APPLICATIONS OF SUP-LATTICE ENRICHED CATEGORY THEORY TO SHEAF THEORY

ANDREW M. PITTS

[Received 28 November 1986—Revised 16 October 1987]

ABSTRACT

Grothendieck toposes are studied via the process of taking the associated **Sl**-enriched category of relations. It is shown that this process is adjoint to that of taking the topos of sheaves of an abstract category of relations. As a result, pullback and comma toposes are calculated in a new way. The calculations are used to give a new characterization of localic morphisms and to derive interpolation and conceptual completeness properties for a certain class of interpretations between geometric theories. A simple characterization of internal sup-lattices in terms of external **Sl**-enriched category theory is given.

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0. *Introduction*

This paper introduces and applies a new method of making calculations about *Grothendieck toposes*—the categories of sheaves first devised by Grothendieck and his collaborators [**7**] in their work in algebraic geometry. To motivate the new approach of this paper, we first review briefly the reasons for the development of this kind of sheaf theory and the existing tools that are available to study it.

0.1. *Grothendieck toposes*

The original motivation for the concept of topos was to have sheaf cohomology theories for non-topological 'spaces' arising in algebraic geometry. These 'spaces', or *sites* as they were called, retain the characteristic feature of topological spaces of knowing when an (open) part of the space is *covered* by other parts, but refine it by allowing the covering to be specified by morphisms which need not just be inclusions of one part of the space in another. The notion of 'sheaf on a topological space' (given in terms of a presheaf on the open sets of the space satisfying unique glueing conditions, rather than in terms of a local homeomorphism over the space) extends easily to the notion of 'sheaf on a site'; and a Grothendieck topos is then the category of such (set-valued) sheaves on a site. (See [**6**, Chapter V] for a more detailed general discussion of these concepts.) It is important to note that it is not the site so much as the topos to which it gives rise which is claimed to be a new and useful concept of generalized space. This is

A.M.S. (1980) subject classification: primary 18F20, 18D20; secondary 06A23, 03C75.

because two quite different looking sites can give rise to equivalent toposes—in contrast to the situation for topological spaces where, under only very mild separation assumptions, a space is recoverable from its category of sheaves.

Grothendieck wrote in [7] of the role of toposes as giving a notion of space (specified indirectly via its sheaves) sufficiently general to allow important topological ideas to be applied usefully in many new situations. The new situation at the time was algebraic geometry, where there are many significant Grothendieck toposes (see [27], for example). In the years that have followed, interesting toposes have been found and used in such fields as differential geometry [22], real algebraic geometry [3], general topology itself [9], model theory [20, 25], and recursion theory [23]. Has this allowed important topological ideas to be applied usefully in new situations, as Grothendieck hoped? The answer is 'yes', but not as easily or extensively as one would like, because of the difficulties of making calculations with toposes. There are, for example, interesting results about the (co)homology and homotopy of toposes [16, 21, 26], but much remains unknown. Much of what we do know of the properties of toposes is due to the combined use of three powerful techniques.

The first is the use of *locale* theory, or 'pointless topology' as Johnstone terms it in his survey [13]. Locales are a generalization of (sober) topological spaces via the lattice of open subsets of a space. Compared with toposes, they seem a mild generalization, and certainly they have a very rich theory closely linked to general topology; see [12]. However, recent work of Joyal and Tierney [15] shows that the jump in generality from locales to toposes is not so very great and can be summed up in the slogan: 'topos = locale + groupoid'. (See [14] for a survey of this and related results.)

The second important technique for studying toposes lies in the realm of mathematical logic and has to do with the model theory of a certain class of theories in infinitary, first-order predicate logic—called *geometric theories*. These are the theories which can be axiomatized by statements of the form: 'for all x satisfying $\varphi(x)$ there is some y and some $i \in I$ such that $\psi_i(x, y)$ holds', where I is some set of indices and the φ, ψ_i are finite conjunctions of atomic predicates asserting equalities or basic relations. Hand in hand with the development of topos theory over the last 25 years, there has emerged a category-theoretic treatment of logic. A key part of this treatment is the extension of Tarski's truth definition to structures valued in categories other than the category of sets; see [19]. In particular, Grothendieck toposes provide precisely the right structure to support models of geometric theories. (For example, from this perspective a sheaf of local rings in the usual sense can be viewed as a model in a topos of sheaves of the theory of local rings—which is a geometric theory.) It was observed by Joyal and Makkai and Reyes [19] that the consideration of the models of a particular geometric theory not just in the category of sets, but in arbitrary Grothendieck toposes, opens up the possibility of constructing a *generic* model for the theory. Thus if **T** is a geometric theory, one can construct a Grothendieck topos **Set**[**T**], called the *classifying topos* of **T**, with the property that there is a natural correspondence between models of **T** in a Grothendieck topos **E** and geometric morphisms from **E** into **Set**[**E**]. (The latter are the topos-theoretic version of continuous maps between topological spaces.) The correspondence is mediated by a particular model G of **T** in **Set**[**T**] (the generic model) with the property that any other model in any other topos can be obtained uniquely up to unique isomorphism as the inverse image of G along a geometric morphism into the classifying topos. So geometric theories give rise to Grothendieck toposes. The other half of the story is that any Grothendieck topos is the classifier of some geometric theory. From this point of view, geometric theories are like sites of definition in that they give us another means of describing toposes in terms of 'generators and relations'. Indeed there is a very close connection between sites and geometric theories under which the covers in the site correspond to the axioms of the theory; see [**8**, § 7.4]. The use of geometric theories rather than sites for constructing toposes with specified properties represents an advance. This is because in contrast to the sites, the theories involved are often quite natural ones with readily apparent properties. (A good example of this occurs in Joyal's covering theorems, which involve geometric theories of subenumeration; see [**14**].)

The third important method used in topos theory has its roots in the seminal work of Lawvere and Tierney on *elementary* toposes (of which Grothendieck toposes are a special case) and the subsequent realization by Bénabou, Mitchell, and others of the correspondence between toposes and a certain kind of constructive set theory. (See [**8**, Introduction] for history and references.) Many mathematical constructions and arguments can be formalized in this set theory and hence can be *relativized* from the category of sets to an arbitrary Grothendieck topos. For example, a lot of locale theory can be so relativized; and it is this technique which is used extensively in [**15**] to develop the close relationship between locales and toposes mentioned above. Another important example has an amusing tinge of self-reference: the notion of Grothendieck topos itself can be relativized! It turns out that specifying a Grothendieck topos (via some site of definition) in a Grothendieck topos **E** amounts to specifying a geometric morphism whose codomain is **E**. In this way the technique of relativization allows the proof of certain properties of geometric morphisms to be reduced to simpler properties of toposes themselves, but in a different set-theoretic universe. To give a simple illustration of this, suppose we had two geometric morphisms $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$, $\mathbf{g}\colon \mathbf{G} \to \mathbf{E}$ and wished to prove things about their pullback (or 'fibre product'):

$$\begin{tikzcd} \mathbf{F} \times_{\mathbf{E}} \mathbf{G} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\cong"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

Relativizing to **E**, we find that the geometric morphisms **f** and **g** correspond to toposes in the set-theoretic universe determined by **E**; and the product of those toposes there is another topos in the world of **E** whose corresponding geometric morphism into **E** turns out to be $\mathbf{F} \times_{\mathbf{E}} \mathbf{G} \to \mathbf{E}$. Thus pullbacks are reduced to products and the latter are easier to deal with in most circumstances.

Unfortunately, not every property of geometric morphisms is reducible to a property of toposes via the technique of relativization. This is particularly so when one seriously considers the 'two-dimensional' aspect of the collection **GTOP** of all Grothendieck toposes—whose '0-cells' are toposes (generalized spaces), whose '1-cells' are geometric morphisms (generalized continuous maps), and whose '2-cells' are given by natural transformations between inverse image functors (generalizing the specialization ordering for points in a topological space). Thus the collection of geometric morphisms from one topos to another naturally forms a *category* rather than a set, and, in particular, a geometric morphism can possess a highly non-trivial automorphism group.

Returning to the idea of relativization, let us give an example of a situation which will be of concern in the sequel and for which the technique of relativization described above is of no help. Consider the so-called *comma square* formed from two geometric morphisms $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$, $\mathbf{g}\colon \mathbf{G} \to \mathbf{E}$:

$$\begin{tikzcd} \mathbf{G} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[r, phantom, shift right=6, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

This is like the pullback square above, except that the square commutes only up to a 2-cell (and is universal amongst such squares for $\mathbf{f}$ and $\mathbf{g}$). Thus there are two different morphisms into $\mathbf{E}$ from the comma topos $\mathbf{G} <_{\mathbf{E}} \mathbf{F}$ (viz. $\mathbf{fp}$ and $\mathbf{gq}$) and they give rise to two different toposes in the world of $\mathbf{E}$. Consequently, relativization to $\mathbf{E}$ does not produce a conceptually simpler description of the comma topos $\mathbf{G} <_{\mathbf{E}} \mathbf{F}$ in the same way that it did for the pullback topos $\mathbf{F} \times_{\mathbf{E}} \mathbf{G}$. Now we know that such a comma topos exists because we can construct it as the classifying topos of a certain theory of homomorphisms built from theories which are classified by $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$. (A site of definition could also be given in terms of sites for $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$.) This is because of the way toposes act as classifiers of geometric theories, whereby geometric morphisms amount to models of a theory in a topos; and then a 2-cell between geometric morphisms corresponds to the usual notion of *homomorphism* between models. In particular, Grothendieck toposes can be used to study the model theory of geometric logic and other related logics: see **[24]** and **[25]**. The results in **[24]**, in particular, require an analysis of properties of comma toposes which existing techniques using locales, sites or classifying toposes do not seem to provide (mainly because they cannot be used in combination with relativization). It was these properties of comma toposes and their applications to model theory which provided the main motivation for the development of the theory presented in this paper, and to a description of which we now turn.

0.2. *Categories of relations*

In this paper we study properties of a Grothendieck topos $\mathbf{E}$ via its associated *category of relations*, $\mathrm{Rel}(\mathbf{E})$, whose objects are just those of $\mathbf{E}$, that is, sheaves, but whose morphisms from X to Y are *subsheaves* of the product $X \times Y$. The study of the calculus of relations between *sets* has a long history. More recently, the calculus of relations in various kinds of *category* (regular, coherent, logos, topos) has been studied by several people, but particularly by Freyd in his so far unpublished work on *allegories* (to appear in a forthcoming book by Freyd and Scedrov). Recently Carboni and Walters have formulated a notion of *cartesian bicategory* which in the locally ordered case led them to an abstract notion of *bicategory of relations* **[2]**. Whilst such structures are closely related to Freyd's allegories, they differ in the way that the presence of finite limits is ensured in the associated *categories of maps*: they eliminate intersection and involution of relations as primitive notions in favour of finite products of objects and relations. This difference is crucial to us here, since it allows us to generalize the treatment of locales by Joyal and Tierney in [15] using the category **Sl** of sup-lattices, whose objects are complete lattices and whose morphisms are all sup-preserving maps. Here we get a treatment of Grothendieck toposes using *categories enriched over* **Sl**, as follows.

In op. cit., a locale is characterized as a sup-lattice A together with an associative, unitary product $A \otimes A \to A$ satisfying some simple conditions which amongst other things ensure that the product is binary meet in the locale; this observation is the starting point for a development of locale theory as part of the 'commutative algebra' of sup-lattices. Generalizing this view point, we consider a sup-lattice enriched category $\mathbf{A}$ together with a coherently associative, unitary product $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ satisfying the simple but rather subtle conditions taken from [2], which amongst other things ensure that the product gives cartesian products in the associated category of maps. Such an $\mathbf{A}$ is termed a *distributive category of relations,* or *dcr* for short. (See Definition 2.1.) It generalizes the notion of locale in the sense that *locales are precisely dcr's with one object.*

The connection with topos theory is that the category of relations of a Grothendieck topos (see 1.1) is a dcr which is both *bounded* (has a small set of generators) and *complete* (has small coproducts and splitting of symmetric idempotents); conversely, the category of maps (see 1.6) of a bounded, complete dcr is a Grothendieck topos. These facts are proved by Carboni and Walters in [2] (and prior to them, similar facts for allegories were proved by Freyd). Here we show (in Proposition 2.5) that this result extends to a duality between the 2-category **GTOP** of Grothendieck toposes, geometric morphisms and natural transformations, and a 2-category whose objects are bounded, complete dcr's. To do this, first we must formulate (in Definition 2.4) the correct notion of 2-cell between morphisms of dcr's—which notion involves considering *lax natural transformations* between sup-lattice enriched functors. (See Definition 1.3.)

However, as a tool for making calculations in **GTOP**, it is not so much the above duality that is important, but rather an associated (bi)adjunction between **GTOP** and the 2-category of dcr's which are bounded but not necessarily complete. This involves the process of *completion* of a dcr (whose explicit description and well-behaved properties were developed by Freyd in the context of allegories). Starting from a bounded dcr $\mathbf{A}$, taking the category of maps of its completion yields a Grothendieck topos which we denote by $\mathrm{Sh}(\mathbf{A})$ and call the *topos of sheaves* on $\mathbf{A}$. This terminology is justified by the fact that when $\mathbf{A}$ has just one object, i.e. is a locale, then $\mathrm{Sh}(\mathbf{A})$ is just Higg's category of $\mathbf{A}$-*valued sets,* well known to be equivalent to the topos of sheaves in the usual sense. (See [5].) Passing from the 'one object' case to the more general 'many object' one, we see (in Theorem 2.9) that *forming toposes of sheaves* (on bounded dcr's) *is biadjoint to taking categories of relations* (of Grothendieck toposes).

It is this description of Grothendieck toposes as categories of sheaves on bounded dcr's which is the key aspect of the theory presented in this paper and it is worth comparing it with the traditional ways of describing toposes mentioned in 0.1. As for sites and geometric theories, bounded dcr's do not give a *canonical* means of presenting a topos—several different bounded dcr's may yield equivalent categories of sheaves. However, unlike sites or theories, bounded dcr's are *algebraic* gadgets and—as we will demonstrate—are susceptible to algebraic methods. We mean something very specific here, namely the analogue of commutative algebra initiated by Joyal and Tierney in [**15**], where the role of additivity is played by (infinitary) sups, abelian groups are replaced by sup-lattices and rings by locales. The step we are taking here is then like that from abelian groups to additive categories. It is tempting to speculate that some of the methods of homological algebra would have analogues in this setting, with useful consequences for the geometric (rather than logical) applications of topos theory.

Our aims here are somewhat more modest. We exploit the fundamental and elegant adjunction between taking sheaves and taking categories of relations mentioned above. It is a simple consequence of the existence of such an adjunction that limits in **GTOP** can be described in terms of colimits of dcr's. But the latter can be calculated using the 'commutative algebra' of sup-lattice enriched categories in a way that extends the calculations in [**15**]. Specifically in § 3 we calculate *pushout* and *cocomma* dcr's in terms of tensor products and 'lax tensor products' of *modules over dcr's.* On taking sheaves, this yields a description of *pullback* and *comma* squares in **GTOP** quite different in character from that afforded by using sites of definition or classifying toposes. As a result we obtain (in Theorem 4.5) the following property of a comma square

$$\begin{tikzcd} \mathbf{G} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

in **GTOP** which was used in [**24**] to derive an interpolation property of pretoposes (which in turn was the key tool of the analysis in op. cit. of the Makkai–Reyes 'conceptual completeness' theorem for pretoposes):

Suppose in the above comma square that $\mathbf{f}^*$ preserves arbitrary intersections of subobjects. Then $\mathbf{q}$ is open; moreover, for any $X \in \mathbf{E}$ and subobjects $B \rightarrowtail \mathbf{f}^*(X)$, $C \rightarrowtail \mathbf{g}^*(X)$, if

$$\mathbf{p}^*(B) \leqslant \varphi_X^{-1}(\mathbf{q}^*C)$$

in $\mathrm{Sub}(\mathbf{p}^*\mathbf{f}^*X)$, then there is $A \rightarrowtail X$ with

$$B \leqslant \mathbf{f}^*(A) \quad \text{and} \quad \mathbf{g}^*(A) \leqslant C.$$

We also show (in Proposition 4.9) that:

$\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ is localic if and only if the associated diagonal geometric morphism $\mathbf{d}\colon \mathbf{F} \to (\mathbf{F} <_{\mathbf{E}} \mathbf{F})$ is an inclusion.

Combining these properties of comma toposes, we derive a 'conceptual completeness' result (Corollary 4.11) for certain kinds of geometric morphism and tie it in with the infinitary generalizations of pretopos conceptual completeness considered by Makkai and Reyes in §§ 7.3 and 7.4 of [**19**].

Finally, in § 5, we give a strikingly simple characterization (Proposition 5.2) of *internal* sup-lattices in a Grothendieck topos **E** as **Sl**-enriched functors

Rel(**E**)$\to$**Sl**. This allows a reformulation of the description of comma and pullback toposes of § 3 in terms of composition and 'lax composition' of **Sl**-enriched *profunctors*.

Acknowledgements

I originally obtained the purely topos-theoretic results in this paper via an analysis of internal sup-lattices (and locales) in Grothendieck toposes which begins with the results of § 5. The account given here using distributive categories of relations and building on the work of Carboni and Walters in **[2]**, apart from its intrinsic interest, is conceptually clearer (to the author, at least) since it can be seen as directly generalizing the techniques of **[15]** from the 'one object' to the 'many object' case. The approach was developed during a visit to the Università di Milano, and I am particularly grateful to Aurelio Carboni for many valuable conversations. Work on the paper was completed during a visit to the University of Sydney, where I benefited from the helpful comments and the hospitality of members of the Sydney Category Seminar. I would also like to thank the editorial advisor and referee for their very helpful comments on the first version of this paper. Finally, I gratefully acknowledge the financial support of the Royal Society.

1. **Sl**-*enriched categories*

1.1. *Relations*

Let **E** be a Grothendieck topos. For X and Y objects of **E**, a *relation* from X to Y is a subobject $R \rightarrowtail X \times Y$; we shall use the notation R: $X \leftrightarrow Y$ to indicate this. Relations are composed using pullbacks and image factorizations in **E**, and the identity on X for this composition is the diagonal subobject Δ: $X \rightarrowtail X \times X$. In this way one gets a category, denoted by Rel(**E**), with the same objects as **E**, but with relations for morphisms.

Given R: $X \leftrightarrow Y$, the *reciprocal* relation $R^0 \rightarrowtail X \times Y \cong Y \times X$ will be denoted R^0: $Y \leftrightarrow X$. The topos **E** becomes a subcategory of Rel(**E**) if we identify a morphism f: $X \to Y$ in **E** with the relation $X \leftrightarrow Y$ given by its graph $\langle 1, f \rangle$: $X \rightarrowtail X \times Y$. Evidently each relation can be decomposed as $R = ba^0$ with a and b morphisms in **E**, for example, by choosing a monomorphism $\langle a, b \rangle$ representing the subobject R.

Of course there is more structure present in Rel(**E**) than just that of being a category: each hom set Rel(**E**)(X, Y) has a partial order (the inclusion relation for subobjects) which is complete (since **E** is a Grothendieck topos); and the operations of pre- and post-composition by a relation are sup-preserving maps between the hom sets of Rel(**E**) (but not in general inf-preserving ones). Thus we see that Rel(**E**) *is enriched over the category of complete posets and sup-preserving maps*. Following Joyal and Tierney **[15]**, we name the latter category after its morphisms as the *category of sup-lattices* and denote it **Sl**.

1.2. *Sup-lattices*

We refer the reader to Chapter I of **[15]** for a development of the properties of **Sl** relevant to us here. In particular, one can build objects in **Sl** in terms of generators and relations. For example, the *tensor product* $M \otimes N$ of two sup-lattices can be described as freely generated by pairs $m \in M$, $n \in N$, denoted $m \otimes n$, subject to the relations

$$\left(\bigvee_i m_i\right) \otimes n = \bigvee_i (m_i \otimes n)$$

and

$$m \otimes \left(\bigvee_i n_i\right) = \bigvee_i (m \otimes n_i).$$

Thus sup-lattice morphisms $M \otimes N \to L$ are in natural bijection with maps $M \times N \to L$ preserving sups in each variable separately. They are then also in natural bijection with morphisms $M \to \mathrm{Hom}(N, L)$, where $\mathrm{Hom}(N, L)$ is the sup-lattice of morphisms $N \to L$ partially ordered pointwise from L.

The free sup-lattice on a set X is just its powerset $P(X)$ partially ordered by subset inclusion. In particular, $P(1)$ is a unit for $\otimes$. In this way **Sl** becomes a symmetric monoidal closed category and one can consider categories *enriched* over **Sl**, using the general theory set out in [**17**], for example. However, we will need to go a little bit outside the usual context of enriched category theory, since the 2-category structure of **Sl** (which it has by virtue of the partial orders on its hom sets) will play an important rôle. (It is involved in the definition of lax natural transformation given below, for example.)

1.3. Definitions. (i) A **Sl**-*category* is a locally small category **C** whose hom sets are sup-lattices and whose composition preserves sups in each variable separately:

$$\left(\bigvee_i s_i\right) \circ r = \bigvee_i (s_i \circ r)$$

and

$$s \circ \left(\bigvee_i r_i\right) = \bigvee_i (s \circ r_i).$$

(ii) A **Sl**-*functor* F: $\mathbf{C} \to \mathbf{D}$ between **Sl**-categories is a functor between the underlying categories which preserves sups of morphisms:

$$F\left(\bigvee_i r_i\right) = \bigvee_i F(r_i).$$

(iii) A *lax natural transformation* ρ: $F \to G$ between **Sl**-functors $\mathbf{C} \to \mathbf{D}$ is specified by a family of morphisms in **D**,

$$\rho_X\colon F(X) \to G(X) \quad (X \in \mathbf{C}),$$

such that, for each r: $X \to X'$ in **C**,

$$\rho_{X'} \circ F(r) \leq G(r) \circ \rho_X$$

in $\mathbf{D}(FX, GX')$.

(iv) For **Sl**-categories **C** and **D**, let

$$\mathbf{Sl\text{-}CAT}_{<}(\mathbf{C}, \mathbf{D})$$

denote the category whose objects are **Sl**-functors $\mathbf{C} \to \mathbf{D}$ and whose morphisms are lax natural transformations (composition and identities being as for ordinary natural transformations). Note that a lax natural transformation with a lax natural inverse is necessarily natural; thus the isomorphisms in $\mathbf{Sl}\text{-}\mathbf{CAT}_<(\mathbf{C},\mathbf{D})$ are just natural isomorphisms between the underlying functors.

Warning. **Sl**-categories, **Sl**-functors and lax natural transformations do not form a 2-category. This is because whenever one has $\sigma\colon F\to G$ in $\mathbf{Sl}\text{-}\mathbf{CAT}_<(\mathbf{A},\mathbf{B})$ and $\rho\colon H\to K$ in $\mathbf{Sl}\text{-}\mathbf{CAT}_<(\mathbf{B},\mathbf{C})$, then, for each $X\in\mathbf{A}$,

$$\begin{tikzcd} HF(X) \arrow[r, "\rho_{FX}"] \arrow[d, "H\sigma_X"'] \arrow[dr, phantom, "\leqslant"] & KF(X) \arrow[d, "K\sigma_X"] \\ HG(X) \arrow[r, "\rho_{GX}"'] & KG(X) \end{tikzcd}$$

only commutes up to an inequality (since ρ is only lax natural): choosing either way round the above square to define the 'horizontal' composition $\rho * \sigma\colon HF\to KG$ gives an associative operation, but the 'interchange law' connecting $*$ and $\circ$ is, in either case, only an inequality ($(\delta\circ\gamma)*(\beta\circ\alpha)\leqslant(\delta*\beta)\circ(\gamma*\alpha)$, if one defines $\rho*\sigma$ to be $\rho_G\circ H\sigma$).

1.4. EXAMPLES. We record the kind of **Sl**-categories, **Sl**-functors and lax natural transformations that concern us here.

(i) If $\mathbf{E}$ is a Grothendieck topos, we noted above that $\mathrm{Rel}(\mathbf{E})$ is a **Sl**-category.

(ii) Suppose that $\mathbf{f}\colon \mathbf{F}\to\mathbf{E}$ is a geometric morphism between Grothendieck toposes. For $X, X'\in\mathbf{E}$, the operation sending a relation $R\rightarrowtail X\times X'$ to

$$\mathbf{f}^*(R)\rightarrowtail \mathbf{f}^*(X\times X')\cong\mathbf{f}^*(X)\times\mathbf{f}^*(X')$$

is a sup-lattice morphism

$$\mathrm{Rel}(\mathbf{E})(X, X')\to\mathrm{Rel}(\mathbf{F})(\mathbf{f}^*(X), \mathbf{f}^*(X')).$$

In this way one gets a **Sl**-functor $\mathrm{Rel}(\mathbf{E})\to\mathrm{Rel}(\mathbf{F})$ which we shall also denote by $\mathbf{f}^*$.

(iii) Now suppose that $\mathbf{f}$, $\mathbf{g}\colon \mathbf{F}\to\mathbf{E}$ are geometric morphisms and that $\varphi\colon \mathbf{f}^*\to\mathbf{g}^*$ is a natural transformation. Given $R\colon X\nrightarrow X'$ in $\mathrm{Rel}(\mathbf{E})$, since

$$\begin{tikzcd} \mathbf{f}^*(R) \arrow[r, tail] \arrow[d, "\varphi_R"'] & \mathbf{f}^*(X\times X')\cong\mathbf{f}^*(X)\times\mathbf{f}^*(X') \arrow[d, "\varphi_X\times\varphi_{X'}"] \\ \mathbf{g}^*(R) \arrow[r, tail] & \mathbf{g}^*(X\times X')\cong\mathbf{g}^*(X)\times\mathbf{g}^*(X') \end{tikzcd}$$

commutes in $\mathbf{F}$, one has $\varphi_{X'}\circ\mathbf{f}^*(R)\leqslant\mathbf{g}^*(R)\circ\varphi_X$ in $\mathrm{Rel}(\mathbf{F})$. Thus φ is a lax natural transformation from $\mathbf{f}^*$ to $\mathbf{g}^*$ regarded as **Sl**-functors as in (ii). (Indeed a simple calculation shows that conversely, a collection $(\varphi_X\colon \mathbf{f}^*(X)\to\mathbf{g}^*(X) \mid X\in\mathbf{E})$ is natural for morphisms in $\mathbf{E}$ if it is lax natural for relations in $\mathbf{E}$.)

1.5. *Tensor of* **Sl**-*categories*

As usual for enriched categories, the tensor product $\mathbf{C}\otimes\mathbf{D}$ of two **Sl**-categories is the **Sl**-category whose objects are pairs $X\in\mathbf{C}$, $Y\in\mathbf{D}$, denoted $X\otimes Y$, and whose hom sup-lattices are given by the tensor product of sup-lattices:

$$(\mathbf{C} \otimes \mathbf{D})(X \otimes Y, X' \otimes Y') = \mathbf{C}(X, X') \otimes \mathbf{D}(Y, Y').$$

The usual universal property of $\mathbf{C} \otimes \mathbf{D}$ extends to one involving lax natural transformations as follows.

For any **Sl**-category $\mathbf{B}$, let $[\mathbf{C}, \mathbf{D}; \mathbf{B}]_<$ denote the category whose objects are functors $\mathbf{C} \times \mathbf{D} \to \mathbf{B}$ which preserve sups of morphisms in each variable separately, and whose morphisms are lax natural transformations. Then, composition with the evident functor $\otimes$: $\mathbf{C} \times \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$ induces a functor

$$\mathbf{Sl\text{-}CAT}_<(\mathbf{C} \otimes \mathbf{D}, \mathbf{B}) \to [\mathbf{C}, \mathbf{D}; \mathbf{B}]_<,$$

which is an isomorphism of categories for each $\mathbf{B}$.

1.6. *Maps*

A *map* in a **Sl**-category $\mathbf{C}$ is a morphism f: $X \to Y$ which possesses a right adjoint, i.e. for which there exists a morphism f^*: $Y \to X$ with $ff^* \leqslant 1_Y$ and $1_X \leqslant f^*f$. The collection of maps contains the identity morphisms (and more generally, all isomorphisms) and is closed under composition. There is thus a category, denoted Map($\mathbf{C}$), whose objects are those of $\mathbf{C}$ and whose morphisms are the maps in $\mathbf{C}$.

For a Grothendieck topos $\mathbf{E}$ (or more generally, just a regular category), it is easily verified that the maps in Rel($\mathbf{E}$) are precisely the (graphs of) morphisms from $\mathbf{E}$, with the right adjoint being given by the reciprocal relation. Thus Map(Rel($\mathbf{E}$)) and $\mathbf{E}$ are isomorphic categories.

If F: $\mathbf{C} \to \mathbf{D}$ is a **Sl**-functor, it necessarily preserves adjoint pairs of morphisms and hence yields a functor

$$F\colon \mathrm{Map}(\mathbf{C}) \to \mathrm{Map}(\mathbf{D}).$$

If ρ: $F \to G$ is a lax natural transformation between **Sl**-functors from $\mathbf{C}$ to $\mathbf{D}$, then for any map f: $X \to X'$ in $\mathbf{C}$, using the lax naturality condition for both f and f^* together with their adjointness, one has

$$\begin{aligned}\rho_{X'} \circ Ff &\leqslant Gf \circ \rho_X \\ &\leqslant Gf \circ \rho_X \circ Ff^* \circ Ff \\ &\leqslant Gf \circ Gf^* \circ \rho_{X'} \circ Ff \\ &\leqslant \rho_{X'} \circ Ff.\end{aligned}$$

Therefore the ρ_X: $F(X) \to G(X)$ are actually *natural for maps* in $\mathbf{C}$.

1.7. *Bounded* **Sl**-*categories*

We shall say that a collection $\mathbf{G}$ of objects of a **Sl**-category $\mathbf{C}$ is **Sl**-*generating* for $\mathbf{C}$ if, for each $X \in \mathbf{C}$,

$$1_X = \bigvee \{s \circ r \mid \mathrm{cod}(r) = \mathrm{dom}(s) \in \mathbf{G} \text{ and } s \circ r \leqslant 1_X\}$$

(from which it follows that every morphism in $\mathbf{C}$ can be expressed as a sup of morphisms which factor through some object in $\mathbf{G}$). Then say that $\mathbf{C}$ is *bounded* if it possesses a small collection of **Sl**-generators.

Evidently a small **Sl**-category is automatically bounded. The large **Sl**-categories by which we are motivated (i.e. those of the form Rel($\mathbf{E}$) for $\mathbf{E}$ a Grothendieck topos) are bounded, as the following result shows:

1.8. LEMMA. *Let* **E** *be a Grothendieck topos and* **G** *a collection of objects of* **E**. *Then the following are equivalent:*

(i) *the objects of* **G** *are* **Sl**-*generators for* Rel(**E**);

(ii) *each* $X \in \mathbf{E}$ *can be expressed as the subquotient of a small coproduct of objects of* **G**, *that is, there is a diagram of the form*

$$\begin{tikzcd} \bullet \arrow[r, two heads, "e"] \arrow[d, tail, "m"'] & X \\ \coprod_{i \in I} U_i & \end{tikzcd}$$

with $U_i \in \mathbf{G}$, *m mono and e epi;*

(iii) *the collection* $\bar{\mathbf{G}}$ *of subobjects of objects of* **G** *generates the topos* **E** *in the usual sense, that is, a parallel pair of morphisms of* **E** *are equal just when they have equal compositions with any morphism whose domain is in* $\bar{\mathbf{G}}$.

Proof. The equivalence of (ii) and (iii) follows from standard properties of Grothendieck toposes. The equivalence of (i) and (ii) follows from two facts about relations in a Grothendieck topos **E**:

(a) X is a subquotient of Y in **E** if and only if X is a retract of Y in Rel(**E**) (see [**11**, Lemma 2.5]);

(b) given relations $R_i\colon X \leftrightarrow U_i$ and $S_i\colon U_i \leftrightarrow Y$ in **E**, and letting

$$U = \coprod_{i \in I} U_i$$

be the coproduct of the U_i with coproduct insertions $m_i\colon U_i \rightarrowtail U$, one has that

$$\bigvee_{i \in I} S_i \circ R_i = \Big(\bigvee_{j \in I} S_j (m_j)^0\Big) \circ \Big(\bigvee_{k \in I} m_k R_k\Big)$$

in Rel(**E**)(X, Y).

(a) holds merely because **E** is a regular category and (b) holds because moreover, small coproducts in **E** are disjoint and stable under pullback.

2. *Distributive categories of relations*

We noted in the previous section that if $\mathbf{C} = \mathrm{Rel}(\mathbf{E})$ with **E** a Grothendieck topos, then $\mathrm{Map}(\mathbf{C}) \simeq \mathbf{E}$. What kind of **Sl**-categories **C** are such that Map(**C**) is a Grothendieck topos? Freyd has given an answer in terms of his theory of 'allegories' (unpublished, but an account of which will appear in the forthcoming book by Freyd and Scedrov): these abstract the ability to form reciprocals and finite meets of relations, together with certain properties they possess with respect to composition. With this structure it is possible to ensure that Map(**C**) has pullbacks by stipulating that every morphism in **C** has a 'tabulation' (which is a particular kind of factorization of the morphism as the reciprocal of a map followed by a map: see [**2**, 3.4]). Furthermore, the existence of a terminal object in Map(**C**) translates in a straightforward way into a condition on **C**. Next, Freyd has shown that with these hypotheses on **C**, the presence of pullback stable, effective coequalizers of equivalence relations in Map(**C**) is equivalent to the splitting in **C** of all symmetric (i.e. equal to their own reciprocal) monads; and the presence of pullback stable, disjoint coproducts in Map(**C**) is equivalent simply to **C** having coproducts. Boundedness of **C** in the sense of 1.7 gives the existence of a small set of generators in Map(**C**). With all these conditions on **C**, we can apply Giraud's Theorem (see [**8**, 0.4]) to see that Map(**C**) is a Grothendieck topos.

The work of Carboni and Walters in [**2**] provides, amongst other things, an analysis of when Map(**C**) is a Grothendieck topos which differs from the one indicated above in the way that it ensures that Map(**C**) has finite limits. They analyse the requirement of having finite limits into the equivalent one of having both finite meets of subobjects and finite products: they impose a simple, global 'product' structure on **C** (see Definition 2.1(i) below) which is equivalent to Map(**C**) having these two properties. Moreover, a certain axiom for this structure (that of 'discreteness'; see Definition 2.1(ii)) makes reciprocals of relations definable and one can then proceed as above with further requirements on **C** to ensure that Map(**C**) is a Grothendieck topos.

We are going to combine the Carboni–Walters approach to categories of relations with the work of Joyal and Tierney in [**15**] on sup-lattices and locales. By forgetting about reciprocation and hiding meets of relations in a global product structure on a **Sl**-category **C**, one has a direct generalization, from sup-lattices (which can be regarded as one-object **Sl**-categories) to arbitrary **Sl**-categories, of Joyal and Tierney's analysis of locales as certain monoids in $(\mathbf{Sl}, \otimes, P(1))$. The analogue of a locale in the 'many-object' case is what Carboni and Walters term a distributive (bi)category of relations and is defined as follows:

2.1. Definitions (cf. [**2**]). (i) A **Sl**-category **C** is *cartesian* if there is a **Sl**-functor

$$\times\colon \mathbf{C}\otimes\mathbf{C}\to\mathbf{C}$$

and an object I of **C**, together with isomorphisms

$$\begin{aligned} a_{XYZ}&\colon X\times(Y\times Z)\cong(X\times Y)\times Z,\\ s_{XY}&\colon X\times Y\cong Y\times X,\\ r_X&\colon X\cong X\times I,\end{aligned}$$

and morphisms

$$\begin{aligned} \Delta_X&\colon X\longrightarrow X\times X,\\ t_X&\colon X\longrightarrow I,\end{aligned}$$

in **C** satisfying the following axioms:

(a) the isomorphisms a_{XYZ}, s_{XY}, r_X are natural in X, Y, $Z\in\mathbf{C}$ and satisfy the classical symmetric, monoidal coherence conditions (see [**17**, 1.4], for example);

(b) the morphisms Δ_X, t_X are maps (in the sense of 1.6) and lax natural in $X\in\mathbf{C}$;

(c) for each $X\in\mathbf{C}$, (X, Δ_X, t_X) is a commutative comonoid object for

$(\mathbf{C}, \times, I)$; that is, the diagrams

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ X \times X \arrow[d, "1\times\Delta"'] & & X \times X \arrow[d, "\Delta\times 1"] \\ X \times (X \times X) \arrow[rr, phantom, "\cong_a"] & & (X \times X) \times X \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ X \times X \arrow[rr, phantom, "\cong_s"] & & X \times X \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "1"'] & X \times X \arrow[d, "1\times t"] \\ X \arrow[r, phantom, "\cong_r"] & X \times I \end{tikzcd}$$

commute.

(ii) If $\mathbf{C}$ is a cartesian **Sl**-category, an object X of $\mathbf{C}$ is called *discrete* by Carboni and Walters if

$$\begin{tikzcd} X \times (X \times X) \arrow[rr, phantom, "\cong"] \arrow[drr, "\Delta^*\times 1", start anchor=east, shift left=20] & & (X \times X) \times X \arrow[d, "\Delta^* \times 1"] \\ X \times X \arrow[u, "1\times\Delta"] \arrow[dr, "\Delta^*"'] & & X \times X \\ & X \arrow[ur, "\Delta"'] & \end{tikzcd}$$

commutes (where Δ^* is the right adjoint to Δ provided by Axiom (b) above). A cartesian **Sl**-category in which every object is discrete will be called a *distributive category of relations,* or *dcr* for short.

2.2. REMARKS. In [2] the following facts about the above definitions are proved.

(i) If $\mathbf{C}$ is a cartesian **Sl**-category, then Map($\mathbf{C}$) has finite products: the terminal object is I and the binary product of X and Y is $X \times Y$ with product projections

$$\pi_1\colon\ X \times Y \xrightarrow{1\times t} X \times I \cong X,$$

$$\pi_2\colon\ X \times Y \xrightarrow{t\times 1} I \times Y \cong Y.$$

(And the isomorphisms a, s, r are then the canonical ones associated to products in Map($\mathbf{C}$).) Moreover, each hom sup-lattice $\mathbf{C}(X, Y)$ is actually a locale, with the binary meet of $r,s\colon X \to Y$ being

$$r \wedge s\colon\ X \xrightarrow{\Delta} X \times X \xrightarrow{r\times s} Y \times Y \xrightarrow{\Delta^*} Y$$

and the top element being

$$\top\colon X \xrightarrow{t} I \xrightarrow{t^*} Y.$$

In particular, the top element of $\mathbf{C}(I, I)$ is the identity on I.

Conversely, if a **Sl**-category $\mathbf{C}$ is such that

(a) Map($\mathbf{C}$) has finite products,

(b) each $\mathbf{C}(X, Y)$ is a locale, with $1\colon I \to I$ the top element of $\mathbf{C}(I, I)$, and

(b) the assignment $X, Y \mapsto X \times Y$ extends *functorially* to morphisms in $\mathbf{C}$ via the formula

$$r \times s = ((\pi_1)^* \circ r \circ \pi_1) \wedge ((\pi_2)^* \circ s \circ \pi_2),$$

then $\mathbf{C}$ is cartesian. (So in particular, being cartesian is a categorical *property* of **Sl**-categories.)

(ii) If now $\mathbf{C}$ is a dcr, one can define a **Sl**-functor $(-)^0\colon \mathbf{C} \to \mathbf{C}$ which is the identity on objects and is given on morphisms $r\colon X \to Y$ by

$$r^0 = (Y \cong Y \times I \xrightarrow{1 \times \Delta t^*} Y \times X \times X \xrightarrow{1 \times r \times 1} Y \times Y \times X \xrightarrow{t\Delta^* \times 1} I \times X \cong X).$$

This **Sl**-functor is an involution ($r^{00} = r$), preserves finite meets of morphisms, and satisfies Freyd's 'law of modularity':

$$(r_1 \circ s) \wedge r_2 \leq (r_1 \wedge (r_2 \circ s^0)) \circ s.$$

(iii) If $\mathbf{C}$ is a dcr, then $f\colon X \to Y$ in $\mathbf{C}$ is a map if and only if f^0 is right adjoint to f. More importantly, the property of being a map can be expressed in a dcr by *equations* involving the cartesian structure of $\mathbf{C}$. For one has that $f\colon X \to Y$ is a map if and only if

$$\Delta_Y \circ f = (f \times f) \circ \Delta_X$$

and

$$t_Y \circ f = t_X.$$

(Regarding each of the above equalities as given by two inequalities, note that in each case one of the inequalities is automatic, being the lax naturality of Δ or t.) When $\mathbf{C} = \mathrm{Rel}(\mathbf{E})$ (with $\mathbf{E}$ a Grothendieck topos), the first of these equations says of a relation $F\colon X \leftrightarrow Y$ that it is 'single-valued' ($F(x, y) \wedge F(x, y') \to y = y'$) whilst the second says that it is 'entire' ($\exists x F(x, y)$).

(iv) If $\mathbf{C}$ is a dcr, then

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

is a product diagram in Map($\mathbf{C}$) if and only if

$$qp^0 = \top \quad \text{and} \quad (p^0p) \wedge (q^0p) = 1.$$

2.3. Examples. (i) Suppose that $\mathbf{E}$ is a Grothendieck topos (or more generally, a regular category with stable sups of subobjects) and that $\mathbf{C}$ is a full subcategory of Rel($\mathbf{E}$) whose objects are closed under finite (including empty) products in $\mathbf{E}$: then $\mathbf{C}$ is a dcr. We shall see below that every dcr arises in this way.

(ii) Distributive categories of relations with just one object (which is necessarily I) correspond precisely to locales: given a locale A, the corresponding one-object dcr $\mathbf{A}$ has $\mathbf{A}(I, I) = A$, with the operations of composition and $\times$ on morphisms both being given by binary meet in A.

2.4. DEFINITIONS. (i) A *morphism* of dcr's, F: $\mathbf{C} \to \mathbf{D}$, is a **Sl**-functor which preserves $\times$, I (up to coherent isomorphisms), Δ, and t. However, by 2.2(i) this is the same as requiring the induced functor F: Map($\mathbf{C}$)$\to$Map($\mathbf{D}$) to preserve finite products (in the usual, up to isomorphism sense). Note that by 2.2(iv) this is also the same as requiring the **Sl**-functor F: $\mathbf{C} \to \mathbf{D}$ to preserve finite meets of morphisms and preserve I up to isomorphism.

(ii) Given dcr's $\mathbf{C}$ and $\mathbf{D}$, let $\mathbf{DCR}(\mathbf{C}, \mathbf{D})$ denote the category whose objects are the morphisms F: $\mathbf{C} \to \mathbf{D}$ of (i) and whose arrows φ: $F \to G$ are lax natural transformations *whose components* φ_X: $F(X) \to G(X)$ *are maps in* $\mathbf{D}$.

Unlike the situation for the categories $\mathbf{Sl}\text{-}\mathbf{CAT}_{<}(\mathbf{C}, \mathbf{D})$ noted in 1.3(iv), here it is the case that the $\mathbf{DCR}(\mathbf{C}, \mathbf{D})$ are the hom categories of a 2-category $\mathbf{DCR}$. This is because, given φ: $F \to G$ in $\mathbf{DCR}(\mathbf{A}, \mathbf{B})$ and ψ: $H \to K$ in $\mathbf{DCR}(\mathbf{B}, \mathbf{C})$, then, for each $X \in A$,

$$\begin{array}{ccc} HF(X) & \xrightarrow{\psi_{FX}} & KF(X) \\ {\scriptstyle H\varphi_X}\downarrow & & \downarrow{\scriptstyle K\varphi_X} \\ HG(X) & \xrightarrow{\psi_{GX}} & KG(X) \end{array}$$

commutes, since, by 1.6, ψ is natural for *maps* in $\mathbf{B}$ and, by hypothesis, φ_X is such. Therefore, defining the horizontal composition in $\mathbf{DCR}$ by

$$\psi * \varphi = \psi_G \circ H\varphi = K\varphi \circ \psi_F,$$

the interchange law holds:

$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha).$$

If $\mathbf{E}$ is a Grothendieck topos, then, as noted in 2.3(i), Rel($\mathbf{E}$) is a dcr. But now if $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ is a geometric morphism, then

$$\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$$

is a morphism in $\mathbf{DCR}$. And if φ: $\mathbf{f}^* \to \mathbf{g}^*$ is a natural transformation between the inverse image parts of two geometric morphisms, then it is necessarily also a 2-cell in $\mathbf{DCR}$. In this way one gets a contravariant 2-functor

$$\mathrm{Rel}\colon \mathbf{GTOP}^{op} \to \mathbf{DCR}$$

from the 2-category of Grothendieck toposes to the 2-category of distributive categories of relations.

Given Grothendieck toposes $\mathbf{E}$ and $\mathbf{F}$, suppose that F: Rel($\mathbf{E}$)$\to$Rel($\mathbf{F}$) is a morphism in $\mathbf{DCR}$. Then F restricts to a functor on maps and hence gives a functor

$$\mathbf{f}^*\colon \mathbf{E} \cong \mathrm{Map}(\mathrm{Rel}(\mathbf{E})) \xrightarrow{F} \mathrm{Map}(\mathrm{Rel}(\mathbf{F})) \cong \mathbf{F}$$

which preserves finite products. But $\mathbf{f}^*$ also preserves finite meets of subobjects (since F preserves finite meets of relations) and hence it preserves all finite limits. It also preserves sups of subobjects (since F preserves sups) and preserves epimorphisms (since $q\colon X \to X'$ is an epi in $\mathbf{E}$ if and only if $qq^0 = 1$ and $1 \leq q^0 q$ in $\mathrm{Rel}(\mathbf{E})$). Therefore $\mathbf{f}^*$ is the inverse image part of a geometric morphism $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$. By construction it is sent by Rel to F, and is the unique such geometric morphism. Moreover, given two morphisms $F, G\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ in **DCR**, with corresponding geometric morphisms $\mathbf{f}, \mathbf{g}\colon \mathbf{F} \to \mathbf{E}$, our choice of 2-cells for **DCR** means that specifying a 2-cell $F \to G$ amounts to specifying a collection of morphisms $\varphi_X\colon \mathbf{f}^*(X) \to \mathbf{g}^*(X)$ in $\mathbf{F}$ which are lax natural for relations in $\mathbf{E}$, or equivalently, natural for morphisms in $\mathbf{E}$ (see 1.4(iii)). Thus the 2-functor Rel: $\mathbf{GTOP}^{op} \to \mathbf{DCR}$ *is full and faithful*. Adapting the work of Freyd to their cartesian bicategories, Carboni and Walters [**2**] have identified the essential image of this 2-functor. Say that a dcr $\mathbf{C}$ is *complete* if it has small coproducts (as a category) and if all *symmetric idempotents* (those $e\colon X \to X$ with $e^0 = e = ee$) in $\mathbf{C}$ *split* (that is, $e = sr$ for some r, s with $rs = 1$; in which case, necessarily $s = r^0$). Then:

2.5. PROPOSITION. *A dcr is isomorphic to one in the image of the full and faithful 2-functor* Rel: $\mathbf{GTOP}^{op} \to \mathbf{DCR}$ *if and only if it is both bounded (in the sense of* 1.7) *and complete. Thus* Rel *gives an equivalence of 2-categories between* $\mathbf{GTOP}^{op}$ *and the full sub-2-category* **bcDCR** *of* **DCR** *comprising the bounded, complete dcr's:*

$$\mathbf{GTOP}^{op} \simeq \mathbf{bcDCR}.$$

The inverse equivalence is given by taking categories of maps (1.6)*: when* $\mathbf{C}$ *is bounded and complete,* $\mathrm{Map}(\mathbf{C})$ *is a Grothendieck topos and* $\mathrm{Rel}(\mathrm{Map}(\mathbf{C})) \cong \mathbf{C}$.

Proof. See Theorem 6.3 of [**2**]. The above condition of 'completeness' for $\mathbf{C}$ includes the splitting of both 'coreflexives' ($e\colon X \to X$ with $e \leq 1$) and symmetric monads in $\mathbf{C}$. The former implies that $\mathbf{C}$ is 'functionally complete' (morphisms $X \to I$ in $\mathbf{C}$ correspond to subobjects of X in $\mathrm{Map}(\mathbf{C})$) and the latter that $\mathbf{C}$ is 'effective' (equivalence relations in $\mathrm{Map}(\mathbf{C})$ have quotients).

We now consider the process of completing a dcr by adding coproducts and splitting the symmetric idempotents. With the definition of 2-cells for dcr's given above, this process enjoys a fully bicategorical universal property:

2.6. PROPOSITION. *Let* **cDCR** *denote the full sub-2-category of* **DCR** *whose objects are complete (as defined in* 2.5). *Then the inclusion*

$$\mathbf{cDCR} \hookrightarrow \mathbf{DCR}$$

has a left biadjoint: for each $\mathbf{C} \in \mathbf{DCR}$ *there is a morphism* $\eta\colon \mathbf{C} \to \hat{\mathbf{C}}$ *in* **DCR** *with* $\hat{\mathbf{C}}$ *complete, and such that for each complete dcr* $\mathbf{D}$ *the functor*

$$\eta^*\colon \mathbf{DCR}(\hat{\mathbf{C}}, \mathbf{D}) \to \mathbf{DCR}(\mathbf{C}, \mathbf{D})$$

is an equivalence of categories.

Moreover, $\eta\colon \mathbf{C} \to \hat{\mathbf{C}}$ *is full and faithful and the objects in the image of* η *are* **Sl***-generators for* $\hat{\mathbf{C}}$ *(see* 1.7). *Hence in particular,* $\hat{\mathbf{C}}$ *is bounded when* $\mathbf{C}$ *is.*

Proof. The explicit construction of $\hat{\mathbf{C}}$ has been given by Freyd in the context of his 'allegories': it can be constructed as a category by first adjoining small coproducts and then by splitting the class of symmetric idempotents. Thus the objects of $\hat{\mathbf{C}}$ are triples (I, X, e) where I is a (small) set, $X = (X(i)|\ i \in I)$ is an I-indexed collection of objects of $\mathbf{C}$ and

$$e = (e(i, j)\colon\ X(i) \to X(j)|\ (i, j) \in I \times I)$$

is a collection of morphisms in $\mathbf{C}$ satisfying

$$\bigvee_{j \in I} e(j, k) \circ e(i, j) = e(i, k) = e(k, i)^0$$

for all $i, k \in I$. Morphisms $r\colon\ (I, X, e) \to (I', X', e')$ in $\hat{\mathbf{C}}$ are collections

$$r = (r(i, i')\colon\ X(i) \to X'(i')|\ i \in I, i' \in I')$$

satisfying

$$\bigvee_{j \in I} r(j, i') \circ e(i, j) = r(i, i') = \bigvee_{j' \in I'} e(j', i') \circ r(i, j'),$$

for all $i \in I$ and $i' \in I'$. The composition of r with $s\colon\ (I', X', e') \to (I'', X'', e'')$ is given by

$$(s \circ r)(i, i'') = \bigvee_{i' \in I'} s(i', i'') \circ r(i, i'),$$

whilst the identity on (I, X, e) is just e itself. The sup of morphisms in $\hat{\mathbf{C}}$ is given componentwise by the sup in $\mathbf{C}$. With these definitions $\hat{\mathbf{C}}$ becomes a **Sl**-category. The proof that it is moreover a complete dcr can be extracted from [**2**]. The dcr morphism $\eta\colon\ \mathbf{C} \to \hat{\mathbf{C}}$ sends X to $\eta(X) = (1, X, 1_X)$ and $r\colon\ X \to X'$ to itself, regarded as a 1×1-indexed family. (Here 1 denotes any one-element set.) Evidently η is full and faithful. The image of $\mathbf{C}$ under η is **Sl**-generating in the sense of 1.7 since the identity on an object (I, X, e) in $\hat{\mathbf{C}}$ can be expressed as the sup

$$1 = \bigvee_{j \in I} (r_j)^0 r_j,$$

where $r_j\colon\ (I, X, e) \to \eta(X(j))$ has components

$$r_j(i, *) = e(i, j)$$

for $i \in I$ and $* \in 1$.

If now $\mathbf{D}$ is a complete dcr, each morphism $F\colon\ \mathbf{C} \to \mathbf{D}$ extends to $\bar{F}\colon\ \hat{\mathbf{C}} \to \mathbf{D}$ with $\bar{F} \circ \eta \cong F$; here $\bar{F}$ is given on objects (I, X, e) by splitting the symmetric idempotent on

$$\coprod_{i \in I} F(X(i))$$

given by the $F(e(i, j))$. Similarly, given $G, H\colon\ \hat{\mathbf{C}} \to \mathbf{D}$ and a 2-cell $\varphi\colon\ G\eta \to H\eta$ in $\mathbf{DCR}(\mathbf{C}, \mathbf{D})$, for each (I, X, e) in $\hat{\mathbf{C}}$ define $\bar{\varphi}_{(I, X, e)}$ to be

$$\bar{\varphi}_{(I,X,e)} = \bigvee_{i \in I} H(r_i)^0 \varphi_{X(i)} G(r_i)$$

(where r_i is as defined above); then one can show that $\bar{\varphi}_{(I,X,e)}$ is lax natural in (I, X, e), that it is a map in $\mathbf{D}$ (because the components of φ are), that $\bar{\varphi}\eta = \varphi$, and that $\bar{\varphi}$ is uniquely determined by these properties.

2.7. REMARK. Underlying the proof of 2.6 are properties of **DCR** which are worth stating explicitly. Let us say that a morphism $F\colon \mathbf{A}\to\mathbf{B}$ in **DCR** is *dense* if the objects in the image of F are **Sl**-generators for **B**. For any **C** in **DCR** consider the functor 'precomposition with F', $F^*\colon \mathbf{DCR}(\mathbf{B},\mathbf{C})\to\mathbf{DCR}(\mathbf{A},\mathbf{C})$. Then:

(i) if F is dense, F^* is faithful;
(ii) if F is dense and full (as a functor), then F^* is full and faithful;
(iii) if F is dense, full and faithful and if **C** is complete, then F^* is an equivalence.

If **A** is the one-object dcr corresponding as in Example 2.3(ii) to a locale A, then $\hat{\mathbf{A}}$ is just the category of 'A-valued sets' and 'A-valued relations', so that $\mathrm{Map}(\hat{\mathbf{A}})$ is the category of A-valued sets and A-valued functional relations of Higgs (cf. [5]): this is well known to be equivalent to the topos of sheaves on the locale A. Generalizing from the 'one-object' to the 'many-object' case, we make the following definition:

2.8. DEFINITION. Let **A** be a bounded dcr. Then by 2.7, $\hat{\mathbf{A}}$ is both bounded and complete; hence by Proposition 2.5, $\mathrm{Map}(\hat{\mathbf{A}})$ is a Grothendieck topos: we shall call it the *topos of sheaves on* **A** and write

$$\mathrm{Sh}(\mathbf{A}) = \mathrm{Map}(\hat{\mathbf{A}}).$$

By Proposition 2.5 again, $\mathrm{Rel}(\mathrm{Sh}(\mathbf{A})) = \mathrm{Rel}(\mathrm{Map}(\hat{\mathbf{A}})) \cong \hat{\mathbf{A}}$. Combining this isomorphism with the morphism $\eta\colon \mathbf{A}\to\hat{\mathbf{A}}$ of 2.6, yields a morphism

$$y_{\mathbf{A}}\colon \mathbf{A}\to \mathrm{Rel}(\mathrm{Sh}(\mathbf{A}))$$

which is full, faithful and dense (in the sense of 2.7). Combining Propositions 2.5 and 2.6 yields the following universal property of $y_{\mathbf{A}}$:

2.9. THEOREM. *Let* **A** *be a bounded dcr. Then for any Grothendieck topos* **E**, *there is an equivalence of categories*

$$\mathbf{GTOP}(\mathbf{E}, \mathrm{Sh}(\mathbf{A})) \cong \mathbf{DCR}(\mathbf{A}, \mathrm{Rel}(\mathbf{E})),$$

given on objects by sending $\mathbf{f}\colon \mathbf{E}\to\mathrm{Sh}(\mathbf{A})$ *to the composition*

$$\mathbf{A} \xhookrightarrow{y_{\mathbf{A}}} \mathrm{Rel}(\mathrm{Sh}(\mathbf{A})) \xrightarrow{\mathbf{f}^*} \mathrm{Rel}(\mathbf{E})$$

and given on morphisms similarly. Thus, forming toposes of sheaves on bounded dcr's (gives a homomorphism of bicategories which) is biadjoint to taking categories of relations on Grothendieck toposes.

2.10. REMARKS. (i) By Proposition 2.5, for each Grothendieck topos **E** there is, up to equivalence, a unique complete bounded dcr **C** with $\mathrm{Sh}(\mathbf{C})\simeq\mathbf{E}$, namely $\mathbf{C} = \mathrm{Rel}(\mathbf{E})$. Indeed the counit of the biadjunction in 2.9 is an equivalence

$$\varepsilon_{\mathbf{E}}\colon \mathrm{Sh}(\mathrm{Rel}(\mathbf{E})) \simeq \mathbf{E}$$

(which, in particular, is pseudonatural in **E**).

(ii) Dropping the condition of completeness, we note that there are many bounded, indeed small, **C** with $\mathrm{Sh}(\mathbf{C})\simeq\mathbf{E}$. In particular, let **C** be any small, full subcategory of $\mathrm{Rel}(\mathbf{E})$ whose objects are closed under finite products in **E** and satisfy any of the equivalent conditions of Lemma 1.8. Then **C** is a small dcr, the inclusion $\mathbf{C} \hookrightarrow \mathrm{Rel}(\mathbf{E})$ is a morphism in **DCR** which is full, faithful and dense, and Rel(**E**) is complete; hence by 2.7(iii), $\hat{\mathbf{C}} \simeq \mathrm{Rel}(\mathbf{E})$ and thus $\mathrm{Sh}(\mathbf{C}) \simeq \mathbf{E}$.

It is immediate from Theorem 2.9 that the process of taking sheaves sends (indexed) colimits in **DCR** to (indexed) limits in **GTOP**. Combining this with Remark 2.10(i), one sees that the calculation of limits in **GTOP** can be reduced to that of colimits of (bounded) dcr's: given a diagram in **GTOP** apply the 2-functor Rel, calculate the colimit in **DCR** (which will remain bounded) and then take sheaves. The point is that the 'linear algebra' of sup-lattices developed by Joyal and Tierney in **[15]** can readily be extended to the wider context of **Sl**-enriched categories and used to make colimit calculations about dcr's. In the next section we will calculate pushouts and cocomma squares in **DCR** using this method. We conclude this section by giving the simple case of coproducts.

2.11. Example. The coproduct of two locales coincides with their tensor product as sup-lattices. More generally, given two dcr's **A** and **B**, their tensor product as **Sl**-categories $\mathbf{A} \otimes \mathbf{B}$ (as defined in 1.5) is easily seen to be a dcr with product given on objects by

$$(X \otimes Y) \times (X' \otimes Y') = (X \times X') \otimes (Y \times Y').$$

The **Sl**-functors

$$P = (-) \otimes I\colon\ \mathbf{A} \to \mathbf{A} \otimes \mathbf{B},$$
$$Q = I \otimes (-)\colon\ \mathbf{B} \to \mathbf{A} \otimes \mathbf{B}$$

are morphisms in **DCR** and make $\mathbf{A} \times \mathbf{B}$ into the coproduct of **A** and **B** in **DCR**, in the sense that for each $\mathbf{C} \in \mathbf{DCR}$ the functor

$$\mathbf{DCR}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \to \mathbf{DCR}(\mathbf{A}, \mathbf{C}) \times \mathbf{DCR}(\mathbf{B}, \mathbf{C}),$$

given by $F \mapsto (FP, FQ)$, is an equivalence of categories.

Evidently $\mathbf{A} \otimes \mathbf{B}$ is bounded if **A** and **B** are; therefore Theorem 2.9 implies that for such **A** and **B**, $\mathrm{Sh}(\mathbf{A} \otimes \mathbf{B})$ is the product in **GTOP** of the toposes Sh(**A**) and Sh(**B**):

$$\mathrm{Sh}(\mathbf{A} \otimes \mathbf{B}) \simeq \mathrm{Sh}(\mathbf{A}) \times \mathrm{Sh}(\mathbf{B}).$$

Thus given Grothendieck toposes **E** and **F**, and putting $\mathbf{A} = \mathrm{Rel}(\mathbf{E})$ and $\mathbf{B} = \mathrm{Rel}(\mathbf{F})$, we see that $\mathrm{Sh}(\mathbf{A}) \simeq \mathbf{E}$ and $\mathrm{Sh}(\mathbf{B}) \simeq \mathbf{F}$, and hence

$$\mathbf{E} \times \mathbf{F} \simeq \mathrm{Sh}(\mathrm{Rel}(\mathbf{E}) \otimes \mathrm{Rel}(\mathbf{F}))$$

and

$$\mathrm{Rel}(\mathbf{E} \times \mathbf{F}) \simeq (\mathrm{Rel}(\mathbf{E}) \otimes \mathrm{Rel}(\mathbf{F}))\hat{}$$

(where $\mathbf{E} \times \mathbf{F}$ denotes the product of **E** and **F** in **GTOP**).

3. *Comma and pullback constructions*

Recall from **[15]** that the pushout of locale homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{g^*} & C \\ {\scriptstyle f^*}\downarrow & & \\ B & & \end{array}$$

is given by the tensor product $C \otimes_A B$ of C and B as A-modules (via g^* and f^* respectively). Thus $C \otimes_A B$ is given as a coequalizer in **Sl** of the pair of morphisms induced by the actions of A on C and on B:

$$C \otimes A \otimes B \underset{1 \otimes \cdot}{\overset{\cdot \otimes 1}{\rightrightarrows}} C \otimes B \xrightarrow{\otimes_A} C \otimes_A B.$$

A minor modification of this construction yields the construction of *cocomma squares* of locales: let

$$C \otimes B \xrightarrow{\circledcirc_A} C \circledcirc_A B$$

be a 'lax coequalizer' in **Sl** of the morphisms

$$C \otimes A \otimes B \underset{1 \otimes \cdot}{\overset{\cdot \otimes 1}{\rightrightarrows}} C \otimes B$$

in the sense that

$$\circledcirc_A \circ (1 \otimes \cdot) \leq \circledcirc_A \circ (\cdot \otimes 1)$$

and $\circledcirc_A$ is universal with this property. Explicitly, $C \circledcirc_A B$ is the sup-lattice generated by pairs

$$c \circledcirc_A b \quad (c \in C, b \in B)$$

subject to the relations

$$\Big(\bigvee_i c_i\Big) \circledcirc_A b = \bigvee_i (c_i \circledcirc_A b), \quad c \circledcirc_A \Big(\bigvee_i b_i\Big) = \bigvee_i (c \circledcirc_A b_i),$$

and

$$c \circledcirc_A (f^*(a) \wedge b) \leq (g^*(a) \wedge c) \circledcirc_A b.$$

Then $C \circledcirc_A B$ is a locale, the maps

$$p^*(b) = \top \circledcirc_A b \quad (b \in B),$$
$$q^*(c) = c \circledcirc_A \top \quad (c \in C)$$

are locale homomorphisms (where $\top$ denotes the top element of a locale), $p^* f^* \leq q^* g^*$, and p^*, q^* are universal with this property. In other words

$$\begin{tikzcd} A \arrow[r, "g^*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "q^*"] \\ B \arrow[r, "p^*"'] & C \circledcirc_A B \end{tikzcd}$$

is a cocomma square of locales.

We shall see that the above constructions for locales generalize to give constructions for pushout and cocomma squares of bounded distributive categories of relations via tensor products and 'lax' tensor products of 'modules' over dcr's. The methods of § 2 then allow us to express pullback and comma squares of Grothendieck toposes in terms of these tensor products on the associated categories of relations.

3.1. DEFINITION. Let $\mathbf{A}$ be a dcr. An $\mathbf{A}$-*module* is a **Sl**-category $\mathbf{M}$ together with a **Sl**-functor $\mathbf{A}\otimes\mathbf{M}\to\mathbf{M}$ (whose action on objects we shall denote by $X\otimes M\mapsto X\cdot M$ and similarly for morphisms), and isomorphisms

$$c_{XYM}\colon (X\times Y)\cdot M\cong X\cdot(Y\cdot M),\quad u_M\colon I\cdot M\cong M,$$

which are natural in X, $Y\in\mathbf{A}$, $M\in\mathbf{M}$ and satisfy the coherence conditions that

$$\begin{tikzcd}[column sep=small] (X\times(Y\times Z))\cdot M \arrow[rr, "a\cdot 1"] \arrow[d, "c"'] & & ((X\times Y)\times Z)\cdot M \arrow[d, "c"] \\ X\cdot((Y\times Z)\cdot M) \arrow[dr, "1\cdot c"'] & & (X\times Y)\cdot(Z\cdot M) \arrow[dl, "c"] \\ & X\cdot(Y\cdot(Z\cdot M)) & \end{tikzcd}$$

and

$$\begin{tikzcd} X\cdot M \arrow[r, "r\cdot 1"] \arrow[ddr, "1"'] & (X\times I)\cdot M \arrow[d, "c"] \\ & X\cdot(I\cdot M) \arrow[d, "1\cdot u"] \\ & X\cdot M \end{tikzcd}$$

commute.

3.2. EXAMPLE. If $F\colon \mathbf{A}\to\mathbf{B}$ is a morphism in **DCR**, then $\mathbf{B}$ can be given the structure of an $\mathbf{A}$-module by defining

$$(X\xrightarrow{r}X')\cdot(Y\xrightarrow{s}Y')=F(X)\times Y\xrightarrow{F(r)\times s}F(X')\times Y.$$

3.3. *Lax tensor product of modules*

Let $\mathbf{A}$ be a bounded dcr and let $\mathbf{M}$ and $\mathbf{N}$ be $\mathbf{A}$-modules. We shall describe a certain $\mathbf{A}$-module $\mathbf{M}\circledcirc_{\mathbf{A}}\mathbf{N}$, to be called the *lax tensor product* of $\mathbf{M}$ and $\mathbf{N}$ (over $\mathbf{A}$). The underlying **Sl**-category of $\mathbf{M}\circledcirc_{\mathbf{A}}\mathbf{N}$ is defined as follows.

(i) The objects of $\mathbf{M}\circledcirc_{\mathbf{A}}\mathbf{N}$ are pairs of objects $M\in\mathbf{M}$, $N\in\mathbf{N}$, and such a pair will be denoted $M\circledcirc N$.

(ii) The hom sup-lattices of $\mathbf{M}\circledcirc_{\mathbf{A}}\mathbf{N}$ will be defined in terms of generators and relations. Since $\mathbf{A}$ is bounded, there is a set $\mathbf{G}$ of objects which are **Sl**-generators for $\mathbf{A}$ in the sense of 1.7. Then the generators of $(\mathbf{M}\circledcirc_{\mathbf{A}}\mathbf{N})(M\circledcirc N, M'\circledcirc N')$ comprise the set of triples $X\in\mathbf{G}$, $m\in\mathbf{M}(X\cdot M, M')$, $n\in\mathbf{N}(N, X\cdot N')$, and such a triple will be denoted $m\circledcirc_X n$. The relations on these generators are of three forms:

$$\Big(\bigvee_i m_i\Big)\circledcirc_X n=\bigvee_i (m_i\circledcirc_X n), \tag{1}$$

$$m\circledcirc_X\Big(\bigvee_i n_i\Big)=\bigvee_i (m\circledcirc_X n_i), \tag{2}$$

$$m\circledcirc_Y(r\cdot 1_{N'})n\leqslant m(r\cdot 1_M)\circledcirc_X n \tag{3}$$

(where in (3), r: $X \to Y$ in $\mathbf{A}$, m: $Y \cdot M \to M'$ in $\mathbf{M}$, and n: $N \to X \cdot N'$ in $\mathbf{N}$). The resulting sup-lattice contains elements $m \otimes_X n$ where X is any object of $\mathbf{A}$ (and where m: $X \cdot M \to M'$ in $\mathbf{M}$ and n: $N \to X \cdot N'$ in $\mathbf{N}$), which are defined as follows: since $\mathbf{G}$ is **Sl**-generating, one can find

$$\left(X \xrightarrow{r_i} U_i \xrightarrow{s_i} X \mid i \in I\right)$$

with $U_i \in \mathbf{G}$ and

$$1_X = \bigvee_{i \in I} s_i \circ r_i;$$

then define

$$m \otimes_X n = \bigvee_{i \in I} m(s_i \cdot 1) \otimes_{U_i} (r_i \cdot 1) n.$$

It is easily checked, using (1), (2), and (3), that this definition of $m \otimes_X n$ is independent of which representation is taken for 1_X as a sup of morphisms factoring through objects in $\mathbf{G}$. Moreover, the elements $m \otimes_X n$ with X unrestricted also satisfy (1), (2), and (3). Thus the hom sup-lattice $(\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N})(M \otimes N, M' \otimes N')$ is, in an evident sense, freely generated by this (possibly large) collection of elements $m \otimes_X n$ subject to the above relations. In particular, it is independent (up to isomorphism) of which set of **Sl**-generators $\mathbf{G}$ is chosen for $\mathbf{A}$.

(iii) Composition in $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$ is defined on generators

$$m \otimes_X n \in (\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N})(M \otimes N, M' \otimes N'),$$

$$m' \otimes_Y n' \in (\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N})(M' \otimes N', M'' \otimes N'')$$

by defining

$$(m' \otimes_Y n') \circ (m \otimes_X n) = m'' \otimes_{Y \times X} n'',$$

where m'' is

$$(Y \times X) \cdot M \overset{c}{\cong} Y \cdot (X \cdot M) \xrightarrow{1 \cdot m} Y \cdot M' \xrightarrow{m'} M''$$

and n'' is

$$N \xrightarrow{n} X \cdot N' \xrightarrow{1 \cdot n'} X \cdot (Y \cdot N'') \overset{c}{\cong} (X \times Y) \cdot N'' \overset{s \cdot 1}{\cong} (Y \times X) \cdot N''.$$

One checks that this operation preserves the relations (1), (2), (3) in each variable separately; hence it induces a composition on the morphisms of $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$ preserving sups in each variable separately. That the composition is associative follows using the observation that (3) *becomes an equality,*

$$m \otimes_Y (r \cdot 1_{N'}) n = m(r \cdot 1_M) \otimes_X n, \tag{4}$$

when r: $X \to Y$ *is a map in* $\mathbf{A}$ (and hence, in particular, when it is an isomorphism). (This follows because

$$\begin{aligned} m(r \cdot 1) \otimes_X n &\leq m(r \cdot 1) \otimes_X (r^* r \cdot 1) n \quad \text{(by (2))} \\ &\leq m(r r^* \cdot 1) \otimes_Y (r \cdot 1) n \quad \text{(by (3))} \\ &\leq m \otimes_Y (r \cdot 1) n \quad \text{(by (1)).)} \end{aligned}$$

(iv) Using (4), we see that the identity on $M \circledcirc N$ in $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$ is

$$1_{M \circledcirc N} = u_M \circledcirc_I u_N^{-1},$$

where u_M: $I \cdot M \cong M$ and u_N: $I \cdot N \cong N$ are as given in Definition 3.1.

Thus $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$ is a **Sl**-category. (Indeed it is an **A**-module, with the action of **A** given on objects by $X \cdot (M \circledcirc N) = (X \cdot M) \circledcirc N$, but we will not need to use this fact.)

3.4. REMARKS. (i) Given a category **C** and a functor F: $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Sl}$, one can weaken the usual notion of *coend* of F (see [**18**, IX.6]) to that of a 'lax coend':

$$\ell \int F = \ell \int^{X \in \mathbf{C}} F(X, X).$$

It is a sup-lattice equipped with **Sl** morphisms

$$\left(v_X \colon F(X) \to \ell \int F \mid X \in \mathbf{C} \right)$$

satisfying 'lax dinaturality' conditions,

$$v_Y \circ F(1, f) \leqslant v_X \circ F(f, 1) \quad (\text{for all } f\colon X \to Y \text{ in } \mathbf{C}),$$

and universal amongst such. Then one can summarize the definition of $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$ by saying that *its objects are pairs* $M \circledcirc N$ $(M \in \mathbf{M}, N \in \mathbf{N})$ *and that its hom sup-lattices are given by lax coends*:

$$(\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N})(M \circledcirc N, M' \circledcirc N') = \ell \int^{X \in \mathbf{A}} \mathbf{M}(X \cdot M, M') \otimes \mathbf{N}(N, X \cdot N').$$

(ii) For m: $M \to M'$ in **M** and n: $N \to N'$ in **N**, define

$$m \circledcirc n \colon M \circledcirc N \to M' \circledcirc N'$$

in $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$ to be $m u_M \circledcirc_I u_N^{-1} n$. Being functorial and sup-preserving in each variable separately, this gives a **Sl**-functor,

$$\circledcirc \colon \mathbf{M} \otimes \mathbf{N} \to \mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}.$$

For $M \in \mathbf{M}$, $X \in \mathbf{A}$, and $N \in \mathbf{N}$, define

$$\lambda_{MXN} \colon M \circledcirc (X \cdot N) \to (X \cdot M) \circledcirc N$$

to be $1_{X \cdot M} \circledcirc_X 1_{X \cdot N}$; it is easily checked to be lax natural in M, X, and N. Then $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$ is the 'lax coequalizer' of the **Sl**-functors

$$(1 \otimes \cdot), (\cdot \otimes 1) \colon \mathbf{M} \otimes \mathbf{A} \otimes \mathbf{N} \to \mathbf{M} \otimes \mathbf{N}$$

in the sense that $\circledcirc$, λ enjoy a universal property with respect to **Sl**-functors F out of $\mathbf{M} \otimes \mathbf{N}$ equipped with a lax natural transformation φ: $F(1 \otimes \cdot) \to F(\cdot \otimes 1)$.

(iii) From the definitions of $\circledcirc$ and λ in (ii), together with the definition of composition in $\mathbf{M} \circledcirc_{\mathbf{A}} \mathbf{N}$, it follows that each $m \circledcirc_X n$: $M \circledcirc N \to M' \circledcirc N'$ can be expressed as the composition

$$m \circledcirc_X n = (m \circledcirc 1) \circ \lambda_{MXN'} \circ (1 \circledcirc n).$$

3.5. LEMMA. *Suppose that* F: $\mathbf{A} \to \mathbf{B}$ *and* G: $\mathbf{A} \to \mathbf{C}$ *are morphisms in* **DCR** *and that* **A** *is bounded. Regarding* **B** and **C** *as* **A**-*modules via* F *and* G, *as in Example* 3.2, *form the lax tensor product* $\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}$. *Then:*

(i) $\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}$ *is a distributive category of relations;*

(ii) *the* **Sl**-*functor* $\oslash$ *of* 3.4(ii) *gives* **Sl**-*functors*

$$P = I \oslash (-)\colon\ \mathbf{B} \to \mathbf{C} \oslash_{\mathbf{A}} \mathbf{B},$$

$$Q = (-) \oslash I\colon\ \mathbf{C} \to \mathbf{C} \oslash_{\mathbf{A}} \mathbf{B},$$

which are morphisms in **DCR**;

(iii) *the lax natural transformation* λ *of* 3.4(ii) *gives morphisms*

$$\varphi_X\colon\ PF(X) = I \oslash F(X) \cong I \oslash X \cdot I \xrightarrow{\lambda_{IXI}} X \cdot I \oslash I \cong G(X) \oslash I = QG(X)$$

for each $X \in \mathbf{A}$, *which are the components of a* 2-*cell* φ: $PF \to QG$ *in* **DCR**.

Proof. (i) The cartesian structure on $\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}$ is defined as follows: for objects $Z_1 \oslash Y_1$ and $Z_2 \oslash Y_2$ of $\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}$ define

$$(Z_1 \oslash Y_1) \times (Z_2 \oslash Y_2) = (Z_1 \times Z_2) \oslash (Y_1 \times Y_2). \tag{5}$$

To extend this product to the morphisms of $\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}$, given

$$c_i\colon\ X_i \cdot Z_i \to (Z_i)', \quad b_i\colon\ Y_i \to X_i \cdot (Y_i)' \quad (i = 1, 2),$$

define

$$(c_1 \oslash_{X_1} b_1) \times (c_2 \oslash_{X_2} b_2) = c \oslash_{X_1 \times X_2} b, \tag{6}$$

where c is

$$(X_1 \times X_2) \cdot (Z_1 \times Z_2) \cong (X_1 \cdot Z_1) \times (X_2 \cdot Z_2) \xrightarrow{c_1 \times c_2} (Z_1)' \times (Z_2)'$$

and b is

$$Y_1 \times Y_2 \xrightarrow{b_1 \times b_2} (X_1 \cdot (Y_1)') \times (X_2 \cdot (Y_2)') \cong (X_1 \times X_2) \cdot ((Y_1)' \times (Y_2)').$$

This operation $\times$ preserves the relations (1), (2), (3) in each variable separately; moreover, from the definition of composition and identities given in 3.3(iii) and (iv), one checks that it is functorial. Therefore (5) and (6) give rise to a **Sl**-functor

$$\times\colon\ (\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}) \otimes (\mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}) \to \mathbf{C} \oslash_{\mathbf{A}} \mathbf{B}. \tag{7}$$

The definition (6) of $\times$ on morphisms implies that, for

$$b_i\colon\ Y_i \to (Y_i)', \quad c_i\colon\ Z_i \to (Z_i)' \quad (i = 1, 2),$$

one has

$$(c_1 \oslash b_1) \times (c_2 \oslash b_2) = (c_1 \times c_2) \oslash (b_1 \times b_2), \tag{8}$$

where $\oslash$ is defined for morphisms as in 3.4(ii). Similarly, one calculates that the lax natural transformation

$$\lambda_{ZXY}\colon\ Z \oslash (X \cdot Y) \to (X \cdot Z) \oslash Y$$

of 3.4(ii) is such that

$$\begin{tikzcd}
(Z_1\times Z_2)\otimes((X_1\times X_2)\cdot(Y_1\times Y_2)) \arrow[r, "\lambda"] \arrow[d, "1\otimes\kappa"', "\cong"] & ((X_1\times X_2)\cdot(Z_1\times Z_2))\otimes(Y_1\times Y_2) \arrow[d, "\kappa\otimes 1", "\cong"'] \\
(Z_1\times Z_2)\otimes(X_1\cdot Y_1\times X_2\cdot Y_2) \arrow[r, "\lambda\times\lambda"] & (X_1\cdot Z_1\times X_2\cdot Z_2)\otimes(Y_1\times Y_2)
\end{tikzcd} \tag{9}$$

commutes (where the κ are the evident canonical isomorphisms). The transformation λ also satisfies

$$(u_I\otimes 1)\circ\lambda_{III}=(1\otimes u_I), \tag{10}$$

where u is as in Definition 3.1.

The isomorphisms showing that (7) is a symmetric monoidal functor with unit $I\otimes I$ are $a=a\otimes a$, $s=s\otimes s$, and $r=r\otimes r$, their naturality for morphisms in $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$ following from (8), (9), (10), and 3.4(iii). Similarly, Axioms (b) and (c) of Definition 2.1(i) hold for $\Delta=\Delta\otimes\Delta$ and $t=t\otimes t$ (whose right adjoints are necessarily $\Delta^*\otimes\Delta^*$ and $t^*\otimes t^*$ respectively): the only requirement on them which is not immediate is their lax naturality for all morphisms in $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$ (and not just for morphisms of the form $c\otimes b$), and once more this follows from (8), (9), (10), and 3.4(iii). Finally, each object of $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$ is discrete in the sense of 2.1(ii), since all the objects of $\mathbf{C}$ and $\mathbf{B}$ are. Thus $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$ is a dcr.

(ii) By 2.4(i), to see that the **Sl**-functors P and Q are morphisms in **DCR**, one has to check that they induce finite product-preserving functors between the associated categories of objects and maps. Now $P(I)=Q(I)=I\otimes I$ is terminal; and for $Y,\ Y'\in\mathbf{B}$, by (8) and using the definitions of Δ and t given in (i), one finds that

$$\langle P\pi_1, P\pi_2\rangle\colon\ P(Y\times Y')\to P(Y)\times P(Y')$$

is the isomorphism

$$\Delta\otimes 1\colon\ I\otimes(Y\times Y')\to(I\times I)\otimes(Y\times Y'),$$

and similarly for Q.

(iii) Since λ is lax natural, so is φ. Therefore one just has to check that for each $X\in\mathbf{A}$, φ_X is a map in $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$. In fact each

$$\lambda_{ZXY}=1\otimes 1\colon\ Z\otimes(X\cdot Y)\to(X\cdot Z)\otimes Y$$

is a map; for, using (9), (10), and the naturality of λ for maps, one has

$$(t\otimes t)\circ\lambda=t\otimes t\quad\text{and}\quad(\Delta\otimes\Delta)\circ\lambda=(\lambda\times\lambda)\circ(\Delta\otimes\Delta),$$

so that, by Remark 2.2(iii), λ is a map.

3.6. Proposition. *Suppose that F: $\mathbf{A}\to\mathbf{B}$ and G: $\mathbf{A}\to\mathbf{C}$ are morphisms in* **DCR** *and that* $\mathbf{A}$ *is bounded. Then with notation as in* 3.5, *one has that*

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{C} \arrow[d, "Q"] \\
\mathbf{B} \arrow[r, "P"'] & \mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}
\end{tikzcd}$$

is a cocomma square in **DCR**, *in the sense that for each* $\mathbf{D} \in \mathbf{DCR}$ *precomposition with* P, φ, Q *induces an equivalence of categories from* $\mathbf{DCR}(\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}, \mathbf{D})$ *to the comma category* (F^*, G^*) *whose objects are triples* (H, ψ, K) *where* $H\colon \mathbf{B} \to \mathbf{D}$, $K\colon \mathbf{C} \to \mathbf{D}$, *and* $\psi\colon HF \to KG$ *in* **DCR**, *and whose morphisms* $(H, \psi, K) \to (H', \psi', K')$ *are pairs* (h, k) *where* $h\colon H \to H'$, $k\colon K \to K'$ *in* **DCR** *and* $\psi' \circ h_F = k_G \circ \psi$.

Proof. We first show that the functor

$$\mathbf{DCR}(\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}, \mathbf{D}) \to (F^*, G^*),$$
$$L \mapsto (LP, L\varphi, LQ)$$

is essentially surjective. Suppose we are given an object (H, ψ, K) of (F^*, G^*). For each object $Z \otimes Y$ of $\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}$, define

$$L(Z \otimes Y) = K(Z) \times H(Y);$$

and for each $c\colon X \cdot Z \to Z'$ and $b\colon Y \to X \cdot Y'$ define

$$L(c \otimes_X b)\colon L(Z \otimes Y) \to L(Z' \otimes Y')$$

to be the composition

$$\begin{tikzcd}
KZ \times HY \arrow[d, "1 \times Hb"'] & & KZ' \times HY' \\
KZ \times H(X \cdot Y') \arrow[d, "\cong"'] & & K(X \cdot Z) \times HY' \arrow[u, "Kc \times 1"'] \\
KZ \times HFX \times HY' \arrow[rr, "1 \times \psi_X \times 1"] & & KZ \times KGX \times HY' \arrow[u, "\cong"']
\end{tikzcd}$$

The facts that H and K are **Sl**-functors and that ψ is lax natural imply that this definition respects the relations (1), (2), (3), and hence induces sup-lattice morphisms

$$L\colon (\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B})(Z \otimes Y, Z' \otimes Y') \to \mathbf{D}(L(Z \otimes Y), L(Z' \otimes Y')).$$

Now L preserves composition in $\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}$ because for $X, X' \in \mathbf{A}$, the diagram

$$\begin{tikzcd}
HF(X \times X') \arrow[r, phantom, "\cong"] \arrow[d, "\psi_{X \times X'}"'] & HF(X) \times HF(X') \arrow[d, "\psi_X \times \psi_{X'}"] \\
KG(X \times X') \arrow[r, phantom, "\cong"] & KG(X) \times KG(X')
\end{tikzcd} \tag{11}$$

in Map(**D**) commutes (since $\psi\colon HF \to KG\colon \mathrm{Map}(\mathbf{A}) \to \mathrm{Map}(\mathbf{B})$ is a natural transformation between product-preserving functors). Similarly, L preserves identity morphisms in $\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}$ because the diagram

$$\begin{tikzcd}
HF(I) \arrow[r, phantom, "\cong"] \arrow[d, "\psi_I"'] & I \arrow[d, "1"] \\
KG(I) \arrow[r, phantom, "\cong"] & I
\end{tikzcd} \tag{12}$$

commutes.

Therefore we have a **Sl**-functor L: $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}\to\mathbf{D}$. It preserves I and $\times$ because H and K do, and thus gives a morphism in **DCR**. For $Y\in\mathbf{B}$ and $Z\in\mathbf{C}$, there are isomorphisms

$$LP(Y)=K(I)\times H(Y)\cong H(Y),$$
$$LQ(Z)=K(Z)\times H(I)\cong K(Z),$$

which are easily checked to be natural in Y and Z. Moreover, by definition of φ_X and the action of L on morphisms,

$$\begin{tikzcd} LPF(X) \arrow[r, phantom, "\cong"] \arrow[d, "L\varphi_X"'] & HF(X) \arrow[d, "\psi_X"] \\ LQG(X) \arrow[r, phantom, "\cong"] & KG(X) \end{tikzcd}$$

commutes for each $X\in\mathbf{A}$. Hence $(LP, Lf, LQ)\cong(H, \psi, K)$ in (F^*, G^*), as required.

Finally, we show that the functor

$$\mathbf{DCR}(\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}, \mathbf{D})\to(F^*, G^*),$$
$$L\mapsto(LP, L\varphi, LQ)$$

is full and faithful. Suppose we are given $L, M\in\mathbf{DCR}(\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}, \mathbf{D})$ and (h, k): $(LP, L\varphi, LQ)\to(MP, M\varphi, MQ)$ in (F^*, G^*). If there is a θ: $L\to M$ in $\mathbf{DCR}(\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}, \mathbf{D})$ with $(\theta_P, \theta_Q)=(h, k)$, then it is the unique such, since, for each $Z\otimes Y\in\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$,

$$\begin{tikzcd} L(Z\otimes Y) \arrow[r, phantom, "\cong"] \arrow[d, "\theta_{Z\otimes Y}"'] & LQ(Z)\times LP(Y) \arrow[d, "\theta_{QZ}\times\theta_{PY}=k_Z\times h_Y"] \\ M(Z\otimes Y) \arrow[r, phantom, "\cong"] & MQ(Z)\times MP(Y) \end{tikzcd}$$

commutes (because θ is a natural transformation between product-preserving functors on categories of maps). Conversely, using the above square to define $\theta_{Z\otimes Y}$ in terms of k_Z and h_Y, we see immediately that each $\theta_{Z\otimes Y}$ is a map, that $\theta_{QZ}=k_Z$, $\theta_{PY}=h_Y$, and that $\theta_{Z\otimes Y}$ is lax natural for morphisms of the form $c\otimes b$ (since k and h are lax natural). The lax naturality of θ for *all* morphisms then follows by Remark 3.4(iii), once one verifies that

$$\begin{tikzcd} L(Z\otimes X\cdot Y) \arrow[r, "L(\lambda)"] \arrow[d, "\theta"'] & L(X\cdot Z\otimes Y) \arrow[d, "\theta"] \\ M(Z\otimes X\cdot Y) \arrow[r, "M(\lambda)"'] & M(X\cdot Z\otimes Y) \end{tikzcd}$$

commutes, and this in turn follows from the hypothesis on (h, k) that

$$\begin{tikzcd} LPF(X) \arrow[r, "L\varphi_X"] \arrow[d, "h_{FX}"'] & LQG(X) \arrow[d, "k_{GX}"] \\ MPF(X) \arrow[r, "M\varphi_X"'] & MQG(X) \end{tikzcd}$$

commutes.

This completes the proof of Proposition 3.6.

Given geometric morphisms $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ and $\mathbf{g}\colon \mathbf{G} \to \mathbf{E}$ between Grothendieck toposes, let $\mathbf{G} \olessthan_{\mathbf{E}} \mathbf{F}$ denote the *comma topos* formed from $\mathbf{f}$ and $\mathbf{g}$. Thus there is a diagram

$$\begin{tikzcd} \mathbf{G} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd} \tag{13}$$

in **GTOP** with the property that for each Grothendieck topos $\mathbf{H}$, composition with $\mathbf{p}$, φ, $\mathbf{q}$ induces an equivalence of categories from $\mathbf{GTOP}(\mathbf{H}, \mathbf{G} <_{\mathbf{E}} \mathbf{F})$ to the comma category $(\mathbf{f}, \mathbf{g})$ whose objects are triples $(\mathbf{h}, \psi, \mathbf{k})$, where $\mathbf{h}\colon \mathbf{H} \to \mathbf{F}$, $\mathbf{k}\colon \mathbf{H} \to \mathbf{G}$, and $\psi\colon \mathbf{fh} \to \mathbf{gk}$ in **GTOP**, and whose morphisms $(\mathbf{h}, \psi, \mathbf{k}) \to (\mathbf{h}', \psi', \mathbf{k}')$ are pairs (α, β) where $\alpha\colon \mathbf{h} \to \mathbf{h}'$, $\beta\colon \mathbf{k} \to \mathbf{k}'$, and $\psi' \circ \mathbf{f}\alpha = \mathbf{g}\beta \circ \psi$.

Combining Proposition 3.6 with Theorem 2.9 yields the following representation of comma toposes in terms of the lax tensor product of dcr's:

3.7. THEOREM. *Given* $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ *and* $\mathbf{g}\colon \mathbf{G} \to \mathbf{E}$ *in* **GTOP**, *regarding* $\mathrm{Rel}(\mathbf{F})$ *and* $\mathrm{Rel}(\mathbf{G})$ *as* $\mathrm{Rel}(\mathbf{E})$*-modules via* $\mathbf{f}^*$ and $\mathbf{g}^*$ *respectively, form the lax tensor product* $\mathrm{Rel}(\mathbf{G}) \olessthan_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})$ *(which one can do since* $\mathrm{Rel}(\mathbf{E})$ *is bounded). It is a bounded dcr whose associated topos of sheaves is equivalent to the comma topos formed from* $\mathbf{f}$ *and* $\mathbf{g}$, *that is,*

$$\mathrm{Sh}(\mathrm{Rel}(\mathbf{G}) \olessthan_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})) \simeq \mathbf{G} <_{\mathbf{E}} \mathbf{F}.$$

Proof. Referring to the construction of $\mathbf{M} \olessthan_{\mathbf{A}} \mathbf{N}$ given in 3.3, it is clear that $\mathbf{M} \olessthan_{\mathbf{A}} \mathbf{N}$ is a bounded **Sl**-category when $\mathbf{M}$ and $\mathbf{N}$ are. (Indeed $\{M_i \olessthan N_j \mid i \in I, j \in J\}$ is **Sl**-generating for $\mathbf{M} \olessthan_{\mathbf{A}} \mathbf{N}$ if $\{M_i \mid i \in I\}$ and $\{N_j \mid j \in J\}$ are **Sl**-generating for $\mathbf{M}$ and $\mathbf{N}$ respectively.) Hence, by Proposition 3.6,

$$\begin{tikzcd} \mathrm{Rel}(\mathbf{E}) \arrow[r, "\mathbf{g}^*"] \arrow[d, "\mathbf{f}^*"'] \arrow[dr, phantom, "\Longrightarrow"] & \mathrm{Rel}(\mathbf{G}) \arrow[d, "Q"] \\ \mathrm{Rel}(\mathbf{F}) \arrow[r, "P"'] & \mathrm{Rel}(\mathbf{G}) \olessthan_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F}) \end{tikzcd}$$

is a cocomma square in **DCR** whose corners are each bounded. Therefore Theorem 2.9, combined with the fact (2.10(i)) that $\mathrm{Sh}(\mathrm{Rel}(\mathbf{E})) \simeq \mathbf{E}$ naturally in $\mathbf{E}$, implies that on taking toposes of sheaves one has a comma square

$$\begin{tikzcd} \mathbf{E} & \mathbf{G} \arrow[l, "\mathbf{g}"'] \\ \mathbf{F} \arrow[u, "\mathbf{f}"] \arrow[ur, phantom, "\Longrightarrow"] & \mathrm{Sh}(\mathrm{Rel}(\mathbf{G}) \olessthan_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})) \arrow[l] \arrow[u] \end{tikzcd}$$

in **GTOP**, as required.

On taking categories of relations, the equivalence of 3.7 becomes

$$(\mathrm{Rel}(\mathbf{G}) \olessthan_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F}))\hat{} \simeq \mathrm{Rel}(\mathbf{G} \olessthan_{\mathbf{E}} \mathbf{F}).$$

This equivalence corresponds via 2.6 to a full, faithful and dense morphism

$$L\colon\ \mathrm{Rel}(\mathbf{G}) \oslash_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F}) \to \mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})$$

in **DCR**, which by construction is derived from

$$\varphi\colon\ (\mathbf{fp})^* \to (\mathbf{gq})^*\colon\ \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})$$

as in the proof of 3.6. In particular, for $Y \in \mathbf{F}$ and $Z \in \mathbf{G}$,

$$L\colon\ (\mathrm{Rel}(\mathbf{G}) \oslash_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F}))(I \oslash Y,\ Z \oslash I) \to \mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y,\ \mathbf{q}^*Z)$$

sends $C \oslash_X B$ to

$$\mathbf{p}^*(Y) \xrightarrow{\mathbf{p}^*B} \mathbf{p}^*(\mathbf{f}^*X \times I) \cong (\mathbf{fp})^*X \xrightarrow{\varphi_X} (\mathbf{gq})^*X \cong \mathbf{q}^*(\mathbf{g}^*X \times I) \xrightarrow{\mathbf{q}^*C} \mathbf{q}^*(Z).$$

This map is an isomorphism since L is full and faithful. Employing the 'lax coend' notation of 3.4(i), one also has

$$\begin{aligned}(\mathrm{Rel}\,(\mathbf{G}) &\oslash_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F}))(I \oslash Y,\ Z \oslash I)\\ &= \ell\!\int^{X \in \mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{G})(X \cdot I,\ Z) \otimes \mathrm{Rel}(\mathbf{F})(Y,\ X \cdot I)\\ &\cong \ell\!\int^{X \in \mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{G})(\mathbf{g}^*X,\ Z) \otimes \mathrm{Rel}(\mathbf{F})(Y,\ \mathbf{f}^*X).\end{aligned}$$

Composing this isomorphism with that induced by L gives:

3.8. COROLLARY. *For the comma square* (13), *given* $Y \in \mathbf{F}$ *and* $Z \in \mathbf{G}$, *consider the sup-lattice*

$$\ell\!\int^{X \in \mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{G})(\mathbf{g}^*X,\ Z) \otimes \mathrm{Rel}(\mathbf{F})(Y,\ \mathbf{f}^*X),$$

which is generated by triples

$$C \oslash_X B \quad (X \in \mathbf{E},\ B\colon\ Y \nrightarrow \mathbf{f}^*X \textit{ in } \mathrm{Rel}(\mathbf{E}) \textit{ and } C\colon\ \mathbf{g}^*X \nrightarrow Z \textit{ in } \mathrm{Rel}(\mathbf{G}))$$

subject to the relations

$$\Big(\bigvee_i C_i\Big) \oslash_X B = \bigvee_i\,(C_i \oslash_X B),$$

$$C \oslash_X \Big(\bigvee_i B_i\Big) = \bigvee_i\,(C \oslash_X B_i),$$

$$C \oslash_{X'} (\mathbf{f}^*A \circ B) \leqslant (C \circ \mathbf{g}^*A) \oslash_X B \quad (A\colon\ X \nrightarrow X' \textit{ in } \mathrm{Rel}(\mathbf{E})).$$

(*As in* 3.3, *the above definition is legitimate despite the possibly large number of generators, because* Rel(**E**) *is bounded.*) *Then the sup-preserving map*

$$\ell\!\int^{X} \mathrm{Rel}(\mathbf{G})(\mathbf{g}^*X,\ Z) \otimes \mathrm{Rel}(\mathbf{F})(Y,\ \mathbf{f}^*X) \to \mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y,\ \mathbf{q}^*Z),$$

which is given on generators by

$$C \oslash_X B \mapsto \mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(B),$$

is an isomorphism.

3.9. REMARK. Corollary 3.8 shows that each element of

$$\mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z)$$

is a sup of elements of the form $\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(B)$, where $X \in \mathbf{E}$, B: $Y \leftrightarrow \mathbf{f}^*X$, and C: $\mathbf{g}^*X \leftrightarrow Z$. However, using the disjoint coproducts of $\mathbf{E}$, we see that any such sup

$$\bigvee_{i \in I} \mathbf{q}^*(C_i) \circ \varphi_{X_i} \circ \mathbf{p}^*(B)$$

can be expressed as a single element $\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(B)$, where

$$X = \coprod_{i \in I} X_i \quad \text{with coproduct insertions } v_i\colon\ X_i \rightarrowtail X,$$

$$C = \bigvee_{i \in I} C_i \circ \mathbf{g}^*(v_i)^0$$

and

$$B = \bigvee_{i \in I} \mathbf{f}^*(v_i) \circ B_i.$$

(Cf. (b) in the proof of Lemma 1.8.)

A simple modification of the constructions we have made so far in this section yields a description of pullbacks in **GTOP** via tensor products.

3.10. *Tensor product of modules*

Given a bounded dcr $\mathbf{A}$ and $\mathbf{A}$-modules $\mathbf{M}$ and $\mathbf{N}$, the *tensor product* of $\mathbf{M}$ and $\mathbf{N}$ over $\mathbf{A}$, $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$, is constructed in exactly the same way as the lax tensor product except that the relation (3) in 3.3 is changed to an *equality*. We shall denote a pair $M \in \mathbf{M}$, $N \in \mathbf{N}$ by $M \otimes N$ when it is regarded as an object of $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$; similarly, we shall denote the generators of its hom sup-lattices by

$$m \otimes_X n \quad (m\colon\ X \cdot M \to M',\ n\colon\ N \to X \cdot N').$$

These hom sup-lattices are now given by coend formulae:

$$(\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N})(M \otimes N, M' \otimes N') = \int^{X \in \mathbf{A}} \mathbf{M}(X \cdot M, M') \otimes \mathbf{N}(N, X \cdot N').$$

As before, there is a **Sl**-functor

$$\otimes\colon\ \mathbf{M} \otimes \mathbf{N} \to \mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$$

and a lax natural transformation

$$\lambda = 1 \otimes_X 1\colon\ M \otimes (X \cdot N) \to (X \cdot M) \otimes N,$$

which is now an isomorphism (its inverse being $m \otimes_{X^2} n$, where m is

$$X \cdot (X \cdot M) \cong (X^2) \cdot M \xrightarrow{(t\Delta^* s) \cdot 1} I \cdot M \cong M$$

and n is

$$N \cong I \cdot N \xrightarrow{(\Delta t^*) \cdot 1} (X^2) \cdot N \cong X \cdot (X \cdot N)\ ).$$

Then $\otimes$ and λ make $\mathbf{M} \otimes_{\mathbf{A}} \mathbf{N}$ into the 'pseudo-coequalizer' of the **Sl**-functors

$$(1 \otimes \cdot), (\cdot \otimes 1)\colon\ \mathbf{M} \otimes \mathbf{A} \otimes \mathbf{N} \to \mathbf{M} \otimes \mathbf{N}.$$

If now F: $\mathbf{A} \to \mathbf{B}$ and G: $\mathbf{A} \to \mathbf{C}$ are morphisms in **DCR**, then, arguing just as in Lemma 3.5, we see that $\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}$ is a dcr, that there are morphisms

$$P = I \otimes (-)\colon \mathbf{B} \to \mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}, \quad Q = (-) \otimes I\colon \mathbf{C} \to \mathbf{C} \otimes_{\mathbf{A}} \mathbf{B},$$

and a natural *isomorphism* φ: $PF \cong QG$. Then one proves in exactly the same way as for 3.6 that:

3.11. PROPOSITION. *The square*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{C} \arrow[d, "Q"] \\ \mathbf{B} \arrow[r, "P"'] & \mathbf{C} \otimes_{\mathbf{A}} \mathbf{B} \end{tikzcd}$$

is a pushout square in **DCR**, *in the sense that, for each* $\mathbf{D} \in \mathbf{DCR}$, *precomposition with* P, φ, Q *induces an equivalence of categories from* $\mathbf{DCR}(\mathbf{C} \otimes_{\mathbf{A}} \mathbf{B}, \mathbf{D})$ *to the pullback category whose objects are triples* (H, ψ, K) *with* H: $\mathbf{B} \to \mathbf{D}$, K: $\mathbf{C} \to \mathbf{D}$, *and* ψ: $HF \cong KG$ *in* **DCR**, *and whose morphisms* $(H, \psi, K) \to (H', \psi', K')$ *are pairs* (h, k) *with* h: $H \to H'$, k: $K \to K'$ *in* **DCR** *and* $\psi' \circ h_F = k_G \circ \psi$.

From this one obtains the analogues of 3.7 and 3.8 for pullback squares in **GTOP**.

3.12. THEOREM. *For a pullback square in* **GTOP**

$$\begin{tikzcd} \mathbf{G} \times_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\cong"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}.$$

one has that

$$\mathbf{G} \times_{\mathbf{E}} \mathbf{F} \simeq \mathrm{Sh}(\mathrm{Rel}(\mathbf{G}) \otimes_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})).$$

In particular, for $Y \in \mathbf{F}$ *and* $Z \in \mathbf{G}$, $\mathrm{Rel}(\mathbf{G} \times_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z)$ *is freely generated as a sup-lattice by the elements*

$$\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(B) \quad (X \in \mathbf{E}, B\colon Y \nrightarrow \mathbf{f}^*X,\ C\colon \mathbf{g}^*X \nrightarrow Z)$$

subject to the relations

$$\mathbf{q}^*(\bigvee C_i) \circ \varphi_X \circ \mathbf{p}^*(B) = \bigvee (\mathbf{q}^*(C_i) \circ \varphi_X \circ \mathbf{p}^*(B)),$$

$$\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(\bigvee B_i) = \bigvee (\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(B_i)),$$

and

$$\mathbf{q}^*(C) \circ \varphi_X \circ \mathbf{p}^*(\mathbf{f}^*(A) \circ B) = \mathbf{q}^*(C \circ \mathbf{g}^*(A)) \circ \varphi_X \circ \mathbf{p}^*(B).$$

Thus

$$\mathrm{Rel}(\mathbf{G} \times_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z) \cong \int^{X \in \mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{G})(\mathbf{g}^*X, Z) \otimes \mathrm{Rel}(\mathbf{F})(Y, \mathbf{f}^*X).$$

4. *Applications*

We are going to use the results of the previous section to derive properties, with respect to the formation of comma squares in **GTOP**, of geometric morphisms *whose inverse image functors preserve arbitrary intersections of subobjects*. The results we obtain are applied in [24] to derive interpolation and conceptual completeness results for pretoposes. They are also related to the infinitary generalizations of conceptual completeness considered by Makkai and Reyes in §§ 7.3 and 7.4 of [19]. (Specializing from the preservation of intersections to open geometric morphisms, we see that the results of the previous section also yield (new) proofs of known properties of such geometric morphisms with respect to formation of pullbacks in **GTOP**.)

4.1. DEFINITIONS. Let F: $\mathbf{A}\to\mathbf{B}$ be a morphism of distributive categories of relations.

(i) The map F will be called *meet* if for all $X, X' \in \mathbf{A}$ the sup-preserving maps

$$F\colon\ \mathbf{A}(X, X')\to\mathbf{B}(F(X), F(X'))$$

also preserve arbitrary infs:

$$F\Big(\bigwedge_i a_i\Big) = \bigwedge_i F(a_i).$$

In this case these maps have left adjoints, which will be denoted by

$$F_!\colon\ \mathbf{B}(F(X), F(X'))\to\mathbf{A}(X, X'). \tag{14}$$

A geometric morphism $\mathbf{f}$: $\mathbf{F}\to\mathbf{E}$ between Grothendieck toposes will be called *meet* when $\mathbf{f}^*$: $\mathrm{Rel}(\mathbf{E})\to\mathrm{Rel}(\mathbf{F})$ is meet, which is to say that $\mathbf{f}^*$ preserves arbitrary infs of subobjects.

(ii) The map F will be called *open* if it is meet and the left adjoints (14) are natural in X and X':

$$F_!(Fa\circ b) = a\circ F_!(b),$$
and
$$F_!(b\circ Fa) = F_!(b)\circ a. \tag{15}$$

(Note that the second of these equations is implied by the first, since one can show for meet F that $F_!(b^0) = (F_!b)^0$.)

4.2. REMARK. Every dcr $\mathbf{A}$ has *right* (*Kan*) *extensions*: for any a: $X\to Y$ and Z in $\mathbf{A}$, the map

$$(-)\circ a\colon\ \mathbf{A}(Y, Z)\to\mathbf{A}(X, Z)$$

preserves sups and hence has a right adjoint, which, following Freyd, we shall denote by

$$(-)/a\colon\ \mathbf{A}(X, Z)\to\mathbf{A}(Y, Z).$$

The *right extension* b/a *of* b *along* a is thus given by

$$b/a = \bigvee\{c \mid (c\circ a)\leqslant b\}$$

and satisfies

$$c\leqslant(b/a) \text{ if and only if } (c\circ a)\leqslant b.$$

The condition (15) in 4.1(ii) is easily seen to be equivalent to requiring that, for all a and b,

$$F(b/a) = Fb/Fa.$$

Thus F: $\mathbf{A} \to \mathbf{B}$ *is open if and only if it is meet and preserves right extensions.* In the case that $\mathbf{A}$ has small coproducts, preservation of right extensions actually implies that F is meet, since given $(a_i\colon X \to Y \mid i \in I)$, we have

$$\bigwedge_i a_i = \bar{a}/\nabla,$$

where $\bar{a} = \bigvee_i a_i(v_i)^0$, $\nabla = \bigvee_i (v_i)^0$, and the v_i are the insertions $X \to \coprod_i X$ of X into its copower by I.

4.3. LEMMA. *Let* F: $\mathbf{A} \to \mathbf{B}$ *be a morphism of bounded dcr's and let* $\mathbf{f}$: $\mathrm{Sh}(\mathbf{B}) \to \mathrm{Sh}(\mathbf{A})$ *be the corresponding geometric morphism between toposes of sheaves. Then:*

(i) $\mathbf{f}$ *is meet if and only if* F *is;*

(ii) $\mathbf{f}$ *is an open geometric morphism if and only if* F *is open;*

(iii) $\mathbf{f}$ *is a surjection if and only if* F *is faithful (as a functor).*

Proof. (i) It suffices to show that F is meet if and only if the induced morphism between completions, $\hat{F}$: $\hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is meet. One implication is immediate since F is the restriction of $\hat{F}$ to the full subcategories $\mathbf{A}$ and $\mathbf{B}$. The converse implication is a consequence of the following properties of infs with respect to coproducts and splitting of symmetric idempotents in a dcr $\mathbf{A}$.

(a) Suppose that

$$X = \coprod_i X_i, \quad Y = \coprod_j Y_j,$$

and we have some morphisms a_k: $X \to Y$. Since coproducts are also products in a dcr (see [2, § 6]), each a_k corresponds to a matrix of components a_{kij}: $X_i \to Y_j$. Then $\bigwedge_k a_k$: $X \to Y$ has components $\bigwedge_k a_{kij}$: $X_i \to Y_j$.

(b) Suppose e: $X \to X$ and f: $Y \to Y$ are symmetric idempotents with splittings

$$e = \left(X \xrightarrow{p} \bar{X} \xrightarrow{p^0} X\right)$$

and

$$f = \left(Y \xrightarrow{q} \bar{Y} \xrightarrow{q^0} Y\right).$$

Given a_k: $X \to Y$ satisfying $f \circ a_k = a_k = a_k \circ e$, one has that $\bigwedge_k (q a_k p^0) = q(\bigwedge_k a_k)p^0$.

(ii) Recall that a geometric morphism $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ is open if $\mathbf{f}^*$: $\mathbf{E} \to \mathbf{F}$ preserves universal quantification of subobjects along morphisms in $\mathbf{E}$, and hence also preserves Heyting implication and infs of subobjects. (See [10]. Equivalently, $\mathbf{f}$ is open if and only if its localic reflection is an open locale in $\mathbf{E}$, which is so if and only if there is a site of definition for $\mathbf{F}$ in $\mathbf{E}$ with inhabited covers; see [15].) Since universal quantification of subobjects in $\mathbf{E}$ and right extensions in Rel($\mathbf{E}$) are interdefinable, by Remark 4.2, $\mathbf{f}$ is open if and only if $\mathbf{f}^*$: Rel(**E**)→Rel(**F**) is an open morphism in **DCR**. Thus for (ii) it suffices to show that F: $\mathbf{A} \to \mathbf{B}$ is open if and only if $\hat{F}$: $\hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is.

If $\hat{F}$ is open, so is

$$\mathbf{A} \overset{\eta}{\hookrightarrow} \hat{\mathbf{A}} \xrightarrow{\hat{F}} \hat{\mathbf{B}}$$

(since η, being full and faithful, necessarily preserves infs and right extensions). Hence $\eta \circ F\ (\cong \hat{F} \circ \eta)$ is open and thus F is also.

Conversely, if F is open, $\hat{F}$ is meet by (i) and it also preserves right extensions because of the following properties which the latter have with respect to coproducts and splitting of symmetric idempotents in a dcr $\mathbf{A}$.

(c) Suppose that

$$X = \coprod_i X_i, \quad Y = \coprod_j Y_j, \quad Z = \coprod_k Z_k$$

and we have a: $X \to Y$ and b: $X \to Z$ with components

$$a_{ij}\colon X_i \to Y_j, \quad b_{ik}\colon X_i \to Z_k.$$

Then b/a: $Y \to Z$ has components

$$\bigwedge_i (b_{ik}/a_{ij})\colon Y_j \to Z_k.$$

(d) Suppose e: $X \to X$, f: $Y \to Y$, and g: $Z \to Z$ are symmetric idempotents with splittings

$$e = \left(X \xrightarrow{p} \bar{X} \xrightarrow{p^0} X\right),$$

$$f = \left(Y \xrightarrow{q} \bar{Y} \xrightarrow{q^0} Y\right),$$

$$g = \left(Z \xrightarrow{r} \bar{Z} \xrightarrow{r^0} Z\right).$$

Given a: $X \to Y$ and b: $X \to Z$ with $f \circ a = a = a \circ e$ and $g \circ b = b = b \circ e$, then

$$(rbp^0)/(qap^0) = r(b/a)q^0.$$

(iii) A geometric morphism $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ is a surjection if and only if $\mathbf{f}^*$: $\mathbf{E} \to \mathbf{F}$ reflects isomorphisms, which is so if and only if $\mathbf{f}^*$: Rel(**E**)→Rel(**F**) is faithful. So it suffices to prove that $\hat{F}$: $\hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is faithful if and only if F: $\mathbf{A} \to \mathbf{B}$ is. But this follows directly from the construction of the completion $\hat{\mathbf{A}}$.

4.4. PROPOSITION. *Let* $\mathbf{A}$ *be a bounded dcr and* F: $\mathbf{A} \to \mathbf{B}$, G: $\mathbf{A} \to \mathbf{C}$ *be morphisms in* **DCR**. *Form the cocomma square*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{C} \arrow[d, "Q"] \\ \mathbf{B} \arrow[r, "P"'] & \mathbf{C} \circledS_{\mathbf{A}} \mathbf{B} \end{tikzcd}$$

as in Proposition 3.6. *Then if F is meet, Q is open. In this case one also has*

(i) *for all b: $F(X)\to F(X')$ in* **B** *(with $X, X'\in$* **A**),

$$Q_!(\varphi_{X'}\circ Pb\circ\varphi_X^0)=G(F_!b)\colon\ G(X)\to G(X') \tag{16}$$

in **C**;

(ii) *given $X, X'\in$* **A**, *b: $F(X)\to F(X')$ in* **B**, *and c: $G(X)\to G(X')$ in* **C**, *if*

$$\varphi_{X'}\circ Pb\leqslant Qc\circ\varphi_X,$$

then there is a: $X\to X'$ in **A** *with $b\leqslant Fa$ and $Ga\leqslant c$;*

(iii) *Q is faithful if F is.*

Proof. We first show that Q is meet by exhibiting left adjoints to the sup-lattice morphisms

$$Q\colon\ \mathbf{C}(Z,Z')\to(\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B})(Q(Z),Q(Z')). \tag{17}$$

Recall that $Q(Z)=Z\otimes I$ and that the sup-lattice $(\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B})(Z\otimes I, Z'\otimes I)$ is generated by elements

$$c\otimes_X b\quad\text{where } X\in\mathbf{A},\ c\colon\ X\cdot Z\to Z' \text{ and } b\colon\ I\to X\cdot I,$$

subject to the relations (1), (2), (3) of 3.3. Now from b we get

$$F(I)\overset{t}{\cong}I\xrightarrow{b}X\cdot I=F(X)\times I\overset{\pi_1}{\cong}F(X)$$

and hence

$$\bar b=F_!(\pi_1 bt)\colon\ I\to X.$$

Then define $Q_!(c\otimes_X b)$ to be

$$Z\cong I\cdot Z\xrightarrow{\bar b\cdot 1}X\cdot Z\xrightarrow{c}Z'.$$

Routine calculations show that this definition respects the relations (1), (2), (3) of 3.3 and hence induces a sup-preserving map

$$Q_!\colon\ (\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B})(Q(Z),Q(Z'))\to\mathbf{C}(Z,Z'). \tag{18}$$

(For (3) one uses the fact that $F_!(Fa\circ b)\leqslant F_!Fa\circ F_!b\leqslant a\circ F_!b$.) Similar calculations give $c\otimes_X b\leqslant QQ_!(c\otimes_X b)$ and $Q_!Qc\leqslant c$, so that (18) is indeed a left adjoint for (17).

To verify that Q is open, it is sufficient to show for $c\otimes_X b$: $Q(Z)\to Q(Z')$ in $\mathbf{C}\otimes_{\mathbf{A}}\mathbf{B}$ and c': $Z'\to Z''$ in **C** that one has

$$Q_!(Qc'\circ(c\otimes_X b))=c'\circ Q_!(c\otimes_X b).$$

But since

$$Qc'\circ(c\otimes_X b)=(c'\otimes 1)\circ(c\otimes_X b)=(c'c)\otimes_X b,$$

this is immediate from the definition of $Q_!$.

We now turn to the proofs of (i), (ii), and (iii).

(i) Calculating $\varphi_{X'}\circ Pb\circ\varphi_X^0$: $G(X)\otimes I\to G(X')\otimes I$, one finds that it is $c'\otimes_{X'\times X}b'$ where

$$c'=((X'\times X)\cdot GX\cong GX'\times(GX\times GX)\xrightarrow{1\times t\Delta^0}GX'\times I\cong GX')$$

and

$$b' = (I \xrightarrow{\Delta t^0} FX \times FX \xrightarrow{b \times 1} FX' \times FX \cong (X' \times X) \cdot I).$$

Then the corresponding $\bar{b}' = F_!(\pi_1 b' t)$ is

$$\bar{b}' = (F_! b \times 1) \circ \Delta t^0\colon\ I \to X' \times X,$$

and hence $Q_!(c' \otimes_{X' \times X} b')$ is

$$GX \cong I \cdot GX \xrightarrow{\bar{b}' \cdot 1} (X' \times X) \cdot GX \xrightarrow{c'} GX',$$

which on substituting the above values of $\bar{b}'$ and c' simplifies to $G(F_! b)$.

(ii) If $\varphi_{X'} \circ Pb \leqslant Qc \circ \varphi_X$, then

$$\varphi_{X'} \circ Pb \circ \varphi_X^0 \leqslant Qc$$

since $\varphi_X \dashv \varphi_X^0$. Hence

$$Q_!(\varphi_{X'} \circ Pb \circ \varphi_X^0) \leqslant c$$

since $Q_! \dashv Q$. Thus by (i), $G(F_! b) \leqslant c$; and $b \leqslant F(F_! b)$, since $F_! \dashv F$. Hence one can take $a = F_!(b)$ in (ii).

(iii) To see that Q is faithful, it suffices to show, for any $c\colon Z \to Z'$, that $Q_!(Qc) = c$. Now

$$Qc = c \otimes 1 = c' \otimes_I b',$$

where

$$c' = \left(I \cdot Z \cong Z \xrightarrow{c} Z'\right)$$

and

$$b' = (I \cong I \cdot I).$$

Hence

$$\begin{aligned} \bar{b}' &= F_!(\pi_1 b' t) \\ &= F_!(1_{FI}) \\ &= F_! F(1_I) \\ &= 1_I \end{aligned}$$

since F is faithful. Thus

$$Q_!(Qc) = \left(Z \cong I \cdot Z \xrightarrow{c'} Z'\right) = c,$$

as required.

On taking toposes of sheaves, we find that the above result about dcr's yields a corresponding result for Grothendieck toposes:

4.5. THEOREM. *Given geometric morphisms* $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ *and* $\mathbf{g}\colon \mathbf{G} \to \mathbf{E}$, *form the comma square in* **GTOP***:*

$$\begin{tikzcd} \mathbf{G} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

Then if $\mathbf{f}$ *is meet,* $\mathbf{q}$ *is open. In this case* $\mathbf{q}$ *is a surjection when* $\mathbf{f}$ *is and the comma square has the following interpolation property: given* $X \in \mathbf{E}$ *and subobjects* $B \rightarrowtail \mathbf{f}^*(X)$ *in* $\mathbf{F}$ *and* $C \rightarrowtail \mathbf{g}^*(X)$ in $\mathbf{G}$, *if*

$$\mathbf{p}^*(B) \leqslant \varphi_X^{-1}(\mathbf{q}^*C)$$

as subobjects of $\mathbf{p}^*\mathbf{f}^*(X)$, *then there is a subobject* $A \rightarrowtail X$ *in* $\mathbf{E}$ *with*

$$B \leqslant \mathbf{f}^*(A) \quad \textit{in } \mathrm{Sub}_{\mathbf{F}}(\mathbf{f}^*X)$$

and

$$\mathbf{g}^*(A) \leqslant C \quad \textit{in } \mathrm{Sub}_{\mathbf{G}}(\mathbf{g}^*X).$$

Proof. By hypothesis, $\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ is meet and hence, by Proposition 4.4,

$$Q\colon \mathrm{Rel}(\mathbf{G}) \to \mathrm{Rel}(\mathbf{G}) \otimes_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})$$

is open. But by Theorem 3.7, the geometric morphism induced by Q between toposes of sheaves is

$$\mathrm{Sh}(\mathrm{Rel}(\mathbf{G})) \simeq \mathbf{G} \xleftarrow{\mathbf{q}} (\mathbf{G} <_{\mathbf{E}} \mathbf{F}) \simeq \mathrm{Sh}(\mathrm{Rel}(\mathbf{G}) \otimes_{\mathrm{Rel}(\mathbf{E})} \mathrm{Rel}(\mathbf{F})).$$

Hence, by Lemma 4.3(ii), $\mathbf{q}$ is also open. Similarly, if $\mathbf{f}$ is a surjection, then $\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ is faithful, and hence by 4.4(iii) so is Q and thus $\mathbf{q}$ is a surjection by Lemma 4.3(iii). Finally, the interpolation property of the comma square becomes, under the equivalence of Theorem 3.7, a special case of 4.4(ii) (by taking $X' = I$).

4.6. Remarks. (i) The interpolation property 4.4(ii) was deduced from the equation (16) and is in fact equivalent to it. Similarly the interpolation property in Theorem 4.5 is equivalent to assuming that, for $B \rightarrowtail \mathbf{f}^*(X)$,

$$q_!(\exists\varphi_X(\mathbf{p}^*B)) = \mathbf{g}^*(\mathbf{f}_!B),$$

where

$$\mathbf{f}_!\colon \mathrm{Sub}_{\mathbf{F}}(\mathbf{f}^*X) \to \mathrm{Sub}_{\mathbf{E}}(X)$$

and

$$\mathbf{q}_!\colon \mathrm{Sub}_{\mathbf{G}<_{\mathbf{E}}\mathbf{F}}(\mathbf{q}^*\mathbf{g}^*X) \to \mathrm{Sub}_{\mathbf{G}}(\mathbf{g}^*X)$$

are the left adjoints of $\mathbf{f}^*$ and $\mathbf{q}^*$ applied to subobjects.

(ii) Theorem 4.5 should be compared with the fact that open (surjective) geometric morphisms are stable under pullback. (See [**10**, § 4] and [**15**, Chapter VII].) Such pullback squares in **GTOP** also enjoy an interpolation property; see Proposition 3.3 of [**25**]. Clearly these properties of open geometric morphisms under pullback correspond by Theorem 3.12 to properties of open morphisms in **DCR** under formation of tensor products; and the latter can be proved directly by arguments like those in 4.4.

In [**24**] the property of cocomma squares of pretoposes analogous to the interpolation property in Theorem 4.5 was used to deduce a version of the Makkai–Reyes 'conceptual completeness' theorem for pretopos morphisms. Here we shall derive similar results for certain kinds of geometric morphisms. These results closely parallel the infinitary generalizations of conceptual completeness considered by Makkai and Reyes in §§ 7.3 and 7.4 of [19], except that we only consider the limiting, fully infinitary case. (However, the method by which we obtain the results appears to be quite different from theirs.)

Let us fix a collection $\mathbb{B}$ of Grothendieck toposes with the following property: for all $\mathbf{E} \in \mathbf{GTOP}$, the collection of functors

$$\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{B}) \quad (\mathbf{B} \in \mathbb{B}, \mathbf{f}\colon \mathbf{B} \to \mathbf{E} \text{ in } \mathbf{GTOP}) \tag{19}$$

is jointly faithful, so that $a \leqslant a'\colon X \nrightarrow X'$ in $\mathrm{Rel}(\mathbf{E})$ if and only if $\mathbf{f}^*(a) \leqslant \mathbf{f}^*(a')$ for all such $\mathbf{f}$. (By a theorem of Barr [8, 7.57],

$$\mathbb{B} = \{\mathrm{Sh}(B) \mid B \text{ a complete Boolean algebra}\}$$

is an example of such a collection.) The following result should be compared with Theorem 7.3.5 of [19]:

4.7. PROPOSITION. *Let* $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ *be a meet geometric morphism between Grothendieck toposes. Suppose that* $\mathbb{B}$ *is as above and that for all* $\mathbf{B} \in \mathbb{B}$ *the functor*

$$f \circ (-)\colon \mathbf{GTOP}(\mathbf{B}, \mathbf{F}) \to \mathbf{GTOP}(\mathbf{B}, \mathbf{E})$$

is full. Then $\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ *is full.*

Proof. Form the comma square

$$\begin{tikzcd} \mathbf{F} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{F} \arrow[d, "\mathbf{f}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

To see that $\mathbf{f}^*\colon \mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ is full, it suffices to show that for each $X \in \mathbf{E}$ and $B \rightarrowtail \mathbf{f}^*(X)$ in $\mathbf{F}$, there is $A \rightarrowtail X$ in $\mathbf{E}$ with $\mathbf{f}^*(A) = B$ in $\mathrm{Sub}_{\mathbf{F}}(\mathbf{f}^*X)$, that is, that $B \leqslant \mathbf{f}^*(A)$ and $\mathbf{f}^*(A) \leqslant B$ for some A. By Theorem 4.4, for this it suffices to show that $\mathbf{p}^*(B) \leqslant \varphi_X^{-1}(\mathbf{q}^*B)$; and by hypothesis on $\mathbb{B}$, the latter holds just in the case where, for all $\mathbf{B} \in \mathbb{B}$ and all $\mathbf{g} \in \mathbf{GTOP}(\mathbf{B}, \mathbf{F})$,

$$\mathbf{g}^*(\mathbf{p}^*B) \leqslant \mathbf{g}^*(\varphi_X^{-1}(\mathbf{q}^*B)).$$

But by assumption, $\varphi_{\mathbf{g}}\colon \mathbf{f}(\mathbf{pg}) \to \mathbf{f}(\mathbf{qg})$ is of the form $\varphi_{\mathbf{g}} = \mathbf{f}\psi$ for some $\psi\colon \mathbf{pg} \to \mathbf{qg}$. Thus

$$\begin{tikzcd} (\mathbf{pg})^*B \arrow[r, tail] \arrow[d, "\psi_B"'] & (\mathbf{pg})^*\mathbf{f}^*X \arrow[d, "\psi_{\mathbf{f}^*X} = \mathbf{g}^*(\varphi_X)"] \\ (\mathbf{qg})^*B \arrow[r, tail] & (\mathbf{qg})^*\mathbf{f}^*X \end{tikzcd}$$

commutes and hence

$$\begin{aligned} \mathbf{g}^*(\mathbf{p}^*B) &\leqslant (\mathbf{g}^*\varphi_X)^{-1}(\mathbf{g}^*(\mathbf{q}^*B)) \\ &= \mathbf{g}^*(\varphi_X^{-1}(\mathbf{q}^*B)), \end{aligned}$$

as required.

Next we consider the hypothesis on $\mathbf{f}\colon \mathbf{F} \to \mathbf{E}$ that the functors

$$\mathbf{f} \circ (-)\colon \mathbf{GTOP}(\mathbf{B}, \mathbf{F}) \to \mathbf{GTOP}(\mathbf{B}, \mathbf{E}) \quad (\mathbf{B} \in \mathbb{B})$$

are faithful, and its relation to the property of $\mathbf{f}$ that it be localic. Recall (from [**11**] and [**15**, VI.5]) that $\mathbf{f}$ is *localic* if for all $Y \in \mathbf{F}$ there is an $X \in \mathbf{E}$ and a diagram of the form

$$\begin{tikzcd} \bullet \arrow[r, tail, two heads, "e"] \arrow[d, tail, "m"'] & Y \\ \mathbf{f}^*(X) & \end{tikzcd}$$

in $\mathbf{F}$ with m mono and e epi. As remarked in (a) of the proof of Lemma 1.8, the existence of such a diagram in $\mathbf{F}$ is equivalent to the existence in Rel($\mathbf{F}$) of a retraction

$$Y \overset{B}{\longrightarrow} \mathbf{f}^*(X) \overset{C}{\longrightarrow} Y, \quad C \circ B = 1_Y.$$

Indeed, given such m and e, we can take $B = m \circ e^0$ and $C = e \circ m^0$; and conversely, given such B and C, we can take

$$m = (B^0 \wedge C \rightarrowtail \mathbf{f}^*(X) \times Y \xrightarrow{\pi_1} \mathbf{f}^*(X))$$

and

$$e = (B^0 \wedge C \rightarrowtail \mathbf{f}^*(X) \times Y \xrightarrow{\pi_2} Y).$$

Thus *the geometric morphism* $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ *is localic if and only if the functor* $\mathbf{f}^*$: Rel($\mathbf{E}$) $\to$ Rel($\mathbf{F}$) *has the property that each object in the codomain is a retract of one in the image of the functor.* More generally, one has:

4.8. LEMMA. *Let* F: $\mathbf{A} \to \mathbf{B}$ *be a morphism of bounded dcr's and let* $\mathbf{f}$: Sh($\mathbf{B}$) $\to$ Sh($\mathbf{A}$) *be the corresponding geometric morphisms between toposes of sheaves. Then:*

(i) $\mathbf{f}$ *is localic if and only if* F *is dense, that is, the objects in the image of* F *are* **Sl**-*generating in* $\mathbf{B}$ (*see* 1.7);

(ii) $\mathbf{f}$ *is an inclusion if and only if* F *is both dense and full* (*as a functor*).

Proof. (i) Since the inclusions $\mathbf{A} \hookrightarrow \hat{\mathbf{A}}$ and $\mathbf{B} \hookrightarrow \hat{\mathbf{B}}$ are dense, evidently $\hat{F}$: $\hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is dense if and only if F is. But since $\hat{\mathbf{A}}$ has, and $\hat{F}$ preserves, coproducts, the density of $\hat{F}$ reduces to asserting that each object of $\hat{\mathbf{B}}$ is a retract of one in the image of $\hat{F}$ (because Y is a retract of $\coprod X_i$ if and only if there is a family

$$\left(Y \xrightarrow{b_i} X_i \xrightarrow{c_i} Y \mid i \in I\right)$$

with $1_Y = \bigvee_i c_i \circ b_i$). But by the remark above, this is equivalent to asserting that $\mathbf{f}$ is localic.

(ii) It is easily verified that a geometric morphism $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ is an inclusion (that is, $\mathbf{f}_*$: $\mathbf{F} \to \mathbf{E}$ is a full and faithful functor) if and only if both $\mathbf{f}$ is localic and $\mathbf{f}^*$ is *full on subobjects* (that is, for all $B \rightarrowtail \mathbf{f}^*(X)$ there is $A \rightarrowtail X$ with $\mathbf{f}^*(A) = B$ in $\mathrm{Sub}_{\mathbf{F}}(\mathbf{f}^*X)$); evidently the latter condition is equivalent to asserting that $\mathbf{f}^*$: Rel($\mathbf{E}$) $\to$ Rel($\mathbf{F}$) is full. Thus $\mathbf{f}$: Sh($\mathbf{B}$) $\to$ Sh($\mathbf{A}$) is an inclusion if and only if $\hat{F}$: $\hat{\mathbf{A}} \to \hat{\mathbf{B}}$ is both dense and full, which is so if and only if (by (i)) F is dense and $\hat{F}$ full. But the construction of the completion ensures that $\hat{F}$ is full if and only if F is.

4.9. PROPOSITION. *Given* $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ in **GTOP**, *form the comma square*

$$\begin{tikzcd} \mathbf{F} <_{\mathbf{E}} \mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & \mathbf{F} \arrow[d, "\mathbf{f}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

and the diagonal geometric morphism $\mathbf{d}$: $\mathbf{F} \to (\mathbf{F} <_{\mathbf{E}} \mathbf{F})$ (*satisfying* $\mathbf{pd} \cong \mathbf{1_F} \cong \mathbf{qd}$ *and* $1_{\mathbf{f}} = (\mathbf{f} \cong \mathbf{fpd} \xrightarrow{\varphi_{\mathbf{d}}} \mathbf{fqd} \cong \mathbf{f})$).
Then $\mathbf{f}$ *is localic if and only if* $\mathbf{d}$ *is an inclusion.*

Proof. Since $\mathbf{d}$ is split by $\mathbf{p}$ (and $\mathbf{q}$), it is necessarily localic. Hence by 4.8(ii), $\mathbf{d}$ is an inclusion if and only if $\mathbf{d}^*$: $\mathrm{Rel}(\mathbf{F} <_{\mathbf{E}} \mathbf{F}) \to \mathrm{Rel}(\mathbf{F})$ is full, and for this it is necessary and sufficient that the maps

$$\mathbf{d}^*: \ \mathrm{Sub}_{\mathbf{F}<_{\mathbf{E}}\mathbf{F}}(W) \to \mathrm{Sub}_{\mathbf{F}}(\mathbf{d}^*W)$$

be surjective for any collection of objects W which are **Sl**-generating in $\mathrm{Rel}(\mathbf{F} <_{\mathbf{E}} \mathbf{F})$. Now $\{\mathbf{p}^*(Y) \times \mathbf{q}^*(Z) \mid Y, Z \in \mathbf{F}\}$ is such a collection. By 3.8 each element of

$$\mathrm{Sub}_{\mathbf{F}<_{\mathbf{E}}\mathbf{F}}(\mathbf{p}^*Y \times \mathbf{q}^*Z) = \mathrm{Rel}(\mathbf{F} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z)$$

is of the form

$$\bigvee_i \mathbf{q}^*(C_i) \circ \varphi_{X_i} \circ \mathbf{p}^*(B_i)$$

for some $X_i \in \mathbf{E}$ and B_i: $Y \leftrightarrow \mathbf{f}^*(X_i)$, C_i: $\mathbf{f}^*(X_i) \leftrightarrow Z$ in $\mathrm{Rel}(\mathbf{F})$. The map

$$\mathrm{Rel}(\mathbf{F} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z) \xrightarrow{\mathbf{d}^*} \mathrm{Rel}(\mathbf{F})((\mathbf{pd})^*Y, (\mathbf{qd})^*Z) \cong \mathrm{Rel}(\mathbf{F})(Y, Z)$$

sends $\mathbf{q}^*(C_i) \circ \varphi_{X_i} \circ \mathbf{p}^*(B_i)$ to $C_i \circ B_i$. Hence $\mathbf{d}$ is an inclusion if and only if, for all $Y, Z \in \mathbf{F}$, every relation $Y \leftrightarrow Z$ is of the form $\bigvee_i C_i \circ B_i$ for some B_i, C_i as above. But this is just the condition that $\mathbf{f}^*$: $\mathrm{Rel}(\mathbf{E}) \to \mathrm{Rel}(\mathbf{F})$ be dense, which by 4.8(i) is equivalent to $\mathbf{f}$ being localic, as required.

If $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ in **GTOP** is localic, then for any $\mathbf{B} \in \mathbf{GTOP}$ the functor

$$\mathbf{f} \circ (-): \ \mathbf{GTOP}(\mathbf{B}, \mathbf{F}) \to \mathbf{GTOP}(\mathbf{B}, \mathbf{E})$$

is faithful. (In view of 4.8(i), this fact is equivalent to 2.7(i).) But the converse does not hold: it is possible for all the induced functors $\mathbf{f} \circ (-)$ to be faithful without $\mathbf{f}$ being localic. An example of this due to Makkai is given in 2.11 of [**24**]. The following result gives a sufficient condition on $\mathbf{f}$ for the faithfulness of the induced functors between categories of points to imply $\mathbf{f}$ localic:

4.10. PROPOSITION. *Given* $\mathbf{f}$: $\mathbf{F} \to \mathbf{E}$ *in* **GTOP**, *suppose that the associated diagonal geometric morphism* $\mathbf{d}$: $\mathbf{F} \to (\mathbf{F} <_{\mathbf{E}} \mathbf{F})$ *is meet. Let* $\mathbb{B}$ *be a collection of Grothendieck toposes satisfying the condition* (19) (*see Proposition* 4.7). *Then* $\mathbf{f}$ *is localic if and only if for all* $\mathbf{B} \in \mathbb{B}$,

$$\mathbf{f} \circ (-): \ \mathbf{GTOP}(\mathbf{B}, \mathbf{F}) \to \mathbf{GTOP}(\mathbf{B}, \mathbf{E})$$

is faithful.

Proof. Using the universal property of the comma topos $\mathbf{F}<_{\mathbf{E}}\mathbf{F}$, the fact that each $\mathbf{f}\circ(-)$ is faithful implies that each

$$\mathbf{d}\circ(-)\colon \mathbf{GTOP}(\mathbf{B},\mathbf{F})\to\mathbf{GTOP}(\mathbf{B},\mathbf{F}<_{\mathbf{E}}\mathbf{F})$$

is full. Hence by Proposition 4.7, $\mathbf{d}^*$: Rel($\mathbf{F}<_{\mathbf{E}}\mathbf{F}$)$\to$Rel($\mathbf{F}$) is full. Since the latter is automatically dense (being split), it follows as in 4.8(ii) that $\mathbf{d}$ is an inclusion. Hence $\mathbf{f}$ is localic by Proposition 4.9.

Propositions 4.7 and 4.10 together give a 'conceptual completeness' result for geometric morphisms $\mathbf{f}\colon \mathbf{F}\to\mathbf{E}$ for which both $\mathbf{f}$ and $\mathbf{d}\colon \mathbf{F}\to(\mathbf{F}<_{\mathbf{E}}\mathbf{F})$ are meet:

4.11. Corollary. *Given* $\mathbf{f}\colon \mathbf{F}\to\mathbf{E}$ *in* **GTOP**, *suppose that both* $\mathbf{f}$ *and* $\mathbf{d}\colon \mathbf{F}\to(\mathbf{F}<_{\mathbf{E}}\mathbf{F})$ *are meet. Let* $\mathbb{B}$ *be as in Propositions* 4.7 *and* 4.10. *Then:*

(i) $\mathbf{f}$ *is an inclusion if and only if, for all* $\mathbf{B}\in\mathbb{B}$,

$$\mathbf{f}\circ(-)\colon \mathbf{GTOP}(\mathbf{B},\mathbf{F})\to\mathbf{GTOP}(\mathbf{B},\mathbf{E})$$

is full and faithful;

(ii) $\mathbf{f}$ *is an equivalence if and only if, for all* $\mathbf{B}\in\mathbb{B}$,

$$\mathbf{f}\circ(-)\colon\mathbf{GTOP}(\mathbf{B},\mathbf{F})\to\mathbf{GTOP}(\mathbf{B},\mathbf{E})$$

is an equivalence.

Proof. (i) Combine 4.7 and 4.10.

(ii) If each $\mathbf{f}\circ(-)$ is essentially surjective, then by hypothesis on $\mathbb{B}$, $\mathbf{f}$ is a surjection. (Cf. Theorem 2.13 of [**24**].) Since by (i) it is also an inclusion, it is an equivalence.

4.12. Remark. Since being *open* is a special case of being *meet*, the condition on $\mathbf{f}$ in 4.11 is reminiscent of the condition in Chapter VII of [**15**] which Joyal and Tierney show characterizes the *atomic* toposes of Barr and Diaconescu [**1**], viz. the condition that both $\mathbf{f}$ and the diagonal $\mathbf{F}\to\mathbf{F}\times_{\mathbf{E}}\mathbf{F}$ be open.

Proposition 4.10 should be compared with Theorem 7.4.2 in [**19**], which draws a similar conclusion, but from different hypotheses: Makkai and Reyes use a choice principle [**19**, 7.4.1(ii)] which in our context becomes the following assumption on a morphism $\mathbf{f}\colon \mathbf{F}\to\mathbf{E}$ in **GTOP**:

4.13. Assumption. For any small family $(X_i|\ i\in I)$ of objects in $\mathbf{E}$, there is a family $(p_i\colon X\to X_i|\ i\in I)$ of morphisms in $\mathbf{E}$ with the following property:

for all families $(B_i\rightarrowtail\mathbf{f}^*(X_i)\times Y|\ i\in I)$ of subobjects in $\mathbf{F}$, the sentence
$\forall y\in Y\,[\bigwedge_{i\in I}\{\exists x_i\in\mathbf{f}^*(X_i)B_i(x_i,y)\}\to\exists x\in\mathbf{f}^*(X)\{\bigwedge_{i\in I}B_i(\mathbf{f}^*(p_i)(x),y)\}]$
is satisfied by $\mathbf{F}$.

We show that this assumption on $\mathbf{f}$ is stronger than the hypothesis of 4.10:

4.14. Proposition. *If* $\mathbf{f}\colon \mathbf{F}\to\mathbf{E}$ *satisfies Assumption* 4.13, *then the diagonal* $\mathbf{d}\colon \mathbf{F}\to(\mathbf{F}<_{\mathbf{E}}\mathbf{F})$ *is meet.*

Proof. The condition on $(p_i|\ i\in I)$ is 4.13 is equivalent to asserting that for all $Y,Z\in\mathbf{F}$ and for any

$$\left(Y\overset{B_i}{\longleftrightarrow}\mathbf{f}^*(X_i)\overset{C_i}{\longleftrightarrow}Z|\ i\in I\right)$$

in Rel(**F**), that

$$\bigwedge_{i\in I} C_i \circ B_i = \Big[\bigwedge_{j\in I} C_j \circ \mathbf{f}^*(p_j)\Big] \circ \Big[\bigwedge_{k\in I} \mathbf{f}^*(p_k^0)(B_k)\Big]. \tag{20}$$

By naturality of φ, one has

$$\begin{aligned} \mathbf{q}^*\Big(\bigwedge_j C_j\mathbf{f}^*p_j\Big)\varphi_X\Big(\bigwedge_k \mathbf{f}^*p_k^0B_k\Big) &\leq \mathbf{q}^*(C_i\mathbf{f}^*p_i)\varphi_X\mathbf{p}^*(\mathbf{f}^*p_i^0B_i)\\ &= \mathbf{q}^*(C_i)\varphi_{X_i}\mathbf{p}^*(\mathbf{f}^*(p_ip_i^0)B_i)\\ &\leq \mathbf{q}^*(C_i)\varphi_{X_i}\mathbf{p}^*(B_i), \end{aligned}$$

so that

$$\mathbf{q}^*\Big[\bigwedge_j C_j\mathbf{f}^*p_j\Big]\varphi_X\Big[\bigwedge_k \mathbf{f}^*p_k^0B_k\Big] \leq \bigwedge_i [\mathbf{q}^*(C_i)\varphi_{X_i}\mathbf{p}^*(B_i)]. \tag{21}$$

Now $\mathbf{d}^*$ sends a relation of the form $\mathbf{q}^*C \circ \varphi_X \circ \mathbf{p}^*B$ to $C \circ B$. Hence (20) and (21) give

$$\bigwedge_i \mathbf{d}^*[\mathbf{q}^*(C_i)\varphi_{X_i}\mathbf{p}^*(B_i)] \leq \mathbf{d}^*\Big[\bigwedge_i \mathbf{q}^*(C_i)\varphi_{X_i}\mathbf{p}^*(B_i)\Big].$$

Since by Remark 3.9 every element of $\mathrm{Rel}(\mathbf{F}<_{\mathbf{E}}\mathbf{F})(\mathbf{p}^*Y, \mathbf{q}^*Z)$ is of the form $\mathbf{q}^*C \circ \varphi_X \circ \mathbf{p}^*B$, we have that $\mathbf{d}^*$ preserves infs of subobjects of all objects of the form $\mathbf{p}^*(Y) \times \mathbf{q}^*(Z)$; but such objects are **Sl**-generating in $\mathrm{Rel}(\mathbf{F}<_{\mathbf{E}}\mathbf{F})$, and hence (as in 4.3(i)) $\mathbf{d}$ is meet.

The implication in Proposition 4.14 is not reversible, since when $\mathbf{f} = \mathbf{1_E}$: $\mathbf{E} \to \mathbf{E}$, then $\mathbf{d}$: $\mathbf{E} \to (\mathbf{E}<_{\mathbf{E}}\mathbf{E})$ is meet (since in this case $\mathbf{p}^*$: $\mathbf{E} \to (\mathbf{E}<_{\mathbf{E}}\mathbf{E})$ is left adjoint to $\mathbf{d}^*$), but Assumption 4.13 is still a non-trivial requirement on the topos **E** (implying, for example, that the infinite product of inhabited objects is inhabited in **E**).

5. *Internal sup-lattices*

In this section we give a strikingly simple characterization of *internal* sup-lattices in a Grothendieck topos, in terms of (external) sup-lattice enriched category theory. Using this characterization, we will reformulate Theorems 3.12 and 3.7 on pullback and comma toposes in terms of composition and 'lax composition' of **Sl**-enriched *profunctors*.

In VI.2 of [**15**], Joyal and Tierney give a characterization of internal sup-lattices in a presheaf topos $[\mathbf{C}^{op}, \mathbf{Set}]$ (with **C** finitely complete). They correspond to functors M: $\mathbf{C}^{op} \to \mathbf{Sl}$ with the property that for each f: $X \to Y$ in **C**, $M(f)$: $M(Y) \to M(X)$ has a left adjoint Σf: $M(X) \to M(Y)$ and that these left adjoints satisfy the equation

$$Mg \circ \Sigma f = \Sigma f' \circ Mg'$$

whenever

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback square in $\mathbf{C}$. (Moreover, a morphism of internal sup-lattices in $[\mathbf{C}^{op}, \mathbf{Set}]$ corresponds to a natural transformation between such functors $\mathbf{C}^{op} \to \mathbf{Sl}$ which commutes with the left adjoints.)

This characterization is a special case of the explanation of cocompleteness in fibred categories (and indexed categories) given by Bénabou (and Paré-Schumacher). Thus for a general Grothendieck topos $\mathbf{E}$ one has the following:

Specifying an internal sup-lattice structure on an object M in $\mathbf{E}$ is equivalent to giving a lifting of the representable functor $\mathbf{E}(-, M)$: $\mathbf{E}^{op} \to \mathbf{Set}$ to a functor $\mathbf{E}^{op} \to \mathbf{Sl}$ in such a way that each f^*: $\mathbf{E}(Y, M) \to \mathbf{E}(X, M)$ has a left adjoint Σf: $\mathbf{E}(X, M) \to \mathbf{E}(Y, M)$ and so that these adjoints satisfy the *Beck-Chevalley condition*

$$g^* \circ \Sigma f = \Sigma f' \circ (g')^* \tag{22}$$

whenever

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback square in $\mathbf{E}$. Moreover, if N is another such internal sup-lattice, then a morphism φ: $M \to N$ in $\mathbf{E}$ preserves internal sups if and only if, for each $X \in \mathbf{E}$, φ_*: $\mathbf{E}(X, M) \to \mathbf{E}(X, N)$ is a morphism in $\mathbf{Sl}$ and for each f: $X \to Y$ in $\mathbf{E}$,

$$\begin{array}{ccc} \mathbf{E}(X, M) & \xrightarrow{\varphi_*} & \mathbf{E}(X, N) \\ {\scriptstyle \Sigma f}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ \mathbf{E}(Y, M) & \xrightarrow[\varphi_*]{} & \mathbf{E}(Y, N) \end{array}$$

commutes.

We can improve on the above description by taking account of the behaviour of such an $M \in \mathbf{E}$ with respect to not just the morphisms in $\mathbf{E}$, but also the *relations*. Thus, given a relation R: $X \leftrightarrow Y$, choose any monomorphism $\langle a, b \rangle$: $R \rightarrowtail X \times Y$ representing it and define a map

$$R \cdot (-)\colon\ \mathbf{E}(X, M) \to \mathbf{E}(Y, M)$$

by

$$R \cdot m = \Sigma b(a^*(m)). \tag{23}$$

This definition is independent of the choice of mono representing R and gives a sup-preserving map (since both Σb and a^* preserve sups). These maps $R \cdot (-)$ are both functorial and sup-preserving:

5.1. LEMMA. (i) *For all* $m \in \mathbf{E}(X, M)$, $1_X \cdot m = m$.
(ii) *Given* R: $X \leftrightarrow Y$ *and* S: $Y \leftrightarrow Z$ *in* Rel($\mathbf{E}$) *and* $m \in \mathbf{E}(X, M)$, *then*

$$(S \circ R) \cdot m = S \cdot (R \cdot m).$$

(iii) *Given* $(R_i$: $X \leftrightarrow Y \mid i \in I)$ *in* Rel($\mathbf{E}$) *and* $m \in \mathbf{E}(X, M)$, *then*

$$\Big(\bigvee_{i \in I} R_i\Big) \cdot m = \bigvee_{i \in I} (R_i \cdot m).$$

Proof. (i) By definition

$$1_X \cdot m = \Sigma 1_X(1_X^*(m)) = m.$$

(ii) The composition $S \circ R$ is defined in terms of pullback and image factorization in **E**. Correspondingly, the identity (ii) follows from the Beck–Chevalley condition (22) and the fact that if $f\colon X \to Y$ is an epi in **E**, then $f^*\colon \mathbf{E}(Y, M) \to \mathbf{E}(X, M)$ is injective and hence $\Sigma f \circ f^* = 1$.

(iii) The domain of a mono representing $\bigvee \{R_i \mid i \in I\}$ is the target of a jointly epimorphic family of morphisms from the domains of the mono's representing the R_i. It suffices to show that for any epimorphic family $(f_i\colon X_i \to X \mid i \in I)$ in **E**, one has

$$\bigvee_i (\Sigma f_i \circ (f_i)^*) = 1\colon\ \mathbf{E}(X, M) \to \mathbf{E}(X, M).$$

But consider the map

$$\mathbf{E}(X, M) \to \prod_i \mathbf{E}(X_i, M),$$

$$m \mapsto ((f_i)^* m \mid i \in I).$$

It is injective (since the f_i are jointly epi) and has a left adjoint given by

$$(m_i \mid i \in I) \mapsto \bigvee_{i \in I} \Sigma f_i(m_i).$$

Hence $\bigvee_{i \in I} \Sigma f_i (f_i)^* m = m$, as required.

The preceding lemma shows that each internal sup-lattice M in **E** determines via (23) a **Sl**-functor

$$\mathbf{E}(-, M)\colon\ \mathrm{Rel}(\mathbf{E}) \to \mathbf{Sl}.$$

Note also that if $\varphi\colon M \to N$ is an internal sup-preserving morphism, then since $\varphi_*\colon \mathbf{E}(-, M) \to \mathbf{E}(-, N)$ commutes with both $(-)^*$ and Σ, one has

$$\varphi_*(R \cdot m) = R \cdot (\varphi_* m),$$

so that φ_* is a natural transformation between the **Sl**-functors $\mathbf{E}(-, M)$, $\mathbf{E}(-, N)\colon \mathrm{Rel}(\mathbf{E}) \to \mathbf{Sl}$.

In this way, for each Grothendieck topos **E**, we obtain a functor

$$\mathbf{Sl}(\mathbf{E}) \to \mathbf{Sl}\text{-}\mathbf{CAT}(\mathrm{Rel}(\mathbf{E}), \mathbf{Sl}) \tag{24}$$

from the category of internal sup-lattices and internal sup-preserving maps in **E** to the category of **Sl**-functors and natural transformations from Rel(**E**) to **Sl**.

5.2. THEOREM. *The functor* (24) *is an equivalence of categories:*

$$\mathbf{Sl}(\mathbf{E}) \simeq \mathbf{Sl}\text{-}\mathbf{CAT}(\mathrm{Rel}(\mathbf{E}), \mathbf{Sl}).$$

Proof. Identifying morphisms in **E** with maps in Rel(**E**), by definition one has for $M \in \mathbf{Sl}(\mathbf{E})$ and $f\colon X \to Y$ in **E**, that

$$f^0 \cdot (-) = f^*\colon\ \mathbf{E}(Y, M) \to \mathbf{E}(X, M) \tag{25}$$

and

$$f \cdot (-) = \Sigma f\colon\ \mathbf{E}(X, M) \to \mathbf{E}(Y, M). \tag{26}$$

In view of (25), a natural transformation

$$\mathbf{E}(-, M) \to \mathbf{E}(-, N)\colon\ \mathrm{Rel}(\mathbf{E}) \to \mathbf{Sl}$$

restricts along the inclusion $\mathbf{E}^{op} \hookrightarrow \mathrm{Rel}(\mathbf{E})$ (sending $f\colon X \to Y$ to $f^0\colon Y \nrightarrow X$), to give a natural transformation between representable functors, which by Yoneda's lemma is of the form φ_* for a unique $\varphi\colon M \to N$; and moreover by (26), φ_* commutes with Σ, so that φ is a morphism in $\mathbf{Sl}(\mathbf{E})$. Thus the functor in (24) is full and faithful.

To see that it is also essentially surjective, suppose we are given a $\mathbf{Sl}$-functor $F\colon \mathrm{Rel}(\mathbf{E}) \to \mathbf{Sl}$. Restricting along $\mathbf{E}^{op} \hookrightarrow \mathrm{Rel}(\mathbf{E})$ gives a functor

$$F\colon\ \mathbf{E}^{op} \to \mathbf{Sl} \tag{27}$$

sending $f\colon X \to Y$ in $\mathbf{E}$ to $F(f^0)\colon F(Y) \to F(X)$ in $\mathbf{Sl}$. Each such map has a left adjoint $F(f)\colon F(X) \to F(Y)$ (since $f \dashv f^0$ in $\mathrm{Rel}(\mathbf{E})$); and these adjoints satisfy the Beck–Chevalley condition for pullback squares in $\mathbf{E}$, since if

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is such a square, then

$$\begin{tikzcd} X' \arrow[r, "f'", "|" marking] & Y' \\ X \arrow[u, "(g')^0", "-" marking] \arrow[r, "f", "|" marking] & Y \arrow[u, "g^0"', "-" marking] \end{tikzcd}$$

commutes in $\mathrm{Rel}(\mathbf{E})$ and hence $F(f') \circ F((g')^0) = F(g^0) \circ F(f)$. Consequently, we just have to show that (27) is representable, i.e. that for some $M \in \mathbf{E}$, $F(-) \cong \mathbf{E}(-, M)\colon \mathbf{E}^{op} \to \mathbf{Set}$. For then M inherits an internal sup-lattice structure from F and $F \cong \mathbf{E}(-, M)$ in $\mathbf{Sl}\text{-}\mathbf{CAT}(\mathrm{Rel}(\mathbf{E}), \mathbf{Sl})$.

To see that $F\colon \mathbf{E}^{op} \to \mathbf{Set}$ is representable, it is sufficient to check that it is a sheaf for the canonical topology on $\mathbf{E}$. Suppose then that $(f_i\colon X_i \to X \mid i \in I)$ is a jointly epimorphic family in $\mathbf{E}$, and that we have $m_i \in F(X_i)$ satisfying

$$F(p_{ij})^0(m_i) = F(q_{ij})^0(m_j)$$

for all i, $j \in I$, where

$$\begin{tikzcd} X_{ij} \arrow[r, "q_{ij}"] \arrow[d, "p_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

are pullback squares in $\mathbf{E}$. Putting $m = \bigvee \{F(f_i)(m_i) \mid i \in I\} \in F(X)$, for each $i \in I$ we have

$$\begin{aligned} F(f_i)^0(m) &= \bigvee_j F((f_i)^0 \circ f_j)(m_j) \\ &= \bigvee_j F(p_{ij}(q_{ij})^0)(m_j) \\ &= \bigvee_j F(p_{ij}(p_{ij})^0)(m_i) \\ &= F\Big(\bigvee_j p_{ij}(p_{ij})^0\Big)(m_i) \\ &= F(1_{X_i})(m_i) \\ &= m_i. \end{aligned}$$

And $m \in F(X)$ is the unique such element, since if m' also satisfies $F(f_i)^0(m') = m_i$ for all $i \in I$, then

$$m' = F(1_X)m' = F\Big(\bigvee_i f_i(f_i)^0\Big)m' = \bigvee_i F(f_i)(m_i) = m.$$

Thus F is a sheaf and hence is representable.

Reciprocation of relations gives an isomorphism of **Sl**-categories, $\mathrm{Rel}(\mathbf{E})^{op} \cong \mathrm{Rel}(\mathbf{E})$. Hence Proposition 5.2 also gives an equivalence

$$\mathbf{Sl}(\mathbf{E}) \cong \mathbf{Sl\text{-}CAT}(\mathrm{Rel}(\mathbf{E})^{op}, \mathbf{Sl}).$$

More generally, the arguments given above easily extend to give an equivalence

$$\mathbf{Sl}(\mathbf{F} \times \mathbf{G}) \simeq \mathbf{Sl\text{-}PROF}(\mathrm{Rel}(\mathbf{F}), \mathrm{Rel}(\mathbf{G})), \tag{28}$$

where $\mathbf{F} \times \mathbf{G}$ denotes the product of $\mathbf{F}$ and $\mathbf{G}$ in **GTOP** and $\mathbf{Sl\text{-}PROF}(\mathrm{Rel}(\mathbf{F}), \mathrm{Rel}(\mathbf{G}))$ is the category of **Sl**-enriched *profunctors* which has **Sl**-functors $\mathrm{Rel}(\mathbf{F})^{op} \otimes \mathrm{Rel}(\mathbf{G}) \to \mathbf{Sl}$ for its objects and natural transformations for its morphisms. The equivalence (28) is given by sending $M \in \mathbf{Sl}(\mathbf{F} \times \mathbf{G})$ to the **Sl**-profunctor

$$F\colon\ \mathrm{Rel}(\mathbf{F})^{op} \otimes \mathrm{Rel}(\mathbf{G}) \to \mathbf{Sl}$$

defined as follows. For objects $Y \otimes Z$ ($Y \in \mathbf{F}$, $Z \in \mathbf{G}$) it is given by

$$F(Y \otimes Z) = (\mathbf{F} \times \mathbf{G})(\pi_1^*(Y) \times \pi_2^*(Z), M)$$

(where $\mathbf{F} \xleftarrow{\pi_1} \mathbf{F} \times \mathbf{G} \xrightarrow{\pi_2} \mathbf{G}$ are the product projections). For morphisms, it is given by sending the generator $R \otimes S\colon\ Y \otimes Z \to Y' \otimes Z'$ (where $R\colon\ Y \leftrightarrow Y'$ in $\mathrm{Rel}(\mathbf{F})$ and $S\colon\ Z \leftrightarrow Z'$ in $\mathrm{Rel}(\mathbf{G})$) to

$$(\pi_1^*(R^0) \times \pi_2^*(S)) \cdot (-)\colon F(Y \otimes Z) \to F(Y' \otimes Z').$$

Returning to the considerations of § 3, given geometric morphisms

$$\mathbf{F} \xrightarrow{\mathbf{f}} \mathbf{E} \xleftarrow{\mathbf{g}} \mathbf{G}$$

in **GTOP**, we see that the standard construction of the pullback

$$\begin{tikzcd} \mathbf{G}\times_{\mathbf{E}}\mathbf{F} \arrow[r, "\mathbf{q}"] \arrow[d, "\mathbf{p}"'] \arrow[dr, phantom, "\cong"] & \mathbf{G} \arrow[d, "\mathbf{g}"] \\ \mathbf{F} \arrow[r, "\mathbf{f}"'] & \mathbf{E} \end{tikzcd}$$

in terms of sites of definition for **E**, **F**, and **G** shows that

$$\mathbf{l}=\langle\mathbf{p},\mathbf{q}\rangle:\ \mathbf{G}\times_{\mathbf{E}}\mathbf{F}\to\mathbf{F}\times\mathbf{G}$$

is a localic geometric morphism. Thus $\mathbf{G}\times_{\mathbf{E}}\mathbf{F}$ is equivalent to the topos of sheaves in $\mathbf{F}\times\mathbf{G}$ on the internal locale $\mathbf{l}_*(\Omega)$ (where Ω is the subobject classifier of $\mathbf{G}\times_{\mathbf{E}}\mathbf{F}$). This internal locale is, in particular, an object of $\mathbf{Sl}(\mathbf{F}\times\mathbf{G})$ and by the remarks above, it corresponds under the equivalence (28) to the **Sl**-profunctor

$$\mathrm{Rel}(\mathbf{G}\times_{\mathbf{E}}\mathbf{F})(\mathbf{p}^*(-),\mathbf{q}^*(-)):\ \mathrm{Rel}(\mathbf{F})^{op}\otimes\mathrm{Rel}(\mathbf{G})\to\mathbf{Sl}.$$

Exactly the same considerations apply to the comma topos $\mathbf{G}<_{\mathbf{E}}\mathbf{F}$ which is localic over $\mathbf{F}\times\mathbf{G}$ with internal locale in $\mathbf{F}\times\mathbf{G}$ corresponding to the **Sl**-profunctor

$$\mathrm{Rel}(\mathbf{G}<_{\mathbf{E}}\mathbf{F})(\mathbf{p}^*(-),\mathbf{q}^*(-)):\ \mathrm{Rel}(\mathbf{F})^{op}\otimes\mathrm{Rel}(\mathbf{G})\to\mathbf{Sl}.$$

Now 3.8 and 3.12 analyse these **Sl**-profunctors in terms of the **Sl**-profunctors

$$\mathbf{f}^{\#}=\mathrm{Rel}(\mathbf{F})(-,\mathbf{f}^*(-)):\ \mathrm{Rel}(\mathbf{F})^{op}\otimes\mathrm{Rel}(\mathbf{E})\to\mathbf{Sl}$$

and

$$\mathbf{g}_{\#}=\mathrm{Rel}(\mathbf{G})(\mathbf{g}^*(-),-):\ \mathrm{Rel}(\mathbf{E})^{op}\otimes\mathrm{Rel}(\mathbf{G})\to\mathbf{Sl}.$$

In general, given **Sl**-categories **A**, **B**, **C** with **A** bounded, and **Sl**-profunctors $F\in\mathbf{Sl\text{-}PROF}(\mathbf{B},\mathbf{A})$ and $G\in\mathbf{Sl\text{-}PROF}(\mathbf{A},\mathbf{C})$, their *composition* is the **Sl**-profunctor

$$G\otimes F\in\mathbf{Sl\text{-}PROF}(\mathbf{B},\mathbf{C})$$

given by the coend formula:

$$(G\otimes F)(B,C)=\int^{A\in\mathbf{A}}G(A,C)\otimes F(B,A)$$

(which exists since **A** is bounded). One also has a 'lax composition':

$$G\circledcirc F\in\mathbf{Sl\text{-}PROF}(\mathbf{B},\mathbf{C})$$

given by lax coends (see 3.4(i)):

$$(G\circledcirc F)(B,C)=\ell\int^{A\in\mathbf{A}}G(A,C)\otimes F(B,A).$$

With these definitions, Theorem 3.12 and Corollary 3.8 give:

$$\mathrm{Rel}(\mathbf{G}\times_{\mathbf{E}}\mathbf{F})(\mathbf{p}^*(-),\mathbf{q}^*(-))\simeq\mathbf{g}_{\#}\otimes\mathbf{f}^{\#}$$

and
$$\mathrm{Rel}(\mathbf{G} <_{\mathbf{E}} \mathbf{F})(\mathbf{p}^*(-), \mathbf{q}^*(-)) \simeq \mathbf{g}_{\#} \otimes \mathbf{f}^{\#}$$
in **Sl-PROF**(Rel(**F**), Rel(**G**)).

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Mathematics Division
University of Sussex
Falmer
Brighton BN1 9QH