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Remarks on quantic nuclei

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1. *Introduction*

Quantales were first introduced by Mulvey[**3**] in order to provide a possible setting for constructive foundations for quantum mechanics, as well as a non-commutative analogue of the maximal spectrum of a C^*-algebra. Quantales have since been studied by several authors: see [**1**, **3**, **4**, **5**].

Niefield and Rosenthal in [**5**], p. 220, raised the question whether, for a quantale Q (in their very general sense), the set NQ of quantic nuclei, together with the & they defined there, necessarily forms a quantale. We will show in this paper that the answer is negative, and further show that their Corollary 2·5 is in error. On the other hand, we point out that the set $\tilde{N}Q$ of localic (quantic) nuclei is a locale, generalizing the classical result that NL is a locale for each locale L.

Following Niefield-Rosenthal, a quantale Q is a complete lattice together with an associative binary operation & satisfying

$$a \mathbin{\&} (\bigvee b_\alpha) = \bigvee (a \mathbin{\&} b_\alpha) \quad \text{and} \quad (\bigvee b_\alpha) \mathbin{\&} a = \bigvee (b_\alpha \mathbin{\&} a),$$

for all $a \in Q$ and $\{b_\alpha\} \subseteq Q$.

As $\bigvee$-preserving mappings, the functions $a\mathbin{\&}-$ and $-\mathbin{\&}a$ have right adjoints which we shall denote by $a \xrightarrow{r} (-)$ and $a \xrightarrow{l} (-)$, respectively.

It is clear that a locale is just a quantale with $\& = \wedge$.

Remark. There are some trivial quantales $(Q, \&)$. For example, let Q be an arbitrary complete lattice and for every $a, b \in Q$, define $a \mathbin{\&_o} b = 0$. Then $(Q, \&_o)$ is a quantale.

Let Q be a quantale. Recall that a quantic nucleus on Q is a closure operator j (i.e. order-preserving, inflationary and idempotent) such that for every $a, b \in Q$,

$$j(a) \mathbin{\&} j(b) \leqslant j(a \mathbin{\&} b).$$

Clearly, for all $a, b \in Q$,

$$j(a \mathbin{\&} b) = j(a \mathbin{\&} j(b)) = j(j(a) \mathbin{\&} b) = j(j(a) \mathbin{\&} j(b)).$$

The set of all quantic nuclei on Q with pointwise ordering will be denoted by NQ. It is known that NQ is a complete lattice with $\bigwedge$ computed pointwise and $\bigvee j_\alpha$ given by the nucleus corresponding to $\bigcap Q_{j_\alpha}$ with $Q_{j_\alpha} = j_\alpha Q$. (See [**5**], proposition 2·4.)

Recall that a quantic nucleus j is localic if for all $a, b \in Q$,

$$j(a \mathbin{\&} b) = j(a) \wedge j(b).$$

It is known that j is localic if and only if Q_j is a locale with $\&_j = \wedge$, where for all $a, b \in Q$,

$$a \&_j b = j(a \& b).$$

The set of all localic quantic nuclei on Q will be denoted by $\tilde{N}Q$. It is clear that in the case where Q is a locale (i.e. $\& = \wedge$), the notions of quantic localic nucleus and the usual nucleus coincide.

The question whether NQ is a quantale was posed in [**5**]. In fact, for this question, we should refer to a particular $\&$ (see the above Remark).

Niefield and Rosenthal suggested the following definition for $\&$:

$$j \& k = \bigwedge\{m \in NQ \mid j(a) \& k(a) \leqslant m(a) \text{ for all } a \in Q\}.$$

Then the above question asks whether $(NQ, \&)$ is a quantale.

We will answer this question in the negative. Our result also shows that Corollary 2·5 in [**5**] is not correct.

2. *Main results*

THEOREM 1. *Let Q be a quantale. If $(NQ, \&)$ is a quantale, then $\& \leqslant \wedge$ in Q.*

Proof. Suppose that $(NQ, \&)$ is a quantale. Then, for all $j, k, m \in NQ$,

$$j \& k \leqslant m \quad \text{if and only if} \quad k \leqslant \; j \xrightarrow{r} m \quad \text{if and only if} \quad j \leqslant \; k \xrightarrow{l} m,$$

where

$$j \xrightarrow{r} m = \bigvee\{n \in NQ \mid j \& n \leqslant m\}$$

and

$$j \xrightarrow{l} m = \bigvee\{n \in NQ \mid n \& j \leqslant m\}.$$

On the other hand

$$\begin{aligned} j \& k \leqslant m \quad &\text{if and only if} \quad j(a) \& k(a) \leqslant m(a) \quad \text{for all } a \in Q,\\ &\text{if and only if} \quad k(a) \leqslant \; j(a) \xrightarrow{r} m(a) \quad \text{for all } a \in Q,\\ &\text{if and only if} \quad j(a) \leqslant \; k(a) \xrightarrow{l} m(a) \quad \text{for all } a \in Q. \end{aligned}$$

Then $\quad k \leqslant \; j \xrightarrow{r} m \quad$ if and only if $\quad k(a) \leqslant \; j(a) \xrightarrow{r} m(a) \quad$ for all $a \in Q$,

and $\quad j \leqslant \; k \xrightarrow{l} m \quad$ if and only if $\quad j(a) \leqslant \; k(a) \xrightarrow{l} m(a) \quad$ for all $a \in Q$.

In particular, for all $a \in Q$,

$$(j \xrightarrow{r} m)(a) \leqslant j(a) \xrightarrow{r} m(a) \quad \text{and} \quad (j \xrightarrow{l} m)(a) \leqslant j(a) \xrightarrow{l} m(a).$$

Next, we claim that, for all $a \in Q$,

$$a \& \tau \leqslant a \quad \text{and} \quad \tau \& a \leqslant a,$$

where τ is the top element in Q. In fact, we take $j = \mathbf{1} \in NQ$ and $m = \mathbf{0} \in NQ$, where for all $a \in Q$,

$$\mathbf{1}(a) = \tau \quad \text{and} \quad \mathbf{0}(a) = a.$$

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Then, for all $a \in Q$, we have

$$a \leqslant (\mathbf{1} \overset{r}{\to} \mathbf{0})(a) \leqslant \mathbf{1}(a) \overset{r}{\to} \mathbf{0}(a) = \tau \overset{r}{\to} a,$$

so that $\tau \,\&\, a \leqslant a$. Similarly $a \,\&\, \tau \leqslant a$, for all $a \in Q$. Furthermore, for all $a, b \in Q$,

$$a \,\&\, b \leqslant \tau \,\&\, b \leqslant b \quad \text{and} \quad a \,\&\, b \leqslant a \,\&\, \tau \leqslant a,$$

and thus $a \,\&\, b \leqslant a \wedge b$, that is, $\& \leqslant \wedge$.

By combining the classical result that NL is a locale for each locale L, we obtain the following corollary.

Corollary 1. *Let Q be a quantale with $\& \geqslant \wedge$. Then $(NQ, \&)$ is a quantale if and only if NQ is a locale with $\& = \wedge$ if and only if Q is a locale.*

In particular, no non-locale quantale in the sense of Borceux and Van den Bossche[**1**], (i.e. a quantale with idempotent which is right unital in the sense of Niefield and Rosenthal) has the property that $(NQ, \&)$ is a quantale.

Remark. In general, the quantum spectrum of a C^*-algebra is a quantale in the sense of Borceux and Van den Bossche but not a locale (see [**1**], p. 214). Thus $(NQ, \&)$ is not a quantale, in general; this answers the question of Niefield and Rosenthal ([**5**], p. 220) negatively.

Corollary 2. *No non-locale quantale in the sense of Borceux and Van den Bossche has the property that both $j \,\&\, —$ and $— \,\&\, j$ have right adjoints for all $j \in NQ$.*

Remark. It follows from Corollary 2 that corollary 2·5 in [**5**] is not correct. In fact, if we define

$$j \overset{r}{\to} m = \bigwedge \{n \in NQ \mid n(a) \leqslant j(a) \overset{r}{\to} m(a) \quad \text{for all } a \in Q\},$$

then there is no reason to say that

$$n \leqslant j \overset{r}{\to} m \quad \text{implies} \quad n(a) \leqslant j(a) \overset{r}{\to} m(a) \quad \text{for all } a \in Q.$$

Indeed, let Q be a non-locale quantale in the sense of Borceux and Van den Bossche. Then Q is not left-sided (cf. [**5**]); that is, there exists an $a_0 \in Q$ such that $\tau \,\&\, a_0 \nleqslant a_0$, or equivalently $a_0 \nleqslant \tau \overset{r}{\to} a_0$. By choosing

$$j = \mathbf{1} \in NQ, \quad m = \mathbf{0} \in NQ, \quad \text{and} \quad n = \mathbf{1} \overset{r}{\to} \mathbf{0},$$

we have $n \leqslant j \overset{r}{\to} m$. But

$$a_0 \leqslant (\mathbf{1} \overset{r}{\to} \mathbf{0})(a_0) \nleqslant \tau \overset{r}{\to} a_0 = \mathbf{1}(a_0) \overset{r}{\to} \mathbf{0}(a_0).$$

Next, we point out that the set $\tilde{N}Q$ of all localic (quantic) nuclei of Q is a locale with $\& = \wedge$; this generalizes the classical result that NL is a locale for each locale L.

Theorem 2. *Let Q be a quantale. Then $\tilde{N}Q$ is a locale.*

The proof of Theorem 2 follows directly from [**5**], 3·12, 3·14 and the classical result for locales. It is possible to give another proof without using the classical result for locales (cf. [**2**], p. 51). In fact, for each $j, k \in \tilde{N}Q$, define

$$(j \overset{l}{\to} k)(a) = \bigwedge \{j(b) \overset{l}{\to} k(b) \mid b \geqslant a\},$$

and

$$(j \overset{r}{\to} k)(a) = \bigwedge \{j(b) \overset{r}{\to} k(b) \mid b \geqslant a\}.$$

Then we can show that $j \xrightarrow{l} k$ is a localic quantic nucleus, i.e. $j \xrightarrow{l} k \in \tilde{N}Q$, and show that $j \xrightarrow{r} k = j \xrightarrow{l} k$ which defines the Heyting implication operation in $\tilde{N}Q$.

COROLLARY (Isbell, Johnstone [**2**]). *If L is a locale, then NL is a locale.*

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