The Algebraic Approach I: The Algebraization of the Chomsky Hierarchy

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Abstract. The algebraic approach to formal language and automata theory is a continuation of the earliest traditions in these fields which had sought to represent languages, translations and other computations as expressions (e.g. regular expressions) in suitably-defined algebras; and grammars, automata and transitions as relational and equational systems over these algebras, that have such expressions as their solutions. The possibility of a comprehensive foundation cast in this form, following such results as the algebraic reformulation of the Parikh Theorem, has been recognized by the Applications of Kleene Algebra (AKA) conference from the time of its inception in 2001.

Here, we take another step in this direction by embodying the Chomsky hierarchy, itself, within an infinite complete lattice of algebras that ranges from dioids to quantales, and includes many of the forms of Kleene algebras that have been considered in the literature. A notable feature of this development is the generalization of the Chomsky hierarchy, including type 1 languages, to arbitrary monoids.

Keywords: Kleene, Language, Context-Free, Regular Expression, Rational, Monoid, Semigroup, Dioid, Quantale, Grammar.

1 The Algebraic Point of View

From its inception, the Applications in Kleene Algebra conference has recognized the possibility of a comprehensive algebraic foundation for formal language and automata theory:

"Recent algebraic versions of classical results in formal language theory, e.g. Parikh's theorem [1], point to the exciting possibility of a general algebraic theory that subsumes classical combinatorial automata and formal language theory [pointing] to a much more general, purely axiomatic theory in the spirit of modern algebra."[1]

An additional step shall be taken in this direction, here, by recasting the the Chomsky hierarchy in algebraic form as an infinite complete lattice of algebras

[1] Programme introduction, Applications of Kleene Algebra, Schloss Dagstuhl, February 2001.

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ranging from dioids to quantales. The synthesis provided by the dioid-quantale hierarchy, introduced here, brings fully to bear the power of monads and adjunctions. Much of these developments were foreshadowed by Kozen [5], where implicit use was made of the monad concept to develop a hierarchical relation between different varieties of Kleene algebras. Earlier work has been carried out by Conway [6] in the study of the algebra that came to be known as the Quantale, the *-continuous Kleene Algebra, and the "countably-closed dioid".

In a separate line of development, the quantale and dioid have also emerged in the 1980's in Quantum Physics (hence the name "quantale"), particularly in the study of C*-algebras and von Neumann algebras, in non-linear dynamics, linear logic, Penrose tilings, discrete event systems ([8,9,10,11,12,13,14,15], see also [4] and references contained therein), and related fields (e.g. *Idempotent Analysis* from Maslov [7], et. al.)

1.1 Preliminaries

The notions of semigroups, monoids, partial orderings, semi-lattices and lattices are standard (e.g. [4,16,17]) and will not be dealt with in great depth here.

In the standard formulation of formal languages and automata, which we will refer to henceforth as the classical theory, a language is regarded as a subset of a free monoid $M = X^*$, though more general monoids may sometimes be considered, e.g. Parikh vectors over commutative monoids, translations and relations over direct products of monoids. Different families of *languages* over an alphabet X are then identified as distinguished families of subsets of a monoid X^*.[2] Along the way, one naturally encounters the issue of closure properties: is a given family closed under substitutions, morphisms, inverse morphisms, products, unions, etc.?

This specificity seems to extend to grammars: curiously, there seems to be an absence of the notion of grammars in the literature other than for free monoids. A formulation suitable for general monoids has therefore been provided in the appendix, where the algebraic concept of free extensions will emerge as a key element.

The monoid product $\cdot : M \times M \to M$ lifts to a product $\cdot : \mathcal{P}M \times \mathcal{P}M \to \mathcal{P}M$ over the power set by $AB = \{ab \in M : a \in A, b \in B\}$. This endows the power-set $\mathcal{P}M$ with the structure of a monoid containing that of M in virtue of the correspondence $\eta_M : a \mapsto \{a\}$ which embeds M into $\mathcal{P}M$ by virtue of the relations $\{a\}\{b\} = \{ab\}$ and $\{a\}\{1\} = \{a1\} = \{a\} = \{1a\} = \{1\}\{a\}$. Whereas the product operation may be thought of as embodying the primitive concept of sequentiality, the additional structure provided by the operators $0 = \emptyset$ and $A+B = A \cup B$ may be thought of as giving us a way to embody non-determinism. The ordering relation $A \geq B \Leftrightarrow A \supseteq B$ may then be identified as a precursor of the notions of derivability or transformation $A \to B$. In this setting, a grammar

[2] The analogous classification of *translations* from an alphabet X to another alphabet Y is then distinguished by the corresponding families of subsets of the product monoid $X^* \times Y^*$. This generalizes further to relations of ternary or higher degree.

or automaton may then be regarded as a way of writing down a system of relations. The principle of *finite derivability* is encoded by the requirement that the object (language, translation, relation, etc.) represented by the grammar or automaton should be the *least solution* in the corresponding relational system.

This is the essence of what may be termed the *Algebraic Approach*.

However, the definitions in the classical theory are cast almost entirely in set-theoretic terms, as are the arguments for the corresponding theorems, even though the ideas and the results frequently have a purely algebraic flavor and can be stated in such fashion often with both an increase in transparency and generality. As a result, the full potential of the results arrived at classically is missed. This discrepancy is what the algebraic approach seeks to rectify.

1.2 Dioids, Quantales and the Relational View

A *dioid* is also known as an idempotent semiring and may be defined by the identities $a\,(bc) = (ab)\,c$, $a1 = a = 1a$, $a + a = a$, $a\,(b + c)\,d = abd + acd$, $a + (b + c) = (a + b)\,c$, $a + 0 = a = 0 + a$, $a + b = b + a$ and $a0b = 0$.

In virtue of idempotency, $a + a = a$, such an algebra may be defined as a partially ordered monoid, with the "natural" partial ordering $a \geq b$ given by $\exists x : a = x + b$, or equivalently by $a = a + b$. Taking the ordering relation as primitive, addition may be defined as the least upper bound, characterizing it by the property that $x \geq a + b$ if and only if $x \geq a$ and $x \geq b$. The minimal element 0 is characterized by the property $x \geq 0$. A consequence of these axioms (see, e.g. [4]) is that both dioid operations $(a, b) \mapsto ab$ and $(a, b) \mapsto a + b$ are monotonic.

The partial ordering enters into formal language theory in various guises as *reducibility*, *derivability*, *transformation*, etc. The addition operator may then be regarded as a representation of the phenomenon of *non-deterministic branching*, the additive identity as that of *failure*.

This view of formal languages as a non-deterministic algebra for words leads to an alternate interpretation of the foregoing. A dioid D is equivalently described as a partially ordered monoid in which every finite subset $A \subseteq D$ has a least upper bound $\sum A \in D$ with $\sum \{a_1, \ldots, a_n\} = 0 + a_1 + \ldots + a_n$, which is assumed to be finitely distributive with respect to the product. Because of the finite distributivity property, the summation operator[3] $\Sigma : \mathcal{F}D \to D$ inherited from the semilattice will turn out to be a dioid homomorphism with $\sum (AB) = (\sum A)\,(\sum B)$, for $A, B \in \mathcal{F}D$; $\sum \{d\} = d$, for $d \in D$; and $a\,(\sum A)\,b = \sum (aAb)$, for $A \in \mathcal{F}D$ and $a, b \in D$.

The least upper bound operator $\Sigma : \mathcal{F}D \to D$ is thus seen to not only be a monoid homomorphism, but the left-adjoint of the monoid embedding $\eta_M : M \to \mathcal{F}M$ into the family $\mathcal{F}M$ of finite subsets of M.

The existence of such an operator for a given monoid M equivalently identifies M as a dioid. Thus $\mathcal{F}M$, itself, is the free dioid extension of M, and $\mathcal{F}X^*$ is

[3] The finite and countable subsets of a given set A will be denoted, respectively, $\mathcal{F}A$ and ωA.

the free dioid generated by a set X. In the context of formal languages, when X is a finite non-empty set representing an alphabet, the family $\mathcal{F}X^*$ may be identified as the family of *finite languages* over the alphabet X. In a more general algebraic context, a family of languages may therefore be regarded as forming a dioid with the additive operator $\cup$, partial ordering relation $\subseteq$, zero element $\emptyset$, multiplicative identity $\{1\}$ and set-wise concatenation as the product.

If least upper bounds exist for arbitrary subsets, with infinite distributivity, the result is the algebra known as a *quantale*.[4] The free quantale extension of a monoid M is just its powerset $\mathcal{P}M$. Correspondingly, the free quantale $\mathcal{P}X^*$ may be regarded as the general family of languages over X.

A similar consideration applies also to the other dioid varieties corresponding to the operators $M \mapsto \mathcal{R}M$ and $M \mapsto \omega M$ respectively to the rational and countable subsets of M. This leads to corresponding adjunction pairs, respectively, to the *-continuous Kleene algebras and closed semirings.

1.3 Kleene Algebras and Regular Expressions

The "process" view is expanded by treating also the notion of *unbounded repetition* or *iteration* by what is known as the Kleene star operator $a \mapsto a^*$. In the classical interpretation over the power set algebra $\mathcal{P}M$ such an operator may be defined as $A^* = \{1\} \cup \bigcup_{n>0} A^n =$ monoid closure of A, This results in what is known as a *Kleene algebra*, which contains the three operations of the product, sum, star; the injection $\eta_M(M)$ of the underlying monoid M of words; and the distinguished constants $\emptyset, \{1\}$. The Kleene star, A^* is the least upper bound of all the powers A^n as $n = 0, 1, 2, \ldots$: $A^* = \sum_{n \geq 0} A^n$. This identity be combined with distributivity to yield what is known as the **-continuity* property: $\sum_{n \geq 0} AB^nC = AB^*C$.

For a given monoid, M, the closure of the family $\mathcal{F}M$ under products, finite unions and the Kleene star yield what are known as the *rational subsets* of M, which we will denote $\mathcal{R}M$. In particular, the families $\mathcal{R}X^*$ and $\mathcal{R}(X^* \times Y^*)$, for finite non-empty alphabets X and Y give us, respectively, the regular languages over X and rational transductions from X to Y.

There are many possible and inequivalent ways to formulate a theory of regular expressions which each have the property of capturing all the identities which hold in the standard set-theoretic interpretation. Two early examples were developed in [2,3].

As shown in [2], *-continuous Kleene algebras are equivalently defined as partially ordered monoids in which the least upper bound property and distributivity property apply to the rational subsets. The corresponding homomorphisms are described equivalently as maps that preserve the Kleene operators, or as monoid homomorphisms that preserve least upper bounds for rational subsets.

[4] Generally, one distinguishes between quantales with or without the multiplicative unit 1. Our focus, here, shall be exclusively on the latter variety, the unital quantales, which we shall, for brevity, refer to as just quantales.

Through a standard construction, by ideals, a *-continuous Kleene algebra may be extended to one which possesses similar properties for its countable subsets (the closed semiring). From here, in turn, a further extension may be formulated, leading to a quantale structure.

2 The Dioid-Quantale Hierarchy

Inevitably, this leads to the question: what other types of "subset families" can we define and incorporate into this hierarchy?

2.1 Monadic Operators

We start by defining a monadic dioid D as a dioid in which the formal sum exists for all members of a distinguished family of subsets $\mathcal{A}D$, with respect to which distributivity also holds. In order to arrive at a consistent formulation, particularly one that admits a construction of adjunctions, we will need to place restrictions on the operator $\mathcal{A}$, as follows:

Definition 1. *A monadic operator $\mathcal{A}$ is a monoid operator satisfying the properties*

- $\mathbf{A}_0$ *$\mathcal{A}M$ is a family of subsets of the monoid M*
- $\mathbf{A}_1$ *$\mathcal{A}M$ contains all the finite subsets of M*
- $\mathbf{A}_2$ *$\mathcal{A}M$ is closed under products, thus making $\mathcal{A}M$ a monoid.*
- $\mathbf{A}_3$ *$\mathcal{A}M$ is closed under unions of subsets from $\mathcal{A}\mathcal{A}M$*
- $\mathbf{A}_4$ *$\mathcal{A}$ respects monoid homomorphisms. If $f : M \to N$ is a monoid homomorphism, then*[5] *$\widetilde{f}(U) \in \mathcal{A}N$ for all $U \in \mathcal{A}M$.*

For any monoid operator $\mathcal{A}$, the following may then be defined:

Definition 2. *Let M be a partially ordered monoid and write $x > A$, if x is an upper bound of a set A. Then*

- $\mathbf{D}_0$ *M is $\mathcal{A}$-additive if every $U \in \mathcal{A}M$ has a least upper bound $\sum U \in M$,*
- $\mathbf{D}_1$ *M is $\mathcal{A}$-separable if for all $x > aUb$ there exists $u > U$ such that $x \geq aub$, where $a, b \in M$ and $U \in \mathcal{A}M$.*
- $\mathbf{D}_2$ *M is strongly $\mathcal{A}$-separable if for all $x > UV$ there exist $u > U, v > V$ such that $x \geq uv$, where $U, V \in \mathcal{A}M$.*
- $\mathbf{D}_3$ *A monoid homomorphism $f : M \to N$ is $\mathcal{A}$-additive if for all $y > \widetilde{f}(U)$ there exists $x > U$ such that $y \geq f(x)$, where $U \in \mathcal{A}M$.*

One may verify that when the monoid is $\mathcal{A}$-additive then both forms of separability become equivalent to each other and to the following conditions

- $\mathbf{D}'_1$ $a, b \in M \& U \in \mathcal{A}M \to \sum(aUd) = a\left(\sum U\right)d$ (*strong distributivity*),
- $\mathbf{D}'_2$ $U, V \in \mathcal{A}M \to \sum(UV) = \sum U \cdot \sum V$ (*distributivity*).

[5] In here, and in the following, we will denote the image of a function f on a set U by $\widetilde{f}(U) \equiv \{f(u) : u \in U\}$.

For order-preserving monoid homomorphisms, $f : M \to M'$, $\mathcal{A}$-additivity reduces equivalently to the condition

$\mathbf{D}_3'$ $U \in \mathcal{A}M \to f\left(\sum U\right) = \sum \widetilde{f}\,(U)$.

Therefore, we are led to the following definitions:

Definition 3. *An $\mathcal{A}$-dioid is a partially ordered monoid M satisfying $\mathbf{D}_0$ and $\mathbf{D}_1$ (or any of its equivalents, $\mathbf{D}_2$, $\mathbf{D}_1'$, $\mathbf{D}_2'$) with respect to $\mathcal{A}$; i.e., a dioid that is both $\mathcal{A}$-additive and $\mathcal{A}$-separable. An $\mathcal{A}$-morphism is an order-preserving monoid homomorphism $f : M \to N$ that satisfies $\mathbf{D}_3$ (or equivalently, $\mathbf{D}_3'$).*

The following results may then be formulated:

Theorem 1. *$\mathcal{A}M$ is an $\mathcal{A}$-dioid for any monoid M.*

Proof. The least upper bound operator in $\mathcal{A}M$ is just set union. Property $\mathbf{A}_3$ guarantees that every member of $\mathcal{A}\mathcal{A}M$ has a union in $\mathcal{A}M$, thus satisfying $\mathbf{D}_0$. Property $\mathbf{D}_2'$ reduces to the requirement that $\bigcup UV = \bigcup U \cdot \bigcup V$, which is verified by the following chain of equivalences

$$x \in \bigcup U \cdot \bigcup V \leftrightarrow \exists A \in U, B \in V : x \in AB \leftrightarrow \exists C \in UV : x \in C \leftrightarrow x \in \bigcup UV\,.$$

Theorem 2. *$\Sigma : \mathcal{A}D \to D$ is an $\mathcal{A}$-morphism for any $\mathcal{A}$-dioid D.*

Proof. Suppose that D is an $\mathcal{A}$-dioid. Then we immediately see that $\Sigma : \mathcal{A}D \to D$ is a monoid homomorphism. Property $\mathbf{D}_3'$ then reduces to the requirement that $\sum \bigcup \mathsf{Y} = \sum_{V \in \mathsf{Y}} \sum V$ for $\mathsf{Y} \in \mathcal{A}\mathcal{A}D$ i.e.,

$$\sup \bigcup \mathsf{Y} = \sup\left\{\sup V : V \in \mathsf{Y}\right\}\,,$$

which is a general property of partially ordered sets. We note that we only need to stipulate the existence of one side of the equation, and of $\sup V$, for each $V \in \mathsf{Y}$. Then both sides will be well-defined.

Theorem 3. *Every monoid homomorphism $f : M \to N$ lifts to an $\mathcal{A}$-morphism $\widetilde{f} : \mathcal{A}M \to \mathcal{A}N$.*

Proof. Let $f : M \to N$ be a monoid homomorphism. Then $\widetilde{f} : \mathcal{A}M \to \mathcal{A}N$ is also one since $\widetilde{f}\left(\{1\}\right) = \{f\left(1\right)\} = \{1\}$ and

$$\widetilde{f}\left(UV\right) = \{f\left(uv\right) : u \in U, v \in V\} = \{f\left(u\right) f\left(v\right) : u \in U, v \in V\} = \widetilde{f}\left(U\right) \widetilde{f}\left(V\right)\,.$$

The requirement that least upper bounds from $\mathcal{A}\mathcal{A}M$ also be preserved is given by $\mathbf{D}_3'$, which takes on the following form here $\widetilde{f}\left(\bigcup \mathsf{Y}\right) = \bigcup \widetilde{\widetilde{f}}\left(\mathsf{Y}\right)$ for $\mathsf{Y} \in \mathcal{A}\mathcal{A}M$.

This is also a general property of sets.

Theorem 4 (*The Universal Property*)**.** *The free $\mathcal{A}$-dioid extension of a monoid M is $\mathcal{A}M$.*

Equivalently, this may be stated as: (a) $\eta_M : M \to \mathcal{A}M, m \mapsto \{m\}$ is a monoid homomorphism, and (b) a monoid homorphism $f : M \to D$ to an $\mathcal{A}$-dioid D extends uniquely to an $\mathcal{A}$-morphism $f^* : \mathcal{A}M \to D$; i.e., such that $f = f^* \circ \eta_M$.

Proof. This is an immediate consequence of theorems 2 and 3 with the homomorphism given by $f^* = \sum \widetilde{f}$. Uniqueness is proven as follows. The equality $f^* = \sum \widetilde{f}$ is already established on finite subsets for any morphism f^* satisfying the property $f = f^* \circ \eta_M$. To show that $f^*(U) = \sum \widetilde{f}(U)$ for a (possibly infinite) $U \in \mathcal{A}M$, consider first the image $\hat{U} = \widetilde{\eta_M}(U) \in \mathcal{A}\mathcal{A}M$. This is a family of singleton subsets. Therefore,

$$\widetilde{f^*}\left(\hat{U}\right) = \left(\widetilde{\sum \widetilde{f}}\right)\left(\hat{U}\right).$$

By $\mathcal{A}$-continuity of f^*, it then follows that

$$f^*(U) = f^*\left(\bigcup \hat{U}\right) = \sum \widetilde{f^*}\left(\hat{U}\right) = \sum \left(\widetilde{\sum \widetilde{f}}\right)\left(\hat{U}\right)$$

while

$$\sum \widetilde{f}(U) = \sum \widetilde{f}\left(\bigcup \hat{U}\right) = \sum \left(\widetilde{\sum \widetilde{f}}\right)\left(\hat{U}\right).$$

Example 1. A hierarchy of $\mathcal{A}$-dioids is provided in the following table

$\mathcal{A}$	**Description**	**Structure**
$\mathcal{F}$	Finite subsets	Dioid
$\mathcal{R}$	Rational subsets	*-Continuous Kleene Algebra
$\mathcal{C}$	Context-free subsets	"Algebraic Dioid"
$\mathcal{S}$	Context-sensitive subsets	"Context-Sensitive Dioid"
$\mathcal{T}$	Turing-computable subsets	"Transcendental Dioid"
ω	Countable subsets	Closed semiring
$\mathcal{P}$	Power set	Quantale (with unit)

More generally, one may readily verify that monadic operators preserve submonoid ordering (in virtue of $\mathbf{A}_4$) and are closed under arbitrary intersections. Therefore, we have the following results.

Theorem 5. *Monadic operators respect submonoid ordering: if $M \subseteq M'$, then $\mathcal{A}M \subseteq \mathcal{A}M'$.*

Proof. This is the direct result of applying $\mathbf{A}_4$ to the inclusion homomorphism $i : m \in M \mapsto m \in M'$.

Theorem 6 (*Hierarchical Completeness*). *Monadic operators form a complete lattice with top $\mathcal{A}M = \mathcal{P}M$ and bottom $\mathcal{A}M = \mathcal{F}M$.*

Proof. Let Z be a family of monadic operators, and define $(\wedge \mathsf{Z})\, M = \bigcap_{\mathcal{A} \in \mathsf{Z}} \mathcal{A}M$. In the special case $\mathsf{Z} = \emptyset$, we define $\wedge \mathsf{Z} = \mathcal{P}$, which trivially satisfies the defining properties for a monadic operator. Otherwise, suppose $\mathsf{Z} \neq \emptyset$. Properties $\mathbf{A}_0$,

$\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_4$ are then easily verified for $\wedge\mathsf{Z}$. Property $\mathbf{A}_3$, however, is not as immediate. For, suppose that $\mathcal{A} \in \mathsf{Z}$, we then have

$$(\wedge\mathsf{Z})\, M = \bigcap_{\mathcal{A}\in\mathsf{Z}} \mathcal{A}M \subseteq \mathcal{A}M\,.$$

To complete the proof, we need to make use of the preservation of submonoid ordering under monadic operators (theorem 5). Then, we may write $\mathcal{A}\,(\wedge\mathsf{Z})\,M \subseteq \mathcal{A}\mathcal{A}M$. Thus, for any family of subsets $\mathsf{Y} \in (\wedge\mathsf{Z})\,(\wedge\mathsf{Z})\,M$, we have that

$$\mathsf{Y} \in \bigcap_{\mathcal{A}\in\mathsf{Z}} \mathcal{A}\,(\wedge\mathsf{Z})\,M \subseteq \mathcal{A}\,(\wedge\mathsf{Z})\,M \subseteq \mathcal{A}\mathcal{A}M\,.$$

Thus, by $\mathbf{A}_3$, $\bigcup\mathsf{Y} \in \mathcal{A}M$. Since $\mathcal{A} \in \mathsf{Z}$ was arbitrarily chosen, this shows that

$$\bigcup\mathsf{Y} \in \bigcap_{\mathcal{A}\in\mathsf{Z}} \mathcal{A}M = (\wedge\mathsf{Z})\,M\,.$$

Thus $\wedge\mathsf{Z}$ satisfies property $\mathbf{A}_3$.

2.2 Closure Under Substitutions

Examples 1 suggest that monadic operators provide us with an algebraic generalization of the classical concept of language family. Properties $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_4$ are well known in the classical setting and are readily established for each of the examples. However, $\mathbf{A}_3$ is decidedly non-classical, and cannot even be expressed in that setting, though it may also be established for ωM by a well-known classical proof, and analogously for $\mathcal{T}M$. The cases $\mathcal{R}M$, $\mathcal{C}M$ and $\mathcal{S}M$ require further elaboration.

In fact, there is a classical analogue closely related to $\mathbf{A}_3$ that also happens to subsume $\mathbf{A}_4$. This relates to the concept of a *substitution*. Given two monoids M, N, a substitution map $\sigma : M \to \mathcal{P}N$ is thought of as a map which replaces each element of M by a *language* in N. Reflecting the hierarchy of $\mathcal{A}$-dioids is a hierarchy of substitutions, determined by the range of the map. This leads to the following definition.

Definition 4. *Let M, N be monoids. A monoid homomorphism $\sigma : M \to \mathcal{P}N$ is called a substitution. In addition, if $\mathcal{A}N \subseteq \mathcal{P}N$ is any family of subsets such that $\sigma(m) \in \mathcal{A}N$, for each $m \in M$, then σ will be called an $\mathcal{A}$-substitution.*

Every substitution $\sigma : M \to \mathcal{P}N$ leads uniquely to a map between the respective power sets of the monoids given, for $A \in \mathcal{P}M$, by $\hat{\sigma}(A) = \bigcup_{m\in A} \sigma(m) \in \mathcal{P}N$. Moreover, it follows directly from this definition that this map distributes over unions: $\hat{\sigma}(\bigcup\mathsf{Y}) = \bigcup_{A\in\mathsf{Y}} \hat{\sigma}(A)$, for all $\mathsf{Y} \subseteq \mathcal{P}M$. Therefore it is a quantale homomophism between the respective power sets. This leads to the following result.

Theorem 7. *A substitution $\sigma : M \to \mathcal{P}N$ determines and is uniquely determined by a quantale homomorphism $\phi : \mathcal{P}M \to \mathcal{P}N$ such that $\phi(\{m\}) = \sigma(m)$ for $m \in M$.*

Proof. In the forward direction, we have

$$\hat{\sigma}\left(\bigcup \mathsf{Y}\right) = \bigcup_{m \in \bigcup \mathsf{Y}} \sigma(m) = \bigcup_{A \in \mathsf{Y}} \bigcup_{m \in A} \sigma(m) = \bigcup_{A \in \mathsf{Y}} \hat{\sigma}(A) \,,$$

for $\mathsf{Y} \subseteq \mathcal{P}M$, and $\hat{\sigma}(\{m\}) = \bigcup_{m' \in \{m\}} \sigma(m') = \sigma(m)$, for $m \in M$. Conversely, suppose $\phi : \mathcal{P}M \to \mathcal{P}N$ is a quantale homomorphism satisfying the stated condition. Then for $A \subseteq M$, making use of the invariance property, we have $\phi(A) = \bigcup_{m \in A} \phi(\{m\}) = \bigcup_{m \in A} \sigma(m) = \hat{\sigma}(A)$.

With these preliminaries, we may then state the following alternative to properties $\mathbf{A}_3$ and $\mathbf{A}_4$

$\mathbf{A}_5$ $\mathcal{A}$ respects $\mathcal{A}$-substitutions. If $\sigma : M \to \mathcal{P}N$ is an $\mathcal{A}$-substitution, then $\hat{\sigma}(U) \in \mathcal{A}N$ for all $U \in \mathcal{A}M$.

We may then establish the equivalence between the two sets of properties as follows:

Theorem 8. *Let $\mathcal{A}$ be a monoid operator satisfying $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{A}_2$. Then $\mathbf{A}_3$ and $\mathbf{A}_4$ are equivalent to $\mathbf{A}_5$.*

Proof. In the following, let M, N be monoids.

First, we will establish $\mathbf{A}_0, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \to \mathbf{A}_5$. Suppose $\sigma : M \to \mathcal{P}N$ is an $\mathcal{A}$-substitution and $U \in \mathcal{A}M$. Then, by $\mathbf{A}_0$, $\mathbf{A}_2$, $\mathcal{A}N \subseteq \mathcal{P}N$ is a monoid with $\sigma : M \to \mathcal{A}N$ a monoid homomorphism. By property $\mathbf{A}_4$, it follows that $\widetilde{\sigma}(U) = \{\sigma(m) : m \in U\} \in \mathcal{A}\mathcal{A}N$. In turn, by $\mathbf{A}_3$, it follows that $\hat{\sigma}(U) = \bigcup \widetilde{\sigma}(U) \in \mathcal{A}N$.

Second, we will prove that $\mathbf{A}_1, \mathbf{A}_5 \to \mathbf{A}_4$. Suppose $f : M \to N$ is a monoid homomorphism and $U \in \mathcal{A}M$. Then by $\mathbf{A}_1$, $\mathcal{A} \geq \mathcal{F}$, therefore $\sigma : m \in M \mapsto \{f(m)\} \in \mathcal{F}N \subseteq \mathcal{A}N$ is an $\mathcal{A}$-substitution. Applying $\mathbf{A}_5$, it follows that $\hat{\sigma}(U) \in \mathcal{A}N$. But

$$\hat{\sigma}(U) = \bigcup_{m \in U} \sigma(m) = \bigcup_{m \in U} \{f(m)\} = \{f(m) : m \in U\} = \widetilde{f}(U) \,.$$

Thus $\widetilde{f}(U) \in \mathcal{A}N$.

Finally, we will show that $\mathbf{A}_0, \mathbf{A}_2, \mathbf{A}_5 \to \mathbf{A}_3$. Suppose $\mathsf{Y} \in \mathcal{A}\mathcal{A}M$. Then, by $\mathbf{A}_2$, the identity map $\sigma = I_{\mathcal{A}M} : \mathcal{A}M \to \mathcal{A}M$ is a monoid homomorphism; and, by $\mathbf{A}_0$, $\sigma : \mathcal{A}M \to \mathcal{A}M \subseteq \mathcal{P}M$ is an $\mathcal{A}$-substitution. Therefore, applying $\mathbf{A}_5$, it follows that $\hat{\sigma}(\mathsf{Y}) \in \mathcal{A}M$. But

$$\hat{\sigma}(\mathsf{Y}) = \bigcup_{U \in \mathsf{Y}} \sigma(U) = \bigcup_{U \in \mathsf{Y}} U = \bigcup \mathsf{Y} \,.$$

Therefore, $\bigcup \mathsf{Y} \in \mathcal{A}M$.

2.3 Closure under Inverse Morphisms

An important application of the universal property (theorem 4) is the following:

Theorem 9 ($\mathbf{A}_6$). *If the monoid homomorphism $f : M \to N$ is surjective, then so is the lift $\widetilde{f} : \mathcal{A}M \to \mathcal{A}N$.*

Proof. Assume the conditions stated hold. The property of surjectivity may be stated solely in terms of the properties of homomorphisms in the following way: given homomorphisms $g, h : N \to P$ to another monoid P, if $g \circ f = h \circ f$ then $g = h$. Surjectivity for the lifting is proven via the analogous property. Assume that $g, h : \mathcal{A}N \to D$ are now $\mathcal{A}$-morphisms to an $\mathcal{A}$-dioid D, such that $g \circ \widetilde{f} = h \circ \widetilde{f}$. Then, we may push this back to a map on the monoid M and write $g \circ \widetilde{f} \circ \eta_M = h \circ \widetilde{f} \circ \eta_M$. But, $\widetilde{f} \circ \eta_M = \eta_M \circ f$, therefore $g \circ \eta_M \circ f = h \circ \eta_M \circ f$. From the surjectivity of f, it follows that $g \circ \eta_M = h \circ \eta_M$. The universal property, theorem 4, asserts that the extension of this map $g \circ \eta_M = h \circ \eta_M : N \to D$ to a map on $\mathcal{A}N$ is unique. Therefore, $g = h$. Thus, $\widetilde{f}$ is surjective.

As a consequence, we find that monadic operators respect inverse morphisms in the following sense:

Theorem 10. *Let $\mathcal{A}$ be a monadic operator. Then if $f : M \to N$ is a surjective monoid homomorphism, and $V \in \mathcal{A}N$ then $V = \widetilde{f}(U)$ for some $U \in \mathcal{A}M$. Moreover, there is a monoid $\hat{N}$, a surjective map $\sigma : \hat{N} \to N$, and a factoring $\sigma = f \circ \phi$ into $\phi : \hat{N} \to M$ and f; such that each $V \in \mathcal{A}N$ may be expressed as $\sigma\left(\hat{V}\right)$ for some $\hat{V} \in \mathcal{A}\hat{N}$ where $\phi\left(\hat{V}\right) \in \mathcal{A}M$.*

Proof. The first statement is a direct consequence of our previous result, theorem 9. For the second part, let $Y \subseteq N$ be any generating subset of the monoid N. The universal property for free monoids then associates a canonical monoid homomorphism $\sigma : \hat{N} = Y^* \to N$ with the inclusion $\sigma : Y \to N$. This maps the free monoid Y^* generated by the set Y onto the closure of that set within N, which (by assumption) is just N, itself.

In its greatest generality, this argument requires the Axiom of Choice. If Y is an infinite set, then for each $y \in Y$, we need to choose an element $m \in M$ such that $f(m) = \sigma(y)$, and then define $\phi(y) = m$. However, for the operators $\mathcal{A} = \mathcal{F}, \mathcal{R}, \mathcal{C}, \mathcal{S}, \mathcal{T}$, we will always be able to express a subset $V \in \mathcal{A}N$ as $V \in \mathcal{A}Y^*$ for some finite subset $Y \subseteq N$. However, this property ($\mathbf{A}_7$: *finite generativity*) will not have any bearing on our results, so we will not further elaborate on it here.

Let $V \in \mathcal{A}N$. Since $\sigma : \hat{N} \to N$ is surjective then there exists $\hat{V} \in \mathcal{A}\hat{N}$ such that $V = \sigma\left(\hat{V}\right)$. The remainder of the theorem then follows by property $\mathbf{A}_4$.

3 Inclusion of the Chomsky Hierarchy

Up to now, we have left the issue unresolved of which of our examples actually constitute monadic operators. Properties $\mathbf{A}_0$ and $\mathbf{A}_1$ are true, by construction, for each operator in example 1. Similarly, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ are well-known and easily proven in the cases of $\mathcal{F}$, ω and $\mathcal{P}$.

However, for $\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{T}$, $\mathbf{A}_3$ is neither obvious nor well-known, while $\mathbf{A}_2$ and $\mathbf{A}_4$ require further clarification. This is what we will resolve here.

3.1 The $\mathcal{R}$ Operator and *-Continuous Kleene Algebras

A *-continuous Kleene algebra is a dioid which is $\mathcal{R}$-additive. By definition, the rational subsets $\mathcal{R}M$ of a monoid M are the closure of $\mathcal{F}M$ under the product, finite union and Kleene star. Therefore $\mathbf{A}_2$ is satisfied, so that we need only prove $\mathbf{A}_3$ and $\mathbf{A}_4$, or equivalently $\mathbf{A}_5$.

Theorem 11. *The operator $\mathcal{R}$ satisfies $\mathbf{A}_5$.*

Proof. This may be shown by induction. Let $\sigma : M \to \mathcal{R}N$ be an $\mathcal{R}$-substitution from a monoid M to the rational subsets of a monoid N. For finite sets $U \in \mathcal{F}M$, we immediately have $\hat{\sigma}(U) = \bigcup_{u\in U} \sigma(U) \in \mathcal{R}N$, since $\mathcal{R}N$ is closed under finite unions. Moreover, we may show that $\hat{\sigma}$ preserves unions and products, since $\hat{\sigma}\left(\bigcup \mathsf{Y}\right) = \bigcup_{u\in U\in \mathsf{Y}} (u) = \bigcup_{U\in\mathsf{Y}} \hat{\sigma}(U)$, for any $\mathsf{Y} \in \mathcal{P}M$, while for $U, V \subseteq M$,

$$\hat{\sigma}(UV) = \bigcup_{u\in U, v\in V} \sigma(uv) = \bigcup_{u\in U} \sigma(u) \bigcup_{v\in V} \sigma(v) = \hat{\sigma}(U)\,\hat{\sigma}(V)\ .$$

From this, it follows that $\hat{\sigma}$ preserve Kleene stars,

$$\hat{\sigma}(U^*) = \hat{\sigma}\left(\bigcup_{n\geq 0} U^n\right) = \bigcup_{n\geq 0} \hat{\sigma}(U^n) = \bigcup_{n\geq 0} \hat{\sigma}(U)^n = \hat{\sigma}(U)^*\ .$$

Consequently, if we let $U, V \in \mathcal{R}M$ and assume by inductive hypothesis that $\hat{\sigma}(U) \in \mathcal{R}N$ and $\hat{\sigma}(V) \in \mathcal{R}N$, then it follows that $\hat{\sigma}(UV) = \hat{\sigma}(U)\,\hat{\sigma}(V) \in \mathcal{R}N$, $\hat{\sigma}(U \cup V) = \hat{\sigma}(U) \cup \hat{\sigma}(V) \in \mathcal{R}N$ and $\hat{\sigma}(U^*) = \hat{\sigma}(U)^* \in \mathcal{R}N$, since $\mathcal{R}N$ is closed under products, finite unions and Kleene stars.

3.2 The $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{T}$ Operators

There remains the issue of $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ with respect to $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{T}$.

For the properties $\mathbf{A}_3$ and $\mathbf{A}_4$, it turns out, again, to be more useful to establish $\mathbf{A}_5$, instead. We do this explicitly here for the operator $\mathcal{C}$, closely following the development of the analogous result in the classical theory (c.f. [20] theorem 9.2.2).

Lemma 1 (*The Composition Lemma). Let M be a monoid $G = (Q, S, H)$ be a context-free grammar over X^* for a finite $X \subseteq M$. Let $\sigma : M \to \mathcal{P}N$ be a context-free substitution to the monoid N. For each $x \in X$, let $G_x = (Q_x, S_x, H_x)$ be a context-free grammar such that $L(G_x) = \sigma(x)$; with the sets Q and Q_x for each $x \in X$ all mutually disjoint.*

Define the composition of the grammars[6] *by*

$$G \circ_\sigma \bigcup_{x\in X} G_x = \left(Q \cup \bigcup_{x\in X} Q_x, \bar{\sigma}(S), \{(q, \bar{\sigma}(\beta)) : (q,\beta) \in H\} \cup \bigcup_{x\in X} H_x\right),$$

where $\bar{\sigma} : X^*[Q] \to N\left[Q \cup \bigcup_{x\in X} Q_x\right]$ *is the monoid homomorphism given by* $\bar{\sigma}(x) = S_x$ *for* $x \in X$ *and* $\bar{\sigma}(q) = q$ *for* $q \in Q$*. Then* $L\left(G \circ \bigcup_{x\in X} G_x\right) = \sigma(L(G))$*.*

Proof: Let G' denote the composition. It is an easy induction to show, for each $x \in X$, that $\alpha \to \beta$ in G_x if and only if $\alpha \to \beta$ in G', where $\alpha, \beta \in N[Q_x]$. This makes use of the mutual disjointness of the sets Q_x. The only rules that can apply here are therefore those from H_x. From this, it follows that $[S_x]_{G'} = [S_x]_{G_x} = L(G_x) = \sigma(x)$, for $x \in X$. In a similar way, one may readily verify that $\alpha \to \beta$ in G if and only if $\bar{\sigma}(\alpha) \to \bar{\sigma}(\beta)$ in G'. Again, making use of the disjointness of the set Q from all the other sets Q_x, it follows that $[q]_{G'} = \bigcup_{w\in[q]_G} [\bar{\sigma}(w)]_{G'}$ since occurrences of variables of Q in a configuration α must be handled by the rules from H.

From $[\bar{\sigma}(x)]_{G'} = [S_x]_{G'} = \sigma(x)$ $(x \in X)$, it follows by inductive argument[7] that $[\bar{\sigma}(w)]_{G'} = \sigma(w)$, for $w \in X^*$. Using this result, we then have

$$[\bar{\sigma}(q)]_{G'} = [q]_{G'} = \bigcup_{w\in[q]_G} [\bar{\sigma}(w)]_{G'} = \bigcup_{w\in[q]_G} \sigma(w) = \hat{\sigma}([q]_G),$$

for all $q \in Q$. Thus, we have

$$L(G') = [S]_{G'} = [\bar{\sigma}(S)]_{G'} = \hat{\sigma}([S]_{G'}) = \hat{\sigma}(L(G)).$$

To fully establish our results, we need to ensure that (i) such mutually disjoint sets can be chosen, as required by the lemma; and (ii) that a (finite) context-free grammar can be presented as a grammar over a finitely generated submonoid. Property (ii) is a consequence of the fact that only a finite subset $X \subseteq M$ will appear on the right-hand side of the rules of H in a grammar $G = (Q, S, H)$ over the monoid M, since H is finite. Property (i) makes use of the following technical lemma.

Lemma 2 (*Substitution Invariance*)**.** *Let* $G = (Q, S, H)$ *be an arbitrary grammar over a monoid* M*,* $\sigma : Q \to R$ *a bijection and*

$$G_\sigma = (R, \sigma(S), \{(\sigma(a), \sigma(b)) : (a,b) \in H\}),$$

where $\sigma : M[Q] \to M[R]$ *is the extension to a monoid homomorphism given by* $\sigma(m) = m$ *for* $m \in M$*. Then* $\alpha \to \beta$ *in* G *iff* $\sigma(\alpha) \to \sigma(\beta)$ *in* G_σ*, for all* $\alpha, \beta \in M[Q]$*. Moreover,* $[\alpha]_G = [\sigma(\alpha)]_{G_\sigma}$ *for all* $\alpha \in M[Q]$*. In particular,* $L(G) = L(G_\sigma)$*.*

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[6] This is the grammar obtained by replacing each terminal x of the grammar G by the start symbol S_x of grammar G_x and combining the non-terminals of G with those of each G_x.

[7] This makes use of the property $[\alpha\beta]_{G'} = [\alpha]_{G'}[\beta]_{G'}$ which may be proven by induction for context-free grammars G'.

Proof. Since the map σ is a bijection, we only need to show that if $\alpha \to \beta$ in G, then $\sigma(\alpha) \to \sigma(\beta)$ in G_σ. This is an easy induction over the structure of derivations.

The converse property follows by considering the inverse σ^{-1}. The remaining statements are then a direct consequence since, for $m \in M$, we have $m \in [\alpha]_G$ iff $\alpha \to m$ in G iff $\sigma(\alpha) \to \sigma(m) = m$ in G_σ iff $m \in [\sigma(\alpha)]_{G_\sigma}$. From this, it will then follow that $L(G) = [S]_G = [\sigma(S)]_G = L(G_\sigma)$.

The proof of $\mathbf{A}_2$ closely follows that of the classical result. Given subsets $L(G_1)$, $L(G_2)$ of M generated by context-free grammars $G_i = (Q_i, S_i, H_i)$ over M $(i = 1, 2)$, one constructs a grammar $G = (Q, S, H)$ for the product by taking $Q = Q_1 \cup Q_2 \cup \{S\}$, $H = H_1 \cup H_2 \cup \{(S, S_1 S_2)\}$, choosing S such that $S \notin Q_1 \cup Q_2 \cup M$. We may then use the property $[\alpha\beta] = [\alpha][\beta]$ to show that $L(G) = [S_1 S_2]_G = [S_1]_{G_1} [S_2]_{G_2} = L(G_1) L(G_2)$.

With these preliminaries established, we then have the following corollary.

Corollary 1. *The operator $\mathcal{C}$ is monadic.*

Though the Composition Lemma and product construction are formulated explicitly for $\mathcal{C}$, it can be refined to make it applicable to $\mathcal{S}$ and $\mathcal{T}$. We'll explain how this may be done for the Composition Lemma. A similar consideration holds for the product construction.

To avoid the need for the property $[\alpha\beta] = [\alpha][\beta]$, the grammar G_x over the monoid N is modified to a grammar over a copy N_x of N. Without loss of generality, we may assume that N is generated by a finite set $Y \subseteq N$, similarly N_x by $Y_x \subseteq N_x$. We must then add rules $n_x \to n$ to map the copy $n_x \in N_x$ of $n \in N$ to n.

For $\mathcal{S}$, in the Composition Lemma, we will also need to prove the context-sensitivity of grammar G'. First, the set X, will be atomic with respect to a given measure over the monoid M. We may also assume that the elements of X are of unit norm or greater, by rescaling the norm. For each $x \in X$, the starting configuration S_x for each grammar G_x will have a norm of at least 1, thereby ensuring the context-sensitivity of the composition of the grammars. In particular, we will have $\|\alpha\| \leq \|\bar{\sigma}(\alpha)\|$, with respect to suitably defined norms. This leads to the following results:

Corollary 2. *The operators $\mathcal{S}$ and $\mathcal{T}$ are monadic.*

4 Concluding Remarks

The Chomsky Hierarchy is the foundation of both the theory of computation and linguistics. What we have shown is that the hierarchy may be encapsulated and generalized in algebraic form as a hierarchy of algebras. At the bottom of the hierarchy is the dioid, or idempotent semiring. Associated with this is the functor $\mathcal{F}$, which maps a given monoid M to its dioid of finite subsets. Thus, the dioid may be regarded as an algebraization of the concept of finite language. At the top of the hierarchy is the unital quantale, which is associated with the functor $\mathcal{P}$ that maps a monoid M to its quantale of subsets. Here, the corresponding classical concept is the general language. In between these two extremes are other algebras, corresponding to other monadic operators, which include operators that generalize the 4 levels of the Chomsky hierarchy: $\mathcal{R} < \mathcal{C} < \mathcal{S} < \mathcal{T}$.

In the sequel paper, we show that this hierarchy is complemented by a hierarchy of adjunctions with the properties that

- if $\mathcal{A} \leq \mathcal{B}$ then there exists an adjunction $\left(\mathbf{Q}_{\mathcal{A}}^{\mathcal{B}}, \mathbf{Q}_{\mathcal{B}}^{\mathcal{A}}\right)$.
- if $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$ then $\mathbf{Q}_{\mathcal{B}}^{\mathcal{C}} \circ \mathbf{Q}_{\mathcal{A}}^{\mathcal{B}} = \mathbf{Q}_{\mathcal{A}}^{\mathcal{C}}$ and $\mathbf{Q}_{\mathcal{B}}^{\mathcal{A}} \circ \mathbf{Q}_{\mathcal{C}}^{\mathcal{B}} = \mathbf{Q}_{\mathcal{C}}^{\mathcal{A}}$.

The functor $\mathbf{Q}_{\mathcal{A}}^{\mathcal{B}}$ extends each $\mathcal{A}$-dioid to its $\mathcal{B}$-completion, and is complemented by the forgetful functor $\mathbf{Q}_{\mathcal{B}}^{\mathcal{A}}$, which maps a $\mathcal{B}$-dioid D to itself, where the least upper bound operator Σ is restricted to the family $\mathcal{A}D$.

Finally, a few additional comments are in order regarding the algebraic representation of context sensitivity. The unusual way in which ϵ-rules enter into the formulation of context-sensitivity indicates that a more natural setting may be found within semigroup theory. This suggests a parallel formulation of monadic semigroup operators with analogous properties $\mathbf{A}_0$-$\mathbf{A}_4$ stated for semigroups. One should then be able to prove that if $\mathcal{A}$ is a monadic semigroup operator, then its ϵ-extension

$$\mathcal{A}_\epsilon M \equiv \mathcal{A}M \cup \{U \cup \{1\} : U \in \mathcal{A}M\}$$

is a monadic monoid operator; particularly, that it satisfies properties $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_5$.

Finally, in classical theory an equivalence between context-sensitive grammars and non-erasing grammars can be proven [18]. Our definition of context-sensitivity is with respect to non-erasing grammars. One needs to separately prove their equivalence within the broader setting provided here. A similar observation holds concerning the need to verify that the normal forms and conversions of the classical theory (e.g., Chomsky and Greibach normal forms, Kuroda normal form) will continue to hold for generalized grammars.

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A Grammars in the Algebraic Approach

The generalization of grammars to arbitrary monoids is, for the most part, straightforward. However, there are few elements which require further elaboration.

Classically, a grammar over the alphabet X affixes a set Q of indeterminates, called either *variables* or (making reference to the notion of parse trees) *nonterminals*. It is assumed that $X \cap Q = \emptyset$. A finite set H of schemes is provided for effecting transitions over configurations in $(X \cup Q)^*$, so that

$$H \subseteq (X \cup Q)^* \times (X \cup Q)^* .$$

A *starting configuration* $S \in (X \cup Q)^*$ is identified and the language is defined as the set of all the words in X^* derivable from S by a finite number of applications of transitions $u \to v$ for $(u, v) \in H$ to subwords in the present configuration. One usually assumes the starting configuration $S \in Q$ to be one of the variables, though this restriction is not essential.

When generalizing to an arbitrary monoid M, one may assume that $X \subseteq M$ is a distinguished subset, though its explicit delineation does not prove to be essential. In place of $(X \cup Q)^*$, one must then take the *free extension* $M [Q]$

of the monoid M by the indeterminates in Q. In the case where $M = X^*$ and $X \cap Q = \emptyset$, the free extension (see below) reduces (up to isomorphism) to $M[Q] \cong (X \cup Q)^*$.

A.1 Free Extensions of Monoids

Thus, in its more general form, a grammar is a structure $G = (Q, S, H)$ over a monoid M composed of a set of variables, Q; a distinguished configuration $S \in M[Q]$; and a set of transition rules $H \subseteq M[Q] \times M[Q]$. In the grammars we consider, H will always be finite. This definition includes, as a special case where $M = X^* \times Y^*$, *translations* from X to Y; and $M = X^*$, for *languages* over alphabet X. More interesting examples might be conceived of where M represents a construction language for graphical or multimedia displays (e.g. a typesetting, hypertext or word processing language); for instance, the commutative monoid that underlies the 2-dimensional symbolic language used in the *Laws of Form* [19] for Boolean algebra.

The monoid $M[Q]$ is the free extension of M by the set Q. It may be thought of as the monoid M, itself, with the set Q of indeterminates added to it. A word $\alpha \in M[Q]$ may be written as $\alpha = m_0 q_1 m_1 \ldots q_n m_n$, its degree being $\deg(\alpha) = n \geq 0$. The monoid product is defined by

$$(m_0 q_1 m_1 \ldots q_n m_n)(n_0 r_1 n_1 \ldots r_n n_p) \equiv m_0 q_1 m_1 \ldots q_n (m_n n_0) r_1 n_1 \ldots r_n n_p \,,$$

with $\deg(\alpha\beta) = \deg(\alpha) + \deg(\beta)$. Classically, one has $M = X^*$ and $M[Q] = (X^*)[Q] = (X \cup Q)^*$, provided that $X \cap Q = \emptyset$. The identity is just the monoid identity $1 \in M$. The monoid M is embedded within $M[Q]$ as the words of degree 0, while the set Q is mapped to the words of degree 1 of the form $(1q1)$.

The free extension $M[Q]$ has the following universal property. Corresponding to a monoid homomorphism $\phi : M \to N$ and map $\sigma : Q \to N$ is a unique monoid homomorphism $\langle\phi, \sigma\rangle : M[Q] \to N$ such that $\langle\phi, \sigma\rangle(m) = \phi(m)$ for words $m \in M \subseteq M[Q]$ of degree 0, and $\langle\phi, \sigma\rangle(1q1) = \sigma(q)$. The map is uniquely given from these criteria by

$$\langle\phi, \sigma\rangle(m_0 q_1 m_1 \ldots q_k m_k) = \phi(m_0)\,\sigma(q_1)\,\phi(m_1) \ldots \sigma(q_k)\,\phi(m_k) \,.$$

A.2 Generalized Grammars

A rule $(\alpha, \beta) \in H$ is then to be thought of as a *one-step transition* $\alpha \to \beta$. More generally, a transition sequence is a sequence of words in $M[Q]$ of the form $\alpha_0 \to \ldots \to \alpha_n$, where $n \geq 0$, such that adjacent members of the sequence are of the form $\gamma\alpha\delta = \gamma\beta\delta$ for some $\gamma, \delta \in M[Q]$, $(\alpha, \beta) \in H$. Corresponding to each $\alpha \in M[Q]$ is the subset $[\alpha] \equiv \{m \in M : \alpha \to m\}$ of elements of M derivable from the configuration α. The language $L(G) \subseteq M$ corresponding to the grammar is that $L(G) \equiv [S] = \{m \in M : S \to m\}$ associated with the starting configuration.

Of particular interest are those grammars where H is restricted to the form $H \subseteq Q \times M[Q]$. Such a grammar is deemed to be *context-free*.

The family $\mathcal{C}M$ of *context-free* subsets of a monoid M shall consist of subsets $L(G) \subseteq M$ generated by a context-free grammar $G = (Q, S, H)$, for finite H. Similarly, the family $\mathcal{T}M$ may be defined, where G is a general grammar (again, with H being finite.)

A.3 Normed Monoids and Context-Sensitive Grammars

A question arises as to how to define *context-sensitive* subsets for monoids other than free monoids $M = X^*$. The classical definition makes explicit reference to the *length* of the elements of $(X \cup Q)^*$, requiring that a restriction be placed on $H \subseteq Q^* \times (X \cup Q)^*$, such that $0 < \ln(\alpha) \leq \ln(\beta)$, for each $(\alpha, \beta) \in H$; where $\ln(\alpha)$ denotes the length of the word $\alpha \in (X \cup Q)^*$. One-step derivations should be restricted to a form where only variables should appear on the left, and the right-hand side should be of length no less than that of the left. Thus, we are able to define a class $\mathcal{S}X^*$ comprising the *context-sensitive* subsets of the free monoid X^*.[8]

A generalization to arbitrary monoids may be found if we require that the monoid operator $M \mapsto \mathcal{S}M$ be well-behaved under monoid homomorphisms. In particular, if $X \subseteq M$ is a generating subset of the monoid M then under the canonical homomorphism $\sigma_X : X^* \to M$, we should expect that $\mathcal{S}M = \{\sigma_X(A) : A \in \mathcal{S}X^*\}$. First, in order for the length restriction to be satisfied, we should require that $1 \notin X$. Second, in order for the definition to be well-behaved, we should also require independence of the selection of a generating subset. In particular, if $Y \subseteq M$ is any other generating subset of M such that $1 \notin Y$, with a canonical homomorphism $\sigma_Y : Y^* \to M$, then there should be a way to convert a context-sensitive grammar $G = (Q, S, H)$ over X^* to one $G' = (Q', S', H')$ over Y^*. Indeed, this can be done by adding new variables $\hat{x}$ for each $x \in X$, replacing each symbol from X in the original grammar with the corresponding variable, and then adding new rules $\hat{x} \mapsto w_x$, where $\sigma_Y(w_x) = \sigma_X(x)$. That is, we define $Q' = Q \cup \{\hat{x} : x \in X\}$, $S' = h(S)$ and

$$H' = \{(h(\alpha), h(\beta)) : (\alpha, \beta) \in H\} \cup \{(\hat{x}, w_x) : x \in X\} ,$$

where $h : (X \cup Q)^* \to Q'^*$ is the monoid homomorphism defined inductively by $h(x) = \hat{x}$, for $x \in X$ and $h(q) = q$, for $q \in Q$. It is not too difficult, then, to show that $\alpha \to \beta$ in G if, and only if, $h(\alpha) \to h(\beta)$ in G' and that $\sigma_X([a]_G) = \sigma_X([h(a)]_{G'})$, for $\alpha \in (X \cup Q)^*$. The length requirement is also satisfied since $\ln(h(\alpha)) = \ln(\alpha)$ and $1 = \ln(\hat{x}) \leq \ln(w_x)$. The latter property is where we specifically require that $1 \notin X$.

[8] This variety of grammar is known, in classical theory, as the *monotonic* or *non-contracting* grammar. A *context-sensitive* grammar, classically, admits rules of the form $\alpha q \beta \to \alpha \gamma \beta$, with the restriction $q \in Q$. This allows production of the empty word, whereas monotonic grammars do not. Therefore, explicit stipulation must be made to allow for the inclusion of the monoid identity $1 \in M$ in the members of the family $\mathcal{S}M$.

The central feature of the context-sensitivity concept is the notion of length. This is what we are actually generalizing to arbitrary monoids. Each generating subset $X \subseteq M$ defines a length function such that the elements of X have minimal length. This leads naturally to the following definitions:

Definition 5 (*Normed Monoids*)**.** *M is a monoid with a length function $m \in M \mapsto \|m\| \in \mathrm{I\!R}$ such that*

Non-negativity $\|m\| \geq 0$, *for* $m \in M$,
Non-degeneracy $\|m\| = 0 \leftrightarrow m = 1$ *for* $m \in M$,
Triangle inequality $\|mm'\| \leq \|m\| + \|m'\|$ *for* $m, m' \in M$.

An element $m \in M - \{1\}$ *is atomic with respect to the norm if*

$$m = m_1 m_2 \rightarrow m_1 = 1 \vee m_2 = 1 \vee \|m\| < \|m_1\| + \|m_2\| .$$

If $\inf_{x \in X} \|x\| > 0$, *where* X *denotes the set of atomic elements, then the norm will be called atomic.*

It follows, by a routine induction, that the atomic elements X corresponding to an atomic norm comprise a generating subset of the monoid M. Conversely, given a generating subset $X \subseteq M - \{1\}$, we may define the length by $\|1\|_X = 0$, $\|m\|_X = n + 1$ for $\left(m \in \sigma_X\left(X^{n+1}\right) - \sigma_X\left(X^n\right)\right)$.

A norm over the monoid M may be extended to a norm over $M[Q]$ by defining $\|q\| = 1$ for $q \in Q$. It will then follow that $X \cup Q$ will comprise the corresponding set of atomic elements. Moreover, the property of atomicity will be preserved by the norm.

The context-sensitive grammar over M is then a "value-reducing" grammar with respect to a given norm; that is, a grammar whose one-step derivations are restricted to the form $(\alpha, \beta) \in H$ where $0 < \|\alpha\| \leq \|\beta\|$, with the prescription that α consist only of variables.