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Note on the Construction of Free Monoids

Stephen Lack

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Abstract We construct free monoids in a monoidal category $(\mathscr{C}, \otimes, I)$ with finite limits and countable colimits, in which tensoring on either side preserves reflexive coequalizers and colimits of countable chains. In particular this will be the case if tensoring preserves sifted colimits.

Keywords Monoidal category · Free monoid · Reflexive coequalizer

Mathematics Subject Classifications (2000) 18D10 · 18D35 · 18A40

1 Background

For any monoidal category $(\mathscr{C}, \otimes, I)$, one can form the category of monoids in $\mathscr{C}$, and for suitable choice of $\mathscr{C}$, this contains many important notions, such as monoids, rings, categories, differential graded algebras, and monads: see [8, Chapter VII]. For each such $\mathscr{C}$, the category Mon$\mathscr{C}$ of monoids in $\mathscr{C}$ has a forgetful functor $U : \text{Mon}\mathscr{C} \to \mathscr{C}$, and this forgetful functor often has a left adjoint, sending an object of $\mathscr{C}$ to the free monoid on that object. In particular, if $\mathscr{C}$ has countable coproducts, and these are preserved by tensoring on either side, then the free monoid on X is given by the well-known "geometric series"

$$I + X + X^2 + X^3 + \ldots$$

where X^n stands for the nth"tensor power" $X \otimes \ldots \otimes X$ of X.

This case includes the free monoid, the free ring (on an additive abelian group), the free category (on a graph) and the free differential graded algebra (on a chain complex), but it does not help with the case of free monads. Conditions for the

S. Lack (✉)
School of Computing and Mathematics, University of Western Sydney,
Locked Bag 1797, Penrith South DC, New South Wales 1797, Australia
e-mail: s.lack@uws.edu.au

existence of free monads were given by Barr in [2]. Further analysis of the free monoid construction was given by Dubuc in [3], of which more will be said below.

The epic paper [5] of Kelly analyzes many constructions of free monoids, free algebras, and colimits, generally requiring transfinite processes. It provides very general conditions for the existence of free monoids, in the case where the category $\mathscr{C}$ is cocomplete and the functors $-\otimes C : \mathscr{C} \to \mathscr{C}$ are cocontinuous, for all objects C (the conditions on $C \otimes -$ are then quite mild). This allows the construction of free monads in many cases, and also the construction of free operads [6, 9]. For example if each $C \otimes -$ preserves filtered colimits then the free monoid is given by a "factorized" version of the geometric series:

$$I + X(I + X(I + X(I + \dots$$

A recent paper of Vallette [10] gave a construction of free monoids under much stronger assumptions on $C \otimes -$ but weaker assumptions on $- \otimes C$, and moreover under the assumption that $\mathscr{C}$ is abelian. These assumptions allowed the construction of free properads and other closely related free structures, in the abelian context.

In this paper we generalize and simplify the construction of Vallette, removing the assumption that $\mathscr{C}$ is abelian. Specifically, we suppose that $\mathscr{C}$ has finite limits and countable colimits, and that the functors $- \otimes C$ and $C \otimes -$ preserve reflexive coequalizers and colimits of countable chains, and we construct free monoids under these assumptions. (This would be the case for example if tensoring preserved sifted colimits [1].) Some examples of such monoidal categories are given in Section 5. The question of whether these free monoids are algebraically free [5] is briefly discussed in Section 6.

Notation

The tensor product of objects will generally be denoted by juxtaposition: XY stands for $X \otimes Y$ (just as X^2 stands for $X \otimes X$). We sometimes write as if the monoidal structure on $\mathscr{C}$ were strict. This is merely for convenience; by the coherence theorem for monoidal categories (see [8, Chapter VII]) it could be avoided. We write $\pi_{m,n} : X^m X^n \cong X^{m+n}$ for the canonical isomorphism built up out of the associativity isomorphisms. (If $\mathscr{C}$ really were strict this would be the identity; otherwise, in order to make sense of tensor powers such as X^n some particular bracketing must be chosen.) We sometimes write X for the identity 1_X on an object X, and row vector notation $(f\ g) : A + B \to C$ for morphisms out of a coproduct. The composite of $f : X \to Y$ and $g : Y \to Z$ is written $g.f$.

2 The Approach of Dubuc

The construction of a free monoid can be broken down into two parts. An object Y is said to be *pointed* if it is equipped with a map $y : I \to Y$; we write $\mathrm{Pt}\mathscr{C}$ for the category of *pointed objects* in $\mathscr{C}$. Then the forgetful functor $U : \mathrm{Mon}\mathscr{C} \to \mathscr{C}$ is the composite of $V : \mathrm{Mon}\mathscr{C} \to \mathrm{Pt}\mathscr{C}$ which forgets the multiplication of a monoid but remembers the unit, and $W : \mathrm{Pt}\mathscr{C} \to \mathscr{C}$, which forgets the point. Since adjunctions compose, to find a left adjoint to $U = WV$, it will suffice to find adjoints to V and to W. But W has a left adjoint sending $C \in \mathscr{C}$ to the coproduct injection $I \to I + C$, provided that coproducts with I exists, so in this case we are reduced to finding a left adjoint to V. This reduction played a key role in [3], which contained a construction that will be important below (as well as various transfinite variants which will not). We describe below one point of view (not contained in [3]) on this construction.

Thus we seek a left adjoint to $V : \mathrm{Mon}\mathscr{C} \to \mathrm{Pt}\mathscr{C}$. In order to motivate the construction, we recall here the connection between monoids and the simplicial category [8, Chapter VII]. We follow Mac Lane in writing $\mathbf{\Delta}$ for the category of finite ordinals and order-preserving maps: this is the "algebraist's simplicial category", as opposed to the "topologist's simplicial category" which omits the empty ordinal, and reindexes the remaining objects. Now $\mathbf{\Delta}$ is monoidal with respect to ordinal sum, and "classifies monoids in monoidal categories", in the sense that for any monoidal category $(\mathscr{C}, \otimes, I)$, the category $\mathrm{Mon}\mathscr{C}$ of monoids in $\mathscr{C}$ is equivalent to the category $\mathscr{M}(\mathbf{\Delta}, \mathscr{C})$ of strong monoidal (=tensor-preserving) functors from $\mathbf{\Delta}$ to $\mathscr{C}$, and monoidal natural transformations. The strong monoidal functor corresponding to a monoid M in $\mathscr{C}$ with multiplication $\mu : M^2 \to M$ and unit $\eta : I \to M$ has image

$$\begin{tikzcd} I \arrow[r, "\eta"] & M \arrow[r, shift left=4, "\eta M"] \arrow[r, shift right=4, "M\eta"'] & M^2 \arrow[l, "m"'] \arrow[r, shift left=8, "\eta M^2"] \arrow[r, "M\eta M"] \arrow[r, shift right=8, "M^2\eta"'] & M^3 \dots \arrow[l, shift right=4, "M\mu"'] \arrow[l, shift left=4, "\mu M"] \end{tikzcd}$$

There is an analogous description of $\mathrm{Pt}\mathscr{C}$: let $\mathbf{\Delta}_{\boldsymbol{mon}}$ be the (non-full) subcategory of $\mathbf{\Delta}$ containing all the objects but only the injective order-preserving maps. This is still monoidal under ordinal sum, and now $\mathrm{Pt}\mathscr{C}$ is equivalent to the category $\mathscr{M}(\mathbf{\Delta}_{\boldsymbol{mon}}, \mathscr{C})$ of strong monoidal functors from $\mathbf{\Delta}_{\boldsymbol{mon}}$ to $\mathscr{C}$ and monoidal natural transformations. Corresponding to the pointed object $(Y, y : I \to Y)$ we have

$$\begin{tikzcd} I \arrow[r, "y"] & Y \arrow[r, shift left=2, "yY"] \arrow[r, shift right=2, "Yy"'] & Y^2 \arrow[r, shift left=4, "yY^2"] \arrow[r, "YyY"] \arrow[r, shift right=4, "Y^2y"'] & Y^3 \dots \end{tikzcd} \tag{$*$}$$

If we identify $\mathrm{Mon}\mathscr{C}$ with $\mathscr{M}(\mathbf{\Delta}, \mathscr{C})$ and $\mathrm{Pt}\mathscr{C}$ with $\mathscr{M}(\mathbf{\Delta}_{\boldsymbol{mon}}, \mathscr{C})$, then the forgetful $V : \mathrm{Mon}\mathscr{C} \to \mathrm{Pt}\mathscr{C}$ is identified with the functor $\mathscr{M}(H, \mathscr{C}) : \mathscr{M}(\mathbf{\Delta}, \mathscr{C}) \to \mathscr{M}(\mathbf{\Delta}_{\boldsymbol{mon}}, \mathscr{C})$ given by composition with the inclusion $H : \mathbf{\Delta}_{\boldsymbol{mon}} \to \mathbf{\Delta}$. If we were dealing with ordinary functors rather than strong monoidal ones, in other words if we sought a left adjoint to $\mathbf{Cat}(H, \mathscr{C}) : \mathbf{Cat}(\mathbf{\Delta}, \mathscr{C}) \to \mathbf{Cat}(\mathbf{\Delta}_{\boldsymbol{mon}}, \mathscr{C})$, then we could simply take the left Kan extension along H. In general this left Kan extension will not send strong monoidal functors to strong monoidal functors, but in special cases it does, and in fact provides the left adjoint to $\mathscr{M}(H, \mathscr{C})$. In such a case, if we form the strong monoidal functor $\mathbf{\Delta}_{\boldsymbol{mon}} \to \mathscr{C}$ corresponding to a pointed object (Y, y), and take its left Kan extension along H, the resulting strong monoidal functor from $\mathbf{\Delta}$ to $\mathscr{C}$ will correspond, via the equivalence $\mathscr{M}(\mathbf{\Delta}, \mathscr{C}) \simeq \mathrm{Mon}\mathscr{C}$, to a monoid in $\mathscr{C}$; and the underlying object of this monoid is precisely the colimit of the diagram $(*)$ above, as a simple calculation involving the coend formula for left Kan extensions shows. This, then, is the construction of Dubuc (in its simplest form where transfinite constructions are not required): if the colimit of $(*)$ exists and is preserved by tensoring on either side, then it has a monoid structure which is free on the pointed object $(Y, y : I \to Y)$.

3 The Construction

Colimits indexed by $\mathbf{\Delta}_{mon}$ can be constructed iteratively using coequalizers and colimits of chains, as we shall do below. Now many important functors do not preserve all coequalizers, but do preserve coequalizers of reflexive pairs (pairs which have a common section). There is also a general way to replace a pair $f, g : A \rightrightarrows B$ by a reflexive pair with the same coequalizer: replace A by $A + B$, and then use the identity map on B, as in

$$\begin{tikzcd} A + B \arrow[r, shift left, "(f\ B)"] \arrow[r, shift right, "(g\ B)"'] & B. \end{tikzcd}$$

The construction given here amounts to an analogous adaptation of the construction of Dubuc described in the previous section. Of course when both constructions work, they agree; in particular this will be the case if coproducts are preserved by tensoring on either side.

Suppose then that $(Y, y : I \to Y)$ is a pointed object, and form the corresponding diagram $(*)$. The colimit of $(*)$ can be constructed as follows. For each n and each $k = 0, \ldots n - 2$, form the coequalizer

$$\begin{tikzcd} Y^{n-1} \arrow[r, shift left, "Y^k y Y^{n-k-1}"] \arrow[r, shift right, "Y^{k+1} y Y^{n-k-2}"'] & Y^n \arrow[r, "r_{n,k}"] & Y^n_k. \end{tikzcd}$$

Now form the cointersection $r_n : Y^n \to Y_n$ of the $r_{n,k} : Y^n \to Y^n_k$ as k runs from 0 to $n - 2$. A straightforward calculation shows that for all j and k, the composites

$$\begin{tikzcd} Y^{n-2} \arrow[r, shift left, "Y^k y Y^{n-k-2}"] \arrow[r, shift right, "Y^{k+1} y Y^{n-k-3}"'] & Y^{n-1} \arrow[r, "Y^j y Y^{n-j-1}"] & Y^n \arrow[r, "r_n"] & Y_n \end{tikzcd}$$

agree, while the composite $r_n.Y^j y Y^{n-j-1}$ is independent of j, and so by the universal property of $r_{n-1} : Y^{n-1} \to Y_{n-1}$ there is a unique map $h_n : Y_{n-1} \to Y_n$ such that the square

$$\begin{tikzcd} Y^{n-1} \arrow[r, "Y^j y Y^{n-j-1}"] \arrow[d, "r_{n-1}"'] & Y^n \arrow[d, "r_n"] \\ Y_{n-1} \arrow[r, "h_n"'] & Y_n \end{tikzcd}$$

commutes for all j. The colimit of $(*)$, corresponding to the construction of Dubuc, is the colimit of the chain consisting of the Y_n with connecting maps h_n.

We modify this at the first step only, replacing coequalizers by reflexive coequalizers, as follows. The original coequalizers can be written as

$$\begin{tikzcd} Y^{n-1} = Y^k Y Y^{n-k-2} \arrow[r, shift left, "Y^k y Y Y^{n-k-2}"] \arrow[r, shift right, "Y^k Y y Y^{n-k-2}"'] & Y^n \arrow[r, "r_{n,k}"] & Y^n_k \end{tikzcd}$$

and our new reflexive coequalizers are

$$\begin{tikzcd} Y^k(Y+Y^2)Y^{n-k-2} \arrow[r, shift left=2, "f_{n,k}"] \arrow[r, shift right=2, "g_{n,k}"'] & Y^n \arrow[l, "d_{n,k}"'] \arrow[r, "q_{n,k}"] & Z^n_k \end{tikzcd}$$

where $f_{n,k} = Y^k(yY \; Y^2)Y^{n-k-2}$, $g_{n,k} = Y^k(Yy \; Y^2)Y^{n-k-2}$, and $d_{n,k} = Y^k b\, Y^{n-k-2}$, with b the coproduct injection $Y^2 \to Y + Y^2$. Then, as before, we shall form the cointersection $q_n : Y^n \to Z_n$ of the $q_{n,k}$, the induced maps $j_n : Z_{n-1} \to Z_n$ (see below) satisfying $j_n.q_{n-1} = q_n.(Y^k y Y^{n-k-1})$, and the colimit Z of the chain consisting of the Z_n and the j_n; we write $z_n : Z_n \to Z$ for the legs of the colimit cocone.

It is, however, worth taking a little more time to justify the existence of the j_n. We must show that for all j and k, the composites

$$\begin{tikzcd} Y^k(Y+Y^2)Y^{n-k-3} \arrow[r, shift left, "Y^k(yY \; Y^2)Y^{n-k-3}"] \arrow[r, shift right, "Y^k(Yy \; Y^2)Y^{n-k-3}"'] & Y^{n-1} \arrow[r, "Y^j y Y^{n-j-1}"] & Y^n \arrow[r, "q_n"] & Z_n \end{tikzcd}$$

are equal. This is completely straightforward if either $j \le k$ or $j \ge k+2$, but the case $j = k+1$ is a bit more complicated; it can be broken down as in the following diagram:

$$\begin{tikzcd}
 & & Y^n \arrow[rddd, "q_n"] & \\
 & Y^{n-1} \arrow[ru, "Y^{k+1} y Y^{n-k-2}"] \arrow[rd, "Y^{k+2} y Y^{n-k-3}"] & & \\
 & & Y^n \arrow[rd, "q_n"] & \\
Y^k(Y+Y^2)Y^{n+k-3} \arrow[ruu, "Y^k(yY \; Y^2)Y^{n+k-3}"] \arrow[r, "Y^k(Y+Y^2)yY^{n-k-3}"'] \arrow[rdd, "Y^k(Yy \; Y^2)Y^{n-k-3}"'] & Y^k(Y+Y^2)Y^{n-k-2} \arrow[ru, "Y^k(yY \; Y^2)Y^{n-k-2}"] \arrow[rd, "Y^k(Yy \; Y^2)Y^{n-k-2}"'] & & Y_n \\
 & & Y^n \arrow[ru, "q_n"] & \\
 & Y^{n-1} \arrow[ru, "Y^{k+2} y Y^{n-k-3}"] \arrow[rd, "Y^{k+1} y Y^{n-k-2}"'] & & \\
 & & Y^n \arrow[ruuu, "q_n"'] &
\end{tikzcd}$$

in which the individual regions are easily seen to commute.

In the following section we prove that Z can be made into a monoid which is free on (Y, y). We record the general result as:

Theorem 1 *Let $\mathscr{C}$ be a monoidal category with finite limits and countable colimits, and the functors $- \otimes C$ and $C \otimes -$ preserve reflexive coequalizers and colimits of countable chains. This includes in particular the case where $\mathscr{C}$ has the stated limits and colimits and $C \otimes -$ and $- \otimes C$ preserve sifted colimits. Then the free monoid on a pointed object (Y, y) exists, and its underlying object Z can be calculated as above. The free monoid on an object X is found by taking Y to be $I + X$ and y to be the coproduct injection.*

4 The Proof

Suppose that $\mathscr{C}$ satisfies the conditions of the theorem. Our construction involved three types of colimit: reflexive coequalizers, finite cointersections of regular epimorphisms, and colimits of chains. By assumption, the first and third of these are preserved by tensoring; we shall see that the second is also preserved. We defer for the moment the proof, merely noting that the case of binary cointersections suffices, and recording:

Lemma 2 *If $q : B \to C$ and $q' : B \to C'$ are regular epimorphisms, then their cointersection (pushout)*

$$\begin{tikzcd} B \arrow[r, "q"] \arrow[d, "q'"'] & C \arrow[d, "r"] \\ C' \arrow[r, "r'"'] & D \end{tikzcd}$$

is preserved by tensoring on either side.

We also need the following form of the "3-by-3 lemma" [4] for reflexive coequalizers, whose proof is once again deferred.

Lemma 3 (3-by-3 lemma) *If*

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "h_1"] \arrow[r, shift right=2, "h_2"'] & A_2 \arrow[l] \arrow[r, "h"] & A_3 \end{tikzcd}$$

$$\begin{tikzcd} B_1 \arrow[r, shift left=2, "k_1"] \arrow[r, shift right=2, "k_2"'] & B_2 \arrow[l] \arrow[r, "k"] & B_3 \end{tikzcd}$$

are reflexive coequalizers, preserved by tensoring on either side, then

$$\begin{tikzcd} A_1 \otimes B_1 \arrow[r, shift left=2, "h_1 \otimes k_1"] \arrow[r, shift right=2, "h_2 \otimes k_2"'] & A_2 \otimes B_2 \arrow[l] \arrow[r, "h \otimes k"] & A_3 \otimes B_3 \end{tikzcd}$$

is also a reflexive coequalizer, and $A_3 \otimes B_3$ is the cointersection of $A_3 \otimes B_2$ and $A_2 \otimes B_3$ (as quotients of $A_2 \otimes B_2$). This shows in particular that regular epimorphisms are closed under tensoring.

We need to construct a multiplication $\mu : ZZ \to Z$. The idea will be first to construct $\mu_{m,n} : Z_m Z_n \to Z_{m+n}$, then show that they pass to the colimit to give the desired μ.

4.1 Construction of $\mu_{m,n}$

Consider first the diagram

$$\begin{tikzcd} Y^m Z_l^n \arrow[r, "\cong"] & Z_{m+l}^{m+n} \\ Y^m Y^n \arrow[u, "Y^m q_l^n"] \arrow[r, "\pi_{m,n}"] \arrow[d, "q_k^m Y^n"'] & Y^{m+n} \arrow[u, "q_{m+l}^{m+n}"'] \arrow[d, "q_k^{m+n}"'] \\ Z_k^m Y^n \arrow[r, "\cong"] & Z_k^{m+n} \end{tikzcd}$$

which we shall build out of the canonical isomorphism $\pi_{m,n} : Y^m Y^n \cong Y^{m+n}$. Since $q_k^m : Y^m \to Z_k^m$ was constructed as the coequalizer of maps $f_{m,k}$ and $g_{m,k}$, thus $q_k^m Y^n : Y^{m+n} = Y^m Y^n \to Z_k^m Y^n$ can be constructed as the coequalizer of $f_k^m Y^n$ and $g_k^m Y^n$; that is, of f_k^{m+n} and g_k^{m+n}; thus we get the induced isomorphism $Z_k^m Y^n \cong Z_k^{m+n}$ at the bottom of the diagram. Similarly the coequalizer defining Z_l^n is preserved by tensoring on the left by Y^m and so we get the induced isomorphism $Y^m Z_l^n \cong Z_{m+l}^{m+n}$ at the top. Thus $Z_m Y^n$ is the cointersection of all the Z_p^{m+n} with $0 \le p \le m-2$, and $Y^m Z_n$ is the cointersection of all the Z_p^{m+n} with $m \le p \le m+n-2$. By the 3-by-3 lemma,

$$\begin{tikzcd} Y^m Y^n \arrow[r, "Y^m q_n"] \arrow[d, "q_m Y^n"'] & Y^m Z_n \arrow[d, "q_m Z_n"] \\ Z_m Y^n \arrow[r, "Z_m q_n"'] & Z_m Z_n \end{tikzcd}$$

is a cointersection, and so $Z_m Z_n$ is the cointersection of all the Z_p^{m+n} with $0 \le p \le m-2$ or $m \le p \le m+n-2$. On the other hand Z_{m+n} is the cointersection of all the Z_p^{m+n} with $0 \le p \le m+n-2$, and so there is a canonical quotient map $\mu_{m,n} : Z_m Z_n \to Z_{m+n}$ fitting into the commutative diagram

$$\begin{tikzcd} Y^m Y^n \arrow[r, "\pi_{m,n}"] \arrow[d, "q_m q_n"'] & Y^{m+n} \arrow[d, "q_{m+n}"] \\ Z_m Z_n \arrow[r, "\mu_{m,n}"'] & Z_{m+n} \end{tikzcd}$$

4.2 Construction of μ

Since tensoring preserves colimits of chains, we have

$$ZZ = Z \otimes Z = (\mathrm{colim}_m Z_m) \otimes (\mathrm{colim}_n Z_n) \cong \mathrm{colim}_m \mathrm{colim}_n (Z_m \otimes Z_n)$$

so there will be a unique map $\mu : ZZ \to Z$ making

$$\begin{tikzcd}
Z_m Z_n \arrow[r, "\mu_{m,n}"] \arrow[d] & Z_{m+n} \arrow[d] \\
ZZ \arrow[r, "\mu"'] & Z
\end{tikzcd}$$

commute provided that the $\mu_{m,n}$ are compatible with the maps $j_n : Z_n \to Z_{n+1}$ (and j_m); in other words that the maps $\mu_{m,n}$ are natural in m and n. We explain the naturality in n; the case of m is similar. In the first diagram below, the left square commutes by definition of $\mu_{m,n}$, and the right square commutes by definition of j_{m+n+1}. In the second diagram, commutativity of the left square follows from the definition of j_{n+1}, while the right square commutes by definition of $\mu_{m,n+1}$.

$$\begin{tikzcd}
Y^m Y^n \arrow[r, "\pi_{m,n}"] \arrow[d, "q_m q_n"'] & Y^{m+n} \arrow[r, "Y^{m+n} y"] \arrow[d, "q_{m+n}"'] & Y^{m+n+1} \arrow[d, "q_{m+n+1}"'] \\
Z_m Z_n \arrow[r, "\mu_{m,n}"'] & Z_{m+n} \arrow[r, "j_{m+n+1}"'] & Z_{m+n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y^m Y^n \arrow[r, "Y^m Y^n y"] \arrow[d, "q_m q_n"'] & Y^m Y^{n+1} \arrow[r, "\pi_{m,n+1}"] \arrow[d, "q_m q_{n+1}"'] & Y^{m+n+1} \arrow[d, "q_{m+n+1}"'] \\
Z_m Z_n \arrow[r, "Z_m j_{n+1}"'] & Z_m Z_{n+1} \arrow[r, "\mu_{m,n+1}"'] & Z_{m+n+1}
\end{tikzcd}$$

Now the composites across the top of the two diagrams agree, by naturality of associativity, and the (common) left vertical $q_m q_n$ is a regular epimorphism, so that the composites across the bottom agree. This gives the desired naturality in n, and so we obtain the required map $\mu : ZZ \to Z$.

4.3 Verification of Associative and Unit Laws

The associative law $\mu.\mu Z = \mu.Z\mu$ will hold provided that

$$\begin{tikzcd}
Z_m Z_n Z_p \arrow[r, "\mu_{m,n} Z_p"] \arrow[d, "Z_m \mu_{n,p}"'] & Z_{m+n} Z_p \arrow[d, "\mu_{m+n,p}"] \\
Z_m Z_{n+p} \arrow[r, "\mu_{m,n+p}"'] & Z_{m+n+p}
\end{tikzcd}$$

commutes for all m, n, and p. Now the two paths around this square will agree provided that they agree when composed with the regular epimorphism $q_m q_n q_p : Y^m Y^n Y^p \to Z_m Z_n Z_p$, and this in turn follows from the evident commutativity of

$$\begin{array}{ccc} Y^m Y^n Y^p & \xrightarrow{\pi_{m,n} Y^p} & Y^{m+n} Y^p \\ {\scriptstyle Y^m \pi_{n,p}} \downarrow & & \downarrow {\scriptstyle \pi_{m+n,p}} \\ Y^m Y^{n+p} & \xrightarrow[\pi_{m,n+p}]{} & Y^{m+n+p}. \end{array}$$

The unit is given by the composite

$$I \xrightarrow{y} Y = Z_1 \xrightarrow{z_1} Z$$

where z_1 is the relevant leg of the colimit cocone; the verification of the unit law is similar to but easier than the verification of associativity.

4.4 Universal Property

The unit of the adjunction will be the map

$$Y = Z_1 \xrightarrow{z_1} Z$$

of pointed objects; we must show that this has the appropriate universal property. In other words, for every monoid $M = (M, \mu, \eta)$ and every morphism $f : (Y, y) \to (M, \eta)$ of pointed objects, we must show that there is a unique monoid morphism $g : (Z, \mu, \eta) \to (M, \mu, \eta)$ with $gz_1 = f$.

For each n, we have the composite f_n as in

$$Y^n \xrightarrow{f^n} M^n \xrightarrow{\mu_{(n)}} M$$

where $\mu_{(n)}$ is the n-ary multiplication operation for the monoid M. We must show that these maps $f_n = \mu_{(n)}.f^n$ pass to the quotient to give $g_n : Z_n \to M$. We check only that the composites in

$$Y + Y^2 \underset{(Yy\ Y^2)}{\overset{(yY\ Y^2)}{\rightrightarrows}} Y^2 \xrightarrow{f^2} M^2 \xrightarrow{\mu} M$$

are equal; the other cases all follow by functoriality of $\otimes$. Now the two displayed composites are maps out of a coproduct, so will agree if their components do; for the components on Y^2 this is trivial, and for the components on Y we have

$$\mu.f^2.yY = \mu.\eta M.f = f = \mu.M\eta.f = \mu.f^2.Yy.$$

Thus the maps $\mu_{(n)}.f^n$ induce maps $g_n : Z_n \to M$, which clearly pass to the colimit to give $g : Z \to M$. We must show that this is a monoid map, and is the unique such which extends f.

Now g preserves the unit by construction, and will preserve the multiplication provided that

$$\begin{tikzcd} Z_m Z_n \arrow[r, "g_m g_n"] \arrow[d, "\mu_{m,n}"'] & MM \arrow[d, "\mu"] \\ Z_{m+n} \arrow[r, "g_{m+n}"'] & M \end{tikzcd}$$

commutes. But $Z_m Z_n$ is a quotient of $Y^m Y^n$, so this in turn restricts to commutativity of

$$\begin{tikzcd} Y^m Y^n \arrow[r, "f_m f_n"] \arrow[d, "\pi_{m,n}"'] & MM \arrow[d, "\mu"] \\ Y^{m+n} \arrow[r, "f_{m+n}"'] & M \end{tikzcd}$$

which holds by construction of the f_n and associativity of μ.

This proves that g is a monoid map; it remains to show the uniqueness. Suppose then that $h : Z \to M$ is a monoid map, with $h.z_1 = f$. In order to show that $h = g$, it will suffice to show that $h.z_n = g_n$ for all n. This in turn will hold if $h.z_n.q_n = f_n$ for all n. Thus we must show that the exterior of the diagram

$$\begin{tikzcd} Y^n \arrow[r, "z_1^n"] \arrow[d, "q_n"'] & Z^n \arrow[r, "h^n"] \arrow[d, "\mu_{(n)}"] & M^n \arrow[d, "\mu_{(n)}"] \\ Z_n \arrow[r, "z_n"'] & Z \arrow[r, "h"'] & M \end{tikzcd}$$

commutes. The right square commutes because h is a monoid homomorphism, so it suffices to show that the left square commutes, and this follows from the definition of $\mu : Z^2 \to Z$ by a straightforward induction.

4.5 Proof of Lemmas

Consider a diagram

$$\begin{tikzcd} A_{11} \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, shift right, "f_2'"'] \arrow[d, shift left, "f_1'"] & A_{12} \arrow[d, shift right, "g_2'"'] \arrow[d, shift left, "g_1'"] \\ A_{21} \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A_{22} \end{tikzcd}$$

in which $g_i . f'_j = g'_j . f_i$ for $i, j \in \{1, 2\}$, and suppose also that there exist $s : A_{12} \to A_{11}$ and $s' : A_{21} \to A_{11}$ with $f_1 .s = f_2 .s = 1$ and $f'_1 .s' = f'_2 .s' = 1$.

Then a map $x : A_{22} \to B$ satisfies $x.g'_1 . f_1 = x.g'_2 . f_2$ if and only if it satisfies $x.g'_1 = x.g'_2$ and $x.g_1 = x.g_2$. For if the former equation holds then we have

$$x.g'_1 = x.g'_1 . f_1 .s = x.g'_2 . f_2 .s = x.g'_2$$

$$x.g_1 = x.g_1 . f'_1 .s' = x.g'_1 . f_1 .s' = x.g'_2 . f_2 .s' = x.g_2 . f'_2 .s' = x.g_2$$

while if the latter two equations hold then

$$x.g'_1 . f_1 = x.g'_2 . f_1 = x.g_1 . f'_2 = x.g_2 . f'_2 = x.g'_2 . f_2 .$$

As a result we have:

Proposition 1 *In the situation above, the coequalizer of $g'_1 . f_1$ and $g'_2 . f_2$ is the cointersection of the coequalizer of g_1 and g_2 and the coequalizer of g'_1 and g'_2.*

To prove the 3-by-3 lemma (Lemma 3), apply this in the case of

$$\begin{tikzcd}
A_1 \otimes B_1 \arrow[r, shift left, "A_1\otimes k_1"] \arrow[r, shift right, "A_1\otimes k_2"'] \arrow[d, shift right, "h_2\otimes B_1"'] \arrow[d, shift left, "h_1\otimes B_1"] & A_1 \otimes B_2 \arrow[d, shift right, "h_2\otimes B_2"'] \arrow[d, shift left, "h_1\otimes B_2"] \\
A_2 \otimes B_1 \arrow[r, shift left, "A_2\otimes k_1"] \arrow[r, shift right, "A_2\otimes k_2"'] & A_2 \otimes B_2
\end{tikzcd}$$

noting that the $A_1 \otimes k_i$ have a common section $A_1 \otimes t$, and the $h_1 \otimes B_1$ have a common section $s \otimes B_1$.

To prove Lemma 2, let $q : B \to C$ and $q' : B \to C'$ be the coequalizers of the reflexive pairs

$$\begin{tikzcd}
A \arrow[r, shift left=2, "h_1"] \arrow[r, shift right=2, "h_2"'] & B \arrow[l, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, shift left=2, "h'_1"] \arrow[r, shift right=2, "h'_2"'] & B \arrow[l, "s'"']
\end{tikzcd}$$

and form the universal object P with morphisms

$$\begin{tikzcd}
P \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] \arrow[d, shift right, "k'_2"'] \arrow[d, shift left, "k'_1"] & A' \arrow[d, shift right, "h'_2"'] \arrow[d, shift left, "h'_1"] \\
A \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & B
\end{tikzcd}$$

satisfying equations as above. In terms of elements, this would be formed as $\{x_1, x_2 \in A, x'_1, x'_2 \in A' \mid h_i(x'_j) = h'_j(x_i)\}$. It is straightforward to show that the relevant pairs are reflexive; for example $x' \mapsto \big(sh'_1(x'), sh'_2(x'), x', x'\big)$ provides a common section to k_1 and k_2. The proposition then reduces the cointersection of q and q' to the reflexive coequalizer of $h'_1.k_1$ and $h'_2.k_2$, which by assumption is preserved by tensoring.

5 Examples

If $\mathscr{C}$ is any variety, equipped with the cartesian product $\times$ as tensor product, then products, reflexive coequalizers, and colimits of chains are all computed as in **Set**, and since the product in **Set** with a fixed object is cocontinuous, it follows that tensoring in $\mathscr{C}$ with a fixed object preserves the relevant colimits.

For a fixed set A, the category **Span**(A, A) of spans from A to A is the category of all sets over $A \times A$. This is monoidal via pullback

$$\begin{tikzcd}
 & & X \times_A Y \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
A & & A & & A
\end{tikzcd}$$

and a monoid in the resulting monoidal category is precisely a category with object-set A. Tensoring on either side is cocontinuous (and in fact has an adjoint) because pullbacks in **Set** are cocontinuous. But now we can move from **Set** to a category $\mathscr{E}$ with finite limits in which pullback may not be cocontinuous, but does preserve reflexive coequalizers and colimits of chains: this is the case, for example, in any variety. If we consider an object $A \in \mathscr{E}$, and the category **Span**$(\mathscr{E})(A, A)$ of internal spans in $\mathscr{E}$ from A to A this is once again monoidal, and once again a monoid is a category in $\mathscr{E}$ with A as its object of objects. But this time tensoring on either side preserves reflexive coequalizers and colimits of chains, but not arbitrary colimits. Thus our construction gives free internal categories in $\mathscr{E}$.

For a more structured example, one could consider not **Span**$(\mathscr{E})$ but **Prof**$(\mathscr{E})$, the bicategory of internal categories in $\mathscr{E}$ and profunctors between them. Fixing an internal category A, we get a monoidal category **Prof**$(\mathscr{E})(A, A)$, and once again the conditions for our construction will be satisfied. Taking $\mathscr{E} =$ **Mon**, the category of monoids, and A to be (a suitable strict version of) the monoidal category $\mathbb{P}$ of finite sets and bijections, we get a monoidal category **Prof**(**Mon**)$(\mathbb{P}, \mathbb{P})$ in which monoids are precisely PROPs (see [7]), and so a different notion of free PROP to that given in [10].

Finally, for a slightly childish example, take the category $\mathscr{C}$ to be the category **Grp** of groups and group homomorphisms, with the cartesian monoidal structure (with the product as tensor product). For a group G the functors $G \times - :$ **Grp** $\to$ **Grp** and $- \times G :$ **Grp** $\to$ **Grp** do not of course preserves all colimits, but they do preserve reflexive coequalizers and colimits of chains, as would be the case with any variety in place of **Grp**. Now by the "Eckmann-Hilton argument", a monoid in **Grp** is precisely an abelian group. So our construction reduces to the abelianization of a group.

6 Algebraically Free Monoids

The free monoid construction we have been looking at involves a pointed object (Y, y) a monoid (Z, μ, η), and a morphism of pointed objects $k : (Y, y) \to (Z, \eta)$. But there is another possible universal property that such data might satisfy. Write $\mathscr{C}^{(Z,\mu,\eta)}$ for the category of objects of $\mathscr{C}$ equipped with an action of the monoid (Z, η, μ), and write $\mathscr{C}^{(Y,y)}$ for the category of objects of $\mathscr{C}$ equipped with an action of the pointed object (Y, y): in other words, a morphism $\alpha : YA \to A$ satisfying $\alpha.yA = 1$. There is a functor $k^* : \mathscr{C}^{(Z,\eta,\mu)} \to \mathscr{C}^{(Y,y)}$ sending A equipped with Z-action $\beta : ZA \to A$ to A equipped with Y-action

$$YA \xrightarrow{kA} ZA \xrightarrow{\beta} A.$$

When this functor k^* is an isomorphism of categories we say that k exhibits (Z, μ, η) as the *algebraically free* monoid on (Y, y) [5]. The algebraically free monoid, if it exists, is free, but it is possible for a free monoid to exist without being algebraically free; nonetheless under fairly general conditions the algebraically free monoid exists (and is free); see [5, Section 23]. This means that the free monoid on (Y, y) can be found by calculating free (Y, y)-actions, an idea that implicitly goes back to [2]. All cases where the free monoid is computed in [5] are done in this way.

Under the hypotheses of our theorem, a necessary and sufficient condition for the free monoid on (Y, y) to be algebraically free is that for any action $\alpha : YA \to A$ of (Y, y) on A, the composites

$$\begin{tikzcd} (Y+Y^2)A \arrow[r, shift left, "(yY\ Y^2)A"] \arrow[r, shift right, "(Yy\ Y^2)A"'] & Y^2A \arrow[r, "Y\alpha"] & YA \arrow[r, "\alpha"] & A \end{tikzcd}$$

agree.

In general there seems to be no reason why this should always be true, although we do not have a specific example where it fails. We therefore conjecture that the hypotheses of our theorem are not sufficient to guarantee that the free monoid is algebraically free.

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