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Characterizing (quasi-)ultrametric finite spaces in terms of (directed) graphs

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ABSTRACT

Let $D = (V, A)$ be a complete directed graph (digraph) with a positive real weight function $d : A \to \{d_1, \dots, d_k\} \subseteq \mathbf{R}_+$ such that $0 < d_1 < \cdots < d_k$. For every $i \in [k] = \{1, \dots, k\}$, let us set $A_i = \{(u, w) \in A \mid d(u, w) \le d_i\}$ and assume that each subgraph $D_i = (V, A_i)$, $i \in [k]$, in the obtained nested family is transitive, that is, $(u, w) \in A_i$ whenever $(u, v), (v, w) \in A_i$ for some $v \in V$. This assumption implies that the considered weighted digraph (D, d) defines a quasi-ultrametric finite space (QUMFS) and, conversely, each QUMFS is uniquely (up to an isometry) is realized by a nested family of transitive digraphs.

These simple observations imply important corollaries. For example, each QUMFS can be realized by a multi-pole flow network. Furthermore, $k \le \binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2)$, where $n = |V|$, and this upper bound for the number k of pairwise distinct distances is precise. Moreover, we characterize all QUMFSes for which the equality holds.

In the symmetric case, $d(u, w) = d(w, u)$, we obtain a canonical representation of an ultrametric finite space (UMFS) together with the well-known bound $k \le n - 1$. Interestingly, due to this representation, a UMFS can be viewed as a positional game structure of k players $\{1, \dots, k\}$ such that, in every play, they make moves in a monotone strictly decreasing order.

1. Introduction

1.1. Finite (quasi-)ultrametric spaces and their (directed) graphs

Given a finite set V and a mapping $d : V \times V \to \mathbf{R}_+$, let us consider the following three standard axioms:

- (i) $d(u, w) = 0$ if and only if $u = w$;
- (ii) $d(u, w) = d(w, u)$; for all $u, w \in V$;
- (iii) $d(u, w) \le \max(d(u, v), d(v, w))$; for all $u, v, w \in V$.

A pair (V, d) satisfying (i) and (iii) is called a *quasi-ultrametric finite space (QUMFS)* and the non-negative real number $d(u, w)$ is the *distance from* u to w. Furthermore, (V, d) is called an *ultrametric finite space (UMFS)* if (ii) holds too; in this case $d(u, w)$ is the *distance between* u and w. See [6] for related concepts and more details.

It is easily seen that in a UMFS the equality in (iii) holds whenever $d(u, v) \ne d(v, w)$; in other words, the largest two distances are equal in any triangle $u, v, w \in V$. For this reason, a UMFS is alternatively called an *isosceles* space; see, for example, [24,25].

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Example 1. Yet, QUMFSes are not necessarily isosceles; let us set, for example, $d(u, v) = 1$, $d(v, w) = 3$, $d(u, w) = 2$, and $d(v, u) = d(w, v) = d(w, u) = 10$.

A UMFS (respectively, QUMFS) can be conveniently represented by a positively weighted complete graph $G = (V, E)$, $d : E \to \mathbf{R}_+$ (respectively, digraph $D = (V, A)$, $d : A \to \mathbf{R}_+$) in which $|V| = n$, $|E| = \frac{1}{2}n(n-1)$, $|A| = n(n-1)$, and $|im(d)| = k$. From now on, we set $im(d) = \{d_1, \ldots, d_k\}$ and assume that $0 < d_1 < \cdots < d_k < \infty$.

Remark 1. A *pseudo*-(Q)UMFS is defined by the relaxation of these inequalities to $0 \le d_1 < \cdots d_k \le \infty$. (For brevity, we write "a (Q)UMFS" to refer to both a UMFS or QUMFS simultaneously.) Respectively, "if and only if" should be replaced by "if" in (i). The edges and arcs of the infinite length $d_k = \infty$ may be just deleted from E and A. Thus, (di)graphs of pseudo-(Q)UMFSes might be not complete.

For simplicity, we will restrict ourselves to the (Q)UMFSes (unless a pseudo-(Q)UMFS is mentioned explicitly, like in Sections 5.1 and 5.2) but keep in mind that almost all statements hold for the pseudo-(Q)UMFSes as well.

Obviously, the ultrametric inequality (iii) is respected by any change of the values of $d_1, \ldots, d_k$ provided their order is preserved. In particular, all statements that hold for the (Q)UMFSes are automatically extended to the pseudo-(Q)UMFSes, whenever the values $d_1 = 0$ and $d_k = \infty$ are allowed. E.g., in Example 1 we could set $d(u, v) = 0$ rather than 1 and/or $d(v, u) = d(w, v) = d(w, u) = \infty$ rather than 10 and get a pseudo-QUMFS with similar properties.

1.2. Main and related results

As it was announced in the title and explained in Abstract, in this paper, we will characterize (Q)UMFSes in terms of their (directed) graphs. In case of UMFSes our characterization is closely related to the recent observation of Demaine, Landau, and Weimann [5] that extends, in its turn, the fundamental results of Gomory and Hu [10,20,21]; see Section 1.3 for more details.

On the other hand, this characterization of the UMFSes can be viewed as a restriction of the one-to-one correspondence between the positional game structures and Π- and Δ-free k-graphs, studied by the first author in [11–14,17]. The UMFSes correspond to the structures in which players $\{1, \ldots, k\}$ move in a strictly monotone decreasing order in every play $p(v)$ from the initial position v_0 to a terminal one $v \in L$; see Section 2.

Remark 2. Let us recall that the semilattices were also characterized as special positional game structures by Libkin and the first author in [30].

We should also mention a significant contribution of Alex and Vladimir Lemin to characterizing ultrametric spaces [24–29]. In these works, mostly, general (infinite) spaces are described in an algebraic language. Yet, according to Lemin, [28], in the late 90s "Israel Gelfand set a problem to describe all *finite* ultrametric spaces up to isometry using *graph theory language*". In this paper we suggest a solution for this Gelfand's problem.

The QUMFSes are considered in the last three sections. In Section 3 they are characterized as nested families of transitive digraphs. This simple observation implies important corollaries.

In Section 4 it is shown that the number of pairwise distinct distances in an n-element QUMFS is at most $\binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2)$ and this bound is precise. Moreover, we characterize all QUMFSes for which the equality holds.

In Section 5 we study realizing (Q)UMFSes by networks. In Sections 5.1 and 5.2 we recall two classical problems (maximum flow and maximum bottleneck directed path) that result in QUMFSes. It is easy to show that each QUMFS is a bottleneck QUMFS. For the symmetric case (and UMFSes) this observation was mentioned by Leclerc in [23].

Even earlier, in [10,20,21], it was shown that each multi-pole maximum flow network defines a QUMFS and that every UMFS can be realized by a symmetric flow network.

Yet, we construct a QUMFS (D^0, d^0) which is not a *flow QUMFS*, that is, it cannot be realized by the set of *all* vertices of a flow network. (In fact, (D^0, d^0) is a unique minimal such QUMFS.) Moreover, we present a polynomial time algorithm recognizing whether a QUMFS (D, d) is a flow QUMFS and outputs a corresponding flow network in case of the positive answer.

In contrast, every QUMFS can be realized by a *subset* of vertices, or in other words, by a multi-pole flow network.

All these results are derived in Section 5.2 from our characterization of a QUMFS (as a nested family of transitive digraphs) obtained in Section 3.

It was shown in [18] (see also [15,16,19]) that the "flow" and "bottleneck" UMFSes both can be realized as the UMFSes of resistances, for an appropriate choice of the two parameters of a conductivity law; see Section 5.3.

Finally, in Section 5.4, we introduce reducible, universal, and complete families of (Q)UMFSes, give examples, and study simple relations between these families.

1.3. Gomory–Hu's representation of UMFSes

The seminal paper [10] begins with the following construction. Given a UMFS defined by a weighted complete graph $(G = (V, E),\ d : E \to \mathbf{R}_+)$, let us choose in (G, d) a lightest spanning tree $(T = (V, E'), d' : E' \to \mathbf{R}_+)$, where d' is the restriction of d to $E' \subseteq E$. Furthermore, for any $u, w \in V$ there is a unique path $p(u, w)$ in T between u and w. (This path consists of a single edge (u, w) if and only if $(u, w) \in E'$.)

Theorem 1 ([10]). *For all $u, w \in V$, the equality $d(u, w) = \max(d(e) \mid e \in p(u, w))$ holds.*

Proof. $\le$ can be easily derived by induction from the ultrametric inequality (iii), while $\ge$ follows immediately from the fact that T is a *lightest* spanning tree of (G, d). □

Let us remark that such a tree may be not unique. Yet, all spanning trees of the graph $G = (V, E)$ have $|V| = n$ vertices and $|E'| = n - 1$ edges. Moreover, all lightest such trees have the same weight distribution, which is uniquely defined by the Kruskal greedy algorithm [22]; see Section 2.2.3 for more details. The next two corollaries are obvious.

Corollary 1 ([10]). *For any weighted tree $T = (V, E'), d' : E' \to \mathbf{R}_+$, formula of Theorem 1 defines a UMFS. Conversely, any UMFS can be obtained from a lightest spanning tree of its weighted graph, by construction of Theorem 1.*

Proof. Given a weighted tree $(T = (V, E'), d')$, let us verify the ultrametric inequality (iii).

For any two vertices u and w of a tree there is a unique path $p(u, w)$ between them.

For any three vertices u, v, w of a tree there is a unique vertex o that belongs to all three paths $p(u, v), p(v, w), p(w, u)$. Then obviously, $p(u, v) = p(u, o) \cup p(o, v)$, $p(v, w) = p(v, o) \cup p(o, w)$, and $p(w, u) = p(w, o) \cup p(o, u)$.

Each of the above three unions consists of two paths that have the unique common vertex o. Let e be a heaviest edge among all edges of $p(u, o)$, $p(v, o)$ and $p(w, o)$. Without any loss of generality, we can assume that e belongs to $p(v, o)$. Then, $d(u, v) = d(v, w) \ge d(u, w)$ and (iii) holds.

All other claims of the Corollary, as well as the next Corollary, are straightforward. □

Corollary 2 ([10]). *In an n-vertex UMFS, the distances take at most $n - 1$ distinct values, that is, $|im(d)| \le n - 1$.* □

1.4. Cartesian binary trees of UMFSes

Recently, Demaine, Landau, and Weimann [5] applied Theorem 1 to assign a binary Cartesian tree to a UMFS as follows: delete from $(T = (V, E'), d' : E' \to \mathbf{R}_+)$ a heaviest edge, repeat the same for each of the obtained two weighted subtrees, etc., until only vertices of T remain. Obviously, this procedure will result in a binary rooted tree $T' = (V', E'', v_0)$ whose leaves $L \subseteq V'$ (respectively, all other vertices $N = V' \setminus L$) are in one-to-one correspondence with V (respectively, with E'). It is easily seen that for any two $u, w \in L = V$ the distance $d(u, w)$ is equal to the weight $d(v(u, w))$ of the lowest common ancestor $v(u, w)$ of u and w in T'. Furthermore, these weights monotone decrease (perhaps, non-strictly) along each path $p(v)$ from the root v_0 to a leaf $v \in L$.

2. Canonical representation of finite ultrametric spaces

2.1. Ultrametric spaces defined by labeled rooted trees

Let $T = (V, E)$ be a finite rooted tree in which $L \subseteq V$ is the set of leaves and $v_0 \in N = V \setminus L$ is the root. For any leaf $v \in L$, there is a unique path $p(v)$ from v_0 to v. Furthermore, let $d : N \to \{d_1, \dots, d_k\} \subseteq \mathbf{R}_+$ be a weight function such that $0 < d_1 < \cdots < d_k$. Then, for each two distinct leaves $u, w \in L$ let us set $d(u, w) = d(v(u, w))$, where $v(u, w) \in N$ is the lowest common ancestor of u and w, or in other words, the last common vertex of the paths $p(u)$ and $p(w)$.

By definition, $d(u, w) = d(w, u) \ge 0$ and standardly, we set $d(u, w) = 0$ if (and only if) $u = w$. Finally, for convenience we will make the following assumption:

- (a) Each vertex $v \in N$ has at least two immediate successors; or in other words, $\deg(v_0) \ge 2$, $\deg(u) \ge 3$ for all $u \in N \setminus \{v_0\}$, and $\deg(v) = 1$ for all leaves $v \in L$, by definition.

Under assumption (a), the following claim holds.

Proposition 1. *The ultrametric inequality, $d(v', v'') \le \max(d(v', v), d(v, v''))$ holds for all $v, v', v'' \in L$ if and only if the weights (non-strictly) monotone decrease along each path $p(v)$, $v \in L$.*

Proof. If all three vertices coincide then all three distances equal 0. If two vertices coincide then distance between them is 0, while two other distances are equal and non-negative. Obviously, the ultrametric inequality holds in both cases. Let us assume that $v, v', v'' \in L$ are pairwise distinct and let $u' = u(v, v')$, $u'' = u(v, v'')$, $u = u(v', v'') \in N$ be the lowest common ancestors of the corresponding three pairs of leaves (see Fig. 1). Obviously, at least two of these three ancestors coincide, say, $u' = u''$. It is also clear that in this case $u' = u'' \ge u$, that is, $u' = u''$ is an ancestor of u. Hence, the ultrametric inequality holds for $v, v', v'' \in L$ if and only if $d(u') = d(u'') \ge d(u)$. Moreover, it holds for any three leaves of L if and only if for any $u, u' \in N$ we have $d(u') \ge d(u)$ whenever u' is an ancestor of u. □

Let us note that Proposition 1 is applicable to the Cartesian binary trees, as well.

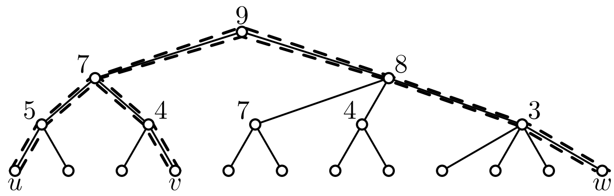

Fig. 1. The canonical representation of a UMFS; for example, $d(u, v) = 7$, $d(u, w) = d(v, w) = 9$.

2.2. Canonical weighted tree of a UMFS and its applications

2.2.1. Main construction

Let us consider a UMFS (G, d) given by a complete graph $G = (V, E)$ and positive weight function $d : E \to \{d_1, \dots, d_k\} \subseteq \mathbf{R}_+$ satisfying the ultrametric inequality. Standardly we assume that $0 < d_1 < \dots < d_k < \infty$.

Remark 3. Yet, all results extend the case of pseudo-QUMFSes, when $0 \le d_1$ and $d_k \le \infty$; see Remark 1.

For each $i \in [k] = \{1, \dots, k\}$, let us set $D_i = \{d_1, \dots, d_i\}$, $E_i = \{e \in E \mid d(e) \in D_i\}$, and $G_i = (V, E_i)$; in other words, G_i is the subgraph of G formed by the edges whose weights are at most d_i.

Proposition 2. *For every $i \in [k] = \{1, \dots, k\}$, the subgraph G_i is transitive, that is, $(u, w) \in E_i$ whenever $(u, v), (v, w) \in E_i$ for some $v \in V$. In other words, G_i is the union of pairwise vertex-disjoint cliques.*

Proof. Transitivity is just a reformulation of the ultrametric inequality (iii). Furthermore, a connected graph is a clique if and only if with every two adjacent edges it contains the whole triangle. $\square$

Remark 4. Let us note that in the above proof, the first claim can be extended to QUMFSes and digraphs (see Section 3), while the second one holds only for the UMFSes and non-directed graphs.

In particular, G_k is the total clique, while G_{k-1} is the union of $m \ge 2$ pairwise vertex-disjoint cliques each of which defines a proper ultrametric subspace (G', d') of (G, d). By construction, the largest weight d_k appears in the latter but not in the former, that is, $d_k \in im(d) \setminus im(d')$; in other words, the d_k-edges form a complete m-partite graph.

Each graph (G', d') can be similarly decomposed, in its turn, and we can proceed until every considered clique becomes a single vertex of V. Obviously, the above procedure results in a rooted tree $T = (V', E')$ whose set of leaves $L \subseteq V'$ is in one-to-one correspondence with V and every other vertex $u \in N = V' \setminus L$ is assigned to an intermediate clique of G and labeled by some d_i; in particular, the root $v_0 \in N$ is labeled by d_k.

By construction, T satisfies the assumption (a) as well as the following property:

- (b) Labels d_i *strictly* monotone decrease along each path $p(v)$ from v_0 to a leaf $v \in L$.

Thus, Propositions 1 and 2 result in the following canonical representation of the UMFSes.

Theorem 2. *The above construction is a one-to-one correspondence between the UMFSes and the labeled rooted trees satisfying the assumptions* (a) *and* (b). $\square$

An example illustrating this theorem is given in Fig. 1.

2.2.2. Canonical and Cartesian trees

Clearly, the above canonical representation of the UMFS (G, d) is related to its Cartesian trees. However, the former is unique, while the latter might be numerous and not satisfy (b). It is also clear that for every $i \in [k]$ the set of vertices labeled by d_i form a forest in a Cartesian tree. Contracting the corresponding subtrees (of every such forest, for all $i \in [k]$), we obtain a tree that satisfies (b) and still defines the same UMFS (G, d). By Theorem 2, it must be the canonical tree of (G, d). Thus, for all Cartesian trees, the above contraction results in the canonical tree. This statement is obvious for the Cartesian trees corresponding to a fixed minimum weight spanning tree of (G, d). Yet, there might be many such spanning trees and, hence, Theorem 2 is essential, in general.

The following bounds result from Theorem 2 immediately:

Corollary 3. *Let T be the canonical tree of a UMFS (G, d), then*

$$k \le |N_T| \le |L_T| - 1 = n - 1.$$

Each of the above two inequalities may turn into the equality: the second one does so if and only if the tree T is binary, while the first one if and only if each label d_i, $i \in [k]$, appears in T only once. $\square$

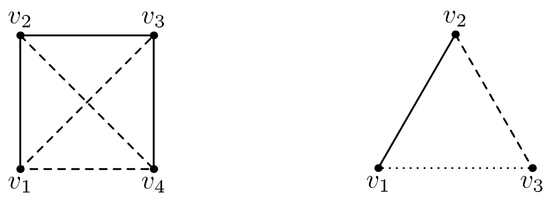

Fig. 2. k-graphs Π and Δ.

2.2.3. Enumerating all minimum weight spanning trees of a UMFS and counting their (unique) weight distribution

The canonical tree $T = (V', E')$ is instrumental for an efficient enumerating all minimum weight spanning trees in $(G = (V, E), d)$. Let us recall that G_{k-1} is the union of at least two pairwise vertex-disjoint cliques $C_1, \ldots, C_m \subseteq V$ of G such that every edge between two distinct cliques is of the largest weight d_k, while each edge within a clique is of a strictly lesser weight; in other words, the d_k-edges form the complete m-bipartite graph with parts $C_1, \ldots, C_m$.

Let us choose in G any $m - 1$ edges that would form a spanning tree in the factor-graph obtained from G by contracting each of the m cliques to a vertex. It is clear that all $m - 1$ chosen edges are of weight d_k and that every spanning tree on V must contain such a selection. Then, let us repeat the same procedure for each of the cliques $C_1, \ldots, C_m$, etc., until every obtained clique becomes a vertex. Obviously, this procedure results in a minimum weight spanning tree and, conversely, each such tree can be obtained in this way.

Combining these arguments with Broder and Mayr [4] algorithm for counting lightest spanning trees in graphs, we obtain a very efficient enumeration procedure. For any given k, it outputs the kth minimum weight spanning tree, with respect to the lexicographic order, in time poly$(\log k, n)$, where $n = |V|$.

Furthermore, the above procedure shows that all minimum weight spanning trees of a given UMFS $(G = (V, E), d)$ have a unique weight distribution, which is explicitly determined by the canonical tree $T = (V', E')$ as follows.

Let $S(v)$ denote the set of all immediate successors of a vertex $v \in V'$ and $s(v) = |S(v)|$. Clearly, $s(v_0) = \deg_T(v_0)$ for the root and $s(v) = \deg_T(v) - 1$ for any other vertex $v \in V'$; in particular, $s(v) = 0$ for a leaf $v \in L(T)$. It is easy to see that for each $i \in [k] = \{1, \ldots, k\}$ the corresponding weight d_i appears $\text{sum}_{v \in N | d(v) = d_i}(s(v) - 1)$ times.

2.3. Canonical trees of UMFSes and positional game structures

We will show that the above representation of a UMFS (G, d) by its canonical rooted labeled tree is a special case of the following one-to-one correspondence between complete edge-colored graphs and positional game structures studied in [11–14,17].

2.3.1. Complementary connected k-graphs

Let us label the edges of G by colors $i \in [k] = \{1, \ldots, k\}$ rather than by weights d_i.

A k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$ is a complete graph on vertices $V = \{v_1, \ldots, v_n\}$ whose $\binom{n}{2}$ edges are partitioned into k subsets (colored by k colors) some of which might be empty.

We assume that $k \geq 2$ and call a k-graph $\mathcal{G}$ *complementary connected* (CC) if $n \geq 2$ and the complement $\overline{G}_i$ to each of the k chromatic components $G_i = (V, E_i)$ of $\mathcal{G}$ is connected on V; or in other words, if for each $u, w \in V$ and $i \in [k]$ there is an i-free path $p(u, w)$ between u and w in $\mathcal{G}$. By convention, we assume that $\mathcal{G}$ is *not* CC when $n = 1$.

It is easy to verify that there is no CC k-graph with $n = 2$, either. Yet, they exist for any $k \geq 2$ and $n \geq 3$. The following two examples Π and Δ in Fig. 2 will play an important role:

Π is defined for any $k \geq 2$ by $V = \{v_1, v_2, v_3, v_4\}$; $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$, and $E_i = \emptyset$ whenever $i > 2$;

Δ is defined for any $k \geq 3$ by $V = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, $E_3 = \{(v_3, v_1)\}$, and $E_i = \emptyset$ whenever $i > 3$.

In other words, Δ is a three-colored triangle, while Π has two non-empty chromatic components each of which is isomorphic to P_4. It is easy to verify that Π and Δ are CC but their proper subgraphs are already not CC, for every $k \geq 2$. In other words, Π and Δ are minimal CC k-graphs. It was shown in [11] that there are no others; see also [13,14] for more details.

Theorem 3 *([11]). Every CC k-graph contains Π or Δ as a subgraph.* □

2.3.2. Canonical decomposition of Π-and Δ-free k-graphs

Given a Π-and Δ-free k-graph $\mathcal{G} = (V; E_1, \ldots, E_k)$, by Theorem 3, there is an $i \in [k]$ such that the complement $\overline{G}_i$ to the chromatic components $G_i = (V, E_i)$ is not connected on V. It is easy to show that such an $i \in [k]$ is unique.

Lemma 1. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on the common vertex-set V such that both complementary graphs $\overline{G}_1 = (V, \overline{E}_1)$ and $\overline{G}_2 = (V, \overline{E}_2)$ are not connected. Then $E_1 \cap E_2 \neq \emptyset$.*

Proof. Let $V_i \subset V$ be a connected component of $\overline{G}_i$, then all edges between V_i and $V \setminus V_i$ belong to E_i, for both $i = 1$ and $i = 2$. Then $E_1 \cap E_2 \neq \emptyset$, since $V_i \neq \emptyset$ and $V_i \neq V$ for both $i = 1$ and $i = 2$. □

So, let $G_i = (V, E_i)$ be the unique not CC component of $\mathcal{G}$. Let us decompose its complement into connected components and consider the corresponding induced k-graphs (note that there are at least two of them). Each such k-graph $\mathcal{G}'$ is still Π-and Δ-free. Hence, there exists a unique $j \in [k]$ (note that $j \neq i$, since $\overline{G_i}$ was decomposed into connected components) such that …etc. Thus, we get a decomposition rooted tree $T = T(\mathcal{G}) = (V', E')$ whose leaves $L \subseteq V'$ are in one-to-one correspondence with V, while all other vertices $N = V' \setminus L$ are labeled by the colors of $[k]$.

By construction, property (a) holds for $T(\mathcal{G})$, yet, (b) should be weakened as follows:

- (b′) The labels are distinct for every two adjacent vertices of N.

The labeled rooted tree $T(\mathcal{G})$ was interpreted in [11–14] as a positional game structure in which $[k]$ is the set of players. Then, the condition (a) means that there is no position with a unique (forced) move, while (b′) means that no player has two successive moves.

2.3.3. UMFSes as positional game structures

Given a complete labeled graph (G, d) that defines a UMFS, it is enough to replace each label d_i by the color i for every $i \in [k]$ to obtain a k-graph $\mathcal{G}$.

Theorem 4. *A k-graph $\mathcal{G} = (V; E_1, \dots, E_k)$ can be realized by a UMFS (G, d) if and only if $\mathcal{G}$ is Δ-free and has no $m \geq 2$ triangles colored $(i_1, i_2, i_2), (i_2, i_3, i_3), \dots (i_{m-1}, i_m, i_m), (i_m, i_1, i_1)$.*

Proof. The existence of a three-colored triangle Δ is in contradiction with the ultrametric inequality, while a two-colored triangle $(i_\ell, i_\ell, i_{\ell+1})$ may exist, yet, only when $d_{i_\ell} > d_{i_{\ell+1}}$, again by the ultrametric inequality. Hence, the distances $d_1, \dots, d_k$, corresponding to the colors $1, \dots, k$, can be ordered if and only if $\mathcal{G}$ contains no cycle of m triangles mentioned in the above statement. □

Let us also notice that the k-graph Π contains two triangles colored $(i_1, i_2, i_2), (i_2, i_1, i_1)$. Hence, by Theorem 4, the k-graph $\mathcal{G}$ is Π-and Δ-free. Then, according to the previous subsection, $\mathcal{G}$ can be represented by a unique tree $T(\mathcal{G})$.

It remains to note that the labeling becomes special, since property (b) is stronger than (b′). One can interpret a UMFS as a positional game structure in every play (a path from the initial position v_0 to a leaf $v \in L$) of which each player makes at most one move.

2.3.4. CIS property of UMFSes

Given a k-graph $\mathcal{G} = (V; E_1, \dots, E_k) = (V; E_i \mid i \in [k])$, let $S_i \subseteq V$ be an inclusion-maximal independent set of the graph $G_i = (V, E_i)$ for each $i \in [k]$ and let $S = \cap_{i=1}^{k} S_i$ be the intersection of all these sets. Obviously, it contains at most one vertex, that is, $|S| \leq 1$. Indeed, if two distinct vertices are in S then the corresponding edge of $\mathcal{G}$ would have no color. Furthermore, we say that $\mathcal{G}$ is a *CIS* k-graph (or equivalently, that it has the *CIS property*) if for every such selection $\{S_1, \dots, S_k\}$ the intersection $S = \cap_{i=1}^{k} S_i$ is not empty.

Theorem 5 (*[11]; see also [14,17]*)**.** *Every Π-and Δ-free k-graph has the CIS property.* □

Remark 5. A new proof was recently given in [17]. It was also conjectured in [11] that no CIS k-graph contains a Δ. This conjecture is still open; see [1] for more details.

It was also shown in [2] that Π-and Δ are the only (locally) minimal non-CIS k-graphs.

Applying Theorem 5 to the UMFSes we obtain the following statement.

Corollary 4. *Given a UMFS (G, d) with $im(d) = \{d_1, \dots, d_k\}$, for each $i \in [k]$, let $S_i \subseteq V$ be an inclusion-maximal vertex set in which no two vertices are at distance d_i. Then, $S = \cap_{i=1}^{k} S_i \neq \emptyset$, that is, every such k sets contain a unique common vertex.* □

3. Representing QUMFSes by nested transitive digraphs

3.1. Transitive directed graphs

A directed graph (digraph) $D = (V, A)$ is called *transitive* if

$$\text{for any } u, w \in V \text{ we have: } (u, w) \in A \text{ whenever } (u, v), (v, w) \in A \text{ for some } v \in V. \tag{1}$$

Proposition 3. *The following three claims hold for a transitive digraph $D = (V, A)$:*

- *If C is a directed cycle then $(u, w) \in A$ for any two vertices u and w of C, or in other words, the vertices of C induce a complete subdigraph in D.*

Furthermore, let $D' = (V', A')$ and $D'' = (V'', A'')$ be two complete subdigraphs of D.

- *If $V' \cap V'' \neq \emptyset$ then a complete digraph is induced by $V' \cup V''$;*
- *If $V' \cap V'' = \emptyset$ and there is an arc $(v', v'') \in A$ such that $v' \in V'$, $v'' \in V''$ then $(w', w'') \in A$ for all $w' \in V'$, $w'' \in V''$.*

Proof. All these three statements result immediately from the transitivity of D. $\square$

The above three claims completely clarify the structure of a transitive digraph $D = (V, A)$.

It is uniquely defined by a partition $V = V_1 \cup \cdots \cup V_m$ and *acyclic* digraph $D' = (V', A')$, where $V' = \{v_1, \ldots, v_m\}$. A complete subdigraph is induced in D by each V_i and $(w_i, w_j) \in A$ whenever $w_i \in V_i$, $w_j \in V_j$, and $(v_i, v_j) \in A'$; here $i, j \in [m] = \{1, \ldots, m\}$ and $i \neq j$. In other words, the vertices of a transitive digraph are partitioned into several pairwise disjoint classes (of equivalent vertices) on which a partial order is defined.

Let us note that in the symmetric, or in other words, non-directed, case ($(u, w) \in A$ whenever $(w, u) \in A$) the above partial order becomes trivial. Indeed, as we already know, a transitive graph is just the union of m cliques; respectively, its complement is a complete m-partite graph.

3.2. Characterizing QUMFSes

Given a complete digraph $D = (V, A)$ and positive weights $d : A \to \{d_1, \ldots, d_k\} \subseteq \mathbf{R}_+$ such that $0 < d_1 < \cdots < d_k$, then, for any $i \in [k] = \{1, \ldots, k\}$ let us set $A_i = \{(u, w) \in A \mid d(u, w) \leq d_i\}$.

Theorem 6. *The weighted digraph $(D = (V, A), d)$ defines a QUMFS if and only if all subdigraphs $D_i = (V, A_i)$, $i \in [k]$, of the corresponding nested family are transitive.*

Proof. *Only if part*. Assume that $D_i = (V, A_i)$ is not transitive for some $i \in [k]$, that is, $d(u, v) \leq d_i$ and $d(v, w) \leq d_i$, while $d(u, w) > d_i$; then $\max(d(u, v), d(v, w)) < d(u, w)$.

If part. Conversely, let us assume that $\max(d(u, v), d(v, w)) < d(u, w)$ for some $u, v, w \in V$. Then $d(u, v) \leq d_i$ and $d(v, w) \leq d_i$, while $d(u, w) > d_i$, where $d_i = \max(d(u, v), d(v, w))$. Hence, transitivity fails for D_i. $\square$

Corollary 5. *Every QUMFS is uniquely (up to an isometry) realized by Theorem 6.*

Proof. Given a QUMFS, let us consider the complete digraph on its elements and introduce the weights equal to the corresponding distances. $\square$

Example 2. For three vertices $V = \{u, v, w\}$ let us define the distances as follows:

$$d_1 = d(u, w) = 1, \quad d_2 = d(u, v) = d(w, v) = 2, \quad d_3 = d(w, u) = 3, \quad d_4 = d(v, u) = d(v, w) = 4.$$

It is not difficult to verify the ultrametric inequality and also that the following four nested arc-sets

$$A_1 = \{(u, w)\}, \quad A_2 = \{(u, w), (u, v), (w, v)\}, \quad A_3 = \{(u, w), (u, v), (w, v), (w, u)\}, \quad \text{and} \quad A_4 = A$$

form transitive digraphs $D_i = (V, A_i)$ for $i = 1, 2, 3, 4$.

4. The upper bound $k \leq \frac{1}{2}(n-1)(n+2)$ for QUMFSes

4.1. Proof of the bound

First, let us recall that $k \leq n - 1$ for an n-element UMFS and that this bound is precise. Here, we will derive a (much larger but also precise) similar upper bound for QUMFSes.

Theorem 7. *The number k of pairwise distinct distances of an n-element QUMFS is at most*

$$\binom{n}{2} + n - 1 = n(n-1) - \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(n-1)(n+2).$$

Proof. Let QUMFS (D, d) be standardly given by a complete digraph $D = (V, A)$ and weighting $d : A \to \{d_1, \ldots, d_k\}$ such that $d_1 < \cdots < d_k$. Each arc (u, w) belongs to $n - 2$ triangles $(u, v), (v, w), (u, w)$, where $v \in V \setminus \{u, w\}$.

Let us assume that $d(u_0, w_0) = d_k$. Then d_k must appear at least $n - 2$ times more among $\{d(u_0, v), d(v, w_0) \mid v \in V \setminus \{u_0, w_0\}\}$. Indeed, by the ultrametric inequality, $d(u, v) = d_k$ or $d(v, w) = d_k$ (or both), for each $v \in V \setminus \{u, w\}$.

In fact, a stronger claim holds. Let us set $V_u = \{v \in V \mid d(u_0, v) < d_k\}$ and $V_w = \{w \in V \mid d(u_0, w) = d_k\}$.

Obviously, both sets are not empty (since $u_0 \in V_u$ and $w_0 \in V_w$) and partition V (that is, $V = V_u \cup V_w$ and $V_u \cap V_w = \emptyset$).

Furthermore, it is easy to derive from the ultrametric inequality that $d(v', v'') = d_k$ whenever $v' \in V_u$ and $v'' \in V_w$.

Let us prove $k \le \frac{1}{2}(n-1)(n+2)$ by induction on n. The base is trivial, since $k = 0$ when $n = 1$. Setting $n' = |V_u|$ and $n'' = |V_w|$, we get $n' > 0$, $n'' > 0$, and $n' + n'' = n = |V|$. Then, by the induction hypothesis,

$$k \le \frac{1}{2}[(n'-1)(n'+2) + (n''-1)(n''+2)] + n'n'' + 1 \equiv \frac{1}{2}(n-1)(n+2). \tag{2}$$

Here the term $n'n''$ appears, because all arcs from V_w to V_u might be of pairwise distinct lengths and 1 is due to d_k. $\square$

Remark 6. Conversely, if $k = \frac{1}{2}(n-1)(n+2)$ then all arcs from V_w to V_u must be of pairwise distinct lengths.

Remark 7. Theorem 7 can be slightly generalized as follows. For any $j = 1, \ldots, n-2$, the number of repetitions among the distances $d_k, d_{k-1}, \ldots, d_{k-j}$ is at least

$$(n-2) + (n-3) + \cdots + (n-2-j) = \sum_{i=0}^{j}(n-2-i) = (j+1)(n-2-j/2).$$

This can be derived from Theorem 6, by induction on j. Substituting $j = n-2$, we conclude that there are at least $\frac{1}{2}(n-1)(n-2)$ repetitions among the distances $d(u, w)$ for $u, w \in V$. Thus, for the number k of pairwise distinct distances we again get the desired upper bound $k \le n(n-1) - \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(n-1)(n+2)$.

4.2. The bound $k \le \frac{1}{2}(n-1)(n+2)$ is tight for QUMFSes

Example 3. Given $V = \{v_1, \ldots, v_n\}$, let us consider the $\binom{n}{2}$ ordered pairs $v_i, v_j \in V$ such that $i < j$ in the lexicographic order and assign the distinct values $1, \ldots, \binom{n}{2}$ to the $\binom{n}{2}$ corresponding distances

$$d(v_1, v_2), \ldots, d(v_1, v_n), d(v_2, v_3), \ldots, d(v_2, v_n), \ldots, d(v_{n-1}, v_n).$$

Then, let us consider the remaining $\binom{n}{2}$ pairs $v_i, v_j \in V$, with $i > j$, in the inverse lexicographic order and set:

$$d(v_n, v_{n-1}) = \binom{n}{2} + 1, d(v_n, v_{n-2}) = d(v_{n-1}, v_{n-2}) = \binom{n}{2} + 2, \ldots, d(v_n, v_1) = \cdots = d(v_2, v_1)$$
$$= \binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2).$$

Remark 8. Moreover, it was shown by Frank and Frisch in [9] that the equality $k = \frac{1}{2}(n-1)(n+2)$ may hold already for the flow QUMFSes; see Section 5.2 for the definitions and also [3,8]

4.3. Characterizing the n-element QUMFSes having $k = \frac{1}{2}(n-1)(n+2)$ pairwise distinct distances

Actually, the last procedure of Example 3 can be generalized as follows. Let us consider a partition $V = V' \cup V''$ such that $i' > i''$ for all $v_{i'} \in V'$, $v_{i''} \in V''$, and set $d(v_{i'}, v_{i''}) = \binom{n}{2} + n - 1$ for all such ordered pairs; then, consider a partition of V', etc., until all sets become singletons. Naturally, such a successive partitioning is represented by a binary ordered (rooted) tree T with n leaves that correspond to the elements of V. Obviously, T contains $n-1$ other vertices, which correspond to the partitions and will be called *interior*. The above procedure numbers them by the integers from $\binom{n}{2} + 1$ to $\binom{n}{2} + n - 1$. This enumeration is monotone decreasing with respect to the partial order over the vertices of T, otherwise arbitrary. In particular, the ground set V, corresponding to the root, gets the number $\binom{n}{2} + n - 1$. It is not difficult to verify that each of such successive partitioning results in a QUMFS on V with $k = \binom{n}{2} + n - 1 = \frac{1}{2}(n-1)(n+2)$ pairwise distinct distances.

Conversely, the proof of Theorem 7 implies that each such QUMFS (with $k = \frac{1}{2}(n-1)(n+2)$) can be realized by such a successive partitioning if we replace the distance ℓ by d_ℓ for all $\ell \in [k] = \{1, \ldots, k\}$, assuming standardly that $d_1 < \cdots < d_k$. To see this, it is enough to notice that $k = \frac{1}{2}((n-1)(n+2))$ if and only if (2) holds with the equality for all partitions in T and the equality in (2) holds if and only if all $n'n''$ distances $\{d(v', v'') \mid v' \in V_w, v'' \in V_u\}$ are pairwise distinct.

4.4. The one-way QUMFSes associated with a UMFS

The above construction can be further generalized as follows. Let us recall that the weighted rooted tree satisfying the assumptions (a) and (b) are in one-to-one correspondence with the UMFSes; see Section 2 and Theorem 2. Let us also suppose that the tree is ordered (that is, the children of each vertex are ordered) and enumerate the leaves $V = \{v_1, \ldots, v_n\}$ in accordance with this order. Finally, let us introduce a QUMFS by the following (asymmetric) distance function on V. See Fig. 3 for an illustration.

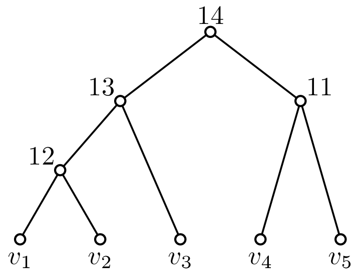

Fig. 3. A one-way QUMFS: $d(v_1, v_2) = 1, d(v_1, v_3) = 2, d(v_1, v_4) = 3, d(v_1, v_5) = 4, d(v_2, v_3) = 5, d(v_2, v_4) = 6, d(v_2, v_5) = 7, d(v_3, v_4) = 8, d(v_3, v_5) = 9, d(v_4, v_5) = 10$, while for $i > j$ the distances $d(v_i, v_j)$ take values 11, 12, 13, 14, in accordance with the weights on the interior vertices.

- If $i = j$; then $d(v_i, v_j) = 0$ (and $d(v_i, v_j) > 0$ otherwise), as in the UMFS.
- If $i > j$; then, $d(v_i, v_j)$ is the weight of the (unique) lowest common ancestor of v_i and v_j, as in the UMFS.
- If $i < j$; then $d(v_i, v_j)$ take $\binom{n}{2}$ (not necessarily distinct) positive values such that
  (i) they satisfy the ultrametric inequality, or in other words, $d(v_i, v_j) \le d(v_{i'}, v_{j'})$ whenever $i \le i'$ and $j \le j'$;
  (j) each new distance is not larger than the weight of any interior vertex, or in other words, $d(v_i, v_j) \le d(v_{i'}, v_{j'})$ whenever $i < j$, while $i' > j'$.

In particular, one can order $d(v_i, v_j)$ lexicographically, as in Example 3 or Fig. 3. One can even set $d(v_i, v_j) = 0$ whenever $i < j$, yet, in this case a pseudo-QUMFS will be obtained. In general, a simple case analysis shows that we obtain a QUMFS whenever all three above conditions hold. Furthermore, the extremal construction of the previous subsection corresponds to the case when the tree is binary and all weights and distances are pairwise distinct.

Theorem 8. *The above construction defines a QUMFS. Furthermore, if the considered tree is binary and the distances $d(v_i, v_j)$ are pairwise distinct for $i < j$ then the resulting QUMFS is extremal, that is, $k = \binom{n}{2} + (n-1) = \frac{1}{2}(n-1)(n+2)$.*

Conversely, each extremal QUMFS can be obtained by this construction; moreover, such a representation is unique up to obvious isomorphisms and isometrics. □

5. Realizing (Q)UMFSes by networks

5.1. Bottleneck QUMFSes

Let (D, c) be a network defined by a strongly connected digraph $(D = (V, A))$ and strictly positive weight function, $c : A \to \{c_1, \ldots, c_k\} \in \mathbf{R}_+$, where $0 < c_1 < \cdots < c_k$. We will interpret $c(u, w)$ as a *width* of the arc $(u, w) \in A$, that is, the largest size of on object that can pass (u, w). Then, obviously, the width of a directed path (dipath) $p(u, w)$ from u to w is the minimum of the widths of its arcs $C(p(u, w)) = \min\{c(e) \mid e \in p(u, w)\}$. Let us define the width $C(u, w)$ as the largest size of on object that can pass from u to w for all $u, w \in V$ (not necessarily $(u, w) \in A$).

Clearly, $C(u, w) = \max\{C(p(u, w)) \mid p(u, w)\}$is the width of a max min (or widest bottleneck) dipath from u to w.

It is also clear that $C : V \times V \to \mathbf{R}_+$ takes only (strictly positive) values $c_1, \ldots, c_k$, but not 0, since D is a strongly connected digraph; see Remarks 1 and 9.

Lemma 2. *The inequality $C(u, w) \ge \min(C(u, v), C(v, w))$ holds for all $u, v, w \in V$.*

Proof. If an object can pass from u to v and from v to w then it can pass from u to w. □

Let $d(u, w) = C^{-1}(u, w)$ be the inverse width and $d(v, v) = 0$ for all $u, v, w \in V$.

Proposition 4.

- *Mapping d is a QUMFS for any network (D, c).*
- *Then, d is a UMFS whenever (D, c) is symmetric, that is, $(u, w) \in A$ whenever $(w, u) \in A$ and $c(u, w) = c(w, u)$.*
- *Conversely, all (Q)UMFSes can be realized in this way.*

Proof. It is easily seen that the inequality of Lemma 2 for C is equivalent with the ultrametric inequality for d and the first statement follows. The second one is obvious.

To realize a given QUMFS d by a network (D, c) it is enough to define $D = (V, E)$ as the complete digraph on V and set $c(u, w) = d^{-1}(u, w)$ for all $u, w \in V$. The obtained network (D, c) is symmetric whenever d is a UMFS. □

For the UMFSes, the last two statements were mentioned by Leclerc in [23].

Remark 9. According to Remark 1, we can easily adjust the above definitions and statements for the case of pseudo-(Q)UMFSes: It is sufficient to allow for functions d, c and C the values 0 and ∞ (assuming standardly that they are mutually inverse, $0^{-1} = \infty$ and $\infty^{-1} = 0$). Also we should include all, not only strongly connected, digraphs $D = (V, A)$ into consideration and set $d(u, w) = \infty$ (and $C(u, w) = 0$) whenever there is no dipath from u to w.

5.2. Flow QUMFSes

5.2.1. The ultrametric inequality for the inverse capacities

Given a network (D, c), let us now interpret $c(u, w)$ as a *capacity* of the arc $(u, w) \in A$, that is, the largest amount of a material that can be transported along (u, w) from u to v per a unit time. Then, obviously, the capacity of a dipath $p(u, w)$ from u to w is again the minimum of the capacities of its arcs, $C(p(u, w)) = \min\{c(e) \mid e \in p(u, w)\}$. Then, for all $u, w \in V$ (not necessarily $(u, w) \in A$), let us define the capacity $C(u, w)$ as the largest amount of the material that can be transported in the unit time from u to w, assuming that all other vertices are transient and the conservation law holds for each of them. Function $C : V \times V \to \mathbf{R}_+$ can take only the strictly positive values (in fact, only $c_1, \ldots, c_k$ and their sums, by [7]); C cannot take value 0, since D is a strongly connected digraph; see Remark 9, yet.

Lemma 3 (*[10]*). *The inequality $C(u, w) \geq \min(C(u, v), C(v, w))$ holds for all $u, v, w \in V$.*

This is an exact copy of Lemma 2. However, the proof cannot be just copied. Indeed, assuming that $C(u, v) \geq x$ and $C(v, w) \geq x$ we have to show that $C(u, w) \geq x$. It would suffice to sum up two x-flows that realize $C(u, v)$ and $C(v, w)$. Yet, by this operation, the capacity of an edge can be exceeded. However, the result can be easily derived from the maximum flow and minimum cut theorem [7]. For the symmetric networks and UMFSes the proof was given by Gomory and Hu [10]. The same arguments work for the digraphs and QUMFSes as follows.

Proof. By the Ford–Fulkerson theorem, $C(u, w)$ is equal to the capacity of a critical (directed) cut (U, V) of u from w, where $V = U \cup W$, $U \cap W = \emptyset$, $u \in U$ and $w \in W$. Obviously, the same (U, V) cuts v from w (respectively, u from v) whenever $v \in U$ (respectively, $v \in W$). It is easily seen that in both cases the inequality follows. □

Again, let $d(u, w) = C^{-1}(u, w)$ be the inverse capacity for all $u, w \in V$ and let $d(v, v) = 0$ for all $v \in V$.

Proposition 5 (*[10]*).

- *Mapping d defines a QUMFS for every network (D, c).*
- *Furthermore, d is a UMFS whenever (D, c) is symmetric.*
- *Moreover, any UMFS (but not any QUMFS) can be realized in this way.*

Proof. As in Proposition 4, the first statement immediately follows from Lemma 3, while the second one is obvious. Finally, the last one results from Corollary 1 as follows. Given a UMFS (G, d), where $G = (V, E)$ is the complete graph on V, let us construct a lightest spanning tree $T = (V, E')$ in (G, d) and set $c(e) = d^{-1}(e)$ for all $e \in E'$. It is easily seen that the obtained symmetric flow network (T, c) defines the original UMFS (G, d). □

Remark 10. Here, we should repeat Remark 9, word to word.

5.2.2. Not every QUMFS is a flow QUMFS

A (Q)UMFS generated by a bottleneck or flow network is called a bottleneck or flow (Q)UMFS, respectively.

By Proposition 4, every (Q)UMFS is a bottleneck (Q)UMFS. Moreover, by Proposition 5, every UMFS is a flow UMFS. Yet, it is not difficult to construct a non-flow QUMFS. For simplicity, let us start with a pseudo-QUMFS.

Example 4. Let us consider pseudo-QUMFS (D^0, d^0) defined by the following (not complete) digraph $D^0 = (V^0, A^0)$, given in Fig. 4, and unit weight function $d^0 : A^0 \to \{1\}$:

$$V^0 = \{u, v', v'', w\}, \qquad A^0 = \{(u, v'), (u, v''), (v', w), (v'', w), (u, w)\}, \qquad d^0(a) \equiv 1 \quad \forall\, a \in A. \tag{3}$$

Digraph D^0 is transitive and, by convention, $d(a, b) = \infty$ (that is, $C(a, b) = 0$) whenever $(a, b) \notin A$. Yet, (D^0, d^0) cannot be realized by a flow network. Indeed, to have $d(u, v') = d(u, v'') = d(v', w) = d(v'', w) = 1$ we must set $c(u, v') = c(u, v'') = c(v', w) = c(v'', w) = 1$. Then, $C(u, w) \geq 2$ and $d(u, w) \leq 0.5$, while $d^0(u, w) = 1$; a contradiction.

Remark 11. Let us notice, however, that we obtain a flow pseudo-QUMFS just replacing $d^0(u, w) = 1$ by any $d^0(u, w) \in [0; 0.5]$ and keeping all other distances as in Example 4.

Example 5. To get a non-flow QUMFS, let us extend D^0 to a complete digraph on V^0 and introduce large enough $d(a, b)$, say $d(a, b) = 10$, for all $a, b \in V^0$ such that $(a, b) \notin A^0$.

5.2.3. Yet, every QUMFS can be realized by a multi-pole flow network

First, let us show that the pseudo-QUMFS (D^0, d^0) of Example 4 can be easily realized by a flow network on a slightly larger vertex-set. To do so, let us replace the vertex u by a new arc (u, v), as in Fig. 5, and consider the network $(D = (V, A), c)$ in which

$$V = \{u, v, v', v'', w\}, \qquad A = \{(u, v), (v, v'), (v, v''), (v', w), (v'', w), (v, w)\}, \quad \text{and} \quad c(a) = 1 \quad \forall\, a \in A.$$

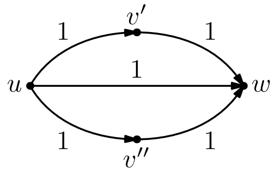

Fig. 4. A non-flow QUMFS.

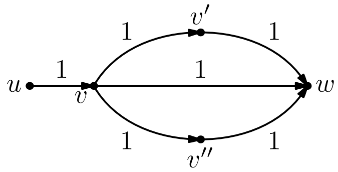

Fig. 5. The corresponding flow QUMFS.

It is not difficult to verify that

$$C(u, v') = C(u, v'') = C(v', w) = C(v'', w) = C(u, w) = 1 \quad \text{and}$$
$$C(a, b) = 0 \quad \text{for all other } a, b \in V^0 = \{u, v, v', v'', w\}.$$

Hence, $d(u, v') = d(u, v'') = d(v', w) = d(v'', w) = d(u, w) = 1$, as requested. Let us notice that $C(v, w) = 3$ and $d(v, w) = 1/3$ but it does not matter.

A still larger multi-pole network is needed to realize the similar QUMFS in which ∞ is replaced, say, by 10. We leave this analysis to the reader. Instead, let us demonstrate how the procedure works in general, implying that every QUMFS is a subspace of a flow QUMFS.

Theorem 9. *Each pseudo-QUMFS given by a weighted digraph $(D = (V, A), d)$ can be realized by a multi-pole flow network $(D' = (V', A'), V \subseteq V', c)$, where $V \subseteq V'$ is a set of poles and $d(u, w) = C^{-1}(u, w)$ for all $u, w \in V$.*

For the beginning, let us consider the case $k = 1$, that is, $0 \le d_1 \le \infty$.

Let us replace in the digraph $D = (V, A_1)$ each vertex $v \in V$ by an arc (v, v^1), every arc $(v, u) \in A_1$ by an arc (v^1, u), and set $c(e) = d_1^{-1}(e)$ for every obtained arc e. Since A_1 is transitive, in the obtained weighted digraph, we get $C(u, w) = d_1^{-1}$ for all $(u, w) \in A_1$ and $C(u, w) = 0$ for all other pairs $u, w \in V$.

Proof of Theorem 9. In general, from $i = k$ to $i = 0$ do: For each vertex $v \in V$ introduce a new vertex v^i and arc (v, v^i), then, replace every arc $(v, u) \in A_i$ by (v^i, u), and set $c(e) = d_i^{-1} - d_{i+1}^{-1}$ for all new arcs. Standardly, we assume that $0 \le d_1 < \cdots < d_k \le \infty$ and set $d_0 = 0$, $d_{k+1} = \infty$. Then, by transitivity of A_i for all $i \in [k]$, we obtain

$$C(u, w) = \sum_{i | (u,w) \in A_i} \left(d_i^{-1} - d_{i+1}^{-1}\right) = d_{i(u,w)}^{-1} = d^{-1}(u, w) \tag{4}$$

for every ordered pair $u, w \in V$, where $i(u, w) = \min(i \mid (u, w) \in A_i)$. $\square$

In particular, these arguments work for a complete digraph D, that is, for a QUMFS.

5.2.4. Recognizing flow QUMFSes and realizing them by flow networks; $k = 1$

Given a QUMFS (D, d) in which $D = (V, A)$ is a complete digraph and $d : A \to \{d_1, \ldots, d_k\}$ is a weight function such that $0 < d_1 < \cdots < d_k < \infty$, assume for the beginning that $k = 1$. In this case, QUMFS (D, d) is defined by d_1 and the corresponding (typically, not complete) digraph $D_1 = (V, A_1)$. Then, the results of Sections 5.2.2 and 5.2.3 provide the following characterization of the flow QUMFSes.

Proposition 6. *A QUMFS (D, d) (of $k = 1$) is realized by a flow network if and only if (transitive) digraph $D_1 = (V, A_1)$ does not contain $D^0 = (V^0, A^0)$ from Example 4 as an induced subdigraph.*

Proof. The "only if" part was already shown in Example 4; let us prove the inverse statement.

By Theorem 6, digraph $D_1 = (V, A_1)$ is transitive and hence, by Proposition 3, its structure is described as follows: D_1 is uniquely defined by a partition $V = V_1 \cup \cdots \cup V_m$ and by an *acyclic* transitive digraph $D' = (V', A')$ such that $V' = \{v_1, \ldots, v_m\}$, a complete subdigraph is induced in D_1 by each V_i; furthermore, $(w_i, w_j) \in A_1$ if and only if $w_i \in V_i$, $w_j \in V_j$, and $(v_i, v_j) \in A'$, for any $i, j \in [m] = \{1, \ldots, m\}$ such that $i \ne j$. In other words, A' defines a partial order P over V'; see Section 3.1.

It is easily seen that D_1 contains D^0 as an induced subdigraph if and only if D' does.

Finally, let $D'' = (V', A'')$ denote the so-called *Hasse diagram* of P; in other words, $D'' = (V', A'')$ is a (unique) subdigraph of $D' = (V', A')$ such that $(v_i, v_j) \in A''$ if and only if v_j is a *cover* of v_i in D', that is, $(v_i, v_j) \in A'$ but $(v_i, v), (v, v_j) \in A'$ holds for no $v \in V'$.

Now, for each arc $(u, w) \in A_1$ let us introduce its capacity $c(u, w)$ as follows:

$$c(u,w) = \begin{cases} (d_1(n_i-1))^{-1} & \text{if } u, w \in V_i \text{ for some } i \in [m], \text{ where } n_i = |V_i|; \\ (d_1 n_i n_j)^{-1} & \text{if } u \in V_i,\ w \in V_j, \text{ and } v_j \text{ is a cover of } v_i; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We will show that the total capacity $C(u, w) = d_1^{-1}$ whenever $(u, w) \in A_1$ and $C(u, w) = 0$ otherwise. Let us assume that $u \in V_i$, $w \in V_j$ and consider the following three cases:

Case 1: $i = j$. In this case $(u, w) \in A_1$ and, hence, there are $n_i - 1$ dipaths from u to w in D_1, one of which consists of 1 arc, while the remaining $n_i - 2$ consist of 2 arcs each.

By (5), each of these $n_i - 1$ dipaths is of capacity $((n_i - 1)d_1)^{-1}$ and, hence, $C(u, w) = d_1^{-1}$.

Case 2: $i \neq j$ and $(u, w) \in A_1$, i.e., v_j is a successor (but not necessarily a cover) of v_i.

Let us suppose that (t) there are (at least) two dipaths from v_i to v_j in D''.

Then, each of them contains at least two arcs, by the definition of the Hasse diagram.

Hence, by transitivity, (tt) D' (or equivalently, D_1) contains D^0 as an induced subdigraph, in contradiction with the main assumption of the theorem.

Remark 12. In fact, the converse is also true, that is, statements (t) and (tt) are equivalent.

Thus, (ttt) there is a unique dipath from v_i to v_j in D'. In this case again $C(u, w) = d_1^{-1}$ for every $u \in V_i$ and $w \in V_j$, by (5). Indeed, cut (V_i, V_j) contains $n_i n_j$ arcs of capacity $(n_i n_j d_1)^{-1}$ each and, by (ttt), there are no other dipaths from V_i to V_j. Hence, $C(u, w) \leq d_1^{-1}$. On the other hand, $n_i(n_i - 1)$ arcs of V_i of capacity $(d_1(n_i - 1))^{-1}$ each, $n_j(n_j - 1)$ arcs of V_j of capacity $(d_1(n_j - 1))^{-1}$ each, and $n_i n_j$ arcs from V_i to V_j of capacity $(n_i n_j d_1)^{-1}$ each are, obviously, sufficient to transport d_1^{-1} from u to w.

Case 3: $i \neq j$ and $(u, w) \notin A_1$, that is, v_j is not a successor of v_i in D'. Then, there is no dipath from v_i to v_j in D'', D', or D and hence $C(u, w) = 0$. □

Remark 13. We extend Proposition 6 to pseudo-QUMFSes just assuming that $0 \leq d_1 \leq \infty$ rather than $0 < d_1 < \infty$.

Let us underline that, in Proposition 6, only an *induced* subdigraph D^0 is an obstruction.

Example 6. If we extend the digraph $D^0 = (V^0, A^0)$ from Example 4 by one new arc (u, v) and define $d(a) = 1$ for all $a \in A'$, while $d(a) = \infty$ for all $a \notin A'$, we will not get a pseudo-QUMFS, since the obtained digraph $D' = (V, A')$ is not transitive. To get a transitive digraph $D = (V, A)$, we have to add one more arc (v', v''). Then, to get a pseudo-QUMFS (D, d), we just extend d by setting $d(v', v'') = 1$, too. By Proposition 6, the obtained pair (D, d) is a flow pseudo-QUMFS and the capacities of the corresponding flow network are defined by (5) as follows:

$$c(u, v') = c(v', u) = c(v'', w) = 1, \qquad c(u, v'') = c(v', v'') = \frac{1}{2} \quad \text{and} \quad c(x, y) = 0 \text{ for other } x, y \in V.$$

Example 7. Similarly, if we extend $D^0 = (V^0, A^0)$ by two new arcs (v', u) and (v'', u) and standardly define $d(a) = 1$ for all $a \in A'$, while $d(a) = \infty$ for all $a \notin A'$, we will not get a pseudo-QUMFS, since the obtained digraph $D' = (V, A')$ is not transitive. To get a transitive digraph $D = (V, A)$, we add two more arcs (v', v'') and (v'', v'). Then, to get a pseudo-QUMFS (D, d), we just extend d by setting $d(v', v'') = d(v'', v') = 1$, too. By Proposition 6, (D, d) is a flow pseudo-QUMFS. The capacities of the corresponding flow network are defined by (5) as follows: $c(x, y) = \frac{1}{2}$, $c(x, w) = \frac{1}{3}$ for any distinct $x, y \in \{u, v', v''\}$, while $c = 0$ for all remaining arcs.

Example 8. Finally, let us extend $D^0 = (V^0, A^0)$ by two new arcs (v', u) and (w, v'') and in the obtained digraph $D' = (V, A')$ standardly define $d(a) = 1$ for all $a \in A'$ and $d(a) = \infty$ for all $a \notin A'$. Again, to get a transitive digraph $D = (V, A)$, we add the arc (v', v'') and, to get a pseudo-QUMFS (D, d), we extend d by setting $d(v', v'') = 1$. Then, by Proposition 6, (D, d) is a flow pseudo-QUMFS and the capacities of the corresponding flow network are defined by (5) as follows:

$c(x, y) = \frac{1}{4}$ for all $x \in \{u, v'\}$, $y \in \{v'', w\}$, while $c(u, v') = c(v', u) = c(v'', w) = c(w, v'') = 1$, and $c = 0$ for all remaining arcs.

In three above examples, it is not difficult to compute the effective capacities and verify the equality $C(x, y) = d^{-1}(x, y)$ for every pair of distinct vertices $x, y \in V = \{u, v', v'', w\}$.

5.2.5. Recognizing flow QUMFSes and realizing them by flow networks

Now, let us consider the general case: $k \geq 1$. Given a pseudo-QUMFS $(D = (V, A), d)$ in which $d : A \to \{d_1, \ldots, d_k\}$ and $0 \leq d_1 < \cdots < d_k \leq \infty$, we wish either to construct on the same digraph a flow network $(D = (V, A), c)$ whose effective capacities $C(u, w)$ are equal to the inverse distances $d^{-1}(u, w)$ for all $u, w \in V$, or to prove that there is no such network. We will need several iterations. The first one is as follows. Let us consider the (transitive) digraph $D_k = (V, A_k)$ and define the acyclic transitive digraph $D' = (V', A')$ and Hasse diagram $D'' = (V', A'')$ as in the previous subsection, in which we had $k = 1$. Furthermore, let us assign the capacity $c(a) = d_k^{-1}$ to each $a \in A''$ and compute the effective capacity $C'(v_i, v_j)$ for all $v_i, v_j \in V'$ in the obtained flow network (D'', c). Then, let us recall the original digraph D_k, define $c(u, w)$ for all $(u, w) \in A_k$ by formula (5) (in which $k = 1$), and compute the effective capacities $C(u, w)$ for all $u, w \in V$. Finally, in the QUMFS (D, d), let us compare $C(u, w)$ and $C'(v_i, v_j)$ for all $u \in V_i$, $w \in V_j$. If $C(u, w) < C'(v_i, v_j)$ for some $u, w \in V$ then, obviously, (D, d) is not a flow QUMFS.

Otherwise, let us update $d(u, w)$ by setting $d^1(u, w) = (C(u, w) - C'(v_i, v_j))^{-1}$. In particular, $d^1(u, w) = \infty$ if and only if $C(u, w) = C'(v_i, v_j)$. Obviously, this equality holds whenever $(v_i, v_j) \in A''$. Then, let us repeat the whole procedure for the obtained pseudo-QUMFS $(D^1 = (V, A^1), d^1)$, etc., getting $(D^\ell = (V, A^\ell), d^\ell)$ after each iteration $\ell = 0, 1, \ldots, L$ (assuming that $d^0 = d_k$, $D^0 = D_k$, and $A^0 = A_k$ for the initial iteration).

Let us note that the distances $d^\ell(u, w)$ are monotone non-decreasing in ℓ and at least one of them becomes ∞ in each step. Hence, the arc-sets A^ℓ are strictly monotone decreasing in ℓ implying that $L < n(n-1)$ where $n = |V|$. After L iterations we either prove that (D, d) is not a flow pseudo-QUMFS, or realize it by a flow network introducing the cumulative capacities: $c(a) = \sum_{\ell=0}^{L} c^\ell(a)$ for all $a \in A$. Obviously, the obtained algorithm is polynomial for the pseudo-QUMFSes and for QUMFSes, in particular. $\square$

Remark 14. Let us notice that, unlike the arc-sets A^ℓ, the numbers of pairwise distinct distances may not decrease in ℓ, that is, strict inequalities $k^\ell < k^{\ell+1}$ may hold.

5.3. Finite ultrametric spaces of resistances

Both bottleneck and flow UMFS can be realized as resistance distances [18]; see also [16,19].

Given a (non-directed) *connected* graph $G = (V, E)$ in which each edge $e \in E$ is an isotropic conductor with the monomial conductivity law

$$y_e^* = y_e^r / \mu_e^s.$$

Here y_e is the voltage, or potential difference, y_e^* current, and μ_e is the resistance of e, while r and s are two strictly positive real parameters independent of $e \in E$. In particular, the case $r = 1$ corresponds to Ohm's low in electricity, while $r = 0.5$ is the standard square law of resistance typical for hydraulics or gas dynamics. The parameter s, in contrast to r, is redundant; yet, it will play an important role too.

It is not difficult to see that for any two arbitrary nodes $u, w \in V$ the obtained two-pole circuit (G, u, w) satisfies the same monomial conductivity law $y_{u,w}^* = y_{u,w}^r / \mu_{u,w}^s$, where $y_{u,w}^*$ is the total current and $y_{u,w}$ voltage between u and w, while $\mu_{u,v}$ is the *effective resistance* of (G, u, w). In other words, (G, u, w) can be effectively replaced by a single edge $e = (u, w)$ of resistance $\mu_{u,w}$ with the same r and s. Obviously, $\mu_{u,w} = \mu_{w,u}$, due to symmetry (isotropy) of the conductivity functions; it is also clear that $\mu_{u,w} > 0$ whenever $u \neq w$; finally, by convention, we set $\mu_{u,w} = 0$ for $u = w$.

In [18], it was shown that for arbitrary three nodes u, v, w the following inequality holds

$$\mu_{u,w}^{s/r} \leq \mu_{u,v}^{s/r} + \mu_{v,w}^{s/r}.$$

In [19], it was also shown that it holds with equality if and only if node v belongs to every path between u and w; see [16] for more details.

Clearly, if $s \geq r$ then we obtain the standard triangle inequality $\mu_{u,w} \leq \mu_{u,v} + \mu_{v,w}$ and the ultrametric inequality $\mu_{u,w} \leq \max(\mu_{u,v}, \mu_{v,w})$ appears when $s/r \to \infty$. Thus, a circuit can be viewed as a metric space in which the distance between any two nodes u and w is the effective resistance $\mu_{u,w}$. Playing with parameters r and s, one can get several interesting examples. Let $r = r(t)$ and $s = s(t)$ depend on a real parameter t; in other words, these two functions define a curve in the positive quadrant $r \geq 0$, $s \geq 0$. It is shown in [16,18] that for the next four limit transitions, as $t \to \infty$, for all pairs of poles $u, w \in V$, the limits $\mu_{u,w} = \lim_{t\to\infty} \mu_{u,w}(t)$ exist and can be interpreted as follows:

- (i) The effective Ohm resistance between poles u and w, when $s(t) = r(t) \equiv 1$, or more generally, whenever $s(t) \to 1$ and $r(t) \to 1$.
- (ii) The standard length (travel time or cost) of a shortest route between terminals u and w, when $s(t) = r(t) \to \infty$, or more generally, $s(t) \to \infty$ and $s(t)/r(t) \to 1$.
- (iii) The inverse width of a maxmin path between terminals u and w when $s(t) \to \infty$ and $r(t) \equiv 1$, or more generally, $r(t) \leq$ const, or even more generally $s(t)/r(t) \to \infty$.
- (iv) The inverse capacity between terminals u and w, when $s(t) \equiv 1$ and $r(t) \to 0$; or more generally, when $s(t) \to 1$, while $r(t) \to 0$.

Obviously, all four example define metric spaces, since in all cases $s(t) \geq r(t)$ for any sufficiently large t. Moreover, for the last two examples the ultrametric inequality holds for any $u, v, w \in V$, because $s(t)/r(t) \to \infty$, as $t \to \infty$, in the cases (iii) and (iv).

These examples allow us to interpret s and r as parameters of a transportation problem.

In particular, s can be viewed as a measure of divisibility of a transported material; $s(t) \to 1$ in examples (i) and (iv), because liquid, gas, or electrical charge are fully divisible; in contrast, $s(t) \to \infty$ for (ii) and (iii), because a car, ship, or individual transported from u to w are indivisible.

Furthermore, the ratio s/r can be viewed as a measure of subadditivity of the transportation cost; so $s(t)/r(t) \to 1$ in examples (i) and (ii), because in these cases the cost of transportation along a path is additive (is the sum of the costs of the edges that form the path); in contrast, $s(t)/r(t) \to \infty$ for (iii) and (iv), because in these cases only edges of the maximum cost (that is, of the minimum capacity or of the bottleneck width) matter.

Other values of parameters s and s/r, between 1 and ∞, correspond to an intermediate divisibility of the transported material and subadditivity of the transportation cost, respectively.

5.4. Reducible, universal, and complete families of (Q)UMFSes

Given two (Q)UMFSes $(D' = (V', A'), d')$ and $(D'' = (V'', A''), d'')$ and a common vertex-set $V \subseteq V' \cap V''$, let us call (D', d') and $(D'', d'')V$-isometric if $d'(u, w) = d''(u, w)$ for all ordered pairs $u, w \in V$.

A family F of (Q)UMFSes will be called:

- *reducible* if for any $(D' = (V', A'), d') \in F$ and $V \subseteq V'$ there is a $(D = (V, A), d) \in F$ such that (D, d) and (D', d') are V-isometric;
- *universal* if for any (Q)UMFS $(D = (V, A), d)$ there is a (Q)UMFS $(D' = (V', A'), d') \in F$ such that $V \subseteq V'$ and restriction of the latter (Q)UMFS to V is the former one;
- *complete* if F contains all (Q)UMFSes.

Obviously, any complete family is universal, any universal one is reducible, and both these containments are strict.

In the last section of [16], it was shown that if $r = s = 1$ then every k-pole n-vertex network, $k \le n$, can be replaced by an equivalent k-vertex network; in other words, the corresponding family of the resistance UMFSes is reducible.

In fact, the same arguments work for $r = 1$ and any $s > 0$. In particular, the symmetric bottleneck networks (for which $r = 1, s \to \infty$) generate reducible UMFSes too. Moreover, this family of UMFSes is complete, as well as the family of UMFSes generated by the symmetric flow networks (for which $s = 1, r \to 0$); see Sections 1.3 and 5.1.

As for the families F_B and F_C of QUMFSes realized by the general (not necessarily symmetric) bottleneck and flow networks, respectively, the results of Sections 5.1 and 5.2 immediately imply the following.

Proposition 7. *Family F_B is complete, while F_C is universal but not complete.* $\square$

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