CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

JOHN L. MACDONALD
ARTHUR STONE

The tower and regular decomposition

Cahiers de topologie et géométrie différentielle catégoriques, tome 23, nº 2 (1982), p. 197-213

<http://www.numdam.org/item?id=CTGDC_1982__23_2_197_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
Vol. XXIII-2 (1982)

3^e COLLOQUE SUR LES CATÉGORIES
DÉDIÉ À CHARLES EHRESMANN
Amiens, Juillet 1980

THE TOWER AND REGULAR DECOMPOSITION

by John L. MACDONALD and Arthur STONE

Composition and its infinitary analogues are among the partially defined operations on the morphisms of a category. Thus a given morphism may have an infinitary decomposition corresponding to a given ordinal. Furthermore, such a decomposition may have certain universal properties.

We first examine the regular decomposition of morphisms described by Isbell [13] (cf. Kelly [14]) and point out the universal properties of that decomposition. An example suggests the relationship between partially defined algebraic structures and regular decomposition length.

Secondly, in the same light, we consider the adjoint tower decomposition of an adjunction. This decomposition is already implicit in Beck's [16] factorization of an adjunction through its category of algebras and is more explicitly described in Applegate-Tierney [1] (cf. Day [5]). The universal properties of this decomposition are entirely analogous to those of the regular decomposition. However, the adjoint tower is not in general a regular decomposition, as shown by an example in the second section.

Finally, we give a class of examples illustrating the nontriviality and interrelatedness of these two decompositions for each ordinal. The examples are *essentially algebraic* in the sense of Freyd [6] and show how the regular length of the counit is related to the essential length of the tower.

1. CHAIN COMPOSITES AND REGULAR LENGTH.

In this section the chain composite f of a family

$$f^{\alpha\beta} \ (\alpha < \lambda, \ \beta = \alpha + 1)$$

of morphisms is defined for given ordinal λ. Then the regular epic component (or largest regular epic «bite») of a morphism f is described and conditions are given for its existence as well as for that of the canonical regular epic decomposition of f, the latter being a specifically defined chain composite, for a certain ordinal λ, of regular epics followed by a monic (cf. Isbell [13]; Kelly [14]). The ordinal λ (when it exists) is called the regular length of f.

We identify an ordinal λ with the ordered set (considered as a category) of ordinals less than λ and let $\alpha\beta$ denote the unique morphism from α to β when $\alpha < \beta$.

DEFINITION 1.0. A *chain* of *object length* λ in a category $\underset{=}{C}$ is a diagram (= functor) $\lambda \to \underset{=}{C}$ where λ is an ordinal. The *morphism length* of a chain

$$* \qquad C^0 \xrightarrow{c^{01}} C^1 \xrightarrow{c^{12}} C^2 \;\dots\; C^\alpha \xrightarrow{c^{\alpha\beta}} C^\beta \;\dots$$

is the order type of the family of morphisms $c^{\alpha\beta}$ with $\beta < \lambda$ and $\beta = \alpha + 1$. So morphism length equals object length when this is 0 or a limit ordinal; otherwise morphism length equals object length minus one. A *cochain* of *object colength* λ in $\underset{=}{C}$ is a contravariant functor $\lambda \to \underset{=}{C}$.

Note that in an ordinal λ (regarded as a category) each object γ is the colimit of the diagram of morphisms $\alpha\beta$ with $\alpha < \gamma$ and $\beta = \alpha + 1$. Call a chain *cocontinuous* (of course) if it is cocontinuous as a functor. For each object C^α in a cocontinuous chain $*$ the family of morphisms $c^{\theta\alpha}$ $(\theta < \alpha)$ forms a colimit cone.

Note that finite chains are always cocontinuous.

DEFINITION 1.1. A morphism f is a *chain composite* of the morphisms $f^{\alpha\beta}$ $(\alpha < \lambda,\ \beta = \alpha + 1)$ if there is a cocontinuous chain C of object length $\lambda + 1$ with

$$C(\alpha\beta) = f^{\alpha\beta} \quad \text{and} \quad f = C(0\,\lambda).$$

When λ is finite chain composite equals composite.

Recall that an epimorphism e is:

(1) *regular* if for some set $\{x_\alpha, y_\alpha\}_{\alpha\in I}$ of ordered pairs of morphisms satisfying $ex_\alpha = ey_\alpha$ we have that $hx_\alpha = hy_\alpha$ (for all α) implies the existence of a morphism d with $de = h$;

$$\begin{tikzcd} & & \bullet \arrow[dr, dashed, "d"] & \\ \bullet \arrow[r, shift left, "x_a"] \arrow[r, shift right, "y_a"'] & \bullet \arrow[ur, "e"] \arrow[rr, "h"] & & \bullet \end{tikzcd}$$

and

(2) *strong* if whenever $be = ma$ with m monic, there is a «diagonal» d with $de = a$ and $md = b$.

The class of regular epics in a category need not be closed under composition (Herrlich-Strecker [11], p. 262). However, a regular epic is strong and

PROPOSITION 1.2. *A chain composite of strong epics is strong epic.*

PROOF. By induction on the length of the chain. For the induction step consider the diagram

$$\begin{tikzcd} \bullet \arrow[r, "e'"] \arrow[d, "a"'] & \bullet \arrow[r, "e"] \arrow[dl, "d'"'] & \bullet \arrow[d, "b"] \arrow[dll, "d"] \\ \bullet \arrow[rr, "m"] & & \bullet \end{tikzcd}$$

where e' and e are strong epic, m is monic, and $bee' = ma$. We have a diagonal d' since e' is strong epic. Then the strong epic e gives us a diagonal d for the square $md' = be$. At a limit ordinal the (infinite) diagram is similar, and the diagonal d exists by the definition of colimit. □

DEFINITION 1.3.

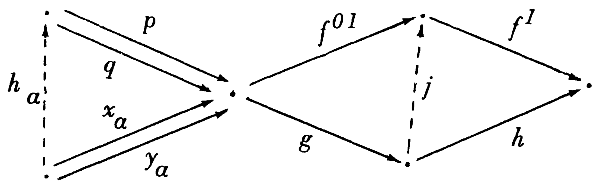
$$\begin{tikzcd} & \bullet \arrow[dr, "f^1"] & \\ \bullet \arrow[ur, "f^{01}"] \arrow[dr, "g"'] & & \bullet \\ & \bullet \arrow[uu, "j"] \arrow[ur, "h"'] & \end{tikzcd}$$

The regular epimorphism f^{01} is a *regular epic component* of morphism f if $f = f^1 . f^{01}$ for some (necessarily unique) f^1 and whenever $f = h . g$ with g a regular epic, there is a (necessarily unique) j with $j . g = f^{01}$.

Note that the usual «uniqueness» proof applies, so that when g is another regular epic component of f, above, j is an isomorphism. A regular epic component factors out of f as much as we can get in one regular epic «bite».

PROPOSITION 1.4. *If f^{01} is a coequalizer morphism for a kernel pair for f, then f^{01} is a regular epic component of f.*

PROOF.



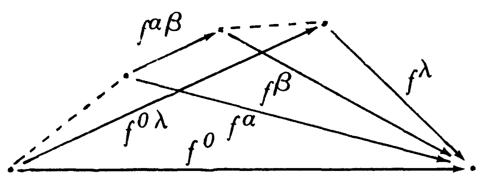
Let p, q be a kernel pair for f, with coequalizer morphism f^{01}, and $f = f^1 . f^{01}$. Let g be a regular epic for the family $\{x_a, y_a\}_{a \in I}$ with $f = h . g$. Then $f . x_a = f . y_a$ $(a \in I)$ so by the definition of kernel pair we get a family $\{h_a\}_{a \in I}$ satisfying

$$p . h_a = x_a \quad \text{and} \quad q . h_a = y_a \quad (a \in I).$$

Since $f^{01} . p = f^{01} . q$ this gives us $f^{01} . x_a = f^{01} . y_a$ and hence a morphism j. □

DEFINITION 1.5. A *canonical regular epic decomposition* of a morphism $f\colon Sce\, f \to Tgt\, f$ in a category $\underline{\underline{C}}$ is a chain

(1.6)



in the comma category $\underline{\underline{C}}/Tgt\, f$ in which

(0) $f = f^0$,

(1) when $\beta = a + 1$ the morphism $f^{a\beta}$ is a regular epic component of f^a,

(2) the chain is cocontinuous, and

(3) f^a is non-monic for $a < \lambda$; f^λ is monic.

Such a decomposition is easy to construct given enough limits and colimits, for example we have:

PROPOSITION 1.7. *Let the category $\underline{\underline{C}}$ have kernel pairs, coequalizers of kernel pairs and colimits of (possibly large) chains of regular epimorphisms. Then every morphism f in $\underline{\underline{C}}$ has a canonical regular epic decomposition.*

PROOF. If f is monic, then f has a canonical regular epic decomposition in which the chain has morphism length 0. When f is not monic, proceed by induction using 1.4 and colimits at limit ordinals. We get a chain composite of coequalizer morphisms $f^{\alpha\beta}$ as in 1.6 which proceeds through the ordinals. However, if γ is least ordinal with $card\gamma > card\,\underline{\underline{C}}$ then $f^{\alpha} = f^{\beta}$ for some pair (α, β) with $\alpha < \beta < \gamma$. It is straightforward to show that $f^{\alpha} = f^{\beta}$ for $\alpha < \beta$ implies f^{α} monic. Let λ be the smallest ordinal such that f^{λ} is monic.

PROPOSITION 1.8. *Canonical regular epic decompositions of morphisms (when they exist) are unique up to unique isomorphism of diagrams.*

PROOF. Follows inductively from the note following 1.3.

The *regular length* of a morphism f is the morphism length of a canonical regular epic decomposition of f. (The regular length is undefined when there is no such decomposition.) The *regular length of* $\underline{\underline{C}}$, if it is defined, is the supremum of the regular lengths of the morphisms of $\underline{\underline{C}}$.

So monomorphisms have regular length zero. Categories with (regular epic)-monic factorizations (and at least one non-monic) have regular length one.

It can be shown that *Cat* has regular length 2. A basic typical example of a regular length 2, strong epimorphism in *Cat* (in notation suggesting it to be the value of the counit of an adjunction) is the natural functor $\epsilon^{\circ}\underline{\underline{3}} : G^{\circ}\underline{\underline{3}} \to \underline{\underline{3}}$. Here $\underline{\underline{3}}$ is the ordinal number $\lambda = 3$ regarded as a category and $G^{\circ}\underline{\underline{3}}$ is the disjoint union of six copies of $\underline{\underline{2}}$, one for each morphism of $\underline{\underline{3}}$. If $a(x)$ denotes the copy of a in $\underline{\underline{2}}$ indexed by x in $\underline{\underline{3}}$ then $\epsilon^{\circ}\underline{\underline{3}}$ maps

$$0(x) \mapsto Sce\ x, \quad 01(x) \mapsto x \quad \text{and} \quad 1(x) \mapsto Tgt\ x.$$

Let $G^{1}\underline{\underline{3}}$ differ from $\underline{\underline{3}}$ by having an additional morphism introduced by requiring that $02 \neq 12.01$. It is a straightforward exercise to show that $\epsilon^{\circ}\underline{\underline{3}}$ has a canonical regular epic factorization

$$G^{\circ}\underline{\underline{3}} \longrightarrow G^{1}\underline{\underline{3}} \longrightarrow \underline{\underline{3}}\,.$$

2. THE TOWER AND MONADIC LENGTH.

In this section we show how an adjunction has a decomposition (when enough colimits exist) with the same type of universal properties as regular decomposition. To bring out this point we use well known results of Beck [16] on adjunctions and monadicity and of Applegate-Tierney [1] on towers freely throughout omitting proofs of standard results.

More specifically, the monadic component (or largest monadic «bite») of an adjunction is described and conditions given for its existence as well as for that of the canonical monadic decomposition, the latter being a specifically defined chain composite, for a certain ordinal δ, of monadic adjunctions (the successive «bites») followed by an idempotent adjunction with invertible unit (cf. Applegate-Tierney [1]). The ordinal δ is called the *monadic length* of the adjunction.

When δ is a successor ordinal it happens in some (but not all) cases that in the canonical monadic decomposition the composite of the last adjunction (with invertible unit) with the preceding monadic adjunction is itself idempotent. Then we say that the *essential length* λ is one less than the monadic length δ. Otherwise we let $\lambda = \delta$.

This decomposition is not the same as the canonical regular epic decomposition as we show by presenting an example of a monadic component which is not epic, even in a rather weak sense.

NOTATION. An adjunction $\underline{N}$ consists of left and right adjoint functors F and U together with a choice of unit and counit η and ϵ. (The choice of one of η and ϵ determines the other.) The monad = triple determined by $\underline{N}$ is

$$\underline{T} = \langle T = UF,\ \eta,\ \mu = F\epsilon U\rangle \quad \text{and} \quad G = FU.$$

Following MacLane [16] we take the source and target of $\underline{N}$ to be those of F and say that $\underline{N}$ is the *composite* $\underline{N}^\beta \Delta \underline{N}^\alpha$ of $\underline{N}^\beta$ and $\underline{N}^\alpha$ when

$$F = F^\beta F^\alpha,\quad U = U^\alpha U^\beta,\quad \eta = U^\alpha \eta^\beta F^\alpha . \eta^\alpha \quad \text{and} \quad \epsilon = \epsilon^\beta . F^\beta \epsilon^\alpha U^\beta .$$

Then we have:

LEMMA 2.1. *If* $\underline{N} = \underline{N}^\beta \Delta \underline{N}^\alpha$, *then*

$$\epsilon^{\alpha} U^{\beta} = U^{\beta}\epsilon . \eta^{\beta} F^{\alpha} U \quad \textit{and} \quad F^{\beta}\epsilon^{\alpha} F^{\alpha} = \epsilon F . F U^{\alpha} \eta^{\beta} F^{\alpha} .$$

In the adjoint tower the composite of certain adjunctions satisfy the hypotheses of

LEMMA 2.2. *When* $\underline{N} = \underline{N}^{\beta} \Delta \underline{N}^{\alpha}$, $\eta = \eta^{\alpha}$ *and* $U^{\beta} F = F^{\alpha}$ *then:*

(1) $\eta^{\beta} F^{\alpha} = 1$ *;*

(2) $\epsilon^{\beta} F = 1$ *;*

(3) $\epsilon^{\alpha} U^{\beta} = U^{\beta} \epsilon$ *;*

(4) $F^{\beta} \epsilon^{\alpha} F^{\alpha} = \epsilon F$ *and*

(5) $\mu^{\alpha} = \mu$ *hence* $\underline{T}^{\alpha} = \underline{T}$.

Call an adjunction $\underline{N}$ monadic (= tripleable) if its right adjoint U is monadic. Recall (e.g., MacLane [16]):

PROPOSITION 2.3. *If the adjunction* $\underline{N}$ *is monadic, then the counit* ϵ *is regular epic; it is a coequalizer for* ϵG *and* $G\epsilon$ *(where* $G = FU$ *).*

DEFINITION 2.4. A monadic adjunction $\underline{N}^{01}$ is a first *monadic component* of an adjunction $\underline{N}$ if

$$\underline{N} = \underline{N}^{1} \Delta \underline{N}^{01} \quad \textit{for some } \underline{N}^{1}, \quad \eta = \eta^{01}, \quad \textit{and} \quad F^{01} = U^{1} F .$$

PROPOSITION 2.5 *(cf. Beck [16]). If* $\underline{\underline{A}}$ *has coequalizers, then every adjunction has a first monadic component.*

PROOF.

$$\begin{tikzcd} & \underline{\underline{X}}^{1} \arrow[dr, "\underline{N}^{1}"] & \\ \underline{\underline{X}} \arrow[ur, "\underline{N}^{01}"] \arrow[rr, "\underline{N}"] & & \underline{\underline{A}} \end{tikzcd}$$

Let $\underline{\underline{X}}^{1}$ denote the category of Eilenberg-Moore algebras $X^{1} = \langle X, a \rangle$ where $a : TX \to X$ is an Eilenberg-Moore structure morphism in $\underline{\underline{X}}$, and let F^{01} and U^{01} be the Eilenberg-Moore free and underlying functors (thus $F^{01} X = \langle UFX, U\epsilon FX \rangle$). Choosing the unit by setting $\eta^{01} = \eta$ fixes the adjunction. The counit ϵ^{01} maps $\langle X, a \rangle \mapsto a$ (regarded now as a morphism in $\underline{\underline{X}}^{1}$).

The Beck comparison functor U^{1} maps $A \mapsto \langle UA, U\epsilon A \rangle$. Then clearly $F^{01} = U^1 F$. It is then routine to verify that $\underline{N} = \underline{N}^1 \Delta \underline{N}^{01}$ using the well known [16] description of the left adjoint F^1 of U^1 and its associated unit η^1 and counit ϵ^1. □

We have the following uniqueness property for the adjunction $\underline{N}^1$ of 2.4:

PROPOSITION 2.6. *If*

$$\underline{N} = \underline{N}^1 \Delta \underline{N}^{01} = \underline{N}^\gamma \Delta \underline{N}^{01}$$

with $\underline{N}^{01}$ *monadic,* $\eta = \eta^{01}$ *and* $F^{01} = U^1 F = U^\gamma F$, *then there is a unique natural isomorphism* $F^1 \to F^\gamma$ *(and a unique natural isomorphism* $U^\gamma \to U^1$*).*

PROOF. If $\underline{N}^\alpha$ is monadic, then ϵ^α and 1 are a coequalizer morphism and object for the pair $G^\alpha \epsilon^\alpha$, $\epsilon^\alpha G^\alpha$ by 2.3. Left adjoints F^1 and F^γ preserve the coequalizer diagram. If

$$F = F^1 F^{01} = F^\gamma F^{01}, \quad \eta = \eta^{01} \quad \text{and} \quad F^{01} = U^1 F = U^\gamma F,$$

then the resulting pair when $\alpha = 01$ is

$$F^1 G^{01} \epsilon^{01} = F U^{01} \epsilon^{01} = F^\gamma G^{01} \epsilon^{01},$$

$$F^1 \epsilon^{01} G^{01} \underset{\text{by } 2.2\,(4)}{=} \epsilon F U^{01} \underset{\text{by } 2.2\,(4)}{=} F^\gamma \epsilon^{01} G^{01}.$$

So F^1 and F^γ are coequalizer objects for the same pair. □

A first monadic component $\underline{N}^{01}$ of an adjunction $\underline{N}$ is maximal (in an obvious sense) among the monadic adjunctions $\underline{N}^\alpha$ through which $\underline{N}$ factors; it takes out of $\underline{N}$ as much as can be had in one monadic «bite», just as the regular epic component of a morphism f factors out as much as we can get in one regular epic «bite».

PROPOSITION 2.7. *Let* $\underline{N} = \underline{N}^1 \Delta \underline{N}^{01} = \underline{N}^\beta \Delta \underline{N}^\alpha$ *with* $\underline{N}^\alpha$ *monadic and* $\underline{N}^{01}$ *a first monadic component of* $\underline{N}$. *Then there is an adjunction* $\underline{N}^\theta$ *with* $\underline{N}^{01} = \underline{N}^\theta \Delta \underline{N}^\alpha$ *and (up to isomorphism)* $\underline{N}^\beta = \underline{N}^1 \Delta \underline{N}^\theta$, *provided* $\underline{\underline{X}}^1$ *has coequalizers.*

PROOF. For simplicity assume that the categories $\underline{\underline{X}}^1$ and $\underline{\underline{X}}^\beta$ are (and not just equivalent to) the Eilenberg-Moore categories for the monads $\underline{T}^{01}$ and $\underline{T}^\alpha$, and let U^1 be (not just isomorphic to) the comparison functor.

$$\begin{tikzcd}
 & \underline{\underline{X}}^1 \arrow[dr, "\underline{N}^1"] & \\
\underline{\underline{X}} \arrow[ur, "\underline{N}^{01}"] \arrow[dr, "\underline{N}^{\alpha}"'] & & \underline{\underline{A}} \\
 & \underline{\underline{X}}^{\beta} \arrow[uu, "\underline{N}^{\theta}"] \arrow[ur, "\underline{N}^{\beta}"'] &
\end{tikzcd}$$

If $\underline{N} = \underline{N}^{\beta} \Delta \underline{N}^{\alpha}$ then there is a natural transformation

$$\phi = U^{\alpha} \eta^{\beta} F^{\alpha} : T^{\alpha} \to T$$

(which is in fact a morphism of monads). The right adjoint U^{θ} maps

$$<X, a> \mapsto <X, a.\phi_X> \quad \text{and} \quad h \mapsto h$$

(where $<X, a>$ is a $\underline{T}$-algebra, $<X, a.\phi_X>$ is a $\underline{T}^{\alpha}$-algebra; if the morphism h in $\underline{\underline{X}}$ is a $\underline{T}$-homomorphism, then it is also a $\underline{T}^{\alpha}$-homomorphism). The comparison functor for $\underline{N}^{\alpha} : \underline{\underline{X}} \to \underline{\underline{X}}^{\beta}$ is the identity. Thus for Y in $|\underline{\underline{X}}^{\beta}|$ we have $Y = (U^{\alpha} Y, U^{\alpha} \epsilon^{\alpha} Y)$. Hence

$$\begin{aligned} U^{\beta} X &= (U^{\alpha} U^{\beta} X,\ U^{\alpha} \epsilon^{\alpha} U^{\beta} X) \underset{(2.1)}{=} (UX, U^{\alpha} U^{\beta} \epsilon X . U^{\alpha} \eta^{\beta} F^{\alpha} UX) \\ &= U^{\theta}(UX, U\epsilon X) = U^{\theta} U^1 X \end{aligned}$$

and $U^{\beta} = U^{\theta} U^1$. Thus

$$U^{\beta} \epsilon F = U^{\theta} U^1 \epsilon F \underset{\text{by } 2.2(3)}{=} U^{\theta} \epsilon^{01} U^1 F = U^{\theta} \epsilon^{01} F^{01} .$$

In particular,

$$U^{\beta} F = U^{\theta} F^{01} \quad \text{and} \quad U^{01} = U^{\alpha} U^{\theta} .$$

Furthermore

$$\epsilon^{\alpha} U^{\theta} = U^{\theta} \epsilon^{01} . \eta^{\beta} F^{\alpha} U^{01}$$

by definition of U^{θ}, using $U^{01} = U^{\alpha} U^{\theta}$ and U^{α} faithful.

The natural transformation τ and the left adjoint F^{θ} are defined by this coequalizer diagram in $(\underline{\underline{X}}^{\beta}, \underline{\underline{X}}^1)$:

$$\begin{tikzcd}[column sep=huge]
\cdot \arrow[r, shift left, "F^{01} U^{\alpha} \epsilon^{\alpha}"] \arrow[r, shift right, "\epsilon^{01} F^{01} U^{\alpha} . F^{01} \phi U^{\alpha}"'] & \cdot \arrow[r, "\tau"] & F^{\theta}
\end{tikzcd}$$

and the counit ϵ^{θ} is the unique morphism from the coequalizer object $F^{\theta} U^{\theta}$ so that $\epsilon^{\theta} . \tau U^{\theta} = \epsilon^{01}$. The unit η^{θ} is the unique morphism from the coequalizer object $1 = Tgt\,\epsilon^{\alpha}$ for which $\eta^{\theta} . \epsilon^{\alpha} = U^{\theta}\tau . \eta^{\beta} G^{\alpha}$ so, by the proof that ϵ^{α} is a coequalizer, η^{θ} is determined by

$$U^{\alpha}\eta^{\theta} = U^{01}\tau . \eta^{01} U^{\alpha} . \quad \square$$

We remark that the usual «uniqueness» proof applies, so that if $\underline{N}^{\alpha}$ is another first monadic component of $\underline{N}$, then $\underline{N}^{\theta}$ is an isomorphism (or more accurately, there is an adjunction $\underline{N}^{\bar{\theta}}$ such that the right adjoints U^{θ} and $U^{\bar{\theta}}$ are inverses).

Proposition 2.6 shows that a first monadic component of an adjunction has certain characteristics of an epimorphism. However, monadic adjunctions are not necessarily epic, not even *pseudo-epic :* $\underline{N}^{\alpha}$ monadic and $\underline{N}^{\beta} \Delta \underline{N}^{\alpha} = \underline{N}^{\gamma} \Delta \underline{N}^{\alpha}$ need not imply F^{β} isomorphic to F^{γ}. For example, in

$$\begin{tikzcd} Ens \arrow[r, "\underline{N}^{\alpha}"] & \underline{\underline{A}} \arrow[r, shift left, "\underline{N}^{\beta}"] \arrow[r, shift right, "\underline{N}^{\gamma}"'] & \underline{\underline{B}} \end{tikzcd}$$

let $\underline{N}^{\alpha}$ be monadic and let the algebras of $\underline{\underline{B}}$ have exactly twice the algebraic structure of the algebras of $\underline{\underline{A}}$. To be concrete let the algebras of $\underline{\underline{A}}$ have, say, one unary operation (no axioms) and let the algebras of $\underline{\underline{B}}$ have two. Then there are non-isomorphic monadic (somewhat forgetful) functors U^{β} and U^{γ} for which the composites with the forgetful functor U^{α} are equal.

A monad $\underline{T} = (T, \eta, \mu)$ is said to be *idempotent* if μ is an isomorphism and *trivial* if, in addition, η is an isomorphism. An adjunction $\underline{N}$ is *idempotent* or *trivial* if its associated monad is.

PROPOSITION 2.8. *The adjunction* $\underline{N}$ *is idempotent if and only if any one of the natural transformations*

$$(2.9) \qquad \epsilon F,\ U\epsilon,\ \eta U,\ F\eta,\ \epsilon FU,\ U\epsilon F,\ FU\epsilon,\ \eta UF,$$
$$F\eta U,\ UF\eta,\ \epsilon FUFU,\ \ldots$$

is an isomorphism, if and only if any two distinct natural transformations 2.9 with the same target and source are equal (say $\epsilon FU = FU\epsilon$ *).*

In particular $\underline{N}$ is idempotent if ϵ is monic (since $\epsilon . \epsilon FU = \epsilon . FU\epsilon$). The proof of the proposition uses only repeated applications of the adjunction equations, the naturality of η and ϵ and the fact that

$$\sigma \cdot \rho = 1 \quad \text{and} \quad \tau \cdot \sigma = 1 \quad \text{imply} \quad \rho = \tau \cdot \sigma \cdot \rho = \tau \, .$$

DEFINITION 2.10. A *canonical monadic decomposition* of an adjunction $\underline{N}$ is a chain

(2.11)
$$\begin{tikzcd}
 & & & \underline{\underline{X}}^{\delta} \arrow[dddr, "\underline{N}^{\delta}"] & \\
 & & \underline{\underline{X}}^{\beta} \arrow[ur, dashed] \arrow[ddrr, "\underline{N}^{\beta}"] & & \\
 & \underline{\underline{X}}^{\alpha} \arrow[ur, "\underline{N}^{\alpha\beta}"] \arrow[drrr, "\underline{N}^{\alpha}"] & & & \\
\underline{\underline{X}}^{0} \arrow[ur, dashed, dash] \arrow[uuurrr, "\underline{N}^{0\delta}"] \arrow[rrrr, "\underline{N}^{0}"'] & & & & \underline{\underline{A}}
\end{tikzcd}$$

in the comma category of adjunctions over $\underline{\underline{A}}$ in which :

(0) $\underline{N} = \underline{N}^0$;

(1) when $\beta = \alpha + 1$ the adjunction $\underline{N}^{\alpha\beta}$ is a first monadic component of $\underline{N}^{\alpha}$, hence $F^{\alpha\beta} = U^{\beta} F^{\alpha}$ and $\eta^{\alpha\beta} = \eta^{\alpha}$;

(2) when κ is a limit ordinal $\underline{\underline{X}}^{\kappa}$ is the limit in *Cat* of the diagram of right adjoints $U^{\alpha\beta}$ ($\alpha < \beta < \kappa$), with

$$F^{\alpha\kappa} = U^{\kappa} F^{\alpha} \;\; (\alpha < \kappa) \quad \text{and} \quad \eta^{\alpha\kappa} = \eta^{\alpha} \; ; \quad \text{and}$$

(3) $\underline{N}^{\alpha}$ is nontrivial for $\alpha < \delta$; $\underline{N}^{\delta}$ is trivial.

We refer to the adjunctions $\underline{N}^{\alpha\beta}$ in 2.11 as *monadic components* of $\underline{N}$. Monadic decompositions are examples of *adjoint towers*. Conditions for the existence of auch a decomposition are given by the following result (cf. Applegate-Tierney [1]).

THEOREM 2.12. *If $\underline{\underline{A}}$ has coequalizers and colimits of (perhaps large) chains of strong epimorphisms, then every adjunction $\underline{N} : \underline{\underline{X}} \to \underline{\underline{A}}$ has a canonical monadic decomposition.*

PROOF. By induction on α. When $\beta = \alpha + 1$ apply 2.5. When κ is a limit ordinal we must define $\underline{N}^{\alpha\kappa}$ and $\underline{N}^{\kappa}$ in such a way that $\underline{N}^{\alpha} = \underline{N}^{\kappa} \Delta \underline{N}^{\alpha\kappa}$, (2) holds. We present only a few of the important details. An object X^{κ} of $\underline{\underline{X}}^{\kappa}$ is an indexed family $< X^{\alpha} >_{\alpha < \kappa}$ of objects of the categories $\underline{\underline{X}}^{\alpha}$ satisfying $U^{\alpha\beta} X^{\beta} = X^{\alpha}$ ($\alpha < \beta < \kappa$). Furthermore, $U^{\alpha\kappa}(< X^{\theta} >_{\theta < \kappa}) = X^{\alpha}$ and the *limit comparison* $U^{\kappa} A = < U^{\alpha} A >_{\alpha < \kappa}$. Furthermore, $F^{\alpha\kappa} = U^{\kappa} F^{\alpha}$

and F^{κ} is a colimit of the diagram

$$F^{\beta} F^{\alpha\beta} U^{\alpha\beta} U^{\beta\kappa} = F^{\alpha} U^{\alpha\kappa} \xrightarrow{F^{\beta}\epsilon^{\alpha\beta} U^{\beta\kappa}} F^{\beta} U^{\beta\kappa} \ldots \quad (\alpha < \beta < \kappa)$$

of (componentwise) strong epimorphisms.

Thus if we do not stop when $\underline{N}^{\alpha}$ becomes trivial we can push the decomposition 2.11 through *all* ordinals α. But we have assumed that the categories of this paper are at most large (relative to a universe U [18]). Consequently there are at most $2^{card\, \mathrm{U}}$ distinct natural transformations ϵ^{α} in 2.11. Let λ be the first ordinal for which we have

$$\epsilon^{\lambda} = \epsilon^{\beta} \quad \text{for some} \quad \beta > \lambda \, .$$

Then by 2.2 (1) $F^{\beta}\eta^{\beta} F^{\lambda\beta} = 1$ and (since $\epsilon^{\beta} F^{\beta}$ is a one-sided inverse for $F^{\beta}\eta^{\beta}$)

$$\epsilon^{\beta} F^{\lambda} = \epsilon^{\beta} F^{\beta} F^{\lambda\beta} = 1 .$$

So $\epsilon^{\lambda} F^{\lambda} U^{\lambda} = 1$ - which by 2 implies that $\underline{N}^{\lambda}$ is idempotent.

If, on the other hand, we let λ be the first ordinal for which $\underline{N}^{\lambda}$ is idempotent, then ϵ^{λ} is isomorphic to ϵ^{β} for all $\beta > \lambda$. This follows, by induction, from the equation

$$\epsilon^{\alpha} = \epsilon^{\beta} . \; F^{\beta}\epsilon^{\alpha\beta} U^{\beta} \quad (\alpha < \beta)$$

and the fact that $F^{\beta}\epsilon^{\alpha\beta} U^{\beta}$ is a coequalizer when $\beta = \alpha + 1$ for the pair $G^{\alpha}\epsilon^{\alpha}$, $\epsilon^{\alpha} G^{\alpha}$ - which is non-distinct when $\lambda \leq \alpha$.

So $\underline{N}^{\beta}$ is idempotent for $\beta \geq \lambda$.

Claim: if $\underline{N}^{\lambda}$ is idempotent and $\beta = \lambda + 1$ then $\underline{N}^{\beta}$ is trivial. For if $\epsilon^{\lambda} F^{\lambda}$ is an isomorphism, then so is $\epsilon^{\lambda\beta}$ (since it is a coequalizer of the pair $G^{\lambda\beta}\epsilon^{\lambda\beta}$, $\epsilon^{\lambda\beta} G^{\lambda\beta} = U^{\beta}\epsilon^{\lambda} F^{\lambda} U^{\lambda\beta}$ - with common one-sided inverse $F^{\lambda\beta}\eta^{\lambda\beta} U^{\lambda\beta}$). Hence, so are $U^{\lambda} F^{\beta}\epsilon^{\lambda\beta}$ and $\eta^{\lambda} U^{\lambda\beta}$ (one-sided inverse of $U^{\lambda\beta}\epsilon^{\lambda\beta}$) - which by definition of η^{β} have $U^{\lambda\beta}\eta^{\beta}$ as their composite. Since $U^{\lambda\beta}$ is faithful, η^{β} is an isomorphism.

The ordinal δ of 2.10 is λ if $\underline{N}^{\lambda}$ is trivial; otherwise $\delta = \lambda + 1$. □

The uniqueness of canonical monadic decompositions (up to isomorphism) follows inductively from the remark following 2.7. In particular:

PROPOSITION 2.13. *In 2.11 the ordinal δ is uniquely determined, the* $\underline{\underline{X}}^{\alpha}$

are determined up to equivalence of categories and once the $\underline{\underline{X}}^{\alpha}$ *are chosen the* $\underline{T}^{\alpha}$ *(associated to* $\underline{N}^{\alpha}$ *) are determined up to isomorphism of monads.*

DEFINITION 2.14. When an adjunction $\underline{N}$ has a canonical monadic decomposition, the *essential length (monadic* length) of $\underline{N}$ is the smallest ordinal λ in 2.11 such that $\underline{T}^{\lambda}$ is idempotent (trivial).

In the proof of 2.12 we have already seen that for a given adjunction monadic length equals either essential length or essential length plus one. An example with essential length *1* and monadic length *2* is the usual adjunction from sets to torsion free abelian groups.

The torsion free abelian groups is typical of a general phenomenon: where λ and δ denote the essential and monadic lengths, all the «visible» operations of the algebras in $\underline{\underline{X}}^{\delta}$ already appear in $\underline{\underline{X}}^{\lambda}$.

PROPOSITION 2.15. *If* $\underline{\underline{A}}$ *is at most large then the monadic length of an adjunction* $\underline{N} : \underline{\underline{X}} \to \underline{\underline{A}}$ *(when defined) has cardinality at most that of the universe* U.

The proof calls for a reexamination of the proof of 2.12. There, to avoid clutter, we spoke of natural transformations ϵ^{α} in place of morphisms $\epsilon^{\alpha} A$. If $\underline{\underline{A}}$ is at most large, then for each object A there are at most *card* U distinct morphisms $\epsilon^{\alpha} A$. Now in the proof of 2.12 let λ_A be the smallest ordinal λ for which we have $\epsilon^{\lambda} A = \epsilon^{\beta} A$ for some $\beta > \lambda$ and let $\lambda = sup_{A \epsilon \underline{\underline{A}}} \lambda_A$. □

Note that in the proof of 2.12 we do not need all the cocompleteness that we ask for in the statement of the proposition. We need only the coequalizers $\tau^{\alpha\beta} X^{\beta}$ of pairs $F^{\alpha} U^{\alpha\beta} \epsilon^{\alpha\beta} X^{\beta}$, $\epsilon^{\alpha} F^{\alpha} U^{\alpha\beta} X^{\beta}$ for Eilenberg-Moore algebras X^{β} ($\beta = \alpha + 1$) and colimits of the resulting chains

$$\cdots \xrightarrow{\tau^{\alpha\beta} X^{\beta}} \cdots \qquad (\alpha < \beta < \kappa)$$

for limit ordinals $\kappa \leq \lambda$.

REMARK 2.16. Monadic decompositions are a refinement of the Day factorizations [5] of adjunctions, analogous to the refinement of strong epic-monic factorizations by regular decompositions.

3. AN EXAMPLE.

A simple example that will produce adjunctions of any desired length λ and epimorphisms of regular length λ is obtained by letting $\underline{\underline{A}} = \underline{\underline{A}}^\lambda$ be the category of algebras A for a sequence of partial operations m_β $(\beta < \lambda)$ where

$$m_\beta \text{ defined at } a \in A \quad \text{iff} \quad m_\alpha a = a \text{ for all } \alpha < \beta.$$

(So $m = m_0$ is everywhere defined.) Homomorphisms $f: A \to B$ are functions satisfying $m_\alpha f(a) = f(m_\alpha a)$ whenever $m_\alpha a$ is defined.

For consideration of this example, which is essentially algebraic in the sense of Freyd [6], let the *α-sequence* of an element a (when defined) be the sequence

$$m_\alpha a,\ m m_\alpha a,\ m^2 m_\alpha, a,\ \ldots,\ m^n m_\alpha a,\ \ldots \quad (n < \omega)$$

and call this sequence *free* if its elements are distinct. There is an adjunction $\underline{N}^{\alpha\gamma}: \underline{\underline{A}}^\alpha \to \underline{\underline{A}}^\gamma$ $(\alpha < \gamma)$ in which $U^{\alpha\gamma}$ forgets the operations m_θ $(\alpha \leq \theta < \gamma)$ and $F^{\alpha\gamma} A$ is the algebra A with a new free α-sequence adjoined at a for each element a satisfying $m_\theta a = a$ $(\theta < \alpha)$. It is easy to see that the Eilenberg-Moore category for $\underline{N}^{\alpha\gamma}$ is isomorphic to $\underline{\underline{A}}^\beta$ where $\beta = \alpha + 1$ - and that $\underline{N}^{0\lambda}$ has essential and monadic length λ.

The example can be modified to give

$$\text{monadic length} = \text{essential length} + 1.$$

For example we might impose upon the algebras A of $\underline{\underline{A}}^\alpha$ $(0 < \alpha)$ the requirement that they satisfy an axiom of the form

$$m_\theta b = a \Rightarrow m_0 a = a$$

(m_0 leaves fixed the elements of $Im\, m_\theta$) for $\alpha = \theta + 1$, and for α a limit ordinal we might let the axiom be

$$\Lambda_{\theta < \alpha} (m_\theta b_\theta = a) \Rightarrow m_0 a = a$$

(m_0 leaves fixed the elements of $\bigcap_{\theta < \alpha} Im\, m_\theta$).

We next show how epimorphisms of regular length λ arise as values of the counit ϵ of $\underline{N}^{0\lambda}$. The unit $\eta^{\alpha\gamma}$ of $\underline{N}^{\alpha\gamma}$ $(\alpha < \gamma)$ consists of the obvious embeddings $A^\alpha \to U^{\alpha\gamma} F^{\alpha\gamma} A^\alpha$. The counit morphisms

$$\epsilon^{\alpha\beta} A^{\beta} : F^{\alpha\beta} U^{\alpha\beta} A^{\beta} \to A^{\beta} \quad (\text{for } \beta = \alpha + 1\ )$$

maps $a \mapsto a$ for $a \in \underline{\underline{A}}^{\beta}$ and maps $m^n m_{\alpha} a \mapsto m^n m_{\alpha} a$ (defined freely on the left and using the structure of A^{β} on the right) whenever m_{α} is defined at a.

For $\lambda = 2$ the canonical monadic decomposition and regular conuit decomposition look as follows:

$$\begin{tikzcd}
 & & \underline{\underline{A}}^2 \arrow[dd, "\underline{N}^2 = 1"] \\
 & \underline{\underline{A}}^1 \arrow[ur, "\underline{N}^{12}"] \arrow[dr, "\underline{N}^1 = \underline{N}^{12}"'] & \\
Sets = \underline{\underline{A}}^0 \arrow[ur, "\underline{N}^{01}"] \arrow[rr, "\underline{N}^0 = \underline{N}"'] & & \underline{\underline{A}}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & F^2U^2 = 1_{\underline{\underline{A}}2} \arrow[dd, "\epsilon^2 = 1"] \\
 & F^1U^1 \arrow[ur, "F^2\epsilon^{12}U^2"] \arrow[dr, "\epsilon^1"] & \\
FU \arrow[ur, "F^1\epsilon^{01}U^1"] \arrow[rr, "\epsilon = \epsilon^0"'] & & 1_{\underline{\underline{A}}2}
\end{tikzcd}$$

Using this we construct an explicit length 2 regular decomposition of the counit ϵ at the object $\bar{2}$ of $\underline{\underline{A}}^2$ described as follows: the underlying set $|\bar{2}|$ of $\bar{2}$ is that of the category $\underline{\underline{2}}$, which is just the set $\{0, 1\}$. The operations of $\bar{2}$ are

$$m_0 \quad \begin{tikzcd} 0 \arrow[dr] & 0 \\ 1 \arrow[r] & 1 \end{tikzcd}
\qquad\qquad
m_1 \quad \begin{tikzcd} 0 & 0 \\ 1 \arrow[ur] & 1 \end{tikzcd}$$

and $\epsilon\bar{2}$ has regular decomposition

$$\begin{tikzcd}[column sep=small]
 & F^1U^1\bar{2} = \boxed{\begin{array}{c} 0,\ 1 = m_0 0 = m_0 1 \\ m_1 1,\ m_0 m_1 1,\ \dots,\ m_0^n m_1 1,\ \dots \end{array}} = F^{12}U^{12}\bar{2} \arrow[dr, "F^2\epsilon^{12}U^2 = \epsilon^{12}\bar{2}"] & \\
FU\bar{2} = \boxed{\begin{array}{l} 0,\ m_0 0,\ \dots,\ m_0^n 0,\ \dots \\ 1,\ m_0 1,\ \dots,\ m_0^n 1,\ \dots \end{array}} \arrow[ur, "F^1\epsilon^{01}U^1\bar{2} = F^{12}\epsilon^{01}U^{12}\bar{2}"] \arrow[rr, "\epsilon\bar{2} = \epsilon^0\bar{2}"] & & \bar{2} = \boxed{\begin{array}{c} 0 = m_1 1 \\ 1 = m_0 0 = m_0 1 \end{array}}
\end{tikzcd}$$

In the same way for an ordinal λ an object $\bar{\lambda}$ of $\underline{\underline{A}}^{\lambda}$ may be described for which $\epsilon\bar{\lambda}$ has length λ regular decomposition. More explicitly, the underlying set $|\bar{\lambda}|$ of $\bar{\lambda}$ is that of the category λ, in other words, the set of ordinals less than λ. Furthermore, a partial operation m_{α} is defined on $|\tilde{\lambda}|$ for each $\alpha < \lambda$ as follows: given $\beta \in |\tilde{\lambda}|$ the element $m_\alpha \beta$ is undefined for $\beta < \alpha$,

$$m_\alpha \alpha = \alpha + 1 \text{ for } \alpha + 1 \neq \lambda, \quad m_\alpha \alpha = 0 \text{ for } \alpha + 1 = \lambda$$
$$\text{and } m_\alpha \beta = \beta \text{ for } \beta > \alpha .$$

An adjunction *model induced* by a set $\underline{\underline{L}}$ of cardinality m of objects of $\underline{\underline{A}}$ is one whose right adjoint $U: \underline{\underline{A}} \to Ens^m$ maps

$$A \mapsto \langle \underline{\underline{A}}(L, A) \rangle_{L \in \underline{\underline{L}}} .$$

In this terminology we see that $\underline{N}^{0\lambda}: \underline{\underline{A}}^0 \to \underline{\underline{A}}^\gamma$ is model induced by the set $\underline{\underline{L}}$ whose only member is $F^{0\gamma}$ of the one element set, i.e., the free 0-sequence on one element.

J. MACDONALD: Mathematics Department
University of British Columbia
VANCOUVER, B.C. V6T 1W5
CANADA

A. STONE: Mathematics Department
University of California, Davis
DAVIS, CA 95678
U.S.A.

REFERENCES.

1. H. APPLEGATE & M. TIERNEY, Categories with models, *Lecture Notes in Math.* 80, Springer (1969), 156-244.
2. H. APPLEGATE & M. TIERNEY, Iterated cotriples, *Lecture Notes in Math.* 137 Springer (1970), 56-99.
3. M. BARR, The point of the empty set, *Cahiers Top. et Géo. Diff.* XIII-1 (1972), 357-368.
4. J. BECK, Distributive laws, *Lecture Notes in Math.* 80, Springer (1969), 119-140.
5. B. DAY, On adjoint-functor factorization, *Lecture Notes in Math.* 420, Springer (1974), 1-19.
6. P. FREYD, Aspects of topoi, *Bull. Austral. Math. Soc.* 7 (1972), 1-76.
7. P. FREYD & M. KELLY, Categories of continuous functors, I, *J. Pure & App. Algebra* 2 (1972), 169-191.
8. P. GABRIEL & G. ULMER, Lokal präsentierbare Kategorien, *Lecture Notes in Math.* 221, Springer (1971).
9. J. GRAY, Formal category theory: Adjointness for 2-categories, *Lecture Notes in Math.* 391, Springer (1974).
10. S. HAYDEN & J. KENNISON, *Zermelo-Fraenkel set theory*, Merrill Press 1968.
11. H. HERRLICH & G. STRECKER, *Category theory*, Heldermann, 1979.
12. J. ISBELL, General functorial semantics, I, *Amer. J. Math.* 94 (1972), 535-596.
13. J. ISBELL, Structure of categories, *Bull. A.M.S.* 72 (1966), 619-655.
14. M. KELLY, Monomorphisms, epimorphsims and pullbacks, *J. Austral. Math. Soc.* 9 (1969), 124-142.
15. J. MACDONALD, Cohomology operations in a category, *J. Pure & App. Alg.* 19 (1980), 275-297.
16. S. MACLANE, *Categories for the working mathematician*, Springer, 1971.
17. J. MONK, *Mathematical Logic*, Springer, 1976.
18. H. SCHUBERT, *Categories*, Springer, 1972.
19. A. STONE, The heights of adjoint towers, *A.M.S. Notices* 21 (1974), A-81.
20. R. STREET, Cosmoi of internal categories, *Trans. A.M.S.* 258 (1980), 271-318.
21. F. ULMER, Locally α-presentable and locally α-generated categories, *Lecture Notes in Math.* 195, Springer (1971), 230-247.