit element. In particular, $M_3 \,\square\, B$ always has a unit element, so that it is not isomorphic to the lattice $M_3 \langle B \rangle$ of Boolean triples of B, see Section 4. Thus we shall define an *ideal* of $A \,\square\, B$.

For arbitrary lattices A and B, we can modify the definition of $\perp_{A,B}$, introduced in Section 1, as follows:

$$\perp_{A,B} = (A \times \perp_B) \cup (\perp_A \times B),$$

where

$$\perp_L = \begin{cases} \{0_L\}, & \text{if } L \text{ has a zero,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

For $\langle a, b \rangle \in A \times B$, define

$$a \boxtimes b = \{\, \langle x, y \rangle \in A \times B \mid x \leq a \text{ and } y \leq b \,\} \cup \perp_{A,B}.$$

If both A and B have a zero element, then $a \boxtimes b$ is an element of $A \,\square\, B$, namely,

$$a \boxtimes b = (a \,\square\, 0_B) \cap (0_A \,\square\, b) = a \otimes b.$$

An element H of $A\square B$ is *confined*, if it is contained in $a\boxtimes b$ for some $\langle a, b \rangle \in A\times B$. We define $A \boxtimes B$, the *lattice tensor product* of A and B, as the ideal of $A \,\square\, B$ consisting of all confined elements of $A \,\square\, B$.

If A has a zero element and B has no zero element, then $a \boxtimes b$ does not contain any element of $A\square B$ unless A has a unit element (so that A is bounded), in which case a equals this unit, thus $a \boxtimes b$ equals $0_A \,\square\, b$, so that it belongs to $A \,\square\, B$. In particular, if A has a zero but no unit and B has no zero, then $A \boxtimes B = \varnothing$. In fact, it is easy to see exactly when $A \boxtimes B$ is nonempty:

Lemma 6.1. *Let A and B be lattices. Then $A \boxtimes B$ is nonempty iff one of the following conditions holds:*

    (i) *both A and B have zero;*
    (ii) *either A or B is bounded;*
    (iii) *both A and B have unit.*

In case (iii), that is, if both A and B have unit, then every element of $A \,\square\, B$ is bounded, so that $A \boxtimes B = A \,\square\, B$. For a lattice L, denote by L^{d} the dual lattice of L.

As one would expect, cases (i) and (iii) correspond to each other *via* lattice dualization:

Theorem 9. *Let A and B be lattices with zero. Then the following isomorphism holds:*

$$(A \boxtimes B)^{\mathrm{d}} \cong A^{\mathrm{d}} \,\square\, B^{\mathrm{d}}.$$

Interestingly, the main observation on the Isomorphism Theorem for lattice tensor products concerns lattices *with zero* (as opposed to lattices with unit):

Theorem 10. *Let A and B be lattices with zero. Then $A \boxtimes B$ is a capped sub-tensor product of A and B. Furthermore, $A \boxtimes B$ is the* smallest *capped sub-tensor product of A and B, with respect to containment.*

The Isomorphism Theorem for Capped Sub-Tensor Products, see Section 1.4, implies then that the isomorphism $\mathrm{Con_c}(A \boxtimes B) \cong \mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B$ holds, for lattices A and B with zero. A direct limit argument and some extra work makes it then possible to obtain the following general result:

Theorem 11. *Let A and B be lattices. If $A \boxtimes B$ is nonempty, then the following isomorphism holds:*

$$\mathrm{Con_c}(A \boxtimes B) \cong \mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B.$$

Theorem 11 is proved by constructing a map,

$$\mu\colon \mathrm{Con_c}\, A \otimes \mathrm{Con_c}\, B \longrightarrow \mathrm{Con_c}(A \boxtimes B),$$

and proving that μ is an isomorphism. The isomorphism μ is easy to describe. Since μ is a join homomorphism, it is sufficient to describe the image of a pure tensor $\alpha \otimes \beta$, where $\alpha = \Theta_A(a_0, a_1)$ and $\beta = \Theta_B(b_0, b_1)$ (with $a_0 \leq a_1$ in A and $b_0 \leq b_1$ in B). According to Lemma 6.1, we split the description into three cases:

(i) A and B are lattices with zero:

$$\mu(\alpha \otimes \beta) = \Theta_{A \boxtimes B}((a_0 \boxtimes b_1) \vee (a_1 \boxtimes b_0), a_1 \boxtimes b_1).$$

(ii) A is bounded (or symmetrically, B is bounded):

$$\mu(\alpha \otimes \beta) = \Theta_{A \boxtimes B}((a_0 \,\square\, b_0) \cap (0_A \,\square\, b_1), (a_1 \,\square\, b_0) \cap (0_A \,\square\, b_1)).$$

(iii) A and B are lattices with unit:

$$\mu(\alpha \otimes \beta) = \Theta_{A \boxtimes B}(a_0 \,\square\, b_0, (a_0 \,\square\, b_1) \cap (a_1 \,\square\, b_0)).$$

Of course, formula (iii) can be obtained from formula (i) and the canonical isomorphism given in Theorem 9.

The lattice tensor product construction $A \boxtimes B$ can be easily related to the constructions $M_3\langle L \rangle$ and $N_5\langle L \rangle$ described in Section 4:

Theorem 12. *Let L be a lattice. Then the following isomorphisms hold:*

$$M_3 \boxtimes L \cong M_3\langle L \rangle,$$
$$N_5 \boxtimes L \cong N_5\langle L \rangle.$$

An isomorphism $\alpha\colon M_3\langle L \rangle \to M_3 \boxtimes L$ is given by

$$\alpha(\langle v \wedge w, u \wedge w, u \wedge v \rangle) = (p \,\square\, u) \cap (q \,\square\, v) \cap (r \,\square\, w),$$

for all u, v, $w \in L$, where p, q, and r are the atoms of M_3.

An isomorphism $\beta\colon N_5\langle L\rangle \to N_5 \boxtimes L$ is given by

$$\beta(\langle v \wedge w, u \wedge w, v\rangle) = (a \,\square\, u) \cap (b \,\square\, v) \cap (c \,\square\, w),$$

where $a > c$ and b are the join-irreducible elements of N_5.

Much more general is the following corollary of Theorem 11 and of the formulas describing the isomorphism μ:

Corollary 6.2. *Let S and L be a lattices; let S be simple.*

  (i) *If S is bounded, then the map $j\colon L \to S \boxtimes L$ defined by*

$$j(x) = 0_S \,\square\, x,$$

    for all $x \in L$, is a congruence-preserving lattice embedding.

  (ii) *If both S and L have zero, then for every $s \in S^-$, the map $j_s\colon L \to S \boxtimes L$ defined by*

$$j_s(x) = s \boxtimes x,$$

    for all $x \in L$, is a congruence-preserving lattice embedding.

For $S = M_3$ and *via* the identification of $M_3 \boxtimes L$ with $M_3\langle L\rangle$, the first embedding is the map $x \mapsto \langle x, x, x\rangle$, while the second embedding is, for example, for $s = p$, the map $x \mapsto \langle x, 0, 0\rangle$. For more general S, this can be used to prove statements stronger than Theorem 7, such as the *Strong Independence Theorem*, see Section 7.2.

7. Some applications

7.1. Congruence representations of distributive semilattices with zero. Let us say that a $\{\vee, 0\}$-semilattice S is *representable* (*$\{0\}$-representable*, *$\{0, 1\}$-representable*, respectively), if there exists a lattice L (a lattice L with zero, a bounded lattice L, respectively) such that $\mathrm{Con}_c L \cong S$. It is an open problem, dating back to the forties, whether every distributive $\{\vee, 0\}$-semilattice is representable or $\{0\}$-representable. Similarly, it is an open problem whether every bounded distributive $\{\vee, 0\}$-semilattice is representable, or $\{0\}$-representable, or $\{0, 1\}$-representable. We refer to G. Grätzer and E. T. Schmidt [23] for a detailed history of this problem. We recall here some partial answers:

  (i) If S satisfies one of the following conditions, then S is representable (see [23], Theorem 13):
    (a) $\mathrm{Id}\,S$ is completely distributive (R. P. Dilworth);
    (b) S is a lattice (E. T. Schmidt);
    (c) S is locally countable, that is, every element of S generates a countable principal ideal (A. P. Huhn for S countable, H. Dobbertin in general).
    (d) $|S| \leq \aleph_1$ (A. P. Huhn).
      In all four cases, the representability of S can be obtained *via* E. T. Schmidt's condition (see [36]) that S is a distributive image of a generalized Boolean semilattice. A closer look at the proofs shows that, in fact, Schmidt's condition implies $\{0\}$-representability.
  (ii) If S is countable, then S is representable by a *sectionally complemented modular lattice* L (G. M. Bergman [3], see also K. R. Goodearl and F. Wehrung [13]). Furthermore, if S is bounded, then one can take L to be bounded.

(iii) If $|S| \leq \aleph_1$, then S is representable by a relatively complemented (not modular *a priori*) lattice with zero. The proof of this result is based on an amalgamation result of J. Tůma [38], see also G. Grätzer, H. Lakser, and F. Wehrung [18]. However, the method fails to produce a bounded lattice L even if S is bounded.

New consequences can be obtained about the class $\mathcal{R}$ of representable $\{\vee, 0\}$-semilattices, the class $\mathcal{R}_0$ of $\{0\}$-representable $\{\vee, 0\}$-semilattices and the class $\mathcal{R}_{0,1}$ of $\{0, 1\}$-representable $\{\vee, 0\}$-semilattices, by using Theorem 11:

Corollary 7.1.

  (i) *The classes $\mathcal{R}_0$ and $\mathcal{R}_{0,1}$ are closed under tensor product.*
  (ii) *Let $A \in \mathcal{R}_{0,1}$ and let $B \in \mathcal{R}$. Then $A \otimes B \in \mathcal{R}$.*

This result can be extended to *iterated tensor products*. If $\langle S_i \mid i \in I \rangle$ is a family of bounded $\{\vee, 0\}$-semilattices, then their *iterated tensor product* is the direct limit of the family $\bigotimes_{i \in J} S_i$, where J ranges over all finite subsets of I, and the transition homomorphisms are defined by $\otimes_{i \in J} x_i \mapsto \otimes_{i \in K} x_i$, where $x_i = 1_{S_i}$, for $i \in K - J$, and $J \subseteq K$ are finite subsets of I.

Corollary 7.2. *The class $\mathcal{R}_{0,1}$ is closed under iterated tensor products.*

Further results can be obtained for other subclasses of $\mathcal{R}$. Let us mention, for example, the following. If L is a lattice, we say that L has *permutable congruences*, if any two congruences of L commute.

Lemma 7.3. *Let A and B be lattices such that $A \boxtimes B$ is nonempty. If A and B have permutable congruences, then $A \boxtimes B$ has permutable congruences.*

By the known representation results, the class of all $\{\vee, 0\}$-semilattices that are representable by lattices with zero and with permutable congruences contains all distributive semilattices of size at most $\aleph_1$—this is because every relatively complemented lattice has permutable congruences. Denote by $\mathcal{R}^c$ ($\mathcal{R}_0^c$, $\mathcal{R}_{0,1}^c$, respectively) the class of all $\{\vee, 0\}$-semilattices that are representable by lattices (lattices with zero, bounded lattices, respectively) with permutable congruences. It is proved in J. Tůma and F. Wehrung [39], using the main result of M. Ploščica, J. Tůma and F. Wehrung [33], that $\mathcal{R}^c$ is a proper subclass of $\mathcal{R}$.

Corollary 7.4.

  (i) *The classes $\mathcal{R}_0^c$ and $\mathcal{R}_{0,1}^c$ are closed under tensor product.*
  (ii) *Let $A \in \mathcal{R}_{0,1}^c$ and let $B \in \mathcal{R}^c$. Then $A \otimes B \in \mathcal{R}^c$.*
  (iii) *The class $\mathcal{R}_{0,1}^c$ is closed under iterated tensor product.*

There is an intriguing similarity between these preservation results and known representation results of dimension groups as ordered K_0 groups of locally matricial rings, see K. R. Goodearl and D. E. Handelman [12].

7.2. **Strong independence of the congruence lattice and the automorphism group.** The *Independence Theorem* for the congruence lattice and the automorphism group of a finite lattice was proved by V. A. Baranskiĭ [2] and A. Urquhart [40] (solving Problem II.19 of [15]) :

The Independence Theorem for Finite Lattices. *Let G be a finite group and let D be a finite distributive lattice. Then there exists a finite lattice L such*

that $\mathrm{Aut}\,L$, *the automorphism group of* L, *is isomorphic to* G, *while* $\mathrm{Con}\,L$, *the congruence lattice of* L, *is isomorphic to* D.

Both proofs utilize the characterization theorem of congruence lattices of finite lattices (as finite distributive lattices) and the characterization theorem of automorphism groups of finite lattices (as finite groups).

In G. Grätzer and E. T. Schmidt [22], a new, stronger form of independence is introduced.

A finite lattice K is an *automorphism-preserving extension* of L, if K is an extension and every automorphism of L has exactly one extension to K, and in addition, every automorphism of K is the extension of an automorphism of L. Of course, then the automorphism group of L is isomorphic to the automorphism group of K.

The following result has been established in G. Grätzer and E. T. Schmidt [22]:

The Strong Independence Theorem for Finite Lattices. *Let* L_{C} *and* L_{A} *be finite lattices, let* L_{C} *have more than one element, and let* $L_{\mathrm{C}} \cap L_{\mathrm{A}} = \{0\}$. *Then there exists a finite atomistic lattice* L *that is a congruence-preserving extension of* L_{C} *and an automorphism-preserving extension of* L_{A}. *In fact, both extensions preserve the zero.*

Of course, the congruence lattice of L is isomorphic to the congruence lattice of L_{C}, and the automorphism group of L is isomorphic to the automorphism group of L_{A}. Therefore, indeed, for finite lattices, independence follows from strong independence. This is because every finite distributive lattice can be obtained as $\mathrm{Con}\,L_{\mathrm{C}}$ for some finite lattice L_{C} (R. P. Dilworth; see G. Grätzer and E. T. Schmidt [19]) and every finite group can be obtained as $\mathrm{Aut}\,L_{\mathrm{A}}$ for some finite lattice L_{A} (see G. Birkhoff [4]).

The question of a possible generalization of the Independence Theorem or the Strong Independence Theorem to infinite lattices was raised in Problems 1 and 2 of G. Grätzer and E. T. Schmidt [22] (Problem 3, whether every lattice with more than one element has a proper congruence-preserving extension, is solved in our paper [24], see Theorem 7). The statement of independence for arbitrary lattices is by itself a problem, because it is not known which distributive $\{\vee, 0\}$-semilattices S are representable as $\mathrm{Con}\,L$ for a lattice L—that is, which S belong to the class $\mathcal{R}$, see Section 7.1. On the other hand, Birkhoff's result extends to all groups: *every group is isomorphic to the automorphism group of some lattice.* Thus a possible formulation of independence for infinite lattices would be with *representable* $\{\vee, 0\}$-semilattices, on the one hand, and arbitrary groups, on the other. Again, such a statement would follow from strong independence.

We proved strong independence in G. Grätzer and F. Wehrung [30], thus solving Problem II.18 of [15] and Problems 1 and 2 of [22]:

The Strong Independence Theorem for Lattices with Zero. *Let* L_{A} *and* L_{C} *be lattices with zero, let* L_{C} *have more than one element. Then there exists a lattice* L *that is a* $\{0\}$-*preserving extension of both* L_{A} *and* L_{C}, *an automorphism-preserving extension of* L_{A}, *and a congruence-preserving extension of* L_{C}.

The Strong Independence Theorem for Lattices. *Let* L_{A} *and* L_{C} *be lattices, let* L_{C} *have more than one element. Then there exists a lattice* L *that is an automorphism-preserving extension of* L_{A} *and a congruence-preserving extension of* L_{C}.

The main ingredients of the proof are direct limits, gluings, and box products (in fact, lattice tensor products).

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