Arbitrary products are coproducts in complete ($\vee$-) semilattices

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It is well known that in the category of sets, products and coproducts fail to be isomorphic since coproduct is disjoint union. Certain categories of universal algebras (e.g. modules, abelian groups, etc.) provide examples of set-based categories where finite products and coproducts coincide, but arbitrary products and coproducts need not be isomorphic [cf. 2, p. 86]. Here we present an example of a set-based category (which is 'algebraic', although it involves a proper class of operations) where *arbitrary* products and coproducts coincide.

Note that one cannot expect to find such an example among the usual categories of (universal) algebras, possibly with infinitary operations allowed. Suppose $\mathscr{A}$ is a nontrivial complete (i.e. closed under the formation of arbitrary products and subobjects) category of universal algebras. Let $\mathscr{M}$ be the smallest infinite regular cardinal greater than the arities of all operations in $\mathscr{A}$ and greater than the number of primitive or defining operations. Suppose A is a finitely generated free algebra on more than one generator; then $1 \leqslant \|A\| \leqslant \mathscr{M}$. Assume that $A^{\mathscr{M}}$ represents the $\mathscr{M}$-fold coproduct of copies of A with natural injections $u_\alpha : A \to A^{\mathscr{M}}$ for $\alpha < \mathscr{M}$. Then $A^{\mathscr{M}}$ would be generated by $\bigcup u_\alpha(A)$, and hence has at most $\mathscr{M}$ elements, since any $\mathscr{M}$-generated algebra in A has at most $\mathscr{M}$ elements (from the choice of $\mathscr{M}$). If products and coproducts were isomorphic in $\mathscr{A}$, then $2^{\mathscr{M}} = \|A^{\mathscr{M}}\| = \mathscr{M}$, a contradiction.

The promised example is the category $\mathscr{S}$ of complete ($\vee$-) semilattices, i.e. the objects of $\mathscr{S}$ are complete posets [1, p. 29] and the morphisms are the (poset) maps preserving all suprema. Products in $\mathscr{S}$ are set-theoretic products with order defined 'component-wise'. For a family $\{S_i \mid i \in I\}$ of complete semilattices and each $j \in I$, define $u_j : S_j \to \prod : S_i$ to be the map such that

$$p_i u_j = \begin{cases} \mathrm{id} & \text{if } i = j \\ \theta_i & \text{if } i \neq j; \end{cases}$$

where $p_i \prod S_i \to S_i$ is the ith projection map and $\theta_i : S_j \to S_i$ is the constant map with value 0_i (the supremum in S_i of ϕ). It is clear that u_j preserve suprema. Furthermore, $\prod S_i$ is generated, as a complete semilattice, by $\bigcup u_j\ (S_j)$; i.e. $x \in \prod S_i$ implies $x = \bigvee u_j p_j x$. Suppose $\{f_i : S_i \to S \mid i \in I\}$ is any collection of morphisms in $\mathscr{S}$, then $f : \prod S_i \to \mathscr{S}$ given by $f((x_i)_{i \in I}) = \bigvee f_i(x_i)$ is easily seen to be the *unique* morphism in $\mathscr{S}$ such that $fu_i = f_i$ for every $i \in I$. Hence arbitrary products and coproducts coincide in $\mathscr{S}$.

Presented by G. Grätzer. Received October 2, 1974. Accepted for publication in final form August 12, 1975.

Note that the same argument yields a similar result for the category $\mathscr{S}_{\mathscr{M}}$ of $\mathscr{M}$-complete semilattices, where $\mathscr{M}$ is an infinite regular cardinal. $\mathscr{S}_{\mathscr{M}}$ is the category whose objects are posets in which every set with fewer than $\mathscr{M}$ elements has a supremum, and whose morphisms are maps which preserve suprema of sets with fewer than $\mathscr{M}$ elements. Then in $\mathscr{S}_{\mathscr{M}}$, the product of any family which is indexed by fewer than $\mathscr{M}$ elements is the coproduct of that family. In view of the above remarks, this is the best one can do.

Statement of priority

The principal result of this note was obtained by the author early in 1967 while a thesis student of Alex Heller. She is indebted to the referee for the information that A. G. Waterman lectured on a similar result in the fall of 1966.

REFERENCES

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