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Unifying Exact Completions

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Abstract We define the notion of exact completion with respect to an existential elementary doctrine. We observe that the forgetful functor from the 2-category of exact categories to existential elementary doctrines has a left biadjoint that can be obtained as a composite of two others. Finally, we conclude how this notion encompasses both that of the exact completion of a regular category as well as that of the exact completion of a category with binary products, a weak terminal object and weak pullbacks.

Keywords Exact category · Elementary existential doctrine · Free construction · Tripos

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1 Introduction

The notion of completion by quotients, and in particular that of exact completion, has been widely studied in category theory, see for example [5, 7, 14]. The concept of quotient completion is pervasive not only in mathematics but also in computer science, in particular for what concerns how proofs are formalized in a computer-assisted way in an intensional set theory that does not carry quotient sets as primitive notion.

In [22] the authors began to study a categorical structure involved with quotient completions, relativizing the basic concept to a doctrine equipped with a logical structure sufficient

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to describe the notion of an equivalence relation. The notion of quotient within an elementary doctrine and that of elementary quotient completion producing a quotient completion that is not generally exact but encompasses relevant examples used in type theory were introduced in [21].

In the present paper, that analysis of quotient completion is pushed further viewing the exact completion of a regular category or the exact completions of a category with binary products, a weak terminal object and weak pullbacks as instances of a more general "exact completion" with respect to an elementary existential doctrine.

Indeed, for an exact category $\mathcal{X}$, the indexed inf-semilattice $\mathrm{Sub}_{\mathcal{X}}: \mathcal{X}^{\mathrm{op}} \longrightarrow$ **InfSL** of subobjects, which assigns to an object A in $\mathcal{X}$ the poset $S(A)$ of subobjects of A in $\mathcal{X}$, constitutes the archetypal example of a fibrations of sets and functions as all known frameworks for modelling a constructive theory of sets produce exact categories, *e.g.* toposes as models of IZF or arising from a tripos, categories of classes for CZF, total setoids à la Bishop on Martin-öf's type theory [24]. Since, within a set theory, functions are defined from the logic, it is of little surprise that the models are obtained from indexed inf-semilattices which are existential elementary doctrines.

We show that many of the models are obtained as a free construction. Indeed, the forgetful functor from the 2-category of exact categories to that of existential elementary doctrines has a left biadjoint that can be obtained as a composite of two others: the first adds (full) comprehensions to an existential elementary doctrine, the other turns an existential elementary doctrine with full comprehension into (the fibration of subobjects of) an exact category, universally so. In particular, when the second is applied to the doctrine of subobjects of a regular category, it gives rise to its exact completion, see [11].

For an existential elementary doctrine P, the elementary quotient completion of P presented in [21] appears as a subcategory of the exact completion of P by the universal properties of the various constructions involved. There are interesting cases when that inclusion is an equivalence; for instance, when P is the poset indexed doctrine $\Psi_{\mathcal{C}}$ of weak subobjects of a category $\mathcal{C}$ with binary products and weak pullbacks. Thus also the exact completion on a category with binary products, a weak terminal object and weak pullbacks is an instance of the exact completion of an elementary existential doctrine as the elementary quotient completion of $\Psi_{\mathcal{C}}$ coincides with the exact completion of $\mathcal{C}$ as a weakly lex category, see *loc.cit.*.

2 Elementary Existential Doctrines

A doctrine subsumes the basic categorical concept of a logic. The notion was introduced, in a series of seminal papers, by F.W. Lawvere to synthetize the structural properties of logical systems, see [16–18], see also [14, 19] for a unified survey. Lawvere's crucial intuition was to consider logical languages and theories as fibrations to study their 2-categorical properties, *e.g.* connectives and quantifiers are determined by structural adjunctions, see for instance [3].

Recall that, for a category $\mathcal{C}$ with binary products, an ***elementary doctrine*** (on $\mathcal{C}$) is a $\mathcal{C}$-indexed inf-semilattice, that is a functor $P: \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** from the opposite category of $\mathcal{C}$ to the category of inf-semilattices and homomorphisms such that, for every object A in $\mathcal{C}$, there is an object δ_A in $P(A \times A)$ and

(i) the assignment

$$\mathit{\exists}_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$

for α in $P(A)$ determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \to P(A)$—the action of a doctrine P on an arrow is written as P_f

(ii) for every map $e := \langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle\colon X \times A \to X \times A \times A$ in $\mathcal{C}$, the assignment

$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \to P(X \times A)$.

Also recall from *loc.cit.* that an ***existential doctrine*** is an indexed inf-semilattice $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ such that, for A_1 and A_2 in $\mathcal{C}$ and projections $\mathrm{pr}_i\colon A_1 \times A_2 \to A_i$, $i = 1, 2$, the functors $P_{\mathrm{pr}_i}\colon P(A_i) \to P(A_1 \times A_2)$ have a left adjoint $\exists_{\mathrm{pr}_i}$ which satisfy

Beck-Chevalley condition: for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"] & A \end{tikzcd}$$

with pr a projection (hence also pr′ a projection), for any β in $P(X)$, the canonical arrow $\exists_{\mathrm{pr}'} P_{f'}(\beta) \leq P_f \exists_{\mathrm{pr}}(\beta)$ in $P(A')$ is iso;

Frobenius reciprocity: for $\mathrm{pr}\colon X \to A$ a projection, α in $P(A)$, β in $P(X)$, the canonical arrow $\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{\mathrm{pr}}(\beta)$ in $P(A)$ is iso.

Remark 2.1 Note for an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ that, in case $\mathcal{C}$ has a terminal object, conditions (ii) entails condition (i).

Also, given α_1 in $P(X_1 \times Y_1)$ and α_2 in $P(X_2 \times Y_2)$, if one writes $\alpha_1 \boxtimes \alpha_2$ for the object

$$P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\alpha_1) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_4 \rangle}(\alpha_2)$$

in $P(X_1 \times X_2 \times Y_1 \times Y_2)$ where pr_i, $i = 1, 2, 3, 4$, are the projections from $X_1 \times X_2 \times Y_1 \times Y_2$ to each of the four factors, then condition (ii) is to require that $\delta_{A \times B} = \delta_A \boxtimes \delta_B$ for every pair of objects A and B in $\mathcal{C}$.

Beyond the standard example of the elementary existential doctrine of subobjects of a regular category $\mathcal{X}$, one can consider examples directly from logic such as the indexed Lindenbaum-Tarski algebras $LT\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of well-formed formulae of a theory $\mathscr{T}$ with equality in a first order language $\mathscr{L}$ where the domain category $\mathcal{V}$ has lists of variables as objects and term substitutions as arrows, with composition given by simultaneous substitution; the functor $LT\colon \mathcal{V}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ takes a list of variables to the Lindenbaum-Tarski algebra of equivalence classes of well-formed formulae of $\mathscr{L}$ whose free variables are within $x_1,\ldots,x_n$, see [22] for more details.

An important example for theories developed for formalizing constructive mathematics is the following: Consider a category $\mathcal{S}$ with (strong) binary products and weak pullbacks and the functor of *weak subobjects* $\Psi\colon \mathcal{S}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ which evaluates, at an object A of $\mathcal{S}$, as the poset reflection of each comma category $\mathcal{S}/A$. The left adjoints are computed by post-composition. We refer the reader to [21, 22] for further details.

We consider the 2-category **ED**: its objects are elementary doctrines, 1-arrows are pairs (F, b) consisting of a functor F and a natural transformation b as indicated

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"'] \arrow[rr, "P"] & {} \arrow[d, phantom, "b \downarrow\cdot"] & \mathbf{InfSL} \\
\mathcal{D}^{\mathrm{op}} \arrow[urr, "R"'] & {} &
\end{tikzcd}$$

such that F preserves binary products, and $b_{A\times A}(\delta_A) = R_{\langle F(\mathrm{pr}_1), F(\mathrm{pr}_2)\rangle}(\delta_{F(A)})$ for every object A in $\mathcal{C}$. The 2-arrows of **ED** are natural transformations $\theta\colon F \dot\to G$ such that

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[d, bend right, "F^{\mathrm{op}}"'] \arrow[d, bend left, "G^{\mathrm{op}}"] \arrow[d, phantom, "\overset{\cdot}{\longleftarrow}\ \theta^{\mathrm{op}}"] \arrow[rr, "P"] & {} \arrow[d, phantom, "b\cdot \left(\le \right) \cdot c"] & \mathbf{InfSL} \\
\mathcal{D}^{\mathrm{op}} \arrow[urr, "R"'] & {} &
\end{tikzcd}$$

so that, for every object A in $\mathcal{C}$ and every α in $P(A)$, one has $b_A(\alpha) \le R_{\theta_A}(c_A(\alpha))$.

The 2-category **EED** is the 1-full subcategory of **ED** on elementary existential doctrines where 2-arrows have each component b_A preserving the existential adjoints.

As we already mentioned in the Introduction, since the indexed inf-semilattice $\mathrm{Sub}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ of subobjects for an exact category $\mathcal{X}$ is elementary existential, that construction induces an obvious forgetful functor from the 2-category **Xct** of exact categories and exact functors to **EED**, as in [1, 7].

In [21] the authors presented a construction to add quotients to an elementary doctrine freely. A similar construction is that used to produce a topos from a tripos, see [13, 26, 28], and it produces a left biadjoint to the forgetful 2-functor from **Xct** to **EED**. The two constructions will be compared in Section 4.

Definition 2.2 Given an elementary existential doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, consider the category $\mathcal{T}_P$, called ***exact completion of the e.e.d.*** P, whose

objects are pairs (A, ρ) such that ρ is in $P(A \times A)$ and satisfies

(a) $\rho \le P_{\langle p_2, p_1\rangle}(\rho)$ in $P(A \times A)$
where $p_1, p_2\colon A \times A \to A$ are the two projections

(b) $P_{\langle p_1, p_2\rangle}(\rho) \wedge P_{\langle p_2, p_3\rangle}(\rho) \le P_{\langle p_1, p_3\rangle}(\rho)$ in $P(A \times A \times A)$
where $p_1, p_2, p_3\colon A \times A \to A$ are the projections

an arrow $\phi\colon (A, \rho) \to (B, \sigma)$ is an object ϕ in $P(A \times B)$ such that

(i) $\phi \le P_{\langle p_1, p_1\rangle}(\rho) \wedge P_{\langle p_2, p_2\rangle}(\sigma)$

(ii) $P_{\langle p_1, p_2\rangle}(\rho) \wedge P_{\langle p_2, p_3\rangle}(\phi) \le P_{\langle p_1, p_3\rangle}(\phi)$ in $P(A \times A \times B)$
where the p_i's are appropriate projections

(ii) $P_{\langle p_1, p_2\rangle}(\phi) \wedge P_{\langle p_2, p_3\rangle}(\sigma) \le P_{\langle p_1, p_3\rangle}(\phi)$ in $P(A \times B \times B)$
where, again, the p_i's are appropriate projections

(iv) $P_{\langle p_1, p_2\rangle}(\phi) \wedge P_{\langle p_1, p_3\rangle}(\phi) \le P_{\langle p_2, p_3\rangle}(\sigma)$ in $P(A \times B \times B)$
where the p_i's are as before

(v) $P_{\langle p_1, p_1\rangle}(\rho) \le \exists_{p_2}(\phi)$ in $P(A)$
where $p_1\colon A \times B \to A$ and $p_2\colon A \times B \to B$ are the projections

where composition $(A,\rho)\xrightarrow{\phi}(B,\sigma)\xrightarrow{\psi}(C,\tau)$ is defined as

$$\exists_{p_2}\left(P_{\langle p_1,p_2\rangle}(\phi)\wedge P_{\langle p_2,p_3\rangle}(\psi)\right)$$

and identity is $(A,\rho)\xrightarrow{\rho}(A,\rho)$

Example 2.3 The main examples of this construction are toposes obtained from a tripos, see [13, 26, 28].

Remark 2.4 It is quite apparent that the elementary structure plays no role in the definitions in 2.2—but it will be crucial for 3.3. We refer the reader to [25] for an analysis of that.

Remark 2.5 The logical relevance of 2.2 is exposed if one considers the allegory $\mathcal{A}_P$ of relations of an elementary existential doctrine $P\colon\mathcal{C}^{\mathrm{op}}\longrightarrow$ **InfSL**, see [11], whose objects are those of $\mathcal{C}$ and the poset of 1-arrows from A to B is $P(A\times B)$. Composition of 1-arrows $A\xrightarrow{\theta}B\xrightarrow{\zeta}C$ is

$$\exists_{p_2}\left(P_{\langle p_1,p_2\rangle}(\theta)\wedge P_{\langle p_2,p_3\rangle}(\zeta)\right)$$

with identities given by δ_A. The opposite θ of a 1-arrow $A\xrightarrow{\theta}B$ is given by $B\xrightarrow{P_{\langle p_2,p_1\rangle}(\theta)}A$.

If one then takes maps in the splitting (allegory) of the "symmetric idempotents" of $\mathcal{A}_P$, one gets exactly the category $\mathcal{T}_P$, see [6].

The locally posetal category $\mathcal{A}_P$ is also a cartesian bicategory, see [8]. The product functor of the base extends to a symmetric tensor $\boxtimes$ as in 2.1. The structure of commutative comonoid on each object A is given by

$$1 \xleftarrow{\top} A \xrightarrow{P_{\langle p_1,p_2\rangle}(\delta)\wedge P_{\langle p_1,p_3\rangle}(\delta)} A\times A$$

Note that the computation of the opposite θ° of a 1-arrow $A\xrightarrow{\theta}B$ in the cartesian bicategory gives precisely the 1-arrow $B\xrightarrow{P_{\langle p_2,p_1\rangle}(\theta)}A$, see [8].

The conditions (ii)–(v) in 2.2 are written in the notation of the bicategory respectively as

$$\begin{tikzcd}
A \arrow[r, bend left, "\rho"] \arrow[rr, bend right, "\phi"'] \arrow[rr, phantom, "\rotatebox{-90}{$\leq$}"] & A \arrow[r, bend left, "\phi"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, bend left, "\phi"] \arrow[rr, bend right, "\phi"'] \arrow[rr, phantom, "\rotatebox{-90}{$\leq$}"] & B \arrow[r, bend left, "\sigma"] & B
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, bend left, "\phi^\circ"] \arrow[rr, bend right, "\sigma"'] \arrow[rr, phantom, "\rotatebox{-90}{$\leq$}"] & A \arrow[r, bend left, "\phi"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, bend left, "\rho"] \arrow[r, bend right, "\phi"'] \arrow[rr, phantom, "\rotatebox{-90}{$\leq$}"] & B \arrow[r, bend right, "\phi^\circ"'] & A
\end{tikzcd}$$

We shall find it easy to obtain the construction of $\mathcal{T}_P$ as the composite of two left biadjoints to forgetful functors:

(i) the left biadjoint to the inclusion of the 1-full 2-subcategory **CEED** of **EED** on those elementary existential doctrines with full comprehensions;

(ii) the left biadjoint to the forgetful functor from **Xct** to **CEED** which takes an exact category $\mathcal{X}$ to the doctrine $\mathrm{Sub}_{\mathcal{X}}\colon\mathcal{X}^{\mathrm{op}}\longrightarrow$ **InfSL** of subobjects of $\mathcal{X}$.

3 The Left Biadjoints

Recall that, for a doctrine $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ and for an object α in some $P(A)$, a ***comprehensions*** of α is a map $\{\!\!\{\alpha\}\!\!\}: X \to A$ in $\mathcal{C}$ such that $P_{\{\!\!\{\alpha\}\!\!\}}(\alpha) = \top_X$ and, for every $f: Z \to A$ such that $P_f(\alpha) = \top_Z$ there is a unique map $g: Z \to X$ such that $f = \{\!\!\{\alpha\}\!\!\} \circ g$. One says that P ***has comprehensions*** if every α has a comprehension, and that P ***has full comprehensions*** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\!\!\{\alpha\}\!\!\}$ factors through $\{\!\!\{\beta\}\!\!\}$.

As we may need also the weakened form of comprehension, recall that a ***weak comprehension*** of α is a map $c: W \to A$ in $\mathcal{C}$ such that $P_c(\alpha) = \top_W$ and, for every $f: Z \to A$ such that $P_f(\alpha) = \top_Z$ there is a (not necessarily unique) map $g: Z \to X$ such that $f = c \circ g$.

Recall from [14] that the fibration of vertical maps on the category of points freely adds comprehensions to a given fibration producing an indexed poset in case the given fibration is such. For a doctrine $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, the indexed poset consists of the base category $\mathcal{G}_P$ of points where

an object is a pair (A, α) where A is in $\mathcal{C}$ and α is in $P(A)$
an arrow $f: (A, \alpha) \to (B, \beta)$ is an arrow $f: A \to B$ in $\mathcal{C}$ such that $\alpha \leq P_f(\beta)$.

Since the fibres of P are inf-semilattices, the category $\mathcal{G}_P$ has binary products and there is a natural embedding $I: \mathcal{C} \to \mathcal{G}_P$ which maps A to $(A, \top_A)$. The indexed functor extends to $(P)_{\mathrm{c}}: {\mathcal{G}_P}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ along I by setting $(P)_{\mathrm{c}}(A, \alpha) := \{\gamma \in P(A) \mid \gamma \leq \alpha\}$. Moreover, the comprehensions in $(P)_{\mathrm{c}}$ are full.

Theorem 3.1 *There is a left bi-adjoint to the inclusion of* **CEED** *into* **EED**.

Proof It is enough to check that, when $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential, the doctrine $(P)_{\mathrm{c}}: {\mathcal{G}_P}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$ is existential and the pair $(I, \mathrm{id}_P): P \to (P)_{\mathrm{c}}$ preserves them. □

For the next step it is useful to recall three results about fibrations with full comprehensions and about regular and exact categories:

The first is in [12]: in the notation introduced above, it states that there is a biequivalence between **CEED** and the 2-category **LFS** of categories with finite limits and a proper stable factorization system (with left exact functors preserving the factorization).

The second is in [15] and shows that the inclusion of the 2-category **Reg** of regular categories (with exact functors) into **LFS** has a left biadjoint: the left biadjoint to the inclusion is computed on a category $\mathcal{B}$ with stable proper factorization system $(\mathcal{E}, \mathcal{M})$ as the category of maps for the cartesian bicategory of $\mathcal{M}$-relations in $\mathcal{B}$.

The third is the result from [11, 27] that the inclusion into **Reg** of the full 2-subcategory **Xct** on exact categories has a left biadjoint, which we shall denote as $(-)_{\mathrm{ex/reg}}: \mathbf{Reg} \longrightarrow \mathbf{Xct}$.

The composite of the three left biadjoints is a 2-functor $\mathbf{CEED} \longrightarrow \mathbf{Xct}$ which, for an elementary existential doctrine $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{InfSL}$, gives the full subcategory $\mathcal{E}_P$ of $\mathcal{T}_P$ on those objects (A, ρ) such that

$$\top_A \leq P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle} \rho$$

—or, equivalently, $\delta_A \leq \rho$.

Following [22] we shall refer to such an object ρ in $P(A \times A)$ as a ***P-equivalence relation on*** A. Condition 2.2(i) for arrows in $\mathcal{E}_P$ becomes redundant and condition 2.2(v) can be reduced to $\top_A \leq \exists_{p_2}(\phi)$. For each object A in $\mathcal{C}$, one can consider the object (A, δ_A) in $\mathcal{E}_P$, and such assignment extends to a functor $D: \mathcal{C} \to \mathcal{E}_P$ mapping an arrow $f: A \to B$

to the relation $\mathcal{I}_{\langle \mathrm{id}_A, f\rangle}(\top_A) = P_{(f\times \mathrm{id}_B)}(\delta_B)$. In turn, it gives rise to a 1-arrow from P to the indexed inf-semilattice of subobjects $\mathrm{Sub}_{\mathcal{E}_P}\colon {\mathcal{E}_P}^{\mathrm{op}} \longrightarrow$ **InfSL** since $\mathrm{Sub}_{\mathcal{E}_P}(A, \delta_A) \xrightarrow{\sim} P(A)$.

Example 3.2 The leading example of the above construction $\mathcal{E}_P$ is the exact completion $\mathcal{X}_{\mathrm{ex/reg}}$ [4, 7, 11] of a regular category $\mathcal{X}$, which coincides with $\mathcal{E}_{\mathrm{Sub}_{\mathcal{X}}}$ for the doctrine $\mathrm{Sub}_{\mathcal{X}}\colon \mathcal{X}^{\mathrm{op}} \longrightarrow$ **InfSL** of subobjects of $\mathcal{X}$.

Other examples come from theories apt to formalize constructive mathematics: the category of total setoids à la Bishop and functional relations based on the minimalist type theory in [20], which coincides with the construction $\mathcal{E}_{G^{\mathbf{mtt}}}$ where the doctrine $G^{\mathbf{mtt}}$ is defined as in [22], or the category of total setoids à la Bishop and functional relations based on the Calculus of Constructions [9], which forms a topos as mentioned in [2] and coincides with $\mathcal{E}_{G^{\mathrm{CoC}}}$ where the doctrine G^{CoC} is constructed from the Calculus of Construction as $G^{\mathbf{mtt}}$ in [22].

Theorem 3.3 *For every elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** with full comprehensions, pre-composition with the 1-arrow*

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[dd, "D^{op}"'] \arrow[dr, "P"] & \\ {} \arrow[r, phantom, "\Downarrow \mathrm{id}_P"] & \mathbf{InfSL} \\ {\mathcal{E}_P}^{op} \arrow[ur, "\mathrm{Sub}_{\mathcal{E}_P}"'] & \end{tikzcd}$$

in **CEED** *induces an equivalence of categories*

$$- \circ (D, \mathrm{id}_P)\colon \mathbf{CEED}(\mathrm{Sub}_{\mathcal{E}_P}, \mathrm{Sub}_{\mathcal{X}}) \equiv \mathbf{CEED}(P, \mathrm{Sub}_{\mathcal{X}})$$

for every $\mathcal{X}$ in **Xct**.

Corollary 3.4 *The action of the left biadjoint to the 2-functor* **Xct** $\longrightarrow$ **EED** *that takes an exact category to the elementary existential doctrine of its subobjects is given by $\mathcal{T}_P$ on each elementary existential doctrine P.*

Proposition 3.5 *If the elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** has full comprehensions, then the inclusion of $\mathcal{E}_P$ into $\mathcal{T}_P$ is an equivalence of categories.*

Proof It is sufficient to note that, since P has full comprehensions, for any α in $P(A)$, one has $\alpha = \mathcal{I}_{\{\alpha\}}\top$. Hence

$$\mathbf{EED}(P, \mathrm{Sub}_{\mathcal{X}}) \equiv \mathbf{CEED}(P, \mathrm{Sub}_{\mathcal{X}})$$

for any regular category $\mathcal{X}$. □

Remark 3.6 The statement in 3.5 holds also when the elementary existential doctrine P has just weak full comprehension. We suspect that this is related to the analysis carried out by Jonas Frey on pre-equipments of triposes in [10].

4 Comparing Quotient Completions

In [21], the authors considered a completion for quotients of an elementary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **InfSL** which compares with the one presented in the previous section when

P is also existential. Recall from *loc.cit.* that the elementary quotient completion $\mathcal{Q}_P$ of P consists of

objects which are pairs (A, ρ) such that ρ is a P-equivalence relation on A,
an arrow $[f]: (A, \rho) \to (B, \sigma)$ is an equivalence class of arrows $f: A \to B$ in $\mathcal{C}$ such that $\rho \leq P_{f \times f}(\sigma)$ in $P(A \times A)$ with respect to the relation determined by the condition that $\rho \leq P_{f \times g}(\sigma)$

Composition is given by that of $\mathcal{C}$ on representatives, and identities are represented by identities of $\mathcal{C}$.

The indexed partial inf-semilattice $\widehat{P}: \mathcal{Q}_P{}^{\mathrm{op}} \longrightarrow \textsf{InfSL}$ on $\mathcal{Q}_P$ is defined on an object (A, ρ) as

$$\widehat{P}(A, \rho) := \mathcal{D}es_\rho$$

where $\mathcal{D}es_\rho$ is the sub-order of $P(A)$ on those α such that $P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq P_{\mathrm{pr}_2}(\alpha)$, where $\mathrm{pr}_1, \mathrm{pr}_2: A \times A \to A$ are the projections.

By Theorem 6.1 in [22], when P is existential with (weak) full comprehensions, also $\widehat{P}$ is existential. Since clearly $\mathrm{Sub}_{\mathcal{E}_P}$ has quotients, there is a canonical arrow

$$\begin{tikzcd}
\mathcal{Q}_P{}^{\mathrm{op}} \arrow[dr, "\widehat{P}"] \arrow[dd, "L^{\mathrm{op}}"'] \arrow[dd, phantom, shift left=6ex, "\ell\,\Downarrow\cdot"] & \\
 & \textsf{InfSL} \\
\mathcal{E}_P{}^{\mathrm{op}} \arrow[ur, "\mathrm{Sub}_{\mathcal{T}_P}"'] &
\end{tikzcd}$$

of elementary existential doctrines which preserves quotients.

It is easy to see that the action of L on objects is the identity and that the components of ℓ are identity homomorphisms. For an arrow $[f]: (A, \rho) \to (B, \sigma)$ in $\mathcal{Q}_P$

$$L[f] = \mathcal{I}_{(\mathrm{pr}_1, \mathrm{pr}_3)} \left(P_{(\mathrm{pr}_1, \mathrm{pr}_2)}(\rho) \wedge P_{(f \circ \mathrm{pr}_2, \mathrm{pr}_3)}(\sigma)\right)$$

where pr denotes a projection from $A \times A \times B$. Note that the construction of L can be performed for any elementary existential doctrine P and that clearly L is faithful.

Example 4.1 An interesting example of the comparison above appears in [10] applied to the doctrine $(P)_{\mathrm{c}}: \mathcal{G}_P{}^{\mathrm{op}} \longrightarrow \textsf{InfSL}$ for $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{InfSL}$ a tripos and it is used to analyze the tripos-to-topos construction in a refined 2-categorical setup of pre-equipments.

Theorem 4.2 *Suppose that $P: \mathcal{C}^{op} \longrightarrow$ $\textsf{InfSL}$ is an elementary existential doctrine with weak full comprehensions. Suppose moreover that, for every A and B in $\mathcal{C}$ and for every α in $P(A \times B)$ such that $\top_A \leq \mathcal{I}_{\mathrm{pr}}(\alpha)$ where* pr: $A \times B \to A$ *is the first projection, there is an arrow $w: A \to B$ in $\mathcal{C}$ such that $\top_A \leq P_{(\mathrm{id}_A, w)}(\alpha)$. Then the functor $L: \mathcal{Q}_P \to \mathcal{E}_P$ is an equivalence.*

Proof There is only to prove that L is full. So, given an arrow $\phi: (A, \rho) \to (B, \sigma)$ in $\mathcal{E}_P$, it is $\top_A \leq \mathcal{I}_{p_2}(\phi)$. By hypothesis, there is $f: A \to B$ in $\mathcal{C}$ such that $\top_A \leq P_{(\mathrm{id}_A, f)}(\phi)$, or equivalently $\mathcal{I}_{\mathrm{id}_A \times f}(\delta_A) \leq \phi$. It is then easy to see that $\phi = L[f]$. $\square$

Remark 4.3 For $P: \mathcal{C}^{\mathrm{op}} \longrightarrow \textsf{InfSL}$ an elementary existential doctrine with full comprehensions, it is possible to prove a converse to 4.2 under the further hypothesis that every reflexive P-relation has a smallest transitive extension, *i.e.* for every object C in $\mathcal{C}$ and every object ζ in $P(C \times C)$ such that $\delta_C \leq \zeta$, there is an object ζ^t in $P(C \times C)$ such that

$$\zeta \leq \zeta^t \qquad P_{\langle p_1,p_2\rangle}(\zeta^t) \wedge P_{\langle p_2,p_3\rangle}(\zeta^t) \leq P_{\langle p_1,p_3\rangle}(\zeta^t)$$

where $p_1, p_2, p_3: C \times C \to C$ are the projections, and ζ^t is smallest with those three properties.

It is easy to see that ζ^t is symmetric when ζ is such.

Given α in $P(A \times B)$ such that $\top_A \leq \exists_{\mathrm{pr}_1}(\alpha)$, we may assume with no loss of generality that $\top_B \leq \exists_{\mathrm{pr}_2}(\alpha)$ since P has full comprehensions—$\mathrm{pr}_1: A \times B \to A$ and $\mathrm{pr}_2: A \times B \to B$ are the two projections. The P-relation $\zeta := \exists_{\mathrm{pr}'_1}\left(P_{(\mathrm{pr}'_1,\mathrm{pr}'_2)}(\alpha) \wedge P_{(\mathrm{pr}'_1,\mathrm{pr}'_3)}(\alpha)\right)$ is reflexive and symmetric in $P(B \times B)$. Hence $\alpha: (A, \delta_A) \to (B, \zeta^t)$ is an arrow in $\mathcal{E}_P$. Since L is an equivalence, there is $[w]: (A, \delta_A) \to (B, \zeta^t)$ in $\mathcal{Q}_P$ such that $L[w] = \alpha$, thus $\top_A \leq P_{(\mathrm{id}_A,w)}(\alpha)$.

Remark 4.4 The leading example of exact completion satisfying the hypothesis of 4.2 is that of exact completion of a category with binary products, a weak terminal object and weak pullbacks [4, 5, 7]. It is $\mathcal{E}_\Psi$ where $\Psi: \mathcal{S}^{\mathrm{op}} \longrightarrow$ **InfSL** is the functor of weak subobjects.

Note that the problem pointed out in [7] *3.3*, p. 103, remains in the present context of doctrines with just *weak* comprehensions as the result in 4.2 does not extend to something similar to 3.3.

Example 4.5 Another relevant application of 4.2 is for the doctrine F^{ML} in [22] giving rise to the total setoid model of Martin-Löf's type theory in [24].

Note also that the second stage of the construction of Joyal's arithmetic universes in [23], which is the category of decidable predicates $\mathrm{Pred}(\mathcal{S})$ on a Skolem theory $\mathcal{S}$, is a regular category and coincides with the base category of the doctrine obtained by adding full comprehension and forcing extensionality in the sense of [21] to the elementary doctrine of decidable predicates on the Skolem category $\mathcal{S}$. Since epis split in $\mathrm{Pred}(\mathcal{S})$, this is an example where the hypothesis of 4.2 holds for the doctrine of subobjects of the regular category $\mathrm{Pred}(\mathcal{S})$.

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