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Categorical Structure of Closure Operators

With Applications to Topology, Algebra and Discrete Mathematics

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In memory of our parents

Nishan Dikranjan and Elise O. Shoulian
Bernhard J. Tholen and Hildegard M. Kennepohl

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Preface

Our motivation for gathering the material for this book over a period of seven years has been to unify and simplify ideas which appeared in a sizable number of research articles during the past two decades. More specifically, it has been our aim to provide the categorical foundations for extensive work that was published on the epimorphism– and cowellpoweredness problem, predominantly for categories of topological spaces. In doing so we found the categorical notion of closure operators interesting enough to be studied for its own sake, as it unifies and describes other significant mathematical notions and since it leads to a never-ending stream of examples and applications in all areas of mathematics. These are somewhat arbitrarily restricted to topology, algebra and (a small part of) discrete mathematics in this book, although other areas, such as functional analysis, would provide an equally rich and interesting supply of examples. We also had to restrict the themes in our theoretical exposition. In spite of the fact that closure operators generalize the universal closure operations of abelian category theory and of topos– and sheaf theory, we chose to mention these aspects only *en passant*, in favour of the presentation of new results more closely related to our original intentions. We also needed to refrain from studying topological concepts, such as compactness, in the setting of an arbitrary closure-equipped category, although this topic appears prominently in the published literature involving closure operators.

Readers of the book are expected to know the basic notions of category theory (such as functor, natural transformation, limit), although many standard notions are being recalled in the text or in the exercises. Some of the exercises should be considered part of the exposition of the general material and should therefore not be omitted, while others deal with specific applications and can be selected according to the Reader's background and interest. Each section contains at most one Theorem, one Proposition, one Lemma, one Corollary, and one set of Remarks and Examples, with very few exceptions. Hence "Proposition n.m" refers to *the* Proposition of Section n.m; in the exceptional case that Section n.m does contain a second proposition, this will be labelled as Proposition* of n.m. Readers interested in new results on (non-)cowellpowered subcategories of topological spaces as presented in Chapter 8 might be able just to browse through Chapters 2,4,6,7 and still understand the material.

We have, over the past seven years, benefitted from the interest in and advice on our work from many colleagues, including Jiří Adámek, Alessandro Berarducci, Reinhard Börger, Francesca Cagliari, Gabriele Castellini, Maria Manuel Clementino, Eraldo Giuli, David Holgate, Jürgen Koslowski, Hans-Peter Künzi, Bob Lowen, Sandra Mantovani, Jan Pelant, Nico Pumplün, Jiří Rosický, Alberto Tonolo, Anna Tozzi, Vladimir Uspenskij and Stephen Watson. We also thank the institutions that made possible our joint work: the Bulgarian Academy of Sciences, the Natural Sciences and Engineering Council of Canada, York University, Fernuniversität Hagen, the Universities of L'Aquila, Sydney, Trieste and Udine. Finally we thank Xiaomin Dong and Sandro Fusco for proof-reading early versions of the text, Angelo Montanari for his computer support, Luis Molina for his help in providing the typescript of chapters 1-5, Francis Borceux for letting us use his diagram package, and the publishers for their patience with the authors, who still feel that they have just started their project but would probably never have finished it without the moral support of their best friends Giovanna D'Agostino, Alberto Policriti and Hélène Massam.

Udine and Toronto, February 1995

Introduction

Closure operators (also closure operations, systems, functions, or relations) have been used intensively in Algebra (Birkhoff [1937], [1940], Pierce [1972]) and Topology (Kuratowski [1922], [1933], Čech [1937], [1966]). But their origins seem to go back to foundational work in Analysis by Moore [1909] and Riesz [1909] who both presented related notions at the "IV Congresso Internazionale dei Matematici" in Rome in 1908 (as was pointed out recently by Germano and Mazzanti [1991]). Early appearances of closure operators are also to be found in Logic (see Hertz [1922] and Tarski [1929]) before Birkhoff's book on Lattice Theory [1940] led to more concentrated investigations on the subject, particularly by Ward [1942], Monteiro and Ribeiro [1942], Ore [1943a, b], and Everett [1944].

Category Theory provides a variety of notions which expand on the lattice-theoretic concept of closure operator most notably through the notion of reflective subcategory (see Freyd [1964], [1972], Kennison [1965], Herrlich [1968]), predecessors of which are present in the works of Samuel [1948], Bourbaki [1957], and Sonner [1963]. The notions of Grothendieck topology and Lawvere-Tierney topology (see Johnstone [1977] and Mac Lane and Moerdijk [1992]) provide standard tools in Sheaf- and Topos Theory and are most conveniently described by particular closure operators.

Both lattice-theoretic and categorical views of closure operators play an important role in Theoretical Computer Science, again in a variety of ways. We mention only Scott's work [1972], [1982] which laid the foundations of domain theory, and we point to the vast literature on generalized functorial Tarski-type least-fixed-point constructions (see, in particular, Wand [1979], Koubek-Reiterman [1979], Kelly [1980]).

The immediate aim of introducing closure operators is to describe conveniently the closure of a substructure with respect to a certain desirable additional property. Well-known examples are the (usual Kuratowski) closure of a subspace of a topological space, or the normal closure of a subgroup of a group, or the Scott closure in a directed-complete partially ordered set. Lattice theorists usually define a *closure operation* c of a lattice L (with bottom element 0) to be a function $c : L \longrightarrow L$ which is

- extensive $(m \leq c(m))$
- monotone $(m \leq n \Rightarrow c(m) \leq c(n))$
- idempotent $(c(c(m)) = c(m))$,

and sometimes require c to be also

- grounded $(c(0) = 0)$
- additive $(c(m \vee n) = c(m) \vee c(n))$

From the categorical point of view, these systems of axioms turn out to be both insufficient and too restrictive. They ignore the important fact that, in the examples mentioned before, the closure operation c is available in *each* subobject lattice, and that every morphism $f : X \to Y$ is *continuous* with respect to the closure operation:

$$f(c_X(m)) \le c_Y(f(m))$$

for every subobject m of X. Like Čech [1959] in his Topology book, we do not assume idempotency of a closure operation a priori. Therefore, when calling a subobject m with $c(m) = m$ to be c-closed, in general, we do not expect the closure $c(m)$ of m to be c-closed. However, normally c-closedness may eventually be achieved by repeated (transfinite) application of c: $m \le c(m) \le c^2(m) \le \cdots \le c^\infty(m)$.

In an arbitrary category $\mathcal{X}$ with a suitable axiomatically defined notion of subobject, a (categorical) *closure operator* C is defined to be a family $(c_X)_{X\in\mathcal{X}}$ satisfying the properties of extension, monotonicity and continuity (see Dikranjan and Giuli [1987a], Dikranjan, Giuli and Tholen [1989]). The first five chapters of this book give a comprehensive introduction to the most important special properties and constructions involving closure operators. In addition to idempotency and additivity, these include hereditariness (for subobjects $m \le y$ of X with $y : Y \to X$, the closure of m in Y is obtained by intersecting its closure in X with y) and productivity (c preserves direct product of subobjects). Closure operators may be ordered like subobjects, and the properties of closure operators that we are interested in are either stable under taking infima or suprema. This is the reason why each closure operator has, for instance, an *idempotent hull* (which, in most cases, may be "computed" by an iterative process, as indicated above) and an *additive core*. Under the transition from C to its idempotent hull, say, other properties may or may not be preserved. Here, for instance, additivity survives the passage, but hereditariness does not in general.

We examine all these properties and constructions carefully, both in terms of theory and of examples, taken predominantly from topology, algebra, and discrete mathematics. This enables us not only to detect common features and construction principles, but also to point to striking dissimilarities. For instance, with respect to the seemingly harmless condition of groundedness, one shows easily that in the category of topological spaces each non-trivial closure operator is grounded, whereas in the category of R-modules only trivial closure operators are grounded. Similarly, additivity is a common property for closure operators in topology but extremely restrictive for R-modules.

While Chapters 1-4 keep the needed categorical apparatus limited, in Chapter 5 we give various functorial descriptions and constructions with closure operators which underline the naturality of the notion. First of all, a closure operator of a category $\mathcal{X}$ is nothing but a *pointed endofunctor* of the category of all subobjects of $\mathcal{X}$. Iterations of this endofunctor as used in functorial fixed-point constructions lead to its idempotent hull (if they "converge"). Closure operators may also be interpreted as generalized *factorization systems*: a morphism f gets factored through the closure of its image,

$$X \to c(f(X)) \to Y .$$

But only if the closure operator is idempotent and if it satisfies a *weak hereditariness property* does one obtain a (dense, closed)-factorization. However, these two special classes of morphisms enjoy the important properties of colimit- and limit stability even in the general case. In the category of R-modules, closure operators (and their properties) correspond to (the theory of) *preradicals*. Not only does this correspondence offer a rich supply of examples, but it is also extendable to our general context and turns out to be useful in non-Abelian structures as well. Hence preradicals offer a third (but more restrictive) interpretation of closure operators.

Categories which come equipped with a fixed closure operator behave like (large) "spaces" the interaction of which is described by *continuous functors*. Similarly to the weak (=initial) topology, for any functor $F : \mathcal{X} \to \mathcal{Y}$ and any closure operator D of $\mathcal{Y}$, one has a coarsest (=largest) closure operator C of $\mathcal{X}$ which makes F continuous; analogously for final structures. Hence closure operators may be pulled back or pushed along functors. In important cases (if F is a fibration, or if F is left- or right-adjoint) one obtains concrete construction procedures for closure operators defined this way. For example, in the category of topological groups we are able to establish closure operators very effectively by pulling back closure operators of both the categories of (discrete) groups and of topological spaces along the respective forgetful functors.

Chapters 6-9 are, generally spoken, devoted to the *epimorphism problem*, that is: to the characterization of those morphisms $f : A \to B$ of a category $\mathcal{A}$ which satisfy the cancellation property $(u \cdot f = v \cdot f \Rightarrow u = v)$. Closely related to this is the question whether $\mathcal{A}$ is *cowellpowered*, this is: whether for every object A in $\mathcal{A}$ there is only a small set (not a proper class) of non-isomorphic epimorphisms with domain A. We shall mostly assume that $\mathcal{A}$ is a full subcategory of $\mathcal{X}$, and our aim is to find an effectively defined closure operator C of $\mathcal{X}$ such that the epimorphisms of $\mathcal{A}$ are characterized as the C-dense morphisms in $\mathcal{A}$.

Two typical examples from topology illustrate this approach. In the category of Hausdorff spaces (spaces in which distinct points can be separated by disjoint open neighbourhoods), the epimorphisms are exactly described by the dense maps with respect to the usual Kuratowski closure (i.e., maps whose image is closed in the codomain). Furthermore, since the size of a Hausdorff space Y containing a dense subspace X is bounded by $2^{2^{\mathrm{card}(X)}}$, this category is cowellpowered. In the category of Urysohn spaces (spaces in which distinct points can be separated by disjoint *closed* neighbourhoods), the epimorphism- and cowellpoweredness problem is much harder. It seems natural to consider the so-called θ-closure of a subspace M of a topological space X first introduced by Veličko [1966], which is given by the points $x \in X$ such that every closed neighbourhood of x meets M. Although epimorphisms of the category of Urysohn spaces are not necessarily θ-dense, the θ-closure leads to the right track: they are characterized as the θ^{∞}-dense maps, with θ^{∞} the unbounded transfinite iteration of θ, that is the idempotent hull of θ. Schröder [1983] constructed, for every cardinal κ, an Urysohn space Y_{κ} of cardinality κ which contains the space $\mathbb{Q}$ of rational numbers as a θ^{∞}-dense subspace. Consequently, the category of Urysohn spaces is not cowellpowered.

We begin our investigations on epimorphisms in Chapter 6 which deals with the *regular closure operator* $\mathrm{reg}^{\mathcal{A}}$ induced by a subcategory $\mathcal{A}$ of $\mathcal{X}$. It was introduced in a topological context by Salbany [1976], but the factorization it induces in $\mathcal{A}$ is exactly Isbell's [1966] dominion factorization which led him to characterize the epimorphisms of semigroups and other algebraic categories (see also Cagliari and Chicchese [1982]). Since within the category $\mathcal{A}$, $\mathrm{reg}^{\mathcal{A}}$-dense means epimorphism and $\mathrm{reg}^{\mathcal{A}}$-closed means regular monomorphism, effective computational methods for regular closure operators need to be developed. Often this is achieved by providing a closure operator C of $\mathcal{X}$ such that its idempotent hull coincides with $\mathrm{reg}^{\mathcal{A}}$, at least when restricted to $\mathcal{A}$; for instance θ does this job for $\mathcal{A} = \mathbf{Ury}$. In other instances, one first needs to "modify C along cokernelpairs" before being able to reach the regular closure via the idempotent hull.

An intensive study of the cokernelpair $X +_M X$ in $\mathcal{X}$ of the subobject $m : M \to X$ with $X \in \mathcal{A}$ is in fact the first step in tackling the epimorphism problem in any non-trivial situation. For "most" categories $\mathcal{X}$ it turns out that for m to be regular monomorphism of $\mathcal{A}$, it is necessary and sufficient that $X +_M X$ already belongs to $\mathcal{A}$, as explained by the *Magic Cube Theorem* (6.4) and *Frolík's Lemma* (6.5).

There are various ways of reversing the passage $\mathcal{A} \mapsto \mathrm{reg}^{\mathcal{A}}$ which are being studied in Chapter 7. For example, as Hausdorff spaces are characterized as the topological spaces X with closed diagonal Δ_X in $X \times X$, one can show in a fairly general categorical context that the objects of any regular-epireflective subcategory $\mathcal{A}$ of $\mathcal{X}$ are those objects X with $\mathrm{reg}^{\mathcal{A}}$-closed diagonal (cf. Giuli and Hušek [1986], Giuli, Mantovani and Tholen [1988].) Hence the passage that assigns to any closure operator C the so-called Delta-subcategory of objects with C-closed diagonal is of particular interest in the context of the epimorphism problem. It is used to characterize the additive regular closure operators, which are of particular interest for the epimorphism problem in subcategories of topological categories. In general, having a subcategory $\mathcal{A}$ of $\mathcal{X}$, it is often advantageous to look for intermediate categories $\mathcal{B}$ such that the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ preserves epimorphisms. We present two good candidates for $\mathcal{B}$ and describe them in terms of closure operators which are intimately connected with the regular closure operators.

Chapter 8 contains a variety of known or new results on **Haus** of Hausdorff spaces These are large families of subcategories either containing **Haus** or being contained in **Haus** for which we present unified criteria and constructions for epimorphisms and (non-)cowellpoweredness.

While every "reasonable" ranked category in algebra is cowellpowered (see Isbell [1966], Gabriel and Ulmer [1971], Adámek and Rosický [1994]), the epimorphism problem remains highly interesting. We concentrate our investigations on areas where closure operators are useful in deriving new results. For instance, Theorem 8.9 gives a complete description of subcategories of R-modules with surjective epimorphisms, and Theorem 8.10 provides a closure-theoretic description of epimorphisms in the category of fields.

As indicated above, closure operators may be described by (generalized) factorization systems. On the other hand, factorization systems $(\mathcal{E}, \mathcal{M})$ with special stability properties of the class $\mathcal{E}$ characterize reflective subcategories and localizations (see Cassidy, Hébert and Kelly [1985], Borceux and Kelly [1987]), the latter of which are described in Topos Theory by Grothendieck- and by Lawvere-Tierney topologies. LT-topologies are simply idempotent and weakly hereditary closure operators whose dense subobjects are stable under pullback. We discuss them briefly in Chapter 9, concentrating on the Delta-subcategory which they induce. Under light assumptions on the category, one can effectively construct the reflector into the Delta-subcategory, and its epimorphisms are just the dense morphisms.

Closure operators can be used to study topological concepts, such as separatedness, regularity, connectedness, and compactness, in abstract categories which are endowed with a closure operator (see in particular Manes [1974], Herrlich, Salicrup and Strecker [1987], Giuli [1991], Dikranjan and Giuli [1988b], [1989], [1991], Fay [1988], Castellini [1992], Clementino [1992], Fay and Walls [1994]). We emphasize that it is not the aim of this monograph to pursue these concepts to any extent. However, notions of separatedness and (dis-)connectedness appear throughout Chapters 6-9 to the extent to which they are of interest in conjunction with the epimorphism problem.

1 Preliminaries on Subobjects, Images, and Inverse Images

In this chapter we provide the basic categorical framework on subobjects, inverse images and image factorization as needed throughout the book.

1.1 $\mathcal{M}$-subobjects

A closure operator in the category of sets assigns to every subset M of a set X an intermediate set $c_X(M)$ such that certain properties hold. In the category of topological spaces, we shall be considering subspaces M of a space X, and in the category of groups subgroups M of a group X to which a closure operator can be applied. In an arbitrary category $\mathcal{X}$, we must first provide a suitable notion of subobject. These subobjects are described by special morphisms $M \to X$ in $\mathcal{X}$ which, in concrete categories of interest, may be safely thought of as inclusion maps $M \hookrightarrow X$.

In order to allow for sufficient flexibility, we define subobjects by introducing an additional parameter: *we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathcal{X}$ which will play the role of subobjects.* (That every morphism in $\mathcal{M}$ is assumed to be a monomorphism of $\mathcal{X}$ is not essential, but it facilitates an easy presentation of the framework; see Exercise 1.G.)

For every object X of $\mathcal{X}$, let $\mathcal{M}/X$ be the class of all $\mathcal{M}$-morphisms with codomain X; the relation given by

$$m \leq n \Leftrightarrow (\exists\, j) \;\; n \cdot j = m$$

$$\begin{tikzcd} M \arrow[rr, "j"] \arrow[dr, "m"'] & & N \arrow[dl, "n"] \\ & X & \end{tikzcd} \tag{1.1}$$

is reflexive and transitive, hence $\mathcal{M}/X$ is a *preordered class*. Since n is monic, the morphism j is uniquely determined, and it is an isomorphism of $\mathcal{X}$ if and only if $n \leq m$ holds; in this case m and n are called *isomorphic*, and one writes $m \cong n$. Of course, $\cong$ is an equivalence relation, and $\mathcal{M}/X$ modulo $\cong$ is a *partially ordered class* for which we can use all lattice-theoretic terminology and notations, such as $\wedge$, $\vee$, $\bigwedge$, $\bigvee$, etc. In fact, we shall use these notations for elements of $\mathcal{M}/X$ rather than for their $\cong$-equivalence classes both of which we refer to as *$\mathcal{M}$-subobjects of X*; the prefix $\mathcal{M}$ is often omitted. This means that, for $m, n \in \mathcal{M}/X$, $m \wedge n$ denotes a representative in $\mathcal{M}/X$ of the meet of the $\cong$-equivalence classes (whenever the meet exists).

In other words, with $\underline{m}$ denoting the $\cong$-equivalence class of m , we have the equivalences

$$m \leq n \quad \Leftrightarrow \quad \underline{m} \leq \underline{n}$$
$$m \cong n \quad \Leftrightarrow \quad \underline{m} = \underline{n}$$
$$k \cong m \wedge n \quad \Leftrightarrow \quad \underline{k} = \underline{m} \wedge \underline{n} ,$$

and analogously for $\vee$, $\bigwedge$, $\bigvee$. We will exclusively use the notation given by the left-hand sides of these equivalences.

It is convenient to assume throughout this book that

- $\mathcal{M}$ *is closed under composition with isomorphisms* (so that for every commutative diagram (1.1) with arbitrary morphisms m , n and an isomorphism j , one has $m \in \mathcal{M}$ if and only if $n \in \mathcal{M}$)
- $\mathcal{M}$ *contains all identity morphisms* (hence all isomorphisms in light of the previous requirement).

This may seem like a departure from our original intuition. However, in all concrete examples there will be a natural subclass $\mathcal{M}_0$ of $\mathcal{M}$ available such that, for every $m \in \mathcal{M}/X$, there is exactly one isomorphic copy of m in $\mathcal{M}_0/X$, i.e. $\mathcal{M}_0/X$ is a (categorical) skeleton of $\mathcal{M}/X$ for every X . In this case, $\mathcal{M}_0/X$ is order-isomorphic to $\mathcal{M}/X$ modulo $\cong$, and we call $\mathcal{M}_0$ a *skeleton* of $\mathcal{M}$.

Note that, in general, $\mathcal{M}/X$ is a proper class. $\mathcal{X}$ is called $\mathcal{M}$*-wellpowered* if there is a skeleton $\mathcal{M}_0$ of $\mathcal{M}$ such that each class $\mathcal{M}_0/X$ is small; equivalently, if $\mathcal{M}/X$ modulo $\cong$ can be labeled by a small set for every object X . In the examples which are of interest to us in this chapter, $\mathcal{X}$ is always $\mathcal{M}$-wellpowered. However, we shall encounter many counter-examples in Chapter 8 (in the dual setting).

EXAMPLES

(1) In the category **Set** of all sets and mappings, let $\mathcal{M}$ be the class of all monomorphisms (i.e., all injective maps). In $\mathcal{M}/X$, every injective map $m : M \to X$ is isomorphic to the inclusion map $m(M) \hookrightarrow X$. Hence, the class $\mathcal{M}_0$ of inclusion maps provides a natural skeleton for $\mathcal{M}/X$. In other words, $\mathcal{M}/X$ modulo $\cong$ is isomorphic to the power set 2^X ordered by inclusion.

(2) In the category **Top** of all topological spaces and (continuous) maps, let $\mathcal{M}$ be the class of *embeddings* (i.e. injective maps $m : M \to X$ such that the set of open sets in M is $\{m^{-1}(U) : U \text{ open in } X\}$). A skeleton $\mathcal{M}_0$ of $\mathcal{M}$ is given by the inclusion maps of subspaces (since, for $m : M \to X$ in $\mathcal{M}$, m induces a homeomorphism of M and the subspace $m(M)$ of X).

(3) The class $\mathcal{M}_0$ of inclusion maps of subgroups is a skeleton of the class $\mathcal{M}$ of all injective homomorphism in the category **Grp** of all groups and their homomorphisms.

1.2 Inverse images are $\mathcal{M}$-pullbacks

Inverse images of $\mathcal{M}$-subobjects are given by pullback. More precisely, for our fixed class $\mathcal{M}$ of monomorphisms in the category $\mathcal{X}$, we say that $\mathcal{X}$ *has $\mathcal{M}$-pullbacks* if, for every morphism $f : X \to Y$ and every $n \in \mathcal{M}/Y$, a pullback diagram

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.2}$$

exists in $\mathcal{X}$ with $m \in \mathcal{M}/X$; hence $n \cdot f' = f \cdot m$, and whenever $f \cdot g = n \cdot h$ holds in $\mathcal{X}$, then there is a (necessarily uniquely determined) morphism t with $m \cdot t = g$ (and $f' \cdot t = h$). Of course, as an $\mathcal{M}$-subobject of X , m is uniquely determined up to isomorphism; it is called the *inverse image of n under f* and denoted by $f^{-1}(n) : f^{-1}(N) \to X$. The pullback property of (1.2) yields that

$$f^{-1}(-) : \mathcal{M}/Y \to \mathcal{M}/X$$

is an order-preserving map so that

$$k \leq n \Rightarrow f^{-1}(k) \leq f^{-1}(n)$$

holds.

$$\begin{tikzcd} & f^{-1}(N) \arrow[rr] \arrow[ddl, "f^{-1}(n)"] & & N \arrow[ddl, "n"] \\ f^{-1}(K) \arrow[ur] \arrow[rr] \arrow[d, "f^{-1}(k)"'] & & K \arrow[ur, "j"] \arrow[d, "k"] & \\ X \arrow[rr, "f"'] & & Y & \end{tikzcd} \tag{1.3}$$

1.3 Review of pairs of adjoint maps

Images of subobjects are given by (left-) adjoints to the maps $f^{-1}(-)$. Hence we first review the notion of adjointness in the context of preordered classes (i.e. classes which come equipped with a reflexive and transitive relation $\leq$).

A pair of mappings $\varphi : P \to Q$, $\psi : Q \to P$ between preordered classes P , Q is called *adjoint* if

$$(*) \qquad m \le \psi(n) \Leftrightarrow \varphi(m) \le n$$

holds for all $m \in P$ and $n \in Q$. One says that φ is *left-adjoint* to ψ or ψ is *right-adjoint* to φ in this case and writes $\varphi \dashv \psi$. Adjoints determine each other uniquely, up to the equivalence relation given by $(x \cong y \Leftrightarrow x \le y$ and $y \le x)$:

LEMMA *The following assertions are equivalent for any pair of mappings $\varphi : P \to Q$, $\psi : Q \to P$ of preordered classes:*

(i) $\varphi \dashv \psi$;

(ii) ψ *is order-preserving, and* $\varphi(m) \cong \min\{n \in Q : m \le \psi(n)\}$ *holds for all* $m \in P$;

(iii) φ *is order-preserving, and* $\psi(n) \cong \max\{m \in P : \varphi(m) \le n\}$ *holds for all* $n \in Q$;

(iv) φ *and* ψ *are order-preserving, and*

$$m \le \psi(\varphi(m)) \quad \textit{and} \quad \varphi(\psi(n)) \le n$$

holds for all $m \in P$, $n \in Q$.

Proof (i) $\Rightarrow$ (ii) & (iii) Putting $n := \varphi(m)$ in $(*)$, one obtains $m \le \psi(\varphi(m))$, hence $\varphi(m) \in Q_m := \{n \in Q : m \le \psi(n)\}$. Furthermore, for all $n \in Q_m$, $(*)$ yields $\varphi(m) \le n$, hence $\varphi(m) \cong \min Q_m$. This formula implies immediately that φ is order-preserving. Dually one obtains the formula for ψ as given in (iii), and that ψ is order-preserving.

(ii) $\Rightarrow$ (iv) As mentioned before, the given formula for φ implies its monotonicity. Furthermore, since $\varphi(m) \in Q_m$, one has $m \le \psi(\varphi(m))$, and since $n \in Q_{\psi(n)}$, one has $\varphi(\psi(n)) \le n$ for all $m \in P$ and $n \in Q$. (iii) $\Rightarrow$ (iv) follows dually.

(iv) $\Rightarrow$ (i) $m \le \psi(n)$ implies $\varphi(m) \le \varphi(\psi(n)) \le n$, and $\varphi(m) \le n$ implies $m \le \psi(\varphi(m)) \le \psi(n)$. □

The most important property of adjoint pairs is the preservation of joins and meets:

PROPOSITION *If $\varphi \dashv \psi$, then φ preserves all existing joins (=suprema), and ψ preserves all existing meets (= infima). Hence one has the formulas*

$$\varphi\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} \varphi(m_i) \quad \textit{and} \quad \psi\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} \psi(n_i) .$$

Furthermore, $\varphi \cdot \psi \cdot \varphi = \varphi$ and $\psi \cdot \varphi \cdot \psi = \psi$, so that φ and ψ give a bijective correspondence between $\psi(Q)$ and $\varphi(P)$.

Proof By monotonicity of φ, $\varphi(m)$ is an upper bound of $\{\varphi(m_i) : \; i \in I\}$, with $m \cong \bigvee_{i\in I} m_i$. For any other upper bound n, one has $m_i \le \psi(n)$ for all

$i \in I$ by (*), hence $m \leq \psi(n)$. Application of (*) again yields $\varphi(m) \leq n$. This proves that φ preserves joins. The assertion for ψ follows dually.

Furthermore, when applying the order-preserving map φ to the first inequality of (iv) in the Lemma one obtains $\varphi(m) \leq \varphi(\psi(\varphi(m)))$, and when exploiting the second inequality in case $n = \varphi(m)$ one obtains $\varphi(\psi(\varphi(m))) \leq \varphi(m)$. Hence $\varphi \cdot \psi \cdot \varphi = \varphi$, and $\psi \cdot \varphi \cdot \psi = \psi$ follows dually. □

The converse of the first statement of the Proposition holds if arbitrary meets (and joins) exist in Q (and P):

THEOREM

(1) *Let Q have all meets (regardless of size of the indexing system I). Then a mapping $\psi : Q \to P$ has left-adjoint φ if and only if ψ preserves all meets.*

(2) *Let P have all joins. Then a mapping $\varphi : P \to Q$ has a right-adjoint ψ if and only if φ preserves all joins.*

Proof It suffices to show (1) since (2) follows by dualization. Furthermore, after the Proposition, we just need to show that preservation of meets by ψ yields existence of a left-adjoint φ . Indeed, putting

$$\varphi(m) \cong \bigwedge\{n \in Q : m \leq \psi(n)\} ,$$

one obtains

$$\psi(\varphi(m)) \cong \bigwedge\{\psi(n) : m \leq \psi(n)\} \geq m ,$$

hence $\varphi(m) \in \{n \in Q : m \leq \psi(n)\} = Q_m$ and $\varphi(m) \cong \min Q_m$. As a meet-preserving map, ψ is order-preserving. Hence $\varphi \dashv \psi$ holds by the Lemma.
□

1.4 Adjointness of image and inverse image

Let $\mathcal{X}$ have $\mathcal{M}$-pullbacks, and for every $f : X \to Y$ in $\mathcal{X}$, let $f^{-1}(-) : \mathcal{M}/Y \to \mathcal{M}/X$ have a left-adjoint

$$f(-) : \mathcal{M}/X \to \mathcal{M}/Y .$$

For $m : M \to X$ in $\mathcal{M}/X$, we call $f(m) : f(M) \to Y$ in $\mathcal{M}/Y$ the *image of* m *under* f ; it is uniquely determined (up to isomorphism) by the property.

$$(*) \qquad m \leq f^{-1}(n) \Leftrightarrow f(m) \leq n$$

for all $n \in \mathcal{M}/Y$. Furthermore, 1.3 yields the following formulas:

(1) $m \leq k \Rightarrow f(m) \leq f(k)$;

(2) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$;

(3) $f\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} f(m_i)$;

(4) $f^{-1}\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} f^{-1}(n_i)$.

For more formulas, see Exercises 1.K and 1.L.

Images can be characterized and constructed without reference to inverse images; this is done in the following two sections.

1.5 The right $\mathcal{M}$-factorization of a morphism

PROPOSITION *Let $\mathcal{X}$ have $\mathcal{M}$-pullbacks, and for $f : X \to Y$ in $\mathcal{X}$, let $f^{-1}(-)$ have a left adjoint $f(-)$. Then there are morphisms e, m in $\mathcal{X}$ such that*

(1) *$f = m \cdot e$ with $m : M \to Y$ in $\mathcal{M}$, and*

(2) *whenever one has a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "e"'] & N \arrow[dd, "n"] \\ M \arrow[d, "m"'] & \\ Y \arrow[r, "v"] & Z \end{tikzcd} \tag{1.4}$$

in $\mathcal{X}$ with $n \in \mathcal{M}$, then there is a uniquely determined morphism $w : M \to N$ with $n \cdot w = v \cdot m$ and $w \cdot e = u$.

Proof Consider $m := f(1_X) : f(X) \to Y$. Since $1_X \leq f^{-1}(m)$, one obtains a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "1_X"'] & f^{-1}(M) \arrow[r] \arrow[dl, "f^{-1}(m)"] \arrow[dr, phantom, "pb."] & M \arrow[d, "m"] \\ X \arrow[rr, "f"] & & Y \end{tikzcd} \tag{1.5}$$

With e the composite of the two top arrows of (1.5) one obtains (1). Given diagram (1.4) with $n \in \mathcal{M}$, the pullback-property yields a morphism $t : X \to v^{-1}(N)$ with $v^{-1}(n) \cdot t = f$. By the same property, one has a morphism $s : X \to f^{-1}\left(v^{-1}(N)\right)$

with $f^{-1}\left(v^{-1}(n)\right) \cdot s = 1_X$, i.e. $1_X \leq f^{-1}\left(v^{-1}(n)\right)$. Therefore $m = f(1_X) \leq v^{-1}(n)$ by adjointness. Now w is the composite of the two top arrows of (1.6).

$$\begin{tikzcd} M \arrow[r] \arrow[d, "m"'] & v^{-1}(N) \arrow[r] \arrow[dl, "v^{-1}(n)"] \arrow[dr, phantom, "pb."] & N \arrow[d, "n"] \\ Y \arrow[rr, "v"] & & Z \end{tikzcd} \tag{1.6}$$

By definition, $n \cdot w = v \cdot m$. Since n is monic, w is uniquely determined by this equation, and $w \cdot e = u$ follows from $n \cdot w \cdot e = v \cdot m \cdot e = n \cdot u$. □

Properties (1) and (2) determine e and m uniquely up to isomorphism: if e' and m' satisfy (1) and (2) as well, then there is an *iso*morphism t with $t \cdot e = e'$ and $m' \cdot t = m$ (consider $n = m'$, $u = e'$, $v = 1_Y$).

Any factorization $f = m \cdot e$ such that properties (1) and (2) hold is called *the right $\mathcal{M}$-factorization of f* . Property (2) is called the *diagonalization property* of the factorization.

These notions may be considered for any class $\mathcal{M}$, not just classes of monomorphisms. Dually, for any class $\mathcal{E}$ of morphisms in $\mathcal{X}$, one has the notion of a *left $\mathcal{E}$-factorization of f* in $\mathcal{X}$, that is a right $\mathcal{E}$-factorization of f in the opposite category of $\mathcal{X}$; just reverse the arrows in (1.4) and interchange the roles of e and m .

1.6 Constructing images from right $\mathcal{M}$-factorizations

Let every morphism in $\mathcal{X}$ have a right $\mathcal{M}$-factorization. For $f : X \to Y$ in $\mathcal{X}$ and $m : M \to X$ in $\mathcal{M}$, one defines $f(m) : f(M) \to Y$ to be the $\mathcal{M}$-part of a right $\mathcal{M}$-factorization of the composite $f \cdot m$.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "m"'] & f(M) \arrow[d, "f(m)"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.7}$$

Property (2) of 1.5 implies that the map

$$f(-) : \mathcal{M}/X \to \mathcal{M}/Y$$

is order-preserving. In case $\mathcal{X}$ has $\mathcal{M}$-pullbacks, $f(-)$ is left-adjoint to $f^{-1}(-)$:

THEOREM *The following assertions are equivalent:*

(i) $\mathcal{X}$ *has $\mathcal{M}$-pullbacks, and every morphism has a right $\mathcal{M}$-factorization;*

(ii) $\mathcal{X}$ *has* $\mathcal{M}$*-pullbacks, and* $f^{-1}(-)$ *has a left-adjoint for every morphism* f *;*

(iii) *every morphism has a right* $\mathcal{M}$*-factorization, and* $f(-)$ *has a right-adjoint for every morphism* f *.*

Proof ((i) $\Rightarrow$ (ii)) & ((i) $\Rightarrow$ iii)) For $f(-)$ as defined above, we must show $f(-) \dashv f^{-1}(-)$. In fact, $m \leq f^{-1}(f(m))$ follows from the pullback property and $f(f^{-1}(n)) \leq n$ from the diagonalization property. Hence Lemma 1.3 gives adjointness since both $f(-)$ and $f^{-1}(-)$ are order-preserving.

(ii) $\Rightarrow$ (i) follows from Proposition 1.5.

(iii) $\Rightarrow$ (i) Denote the existing right-adjoint of $f(-)$ by $f^{-1}(-)$. For every $n : N \to Y$ in $\mathcal{M}$ one has a commutative diagram

$$\begin{tikzcd}
f^{-1}(N) \arrow[r] \arrow[d, "f^{-1}(n)"'] & f(f^{-1}(N)) \arrow[r] \arrow[dr, "f(f^{-1}(n))"'] & N \arrow[d, "n"] \\
X \arrow[rr, "f"] & & Y
\end{tikzcd} \tag{1.8}$$

We must show that it is a pullback diagram in $\mathcal{X}$. In order to check the universal property, we consider morphisms $g : Z \to X$ and $h : Z \to N$ with $f \cdot g = n \cdot h$ and form the right $\mathcal{M}$-factorization $g = k \cdot e$ of g with $k : K \to X \in \mathcal{M}$.

The diagonalization property yields a morphism w rendering

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[d, "e"'] & N \arrow[dd, "n"] \\
K \arrow[ur, "w"] \arrow[d, "k"'] & \\
X \arrow[r, "f"] & Y
\end{tikzcd} \tag{1.9}$$

commutative, and by the same property one has $f(k) \leq n$. Therefore, $k \leq f^{-1}(n)$ by adjointness. Hence, there is a morphism $j : K \to f^{-1}(N)$ with $f^{-1}(n) \cdot j = k$. Consequently, for $t := j \cdot e : Z \to f^{-1}(N)$ one has $f^{-1}(n) \cdot t = k \cdot e = g$. Since n and $f^{-1}(n)$ are monic, t is uniquely determined and satisfies also $f' \cdot t = h$ with f' the composite of the upper two horizontal arrows of (1.8). This completes the proof. □

We call $\mathcal{X}$ *finitely* $\mathcal{M}$*-complete* if one (and then all) of the assertions of the Theorem hold.

EXAMPLES

(1) Every category is finitely $\mathcal{M}$-complete for $\mathcal{M}$ the class of isomorphisms.

(2) In each of the Examples 1.1, the category in question is (obviously) finitely $\mathcal{M}$-complete. In each case, the right $\mathcal{M}$-factorization of a morphism $f: X \to Y$ is given by its image $f(X)$ considered as a subobject of Y .

(3) Here is a natural example where $\mathcal{M}$-factorizations are not given by the set-theoretic image: in **Top**, let $\mathcal{M}$ be the class of *closed* embeddings. Then **Top** is finitely $\mathcal{M}$-complete; the right $\mathcal{M}$-factorization of a map $f: X \to Y$ is given by the closure of $f(X) \subseteq Y$.

(4) (*Existence of $\mathcal{M}$-pullbacks does not imply existence of right $\mathcal{M}$-factorizations*) Let now $\mathcal{M}$ be the class of *open* embeddings in **Top**. Obviously, $\mathcal{M}$-pullbacks exist. On the other hand, it is easy to see that $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, does not have a right $\mathcal{M}$-factorization (see Exercise 1.B).

(5) (*Existence of right $\mathcal{M}$-factorizations does not imply existence of $\mathcal{M}$-pullbacks*) In the category **CTop** of all connected topological spaces, every map has obviously a right $\mathcal{M}$-factorization, with $\mathcal{M}$ the class of embeddings. But $\mathcal{M}$-pullbacks fail to exist (see Exercise 1.B).

1.7 Stability properties of $\mathcal{M}$-subobjects

For $\mathcal{M}$-subobjects $m: M \to X$ and $n: N \to Y$ in $\mathcal{M}$, the direct product $M \times N \to X \times Y$ should also be an $\mathcal{M}$-subobject. Here is a much more general result:

PROPOSITION *Let every morphism in $\mathcal{X}$ have a right $\mathcal{M}$-factorization. Then, for any diagram type $\mathcal{D}$, $\mathcal{M}$ is* closed under $\mathcal{D}$-limits*; this means that, for any natural transformation $\mu: H \to K$ with H , $K: \mathcal{D} \to \mathcal{X}$, $k = \varprojlim \mu : \varprojlim H \to \varprojlim K$ belongs to $\mathcal{M}$ if every μ_d , $d \in \mathcal{D}$, belongs to $\mathcal{M}$.*

Proof (Since we shall prove a more general result in 5.2, we just give a sketch here.) Consider a right $\mathcal{M}$-factorization $k = m \cdot e$ and, for every $d \in \mathcal{D}$, apply the diagonalization property to:

$$\begin{tikzcd} \lim\limits_{\leftarrow} H \arrow[r] \arrow[d, "e"'] & Hd \arrow[dd, "\mu_d"] \\ M \arrow[d, "m"'] & \\ \lim\limits_{\leftarrow} K \arrow[r] & Kd \end{tikzcd} \tag{1.10}$$

Then the limit property of $\lim\limits_{\leftarrow} H$ is used to find an inverse of e, hence $k = m \cdot e \in \mathcal{M}$. $\square$

Most other stability properties which one usually expects to hold, follow from closedness under limits:

THEOREM *Let $\mathcal{M}$ be closed under $\mathcal{D}$-limits for every $\mathcal{D}$. Then one has:*

(1) $\mathcal{M}$ *is closed under arbitrary direct products.*

(2) $\mathcal{M}$ *is* stable under pullback, *that is for every pullback diagram*

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.11}$$

$n \in \mathcal{M}$ *implies* $m \in \mathcal{M}$.

(3) *If* $n \cdot m \in \mathcal{M}$ *with* n *monic, then* $m \in \mathcal{M}$.

(4) $\mathcal{M}$ *is* stable under multiple pullback, *that is: for every multiple pullback diagram*

$$\begin{tikzcd} & M_i \arrow[dr, "m_i"] & \\ M \arrow[ur, "j_i"] \arrow[rr, "m"'] & & X \end{tikzcd} \tag{1.12}$$

$m_i \in \mathcal{M}$ for all $i \in I$ implies $m \in \mathcal{M}$.

(Multiple pullbacks are considered in more detail in 1.9 below.)

Proof

(1) Consider discrete diagram types D .

(2) Given the pullback diagram (1.11), consider

$$\begin{tikzcd} & N \arrow[rr, "n"] \arrow[dd, "n"' near start] & & Y \arrow[dd, "1_Y"] \\ M \arrow[ur, "f'"] \arrow[rr, "m" near start] \arrow[dd, "m"'] & & X \arrow[ur, "f"] \arrow[dd, "1_X" near start] & \\ & Y \arrow[rr, "1_Y" near start] & & Y \\ X \arrow[ur, "f"] \arrow[rr, "1_X"'] & & X \arrow[ur, "f"'] & \end{tikzcd} \tag{1.13}$$

Since both top and bottom face are pullback diagrams, M is the limit of the diagram H given by f and n , and X is the limit of the diagram K given by f and 1_Y . The three vertical arrows 1_X , 1_Y , n constitute a natural transformation $\kappa : H \to K$ which belongs pointwise to $\mathcal{M}$. Hence its limit m belongs also to $\mathcal{M}$.

(3) If $n \cdot m \in \mathcal{M}$ with n monic, then one has a pullback diagram

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "m"'] & M \arrow[d, "n \cdot m"] \\ N \arrow[r, "n"'] & X \end{tikzcd} \tag{1.14}$$

Hence $m \in \mathcal{M}$ follows from (2).

(4) Consider the multiple pullback diagram (1.12) with every $m_i \in \mathcal{M}$. Then M is the limit of the diagram H given by $(m_i)_{i \in I}$. Let K be a diagram of the

same type with constant value 1_X . We have a natural transformation $\mu : H \to K$ given by the m_i's, hence pointwise in $\mathcal{M}$. Therefore its limit m is also in $\mathcal{M}$. □

REMARK We emphasize that we distinguish between *stability under pullback* (as defined in the Theorem) and *closedness under $\mathcal{D}$-limits* (as defined in the Proposition), for $\mathcal{D} = \{\cdot \to \cdot \leftarrow \cdot\}$. The proof of (2) of the Theorem shows that the latter property implies the former. The converse proposition holds under mild additional hypotheses.

COROLLARY *Let $\mathcal{X}$ have pullbacks, and let every morphism in $\mathcal{X}$ have a right $\mathcal{M}$-factorization. Then $\mathcal{X}$ is finitely $\mathcal{M}$-complete.*

Proof By the Proposition and the Theorem, $\mathcal{M}$ is stable under pullback. Since $\mathcal{X}$ has pullbacks, $\mathcal{M}$-pullbacks exist in $\mathcal{X}$. □

1.8 $\mathcal{M}$-subobjects of $\mathcal{M}$-subobjects

For $\mathcal{M}$-subobjects $m : M \to N$ and $n : N \to X$, the composite $n \cdot m : M \to X$ should be an $\mathcal{M}$-subobject. But closedness under composition is suspiciously absent from the stability properties of $\mathcal{M}$ listed in 1.7, for a good reason: *finite $\mathcal{M}$-completeness does not imply that $\mathcal{M}$ is closed under composition.*

EXAMPLE In the category **Grp** of groups, let $\mathcal{M}$ be the class of those injective homomorphisms $f : G \to H$ for which $f(G)$ is *normal* in H . Obviously, **Grp** is $\mathcal{M}$-complete (the right $\mathcal{M}$-factorization of a homomorphism $f : G \to H$ is given by the normal closure of $f(G)$ in H). However, if N is a normal subgroup of G and M is a normal subgroup of N , M need not be normal in G (consider the group D_4 of symmetries of the square).

Closedness of $\mathcal{M}$ under composition makes right $\mathcal{M}$-factorizations "symmetric in both factors":

THEOREM *The following two assertions are equivalent:*

(i) *every morphism has a right $\mathcal{M}$-factorization, and $\mathcal{M}$ is closed under composition;*

(ii) *there is a class $\mathcal{E}$ of morphisms in $\mathcal{X}$ such that*

  (1) *every morphism f in $\mathcal{X}$ has a factorization $f = m \cdot e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, and*

  (2) *for every commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "e"'] & M \arrow[d, "m"] \\ Y \arrow[r, "v"] & Z \end{tikzcd} \tag{1.15}$$

with $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$ *, there is a uniquely determined morphism* $w : Y \to M$ *with* $w \cdot e = u$ *and* $m \cdot w = v$ *.*

Proof (i) $\Rightarrow$ (ii) We write $e \perp m$ and we say that c *is orthogonal to* m , if for every commutative diagram (1.15) there is exactly one w with $w \cdot e = u$ and $m \cdot w = v$. Let $\mathcal{E}$ be the class

$$\mathcal{M}^{\perp} := \{e \in \mathrm{Mor}\mathcal{X} : (\forall m \in \mathcal{M})\ e \perp m\} .$$

It then suffices to show that, when forming the right $\mathcal{M}$-factorization of $f = m \cdot e : X \to Y$, one has $e \in \mathcal{E}$. To this end, we form the right $\mathcal{M}$-factorization of $e = n \cdot d : X \to M$ and apply the diagonalization property of the first factorization to

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "e"'] & N \arrow[dd, "m \cdot n"] \\ M \arrow[d, "m"'] & \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd} \tag{1.16}$$

Since $m \cdot n \in \mathcal{M}$ one obtains a morphism $t : M \to N$ with $m \cdot n \cdot t = m$. Since m and n are monic, n is an isomorphism. Now the diagonalization property of the second factorization easily yields $e \in \mathcal{M}^{\perp} = \mathcal{E}$.

(ii) $\Rightarrow$ (i) We first show that $\mathcal{M}$ must coincide with the class

$$\mathcal{E}_{\perp} := \{m \in \mathrm{Mor}\mathcal{X} : (\forall e \in \mathcal{E})\ e \perp m\} .$$

Property (2) gives $\mathcal{M} \subseteq \mathcal{E}_{\perp}$. Vice versa, for $m \in \mathcal{E}_{\perp}$ consider a factorization $m = k \cdot c$ with $k \in \mathcal{M}$ and $c \in \mathcal{E}$ (which exists by (1)).

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "c"'] & M \arrow[d, "m"] \\ K \arrow[r, "k"] & X \end{tikzcd} \tag{1.17}$$

Since $m \in \mathcal{E}_\perp$ there is a morphism $w : K \to M$ with $w \cdot c = 1_M$ and $m \cdot w = k$. Since k is monic, also w is monic. Hence c is an isomorphism, and we have $m \cong k \in \mathcal{M}/X$. Now it is a straight exercise to prove that $\mathcal{E}_\perp$ is closed under composition (see Exercise 1.E). □

One says that $\mathcal{X}$ *has* $(\mathcal{E}, \mathcal{M})$*-factorizations* if condition (1) and (2) of (ii) (and therefore (i)) of the Theorem holds; property (2) is referred to as the $(\mathcal{E}, \mathcal{M})$ *-diagonalization property*. Note that the proof of the Theorem shows that $\mathcal{E}$ and $\mathcal{M}$ determine each other uniquely (provided also $\mathcal{E}$ is assumed to be closed under composition with isomorphisms). $(\mathcal{E}, \mathcal{M})$-factorizations are simultaneous right $\mathcal{M}$- and left $\mathcal{E}$-factorizations (see Exercise 1.N).

In each of the Examples 1.1, the category has $(\mathcal{E}, \mathcal{M})$ -factorizations, and $\mathcal{E}$ is given by the surjective maps. In general, however, $\mathcal{E}$ need not be a class of epimorphisms of the category. For instance, since the class $\mathcal{M}$ of closed embeddings in **Top** is closed under composition, there is a class $\mathcal{E}$ such that **Top** has $(\mathcal{E}, \mathcal{M})$ -factorizations. $\mathcal{E}$ is the class of all *dense* maps $f : X \to Y$ in **Top** (that is: the image $f(X)$ is dense in Y); these are not necessarily epic in **Top**.

All results presented so far could have been established without assuming $\mathcal{M}$ to be a class of monomorphisms (see Exercise 1.G). Without that assumption one has a perfect duality principle: if $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$ -factorizations, $\mathcal{X}^{op}$ has $(\mathcal{M}, \mathcal{E})$ -factorizations. In particular: whatever property holds for $\mathcal{M}$ in general, its dual is valid for $\mathcal{E}$. Therefore, from the Theorem and from Proposition 1.7 one obtains:

COROLLARY *If $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$ -factorizations, then $\mathcal{M}$ is closed under $\mathcal{D}$-limits and $\mathcal{E}$ is closed under $\mathcal{D}$-colimits for every $\mathcal{D}$; both classes are closed under composition.* □

1.9 When the subobjects form a large-complete lattice

If $\mathcal{X}$ has $\mathcal{M}$-pullbacks and if $\mathcal{M}$ is closed under composition, the preordered class $\mathcal{M}/X$ has binary meets for every object X : one obtains the meet

$$m \wedge n : M \wedge N \to X$$

of two $\mathcal{M}$-subobjects $m : M \to X$ and $n : N \to X$ as the diagonal of the pullback diagram

$$\begin{tikzcd} M \wedge N \arrow[r] \arrow[d] & N \arrow[d, "n"] \\ M \arrow[r, "m"] & X \end{tikzcd} \tag{1.18}$$

In general, for any $\mathcal{M}$, we say that $\mathcal{X}$ *has $\mathcal{M}$-intersections* if for every family $(m_i)_{i \in I}$ in $\mathcal{M}/X$ (I may be a proper class, or empty), a multiple pullback diagram

$$\begin{tikzcd} & M_i \arrow[dr, "m_i"] & \\ M \arrow[ur, "j_i"] \arrow[rr, "m"] & & X \end{tikzcd} \tag{1.19}$$

exists in $\mathcal{X}$ with $m \in \mathcal{M}/X$; hence $m_i \cdot j_i = m$ for all $i \in I$, and whenever one has $g : Z \to X$ and $h_i : Z \to M_i$ with $m_i \cdot h_i = g$ for all $i \in I$ then there is uniquely determined morphism $t : Z \to M$ with $m \cdot t = g$ and $j_i \cdot t = h_i$ for all $i \in I$. One easily verifies that m indeed assumes the role of the meet of $(m_i)_{i \in I}$ in $\mathcal{M}/X$. Hence one writes

$$m \cong \bigwedge_{i \in I} m_i \ : \ \bigwedge_{i \in I} M_i \to X \ .$$

But we often call m the *$\mathcal{M}$-intersection* of $(m_i)_{i \in I}$ in order to emphasize its categorical characterization as a multiple pullback. Of course, we speak of *finite $\mathcal{M}$-intersections* if I is finite.

Whereas the assumption that $\mathcal{M}$ be a class of monomorphism was put only for convenience in the previous sections, every morphism in $\mathcal{M}$ is *necessarily* monic if $\mathcal{X}$ has $\mathcal{M}$-intersections (with no restriction on the size of the indexing system I): see Exercise 1.F. In other words, assuming $\mathcal{M}$ to be a class of monomorphisms is no longer a restriction of generality if $\mathcal{X}$ is assumed to have $\mathcal{M}$-intersections.

PROPOSITION *If $\mathcal{X}$ has $\mathcal{M}$-intersections, then every preordered class $\mathcal{M}/X$ has the structure of a large-complete lattice, i.e., class-indexed meets and joins exist in $\mathcal{M}/X$ for every object $X \in \mathcal{X}$*

Proof As usual, one constructs the join of $(m_i)_{i \in I}$ in $\mathcal{M}/X$ as the meet of all upper bounds of $(m_i)_{i \in I}$ in $\mathcal{M}/X$. □

If $\mathcal{X}$ has also $\mathcal{M}$-pullbacks, it is easy to see that the join $m \in \mathcal{M}/X$ of $(m_i)_{i \in I}$ has the following categorical property: there are morphisms j_i $(i \in I)$ such that

(1) $m \cdot j_i = m_i$ for all $i \in I$;

(2) whenever one has commutative diagrams

$$\begin{tikzcd} M_i \arrow[r, "u_i"] \arrow[d, "j_i"'] & N \arrow[dd, "n"] \\ M \arrow[d, "m"'] & \\ X \arrow[r, "v"] & Z \end{tikzcd} \tag{1.20}$$

in $\mathcal{X}$ with $n \in \mathcal{M}$, then there is a uniquely determined morphism $w : M \to N$ with $n \cdot w = v \cdot m$ and $w \cdot j_i = u_i$ for all $i \in I$.

A subobject $m \in \mathcal{M}/X$ is called an *$\mathcal{M}$-union* of $(m_i)_{i \in I}$ if this categorical property holds. Letting $v = 1_X$ in (1.20) one sees that unions are joins in $\mathcal{M}/X$, hence one writes

$$m \cong \bigvee_{i \in I} m_i \ : \ \bigvee_{i \in I} M_i \to X \ .$$

$\mathcal{X}$ is said to *have (finite) $\mathcal{M}$-unions* if for every family $(m_i)_{i \in I}$ in $\mathcal{M}/X$ (with I finite) an $\mathcal{M}$-union exists.

COROLLARY *If $\mathcal{X}$ has $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections, then $\mathcal{X}$ has $\mathcal{M}$-unions.* □

A category may have $\mathcal{M}$-pullbacks and $\mathcal{M}$-unions (hence each $\mathcal{M}/X$ is a large-complete lattice), but fail to have $\mathcal{M}$-intersections (i.e. meets in $\mathcal{M}/X$ may fail to be intersections): **Top** with $\mathcal{M}=$ { open embeddings } (cf. Example (4) of 1.6) has $\mathcal{M}$-pullbacks and $\mathcal{M}$-unions; but the meet of open sets $U_i \subseteq X$ is just the interior of $\bigcap_{i \in I} U_i$, hence is in general properly smaller than the multiple pullback in **Top**.

1.10 The right $\mathcal{M}$-factorization of a sink

If $\mathcal{X}$ has $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections, $f^{-1}(-) : \mathcal{M}/Y \to \mathcal{M}/X$ preserves all meets for $f : X \to Y$ (since the latter are given by limits in $\mathcal{X}$, and limits commute with limits). Consequently, f has a right $\mathcal{M}$-factorization (by Theorem 1.3 and Proposition 1.5), hence $\mathcal{X}$ is finitely $\mathcal{M}$-complete. Next we shall show that a much stronger result can be obtained: any class-indexed family $(f_i : X_i \to Y)_{i \in I}$ of morphisms in $\mathcal{X}$ with common codomain, commonly called a *sink*, can be simultaneously factorized. A *right $\mathcal{M}$-factorization of the sink* $(f_i)_{i \in I}$ consists of morphisms $m \in \mathcal{M}$ and e_i $(i \in I)$ in $\mathcal{X}$ such that

(1) $f_i = m \cdot e_i$ for all $i \in I$ with $m : M \to Y$;

(2) whenever one has commutative diagrams

$$\begin{tikzcd} X_i \arrow[r, "u_i"] \arrow[d, "e_i"'] & N \arrow[dd, "n"] \\ M \arrow[d, "m"'] & \\ Y \arrow[r, "v"] & Z \end{tikzcd} \tag{1.21}$$

in $\mathcal{X}$ with $n \in \mathcal{M}$, then there is a uniquely determined morphism $w : M \to N$ with $n \cdot w = v \cdot m$ and $w \cdot e_i = u_i$ for all $i \in I$.

We refer to (2) as the *simultaneous diagonalization property*. Note that the sink $(f_i : X_i \to Y)_{i \in I}$ may be empty in which case its right $\mathcal{M}$-factorization is given by the least subobject of Y (see 1.11).

THEOREM *The following assertions are equivalent:*

(i) $\mathcal{X}$ *has* $\mathcal{M}$*-pullbacks and* $\mathcal{M}$*-intersections;*

(ii) $\mathcal{X}$ *has* $\mathcal{M}$*-unions, and every morphism in* $\mathcal{X}$ *has a right* $\mathcal{M}$*-factorization;*

(iii) *every sink in* $\mathcal{X}$ *has a right* $\mathcal{M}$*-factorization.*

Proof (i) $\Rightarrow$ (ii) follows from the remarks above and from Corollary 1.9.

(ii) $\Rightarrow$ (iii) Given the sink $(f_i : X_i \to Y)_{i \in I}$, first form the right $\mathcal{M}$-factorization of each $f_i = m_i \cdot d_i$ and then consider the $\mathcal{M}$-union $m : M \to Y$ of $(m_i)_{i \in I}$. Hence one has j_i with $m \cdot j_i = m_i$, so that $e_i := j_i \cdot d_i$ satisfies $m \cdot e_i = f_i$ for all $i \in I$. The simultaneous diagonalization property follows easily from the diagonalization property for each f_i and the characterization of m as a union.

(iii) $\Rightarrow$ (i) For $f : X \to Y$ and $n : N \to Y$ in $\mathcal{M}$ we must show the existence of a pullback diagram

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.22}$$

with $m \in \mathcal{M}$: let $(g_i : Z_i \to X)_{i \in I}$ be the family of all morphisms g_i such that there is a morphism $h_i : Z_i \to N$ with $f \cdot g_i = n \cdot h_i$. Factoring $(g_i)_{i \in I}$ and applying the simultaneous diagonalization property gives morphisms $m \in \mathcal{M}$ and f' with $f \cdot m = n \cdot f'$ as well as morphisms $e_i : Z_i \to M$ with $m \cdot e_i = g_i$ for all $i \in I$. The latter morphisms guarantee that the universal property holds so that (1.22) is a pullback diagram.

The proof that $\mathcal{M}$-intersections exist in $\mathcal{X}$ is very similar to the one just given. We leave it as an exercise to the reader (cf. Börger-Tholen [1990]). □

$\mathcal{X}$ is called *$\mathcal{M}$-complete* if one (and then all) of the properties of the Theorem hold. Trivially, *$\mathcal{M}$-completeness implies finite $\mathcal{M}$-completeness*; the converse implication does not hold in general (see Example (2) below). We also note that $\mathcal{M}$-completeness does not imply closedness under composition for $\mathcal{M}$ (Example 1.8 may be re-employed here).

COROLLARY *Let the complete category $\mathcal{X}$ be $\mathcal{M}$-wellpowered. Then $\mathcal{X}$ is $\mathcal{M}$-complete if and only if $\mathcal{M}$ is stable under pullback and multiple pullback.*

Proof One has to show that any class-indexed family $(m_i)_{i \in I}$ in $\mathcal{M}/X$ has an intersection if $\mathcal{X}$ is complete and $\mathcal{M}$-wellpowered. But since $\mathcal{X}$ is $\mathcal{M}$-wellpowered there is a small subfamily $(m_j)_{j \in J}$ with $J \subseteq I$ such that every m_i is isomorphic to some m_j . Therefore, the multiple pullback of $(m_j)_{j \in J}$ (which exists since $\mathcal{X}$ is assumed to have all small limits) serves also as a multiple pullback of $(m_i)_{i \in I}$. The rest of the proof is trivial. □

EXAMPLES

(1) The right $\mathcal{M}$-factorization of a sink $(f_i : X_i \to Y)_{i \in I}$ in **Set** with $\mathcal{M}$ the class of injective maps is given by

$$X_i \to M = \bigcup_{i \in I} f_i(X_i) \hookrightarrow Y \ .$$

When considering M as a subspace of Y , one obtains the right $\mathcal{M}$-factorization in **Top** with $\mathcal{M}$ the class of embeddings. In **Grp** with $\mathcal{M}$ the class of injective homomorphisms one must exchange M for the subgroup generated by M . In each case, one has an $\mathcal{M}$-complete category.

(2) A poset $(X, \leq)$ can be considered a (small) category $\mathcal{X}$: the set of $\mathcal{X}$-objects is X , and there is a morphism $x \to y$ iff $x \leq y$, and then there is only one such morphism. Hence every morphism in $\mathcal{X}$ is monic (and epic), and we can let $\mathcal{M}$ be the set of all $\mathcal{X}$-morphism. $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$ -factorizations, with $\mathcal{E}$ the set of isomorphisms (= identity maps) in $\mathcal{X}$, and $\mathcal{X}$ has $(\mathcal{M}-)$pullbacks iff $(X, \leq)$ has binary meets. However, $(\mathcal{M}-)$completeness of $\mathcal{X}$ means completeness of $(X, \leq)$ as a lattice. Hence every non-complete semilattice gives an example of a finitely $\mathcal{M}$-complete but not $\mathcal{M}$-complete category.

1.11 The last word on least and last subobjects

$\mathcal{M}/X$ always has a largest element, 1_X . It is the intersection of the empty family of $\mathcal{M}$-subobjects of X . Obviously, $f^{-1}(1_Y) \cong 1_X$ holds for every $f : X \to Y$. The union of the empty family in $\mathcal{M}/X$ (if it exists) is called the *trivial* $\mathcal{M}$-*subobject* of X ; it is the least element of $\mathcal{M}/X$ and therefore denoted by

$$o_X : O_X \to X .$$

Its characteristic categorical property (cf. Diagram (1.20)) reads as follows: for every diagram

$$\begin{tikzcd} O_X \arrow[d, "o_X"'] & N \arrow[d, "n"] \\ X \arrow[r, "v"] & Z \end{tikzcd} \tag{1.23}$$

with $n \in \mathcal{M}$ there is a uniquely determined morphism $w : O_X \to N$ with $n \cdot w = v \cdot o_X$. If $f : X \to Y$ has a right $\mathcal{M}$-factorization, its diagonalization property easily gives $f(o_X) \cong o_Y$. For $f \in \mathcal{M}$ and $\mathcal{M}$ closed under composition, this means $f \cdot o_X \cong o_Y$, hence $O_X \cong O_Y$.

PROPOSITION *Under each of the following hypotheses, an object* X *of* $\mathcal{X}$ *has a trivial* $\mathcal{M}$*-subobject:*

(a) $\mathcal{X}$ *has finite* $\mathcal{M}$*-unions ;*

(b) $\mathcal{X}$ *has* $\mathcal{M}$*-pullbacks, and* $\mathcal{M}/X$ *has a least element;*

(c) $\mathcal{X}$ *has an initial object, and every morphism has a right* $\mathcal{M}$*-factorization.*

Proof (a) By definition. (b) In order to show that the least element o_X in $\mathcal{M}/X$ satisfies the categorical property, obtain the desired arrow w in (1.23) as the composite $O_X \to v^{-1}(N) \to N$. (c) For an initial object I of $\mathcal{X}$, obtain o_X as the $\mathcal{M}$-part of the right $\mathcal{M}$-factorization of the only morphism $I \to X$. □

An object X in $\mathcal{X}$ is called *trivial* if $\mathcal{M}/X$ has only one element, up to isomorphisms; if $o_X : O_X \to X$ exists, this means $o_X \cong 1_X$, or o_X is an isomorphism. In this terminology, O_X is a trivial object of $\mathcal{X}$, provided $\mathcal{M}$ is closed under composition.

Note that if there is a morphism $f : X \to Y$, one has a morphism $O_f : O_X \to O_Y$ (with $o_Y \cdot O_f = f \cdot o_X$).

In general, if there is no morphism $X \to Y$, there may be no morphism $O_X \to O_Y$ either:

EXAMPLE The category **CRng** of commutative unital rings and its homomorphisms is $\mathcal{M}$-complete for $\mathcal{M}=$ { injective homomorphisms }. Every ring R has a least (unital!) subring which can be obtained as the image of $\mathbb{Z} \to R$. For integral domains of characteristic p , one has $O_R \cong \mathbb{Z}_p$. In case $p \neq q$, there are no morphisms $\mathbb{Z}_p \to \mathbb{Z}_q$.

Exercises

1.A (*Images and inverse images of composites*) Prove the formulas

$$(g \cdot f)(m) \cong g(f(m)) \quad \text{and} \quad (g \cdot f)^{-1}(n) \cong f^{-1}\left(g^{-1}(n)\right) \ .$$

Furthermore, for an isomorphism f , verify that $f^{-1}(n)$ can be interpreted as both, the inverse image of n under f or the image of n under f^{-1} .

1.B (*Independence of existence of $\mathcal{M}$-pullbacks and of right $\mathcal{M}$-factorizations,* cf. 1.6)

(a) Prove that maps in **Top** may fail to have right $\mathcal{M}$-factorizations for $\mathcal{M}$ the class of open embeddings.

(b) Prove that the category **CTop** does not have $\mathcal{M}$-pullbacks, for $\mathcal{M}$ the class of embeddings. (Note: it is not enough just to state that a pullback of connected spaces formed in **Top** no longer lives in **CTop**.)

1.C (*Stability properties of monomorphisms and retractions*)

(a) Prove that, in any category, the class of monomorphisms is closed under composition and under $\mathcal{D}$-limits for every $\mathcal{D}$.

(b) Prove that, in any category, the class of *retractions* (morphisms p such that there is j with $p \cdot j = 1$) is closed under composition and stable under pullback and multiple pullback, but fails to be closed under $\mathcal{D}$-limits in general.

1.D (*Extremal monomorphisms*) A monomorphism $m : M \to X$ in a category is called *extremal* if $m = f \cdot e$ with an epimorphism e holds only if e is an isomorphism.

(a) Show that in **Top** the extremal monomorphisms are exactly the embeddings.

(b) Prove that a morphism in a category is an isomorphism if and only if it is both epic and extremally monic. Dualize the statement.

(c) Prove that every monomorphism is extremal if and only if every morphism which is both epic and monic is actually an isomorphism. Dualize the statement.

(d) Prove that extremal monomorphisms are *left cancellable*, that is: if a composite $n \cdot m$ is extremally monic, then also m is extremally monic.

(e) Construct categories in which extremal monomorphisms are not closed under composition or stable under pullback or multiple pullback. (It may be hard to find concrete categories here; see also 1.E below.)

1.E (*Strong monomorphisms*) A morphism $m : M \to X$ in a category is called a *strong* monomorphism if $m \cdot u = v \cdot e$ with e epic implies that there is exactly one morphism w with $m \cdot w = v$.

(a) Prove that every strong monomorphism is in fact monic, even extremally monic (cf. 1.D), and that every section is strongly monic.

(b) Prove that the classes of strong and of extremal monomorphism coincide in categories with pushouts.

(c) Show that a morphism m is strongly monic if and only if m is a monomorphism belonging to $\mathcal{E}_\perp$, with $\mathcal{E}$ the class of all epimorphisms (see 1.8).

(d) For any class $\mathcal{E}$ of morphisms, $\mathcal{E}_\perp$ is closed under composition and under $\mathcal{D}$-limits for every $\mathcal{D}$. Conclude that the class of strong monomorphisms has the same properties.

(e) For arbitrary classes $\mathcal{M}$ and $\mathcal{E}$, show that $\mathcal{M} \cap \mathcal{M}^\perp$ and $\mathcal{E} \cap \mathcal{E}_\perp$ contain only isomorphisms.

1.F (*Discussing the blanket assumptions on* $\mathcal{M}$ *again*)

(a) Show that *any* class $\mathcal{M}$ of morphisms which is stable under pullback and multiple pullback must contain all isomorphisms and be closed under composition with isomorphisms. (Don't forget multiple pullbacks of empty families!)

(b) Let n be a monomorphism in a category. Show that the family $(m_i)_{i \in I}$ with $m_i = n$ for all i and any non-empty class I has a multiple pullback.

(c) Suppose that, for a morphism n in $\mathcal{X}$, the family $(m_i)_{i \in I}$ with $m_i = n$ for all $i \in I$ and $I = \mathrm{Mor}\mathcal{X}$ has a multiple pullback in $\mathcal{X}$. Prove that m is a monomorphism. (*Hint:* Assume $n \cdot x = n \cdot y$ for $x \neq y$ and consider the multiple pullback diagram (1.19). The class $K = \{h : (\forall i \in I)\ j_i \cdot h \in \{x, y\}\}$ is not empty, and one can find a surjective map $\sigma : I \to K$. Now apply the universal property to $(z_i)_{i \in I}$ with

$$z_i = \begin{cases} x & \text{if } j_i \cdot \sigma(i) = y \\ y & \text{if } j_i \cdot \sigma(i) = x \end{cases} \quad .)$$

1.G (*For the category-minded*) Let $\mathcal{M}$ be any class of morphisms, not necessarily a class of monomorphisms. For an object X , let $\mathcal{M}/X$ be the category whose objects are the $\mathcal{M}$-morphisms with codomain X ; a morphism $j : m \to n$

in $\mathcal{M}/X$ is an $\mathcal{X}$-morphism which makes diagram (1.1) commute, and composition is as in $\mathcal{X}$. Reading "adjoint map" as "adjoint functor", show that all results proved in 1.5-1.8 remain valid in this more general setting. Prove that the category **Cat** of small categories and functors is finitely $\mathcal{M}$-complete but not $\mathcal{M}$-complete for $\mathcal{M}=$ { full and faithful functors }. Describe the class $\mathcal{E}$ for which **Cat** has $(\mathcal{E}, \mathcal{M})$ -factorizations.

1.H (*Factorization of small sinks*) Let $\mathcal{X}$ have right $\mathcal{M}$-factorizations of morphisms and (small) coproducts. Show that every sink $(f_i : X_i \to Y)_{i \in I}$ with I small (a set, not a proper class) has a right $\mathcal{M}$-factorization. (Don't forget the case $I = \emptyset$.)

1.I (*Factorization of sinks when $\mathcal{M}$ is closed under composition*) Prove a sink-version of Theorem 1.8, that is: replace in Theorem 1.8 the morphisms $f, e,$ and u by sinks with the same indexing set, and the class $\mathcal{E}$ by a conglomerate $\overline{\mathcal{E}}$ of sinks.

1.J (*Change of universe*) Let $k : X \to Y$ and $m_i : M_i \to X$ $(i \in I)$ be $\mathcal{M}$-morphisms. Prove for $\mathcal{M}$ closed under composition:

(a) For $I \neq \emptyset$, the $\mathcal{M}$-intersection $\bigwedge_{i \in I} k \cdot m_i$ exists if and only if $m \cong \bigwedge_{i \in I} m_i$ exists, and then one has $k \cdot m \cong \bigwedge_{i \in I} k \cdot m_i$. (What happens for $I = \emptyset$?)

(b) If the $\mathcal{M}$-union $m \cong \bigvee_{i \in I} m_i$ exists, then $\bigvee_{i \in I} k \cdot m_i$ exists, and $k \cdot m \cong \bigvee_{i \in I} k \cdot m_i$ holds. The converse proposition holds for $\mathcal{M}$ *stable under pushout* (= dual to stable under pullback).

1.K (*More formulas on image and preimage*) Let $\mathcal{X}$ have $(\mathcal{E}, \mathcal{M})$ -factorizations and $\mathcal{M}$-pullbacks. Prove for $f : X \to Y$:

(a) If $f \in \mathcal{M}$, then $f^{-1}(o_Y) \cong o_X$ (provided the trivial subobjects exist).

(b) $f \in \mathcal{E}$ if and only if $f(1_X) \cong 1_Y$.

(c) If $f \in \mathcal{M}$, then $f^{-1}(f(m)) \cong m$ for all $m \in \mathcal{M}/X$.

(d) If $f \in \mathcal{E}$ and if $\mathcal{E}$ is stable under pullback, then $f(f^{-1}(n)) \cong n$ for all $n \in \mathcal{M}/Y$.

1.L (*More on preservation of unions and intersections*) An object P in $\mathcal{X}$ is said to be *projective* w.r.t. a morphism $f : X \to Y$ if for every $y : P \to Y$ there is an $x : P \to X$ with $f \cdot x = y$. Let $\mathcal{X}$ be $\mathcal{M}$-complete so that $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$ -factorizations, and assume the existence of an object P such that

$$e \in \mathcal{E} \Leftrightarrow P \text{ is projective w.r.t. } e$$

holds for every morphism e in $\mathcal{X}$. Prove for a morphism $f : X \to Y$ and non-empty families $(m_i)_{i\in I}$ in $\mathcal{M}/X$ and $(n_i)_{i\in I}$ in $\mathcal{M}/Y$:

(a) If f is a monomorphism, then $f(\bigwedge_{i\in I} m_i) \cong \bigwedge_{i\in I} f(m_i)$.

(b) If the sink $(j_i : N_i \to N)_{i\in I}$ belonging to a union $n = \bigvee_{i\in I} n_i$ as in (1.20) has the property that, for every $y : P \to N$, there is an $i \in I$ and a morphism $x : P \to N_i$ with $j_i \cdot x = y$, then $f^{-1}(\bigvee_{i\in I} n_i) \cong \bigvee_{i\in I} f^{-1}(n_i)$.

1.M (*Subobjects need not be closed under colimits*)

(a) Show that monomorphisms in **Set** are not closed under the formation of coequalizers.

(b) Show that monomorphisms in **Grp** are not stable under pushout.

1.N (*$\mathcal{E}$ is closed under composition*) (1) Let $\mathcal{X}$ have right $\mathcal{M}$-factorizations, and let $\mathcal{E}$ be the class of morphisms in $\mathcal{X}$ for which the $\mathcal{M}$-part of their right $\mathcal{M}$-factorization is an isomorphism. Show $\mathcal{E} = \mathcal{M}^{\perp}$ and conclude that $\mathcal{E}$ is closed under composition (cf. Theorem 1.8 and Exercise 1.E).
(2) Show that $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorizations if and only if every morphism has a factorization which is simultaneously a right $\mathcal{M}$-factorization and a left $\mathcal{E}$-factorization.

1.O (*Trivial objects*) Consider the only morphism $t : X \to T$ of an object X in $\mathcal{X}$ into a terminal object T , and assume $t^{-1}(o_T) \cong o_X$. Show that X is trivial if and only if there is a morphism $X \to O_T$ in $\mathcal{X}$.

Notes

Finding an adequate notion of factorization system in a category has been a theme in category theory almost from the very beginnings. Early references include Mac Lane [1948] and Isbell [1957], but it was not before the late sixties to early seventies that a generally accepted definition emerged, most comprehensively presented by Freyd and Kelly [1972], but see also Kennison [1968], Herrlich [1968], Ringel [1970], Pumplün [1972], Dyckhoff [1972] and Bousfield [1977]; it is the self-dual notion of $(\mathcal{E}, \mathcal{M})$-factorization system as presented here in 1.8. We have chosen to take a "one-sided" approach to it via right $\mathcal{M}$- factorizations (going back to Ehrbar and Wyler [1968], [1987], Tholen [1979], [1983] and MacDonald and Tholen [1982]) since idempotent closure operators "are" exactly such factorization systems, as will be made precise in 5.3. The notion of finite $\mathcal{M}$-completeness and its characterization by Theorem 1.6 does not seem to have appeared previously in the literature. Theorem 1.7 goes back to Im and Kelly [1986].

2 Basic Properties of Closure Operators

Categorical closure operators as defined in this chapter for any category with a suitable subobject structure provide simultaneously a coherent closure operation for the subobjects of each object of the category. The notions of closedness and denseness associated with a closure operator are discussed from a factorization point of view. This leads to a symmetric presentation of the fundamental properties of idempotency vis-a-vis weak hereditariness and of hereditariness vis-a-vis minimality. Further important properties are given by additivity and productivity which are briefly discussed at the end of the chapter.

2.1 The categorical setting

Throughout this chapter, we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathcal{X}$ which contains all isomorphisms of $\mathcal{X}$. Furthermore, it is assumed that

- $\mathcal{M}$ is closed under composition, and that
- $\mathcal{X}$ is finitely $\mathcal{M}$-complete (see 1.6).

Consequently,

(1) $\mathcal{X}$ has $\mathcal{M}$-pullbacks (i.e., inverse images of $\mathcal{M}$-subobjects along $\mathcal{X}$-morphisms exist, see 1.2);

(2) every morphism in $\mathcal{X}$ has a right $\mathcal{M}$-factorization (in particular, images of $\mathcal{M}$-subobjects under $\mathcal{X}$-morphisms exist, see 1.4/1.5);

(3) there is a class $\mathcal{E}$ of $\mathcal{X}$-morphisms, such that $\mathcal{X}$ has $(\mathcal{E},\ \mathcal{M})$ -factorizations (see 1.8);

(4) $\mathcal{X}$ has finite $\mathcal{M}$-intersections (see 1.9);

(5) $\mathcal{M}$ is closed under $\mathcal{D}$-limits in $\mathcal{X}$ for every diagram type $\mathcal{D}$ (see 1.6).

Often we shall assume that $\mathcal{X}$ is even $\mathcal{M}$-complete. One then has in addition that

(2′) every sink in $\mathcal{X}$ has a right $\mathcal{M}$-factorization (see 1.10), and that

(4′) $\mathcal{X}$ has $\mathcal{M}$-intersections and $\mathcal{M}$-unions (in particular, class-indexed meets and joins exist in every preordered class $\mathcal{M}/X$; see 1.9).

Whenever $\mathcal{M}$-completeness (rather than finite $\mathcal{M}$-completeness) is needed, we shall say so explicitly. $\mathcal{E}$ will always denote the class determined by $\mathcal{M}$ and property (3).

2.2 The local definition of closure operator

A *closure operator* C of the category $\mathcal{X}$ with respect to the class $\mathcal{M}$ of subobjects is given by a family $C = (c_X)_{X \in \mathcal{X}}$ of maps $c_X : \mathcal{M}/X \to \mathcal{M}/X$ such that for every $X \in \mathcal{X}$:

(1) (*Extension*) $m \leq c_X(m)$ for all $m \in \mathcal{M}/X$;

(2) (*Monotonicity*) if $m \leq m'$ in $\mathcal{M}/X$, then $c_X(m) \leq c_X(m')$;

(3) (*Continuity*) $f(c_X(m)) \leq c_Y(f(m))$ for all $f : X \to Y$ in $\mathcal{X}$ and $m \in \mathcal{M}/X$.

By the monotonicity condition, $m \cong m'$ implies $c_X(m) \cong c_X(m')$. Therefore, it suffices to define the maps c_X on a skeleton of $\mathcal{M}/X$.

In the presence of (2), the continuity condition can equivalently be expressed as

(3′) $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$ for all $f : X \to Y$ and $n \in \mathcal{M}/Y$.

In fact, (3) gives that, for $n \in \mathcal{M}/Y$ and $m = f^{-1}(n)$,

$$f(c_X(f^{-1}(n))) \leq c_Y(f(f^{-1}(n))) \leq c_Y(n)$$

by (2), hence (3′) follows with (∗) of 1.4. The proof that, in the presence of (2), (3′) implies (3) is completely analogous.

The conjunction (2) & (3) is equivalent to:

(4) $f(m) \leq n$ implies $f(c_X(m)) \leq c_Y(n)$ for all $f : X \to Y$, $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$.

In fact, if $f(m) \leq n$, (2) & (3) give $f(c_X(m)) \leq c_Y(f(m)) \leq c_Y(n)$. Vice versa, (4) implies (2) (consider $f = 1_X$) and (3) (consider $n = f(m)$). Similarly, (2) & (3′) ($\Leftrightarrow$ (2)&(3)) is also equivalent to:

(4′) $m \leq f^{-1}(n)$ implies $c_X(m) \leq f^{-1}(c_Y(n))$ for all $f : X \to Y$, $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$.

In the particular case that $f : X \to Y$ belongs to $\mathcal{M}$, note that one has $f(m) \cong f \cdot m$ since $\mathcal{M}$ is assumed to be closed under composition. Hence in this case, implication (4) simply reads as:

(4″) $f \cdot m \leq n$ implies $f \cdot c_X(m) \leq c_Y(n)$.

We fix some standard notations. For an $\mathcal{M}$-subobject $m : M \to X$, the domain of its *C-closure* $c_X(m)$ is denoted by $c_X(M)$.

If there is no danger of confusion, we may write c instead of c_X . Because of (1), one has a commutative diagram

$$\begin{tikzcd} M \arrow[rr, "j_m"] \arrow[dr, "m"'] & & c_X(M) \arrow[dl, "c_X(m)"] \\ & X & \end{tikzcd} \tag{2.1}$$

with a uniquely determined morphism j_m which, by Theorem 1.7(3), belongs again to $\mathcal{M}$.

We shall discuss examples of closure operators more systematically in Chapter 3. Here we just mention the most fundamental example which gives guidance for the general terminology:

EXAMPLE (*Kuratowski closure operator*) For a subset M of a topological space X, the (Kuratowski) closure of M in X is defined as usual by

$$k_X(M) = \{x \in X : U \cap M \neq \emptyset \text{ for every open set } U \ni x\} = \overline{M} .$$

This way one obtains a closure operator $K = (k_X)_{X \in \mathbf{Top}}$ of **Top** with respect to the class $\mathcal{M}$ of embeddings. (It suffices to define K on the skeleton $\mathcal{M}_0$ of $\mathcal{M}$ consisting of the inclusion maps of all subspaces; for an arbitrary embedding $m : M \to X$ one may then put $k_X(m) := k_X(m(M))$.)

2.3 Closed and dense subobjects

An $\mathcal{M}$-subobject $m : M \to X$ is called *C-closed* (in X) if it is isomorphic to its C-closure, that is: if $j_m : M \to c_X(M)$ is an isomorphism. m is called *C-dense* in X if its C-closure is isomorphic to 1_X, that is: if $c_X(m) : c_X(M) \to X$ is an isomorphism. The prefix C may be omitted.

It is easy to verify that for the Kuratowski closure operator K of **Top**, K-closed and K-dense for a subspace inclusion $M \hookrightarrow X$ means closed and dense in the usual topological sense, respectively.

We are interested in stability properties of C-closed and of C-dense morphisms. The continuity condition (3) ($\Leftrightarrow$ (3')) implies that C-closedness is preserved by inverse images, and that C-denseness is preserved by images:

PROPOSITION *Let $f : X \to Y$ be a morphism.*

(1) *If n is C-closed in Y, then $f^{-1}(n)$ is C-closed in X.*

(2) *If m is C-dense in X and $f \in \mathcal{E}$, then $f(m)$ is C-dense in Y.*

Proof (1) If $n \cong c_Y(n)$, then $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n)) \cong f^{-1}(n)$.
(2) If $1_X \cong c_X(m)$ and $f \in \mathcal{E}$, then $1_Y \cong f(1_X) \cong f(c_X(m)) \leq c_Y(f(m))$.
□

Let $\mathcal{M}^C$ be the class of C-closed $\mathcal{M}$-subobjects. The Proposition asserts that $\mathcal{M}^C$ is stable under pullback. In fact, a more general property holds:

THEOREM *$\mathcal{M}^C$ is closed under $\mathcal{D}$-limits, for any diagram type $\mathcal{D}$. In particular, direct products and intersections of C-closed subobjects are C-closed* (cf. 1.7).

We leave the proof as an exercise to the reader, since a more general result will be proved in 5.2. Stability under pullback gives in particular:

COROLLARY *If, for monomorphisms* m *and* n *,* $n \cdot m$ *is a* C*-closed* $\mathcal{M}$*-subobject, then* m *is a* C*-closed* $\mathcal{M}$*-subobject.* □

One is tempted to assume that the class of all C-dense subobjects enjoys the dual properties. But this is not true in general: for $C = T$ the trivial operator (see Exercise 2.A), every $\mathcal{M}$-subobject is C-dense; however, the class $\mathcal{M}$ need not be closed under colimits, not even be stable under pushout (see Exercise 1.M). Despite its triviality, this example shows everything that may go wrong: the obstacle is the subobject property, not C-denseness - if defined for arbitrary morphisms, and not just for subobjects.

One calls a morphism $f : X \to Y$ *C-dense* if $f(1_X)$ is C-dense in Y , that is: if $c_Y(f(1_X)) \cong 1_Y$. The class of C-dense morphisms in $\mathcal{X}$ is denoted by $\mathcal{E}^C$. Note that $\mathcal{E}$ is a subclass of $\mathcal{E}^C$. Now one can prove:

THEOREM* $\mathcal{E}^C$ *is closed under* $\mathcal{D}$*-colimits, for any diagram type* $\mathcal{D}$*.*

Again, we postpone the proof of the Theorem* and of the following Corollary (see 5.2 and Exercise 2.F).

COROLLARY* *If, for arbitrary morphisms* e *and* d *,* $d \cdot e$ *is* C*-dense, then* d *is* C*-dense.* □

2.4 Idempotent and weakly hereditary closure operators

A closure operator C may or may not have further important stability properties which were not discussed in 2.3, for instance:

(ID) The C-closure of an $\mathcal{M}$-subobject of X is C-closed, i.e. $c_X(c_X(m)) \cong c_X(m)$ for all $m : M \to X$ in $\mathcal{M}$.

(WH) An $\mathcal{M}$-subobject of X is C-dense in its C-closure, i.e. $c_Y(j_m) \cong 1_Y$ for all $m : M \to X$ in $\mathcal{M}$, with $Y = c_X(M)$.

(CC) Composites of C-closed $\mathcal{M}$-subobjects are C-closed, i.e. if $m : M \to N$ and $n : N \to X$ in $\mathcal{M}$ are C-closed, then $n \cdot m$ is C-closed.

(CD) Composites of C-dense $\mathcal{M}$-subobjects are C-dense, i.e. if $m : M \to N$ and $n : N \to X$ in $\mathcal{M}$ are C-dense, then $n \cdot m$ is C-dense.

One easily shows that the Kuratowski closure operator of **Top** enjoys each of these properties. However, none of these properties holds in general (see, for instance, the θ-closure operator of **Top**, described in 3.3 below). If properties (ID) or

(WH) hold for every $X \in \mathcal{X}$, then C is called *idempotent* or *weakly hereditary*, respectively; in case the conditions are restricted to a specific X , we shall add *for* X . (The reason for choosing the name "weakly hereditary" will become clear in 2.5.) The logical connections between the two properties will become clear once we have proved:

LEMMA *(Diagonalization Lemma) For every commutative diagram*

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "v"] & Y \end{tikzcd} \tag{2.2}$$

with m , $n \in \mathcal{M}$, *there is a uniquely determined morphism* w *rendering the diagram*

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "j_m"'] & N \arrow[d, "j_n"] \\ c_X(M) \arrow[r, "w"] \arrow[d, "c_X(m)"'] & c_Y(N) \arrow[d, "c_Y(n)"] \\ X \arrow[r, "v"] & Y \end{tikzcd} \tag{2.3}$$

commutative.

Proof By the diagonalization property of (right) $\mathcal{M}$-factorizations (cf. 1.5) one has $v(m) \leq n$ in $\mathcal{M}/Y$, hence $v(c_X(m)) \leq c_Y(n)$ by continuity. Hence w is the composite $(c_X(M) \to v(c_X(M)) \to c_Y(N))$. □

COROLLARY

(1) *If* m *in* (2.2) *is* C*-dense, then there is a uniquely determined morphism* $t : X \to c_Y(N)$ *with* $t \cdot m = j_n \cdot u$ *and* $c_Y(n) \cdot t = v$.

(2) *If* n *in* (2.2) *is* C*-closed, then there is a uniquely determined morphism* $s : c_X(M) \to N$ *with* $s \cdot j_m = u$ *and* $n \cdot s = v \cdot c_X(m)$.

(3) *In* (2.2)*, if* m *is* C*-dense and* n *is* C*-closed, then there is a uniquely determined morphism* $d : X \to N$ *with* $d \cdot m = u$ *and* $n \cdot d = v$.

(4) $\mathcal{E}^C \cap \mathcal{M}^C$ *is the class of isomorphisms in* $\mathcal{X}$. $\square$

PROPOSITION *For an idempotent closure operator* C *, every morphism has a right* $\mathcal{M}^C$*-factorization, and* $\mathcal{E}^C$ *is closed under composition.*

Proof Every morphism $f : X \to Y$ has an $(\mathcal{E}, \mathcal{M})$-factorization $f = m \cdot e$. If C is idempotent, then $c_Y(m) \in \mathcal{M}^C$. With the Corollary it is easy to show that $f = c_Y(m) \cdot (j_m \cdot e)$ is a right $\mathcal{M}^C$ -factorization of f .

$$\begin{tikzcd}
 & & c_Y(M) \arrow[dd, "c_Y(m)"] \\
 & M \arrow[ur, "j_m"] \arrow[dr, "m"] & \\
X \arrow[ur, "e"] \arrow[rr, "f"] & & Y
\end{tikzcd} \tag{2.4}$$

From Exercise 1.N we obtain that $\mathcal{E}^C = (\mathcal{M}^C)^{\perp}$ is closed under composition. $\square$

It is easy to show that the factorization (2.4) is a left $\mathcal{E}^C$-factorization of f if C is weakly hereditary; in fact, $d := j_m \cdot e$ is C-dense since $d(1_X) \cong j_m$. Now the dual of Exercise 1.N gives:

PROPOSITION* *For a weakly hereditary closure operator* C *, every morphism has a left* $\mathcal{E}^C$*-factorization, and* $\mathcal{M}^C$ *is closed under composition.* $\square$

THEOREM *The following statements are equivalent for a closure operator* C *:*

(i) C *is idempotent and weakly hereditary;*

(ii) C *is idempotent and* (CC) *holds for every* $X \in \mathcal{X}$ *;*

(iii) C *is weakly hereditary and* (CD) *holds for every* $X \in \mathcal{X}$ *;*

(iv) $\mathcal{X}$ *has* $(\mathcal{E}^C , \mathcal{M}^C)$ *-factorizations.*

Proof (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from the Proposition, Proposition* and Theorem 1.8.

(iv) $\Rightarrow$ (i) One considers an $(\mathcal{E}^C , \mathcal{M}^C)$ - factorization

$$M \xrightarrow{d} N \xrightarrow{n} X$$

of $m = n \cdot d \in \mathcal{M}$. Then $m \leq n$ implies $c_X(m) \leq n$ since $n \in \mathcal{M}^C$. From the Diagonalization Lemma, one obtains a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{1_M} & M \\ {\scriptstyle j_d}\downarrow & & \downarrow{\scriptstyle j_m} \\ c_N(M) & \xrightarrow{w} & c_X(M) \\ {\scriptstyle c_N(d)}\downarrow & & \downarrow{\scriptstyle c_X(m)} \\ N & \xrightarrow{n} & X \end{array} \tag{2.5}$$

Since $c_N(d)$ is an isomorphism, one has $n \leq c_X(m)$, hence $n \cong c_X(m)$. Therefore w is an isomorphism, and $c_X(m) \in \mathcal{M}^C$, $j_m \in \mathcal{E}^C$ follows from $n \in \mathcal{M}^C$, $d \in \mathcal{E}^C$. □

Remarks

(1) The Proposition (and the Theorem), together with 1.7 (and 1.8), yield the stability properties of $\mathcal{M}^C$ (and $\mathcal{E}^C$) as described in 2.3 in case C is idempotent (and weakly hereditary).

(2) Although, in general, $\mathcal{E}^C$ need not be closed under composition, it is always closed under composition with $\mathcal{E}$-morphisms: see Exercise 2.F.

(3) From the Proposition and Proposition* we obtain the implications

$$(WH) \Longrightarrow (CC) \quad \text{and} \quad (ID) \Longrightarrow (CD) .$$

In 4.6 below we shall exhibit a non-trivial example which simultaneously shows that, in general, the converse implications are not true (see Exercise 4.H). A trivial example of this type is provided in Exercise 2.E(c)

2.5 Minimal and hereditary closure operators

Suppose that the $\mathcal{M}$-subobject $m : M \to X$ factors through a larger $\mathcal{M}$-subobject $y : Y \to X$ as

$$\begin{tikzcd} M \arrow[rr, "m_Y"] \arrow[dr, "m"'] & & Y \arrow[dl, "y"] \\ & X & \end{tikzcd} \tag{2.6}$$

Is it then possible to compute $c_Y(m_Y)$ with the help of $c_X(m)$? For the Kuratowski closure operator K of **Top** this is easily done: for subsets $M \subseteq Y \subseteq X$ one has $k_Y(M) = Y \cap k_X(M)$. In the general context, since $m_Y \cong y^{-1}(m)$ in $\mathcal{M}/Y$, one has

$$c_Y(m_Y) \leq y^{-1}(c_X(m))$$

by continuity. But in general, these two subobjects of Y are not isomorphic, even when C is idempotent and weakly hereditary (see Example (2) in 3.4 below). C is called *hereditary for* X if

(HE) $\quad c_Y(m_Y) \cong y^{-1}(c_X(m))$ for all $m \leq y$ in $\mathcal{M}/X$,

and *hereditary* if it is hereditary for all $X \in \mathcal{X}$.

First we want to show that weak hereditariness can be described by a weakened version of (HE):

LEMMA *A closure operator* C *is weakly hereditary if and only if* (HE) *holds under the restriction that* $y = c_X(z)$ *for some* $z \geq m$ *in* $\mathcal{M}/X$.

Proof When considering $z = m$, hence $m_Y = j_m$, the stated condition gives $c_Y(j_m) = c_Y(m_Y) \cong y^{-1}(y)$; but $y^{-1}(y)$ is an isomorphism since y is monic. Thus (WH) follows.

Vice versa, let C be weakly hereditary and consider m , z , y as in the Proposition. $m \leq z$ implies $c_X(m) \leq y$. It follows immediately from the pullback property that the morphism $k : c_X(M) \to Y = c_X(Z)$ with $y \cdot k = c_X(m)$ satisfies $k \cong y^{-1}(c_X(m))$. The Diagonalization Lemma gives a commutative diagram

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d] & M \arrow[d, "j_m"] \\ c_Y(M) \arrow[r, "\ell"] \arrow[d, "c_Y(m_Y)"'] & c_X(M) \arrow[d, "c_X(m)"] \arrow[dl, "k"'] \\ Y \arrow[r, "y"] & X \end{tikzcd} \tag{2.7}$$

Since $k \cdot j_m = m_Y$, we have (with $N = c_X(M)$)

$$k \cdot c_N(j_m) \leq c_Y(m_Y) = k \cdot \ell$$

by continuity, hence $c_N(j_m) \leq \ell$. Since by hypothesis $c_N(j_m)$ is an isomorphism, also ℓ is one. Therefore, $c_Y(m_Y) \cong k \cong y^{-1}(c_X(m))$. □

PROPOSITION

(1) *An idempotent closure operator* C *is weakly hereditary if and only if* (HE) *holds under the restriction that* y *is* C*-closed.*

(2) *A closure operator* C *is hereditary if and only if* C *is weakly hereditary and* (HE) *holds under the restriction that* y *is* C*-dense.*

Proof (1) follows immediately from the Lemma. The Lemma also gives that the two stated conditions of (2) are necessary for hereditariness. We must show that they are also sufficient and consider $m \leq y$ in $\mathcal{M}/X$ as in (HE). Consider the following pullback diagram:

$$\begin{tikzcd}
y^{-1}(c_X(M)) \arrow[r] \arrow[d, "y^{-1}(c_X(m))"'] & w^{-1}(c_X(M)) \arrow[r] \arrow[d, "w^{-1}(c_X(m))"] & c_X(M) \arrow[d, "c_X(m)"] \\
Y \arrow[r, "j_y"'] & W := c_X(Y) \arrow[r, "w := c_X(y)"'] & X
\end{tikzcd} \tag{2.8}$$

Let $m_W : M \to W$ be defined by $w \cdot m_W = m$. By the Proposition, one has

$$c_W(m_W) \cong w^{-1}(c_X(m)) .$$

Since $j_y \cdot m_Y = m_W$, and j_y is C-dense, one also has

$$c_Y(m_Y) \cong j_y^{-1}(c_W(m_W)) .$$

Consequently,

$$c_Y(m_Y) \cong j_y^{-1}(w^{-1}(c_X(m))) \cong y^{-1}(c_X(m)) .$$ □

Next we wish to characterize hereditary closure operators in terms of the following *left-cancellation property of* C*-dense subobjects w.r.t.* $\mathcal{M}$*:*

(LD) For all $m : M \to N$ and $n : N \to X$ in $\mathcal{M}$, if $n \cdot m$ is C-dense, then m is C-dense.

THEOREM *A closure operator* C *is hereditary if and only if* C *is weakly hereditary and* (LD) *holds for all* $X \in \mathcal{X}$.

Proof (LD) is certainly a necessary condition for hereditariness. To wit, let $k := n \cdot m$ be C-dense, so that $c_X(k)$ is an isomorphism. But then, also $c_N(m) \cong n^{-1}(c_X(k))$ is an isomorphism, hence m is C-dense.

Conversely, let us assume that (LD) holds with C weakly hereditary. With $m \leq y$ as in (2.6), we consider the following commutative diagram (with k such that $y^{-1}(c_X(m)) \cdot k = c_Y(m_Y)$):

$$\begin{tikzcd} c_Y(M) \arrow[r, "k"] & y^{-1}(c_X(M)) \arrow[r, "y^{-1}(c_X(m))"] \arrow[d] & Y \arrow[d, "y"] \\ M \arrow[u, "j_{m_Y}"] \arrow[r, "j_m"'] & c_X(M) \arrow[r, "c_X(m)"'] & X \end{tikzcd} \tag{2.9}$$

Since C is weakly hereditary, j_m is C-dense, hence also $k \cdot j_{m_Y}$ is C-dense, due to (LD). Now the Diagonalization Lemma (see Corollary 2.4 (1)) gives immediately an inverse of k , as desired. □

Together with Theorem 2.4, the Theorem implies:

COROLLARY *A closure operator C is hereditary and idempotent if and only if C is weakly hereditary and* (CD) *and* (LD) *hold for every* $X \in \mathcal{X}$. □

Finally, we briefly examine the "dual" of (LD), that is the *right cancellation property of C-closed subobjects w.r.t. $\mathcal{M}$:*

(RC) For all $m : M \to N$ and $n : N \to X$ in $\mathcal{M}$, if $n \cdot m$ is C-closed, then n is C-closed.

Clearly this property fails already for the Kuratowski closure operator K of **Top** (consider, for example, a point of a non-closed subspace of a T_1-space). While topologically meaningless, (RC) becomes interesting in algebraic examples, as we shall see in 3.4/3.5. The Theorem suggests to consider this property in conjunction with idempotency (the "dual" of weak hereditariness).

For an idempotent closure operator C $c_X(o_X)$ is C-closed. Hence (RC) gives that $y \in \mathcal{M}/X$ is C-closed if (and only if) $c_X(o_X) \leq y$. Hence it is clear that C will be determined by the subobjects $c_X(o_X)$, $X \in \mathcal{X}$; in fact, we obtain immediately that (RC) and (ID) imply

$$c_X(y) \cong y \vee c_X(o_X)$$

for all $y \in \mathcal{M}/X$. We may express this equivalently without referring to the trivial subobject, as follows:

(MI) $c_X(y) \cong y \vee c_X(m)$ for all $m \leq y$ in $\mathcal{M}/X$.

A closure operator C satisfying (MI) is called *minimal for* X ; C is *minimal* if it is minimal for all $X \in \mathcal{X}$.

It is an easy exercise that, in turn, a minimal closure operator C is idempotent and satisfies (RC). Hence we have:

THEOREM* *A closure operator* C *is minimal if and only if* C *is idempotent and* (RC) *holds for every* $X \in \mathcal{X}$. □

COROLLARY* *A closure operator* C *is minimal and weakly hereditary if and only if* C *is idempotent and* (CC) *and* (RC) *hold for every* $X \in \mathcal{X}$. □

In summary we may state: hereditary and idempotent closure operators are the weakly hereditary operators with perfect behaviour of dense subobjects, while minimal and weakly hereditary closure operators are the idempotent operators with perfect behaviour of closed subobjects.

2.6 Grounded and additive closure operators

For a closure operator C , intersections of C-closed subobjects are C-closed (see 2.3). Hence, if $\mathcal{X}$ is $\mathcal{M}$-complete, $\mathcal{M}^C/X$ has, like $\mathcal{M}/X$, all meets and joins. But whereas meets are formed by multiple pullback in both cases, joins in $\mathcal{M}^C/X$ and in $\mathcal{M}/X$ differ in general (consider the Kuratowski closure operator of **Top**), even finite (see Example (2) of Section 4.3 below) or just empty joins (consider the trivial closure operator; see Exercise 2.A).

Let $\mathcal{X}$ have finite $\mathcal{M}$-unions. A closure operator C is called *grounded* (*for* $X \in \mathcal{X}$) if

(GR) $\quad c_X(o_X) \cong o_X$

and *additive* (*for* X) if

(AD) $\quad c_X(m \vee n) \cong c_X(m) \vee c_X(n) \qquad$ for m , $n \in \mathcal{M}/X$.

Note that the Kuratowski closure operator of **Top** is both grounded and additive. In Section 3 we shall give examples showing that both properties are logically independent of the properties discussed previously. Here we just mention a trivial fact:

PROPOSITION *Let* $\mathcal{X}$ *have finite* $\mathcal{M}$*-unions. Then an idempotent closure operator* C *is grounded and additive if and only if finite* $\mathcal{M}$*-unions of* C*-closed subobjects are* C*-closed.* □

How do arbitrary joins in $\mathcal{M}^C/X$ look like? For C idempotent, this is easy, provided joins exist in $\mathcal{M}/X$: the join of $(m_i)_{i\in I}$ in $\mathcal{M}^C/X$ is $c_X(\bigvee_{i\in I} m_i)$, with $\bigvee$ denoting the join in $\mathcal{M}/X$ (see Exercise 2.F). For arbitrary C , one can reduce this problem to the idempotent case, as we shall see in 4.6.

For $\mathcal{X}$ with $\mathcal{M}$-unions, one calls C *fully additive* (*for* X) if

$$\text{(FA)} \qquad c_X\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} c_X(m_i)$$

holds for all $m_i \in \mathcal{M}/X$, $i \in I \neq \emptyset$. (Empty unions are *not* permitted here, hence a fully additive closure operator is not a priori required to be grounded!) C is called *directedly additive* (for X) if (FA) holds for every directed family $(m_i)_{i\in I}$ in $\mathcal{M}/X$ (so that for all i , $j \in I \neq \emptyset$ there is $k \in I$ with $m_i \leq m_k$ and $m_j \leq m_k$).

Clearly, there are obvious non-finite versions of the Proposition above characterizing full and directed additivity in terms of C-closed subobjects, for an idempotent closure operator C . The following easy Theorem clarifies the interrelationships between the notions just introduced.

THEOREM *A closure operator of a category with $\mathcal{M}$-unions is fully additive if and only if it is additive and directedly additive.*

Proof For $m_i \in \mathcal{M}/X$, $i \in I \neq \emptyset$, and for every $F \subseteq I$ finite and non-empty, let m_F be the join of $(m_i)_{i\in F}$. Then

$$m \cong \bigvee_{i\in I} m_i \Leftrightarrow m \cong \bigvee_{F} m_F \ .$$

Since $(m_F : \emptyset \neq F \subseteq I \text{ finite})$ is a directed family in $\mathcal{M}/X$, the "if" part of the Theorem follows readily. The "only if" part is trivial. □

We want to stress that full additivity, like minimality, but unlike the other properties discussed previously, is usually not expected to hold. For instance, the Kuratowski closure operator $K = (k_X)_{X\in\mathbf{Top}}$ is *not* fully additive (see Exercise 2.I): a T_1-space X for which k_X satisfies (FA) must be discrete. More precisely, the following statements are equivalent for a topological space X :

(i) K is fully additive for X ;

(ii) each point of X has a least neighbourhood;

(iii) the intersection of an arbitrary family of open sets in X is open.

Spaces X with these properties are called *Alexandroff*. We denote by **Alex** the corresponding full subcategory of **Top**.

An example of a directedly additive but not additive closure operator will be given in 3.4 below (see Example (1)).

REMARK Trivially, *every minimal closure operator is fully additive*, but it is grounded only if it is (isomorphic to) the discrete operator (see Exercise 2.A). In 3.4 below we will encounter important examples of (non-discrete) minimal closure operators; it is for this reason that we excluded groundedness from the requirements for full additivity. For the same reason, we did *not* define additivity in the sense of "finite additivity", so that it would include groundedness.

Full additivity, however, does not imply minimality, even in the presence of idempotency, hereditariness and groundedness, as is shown by the following easy example.

EXAMPLE Define the *point-closure* P of a subset M of a topological space by

$$p_X(M) = \bigcup_{x \in M} k_X(\{x\})$$

(with K the Kuratowski closure). One easily checks that P is indeed a closure operator of **Top** (w.r.t. embeddings) which is obviously fully additive and grounded and which inherits idempotency and hereditariness from K. Every space X which is not T_1 shows that P is not discrete, hence not minimal.

2.7 Productive closure operators

Hereditary closure operators are well-behaved with respect to $\mathcal{M}$-subobjects, that is: for an $\mathcal{M}$-subobject $y : Y \to X$, c_Y is completely determined by the restriction of c_X to Y. Similarly, we now want to investigate closure operators which are well-behaved with respect to direct products. To make this precise, we let $\mathcal{X}$ have direct products and consider $\mathcal{M}$-subobjects $m_i : M_i \to X_i$, $i \in I$. Their direct product

$$m = \prod_{i \in I} m_i : M = \prod_{i \in I} M_i \to X = \prod_{i \in I} X_i$$

is again an $\mathcal{M}$-subobject (see 1.7). By the Diagonalization Lemma, for every $i \in I$ there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{q_i} & M_i \\ \downarrow{\scriptstyle j_m} & & \downarrow{\scriptstyle j_{m_i}} \\ c_X(M) & \xrightarrow{t_i} & c_{X_i}(M_i) \\ \downarrow{\scriptstyle c_X(m)} & & \downarrow{\scriptstyle c_{X_i}(m_i)} \\ X & \xrightarrow{p_i} & X_i \end{array} \tag{2.10}$$

(with product projections p_i, q_i). There is an induced morphism

$$t : c_X(M) \longrightarrow \prod_{i \in I} c_{X_i}(M_i)$$

with $r_i \cdot t = t_i$ (r_i a projection, $i \in I$). One easily checks that $s \cdot t = c_X(m)$ holds with

$$s = \prod_{i \in I} c_{X_i}(m_i) \ : \ \prod_{i \in I} c_{X_i}(M_i) \to X \ ,$$

hence $c_X(m) \leq s$. C is called *(finitely) productive* (*for* X) if, for every (finite) family $(m_i)_{i \in I}$ of $\mathcal{M}$-subobjects (of X), $s \leq c_X(m)$, that is: t is an isomorphism. Equivalently, $c_X(M)$ is a direct product of $(c_{X_i}(M_i))_{i \in I}$ with product projections t_i (as in (2.10)), i.e.

$$\text{(II)} \quad c_X(M) \cong \prod_{i \in I} c_{X_i}(M_i) \ \text{ and } \ c_X(m) \cong \prod_{i \in I} c_{X_i}(m_i) \ .$$

Thus for a productive closure operator, the closure of a "box-shaped" subobject is given by the product of the closure of each edge. The Kuratowski closure operator K of **Top** has this property (see Exercise 2.B).

PROPOSITION *C is (finitely) productive if for every (finite) family $(m_i)_{i \in I}$ the induced morphism t belongs to $\mathcal{E}$; if C is idempotent, it even suffices that t belongs to $\mathcal{E}^C$.*

Proof Since $s \in \mathcal{M}$ we have $t \in \mathcal{M}$ (see 1.7). Hence t is an isomorphism if t belongs to $\mathcal{E}$ (see Exercise 1.E). For C idempotent one has $s \in \mathcal{M}^C$, hence $t \in \mathcal{M}^C$ (see 2.3). Therefore it suffices to have $t \in \mathcal{E}^C$ in order to conclude that t is an isomorphism (see Corollary 2.4). □

THEOREM *If C is (finitely) productive, then the (finite) direct product of C-dense $\mathcal{M}$-subobjects is C-dense, i.e., $\mathcal{E}^C \cap \mathcal{M}$ is closed under (finite) direct products. Conversely, if $\mathcal{E}^C \cap \mathcal{M}$ is closed under (finite) direct products for a weakly hereditary closure operator C , then C is (finitely) productive.*

Proof If each m_i is C-dense, one has $c_X(m_i)$ iso and therefore $c_X(m) \cong \prod_{i \in I} c_{X_i}(m_i)$ iso under condition (II), so that m is C-dense. Conversely, for C weakly hereditary, each j_{m_i} is C-dense, hence

$$k := \prod_{i \in I} j_{m_i} : M \to Z := \prod_{i \in I} c_{X_i}(M_i)$$

is C-dense by hypothesis. Since, in the notation used previously, one has $s \cdot k = m$, the Diagonalization Lemma 2.4 yields a morphism $v : c_Z(M) \to c_X(M)$ with $c_X(m) \cdot v = s$, hence $s \leq c_X(m)$. □

COROLLARY *For C (finitely) productive and $\mathcal{E}$ closed under (finite) direct products, also $\mathcal{E}^C$ is closed under (finite) direct products.*

Proof Every C-dense morphism is the composite of an $\mathcal{E}$-morphism followed by a morphism in $\mathcal{E}^C \cap \mathcal{M}$. □

2.8 Restriction of closure operators to full subcategories

For a full subcategory $\mathcal{Y}$ of a category $\mathcal{X}$, it is natural to ask whether it is possible to restrict a closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ to the category $\mathcal{Y}$, by considering $c_X(m)$ as the closure of the subobject $m : M \to X$ belonging to $\mathcal{Y}$. Obviously, we are implying here that our notion of subobject in the category $\mathcal{Y}$ is given by the class

$$\mathcal{M}_{\mathcal{Y}} := \mathcal{M} \cap \text{ Mor } \mathcal{Y} .$$

But we must make sure that this gives indeed a satisfactory notion of subobject in $\mathcal{Y}$, for instance: does each morphism in $\mathcal{Y}$ have a right $\mathcal{M}_{\mathcal{Y}}$-factorization (so that the continuity condition in $\mathcal{Y}$ makes sense)? Furthermore: does $c_X(m)$ belong to $\mathcal{M}_{\mathcal{Y}}$ for m in $\mathcal{M}_{\mathcal{Y}}$? Obviously, the latter question has a positive answer if the full subcategory $\mathcal{Y}$ *is closed under* $\mathcal{M}$*-subobjects in* $\mathcal{X}$, that is: if for every $m : M \to X$ in $\mathcal{M}$ with $X \in \mathcal{Y}$ one has also $M \in \mathcal{Y}$. Since $\mathcal{M}$ contains all isomorphisms of $\mathcal{X}$, this property implies that $\mathcal{Y}$ is *replete* in $\mathcal{X}$, i.e., closed under isomorphisms. Obviously, the class $\mathcal{M}_{\mathcal{Y}}$ is a class of monomorphisms in $\mathcal{Y}$, contains all isomorphisms of $\mathcal{Y}$, and is closed under composition when $\mathcal{M}$ has the respective properties. Furthermore:

LEMMA *If* $\mathcal{Y}$ *is closed under* $\mathcal{M}$*-subobjects in* $\mathcal{X}$*, then it is closed under the formation of images, inverse images, intersections and unions, that is: when applying these constructions in* $\mathcal{X}$ *to data in* $\mathcal{M}_{\mathcal{Y}}$ *, one obtains data in* $\mathcal{M}_{\mathcal{Y}}$ *; in particular, (finite)* $\mathcal{M}$*-completeness of* $\mathcal{X}$ *implies (finite)* $\mathcal{M}_{\mathcal{Y}}$ *-completeness of* $\mathcal{Y}$ *.*

The easy proof can be left as an exercise. (A more general result will be proved in 5.8.) The following Proposition is now evident:

PROPOSITION *Let* C *be a closure operator of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$*. For a full subcategory* $\mathcal{Y}$ *of* $\mathcal{X}$ *closed under* $\mathcal{M}$*-subobjects and for every* $Y \in \mathcal{Y}$ *, the function* $c_Y : \mathcal{M}/Y \to \mathcal{M}/Y$ *can be restricted to*

$$c_Y^{\mathcal{Y}} : \mathcal{M}_{\mathcal{Y}}/Y \to \mathcal{M}_{\mathcal{Y}}/Y ,$$

and $C|_{\mathcal{Y}} = (c_Y^{\mathcal{Y}})_{Y \in \mathcal{Y}}$ *is a closure operator of* $\mathcal{Y}$ *w.r.t.* $\mathcal{M}_{\mathcal{Y}}$*. Each of the properties discussed in* 2.4 - 2.6 *is inherited by* $C|_{\mathcal{Y}}$ *from* C *; (finite) productivity is inherited if* $\mathcal{Y}$ *is closed under (finite) products in* $\mathcal{X}$*. Furthermore,* $m \in \mathcal{M}_{\mathcal{Y}}$ *is* $C|_{\mathcal{Y}}$*-closed (*$C|_{\mathcal{Y}}$*-dense) if and only if* m *is* C*-closed (*C*-dense).* □

We call $C|_{\mathcal{Y}}$ the *restriction* of C to $\mathcal{Y}$, and C is an *extension* of $C|_{\mathcal{Y}}$ to $\mathcal{X}$.

Recall that a full subcategory $\mathcal{Y}$ of $\mathcal{X}$ is *reflective in* $\mathcal{X}$ if for every X there is an object $RX \in \mathcal{Y}$ and a morphism $\rho_X : X \to RX$ (the *reflexion of* X *into* $\mathcal{Y}$) such that every morphism $f : X \to Y$ with $Y \in \mathcal{Y}$ factors as $f = g \cdot \rho_X$, with a uniquely determined $g : RX \to Y$. $\mathcal{Y}$ is $\mathcal{E}$*-reflective* if all reflexion morphisms ρ_X $(X \in \mathcal{X})$ belong to the class $\mathcal{E}$.

COROLLARY *A closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ can be restricted to every replete $\mathcal{E}$-reflective subcategory.*

Proof It suffices to show that *a replete $\mathcal{E}$- reflective subcategory of $\mathcal{X}$ is closed under $\mathcal{M}$-subobjects.* Indeed, factoring $m : M \to Y$ in $\mathcal{M}$ with $Y \in \mathcal{Y}$ as $m = g \cdot \rho_M$ gives, by the ($\mathcal{E}$, $\mathcal{M}$)-diagonalization property, a morphism $w : RM \to M$ with $w \cdot \rho_M = 1_M$. With the uniqueness property of reflexions, one also has $\rho_M \cdot w = 1_{RM}$ since $(\rho_M \cdot w) \cdot \rho_M = \rho_M$. Hence $M \cong RM \in \mathcal{Y}$. □

For a replete reflective subcategory, closedness under $\mathcal{M}$-subobjects in fact characterizes $\mathcal{E}$-reflectivity. We also note that a replete reflective subcategory is closed under all limits, particularly under direct products. For $\mathcal{X}$ with products and $\mathcal{E}$-*cowellpowered* (= dual to $\mathcal{M}$-wellpowered) and any full replete subcategory, closedness under products and $\mathcal{M}$-subobjects characterizes $\mathcal{E}$-reflectivity (see Exercise 2.K).

Instances of the Corollary will be discussed in Section 3.

REMARK Closedness under $\mathcal{M}$-morphisms is not a necessary condition for being able to restrict a closure operator C of $\mathcal{X}$ to a full subcategory $\mathcal{Y}$. Obviously it suffices that for $m : M \to X$ in $\mathcal{M}_{\mathcal{Y}}$, also $c_X(m)$ belongs to $\mathcal{M}_{\mathcal{Y}}$. We shall give an important example of a subcategory $\mathcal{Y}$ with this property which is not closed under $\mathcal{M}$-morphisms at the end of Section 3.5.

Exercises

2.A *(Discrete and trivial closure operators)*

(a) Show the existence of the following closure operators in a finitely $\mathcal{M}$-complete category $\mathcal{X}$:

$$S = (s_X)_{X \in \mathcal{X}} \quad \text{with } s_X(m) = m \text{ for all } m \in \mathcal{M}/X ,$$
$$T = (t_X)_{X \in \mathcal{X}} \quad \text{with } t_X(m) = 1_X \text{ for all } m \in \mathcal{M}/X ;$$

S is called the *discrete closure operator* and T the *trivial closure operator.*

(b) For the closure operators of (a), characterize the closed and the dense morphisms. Decide which of the properties idempotent, weakly hereditary, hereditary, grounded, minimal, (fully) additive and (finitely) productive are enjoyed by each of the two operators. Prove that these are the only minimal closure operators of **Top**.

2.B *(Properties of the Kuratowski closure operator)* Verify that K of **Top** satisfies all properties discussed in 2.4-2.7, except full additivity, directed additivity, and minimality.

2.C (*Characterization of idempotency and weak hereditariness*) Prove for a closure operator C of $\mathcal{X}$ with respect to $\mathcal{M}$:

(a) C is idempotent if and only if for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$: $c_X(m) \cong \bigwedge\{n \in \mathcal{M}/X : n \geq m$ and n is C-closed$\}$.

(b) C is weakly hereditary if and only if for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$: $c_X(m) \cong \bigvee\{k \in \mathcal{M}/X : k \cdot d = m$ for a C-dense $d\}$.
(Note that it is *not* necessary here to assume $\mathcal{X}$ to be $\mathcal{M}$-complete).

2.D (*Defining closure operators from given closed subobjects*)

(a) In an $\mathcal{M}$-complete category $\mathcal{X}$, let $\mathcal{C} \subseteq \mathcal{M}$ be a subclass containing all isomorphisms, closed under composition with isomorphisms, and stable under pullback. Prove that one can define an idempotent closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ by
$$c_X(m) = \bigwedge\{k \in \mathcal{C}/X \; : \; k \geq m\} \, .$$

(b) Under the assumptions of (a), show that $\mathcal{C} = \mathcal{M}^C$ if and only if $\mathcal{C}$ is stable under multiple pullback. In this case, C is weakly hereditary if and only if $\mathcal{C}$ is closed under composition.

(c) Show that C is hereditary if and only if for every $m : M \to X$ in $\mathcal{M}$ and every $k : K \to M$ in $\mathcal{C}$ there is $l : L \to X$ in $\mathcal{C}$ with $k \cong m^{-1}(l)$.

(d) Guided by 2.C(b), under which conditions can you define a closure operator with dense subobjects in a given class $\mathcal{D} \subseteq \mathcal{M}$?

2.E (*Closure operations of a poset*) For a partially ordered set $(X, \leq)$, we call a function $c : X \to X$ with $m \leq c(m)$ and $(m \leq m' \Rightarrow c(m) \leq c(m'))$ for all m, $m' \in X$ a *closure operation* of X. (Most authors require further properties). $(X, \leq)$ is considered as a (small) category $\mathcal{X}$ in the usual way (see Example (2) of 1.10). Note that every morphism in $\mathcal{X}$ is monic. Let therefore $\mathcal{M}$ be the class of all morphisms in $\mathcal{X}$.

(a) Verify that $\mathcal{X}$ is finitely $\mathcal{M}$-complete if and only if $(X, \leq)$ has finite meets.

(b) If $(X, \leq)$ has finite meets, then every closure operation c of X induces a closure operator $C = (c_x)_{x \in X}$ of the category $\mathcal{X}$ w.r.t. $\mathcal{M}$ with
$$c_x(m) := c(m) \wedge x$$
for all $m \leq x$ in X. Note that c is uniquely determined by C, as $c = c_1$, for 1 the top element in X.

(c) Let $X = [0,1]$ be the closed unit interval with its natural order. With
$$c_x(m) = \frac{m+x}{2}$$
one obtains a closure operator C of the category $\mathcal{X}$. Show that C is *not* induced by a closure operation of the poset $[0,1]$ in the sense of (b).

(d) In the setting of (b), find conditions on c which equivalently describe each of the properties introduced in 2.4-2.7 for the induced operator C.

(e) Find an example of an $\mathcal{M}$-complete category $\mathcal{X}$ and a closure operator C which is not finitely productive.

2.F (*More on closed and dense subobjects*) Prove for a closure operator C of an $\mathcal{M}$-complete category $\mathcal{X}$:

(a) If C is idempotent, then the formula

$$c_X(\bigvee_{i\in I} m_i) \cong c_X(\bigvee_{i\in I} c_X(m_i))$$

holds; describe arbitrary joins in $\mathcal{M}^C/X$ (see 2.6).

(b) For arbitrary morphisms d, e in $\mathcal{X}$ such that the composite $d \cdot e$ exists, one has
1. $d \cdot e \in \mathcal{E} \Rightarrow d \in \mathcal{E}$
2. $e \in \mathcal{E}$, $d \in \mathcal{E}^C \Rightarrow d \cdot e \in \mathcal{E}^C$
3. $e \in \mathcal{E}^C$, $d \in \mathcal{E} \Rightarrow d \cdot e \in \mathcal{E}^C$
4. $d \cdot e \in \mathcal{E}^C \Rightarrow d \in \mathcal{E}^C$ (cf. Corollary* 2.3)
5. $d \cdot e \in \mathcal{E}^C$, $d \in \mathcal{M}$, C hereditary $\Rightarrow e \in \mathcal{E}^C$.

(c) For $k \leq m$ and $k \leq n$ in $\mathcal{M}/X$, let $K \to N$ be C-dense. Then also $M \to M \vee N$ is C-dense.

(d) For $\mathcal{M}$-subobjects $d_i : M_i \to Y_i$, $y_i : Y_i \to X$, the induced $\mathcal{M}$-subobject $d : \bigvee_{i\in I} M_i \to \bigvee_{i\in I} Y_i$ is C-dense if all d_i's are C-dense.

2.G (*Grounded closure operators of R-modules*) For a unital ring R, show that the only grounded closure operator (up to isomorphism) of the category $\mathbf{Mod}_R$ of (left) R-modules with respect to the class of all monomorphisms is the discrete closure operator (cf. Exercise 2.A). *Hint*: For a submodule N of an R-module M, consider the quotient module M/N.

2.H (*Non-grounded closure operators of* **Top**) Prove that the only non-grounded closure operator of **Top** is the trivial closure operator (cf. Exercise 2.A). *Hint*: For every closure operator C of **Top**, there is a functor $F : \mathbf{Top} \to \mathbf{Top}$, $X \mapsto c_X(\emptyset)$; depending on whether $c_1(\emptyset) = \emptyset$ or $c_1(\emptyset) \cong 1$ (for 1 a singleton space), show that either $FX = \emptyset$ for all $X \in \mathbf{Top}$ or $FX \cong X$ for all $X \in \mathbf{Top}$.

2.I (*Fully additive closure operators in the presence of "points"*)

(a) Recall that an object P in a category $\mathcal{X}$ with coproducts is an *$\mathcal{E}$-generator* if the canonical morphism

$$\coprod_{\mathcal{X}(P,X)} P \longrightarrow X$$

belongs to $\mathcal{E}$ for every $X \in \mathcal{X}$. Show that then every $m : M \to X$ in $\mathcal{M}$ has a presentation as "join of its points":

$$m \cong \bigvee_{x \in \mathcal{X}(P\,,\,M)} m \cdot x(1_P)\ .$$

(b) Under the hypothesis of (a), show that two fully additive closure operators C and D on $\mathcal{X}$ coincide if

$$c_X(z(1_P)) \cong d_X(z(1_P))$$

for all $z \in \mathcal{X}(P\,,\,X)$, $X \in \mathcal{X}$. Conclude that, if "points" $z(1_P)$ are C-closed, then a fully additive closure operator C is the discrete closure operator (up to isomorphism, see Exercise 2.A).

(c) Show that the Kuratowski closure operator K of **Top** is *not* fully additive. However, if the subspace inclusion maps $m_i : M_i \hookrightarrow X$ satisfy the property: for every $x \in X$, there is a neighbourhood U of x which meets only finitely many M_i 's, then (FA) holds with $C = K$.

2.J (*Finite productivity in additive categories*) Prove that in the category $\mathbf{Mod}_R$, every closure operator is finitely productive. (*Hint:* For $M_i \le X_i$ and $c_i(M_i) = c_{X_i}(M_i)$, prove $c_i(M_i) \times 0 \le c(M_1 \times M_2)$ and then conclude $c_1(M_1) \times c_2(M_2) = (c_1(M_1) \times 0) + (0 \times c_2(M_2)) \le c(M_1 \times M_2)$.) Extend this result to every additive category with finite products (and coproducts).

2.K (*Characterization of $\mathcal{E}$-reflective subcategories*) Let $\mathcal{Y}$ be a full and replete subcategory of $\mathcal{X}$. Prove:

(a) For $\mathcal{Y}$ reflective in $\mathcal{X}$ one has that $\mathcal{Y}$ is $\mathcal{E}$-reflective if and only if $\mathcal{Y}$ is closed under $\mathcal{M}$-subobjects in $\mathcal{X}$.

(b) If $\mathcal{Y}$ is reflective in $\mathcal{X}$, then it is *closed under* all existing *limits* in $\mathcal{X}$ (i.e., whenever $\varprojlim H$ exists in $\mathcal{X}$ for $H : \mathcal{D} \to \mathcal{X}$ with $Hd \in \mathcal{Y}$ for all $d \in \mathcal{D}$, then $\varprojlim H \in \mathcal{Y}$).

(c) If $\mathcal{X}$ has direct products and is *$\mathcal{E}$-cowellpowered* (so that for every $X \in \mathcal{X}$ there is only a set of isomorphism classes of $\mathcal{E}$-morphisms with domain $\mathcal{X}$), then $\mathcal{Y}$ is $\mathcal{E}$-reflective in $\mathcal{X}$ if and only if $\mathcal{Y}$ is closed under direct products and $\mathcal{M}$-subobjects in $\mathcal{X}$. *Hint*: Study the *Adjoint Functor Theorem* in any book on Category Theory.

2.L (*Hereditariness for free*) Prove that (HE) holds for every closure operator C , provided that y is a *section* (so that there exists a morphism $p : Y \to X$ with $p \cdot y = 1_Y$). *Hint:* First show that for every $k \in \mathcal{M}/X$ with $k \le y$ (so that $k = y \cdot k_Y$) one has $p(k) \cong k_Y$. Then exploit C-continuity of p to show $p(c_X(m)) \le c_Y(m_Y)$.

2.M (*(κ, λ)-additivity*) Let ∞ denote a symbol greater than any element of the ordered class *Card* of all (small) cardinals numbers. For $\lambda \leq \kappa \in Card \cup \{\infty\}$, define a closure operator C of an $\mathcal{M}$-complete category $\mathcal{X}$ (as in 2.1) to be (κ, λ)-*additive*, if for all $X \in \mathcal{X}$ and $m_i \in \mathcal{M}/X$, $i \in I$, with $\operatorname{card} I < \kappa$, the formula

$$c_X(\bigvee_{i \in I} m_i) \cong \bigvee_{J \subseteq I, \operatorname{card} J < \lambda} c_X(\bigvee_{i \in J} m_i)$$

holds true. Confirm for every C:

1. C is additive $\Leftrightarrow$ C is $(\aleph_0, 2)$-additive,
2. C is fully additive $\Leftrightarrow$ C is $(\infty, 2)$-additive,
3. C is grounded $\Leftrightarrow$ C is $(1, 0)$-additive,
4. for every $\kappa \in Card \cup \{\infty\}$, C is (κ, κ)-additive,
5. C is (κ, λ)-additive & $\lambda' \leq \lambda \leq \kappa \leq \kappa' \Rightarrow C$ is (κ', λ')-additive.

Notes

The categorical notion of closure operator as introduced in 2.2 (with the key ingredient given by the continuity condition) goes back to Dikranjan and Giuli [1987a] and includes both, the lattice-theoretic closure operations (see Exercise 2.E) and the Lawvere-Tierney topologies or universal closure operations of Topos- and Sheaf Theory (see Chapter 9), as special instances. Principal properties like idempotency, (weak) hereditariness, and additivity are already discussed in the Dikranjan-Giuli paper, although the "symmetric" approach to idempotency / weak hereditariness and to hereditariness / minimality as given in 2.4 and 2.5 is not yet apparent in that paper. Earlier papers in Categorical Topology are mostly concerned with particular instances of closure operators, predominantly in the category of topological spaces (see Chapters 6-8). To be mentioned particularly is the paper by Cagliari and Cicchese [1983] which introduces for epireflective subcategories of **Top** a stronger notion of closure operator, with the continuity condition stated explicitly as one of the axioms.

3 Examples of Closure Operators

Most of the examples presented in this chapter will be used throughout the book, especially the closure operators for topological spaces, R-modules and for groups presented in sections 3.3, 3.4, and 3.5, respectively. Nevertheless, we begin with structures which generalize topological spaces, namely pretopological spaces and filter convergence spaces, for two reasons. First, additive and grounded closure operators of concrete categories may be interpreted as concrete functors with values in the category of pretopological spaces, as we shall see in Chapter 5 and apply in Chapter 8. Second, the natural closure operators of these generalized topological structures are closely linked to the natural closure operators occurring in the categories of graphs and partially ordered sets presented in 3.6. Hence they provide a unifying view of topological and "discrete" structures.

3.1 Kuratowski closure operator, Čech closure operator

The Kuratowski closure operator $K = (k_X)_{X\in\mathbf{Top}}$ of the category **Top** as described in Example 2.2 determines completely the structure of each space X :

$$M \subseteq X \text{ closed } \Longleftrightarrow k_X(M) = M$$

$$N \subseteq X \text{ is a neighbourhood of } x \in X \Leftrightarrow x \notin k_X(X \setminus N).$$

In what follows, we shall describe extensions of the Kuratowski closure operator to supercategories of **Top**. This approach will allow for common descriptions of fundamental closure operators in some familiar categories.

In this section, we consider the category **PrTop** of *pretopological spaces*: a *pretopology* k_X on a set X is a map $k_X : 2^X \to 2^X$ with

$$k_X(\emptyset) = \emptyset\,,\ M \subseteq k_X(M) \quad \text{and} \quad k_X(M \cup N) = k_X(M) \cup k_X(N);$$

note that then k_X is monotone, but idempotency is *not* required. A *continuous* map $f : (X\,,\ k_X) \to (Y\,,\ k_Y)$ must satisfy $f(k_X(M)) \subseteq k_Y(f(M))$ for all $M \subseteq X$.

Every subset Y of a pretopological space (X, k_X) carries the subspace structure given by $k_Y(M) = Y \cap k_X(M)$.

Let $\mathcal{M}$ be the class of *embeddings*, i.e. of injective maps $f : (X, k_X) \to (Y, k_Y)$ such that (X, k_X) is isomorphic to the subspace $f(X)$ of (Y, k_Y) ; equivalently,

$$k_X(M) = f^{-1}(k_Y(f(M))) \text{ for all } M \subseteq X\ .$$

The existence of $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections is easily verified. Hence **PrTop** is $\mathcal{M}$-complete.

Now, by the very definition of **PrTop** and of the class $\mathcal{M}$, *the Čech closure operator* $K = (k_X)_{X\in\mathbf{PrTop}}$ *of* **PrTop** *is grounded, additive and hereditary but neither idempotent nor fully additive*. It is less obvious that *it is also productive*, since the explicit description of direct products in **PrTop** is a bit cumbersome: see Exercise 3.B. (We shall provide another proof for the productivity of K in 3.2 below.)

Finally we point out that, by means of its Kuratowski closure operator, **Top** is fully embedded into **PrTop**; a pretopological space (X, k_X) is topological iff $k_X k_X = k_X$. Hence, by definition of this embedding, the Čech closure operator of **PrTop** restricted to **Top** gives the Kuratowski closure operator of **Top**. It is important to note here that subspaces of $X \in$ **Top** formed in **PrTop** are in fact subspaces in **Top**, so that **Top** is closed under subobjects in **PrTop**, and Proposition 2.8 applies here.

Furthermore, one easily sees from the description of direct products as given in Exercise 3.B that **Top** is closed under direct products in **PrTop**, hence productivity of the Kuratowski closure operator in **Top** can be formally concluded from the corresponding result in **PrTop**.

Closedness of **Top** in **PrTop** under subobjects and direct products follows also from the fact that **Top** *is a bireflective subcategory of* **PrTop** (so that the reflexions are *bimorphisms* in **PrTop**, i.e., bijective continuous maps): the reflector R takes $(X\ ,\ k_X) \in$ **PrTop** to $(X, k_X^\infty) \in$ **Top** ; here $k_X^\infty = k_X^\alpha$ for the least ordinal number α with $k_X^{\alpha+1} = k_X^\alpha$. (The ordinal powers of k_X are defined by $k_X^0 = id$, $k_X^{\alpha+1} = k_X k_X^\alpha$, and $k_X^\beta(M) = \bigcup_{\gamma<\beta} k_X^\gamma(M)$ for a limit ordinal β). The verification of these claims is left to the reader as Exercise 3.A.

3.2 Filter convergence spaces and Katětov closure operator

In a pretopological space (X, k_X) one has, as in **Top**, a notion of convergence: a filter $\mathcal{F}$ converges to a point $x \in X$ if and only if every neighbourhood of x belongs to $\mathcal{F}$, i.e., if and only if

$$\mathcal{V}_x := \{N \subseteq X : x \notin k_X(X \setminus N)\} \subseteq \mathcal{F} .$$

Often it turns out to be useful to replace (and in fact generalize) pretopologies by *convergence structures*, as follows. A *filter convergence space* (X, q_X) is a set X equipped with a map q_X which assigns to each $x \in X$ a set $q_X(x)$ of filters on X such that, when writing $\mathcal{F} \to x$ instead of $\mathcal{F} \in q_X(x)$, one has

$$\dot{x} \to x\ , \quad \text{with } \dot{x} = \{N \subseteq X : x \in N\}\ ,$$

$$\text{if } \mathcal{F} \to x \quad \text{and} \quad \mathcal{G} \supseteq \mathcal{F}\ , \quad \text{then } \mathcal{G} \to x$$

for every $x \in X$ and all filters $\mathcal{F}$, $\mathcal{G}$ on X . A *continuous* map $f : (X, q_X) \to (Y, q_Y)$ must preserve convergence, i.e.

$$\mathcal{F} \xrightarrow{q_X} x \text{ implies } f\mathcal{F} \xrightarrow{q_Y} f(x)$$

for all $x \in X$ and filters $\mathcal{F}$ on X ; here $f\mathcal{F}$ denotes the filter on Y with filter-base $\{f(F) : F \in \mathcal{F}\}$. This defines the category **FC**.

For a family $f_i : X \to X_i$ with $(X_i, q_i) \in \mathbf{FC}$, $i \in I$, we can define an *initial* (or *weak*) convergence structure q_X on X , as follows:

$$\mathcal{F} \xrightarrow{q_X} x :\Longleftrightarrow (\forall i \in I)\ f_i\mathcal{F} \xrightarrow{q_i} f_i(x) .$$

In particular, subspace and direct product structures are easily described. For a subset Y of (X, q_X) , the subspace structure q_Y on Y is given by

$$\mathcal{F} \xrightarrow{q_Y} y \Longleftrightarrow (\exists \mathcal{G})\ \mathcal{G} \xrightarrow{q_X} y \quad \text{and} \quad \mathcal{F} \supseteq \{G \cap Y \mid G \in \mathcal{G}\} .$$

The direct product (X, q_X) of (X_i, q_i) , $i \in I$, is given by the cartesian product $X = \prod_{i \in I} X_i$ with the convergence structure

$$\mathcal{F} \xrightarrow{q_X} x \Longleftrightarrow (\forall i \in I)\ p_i\mathcal{F} \xrightarrow{q_i} x_i ,$$

for $x = (x_i)_{i \in I}$ and p_i the i-th projection.

One easily shows that for $\mathcal{M}$ the class of (subspace) embeddings, $\mathbf{FC}$ is $\mathcal{M}$-complete. Hence we are ready to define the *Katětov closure operator* $K = (k_X)_{X \in \mathbf{FC}}$, by

$$k_X(M) := \{x \in X : (\exists \mathcal{F})\ \mathcal{F} \xrightarrow{q_X} x \quad \text{and} \quad M \in \mathcal{F}\}$$

PROPOSITION K *is a hereditary, grounded, additive and productive closure operator of* $\mathbf{FC}$ *w.r.t. the class of embeddings.*

Proof That K is a hereditary and grounded closure operator of $\mathbf{FC}$ is easily checked. In order to prove its additivity, let $x \in k_X(M \cup N)$ so that $\mathcal{F} \to x$ for a filter $\mathcal{F}$ on X with $M \cup N \in \mathcal{F}$. Choose an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$; then $M \in \mathcal{U}$ or $N \in \mathcal{U}$, and $\mathcal{U} \to x$. Hence $x \in k_X(M) \cup k_X(N)$.

Finally, consider a subset $M = \prod_{i \in I} M_i$ of $X = \prod_{i \in I} X_i$, and let $x = (x_i)_{i \in I} \in \prod_{i \in I} k_i(M_i)$. Hence there are filters $\mathcal{F}_i$ on X_i with $\mathcal{F}_i \to x_i$ and $M_i \in \mathcal{F}_i$, $i \in I$. By the definition of the convergence structure on X , for the filter $\mathcal{F}$ on X with filterbase

$$\left\{ \prod_{i \in I} F_i : (\forall i \in I)\ F_i \in \mathcal{F}_i \right\} ,$$

one has $\mathcal{F} \to x$, and $\prod_{i \in I} M_i \in F$. Hence $x \in k_X(M)$. □

We wish to show that the Katětov closure operator K *of* $\mathbf{FC}$ *is an extension of the Čech closure operator of* **PrTop** *and of* **Top**. First of all, each pretopological space becomes a filter convergence space, by

$$\mathcal{F} \to x \Longleftrightarrow \mathcal{V}_x \subseteq \mathcal{F}$$

(see the beginning of this section). This gives a full embedding

$$\mathbf{PrTop} \longrightarrow \mathbf{FC},$$

in fact, a *bireflective embedding*. The reflector provides $(X, q_X) \in \mathbf{FC}$ with its pretopology k_X, as described above. Hence, if (X, q_X) originates from a pretopological space, the induced pretopology k_X coincides with the original pretopology.

A continuous map in **PrTop** is an embedding if and only if it is an embedding in **FC**. This can easily be shown directly or, as we shall point out in 5.8, derived from general categorical facts. They also imply that **PrTop** is closed under subobjects and products in **FC**: see Exercise 2.K. The latter fact shows that the productivity of K in **FC** gives in particular:

COROLLARY *The Čech closure operator of* **PrTop** *and the Kuratowski closure operator of* **Top** *are productive.* □

On the other hand, one concludes that K in **FC** is neither idempotent nor fully additive since its restriction to **PrTop** does not have these properties.

We shall resume the discussion of extensions and restrictions of the Kuratowski closure operator in 3.6.

3.3 Sequential closure, b-closure, θ-closure, $\mathfrak{k}$-closure

In this section we discuss four particular closure operators of the category **Top** of topological spaces w.r.t. the class $\mathcal{M}$ of embeddings. Each of them is of interest to topologists and is therefore discussed here in some detail, although they are instances of more general constructions to be discussed later.

For a topological space X and a subspace M, we denote by k_X the usual (Kuratowski) closure and then consider:

(a) the *sequential closure* $\sigma_X(M)$ of M in X, containing all points $x \in X$ such that there exists a sequence (x_n) in M converging to x in X;

(b) the *b-closure* (or *front closure*) $b_X(M)$ of M in X, containing all points $x \in X$ with
$$k_X(\{x\}) \cap M \cap U \neq \emptyset$$
for every neighbourhood U of x;

(c) the *θ-closure* $\theta_X(M)$ of M in X, containing all points $x \in X$ with
$$M \cap k_X(U) \neq \emptyset$$
for every neighbourhood U of x;

(d) the *$\mathfrak{k}$-closure* $\mathfrak{k}_X(M)$ of M in X, containing all points $x \in X$ such that there is a compact subspace B of X with
$$x \in k_X(M \cap B).$$

It is not difficult to check that each of the four operators gives in fact a closure operator of **Top** w.r.t. $\mathcal{M}$; in order to check continuity of the θ-closure, it is useful to observe that

$$\theta_X(M) = \bigcap\{k_X(U) : U \supseteq M\ ,\ U \text{ open}\}.$$

First we relate these closure operators to the usual closure:

PROPOSITION *Let X be a topological space and $M \subseteq X$. Then:*

(1) $b_X(M) \subseteq k_X(M)$ *and* $\sigma_X(M) \subseteq \mathfrak{k}_X(M) \subseteq k_X(M) \subseteq \theta_X(M)$.

(2) (a) $\sigma_X = k_X$ *iff* X *is Frechét-Urysohn;*

(b) $b_X = k_X$ *iff* X *is a topological sum of indiscrete spaces;*

(c) $\theta_X = k_X$ *iff* X *is a regular space;*

(d) $\mathfrak{k}_X = k_X$ *iff* X *is a k-space.*

(3) *Each of the inclusions in* (1) *may be proper.*

Proof

(1) $\sigma_X(M) \subseteq k_X(M)$ holds since $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact whenever $x_n \longrightarrow x$ in X . The other inclusions follow immediately from the definitions.

(2) (a) By definition, a space X is Frechét-Urysohn if a subset $M \subseteq X$ is closed whenever $x_n \longrightarrow x$ with $x_n \in M$ for all n implies $x \in M$. For (b), see Exercise 3.E. (c) Let X be regular, i.e., for every neighbourhood V of x there is a neighbourhood U of x with $k_X(U) \subseteq V$. To show $\theta_X(M) = k_X(M)$, let $x \in \theta_X(M)$ and let V be a neighbourhood of x . With U as above, one has $M \cap k_X(U) \neq \emptyset$ by definition of the θ-closure, hence $M \cap V \neq \emptyset$. Vice versa, given $\theta_X = k_X$, in order to show regularity of X , one shows that for every $x \in X$ and $A \subseteq X$ closed with $x \notin A$, there are disjoint open sets U, V with $x \in U$ and $A \subseteq V$. In fact, by assumption one has $\theta_X(A) = A$, hence $x \notin \theta_X(A)$. Therefore there is an open set $U \ni x$ with $A \cap k_X(U) = \emptyset$, i.e., $A \subseteq V := X \setminus k_X(U)$. (d) By definition, X is a k-space if a subset $M \subseteq X$ is closed whenever $M \cap B$ is closed in B for every compact subspace B of X . The k-space property means that $\mathfrak{k}$-closed sets in X are closed, and the latter property translates into

$$k_X(M) \subseteq k_X(\mathfrak{k}_X(M)) = \mathfrak{k}_X(M)$$

for all $M \subseteq X$.

(3) For any non-discrete Hausdorff space X , $b_X \neq k_X$, by (2)(b). By (2)(c), for a non-regular space, θ-closure and Kuratowski-closure are different. Similarly, for any space which is not a k-space, $\mathfrak{k}$-closure and Kuratowski closure differ. (Consider, for example, the subspace $\mathbb{N} \cup \{x\}$ of $\beta\mathbb{N}$ with any $x \in \beta\mathbb{N} \setminus \mathbb{N}$.) To see that the sequential closure may be properly smaller than the k-closure, consider any compact space which is not Frechét-Urysohn. (For example, $X = \beta\mathbb{N}$ is compact, so that $\mathfrak{k}_X = k_X$, while $\mathbb{N}$ is a σ-closed subspace of X which is dense.) □

THEOREM *The closure operators* σ , b , θ , *and* $\mathfrak{k}$ *of* **Top** *have the following properties:*

(1) b *is idempotent while* σ , θ *and* $\mathfrak{k}$ *are not.*

(2) b *and* σ *are hereditary, while* $\mathfrak{k}$ *is weakly hereditary but not hereditary.* θ *is not even weakly hereditary.*

(3) *All* b , σ , θ *and* $\mathfrak{k}$ *are grounded, additive and productive. None of the four operators is fully additive, but they are all fully additive for Alexandroff spaces.*

Proof

(1) To check idempotency of b_X , consider $x \in X \setminus b_X(M)$. Then for some open neighbourhood U of x , $k_X(\{x\}) \cap M \cap U = \emptyset$. Hence, for each $y \in k_X(\{x\}) \cap U$, $k_X(\{y\}) \subseteq k_X(\{x\})$, and U is an open neighbourhood of y , thus $y \notin b_X(M)$. This proves

$$(k_X(\{x\}) \cap U) \cap b_X(M) = \emptyset ,$$

hence $x \notin b_X(b_X(M))$.

For non-idempotency of θ , we refer to Exercise 3.F. Non-idempotency of σ and $\mathfrak{k}$ is witnessed by the following space X introduced by Arhangel'skii and Franklin [1968]. In $\mathbb{R}^2$, consider the set

$$X = \{(0,0)\} \cup \bigcup_{n=1}^{\infty} X_n$$

with $X_n = \{(1/n, 1/m) : m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$, and provide it with the following topology: each point (x, y) with $xy > 0$ is isolated; the basic neighbourhoods of $x_n = (1/n, 0)$ are $\{x_n\} \cup X_n \setminus \{(1/n, 1/m) : m \leq k\}$ with $k \in \mathbb{N}$; the basic neighbourhoods of $(0,0)$ are $\{(0,0)\} \cup \bigcup_{n \geq k} (X_n \setminus F_n)$ with $k \in \mathbb{N}$ and each $F_n \subseteq X_n$ finite. (Note that this topology τ' is obtained by adding to the usual Euclidean topology τ of X new neighbourhoods of $(0,0)$ in order to make all "diagonal" sequences converging to $(0,0)$ in (X, τ) closed sets in τ' .) Then, for

$$M = \{(x,y) \in X : \; xy > 0\},$$

one has $\sigma_X(M) \cong X \setminus \{(0,0)\}$, whereas $\sigma_X(\sigma_X(M)) = X$. Since always $\sigma_X(N) \subseteq \mathfrak{k}_X(N)$, for $N \subseteq X$, one also has $\mathfrak{k}_X(\mathfrak{k}_X(M)) = X$. On the other hand, $(0,0) \notin \mathfrak{k}_X(M)$. In fact, by definition of the topology of X , any subset $\{(1/n_k \, , \, 1/m_k) : k \in \mathbb{N}\}$ with $n_1 < n_2 < \ldots < n_k < \ldots$ is closed and discrete in X , hence cannot be contained in a compact subset B of X. Hence B meets M in only finitely many "columns" X_n . This shows $(0,0) \notin \mathfrak{k}_X(M \cap B)$.

(2) For hereditariness of b , see Exercise 3E. Hereditariness of σ is trivial. To show that θ is not weakly hereditary is left to the reader: see Exercise 3.F and Example 4.4 below. To check weak hereditariness of $\mathfrak{k}_X$ for a Hausdorff space

X (or, more generally, for any space X in which compact subspaces are closed), consider $M \subseteq X$ and $x \in \mathfrak{k}_X(M)$ hence $x \in Y = k_X(M \cap B)$ for a compact subspace $B \subseteq X$. Since $Y \subseteq \mathfrak{k}_X(M)$, it suffices to show $x \in \mathfrak{k}_Y(M \cap Y)$. Indeed, $C := B \cap Y$ is a compact subset of Y containing $M \cap B$, since Y is closed in X. Hence

$$Y = k_X(M \cap B) = k_Y(M \cap B \cap Y) = k_Y((M \cap Y) \cap C)$$

is contained in $\mathfrak{k}_Y(M \cap Y)$.
To see that k is not hereditary, consider $X = \beta\mathbb{N}$ and any point x in the remainder $X \setminus \mathbb{N}$. By (2)(d), $\mathfrak{k}_X = k_X$, hence $x \in \mathfrak{k}_X(\mathbb{N})$. On the other hand, $x \notin k_Y(\mathbb{N})$ for $Y = \mathbb{N} \cup \{x\}$ since $\mathbb{N}$ contains no infinite compact subsets. (In fact, for every infinite subset $A \subseteq \mathbb{N}$ consider a partition $A = A_1 \cup A_2$ into infinite subsets; then at least one of the two subsets A_i does not meet all neighbourhoods of x, hence A_i has an accumulation point in X.)

(3) Groundedness holds trivially. σ and θ are additive since the intersection of two closed neighbourhoods is a closed neighbourhood. Additivity of $\mathfrak{k}$ follows from additivity of k and the fact that the union of two compact subsets is compact. Additivity of b is a direct consequence of the definition of b-closure. For the assertions on full additivity, see Exercise 3.G. Finally we turn to productivity and consider a family of spaces X_i and $M_i \subseteq X_i$, $i \in I$. With $X = \prod_{i \in I} X_i$ and $M = \prod_{i \in I} M_i$ we must show $\prod_{i \in I} c_{X_i}(M_i) \subseteq M$ for each of the four closure operators.

In case of the sequential closure operator, one just observes that if $x_n^{(i)} \to x_i$ for every $i \in I$, then $(x_n^{(i)})_{i \in I} \to (x_i)_{i \in I}$.
For the $\mathfrak{k}$-closure, let $x_i \in \mathfrak{k}_{X_i}(M_i)$ for all $i \in I$, hence

$$x_i \in k_{X_i}(M_i \cap B_i)$$

for compact sets $B_i \subseteq X_i$, $i \in I$. Since K is productive (see Corollary 3.2),

$$x = (x_i)_{i \in I} \in A = k_X\left(\prod_{i \in I}(M_i \cap B_i)\right)$$

By Tychonoff's Theorem, $B = \prod_{i \in I} B_i$ is compact, hence $A = k_X(M \cap B) \subseteq \mathfrak{k}_X(M)$ in view of the obvious equality $M \cap B = \prod_{i \in I}(M_i \cap B_i)$. Hence $x \in A \subseteq \mathfrak{k}_X(M)$.
Let now $x_i \in \theta_{X_i}(M_i)$ for every $i \in I$. Since K is productive, the (ordinary) closure of a basic neighbourhood of $x = (x_i)_{i \in I}$ in X is of the form

$$W = \prod_{i \in F} k_{X_i}(U_i) \times \prod_{i \in I \setminus F} X_i$$

with $F \subseteq I$ finite and each U_i a neighbourhood of x_i . Since $k_{X_i}(U_i)$ meets M_i for all $i \in F$, one has $W \cap M \neq \emptyset$, hence $x \in \theta_X(M)$.

The argumentation in case of the b-closure is similar and can be left as an exercise (see Exercise 3.E). □

3.4 Preradicals in R-modules and Abelian groups

Preradicals give a rich supply of closure operators in the category $\mathbf{Mod}_R$ of left R-modules and R-linear maps, for a unital ring R . Here $\mathcal{M}$ is the class of monomorphisms, a skeleton of which is given by inclusion maps of submodules.

A *preradical* $\mathbf{r}$ in $\mathbf{Mod}_R$ is a subfunctor of the identity functor of $\mathbf{Mod}_R$; hence for every R-module M one has a submodule $\mathbf{r}(M)$ such that every R-linear map $f : M \to N$ can be restricted as $\mathbf{r}(f) : \mathbf{r}(M) \to \mathbf{r}(N)$. We are interested in closure operators C of $\mathbf{Mod}_R$ with the property that $c_M(0) = \mathbf{r}(M)$ for every R-module. When writing $C \leq D$ iff $c_M(N) \subseteq d_M(N)$ for all submodules $N \leq M$, $M \in \mathbf{Mod}_R$, we easily obtain:

PROPOSITION *For every preradical* $\mathbf{r}$ *of* $\mathbf{Mod}_R$ *, there is a least closure operator* $C_\mathbf{r}$ *of* $\mathbf{Mod}_R$ *with* $(c_\mathbf{r})_M(0) = \mathbf{r}(M)$ *for all R-modules* M *, and a last closure operator* $C^\mathbf{r}$ *of* $\mathbf{Mod}_R$ *with* $c^\mathbf{r}_M(0) = \mathbf{r}(M)$ *for all* M *. For all* $N \leq M$ *, the formulas*

$$(c_\mathbf{r})_M(N) = N + \mathbf{r}(M)$$
$$c^\mathbf{r}_M(N) = \pi^{-1}(\mathbf{r}(M/N))$$

hold; here $\pi : M \to M/N$ *is the canonical projection. One calls* $C_\mathbf{r}$ *and* $C^\mathbf{r}$ *the* minimal *and the* maximal *closure operator induced by* $\mathbf{r}$*, respectively.*
N *is* $C_\mathbf{r}$*-closed* ($C_\mathbf{r}$*-dense*) *in* M *iff* $\mathbf{r}(M) \subseteq N$ ($\mathbf{r}(M)+N = M$*, respectively*) *;* N *is* $C^\mathbf{r}$*-closed* ($C^\mathbf{r}$*-dense*) *in* M *iff* $\mathbf{r}(M/N) = 0$ ($\mathbf{r}(M/N) = M/N$*, respectively*). □

Proof Every closure operator C with $c_M(0) = \mathbf{r}(M)$ for all M must satisfy $\mathbf{r}(M) \leq c_M(N)$ for all $N \leq M$, by monotonicity, hence $N + \mathbf{r}(M) \leq c_M(N)$. Also, by continuity, one has

$$c_M(N) = c_M(\pi^{-1}(0)) \leq \pi^{-1}(\mathbf{r}(M/N)) .$$

Checking that the given formulas define closure operators $C_\mathbf{r}$ and $C^\mathbf{r}$ with the desired properties is routine work now. □

Below we list some properties of $C_\mathbf{r}$ and $C^\mathbf{r}$ which, of course, depend on properties of the preradical $\mathbf{r}$. The definitions of these properties are given in the proof below.

THEOREM *Let* $\mathbf{r}$ *be a preradical of* $\mathbf{Mod}_R$ *. Then:*

(1) *$C_{\mathbf{r}}$ is idempotent, minimal and fully additive, but not grounded, unless $\mathbf{r}(M) = 0$ for every R-module M .*

(2) *$C^{\mathbf{r}}$ is idempotent iff $\mathbf{r}$ is a radical.*

(3) *Equivalent are: $C_{\mathbf{r}}$ is weakly hereditary; $C^{\mathbf{r}}$ is weakly hereditary; $\mathbf{r}$ is idempotent.*

(4) *Equivalent are: $C_{\mathbf{r}}$ is hereditary; $C^{\mathbf{r}}$ is hereditary; $\mathbf{r}$ is hereditary.*

(5) *$C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ are finitely productive. Equivalent are: $C_{\mathbf{r}}$ is productive; $C^{\mathbf{r}}$ is productive; $\mathbf{r}$ is Jansenian.*

(6) *Equivalent are: $C_{\mathbf{r}} = C^{\mathbf{r}}$; $\mathbf{r}$ is cohereditary; $C^{\mathbf{r}}$ is additive. In this case $\mathbf{r}$ is necessarily a radical.*

Proof

(1) is trivial.

(2) For all $N \le M$, one has $c^{\mathbf{r}}_M(c^{\mathbf{r}}_M(N)) = c^{\mathbf{r}}_M(N)$ if and only if $\mathbf{r}(M/c^{\mathbf{r}}_M(N)) = 0$. Since always $\mathbf{r}(M) \le \pi^{-1}(\mathbf{r}(M/N)) = c^{\mathbf{r}}_M(N)$, with equality to hold for $N = 0$, the latter condition is equivalent to

$$\mathbf{r}(M/\mathbf{r}(M)) = 0$$

which, by definition, means that $\mathbf{r}$ *is a radical.*

(3) $\mathbf{r}(\mathbf{r}(M))$ is the $C_{\mathbf{r}}$-closure of 0 in $\mathbf{r}(M)$. This shows that $C_{\mathbf{r}}$ is weakly hereditary if and only if

$$\mathbf{r}(\mathbf{r}(M)) = \mathbf{r}(M)$$

for all M , i.e. iff $\mathbf{r}$ *is idempotent.* Any $N \le M$ is $C^{\mathbf{r}}$-dense in $c^{\mathbf{r}}_M(N)$ iff $\mathbf{r}(c^{\mathbf{r}}_M(N)/N) = c^{\mathbf{r}}_M(N)/N$. Since the latter module is isomorphic to $\mathbf{r}(M/N)$, according to the definition of $c^{\mathbf{r}}_M(N)$, one concludes that $C^{\mathbf{r}}$ is weakly hereditary iff $\mathbf{r}(\mathbf{r}(M/N)) = \mathbf{r}(M/N)$ holds for all $N \le M$. This is clearly equivalent to the idempotency of $\mathbf{r}$.

(4) *Hereditariness of* $\mathbf{r}$ means, by definition,

$$\mathbf{r}(L) = \mathbf{r}(M) \cap L$$

for all $L \le M$. Hence, for $N \le L$, this implies

$$\begin{aligned}(c_{\mathbf{r}})_M(N) \cap L &= (\mathbf{r}(M) + N) \cap L = (\mathbf{r}(M) \cap L) + N \\ &= \mathbf{r}(L) + N = (c_{\mathbf{r}})_L(N)\,,\end{aligned}$$

that is: $C_{\mathbf{r}}$ is hereditary. For the converse proposition, consider $N = 0$. To derive hereditariness of $C^{\mathbf{r}}$, observe that

$$\mathbf{r}(\pi(L)) = \mathbf{r}(M/N) \cap \pi(L)$$

holds when $\mathbf{r}$ is hereditary. Taking inverse images w.r.t. π yields

$$c^{\mathbf{r}}_L(N) = c^{\mathbf{r}}_M(N) \cap L\,,$$

i.e. hereditariness of $C^{\mathbf{r}}$. Again, for the converse proposition, apply this to $N = 0$.

(5) For finite productivity, see Exercise 2.J. For $\mathbf{r}$ *to be Jansenian* means that for all families of modules M_i , $i \in I$,

$$\mathbf{r}(\prod_{i\in I} M_i) = \prod_{i\in I} \mathbf{r}(M_i).$$

To deduce productivity of $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ is a straightforward exercise. On the other hand, this is a necessary condition for the productivity of each $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ as the application to zero-submodules shows.

(6) $C_{\mathbf{r}} = C^{\mathbf{r}}$ holds iff

$$\mathbf{r}(M/N) = (\mathbf{r}(M) + N)/N$$

whenever $N \leq M$; by definition, this means that $\mathbf{r}$ *is cohereditary*. This condition implies that $C^{\mathbf{r}}$ is additive since $C_{\mathbf{r}}$ is always additive. To prove the converse is left as Exercise 3.M. That a cohereditary preradical is a radical follows from (1) and (2).

□

We now turn to three specific preradicals, in fact: radicals, in the category $\mathbf{AbGrp}$ $(= \mathbf{Mod}_{\mathbb{Z}})$ of abelian groups which will be of interest later on.

EXAMPLES

(1) Consider the radical $\mathbf{t}$ defined by the *torsion subgroup*

$$\mathbf{t}(A) = \{a \in A : (\exists n \in \mathbb{Z})\; n > 0 \quad \text{and} \quad na = 0\}$$

of an abelian group A . The corresponding closure operator $C^{\mathbf{t}}$ can be described by

$$c^{\mathbf{t}}_A(B) = \{a \in A : (\exists n \in \mathbb{Z})\; n > 0 \quad \text{and} \quad na \in B\}$$

for every subgroup $B \leq A$. From the Theorem one obtains that $C^{\mathbf{t}}$ *is hereditary, idempotent and finitely productive, but neither productive nor additive*. Furthermore, it is easy to check that $C^{\mathbf{t}}$ *is nevertheless directedly additive.*

(2) The *maximal divisible subgroup* $\mathbf{d}(A)$ of an abelian group A defines a radical $\mathbf{d}$. (Recall that A is *divisible* if for every $y \in A$ and for every positive integer n , there is $x \in A$ with $nx = y$.) From the respective properties of $\mathbf{d}$ one derives with the Theorem that $C^{\mathbf{d}}$ *is idempotent, weakly hereditary and productive but neither hereditary nor additive*. A subgroup $B \leq A$ is $C^{\mathbf{d}}$-closed ($C^{\mathbf{d}}$-dense)

if and only if A/B is reduced (divisible, respectively).
It is easy to see that $C^{\mathbf{d}}$ coincides with the discrete closure operator of **AbGrp** for the groups A with the property that all quotients of A are reduced; see (i) and (ii) of (3) below.

(3) The *Frattini subgroup* $\mathbf{f}(A) = \bigcap\{M : M$ is a maximal (proper) subgroup of $A\}$ of an abelian group A defines a non-idempotent and non-Jansenian radical $\mathbf{f}$ of **AbGrp** . For the corresponding closure operator $C^{\mathbf{f}}$ one has

$$c_A^{\mathbf{f}}(B) = \bigcap\{M : M \text{ is a maximal subgroup of } A \text{ with } B \leq M\}$$

The Theorem gives that $C^{\mathbf{f}}$ *is idempotent, but neither weakly hereditary nor productive*. We note that $B \leq A$ is $C^{\mathbf{f}}$-dense iff there is no maximal subgroup of A containing B .

The following presentation of $\mathbf{f}(A)$ turns out to be useful:

$$\mathbf{f}(A) = \bigcap\{pA : p \text{ prime }\}. \qquad (*)$$

In fact, for every maximal subgroup M of A there is a prime number p with $pA \subseteq M$ (since A/M is a simple Abelian group, hence $p = |A/M|$ is prime). On the other hand, for every prime p , A/pA is a group of exponent p , hence it is a direct sum of cyclic groups of order p . Therefore $\mathbf{f}(A/pA) = 0$, and the $C^{\mathbf{f}}$ -closed subgroup pA of A is an intersection of maximal subgroups. This proves $(*)$.

Since $\mathbf{d}(A) \subseteq pA$ for every prime number p , $(*)$ gives $\mathbf{d}(A) \leq \mathbf{f}(A)$ for every A , hence $C^{\mathbf{d}} \leq C^{\mathbf{f}}$. In particular, $C^{\mathbf{d}}$-dense maps are $C^{\mathbf{f}}$-dense. In fact, the converse proposition is also true (although $C^{\mathbf{d}}$ is weakly hereditary and $C^{\mathbf{f}}$ is not): $(*)$ gives that for a $C^{\mathbf{f}}$-dense subgroup $B \leq A$, the quotient $A' = A/B$ satisfies $A' = pA$ for every prime p . This means that A' is divisible, hence B is $C^{\mathbf{d}}$-dense in A .

For a proof of the following result which relates properties of $C^{\mathbf{d}}$ and $C^{\mathbf{f}}$ further, we refer the reader to Dikranjan-Prodanov [1976].

Equivalent are for an abelian group A :

(i) $C^{\mathbf{d}}$ coincides with the discrete closure operator for A ;

(ii) every quotient of A is reduced;

(iii) no proper subgroup of A is $C^{\mathbf{f}}$-dense ;

(iv) every proper subgroup of A is contained in a maximal subgroup;

(v) there exists a subgroup B of A and non-negative integers k_p for every prime p such that $B \cong \mathbb{Z}^n$ for some $n \geq 0$, and A/B is a torsion group such that its p-torsion component $\mathbf{t}_p(A/B)$ is of exponent p^{k_p} , i.e., $p^{k_p}\mathbf{t}_p(A/B) = 0$.

3.5 Groups, rings and fields

We consider examples of closure operators in the category **Grp** of (multiplicatively written, not necessarily Abelian) groups and their homomorphisms, in the category **CRng** of commutative unital rings and their homomorphisms, both with respect to the class of monomorphisms, and in its full subcategory **Fld** of fields (in which every morphism is monic).

(1) The *normal closure* of a subgroup M of a group G is defined by

$$\nu_G(M) := \bigcap\{N : M \leq N \trianglelefteq G\}$$

with $M \leq N$ standing for "M subgroup of N" and $N \trianglelefteq G$ for "N normal subgroup of G". According to Exercise 2.D(a), (b) and Example 1.8, $\nu = (\nu_G)_{G \in \mathbf{Grp}}$ is an idempotent closure operator of the category **Grp** (with respect to the usual subobjects) which is not weakly hereditary. Normal subgroups are not only stable under intersection, but also under arbitrary joins in the subgroup lattice. Therefore ν is fully additive. It is trivially grounded and also easily seen to be productive.

(2) With the help of the normal closure, one may generate closure operators from preradicals, as in the Abelian case (see 3.4). For a *preradical* $\mathbf{r}$ in **Grp**, that is: for a subfunctor of the identity functor of **Grp**, we first observe that $\mathbf{r}(G)$ is actually a *normal* subgroup of G (since it is invariant under every endomorphism of G). For every subgroup $U \leq G$, we therefore have

$$U \vee \mathbf{r}(G) = U \cdot \mathbf{r}(G) = \mathbf{r}(G) \cdot U \ ,$$

and we can define

$$\begin{aligned} (c_{\mathbf{r}})_G(U) &:= \mathbf{r}(G) \cdot U \ , \\ c^{\mathbf{r}}_G(U) &:= \pi^{-1}(\mathbf{r}(G/\nu_G(U))) \ , \end{aligned}$$

with $\pi : G \to G/\nu_G(U)$ the canonical projection. While $(c_{\mathbf{r}})_G(U)$ is, like U, just a subgroup of G , $c^{\mathbf{r}}_G(U)$ is always normal in G , so that $\nu_G(c^{\mathbf{r}}_G(U)) = c^{\mathbf{r}}_G(U)$, and due to the idempotency of ν one also has $c^{\mathbf{r}}_G(U) = c^{\mathbf{r}}_G(\nu_G(U))$.

We can leave it to the reader to check that Proposition 3.4 remains valid for groups, so that $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ are the least and last closure operators C respectively with $c_G(\{e\}) = \mathbf{r}(G)$, with e the neutral element of G .

Of particular importance is the preradical $\mathbf{k}$ given by the *commutator subgroup*:

$$\mathbf{k}(G) = [G, G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle \ .$$

Here one has $(c_{\mathbf{k}})_G(U) = \mathbf{k}(G) \cdot U = c^{\mathbf{k}}_G(U)$, so that there is (up to isomorphism) only one closure operator C with $c_G(\{e\}) = \mathbf{k}(G)$.

(3) For a subring B of $A \in \mathbf{CRng}$, let $\mathrm{int}_A(B)$ be the *integral closure of B in A* , i.e., the set of all elements $a \in A$ for which there is a monic polynomial $p(x) \in B[x]$ with $p(a) = 0$. It is elementary to verify that int is a closure operator

of **CRng**. int *is hereditary* (since trivially $C \cap \mathrm{int}_A(B) = \mathrm{int}_C(B)$ for $B \leq C \leq A$) *and idempotent* (which follows from elementary facts on finitely generated modules; see, for example, Hungerford [1974], p.397). int *is not grounded*: in $A = \mathbb{Q}(x)/(x^2-2) = \mathbb{Q}(\sqrt{2})$, the least subobject, $\mathbb{Z}$, is int-dense.

Although **Fld** is not closed under subobjects in **CRng**, it is nevertheless possible to restrict int to **Fld**, since for fields $B \leq A$ also $\mathrm{int}_A(B)$ is a field; it coincides with the *algebraic closure of* B *in* A ; i.e., with the subfield of elements of A which are algebraic over B .

(4) Let S be a set of polynomials in $\mathbb{Z}[x]$ of positive degree. For a field F , let $K_F(S)$ be the *splitting field of* S *in* F , i.e. the subfield of F generated by the roots of polynomials from S in F . Then, for every subfield $L \leq F$, let $c_F^S(L) = L \cdot K_F(S)$. We leave it to the reader to verify that C^S *is an idempotent and weakly hereditary closure operator of* **Fld**; see Exercise 3.I. Here we just mention that, in general, C^S *is neither grounded* (by the same example as in (1)) *nor hereditary*: consider $S = \{x^4-2\}$, $L = \mathbb{Q}$, $F = \mathbb{Q}(\sqrt{2})$ and $M = \mathbb{Q}(\sqrt[4]{2})$. Then L is C^S-dense in M , but C^S-closed in F ; if C^S was hereditary, by assertion 5 of Exercise 2.F(b), we would have L also C^S-dense in F , a contradiction.

(5) We now generalize the setting of (4), by allowing S to be a set of pairs (k,f) , with a field k and a polynomial $f(x) \in k[x]$ of positive degree. Let k_f be the subfield of k generated by the coefficients of f . For a fixed field F and a subfield $L \leq F$, consider the set $\Sigma(S)$ of all triples $\alpha = (k,f,\sigma)$ with $(k,f) \in S$ and $\sigma : k_f \to L$ a field homomorphism. For each such α , we have the polynomial $\sigma(f(x)) \in L[x]$, so that we can form the splitting field $K_\alpha = K_F(\{\sigma(f(x))\})$ (see (2)). Now let $c_F^S(L)$ be the composite of L and all K_α , $\alpha \in \Sigma(S)$.

As before, C^S is a closure operator of **Fld**. To check continuity, consider $\varphi : F \to F'$ in **Fld**. For every $\alpha = (k,f,\sigma) \in \Sigma(S)$, by restriction of $\varphi \cdot \sigma$ one obtains a field homomorphism $\sigma' : k_f \to \varphi(L)$, and $\varphi(K_\alpha) \subseteq K'_F(\{\sigma'(f(x))\})$. This implies

$$\varphi(c_F^S(L)) \subseteq c_{F'}^S(\varphi(L)) .$$

It is easy to see that C^S *is weakly hereditary, but in general no longer idempotent*: consider $S = \{(k_1,f_1), (k_2,f_2)\}$ with $k_1 = \mathbb{Q}$, $k_2 = (\mathbb{Q}(\sqrt{2}))$, $f_1(x) = x^2-2$, $f_2(x) = x^2 - \sqrt{2}$; then, for $L = \mathbb{Q}$ and $F = \mathbb{Q}(\sqrt[4]{2})$, one has

$$c_F^S(L) = \mathbb{Q}(\sqrt{2}) \quad \text{and} \quad c_F^S(c_F^S(L) = F).$$

3.6 Graphs and partially ordered sets

A (*directed*) *graph* is a pair (X,E) with a set X and a subset $E \subseteq X \times X$. Elements in X are called *vertices*, while elements in E are *edges* of the graph; an edge of type (x,x) is a *loop*. We often write $x \to y$ (or $x \xrightarrow{E} y$) if $(x,y) \in E$. A morphism $f : (X,E) \to (Y,F)$ of graphs is a map $f : X \to Y$ with the property that

$$x \longrightarrow y \text{ implies } f(x) \longrightarrow f(y)$$

for all x, $y \in X$. This defines the category **Gph** of graphs.

Much of our interest in graphs arises from the fact that each filter convergence space (X, q_X) induces a graph with X as its set of vertices, as follows:

$$x \longrightarrow y :\Longleftrightarrow \dot{x} \xrightarrow{q_X} y$$

In particular, each pretopological space (X, k_X) induces a graph:

$$\begin{aligned} x \longrightarrow y &\iff \mathcal{V}_y \subseteq \dot{x} \\ &\iff (\forall N \text{ nbhd of } y)\ x \in N \\ &\iff y \in k_X(\{x\}). \end{aligned}$$

We note that graphs (X, E) arising this way have a loop for each $x \in X$, i.e. the relation E is reflexive on X. We call graphs (X, E) with this property *spatial*, for reasons which will become clear from the Corollary below. The full subcategory of spatial graphs in **Gph** is denoted by **SGph**.

Each *preordered set* $(X, \leq)$ (with $\leq$ a reflexive and transitive relation) is in particular a spatial graph. Hence the category **PrSet** of preordered sets and monotone maps is a full subcategory of **SGph**. Finally, **PrSet** contains the full subcategory **PoSet** of *partially ordered sets* $(X, \leq)$ (for which $\leq$ is also antisymmetric).

All **Gph**, **SGph**, and **PrSet** (but not **PoSet**) are topological categories over **Set** : for a family $f_i : X \to X_i$ with $(X_i, E_i) \in$ **Gph** one defines the *initial* structure on X by

$$x \to y :\Longleftrightarrow (\forall i \in I) \quad f_i(x) \to f_i(y)\ .$$

As in **FC**, this yields an easy description of substructures and direct products (also for posets). A subset $M \subseteq X$ in a graph (X, E) becomes a *subgraph*, by taking $E \cap (M \times M)$ as its set of edges. Clearly, **Gph**, **SGph**, **PrSet** and **PoSet** are $\mathcal{M}$-complete, for $\mathcal{M}$ the class of *embeddings* (of subgraphs) in the respective category.

For a graph (X, E) and a subset $M \subseteq X$ one introduces the *up-closure of* M by

$$\uparrow_X M := \{x \in X : (\exists a \in M)\, a \to x\},$$

and the *down-closure of* M by

$$\downarrow_X M := \{x \in X : (\exists a \in M)\, x \to a\}.$$

It is clear that $M \subseteq \uparrow_X M$ holds for all $M \subseteq X$ iff (X, E) is spatial; similarly for $\downarrow_X$. The other axioms for a closure operator hold even for arbitrary graphs, hence one has closure operators $\uparrow$ and $\downarrow$ of **SGph** with respect to the class of embeddings.

The *convex closure* $\mathrm{conv}_X(M)$ of a set $M \subseteq X$ in a spatial graph (X, E) is defined as the set of all vertices x in X such that there exists a finite *path* of edges

$$\cdot \to \cdot \to \cdot \ldots \cdot \to x \to \cdot \to \cdot \ldots \cdot \to \cdot$$

with both endpoints in M .

PROPOSITION

(1) $\uparrow$ *and* $\downarrow$ *are hereditary, grounded, fully additive and productive but non-idempotent closure operators of* **SGph**.

(2) conv *is a weakly hereditary, idempotent, grounded and finitely productive closure operators of* **SGph**, *but it is neither hereditary, nor additive, nor productive.*

□

Proof (1) and the positive statements of (2) are checked routinely. To see that conv is not additive, observe that in the spatial graph

$$0 \to 1 \to 2$$

both $\{0\}$ and $\{2\}$ are conv-closed, but $\{0, 2\}$ is not. (When presenting a spatial graph by a diagram, *we do not draw loops.*) In the spatial graph

$$0 \to 1 \to 2 \to 3\,,$$

for $M = \{0, 3\}$ and $Y = \{0, 2, 3\}$ one has $2 \in Y \cap \mathrm{conv}_Y(M)$, but $2 \notin Y \cap \mathrm{conv}_Y(M)$; hence conv is not hereditary. Finally, to show that conv is not productive, let (X_n, E_n) be the spatial graph

$$0 \to 1 \to \cdot \ldots \cdot \to n \to \cdot \ldots \cdot \to 2n\,.$$

For every $n \in \mathbb{N}$, $M_n = \{0, 2n\}$ is conv-dense in X_n , but $x = (x_n)_{n \in \mathbb{N}}$ with $x_n = n$ in $X = \prod_{n \in \mathbb{N}} X_n$ does not belong to $\mathrm{conv}_X(M)$, with $M = \prod_{n \in \mathbb{N}} M_n$. In fact, if

$$a \to \cdot \ldots \cdot \to x \to \cdot \ldots \cdot \to b$$

with a , $b \in M$, then necessarily $a = (0, 0, 0, \ldots)$ and $b = (2, 4, \ldots, 2n, \ldots)$. Let k be the *length* of the above path, i.e. the number of edges. Projecting it onto X_k yields a path of length at most k – a contradiction. □

For a spatial graph (X, E) , both $\uparrow_X$ and $\downarrow_X$ are pretopologies on X . These assignments are functorial, so that there are functors

$$\uparrow : \mathbf{SGph} \to \mathbf{PrTop} \quad \text{and} \quad \downarrow : \mathbf{SGph} \to \mathbf{PrTop}.$$

THEOREM

(1) *Both* $\uparrow$ *and* $\downarrow$ *give full embeddings of the category* **SGph** *as a bicoreflective subcategory of* **PrTop**.

(2) *The closure operators* $\uparrow$ *and* $\downarrow$ *of* **SGph** *are the restrictions of the Čech closure operator* K *of* **PrTop** *along the embeddings* $\uparrow$ *and* $\downarrow$ *, respectively.*

(3) *The following are equivalent for* $(X, E) \in$ **SGph** *:*

(i) $\uparrow_X$ *is idempotent,*

(ii) $(X, \uparrow_X) \in$ **Top**,

(iii) $\downarrow_X$ *is idempotent,*

(iv) $(X, \downarrow_X) \in$ **Top**,

(v) E *is transitive,*

(vi) $(X, E) \in$ **PrSet**.

(4) *The following are equivalent for* $(X, E) \in$ **PrSet** *:*

(i) *each vertex of the graph* (X, E) *is* conv-*closed,*

(ii) $(X, \uparrow_X)$ *is a* T_0*-space,*

(iii) $(X, \downarrow_X)$ *is a* T_0*-space,*

(iv) (X, E) *is a partially ordered set.*

Proof

(1) By repeated use of $(x \to y \Leftrightarrow y \in\uparrow \{x\})$ one shows that $\uparrow$ is a full embedding. Its coreflector is given by the underlying graph of a pretopological space (X, k_X) , as described before:

$$x \xrightarrow{k} y \Leftrightarrow y \in k_X\{x\}.$$

Indeed, we then have that $id_X : (X\ ,\ \uparrow^k_X) \to (X, k_X)$ is continuous, i.e., $\uparrow^k_X M \subseteq k_X(M)$ for every $M \subseteq X$ (since $x \in\uparrow^k_X M$ means $a \xrightarrow{k} x$ for some $a \in M$, hence $x \in k_X\{a\} \subseteq k_X(M)$). One also easily verifies that, for any spatial graph (Y, F) and its induced pretopology $\uparrow_Y$, every continuous map $f : (Y, \uparrow_Y) \to (X, k_X)$ gives a morphism $f : (Y, F) \to (X, \uparrow^k_X)$ of graphs.

This proves the claim for $\uparrow$ which, by dualization, also proves the claim for $\downarrow$: there is an isomorphism of categories $* :$ **SGph** $\to$ **SGph** which sends (X, E) to its *opposite graph* (X, E^{-1}) , and $\downarrow = \uparrow *$.

(2) Apply Proposition 2.8.

(3) is trivial. (Of course, **Top** is being thought of as being embedded into **PrTop**, according to 3.1.)

(4) Statement (i) means $\{x\} =$ conv$(\{x\})$ for all $x \in X$; that is: the existence of a path $x \to \cdot \ldots \cdot \to y \to \cdot \ldots \cdot \to x$ yields $y = x$. In the presence of transitivity,

this simply means that E is antisymmetric. Hence (i) $\Leftrightarrow$ (iv). For the equivalence of these statements with both (ii) and (iii) one just needs to recall that a topological space X is T_0 if and only if $k_X(\{x\}) = k_X(\{y\})$ implies $x = y$. □

Let us call a pretopological space (X, k_X) *Alexandroff* if k_X satisfies condition (FA) of 2.6, i.e.

$$k_X(\bigcup_{i\in I} M_i) = \bigcup_{i\in I} k_X(M_i)$$

for every non-empty family of subsets M_i of X . Hence we have the full subcategory **PrAlex** of **PrTop**, and (with the notation introduced in 2.6), $\mathbf{Alex} = \mathbf{PrAlex} \cap \mathbf{Top}$. By $\mathbf{Top}_0$ we denote the full subcategory of T_0-spaces in **Top**; and $\mathbf{Alex}_0 = \mathbf{Alex} \cap \mathbf{Top}_0$.

COROLLARY *There are isomorphisms of categories*

$$\mathbf{PrAlex} \cong \mathbf{SGph}\,, \qquad \mathbf{Alex} \cong \mathbf{PrSet}\,, \qquad \mathbf{Alex}_0 \cong \mathbf{PoSet}\,.$$

Proof It suffices to show that the bicoreflective modification

$$id_X : (X, \uparrow^k_X) \to (X, k_X)$$

of a pretopological space is an isomorphism if and only if k_X satisfies (FA). Since $\uparrow$ is fully additive, "only if" is clear. On the other hand, if (FA) holds for k_X , one has

$$k_X(M) = \bigcup_{a\in M} k_X(\{a\})$$

for all $M \subseteq X$. Since $k_X(\{a\}) = \uparrow^k_X (\{a\})$ for all $a \in M$, this means $k_X(M) = \uparrow^k_X (M)$ for all $M \subseteq X$. □

We may summarize the relations between subcategories as given in this section, as follows:

$$
\begin{tikzcd}
 & \mathbf{PrTop} & & \\
\mathbf{SGph} \cong \mathbf{PrAlex} \arrow[ur, "bi-c"] & & \mathbf{Top} \arrow[ul, "bi-r"'] & \\
 & \mathbf{PrSet} \cong \mathbf{Alex} \arrow[ul, "bi-r"] \arrow[ur, "bi-c"] & & \mathbf{Top}_0 \arrow[ul, "epi-r"'] \\
 & & \mathbf{PoSet} \cong \mathbf{Alex}_0 \arrow[ul, "epi-r"] \arrow[ur, "bi-c"'] &
\end{tikzcd}
$$

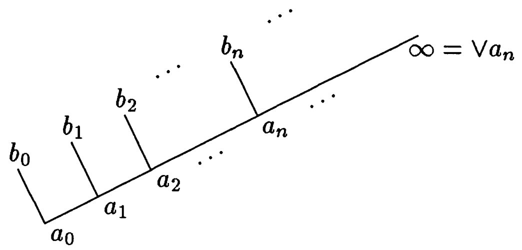
Hence there are full reflective and coreflective embeddings as indicated: see also 3.1 and Exercises 3.A, 3.J, and 3.L.

3.7 Directed-complete posets and Scott closure

A partially ordered set $(X, \leq)$ is *directed-complete* if every directed subset $A \subseteq X$ has a supremum in X, thus $\bigvee A$ exists in X whenever $A \neq \emptyset$ and for all x, $y \in A$ there is $z \in A$ with $z \geq x$ and $z \geq y$. The morphisms of the category **DCPO** of directed-complete posets (dcpos, for short) are given by maps $f : X \to Y$ which preserve directed joins, i.e., $f(\bigvee A) = \bigvee f(A)$ for every directed subset A of X. Note that such a map is necessarily monotone, and $f(A)$ is directed whenever A is. A subdcpo M of $(X, \leq)$ is given by a subset which is closed under directed joins. With the corresponding class $\mathcal{M}$ of embeddings, **DCPO** is $\mathcal{M}$-complete. (We remark that some authors require a dcpo to have a bottom element, without insisting on its preservation by morphisms. However, they assume then the bottom element of a subdcpo to coincide with the bottom element of the given dcpo. The severe disadvantage of these assumptions is that the resulting category has neither equalizers nor pullbacks of subdcpos. Not to insist on the existence of a bottom element does not prevent us from recognizing their importance. In fact, all our examples of dcpos either have bottom elements or can easily be modified to this extent.)

Since **DCPO** is a non-full (!) subcategory of **PoSet** one is tempted to investigate the up- and down-closures of **PoSet** in the new environment. For a subdcpo M of X, the up-closure $\uparrow_X M = \{x \in X : (\exists a \in M) a \leq x\}$ is indeed a subdcpo (since for every non-void subset $A \subseteq \uparrow M$ with existing join X, this join belongs to $\uparrow\uparrow M = \uparrow M$). However the down-closure $\downarrow_X M = \{x \in X : (\exists a \in M) x \leq a\}$ may fail to be closed under directed joins.

EXAMPLE Provide $L = \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\} \cup \{\infty\}$ with the following partial order:



Then X is a dcpo and $M = \{b_n | n \in \mathbb{N}\}$ a subdcpo. However, the directed set $\{a_n | n \in \mathbb{N}\}$ has no join in $\downarrow M = X \setminus \{\infty\}$.

Of course, every subset Z of a dcpo X may be closed under directed joins, i.e., $\mathrm{dir}_X\, Z = \{x \in X : (\exists A \subseteq Z \text{ directed })\bigvee A = x\}$ is a subdcpo (check!). We may now define the *up-directed down-closure* $(\mathrm{dir} \downarrow)_X(M)$ of a subdcpo M of X as $\mathrm{dir}_X(\downarrow_X M)$.

PROPOSITION *The up-directed down-closure* $\mathrm{dir} \downarrow$ *is a weakly hereditary, grounded, additive and productive closure operator of* **DCPO** . *It is neither hereditary, nor idempotent, nor fully additive.*

Proof $\mathrm{dir} \downarrow$ is certainly extensive and monotone. For every map $f : X \to Y$ of dcpos and every directed subset $A \subseteq \downarrow M$ with a subdcpo M , also $f(A) \subseteq f(\downarrow M) \subseteq \downarrow f(M)$ is directed and $f(\bigvee A) = \bigvee f(A)$. Hence $f(\mathrm{dir} \downarrow M) \subseteq \mathrm{dir} \downarrow f(M)$, thus the continuity condition holds.

To show that $\mathrm{dir} \downarrow$ is weakly hereditary, consider $x \in \mathrm{dir}_X \downarrow_X M =: Y$. Then $x = \bigvee A$ for a directed subset $A \subseteq \downarrow_X M$. But since $\downarrow_X M = \downarrow_Y M \subseteq Y$, this shows immediately $x \in \mathrm{dir}_Y \downarrow_Y M$, as desired. Groundedness of $\mathrm{dir} \downarrow$ is obvious since $\mathrm{dir} \downarrow \emptyset = \emptyset$.

In order to establish additivity of $\mathrm{dir} \downarrow$, we first show

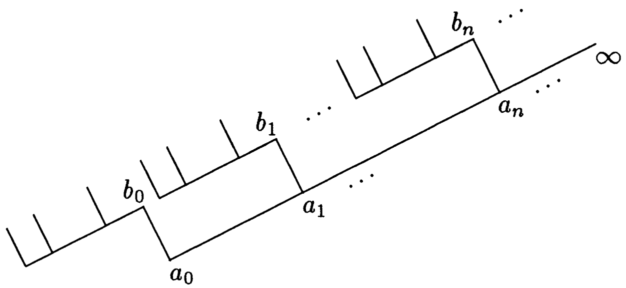
$$\mathrm{dir}\,(Z \cup W) = \mathrm{dir}\, Z \cup \mathrm{dir}\, W \tag{$*$}$$

for arbitrary subsets Z , W of a dcpo X . (In fact, here it suffices that X is just a poset). Since "$\supseteq$" is obvious, we need to show only the inclusion "$\subseteq$" of $(*)$. Hence consider $x = \bigvee A$ for a directed subset $A \subseteq Z \cup W$. For each $a \in A$ the set $A_a := A \cap \uparrow a$ is directed and *cofinal in* A (so that for every $b \in A$ there is $c \in A_a$ with $c \geq b$), hence $x = \bigvee A_a$. Under the assumption $x \notin \mathrm{dir}\, Z$, we then know that no A_a is contained in Z . This means that for every $a \in A$ there is $b \geq a$ in A with $b \notin Z$, hence $wb \in W$. Consequently, $A \cap W$ is cofinal in A and $A \cap W$ is directed. Therefore $x = \bigvee(A \cap W) \in \mathrm{dir}\, W$.

Since for a subdcpo M of X one has $\mathrm{dir}\, M = M$, formula $(*)$ shows that the join of subdcpos M and N is simply given by $M \cup N$. Hence additivity of dir $\downarrow$ follows immediately from $(*)$ and the additivity of $\downarrow$ in **PoSet**.

Checking productivity is a routine matter that can be left to the reader. To see that dir $\downarrow$ is not hereditary one may return to the "ladder" L of the Example and consider $M = \{b_n : n \in \mathbb{N}\}$ and $N = M \cup \{\infty\}$. Then $\infty \in N \cap \mathrm{dir}_L \downarrow_L M$, but $\infty \notin \mathrm{dir}_N \downarrow_N M$.

In order to show non-idempotency of dir $\downarrow$, we again consider the ladder L and glue to each b_n a new copy $L^{(n)}$ of L , identifying $\infty \in L^{(n)}$ with $b_n \in L$. This defines a dcpo L_2 which contains $L_1 := L$ as subdcpo.



Now let M_2 be the set of all maximal points of L_2 , with the exception of b_0 , $b_1, \ldots,$ and ∞ . This is a subdcpo of L_2 with

$$\mathrm{dir} \downarrow M_2 = L_2 \setminus (\{a_n : n \in \mathbb{N}\} \cup \{\infty\}) \neq L_2 = \mathrm{dir} \downarrow (\mathrm{dir} \downarrow M_2) .$$

Finally, in order to show that dir $\downarrow$ is not fully additive, first we remark that the join of subdcpos N_n $(n \in \mathbb{N})$ is given by dir $(\bigcup N_n)$. Now consider the dcpo $L' = L \cup \{c\}$ with $c < \infty$ but c incomparable to any other element in L , and let $N_n := \{a_i , b_i : i \leq n\}$. This defines an ascending chain of subdcpos $N_n = \mathrm{dir} \downarrow N_n$ of L' whose join is L . However, $\mathrm{dir} \downarrow (\bigvee N_n) = \mathrm{dir} \downarrow L = L'$.

□

DEFINITION A subset M of a dcpo X is called *Scott-closed* if it is down-closed ($\downarrow M = M$) and closed under directed joins ($\mathrm{dir} M = M$) . This is the same as to say that M is a (dir $\downarrow$)-closed subdcpo of X . (If $M = \mathrm{dir} \downarrow M$, then $\downarrow M \subseteq \mathrm{dir} \downarrow M = M$ and $\mathrm{dir}\, M \subseteq \mathrm{dir} \downarrow M = M$.)

THEOREM (1) *The Scott-closed subsets of a dcpo X form a topology on X , making X a topological space* Scott$X$.

(2) *The Scott closure* $\mathrm{scott}_X(M)$ *of a subdcpo M of X is the Kuratowski closure of M in* Scott$X$, *i.e.* $\mathrm{scott}_X(M) := k_{\mathrm{Scott}X}(M)$. *This defines an idempotent, weakly hereditary, grounded and additive closure operator of* **DCPO** *which is neither hereditary nor fully additive. Closedness with respect to* scott *means Scott-closed.*

(3) Scott : **DCPO** $\to$ **Top** *is a full and faithful functor which commutes with the underlying* **Set***-functors.*

Proof (1) Scott-closedness is stable under finite union since dir $\downarrow$ is additive. It is trivially stable under arbitrary intersection.

(2) We check the continuity condition for scott. For $f : X \to Y$ in **DCPO** and a subdcpo M of X, one has

$$\begin{aligned} f(\mathrm{scott}_X(M)) &= f(\bigcap\{N : M \subseteq N \subseteq X\,,\ N = \mathrm{dir}\downarrow N\}) \\ &\subseteq \bigcap\{f(N) : M \subseteq N \subseteq X\,,\ N = \mathrm{dir}\downarrow N\} \\ &= \bigcap\{L : f(M) \subseteq L \subseteq Y\,,\ L = \mathrm{dir}\downarrow L\} \\ &= \mathrm{scott}_Y(f(M))\,, \end{aligned}$$

since $N = \mathrm{dir}\downarrow N$ implies $f(N) \subseteq \mathrm{dir}\downarrow f(N)$, see the Proposition. Idempotency, groundedness and additivity of scott in **DCPO** follow immediately from the respective properties of K in **Top**. For weak hereditariness of scott one evokes the respective property of $\mathrm{dir}\downarrow$. Finally, scott is neither hereditary nor fully additive, for the same reason as $\mathrm{dir}\downarrow$, see the Proposition.

(3) The continuity condition for scott means that every map $f : X \to Y$ in **DCPO** gives a map $f : \mathrm{Scott}X \to \mathrm{Scott}Y$ in **Top**. Hence Scott becomes a functor which, at the **Set** level, maps identically. Its fullness follows from Exercise 3.P. □

An element c of a dcpo X is called *compact* (or *finite*) if $c \leq \bigvee A$ for a directed subset A of X implies $c \leq a$ for some $a \in A$. The dcpo X is called *algebraic* if every element $x \in X$ is the join of the compact elements below x, that is:

$$x = \bigvee\{c \in X \ : \ c \leq x\,, \quad c \text{ compact}\}.$$

(Note that this join is directed since finite joins of compact elements are compact.) An algebraic dcpo X in which every pair of elements with an upper bound actually has a least upper bound is called a *domain*. These structures play an important role in Theoretical Computer Science. Although we must leave it to the interested reader to study the closure operators $\mathrm{dir}\downarrow$ and scott in the realm of domains, we observe that the crucial examples $L = L_1$, L_2 and L' used in the proof of the Proposition are actually domains.

Exercises

3.A **(PrTop** *as a topological category***)**

(a) For a family of **Set**-maps $f_i : X_i \to X$ with $(X_i\,,\ k_i) \in$ **PrTop**, show that

$$k_X(M) = \bigcup_{i \in I} f_i\left(k_i(f_i^{-1}(M))\right)$$

gives a pretopology on X with the property: a **Set**-map $g : X \to Y$ with $(Y, k_Y) \in$ **PrTop** belongs to **PrTop** iff each composite $g \cdot f_i$ does.

(b) Conclude that the forgetful functor $U : \mathbf{PrTop} \to \mathbf{Set}$ is *topological*. (Readers not familiar with topological functors are referred to Exercise 5.P). In particular, **PrTop** is complete and cocomplete.

(c) Verify that **Top** is bireflectively embedded into **PrTop**.

3.B (*Direct products in* **PrTop**) For a family (X_i, k_i) , $i \in I$, of pretopological spaces, let $X = \prod_{i \in I} X_i$ and $p_i : X \to X_i$ be the i-th projection. For a subset $M \subseteq X$, let $\mathcal{A}_M$ be the set of families $(A_i)_{i \in F}$ with $F \subseteq I$ finite, $A_i \subseteq X_i$ for each $i \in F$, and

$$M \subseteq \bigcup_{i \in F} p_i^{-1}(A_i) .$$

Finally, let $k_X(M)$ be the set of all $x \in X$ with the property:

$$(\forall (A_i)_{i \in F} \in \mathcal{A}_M)(\exists i \in F)\ x \in p_i^{-1}(k_i(A_i)) .$$

(a) Show that k_X is a pretopology on X .

(b) Prove that (X, k_X) , together with the projections p_i , gives a direct product in the category **PrTop** .

(c) Conclude that the Čech closure operator K of **PrTop** is productive.*Hint*: For $M_i \subseteq X_i$, $i \in I$, first consider the case $I = \{1,2\}$ and show $k_1(M_1) \times k_2(M_2) \subseteq k_X(M)$, hence K is finitely productive. Then use finite productivity to show $\prod_{i \in I} k_i(M_i) \subseteq k_X(M)$ in the general case.

(d) Show that **Top** is closed under direct products in **PrTop**, by showing $k_X k_X = k_X$ whenever $k_i k_i = k_i$ holds for all $i \in I$.

3.C (*House-keeping in* **FC**)

(a) Verify the claims that **FC** is a topological category over **Set** (see Exercise 3.A) and that **PrTop** is bireflectively embedded into **FC**.

(b) Show for the Katětov closure operator K of **FC** and every $M \subseteq X \in \mathbf{FC}$:

$$k_X(M) = \{x \in X : (\exists \mathcal{F})(\mathcal{F} \xrightarrow{q_X} x \text{ and } (\forall F \in \mathcal{F})\ M \cap F \neq \emptyset)\} .$$

3.D (θ*-modification and* θ*-closure in* **FC**)

(a) Show that for $(X, q_X) \in \mathbf{FC}$ one defines a new convergence structure θq_X on X , as follows:

$$\mathcal{F} \xrightarrow{\theta q_X} x :\Longleftrightarrow (\exists \mathcal{G})\ \mathcal{G} \xrightarrow{q_X} x \text{ and } \mathcal{F} \supseteq k_X(\mathcal{G}) ;$$

here $k_X(\mathcal{G})$ denotes the filterbase (check!) $\{k_X(G) : G \in \mathcal{G}\}$, with k_X the Katětov closure operator.

(b) Prove that the assignment $(X, q_X) \mapsto \Theta X := (X, \theta q_X)$ is functorial, i.e. yields a functor $\mathbf{FC} \to \mathbf{FC}$.

(c) Show that

$$\theta_X(M) := k_{\Theta X}(M)$$

is an additive closure operator of **FC** (with respect to the class of embeddings)

(d) Prove that the θ-closure of topological spaces (as defined in 3.3) is a restriction of θ in **FC**, subject to the embedding $\mathbf{Top} \to \mathbf{FC}$. Conclude that the θ-closure in **FC** is neither weakly hereditary nor idempotent (see 3.F below).

3.E (*b-closure*)

(a) Prove that b is hereditary, grounded, additive and productive, but not fully additive.

(b) Prove (2)(b) of Proposition 3.3. *Hint*: If $b_X = k_X$, using hereditariness of b_X , show that point-closures $Y = k_X(\{x\})$ are indiscrete.

(c) Show that, in general, $b_X(M)$ is not comparable with $\sigma_X(M)$ by inclusion. *Hint*: Show $\sigma_X(M) \not\subseteq b_X(M)$ for every non-discrete metrizable space X . For $b_X(M) \not\subseteq \sigma_X(M)$, consider the space X of ordinals $\leq \omega_1$ (the first uncountable ordinal), provided with the following topology: basic neighbourhoods of ω_1 are intervals $(\alpha, \omega_1]$, $\alpha < \omega_1$, and the least neighbourhood of $\alpha < \omega_1$ is $[\alpha, \omega_1]$. Then $X = k_X(\{\omega_1\})$. For $M = X \setminus \{\omega_1\}$, $\omega_1 \in b_X(M)$. On the other hand, M is σ-closed since no sequence (α_n) with $\alpha_n < \omega_1$ converges to ω_1 .

3.F (*Bad properties of the θ-closure in* **Top**)

(a) Show that the θ-closure of **Top** is neither idempotent nor weakly hereditary. *Hint*: To disprove idempotency, provide $X = \{x, a, b\}$ with the topology given by the open sets $\emptyset$, $\{a\}$, $\{b\}$, X , and consider the θ-closure of $\{a\}$. To disprove weak hereditariness, consider the unit interval $X = [0,1]$ equipped with the topology in which points $x \neq 0$ have the usual neighbourhoods while a typical neighbourhood of 0 is of the form $[0, \varepsilon) \setminus F$ with $0 < \varepsilon \leq 1$ and $F = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\theta_X(F) = F \cup \{0\}$ is discrete, hence $0 \notin \theta_{\theta_X(F)}(F)$.

(b) Prove that θ_X satisfies the weak hereditariness condition (WH) for every finite topological space X .

3.G (*Behaviour of b , σ , θ and $\mathfrak{k}$ with respect to full additivity*) Prove:

(a) For every Alexandroff space X (see 2.6), all b , σ , θ and $\mathfrak{k}$ are fully additive.

(b) b is fully additive for a space X iff X is Alexandroff; the corresponding statement for $\mathfrak{k}$ holds true for X Hausdorff.

(c) θ is fully additive for a Hausdorff space X iff X is discrete; there is a non-discrete T_1-space X for which θ is fully additive but K is not (i.e., X is not Alexandroff).

(d) None of b, σ, θ and $\mathfrak{k}$ is fully additive.

3.H (*Properties of maximal closure operators*)

(a) Prove all claims on the maximal closure operators induced by $\mathbf{t}$, $\mathbf{d}$, and $\mathbf{f}$ as given in 3.3.

(b) Show that every hereditary maximal closure operator of $\mathbf{Mod}_R$ is directedly additive. *Hint:* For $C = C^{\mathbf{r}}$ with $\mathbf{r}$ hereditary, show that $x \in M$ belongs to $c_M(N)$ if and only if $x \in c_{Rx}(N \cap Rx)$; here Rx is the cyclic submodule generated by x .

3.I (*Properties of* C^S)

(a) For $S \subseteq \mathbb{Z}[x]$, show that C^S is an idempotent and weakly hereditary closure operator of $\mathbf{Fld}$ (cf. 3.5 (2)).

(b) Let S be as in 3.5 (3), but assume that k_f contains all roots of $f(x)$ in k , for every $(k, f) \in S$. Prove that C^S is hereditary.

3.J (*Some reflections and coreflections*)

(a) Show that $\mathbf{PrSet} \to \mathbf{SGph}$ and $\mathbf{SGph} \to \mathbf{Gph}$ are full bireflective embeddings, and that $\mathbf{PoSet} \to \mathbf{PrSet}$ is epireflective. *Hint*: Every relation E on a set has a "transitive hull". For a preorder $\leq$ on X , consider $X/\cong$ for $(x \cong y \Leftrightarrow x \leq y$ and $y \leq x)$.

(b) Verify that $\mathbf{Top}_0$ $(\mathbf{Alex}_0)$ is epireflective in $\mathbf{Top}$ $(\mathbf{Alex}$, respectively) . *Hint*: For a topological space X , consider $X/\sim$ for $(x \sim y \Leftrightarrow k_X(\{x\}) = k_X(\{y\}))$.

(c) Prove that $\mathbf{Alex}$ $(\mathbf{Alex}_0)$ is bicoreflective in $\mathbf{Top}$ $(\mathbf{Top}_0$, respectively) . *Hint*: The bicoreflector of $\mathbf{SGph} \cong \mathbf{PrAlex} \to \mathbf{PrTop}$ can be restricted accordingly.

3.K (*Undirected graphs*) Characterize the spatial graphs (X, E) for which E is an equivalence relation (a) in terms of the operators $\uparrow_X$, $\downarrow_X$, and conv_X, and (b) by topological properties of the space $(X, \downarrow_X)$ (see Theorem 3.6).

3.L (*Why many coreflective categories are bicoreflective*)

(a) Let $\mathcal{Y}$ be an epicoreflective subcategory $\mathcal{X}$ (so that the coreflexions $\rho_X : RX \to X$ are epic for every $X \in \mathcal{X}$). Show that each ρ_X is also monic, so that $\mathcal{Y}$ is bicoreflective in $\mathcal{X}$.

(b) Prove that a coreflective subcategory of **Top** either contains only the empty space, or is bicoreflective. If $\mathcal{Y}$ is replete, then RX can be chosen to have the same underlying set as X, with ρ_X the identity map in **Set**; RX is called the *coreflective modification of* X . *Hint:* If there is a non-empty space $Y \in \mathcal{Y}$, consider constant maps $Y \to X$ to show that ρ_X is surjective.

(c) Formulate conditions which allow to generalize (b) from $\mathcal{X} = \mathbf{Top}$ to an arbitrary concrete category $\mathcal{X}$ (which comes equipped with a faithful functor $U : \mathcal{X} \to \mathbf{Set}$).

3.M (*Minimality versus (full) additivity*)

(a) Find examples in **Top** (with $\mathcal{M}$ the class of embeddings) which show that for each of the implications

$$C \text{ minimal } \Rightarrow C \text{ fully additive } \Rightarrow C \text{ additive}$$

the converse does not hold true in general.

(b) Prove that in $\mathbf{Mod}_R$ every additive closure operator is minimal (hence here the implications of (a) are reversible), while there exist directedly additive non-minimal (hence non-additive) closure operators. *Hint:* For a submodule N of M , consider the submodule $(N \times N) + \Delta_M$ of $M \times M$ and use Exercise 2.J. For the second part consider the closure operator $C^{\mathbf{t}}$ of **AbGrp**.

(c) Find an example of a fully additive closure operator of **Grp** which is not minimal.

3.N (*Some non-examples in* **Grp** *and* **PoSet**)

(a) For a subgroup $H \leq G$, let $N_G(H) = \{g \in G : gH = Hg\}$ be the *normalizer of* H *in* G . Show that $H \mapsto N_G(H)$ satisfies the axioms of a closure operator (w.r.t. $\mathcal{M} = \{\text{monomorphisms}\}$) , *except* monotonicity.

(b) Show that the following two constructions *fail* to give preradicals of **Grp** (w.r.t. $\mathcal{M} = \{\text{monomorphisms}\}$): $Z(G) = \{g \in G : (\forall x \in G)\ gx = xg\}$, the *centre of* G ; $\mathbf{f}(G)$, the *Frattini subgroup of* G (to be defined as in the abelian case, see Example 3.4(3)). Show that both constructions *are* functorial with respect to *surjective* group homomorphisms.

(c) For every subset M of a partially ordered set X , let $\mathrm{dir}_X(M)$ be the set of elements $x \in X$ for which there is a directed subset $D \subseteq M$ with $\bigvee D = x$. Show that this construction satisfies the conditions of extension and monotonicity for a closure operator of **PoSet** (w.r.t. order embeddings) as well as conditions (ID), (HE), (AD), but that the continuity condition fails.

3.O (*Recognizing total preorders*) Recall that a preorder $\leq$ on X is *total* if any two elements in X are comparable w.r.t. $\leq$. A subset of a preordered set is a *chain* if it is total w.r.t. the induced preorder. For a preordered set $(X, \leq)$, let

$\uparrow(X,\leq)$ be the set of $\uparrow$-closed subsets of X (hence $\uparrow(X,\leq)$ gives a skeleton of $\mathcal{M}^{\uparrow}/(X,\leq)$, while the entire powerset $P(X)$ gives a skeleton of $\mathcal{M}/(X,\leq)$). Prove:

(a) A preorder $\leq$ on X is total if and only if $\uparrow(X,\leq)$ is a chain in $(P(X),\subseteq)$.

(b) A preorder $\leq$ on X is a total order if and only if $\uparrow(X,\leq)$ is a maximal chain in $(P(X),\subseteq)$.

(c) For every chain $\mathcal{C}$ in $(P(X),\subseteq)$, there exists a total preorder $\leq_{\mathcal{C}}$ on X with $\uparrow(X,\leq_{\mathcal{C}})=\mathcal{C}$. (*Hint:* Define $x\leq_{\mathcal{C}}y$ iff $x\in D$ implies $y\in D$ for all $D\in\mathcal{C}$. Then $\uparrow x=\{D\in\mathcal{C}:x\in D\}$ for all $x\in X$.)

(d) For every set X , show that the assignments

$$\psi:\leq\;\mapsto\;\uparrow(X,\leq)\qquad\text{and}\qquad\varphi:\mathcal{C}\mapsto\;\leq_{\mathcal{C}}$$

define a Galois correspondence between preorders on X and subsets of $P(X)$, that is:

$$\varphi\dashv\psi:(\{E\subseteq X\times X: E\text{ preorder on }X\}\,,\,\subseteq)\longrightarrow(P\,P(X)\,,\,\subseteq)^{op}$$

is a pair of adjoint maps (see 1.3). Characterize the preorders $\leq$ on X with $\varphi(\psi(\leq))=\;\leq$ and the subsets $\mathcal{C}\subseteq P(X)$ with $\psi(\varphi(\mathcal{C}))=\mathcal{C}$.

(e) Show that there is a bijective correspondence between total orders on a set X and maximal chains in $(P(X),\subseteq)$

3.P (*Specialization order and Scott topology*)

(a) For a topological space X , define the *specialization preorder* of X by ($x\leq y\Leftrightarrow x\in k_X(y)$ "y specializes x") . Prove that this defines a functor S : **Top** $\to$ **PrSet** which is the coreflector of the embedding $\downarrow$: **PrSet** $\to$ **Top** .

(b) For every preordered set $(X,\leq)$, there is a finest topology on X whose specialization preorder is $\leq$. This topology is given by the *Alexandroff space* $(X,\downarrow_X)$. There is also a coarsest topology X whose specialization preorder is $\leq$, the so-called *upper-interval topology*; a base of its closed sets is given by $\{\downarrow\{x\}:x\in X\}$.

(c) For a dcpo X , show $S(\mathrm{Scott}X)=X$. Conclude that $\mathrm{Scott}X$ is a T_0-space. (*Hint:* Observe that for every $y\in X$, $\downarrow\{y\}=\mathrm{dir}\downarrow\{y\}=\mathrm{scott}\{y\}$.)

(d) Show that for a morphism $f:(X,\leq)\to(Y,\leq)$ in **PrSet**, f need not be continuous with respect to the upper-interval topology. Conclude that "the upper-interval topology is not functorial."

3.Q (*Productivity of Scott*) Is the Scott closure operator productive?

3.R (*Cartesian closedness of* **Alex**)

(a) Show that the category **Alex** is cartesian closed, i.e., find for all spaces $X, Y \in$ **Alex** a space $Y^X \in$ **Alex** and a continuous map $e : Y^X \times X \to Y$ such that every other continuous map $f : Z \times X \to Y$ (with $Z \in$ **Alex**) factors as $f = e \cdot (h \times 1_X)$ for a unique continuous map $h : Z \to Y^X$. *Hint*: Find a suitable topology on the set of continuous maps from X to Y, or work in the category **PrSet**.

(b) Show that for every $X \in$ **Alex** the functor $(-) \times X :$ **Alex** $\to$ **Alex** preserves all limits and colimits.

3.S (*Modification of* $\mathfrak{k}$*-closure*) For a topological space X and $M \subseteq X$ set $\mathfrak{c}_X(M) = \bigcup\{k_X(M \cap k_X(B)) | B \subseteq X \text{ compact}\}$. Show:

(a) For X Hausdorff, $\mathfrak{c}_X = k_X$ iff X is a k-space.

(b) $\mathfrak{c}$ is a closure operator of **Top** which is weakly hereditary for Hausdorff spaces, but not hereditary.

(c) $\mathfrak{c}$ is productive.

Hint: (a) If X has the property that every compact subspace is closed, in particular if X is Hausdorff, the k-space property means that $\mathfrak{c}$-closed sets in X are closed, and the latter property translates into $k_X(M) \subseteq k_X(\mathfrak{c}_X(M)) = \mathfrak{c}_X(M)$. (c) As in the proof of Theorem 3.3, let $x_i \in \mathfrak{c}_{X_i}(M_i)$ for all $i \in I$, hence $x_i \in k_{X_i}(M_i \cap k_{X_i}(B_i))$ for compact sets $B_i \subseteq X_i$, $i \in I$. Since K is productive,

$$x = (x_i)_{i \in I} \in A = k_X\left(\prod_{i \in I}(M_i \cap k_{X_i}(B_i))\right) .$$

By Tychonoff's Theorem, $B = \prod_{i \in I} B_i$ is compact, hence $k_X(M \cap k_X(B)) \subseteq \mathfrak{c}_X(M)$. Since $k_X(B) = \prod_{i \in I} k_{X_i}(B_i)$, one has $M \cap k_X(B) = \prod_{i \in I}(M_i \cap k_{X_i}(B_i))$. Hence $x \in A \subseteq \mathfrak{c}_X(M)$.

3.T (*Some topological concepts based on closure*) For the category **Top** and its Kuratowski closure operator K, and for every continuous map $f : X \to Y$, show the following properties:

(a) f is a closed map if and only if $f(k_X(M)) = k_Y(f(M))$ for all $M \in X$.

(b) f is an open map if and only if $f^{-1}(k_Y(N)) = k_X(f^{-1}(N))$ for all $N \in Y$.

(c) f is a local homeomorphism if and only if f and the induced diagonal map $\delta : X \to X \times_Y X = \{(x, x') : f(x) = f(x')\} \subseteq X \times X$ are open.

(d) X is Hausdorff if and only if the diagonal map $\delta : X \to X \times X$ is closed.

(e) X is irreducible (that is if $X = F \cup G$ with F, G closed subsets of X, then either $F = X$ or $G = X$) if and only if the diagonal map of (d) is dense.

(f) X is discrete if and only if the diagonal map of (d) is a local homeomorphism.

3.U (*Metric spaces*) Metric spaces form the objects of the category **Met** whose morphisms $f : (X, d) \to (Y, d')$ are non-expansive maps, i.e., $d'(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$.

(a) Prove that the class $\mathcal{M}$ of isometries (those f with $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$) makes **Met** an $\mathcal{M}$-complete category, and that **Met** has finite products.

(b) For every fixed real $\varepsilon > 0$ and every $M \subseteq X \in \mathbf{Met}$, define

$$\rho_X^\varepsilon(M) := \{x \in X : \mathrm{dist}(x, M) \leq \varepsilon\},$$

with $\mathrm{dist}(x, M) = \inf\{d(x, y) : y \in M\}$. Show that ρ^ε is a non-idempotent but hereditary, grounded, fully additive and finitely productive closure operator of **Met** w.r.t. $\mathcal{M}$.

(c) For all $M \subseteq X \in \mathbf{Met}$, show

$$\bigcup_{\varepsilon > 0} \rho_X^\varepsilon(M) = X \text{ if } M \neq \emptyset, \text{ and } \bigcap_{\varepsilon > 0} \rho_X^\varepsilon(M) = \overline{M},$$

with $\overline{M} = k_X(M)$ the topological (Kuratowski-) closure of M in X.

Notes

The category **FC** contains the cartesian closed topological category of all *Choquet* spaces or pseudotopological spaces as a full subcategory, and **PrTop** is still fully contained in that subcategory: see Bentley and Lowen [1992] for details. We refer also to Bentley, Herrlich and Lowen-Colebunders [1990] for further reading on generalized convergence structures. Chapters 6-8 contain an abundance of closure operators of Top which make the four given examples of 3.3 look like chosen rather arbitrarily; however, these four are most frequently used as building blocks for new operators. Moreover, the Kuratowski closure operator can be characterized as the only non-trivial hereditary and additive closure operator of the subcategory of T_0-spaces with "good behaviour" on products (called *finite structure property*, see 4.11); this and further characterizations are given in Dikranjan, Tholen and Watson [1995]. Readers interested in (pre)radicals of modules and Abelian groups should consult Fuchs [1970] and Bican, Jambor, Kepka and Nemec [1982]. Finally, we note that domain theory is still a fast-growing branch in the interface between mathematics and computer science. Almost any of the many expository articles and texts will inspire the reader to establish new closure operators in the categories in question.

4 Operations on Closure Operators

Despite the powerful continuity condition, the notion of closure operator is very general. It is therefore important to provide tools for improving a given operator. Fortunately, there is a natural lattice structure for closure operators that allows us to distinguish between properties stable under meet (idempotency, hereditariness, productivity), and those stable under join (weak hereditariness, minimality, additivity). Hence it is clear that each closure operator has an idempotent hull and a weakly hereditary core, and analogously for the other properties. The passage to the hull w.r.t. to a meet-stable property will normally not destroy already existing join-stable properties, and the passage to the core w.r.t. to a join-stable property will normally preserve meet-stable properties.

In order to show this, it is advantageous to have computationally accessible constructions at hand. In the case of the idempotent hull and the weakly hereditary core, they are provided by the composition and cocomposition of closure operators. The computation of the additive core and especially of the fully additive core and the minimal core is fairly easy, while determining the hereditary hull requires more effort and restrictive conditions on the category.

We do not describe (finitely) productive hulls since, contrary to first impressions, (finite) productivity is an extremely weak property which, as we shall see, often comes as a by-product of idempotency. Even in categories with a rich supply of closure operators (like topological spaces), it is in fact hard to find examples of closure operators which fail to be finitely productive.

4.1 The lattice structure of all closure operators

For a category $\mathcal{X}$ and a class $\mathcal{M}$ of monomorphisms as in 2.1, we consider the conglomerate

$$CL(\mathcal{X},\mathcal{M})$$

of all closure operators on $\mathcal{X}$ with respect to $\mathcal{M}$. It is preordered by

$$C \leq D \Longleftrightarrow c_X(m) \leq d_X(m) \text{ for all } m \in \mathcal{M}/X\ ,\ X \in \mathcal{X}\ .$$

This way $CL(\mathcal{X},\mathcal{M})$ inherits a lattice structure from $\mathcal{M}$:

PROPOSITION *For $\mathcal{X}$ $\mathcal{M}$-complete, every family $(C_i)_{i\in I}$ in $CL(\mathcal{X},\mathcal{M})$ has a join $\bigvee_{i\in I} C_i$ and a meet $\bigwedge_{i\in I} C_i$ in $CL(\mathcal{X},\mathcal{M})$. The discrete closure operator is the least element in $CL(\mathcal{X},\mathcal{M})$, and the trivial closure operator is the largest one.* □

Proof It is easily checked that, in case $I \neq \emptyset$, with

$$c_X(m) := \bigvee_{i\in I} c_{i_X}(m) \quad \text{and} \quad d_X(m) := \bigwedge_{i\in I} c_{i_X}(m)$$

for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$, one obtains closure operators C and D on $\mathcal{X}$. (For the continuity condition, use 1.4 (3), (4).)

That C and D assume the role of the join and meet of $(C_i)_{i \in I}$ in $CL(\mathcal{X}, \mathcal{M})$ respectively is checked immediately. It is also obvious that the discrete and the trivial closure operators (as defined in Exercise 2.A) are least and largest respectively in $CL(\mathcal{X}, \mathcal{M})$. □

We say that $CL(\mathcal{X}, \mathcal{M})$ *has the structure of a large-complete lattice* in this situation.

4.2 Composition of closure operators

In this and the following section we shall show that $CL(\mathcal{X}, \mathcal{M})$ has also an interesting algebraic structure which is compatible with its lattice structure. We first define the *composite*

$$DC = ((dc)_X)_{X \in \mathcal{X}}$$

of two closure operators C and D on $\mathcal{X}$ with respect to $\mathcal{M}$ by composing the maps c_X and d_X :

$$(dc)_X(m) := d_X(c_X(m))$$

for all $m \in \mathcal{M}/X$.

$$\begin{tikzcd}
 & & d_X(c_X J(M)) \arrow[dd, "d_X(c_X(m))"] \\
 & c_X(M) \arrow[ur] \arrow[dr, "c_X(m)"'] & \\
M \arrow[ur] \arrow[rr, "m"] & & X
\end{tikzcd} \tag{4.1}$$

One easily proves (see Exercise 4.A):

LEMMA *DC is a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$. The composition provides $CL(\mathcal{X}, \mathcal{M})$ with the structure of a monoid with zero which is compatible with the lattice structure. More precisely, one has the following rules:*

(1) $(A\ B)C = A(B\ C)$ (*associativity*);

(2) $S\ C = C = C\ S$ (*discrete operator is neutral*);

(3) $T\ C = T \cong C\ T$ (*trivial closure operator is absorbing*);

(4) $A \leq B \Rightarrow AC \leq BC$ *and* $CA \leq CB$ (*monotonicity*);

(5) $(\bigwedge_{i\in I} A_i)C \cong \bigwedge_{i\in I}(A_iC)$ *and* $(\bigvee_{i\in I} A_i)C \cong \bigvee_{i\in I}(A_iC)$ *(for* $I \neq \emptyset$ *);*

(6) $A(\bigwedge_{i\in I} C_i) \leq \bigwedge_{i\in I}(AC_i)$ *and* $A(\bigvee_{i\in I} C_i) \geq \bigvee_{i\in I}(AC_i)$. □

For later use we prove:

PROPOSITION *If both* C *and* D *are weakly hereditary (grounded, (fully, directedly) additive, minimal, (finitely) productive, resp.) closure operators, then also* DC *is weakly hereditary (grounded, (fully, directedly) additive, minimal, (finitely) productive, resp.).*

Proof Groundedness, additivity, minimality and productivity are obviously stable under composition. The assertion for weak hereditariness is more complicated. For $m : M \to X$ in $\mathcal{M}$, let

$$j = j_m : M \to c_X(M) \quad \text{and} \quad k := j_{c_X(m)} : c_X(M) \to d_X(c_X(M)) =: Y.$$

For $m_Y := k \cdot j$, we must show that $d_Y(c_Y(m_Y))$ is an isomorphism. Since j is C-dense, from Corollary 2.4(1) one obtains a morphism t rendering the diagram

$$\begin{tikzcd}
M \arrow[rr, "1_M"] \arrow[dd, "j"'] & & M \arrow[d] \\
& & c_Y(M) \arrow[d, "c_Y(m_Y)"] \\
c_X(M) \arrow[rru, "t"] \arrow[rr, "k"] & & Y
\end{tikzcd} \tag{4.2}$$

commutative. Since k is D-dense, we conclude $1_Y \cong d_Y(k) \leq d_Y(c_Y(m_Y))$ and have the desired result. □

The composite of hereditary (idempotent) closure operators need not be hereditary (idempotent, resp.):

EXAMPLES

(1) The Kuratowski closure operator K of the category **PrTop** of pretopological spaces is hereditary (see 3.1), but KK is not. In fact, for a pretopological space X and subspaces $M \subseteq Y \subseteq X$ one has

$$k_Y(k_Y(M)) = k_X(k_X(M) \cap Y) \cap Y,$$

but this set may be properly contained in $k_X(k_X(M)) \cap Y$. For instance, consider $X = \mathbb{Z}$ with $k_X(M) = \{x : (\exists n \in \{-1,0,1\})\ x+n \in M\}$, and take Y the even integers and $M = \{0\}$.

(2) For the (hereditary) sequential closure operator σ of **Top**, $\sigma\sigma$ is not hereditary: see Exercise 4.B.

(3) One defines an idempotent and weakly hereditary closure operator K^* of **Top** by

$$k_X^*(M) = \bigcap\{U \subseteq X : M \subseteq U \text{ , } U \text{ open }\}$$

(see Exercise 2.D). K^* is called the *inverse Kuratowski closure operator* of **Top** . The composite K^*K (with K the Kuratowski closure operator of **Top**) fails to be idempotent. Indeed, for the space $X = \{a,b,c\}$ with $\emptyset, \{a\}$, $\{a,b\}$, $\{a,c\}$, X open and its (K-) closed subspace $M = \{b\}$, one has $k_X^*(k_X(k_X^*(M))) \neq k_X^*(M)$. We also note that K^*K is not hereditary: again consider $X = \{a,b,c\}$, but now with $\emptyset$, $\{b\}$, $\{c\}$, $\{b,c\}$, X open; for $M = \{b\}$ and $Y = \{b,c\}$, one then has $k_Y(M) = k_Y^*(M) = M$ while $k_X^*(k_X(M)) \cap Y = Y$. However, K^*K is weakly hereditary, by the Proposition.

4.3 Cocomposition of closure operators

Taking the D-closure of $c_X(m)$ as in the definition of the composite DC is only one of the two obvious choices to proceed. The other is to form the D-closure of j_m in $c_X(M)$. Hence the *cocomposite*

$$D * C = ((d * c)_X)_{X \in \mathcal{X}}$$

of two closure operators C and D on $\mathcal{X}$ with respect to $\mathcal{M}$ arises by factoring the morphism $j_m : M \to c_X(M) =: Z$ through $d_Z(M)$:

$$(d * c)_X(m) := c_X(m) \cdot d_Z(j_m)$$

for all $m \in \mathcal{M}/X$.

$$\begin{tikzcd}
d_Z(M) \arrow[dr, "d_Z(j_m)"] & & \\
& c_X(M) = Z \arrow[dr, "c_X(m)"] & \\
M \arrow[uu] \arrow[ur, "j_m"] \arrow[rr, "m"] & & X
\end{tikzcd} \tag{4.3}$$

The cocomposition satisfies properties similar to the composition but the respective rules are harder to prove:

LEMMA *$D * C$ is a closure operator on X with respect to $\mathcal{M}$. Like the composition, the cocomposition gives $CL(\mathcal{X}, \mathcal{M})$ the structure of a monoid with* *zero which is compatible with its lattice structure. Specifically, the following rules hold:*

(1) $(A * B) * C = A * (B * C)$ *(associativity);*

(2) $T * C = C = C * T$ *(trivial closure operator is neutral);*

(3) $S * C = S \cong C * S$ *(discrete closure operator is absorbing);*

(4) $A \leq B \Rightarrow A * C \leq B * C$ *and* $C * A \leq C * B$ *(monotonicity);*

(5) $(\bigwedge_{i\in I} A_i) * C \cong \bigwedge_{i\in I}(A_i * C)$ *and* $(\bigvee_{i\in I} A_i) * C \cong \bigvee_{i\in I}(A_i * C)$ *(for* $I \neq \emptyset$*) ;*

(6) $A * (\bigwedge_{i\in I} C_i) \leq \bigwedge_{i\in I}(A * C_i)$ *and* $A * (\bigvee_{i\in I} C_i) \geq \bigvee_{i\in I}(A * C_i)$.

Proof $D * C$ is obviously extensive. Its monotonicity follows easily with the Diagonalization Lemma. In order to show the Continuity Condition, we consider Diagram (4.3) and an $\mathcal{X}$-morphism $f : X \to Y$. One has $\mathcal{E}$-morphisms $e : M \to f(M)$ and $d : Z \to f(Z)$ with $Z = c_X(M)$. Let g be the composite

$$Z \xrightarrow{d} f(Z) \xrightarrow{k} c_X(f(M)) =: W$$

with k given by the continuity condition for C . There is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "j_m"'] & f(M) \arrow[d, "j_{f(m)}"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \tag{4.4}$$

which shows $g(m) \cong f(m)$, more precisely: $g(j_m) \cong j_{f(m)}$ in $\mathcal{M}/W$. Similarly, one easily sees that $g(d_Z(m)) \cong f(d_Z(m))$. Now the Continuity Condition for D applied to g shows

$$g(d_Z(j_m)) \leq d_W(g(j_m)),$$

and this implies the desired inequality

$$f((d * c)_X(m)) \leq (d * c)_Y(f(m)).$$

For rules (1) - (6), see Exercise 4.A. □

PROPOSITION *If both* C *and* D *are idempotent (hereditary, grounded, (finitely) productive) then also* $D * C$ *is idempotent (hereditary, grounded, (finitely) productive resp.).* □

Proof Once again we consider Diagram (4.3) and form the $D * C$-closure of $n := (d * c)_X(m) : N = d_Z(M) \to X$:

$$\begin{tikzcd}
d_W(N) \arrow[dr, "d_W(j_n)"] & & \\
& c_X(N) = W \arrow[dr, "c_X(n)"] & \\
N \arrow[uu, "k"] \arrow[ur, "j_n"] \arrow[rr, "n"] & & X
\end{tikzcd} \tag{4.5}$$

For C and D idempotent, we must show that k is an isomorphism. Since $m \leq n$ one has $c_X(m) \leq c_X(n)$. On the other hand, by definition of n we have $n \leq c_X(m)$, hence $c_X(n) \leq c_X(c_X(m)) \cong c_X(m)$. Therefore there is an isomorphism $i : Z = c_X(M) \to W = c_X(N)$ with $i \cdot d_Z(j_m) = j_n$. Since $d_Z(d_Z(j_m)) \cong d_Z(j_m)$, this gives immediately that k is an isomorphism as well.

The assertions on hereditariness, groundedness and (finite) productivity are easier to prove and are left as Exercise 4.D. □

In general $D * C$ does *not* inherit weak hereditariness, (full) additivity or minimality from D and C : see the Examples below and Exercise 4.C(c).

EXAMPLES

(1) In the category **AbGrp** of abelian groups, we consider the idempotent preradicals given by

$$\begin{aligned}
\mathbf{d}(A) &= \text{maximal divisible subgroup of } A \text{ (see 3.3)},\\
\mathbf{soc}(A) &= \text{sum of cyclic subgroups of prime order (\textit{socle} of } A\text{).}
\end{aligned}$$

Then

$$\mathbf{r}(A) = \mathbf{soc}(\mathbf{d}(A))$$

is a sum of cyclic subgroups of prime order, thus $\mathbf{r}(A)$ contains no divisible subgroups except zero. Hence $\mathbf{d}(\mathbf{r}(A)) = 0$, whence also $\mathbf{soc}(\mathbf{d}(\mathbf{r}(k))) = \mathbf{r}(\mathbf{r}(A)) = 0$. The induced closure operators $C^{\mathbf{d}}$ and $C^{\mathbf{soc}}$ are weakly hereditary ($C^{\mathbf{soc}}$ is even hereditary), and one has

$$C^{\mathbf{soc}} * C^{\mathbf{d}} \cong C^{\mathbf{soc}\,\mathbf{d}} = C^{\mathbf{r}}$$

(see 3.4). But since $\mathbf{r}$ is not idempotent, $C^{\mathbf{r}}$ is not weakly hereditary.

(2) Both K and K^* are additive, idempotent and hereditary closure operators of **Top** (see Example (3) of 4.2). By Exercise 4.A one has

$$K * K^* \cong K \wedge K^* .$$

But $K \wedge K^*$ is not additive. For this, provide $X = [0,1] \cup \{\infty\}$ with the following topology: the unit interval $[0,1]$ with its natural topology is an open subspace of X, and the only neighbourhood of ∞ in X is X. Since $k_X(\{1/2\}) = \{1/2, \infty\}$, for $M = [0,1/2) \cup \{\infty\}$ one has $1/2 \in k_X(M) \cap k^*_X(M)$, hence M is not $K \wedge K^*$ -closed. On the other hand, M is the union of the two $K \wedge K^*$ -closed subsets $[0,1/2)$ and $\{\infty\}$. Therefore $K \wedge K^*$ is not additive (see 2.6).

(3) Both $\uparrow$ and $\downarrow$ are fully additive, idempotent and hereditary closure operators of **PoSet** (see 3.6). But $\uparrow * \downarrow = \uparrow \wedge \downarrow$ (see Exercise 4.A) is not additive (see Exercise 4.E).

4.4 Closedness and density for (co)composites

For arbitrary operators C and D, we wish to describe the DC-closed and the $D*C$-dense morphisms. This will prove useful in the sequel. We first note some easy facts on joins and meets of closure operators which follow immediately from the respective definitions:

PROPOSITION

(1) *If* $C \le D$ *, then* $\mathcal{E}^C \subseteq \mathcal{E}^D$ *and* $\mathcal{M}^D \subseteq \mathcal{M}^C$.

(2) *If* $C \cong \bigvee_{i \in I} C_i$ *, then* $\mathcal{M}^C = \bigcap_{i \in I} \mathcal{M}^{C_i}$.

(3) *If* $D \cong \bigwedge_{i \in I} C_i$ *, then* $\mathcal{E}^D = \bigcap_{i \in I} \mathcal{E}^{C_i}$.

□

In pursuing our goal, we now prove:

LEMMA

(1) $D * C \le D \wedge C \le D \vee C \le DC$.

(2) $\mathcal{M}^{CC} = \mathcal{M}^C$ *and* $\mathcal{E}^{C*C} = \mathcal{E}^C$.

Proof

(1) Immediately from the definitions of $D*C$ and DC one has $D*C \le D$, $C \le DC$, hence (1).

(2) From $C \le CC$ one obtains $\mathcal{M}^{CC} \subseteq \mathcal{M}^C$. On the other hand, $m \cong c_X(m)$ implies $c_X(m) \cong c_X(c_X(m))$, hence $\mathcal{M}^C \subseteq \mathcal{M}^{CC}$. Similarly $C*C \le C$ gives $\mathcal{E}^{C*C} \subseteq \mathcal{E}^C$, and since $c_X(m) \cong 1_X$ implies $c_Y(j_m) \cong 1_Y$ for $Y = c_X(M)$, one also has $\mathcal{E}^C \subseteq \mathcal{E}^{C*C}$. □

THEOREM *For arbitrary closure operators* C *and* D *one has:*

(1) $\mathcal{M}^{CD} = \mathcal{M}^{DC} = \mathcal{M}^{D\vee C} = \mathcal{M}^{D} \cap \mathcal{M}^{C}$.

(2) $\mathcal{E}^{C*D} = \mathcal{E}^{D*C} = \mathcal{E}^{D\wedge C} = \mathcal{E}^{D} \cap \mathcal{E}^{C}$.

Proof

(1) From the Proposition and the Lemma, we obtain

$$\mathcal{M}^{DC} \subseteq \mathcal{M}^{D\vee C} = \mathcal{M}^{(D\vee C)(D\vee C)}.$$

But since D , $C \leq (D \vee C)$, monotonicity of composition gives $DC \leq (D \vee C)(D \vee C)$, hence $\mathcal{M}^{(D\vee C)(D\vee C)} \subseteq \mathcal{M}^{DC}$. With the Proposition, this shows $\mathcal{M}^{DC} = \mathcal{M}^{D\vee C} = \mathcal{M}^{D} \cap \mathcal{M}^{C}$, with the latter term symmetric in D and C ; hence it coincides also with $\mathcal{M}^{CD}$.

(2) follows dually, by reversing the order and by replacing composites by cocomposites. □

We remark that $\mathcal{E}^{C}$ may be a proper subclass of $\mathcal{E}^{CC}$ and $\mathcal{M}^{C}$ a proper subclass of $\mathcal{M}^{C*C}$.

EXAMPLE Consider the θ-closure of **Top** (see 3.3).

(1) Let X be the unit interval $[0,1]$ provided with the least topology such that $F = \{1/n : n \in \mathbb{N}\}$ and every naturally closed subset of $[0,1]$ is closed. Now $\theta_X(F) = F \cup \{0\}$ is discrete, hence $\theta_{F\cup\{0\}}(F) = F$. In other words: F is $\theta * \theta$ -closed in X , but not θ-closed. (See Exercise 3.F.)

(2) As in Example 4.2(3), let now $X = \{a,b,c\}$ be the topological space with $\emptyset$, $\{b\}$, $\{c\}$, $\{b,c\}$, X open. Then $\theta_X(\{b\}) = \{a , b\}$ and $\theta_X(\{a,b\}) = X$. Hence $\{b\}$ is $\theta\theta$ -dense but not θ-dense in X . (For an algebraic example of this type, see Exercise 4.C.)

4.5 Properties stable under meet or join

A property P for closure operators is said to be *stable under arbitrary meet* if for any family $(C_i)_{i\in I}$ of closure operators with property P also $\bigwedge_{i\in I} C_i$ satisfies P (whenever the meet exists); analogously for *joins*. Note that empty families are permitted; hence a property stable under arbitrary meet (join) must necessarily hold for the trivial (discrete) closure operator.

PROPOSITION

(1) *The following properties are stable under arbitrary meet: idempotency, hereditariness, (finite) productivity.*

(2) *Stable under arbitrary join are: weak hereditariness, minimality, groundedness, (full, directed) additivity.*

Proof

(1) A closure operator A is idempotent if and only if $AA \leq A$ (see Exercise 4.A). From Lemma 4.2 one always has $A(\bigwedge_{i\in I} C_i) \leq \bigwedge_{i\in I}(AC_i)$, hence for $A = \bigwedge_{i\in I} C_i$ one obtains

$$AA \leq \bigwedge_{i\in I}(AC_i) \leq \bigwedge_{i\in I}(C_iC_i).$$

Therefore $AA \leq A$ if each C_i is idempotent. Hence idempotency is stable under arbitrary meet.

Stability of hereditariness and productivity under arbitrary meet follows from the principle that "limits commute with limits", hence meets commute with inverse images (cf. 1.4) and products. Indeed, for $A = \bigwedge_{i\in I} C_i$ and for $m \leq y$ in $\mathcal{M}/X$ (as in Diagram (2.7)), one has

$$a_Y(m_Y) \cong \bigwedge_{i\in I} y^{-1}(c_{i_X}(m)) \cong y^{-1}(\bigwedge_{i\in I} c_{i_X}(m)) \cong y^{-1}(a_X(m)),$$

if each C_i is hereditary. Similarly for direct products.

(2) A closure operator A is weakly hereditary if and only if $A * A \geq A$ (see Exercise 4.A). Now stability of weak hereditariness under arbitrary joins follows dually to the first part of proof (1) – replace $\leq$ by $\geq$, $\wedge$ by $\vee$, and composition by cocomposition. The assertion for groundedness and (full, directed) additivity is obvious since "joins commute with joins", and it is trivial in the case minimality. □

We already showed in Example (2) of 4.3 that the meet $C \wedge D$ of additive (and idempotent and hereditary) closure operators C, D may fail to be additive. Exercise 4.E provides an example with C , D even fully additive (and idempotent and hereditary). Likewise, $C \wedge D$ may not be weakly hereditary for both C and D weakly hereditary: this follows from Example (1) of 4.3 in conjunction with Exercise 2.A(c).

Example (3) of 4.3 shows that $C \vee D$ may fail to be idempotent when both C and D are idempotent. Exercise 4.D (b) indicates how to show that hereditariness in general, is *not* preserved by the binary join:

EXAMPLE Let $(X, \leq)$ be a lattice (i.e., a poset with finite meets and joins). It was observed in Exercise 2.E that every closure operation c of the lattice $(X, \leq)$ induces a closure operator C of the induced category $\mathcal{X}$:

$$c_x(m) := c(m) \wedge x$$

for all $m \leq x$ in X . Obviously C is hereditary; in fact, *any closure operator of $\mathcal{X}$ is induced by a closure operation of the lattice $(X, \leq)$ if and only if it is hereditary.*

Assume that $(X, \leq)$ is not distributive, so that there are elements $a,\ b,\ x \in X$ with

$$(a \vee b) \wedge x > (a \wedge x) \vee (b \wedge x) .$$

(Consider, for instance, the lattice of subspaces of a 2-dimensional linear space.) We may define closure operations c , d of the lattice $(X, \leq)$ by

$$c(m) := m \vee a \quad , \quad d(m) := m \vee b .$$

For the induced categorical closure operators C and D we obtain in case $m = 0$:

$$\begin{aligned}
(c \vee d)_x(0) &= c_x(0) \vee d_x(0) \\
&= (a \wedge x) \vee (b \wedge x) \\
&< (a \vee b) \wedge x \\
&= (c_1(0) \vee d_1(0)) \wedge x \\
&= (c \vee d)_1(0) \wedge x .
\end{aligned}$$

Hence $C \vee D$ is not hereditary.

Together with Exercise 4.D(b), this shows that *a lattice is distributive if and only if the join of any two hereditary closure operators of the induced category is hereditary.*

4.6 Idempotent hull and weakly hereditary core

In this section we assume $\mathcal{X}$ to be $\mathcal{M}$-complete. C is a closure operator w.r.t. $\mathcal{M}$. As an immediate consequence of Proposition 4.5 we obtain:

PROPOSITION

(1) *There is a least idempotent closure operator* $\hat{C} \geq C$. *We call* $\hat{C}$ *the* idempotent hull of C .

(2) *There is a largest weakly hereditary closure operator* $\check{C} \leq C$. *We call* $\check{C}$ *the* weakly hereditary core of C .

Proof

(1) With $\mathcal{D} = \{D : D \geq C\ ,\ D \text{ idempotent}\}$, Proposition 4.5 gives that $\hat{C} = \bigwedge \mathcal{D}$ is idempotent, hence $\hat{C}$ is a least element in $\mathcal{D}$.

(2) follows dually. □

In order to be able to "compute" $\hat{C}$ and $\check{C}$ in concrete examples, we need a more concrete description of $\hat{C}$ and $\check{C}$. To this end, one defines *the ascending extended ordinal chain of* C :

$$C^0 \leq C^1 \leq C^2 \leq \ldots \leq C^\alpha \leq C^{\alpha+1} \leq \ldots \leq C^\infty \leq C^{\infty+1}$$

as follows:

$$C^0 = S \,,\; C^{\alpha+1} = CC^\alpha \,,\; C^\beta = \bigvee_{\gamma<\beta} C^\gamma$$

for every (small) ordinal number α and for $\alpha = \infty$, and for every limit ordinal β and for $\beta = \infty$; here $\infty, \infty+1$ are (new) elements with $\infty+1 > \infty > \alpha$ for all $\alpha \in Ord$, the class of small ordinals. (S is the discrete closure operator, see Exercise 4.A.) Dually one can define the *descending extended ordinal chain of* C .

$$C_{\infty+1} \leq C_\infty \leq \ldots \leq C_{\alpha+1} \leq C_\alpha \leq \ldots C_2 \leq C_1 \leq C_0$$

by putting

$$C_0 = T \,,\; C_{\alpha+1} = C * C_\alpha \,,\; C_\beta = \bigwedge_{\gamma<\beta} C_\gamma$$

for all α and β as above, with T the trivial closure operator.

LEMMA *If for each* $m \in \mathcal{M}/X$, $X \in \mathcal{X}$, *there is an* $\alpha \in Ord \cup \{\infty\}$ *with* $c^\alpha(m) \cong c^{\alpha+1}(m)$, *then* C^∞ *is idempotent. Similarly, if for each* m *there is an* α *with* $c_\alpha(m) \cong c_{\alpha+1}(m)$, *then* C_∞ *is weakly hereditary.*

Proof By ordinal induction, one easily shows $c^\beta(m) \cong c^\alpha(m)$ and $c^\gamma(c^\beta(m)) \cong c^\beta(m)$ for all $\beta, \gamma \in Ord \cup \{\infty\}$, $\beta \geq \alpha$. Hence

$$c^\infty(c^\infty(m)) \cong c^\infty(m)$$

for $\beta = \gamma = \infty$. Analogously one proves $(c_\infty * c_\infty)(m) \cong c_\infty(m)$. □

We say that a preordered class P *has no proper ascending extended ordinal chains* if for every ascending extended ordinal chain

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_\alpha \leq x_{\alpha+1} \leq \ldots \leq x_\infty \leq x_{\infty+1}$$

in P there is an $\alpha \in Ord \cup \{\infty\}$ with $x_\alpha \cong x_{\alpha+1}$. In the dual situation one says that P has *no proper descending extended ordinal chains.* Obviously, if there is only a (small) set of $\cong$ -equivalence classes in P , then P has neither proper ascending nor descending extended ordinal chains. In particular, if $\mathcal{X}$ is $\mathcal{M}$-wellpowered, so that for every $X \in \mathcal{X}$, $P = \mathcal{M}/X$ has only a set of $\cong$-equivalence classes, the hypotheses of the Lemma are satisfied.

Independently from $\mathcal{M}$-wellpoweredness, we shall often encounter closure operators C with $C^\alpha \cong C^{\alpha+1}$ for some (*small*) $\alpha \in Ord$, so that one has $c_X^\alpha(m) \cong c_X^{\alpha+1}(m)$ for *all* $m \in \mathcal{M}/X$, $X \in \mathcal{X}$. In that case we shall call C *bounded*, and the least such α is called the *order* of C ; otherwise C is *unbounded*. Similarly, if $C_\alpha \cong C_{\alpha+1}$ for some $\alpha \in Ord$, C is called *cobounded*, and the least such α is the *co-order* of C ; otherwise C is *uncobounded*.

For each $\alpha \in Ord$, there are examples of closure operators with order α or co-order α : see Exercise 4.F. Unbounded and uncobounded closure operators will be considered in detail in Chapter 8.

THEOREM

(1) *If $\mathcal{X}$ is $\mathcal{M}$-wellpowered or if C is bounded, then $C^\infty \cong \hat{C}$ is the idempotent hull of C . One has*

$$\mathcal{M}^{C^\infty} = \mathcal{M}^C \quad \textit{and} \quad \mathcal{E}^{C^\infty} \supseteq \mathcal{E}^C .$$

C^∞ inherits the following properties from C : weak hereditariness, groundedness, (full, directed) additivity.

(2) *If $\mathcal{X}$ is $\mathcal{M}$-wellpowered or if C is cobounded, then $C_\infty \cong \check{C}$ is the weakly hereditary core of C . One has*

$$\mathcal{E}^{C_\infty} = \mathcal{E}^C \quad \textit{and} \quad \mathcal{M}^{C_\infty} \supseteq \mathcal{M}^C$$

C_∞ inherits the following properties from C : idempotency, hereditariness, groundedness, (finite) productivity.

Proof

(1) Idempotency of C^∞ under the given hypotheses follows from the Lemma. By induction one easily shows $C^\alpha \leq D$ for every idempotent closure operator $D \geq C$ and every $\alpha \in Ord \cup \{\infty\}$; hence $C^\infty \leq \hat{C}$, and $C^\infty \cong \hat{C}$ holds if (and only if) C^∞ is idempotent. Since $C \leq C^\infty$, one has $\mathcal{E}^C \subseteq \mathcal{E}^{C^\infty}$. Furthermore, with Proposition and Lemma 4.4 one shows $\mathcal{M}^{C^\alpha} = \mathcal{M}^C$ for all $\alpha \in Ord \cup \{\infty\}$ by induction. Finally, with Propositions 4.2 and 4.5 one obtains that every C^α is weakly hereditary (grounded, (fully) additive), $\alpha \in Ord \cup \{\infty\}$, whenever C has the respective property.

(2) is proved analogously. □

REMARK We shall show in 5.4 that, independently from the validity of $\hat{C} \cong C^\infty$ and $\check{C} \cong C_\infty$, one always has $\mathcal{M}^{\hat{C}} = \mathcal{M}^C \subseteq \mathcal{M}^{\check{C}}$ and $\mathcal{E}^{\check{C}} = \mathcal{E}^C \subseteq \mathcal{E}^{\hat{C}}$ as well as the implications (C weakly hereditary $\Rightarrow$ $\hat{C}$ weakly hereditary) and (C idempotent $\Rightarrow$ $\check{C}$ idempotent).

COROLLARY *For any closure operator C with C^∞ idempotent or C_∞ weakly hereditary, one has $(C_\infty)^\infty \leq (C^\infty)_\infty$. These two closure operators are isomorphic if C is idempotent or weakly hereditary; otherwise, the inequality may be strict.*

Proof $C \leq C^\infty$ implies $C_\infty \leq (C^\infty)_\infty$; if C^∞ is idempotent, also $(C^\infty)_\infty$ is idempotent, hence $(C_\infty)^\infty \leq (C^\infty)_\infty$. Similarly one obtains this inequality when C_∞ is weakly hereditary. If C is idempotent, also C_∞ is idempotent, and one has $C \cong C^\infty$ and $C_\infty \cong (C_\infty)^\infty$. Similarly, when C is weakly hereditary, one has $C \cong C_\infty$ and $C^\infty \cong (C^\infty)_\infty$.

In order to show that one may have $(C_\infty)^\infty \not\cong (C^\infty)_\infty$ we consider $C = C^{\mathbf{r}}$ for the preradical $\mathbf{r} = \mathbf{soc}\ \mathbf{d}$ of Example (1) of 4.3. According to Exercise 4.G, we must show $(\mathbf{r}^\infty)_\infty < (\mathbf{r}_\infty)^\infty$. We already saw in 4.3 that $\mathbf{r}^2 = \mathbf{0}$, hence $(\mathbf{r}^\infty)_\infty = \mathbf{0}$. On the other hand, $\mathbf{soc}_\infty(A) = \mathbf{t}(A)$ is the torsion subgroup of an abelian group A , hence $\mathbf{r}_\infty = \mathbf{t}\ \mathbf{d}$. Since $\mathbf{r}_\infty$ is idempotent, one has $(\mathbf{r}_\infty)^\infty = \mathbf{r}_\infty$. For every non-zero divisible torsion group A one now has $(\mathbf{r}_\infty)^\infty(A) = \mathbf{r}_\infty(A) = A \neq 0$.
□

The θ-closure of **Top** shows that, in general, $\mathcal{E}^C$ is properly contained in $\mathcal{E}^{C^\infty}$ and $\mathcal{M}^C$ is properly contained in $\mathcal{M}^{C_\infty}$: see Example 4.4. $C = K$ in **PrTop** or $C = \sigma$ in **Top** gives a hereditary closure operator for which C^∞ is not hereditary: see Example 4.2. Example (1) below shows that the passage $C \mapsto C^\infty$ does not preserve productivity. Finally there is an example of an additive (in fact: fully additive, see Exercise 2.I) closure operator whose weakly hereditary core is not additive; see Example (2) below.

EXAMPLES

(1) For a fixed prime number p , we consider the closure operator C of **AbGrp** given by

$$c_A(M) = \{x \in A : px \in M\} .$$

It can be presented as the maximal closure operator $C^{\mathbf{s}_p}$, with

$$\mathbf{s}_p(A) = \{x \in A : px = 0\}$$

the *p-socle* of A . $\mathbf{s}_p$ is a hereditary preradical with $\mathbf{s}_p^\infty = \mathbf{s}_p^\omega = \mathbf{t}_p$, with $\mathbf{t}_p$ the preradical given by the p-torsion subgroup of a group. It follows from Theorem 3.4 that C is hereditary. Furthermore, C is productive (since $\mathbf{s}_p$ is Jansenian), but $C^\infty \cong C^\omega = C^{\mathbf{t}_p}$ is not (since $\mathbf{t}_p$ is not Jansenian). More specifically, $\{0\}$ is C^∞-dense in $\mathbb{Z}_{p^n}$, but $\{0\}$ is not C^∞-dense in $\prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$ (cf. Exercise 4.G).

(2) For a prime number p , let $\mathbf{p}$ be the radical given by p-multiples: $\mathbf{p}(A) = pA$ for an abelian group A . $C^{\mathbf{p}}$ is easily seen to be fully additive. One has $(C^{\mathbf{p}})_\infty \cong C^{(\mathbf{p}^\infty)}$ (see Exercise 4.G), and $\mathbf{p}^\infty$ assigns to A its maximal p-divisible subgroup $\mathbf{d}_p(A)$. Now it follows from Theorem 3.4 that $C^{(\mathbf{p}^\infty)}$ is not additive since $C^{(\mathbf{p}^\infty)} \not\cong C_{(\mathbf{p}^\infty)}$. But one can see non-additivity of $C^{(\mathbf{p}^\infty)}$ also without reference to the Theorem:

Consider the group $A = \mathcal{I}_p$ of p-adic integers; it satisfies $\mathbf{d}_p(\mathcal{I}_p) = 0$. Fix a cyclic subgroup Z of $\mathcal{I}_p$ with $Z \not\subseteq p\mathcal{I}_p$; it is $C^{\mathbf{d}_p}$ -dense in $\mathcal{I}_p$, i.e. $\mathbf{d}_p(\mathcal{I}_p/Z) = \mathcal{I}_p/Z$, or $\mathcal{I}_p/Z$ is p-divisible. Now consider

$$X = \Delta_{\mathcal{I}_p} + Z \times Z$$

as a subgroup of $\mathcal{I}_p \times \mathcal{I}_p$, with $\Delta_{\mathcal{I}_p}$ the diagonal in $\mathcal{I}_p \times \mathcal{I}_p$. Both $M_1 = Z \times \{0\}$ and $M_2 = \{0\} \times Z$ are proper $C^{\mathbf{d}_p}$-closed subgroups since

$$X \cong M_1 \times \mathcal{I}_p \cong M_2 \times \mathcal{I}_p$$

and $\mathbf{d}_p(\mathcal{I}_p) = 0$. On the other hand, $X/(Z \times Z) \cong \mathcal{I}_p/Z$ so that $M_1 + M_2 = Z \times Z$ is a proper $\mathbf{d}_p$-dense subgroup, in particular not $\mathbf{d}_p$-closed.

4.7 Indiscrete operator, proper closure operators

As remarked in Proposition 4.5, groundedness is stable under arbitrary join. Trivially, one also has that $\bigwedge_{i \in I} C_i$ is grounded if at least one C_i is grounded. What about the case $I = \emptyset$? Certainly, the largest operator T in $CL(\mathcal{X}, \mathcal{M})$ is not grounded unless $O_X \cong X$ for all $X \in \mathcal{X}$. But is there a largest element in

$$GCL(\mathcal{X}, \mathcal{M}),$$

the conglomerate of grounded closure operators with respect to $\mathcal{M}$?

Proposition *For $\mathcal{X}$ $\mathcal{M}$-complete, $GCL(\mathcal{X}, \mathcal{M})$ has the structure of a large-complete lattice. Arbitrary joins and non-empty meets are formed as in $CL(\mathcal{X}, \mathcal{M})$. The largest element G in $GCL(\mathcal{X}, \mathcal{M})$ is called the* indiscrete closure operator *and is described by*

$$(*) \qquad g_X(m) = \bigwedge \{e^{-1}(o_Z) : (\exists Z \in \mathcal{X}\ \exists e : X \to Z \quad in \quad \mathcal{E})\ e(m) \cong o_Z)\}$$

for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$, with the meet taken in $\mathcal{M}/X$. □

Proof We first show that $(g_X)_{X \in \mathcal{X}}$ defined by $(*)$ is in fact a closure operator, with the help of the formulas given in 1.4 and in Exercise 1.K. First of all, for every $e \in \mathcal{E}$ with $e(m) \cong o_Z$ one has $m \le e^{-1}(e(m)) \cong e^{-1}(o_Z)$, hence $m \le g_X(m)$. In order to check 2.2 (4), let $f(m) \le n$ for $f : X \to Y$, $m \in \mathcal{M}/X$, and $n \in \mathcal{M}/Y$. We then have

$$f(g_X(m)) \le \bigwedge \{f(e^{-1}(o_Z)) : e \in A\ ,\ Z = \text{codomain } (e)\}$$

with $A = \{e \in \mathcal{E} \mid \text{domain } (e) = X\ ,\ e(m) = 0\}$ whereas

$$g_Y(f(m)) = \bigwedge \{d^{-1}(o_W) : d \in B\ ,\ W = \text{ codomain } (d)\}\ ,$$

with $B = \{d \in \mathcal{E} \mid \text{domain}(d) = Y\ ,\ d(n) = 0\}$. Hence to obtain $f(g_X(m)) \le g_Y(n)$ it suffices to show that, for every $d \in B$, there is an $e \in A$ with $f(e^{-1}(o_Z)) \le d^{-1}(o_W)$ (with Z , W the respective codomains). Indeed, $f(m) \le n$ gives $d(f(m)) \le d(n) \cong o_Z$, the $(\mathcal{E}\ ,\ \mathcal{M})$–factorization of $d \cdot f = u \cdot e$ with $e \in \mathcal{E}$ and $u \in \mathcal{M}$, hence $u(e(m)) \cong o_W$ and

$$e(m) \cong u^{-1}(u(e(m))) \cong u^{-1}(o_W) \cong o_Z$$

since $u \in \mathcal{M}$. Thus $e \in A$, and from $e(e^{-1}(o_Z)) \leq o_Z$ one obtains

$$d(f(e^{-1}(o_Z))) \cong u(e(e^{-1}(o_Z))) \leq u(o_Z) = o_W ,$$

hence $f(e^{-1}(o_Z)) \leq d^{-1}(o_W)$ as desired.

Clearly, $G = (g_X)_{X \in \mathcal{X}}$ is grounded (consider $e = 1_X$). For any other grounded closure operator C and every $e \in A$ as above, one has

$$c_X(m) \leq e^{-1}(e(c_X(m))) \leq e^{-1}(c_X(e(m))) \cong e^{-1}(c_X(o_Z)) = e^{-1}(o_Z),$$

hence $C \leq G$. □

EXAMPLES

(1) In **Set** with $\mathcal{M}$ the class of injective maps, one has

$$g_X(M) = \begin{cases} \emptyset & \text{for } M = \emptyset \\ X & \text{else} \end{cases}$$

for all $M \subseteq X$, hence G induces the coarsest or indiscrete topology on X . The same formula holds true in **Top** with $\mathcal{M}$ the class of embeddings.

(2) In $\mathbf{Mod}_R$ with $\mathcal{M}$ the class of monomorphisms, G coincides with the discrete closure operator since this is the only grounded closure operator (up to isomorphisms), see Exercise 2.G.

(3) In **Fld** with $\mathcal{M}$ the class of all (mono)morphisms, the formula

$$g_F(E) = \begin{cases} E & \text{for the prime subfield } E \text{ of } F \\ F & \text{else} \end{cases}$$

holds. Indeed this formula obviously defines a closure operator of **Fld** and therefore gives the largest grounded closure operator of **Fld** .

Since groundedness is stable under arbitrary join, for every closure operator C there is a largest grounded closure operator $C^G \leq C$, called the *grounding of* C . With the Proposition one obtains:

COROLLARY *For a closure operator C of an $\mathcal{M}$-complete category, $C^G \cong C \wedge G$ is the largest grounded operator $\leq C$, called the grounding of C . If C is idempotent or hereditary, C^G has the respective property.* □

Proof $C \wedge G$ is grounded since G is grounded, and for every grounded operator $D \leq C$ one has $D \leq G$, hence $D \leq C \wedge G$. G is idempotent and hereditary: see Exercise 4.I. These properties are stable under meet, hence they are inherited by $C^G \cong C \wedge G$ from C . □

In an $\mathcal{M}$-complete category one always has the discrete, the indiscrete, and the trivial closure operators:

$$S \leq G \leq T .$$

Any closure operator not isomorphic to any of these is called *proper*. There are categories which do not have any proper closure operators:

LEMMA *The category* **Set** *(with* $\mathcal{M}$ *the class of injective maps) has no proper closure operators.*

Proof Let C be a closure operator of **Set**, and for $D = \{0,1\}$ first assume $C_D(\{0\}) = D$. We claim that then $c_X(M) = X$ for all $\emptyset \neq M \subseteq X$. Indeed for $x \in M$ and any $y \in X$, one considers $f : D \to X$ with $f(0) = x$ and $f(1) = y$ and has

$$y \in f(c_D(\{0\})) \subseteq c_X(\{x\}) \subseteq c_X(M) .$$

Hence C must be G or T in this case. The only other case is $c_D(\{0\}) = \{0\}$, in which case one has $C = S$. Indeed, if there was $N \subseteq X$ with $N \neq c_X(N)$, consider $h : X \to D$ with $h(N) = \{0\}$ and $h(X \setminus N) = \{1\}$; then

$$1 \in g(c_X(N)) \subseteq c_D(\{0\}) = \{0\} .$$

We have seen in 3.3 that **Top** (with $\mathcal{M}$ the class of embeddings) has quite a few proper closure operators. (We shall see in Chapter 8, for instance, that the θ-closure in neither bounded nor cobounded, so that all its powers θ^α and copowers θ_α are distinct.) We already know that every non-trivial closure operator C of **Top** must be grounded (see Exercise 2.H), hence $C \leq G$, in fact $C < G$ if C is proper. The next question then is whether there is a *largest proper closure* operator in **Top**.

For this we define $Q = (q_X)_{X \in \mathbf{Top}}$ by

$$q_X(M) = \bigcap\{A \subseteq X : A \text{ open and } (K\text{-})\text{closed} , \ M \subseteq A\} ;$$

i.e., $q_X(M)$ is the *quasicomponent of* M *in* X.

THEOREM Q *is an idempotent, grounded, additive and productive closure operator of* **Top**, *but neither weakly hereditary nor fully additive. It is the largest proper closure operator of* **Top**, *hence every proper* C *satisfies* $S < C \leq Q < G < T$.

Proof That Q is an idempotent closure operator of **Top** follows from Exercise 2.D(a). Checking the other properties of Q claimed in the first part of the Theorem is left as Exercise 4.T. For any proper closure operator C we have $C < G$, and we must show $C \leq Q$. Let $D = \{0,1\}$ be discrete. The first part of the proof of the Lemma shows that we must have $c_D(\{0\}) = \{0\}$. For every closed and open $N \subseteq X$ one considers the characteristic map h as in the second part of the proof of the Lemma. This shows that $N = h^{-1}(\{0\})$ is C-closed. Consequently, for every $M \subseteq X$,

$$c_X(M) \subseteq \bigcap\{N \subseteq X : N \quad C\text{-closed}, M \subseteq N\} \subseteq q_X(M) .$$

□

4.8 Additive core

Since additive closure operators are stable under arbitrary join, for each closure operator C of an $\mathcal{M}$-complete category there is a largest additive closure operator $\leq C$, called the *additive core* of C. Under some additional assumptions, we will give a more concrete description of the additive core below. However, without any additional assumptions we can state:

LEMMA *The additive core of an idempotent (grounded) closure operator is idempotent (grounded, respectively).*

Proof Denoting the additive core of C by $\dot{C}$, we have $\dot{C}\dot{C} \leq CC \leq C$ when C is idempotent. By Proposition 4.2, $\dot{C}\dot{C}$ is additive, hence $\dot{C}\dot{C} \leq \dot{C}$. But this means that $\dot{C}$ is idempotent (cf. Exercise 4.A). The assertion on groundedness is trivial since $\dot{C} \leq C$. □

Let now

$$c_X^+(m) = \bigwedge\{c_X(m_1) \vee \ldots \vee c_X(m_k) : m_i \in \mathcal{M}/X\ ,\ 1 \leq i \leq k\ ,\ m_1 \vee \ldots \vee m_k \geq m\}$$

for every $m \in \mathcal{M}/X$ and consider the following conditions on $\mathcal{X}$ and $\mathcal{M}$:

(a) *Binary joins are preserved by inverse images*, i.e.

$$f^{-1}(n_1 \vee n_2) \cong f^{-1}(n_1) \vee f^{-1}(n_2)$$

for all $f : X \to Y$ and n_1, $n_2 \in \mathcal{M}/Y$.

(b) *Joins distribute over arbitrary meets in each* $\mathcal{M}/X$, i.e.

$$m \vee \bigwedge_{i \in I} m_i \cong \bigwedge_{i \in I} m \vee m_i$$

for all m, $m_i \in \mathcal{M}/X$, $i \in I$. (In other words, $(\mathcal{M}/X)^{op}$ has the structure of a frame.)

REMARKS

(1) Condition (a) entails the distributive law

$$n \wedge (n_1 \vee n_2) \cong (n \wedge n_1) \vee (n \wedge n_2)$$

for $n = f \in \mathcal{M}$, whereas condition (b) entails the distributive law

$$m \vee (m_1 \wedge m_2) \cong (m \vee m_1) \wedge (m \vee m_2).$$

In every lattice, the two (finite) distributive laws are logically equivalent. Therefore, and since f can be factored as $f = m \cdot e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, in the presence of (b) it is sufficient to consider $f \in \mathcal{E}$ in (a).

(2) If the distributive law holds in each $\mathcal{M}/X$, in particular if condition (a) or (b) holds, then

$$c_X^+(m) \cong \bigwedge\{c_X(m_1) \vee \ldots \vee c_X(m_k) : m_i \in \mathcal{M}/X\,,\ 1 \leq i \leq k\,,\ m_1 \vee \ldots \vee m_k \cong m\}.$$

In fact, whenever $m_1 \vee \ldots \vee m_k \geq m$, then $(m_1 \wedge m) \vee \ldots \vee (m_k \vee m) \cong m$ and $c_X(m_1 \wedge m) \vee \ldots \vee c_X(m_k \wedge m) \leq c_X(m_1) \vee \ldots \vee c_X(m_k)$.

THEOREM *Under condition* (a), $C^+ = (c_X^+)_{X \in \mathcal{X}}$ *is a closure operator with* $C^+ \leq C$ *and* $D \leq C^+$ *for every additive closure operator* $D \leq C$. *If, in addition, condition* (b) *holds,* C^+ *is additive and therefore the additive core of* C. *Furthermore, if* C *is idempotent* (*hereditary, grounded*), C^+ *has the respective property.*

Proof Extension and monotonicity are obviously satisfied. To check continuity for C^+, $f(m) \leq n_1 \vee \ldots \vee n_k$ with $m \in \mathcal{M}/X$ and $n_i \in \mathcal{M}/Y$ implies

$$m \leq f^{-1}(f(m)) \leq f^{-1}(n_1 \vee \ldots \vee n_k) \cong f^{-1}(n_1) \vee \ldots \vee f^{-1}(n_k)$$

by Condition (a). Therefore

$$\begin{aligned} c_X^+(m) &\leq c_X(f^{-1}(n_1)) \vee \ldots \vee c_X(f^{-1}(n_k)) \\ &\leq f^{-1}(c_Y(n_1)) \vee \ldots \vee f^{-1}(c_Y(n_k)) \end{aligned}$$

by continuity of C. Since $f(-)$ preserves joins, it follows that

$$\begin{aligned} f(c_X^+(m)) &\leq f(f^{-1}(c_Y(n_1))) \vee \ldots \vee f(f^{-1}c_Y(n_k)) \\ &\leq c_Y(n_1) \vee \ldots \vee c_Y(n_k). \end{aligned}$$

Consequently, $f(c_X^+(m)) \leq c_Y^+(f(m))$.

Obviously, when $D \leq C$ is additive, for any "cover" $m_1 \vee \ldots \vee m_k \geq m$ one has

$$d_X(m) \leq d_X(m_1) \vee \ldots \vee d_X(m_k) \leq c_X(m_1) \vee \ldots \vee c_X(m_k)$$

hence $d_X(m) \leq c_X^+(m)$.

For any "covers" $m_1 \vee \ldots \vee m_k \geq m$ and $n_1 \vee \ldots \vee n_j \geq n$ one has $m_1 \vee \ldots \vee m_k \vee n_1 \vee \ldots \vee n_j \geq m \vee n$, hence

$$c_X^+(m \vee n) \leq (c_X(m_1) \vee \ldots \vee c_X(m_k)) \vee (c_X(n_1) \vee \ldots \vee c_X(n_j))\,.$$

Hence $c_X^+(m \vee n)$ is covered by the meet of all latter terms which, under Condition (b), is isomorphic to $c_X^+(m) \vee c_X^+(n)$.

By the Lemma, C^+ inherits idempotency and groundedness from C. Without reference to Condition (b), we now show that C^+ is hereditary when C is hereditary. For that, let $m = y \cdot m_Y$ and $y : Y \to X$ be in $\mathcal{M}$ and consider any "cover" $n_1 \vee \ldots \vee n_k \geq m_Y$ in $\mathcal{M}/Y$. Then $y \cdot n_1 \vee \ldots \vee y \cdot n_k \geq m$ (see Exercise 1.J) and

$$\begin{aligned} c_Y(n_1) \vee \ldots \vee c_Y(n_k) &\cong y^{-1}(c_X(y \cdot n_1)) \vee \ldots \vee y^{-1}(c_X(y \cdot n_k)) \\ &\cong y^{-1}(c_X(y \cdot n_1) \vee \ldots \vee c_X(y \cdot n_k)) \\ &\geq y^{-1}(c_X^+(m)), \end{aligned}$$

hence $c_Y^+(m_Y) \geq y^{-1}(c_X^+(m))$. □

Under the assumptions of the Theorem one obtains with Proposition 4.5:

COROLLARY *The conglomerate of additive closure operators*

$$ACL(\mathcal{X}, \mathcal{M})$$

has the structure of a large-complete lattice. Joins are formed as in $CL(\mathcal{X}, \mathcal{M})$ *, whereas the meet of* $(C_i)_{i \in I}$ *in* $ACL(\mathcal{X}, \mathcal{M})$ *is given by*

$$(\bigwedge_{i \in I} C_i)^+,$$

with $\bigwedge$ *denoting the meet in* $CL(\mathcal{X}, \mathcal{M})$ *.* □

EXAMPLES (1) With K and K^* as in Example 4.2

$$(K \wedge K^*)^+$$

is the front-closure b in **Top** (see 3.3).

(2) Although in $\mathbf{Mod}_R$ conditions (a) and (b) do not hold, it is easy to determine the additive core of a closure operator C (which, by necessity must be given by the defining formula for C^+ :) it is the minimal closure operator $C_{\mathbf{r}}$, with $\mathbf{r}$ the preradical induced by C (in particular, C^+ as defined above, is indeed a closure operator). For the proof observe that $C_{\mathbf{r}} \leq C \leq C^{\mathbf{r}}$ with $C_{\mathbf{r}}$ additive (see Theorem 3.4) implies $C_{\mathbf{r}} \leq C^+ \leq C^{\mathbf{r}}$. Hence C^+ induces the same preradical as C , and according to Exercise 3.M it is minimal. Hence

$$C^+ \cong C_{\mathbf{r}} .$$

4.9 Fully additive core

Fully additive closure operators are, like additive closure operators, stable under arbitrary joins. Hence each closure operator of an $\mathcal{M}$-complete category has a *fully additive core*, i.e., a largest fully additive closure operator below itself. Replacing finite joins by arbitrary joins, one may construct it analogously to the additive core (see Exercise 4.M). Here, however, we give an alternative construction which is modeled after the point-closure in **Top** (see Example 2.6) which is the fully additive core of the Kuratowski closure operator K .

A subobject $p \in \mathcal{M}/X$ is called *$\bigvee$-prime* if $p \leq \bigvee_{i \in I} m_i$ for any family $(m_i)_{i \in I}$ in $\mathcal{M}/X$ implies $p \leq m_i$ for at least one $i \in I$. Since here $I = \emptyset$ is permitted, one has $p \not\cong o_X$. The class $\mathcal{P}$ of all *$\bigvee$-prime* elements in $\mathcal{M}$ will assume the role of "points", under the following hypotheses:

(A) $f(p) \in \mathcal{P}/Y$ for every $f : X \to Y$ and $p \in \mathcal{P}/X$,

(B) $m \cong \bigvee\{p \in \mathcal{P}/X : p \leq m\}$ for every $m \in \mathcal{M}/X$.

Under hypotheses (A) and (B) we may put

$$c_X^{\oplus}(m) :\cong \begin{cases} \bigvee\{c_X(p) : p \in \mathcal{P}/X\,,\ p \leq m\} & \text{if } m \not\cong o_X\,, \\ c_X(o_X) & \text{else,} \end{cases}$$

and then prove:

THEOREM *In an $\mathcal{M}$-complete category satisfying conditions* (A) *and* (B), *the fully additive core $C^{\oplus}$ of a closure operator C is given by the formula above. $C^{\oplus}$ inherits each of the following properties from C : idempotency, groundedness and hereditariness.*

Proof First we must verify that $C^{\oplus}$ is a closure operator. Property (B) implies that for every $m \not\cong o_X$ there is $p \in \mathcal{P}/X$ with $p \leq m$. With this observation one easily checks the properties of extension and monotonicity. Continuity for $C^{\oplus}$ follows from

$$\begin{aligned} f(c_X^{\oplus}(m)) &\cong \bigvee\{f(c_X(p)) : p \in \mathcal{P}/X\,,\ p \leq m\} \\ &\leq \bigvee\{c_Y(f(p)) : p \in \mathcal{P}/X\,,\ p \leq m\} \\ &\leq \bigvee\{c_Y(q) : q \in \mathcal{P}/Y\,,\ q \leq f(m)\} \cong c_Y^{\oplus}(f(m)) \end{aligned}$$

for $m \not\cong o_X$; here we use successively Axiom (B), preservation of joins by $f(-)$, continuity of C, and Axiom (A). In case $m \cong o_X$, continuity for $C^{\oplus}$ follows immediately from continuity of C since $f(o_X) \cong o_Y$.

Clearly, $C^{\oplus} \leq C$. Full additivity of $C^{\oplus}$ follows from (B) and the definition of $\bigvee$-primeness. Furthermore, any fully additive $D \leq C$ satisfies

$$\begin{aligned} d_X(m) &\cong \bigvee\{d_X(p) : p \in \mathcal{P}/X\,,\ p \leq m\} \\ &\cong \bigvee\{c_X(p) : p \in \mathcal{P}/X\,,\ p \leq m\} \cong c_X^{\oplus}(m) \end{aligned}$$

for $m \not\cong o_X$. The case $m \cong o_X$ is trivial. Therefore $C^{\oplus}$ is the fully additive core of C.

As in Lemma 4.8 one may show that $C^{\oplus}$ is idempotent and grounded if C has the respective property. In order to show the same with respect to hereditariness we need:

LEMMA *In the presence of* (B), *condition* (A) *holds if and only if* $f^{-1}(-)$ *preserves arbitrary joins.*

Proof Under condition (A), for all $f : X \to Y$, $n_i \in \mathcal{M}/Y (i \in I)$ and $p \in \mathcal{P}/X$ one has

$$\begin{aligned} p \le f^{-1}\left(\bigvee_{i\in I} n_i\right) &\Leftrightarrow f(p) \le \bigvee_{i\in I} n_i \\ &\Leftrightarrow (\exists i \in I) \quad f(p) \le n_i \\ &\Leftrightarrow (\exists i \in I) \quad p \le f^{-1}(n_i) \\ &\Leftrightarrow p \le \bigvee_{i\in I} f^{-1}(n_i) \end{aligned}$$

and therefore $f^{-1}(\bigvee_{i\in I} n_i) \cong \bigvee_{i\in I} f^{-1}(n_i)$ from (B). Conversely, the last formula gives the implications

$$\begin{aligned} f(p) \le \bigvee_{i\in I} n_i &\Longrightarrow p \le f^{-1}\left(\bigvee_{i\in I} n_i\right) \cong \bigvee_{i\in I} f^{-1}(n_i) \\ &\Longrightarrow (\exists i \in I) \quad p \le f^{-1}(n_i) \\ &\Longrightarrow (\exists i \in I) \quad f(p) \le n_i \ ; \end{aligned}$$

hence $f(p) \in \mathcal{P}/Y$, and (A) follows. □

Now we can complete the proof of the Theorem and show that $C^{\oplus}$ is hereditary if C is. Indeed, if we consider $m \le y$ in $\mathcal{M}/X$ and denote by $m_Y : M \to Y$ the morphism with $m = y m_Y$, then there is an order-isomorphism

$$\begin{aligned} \{p \in \mathcal{P}/X : p \le m\} &\longleftrightarrow \{q \in \mathcal{P}/Y : q \le m_Y\} \\ p &\mapsto p_Y \\ y \cdot q &\leftarrow\!\shortmid q \end{aligned}$$

(note that, by (A), $y \cdot q \cong y(q)$ belongs to $\mathcal{P}$ if q does). Hence, with the Lemma, one obtains

$$\begin{aligned} y^{-1}(c_X^{\oplus}(m)) &\cong \bigvee\{y^{-1}(c_X(p)) : p \in \mathcal{P}/X\ ,\ p \le m\} \\ &\cong \bigvee\{c_Y(p_Y) : p \in \mathcal{P}/X\ ,\ p \le m\} \\ &\cong \bigvee\{c_Y(q) : q \in \mathcal{P}/Y\ ,\ q \le m_Y\} \\ &\cong c_Y^{\oplus}(m_Y)\ . \end{aligned}$$

EXAMPLES

(1) For the θ-closure operator of **Top** (with $\mathcal{M}$ given by subspace embeddings)

$$\theta_X\{x\} = \bigcap\{k_X(U) : U \text{ open nbhd of } x \text{ in } X\} = adh(x)$$

is the set of *adherence points* of the neighbourhood filter of the point x in X. Since the $\bigvee$-prime subobjects are described by singleton subspaces, one has

$$\theta_X^{\oplus}(M) = \bigcup_{x \in M} adh(x)$$

for every subset M of X.

(2) It is easy to see that the Kuratowski closure K and the sequential closure σ of **Top** coincide on points. Consequently, $\sigma^{\oplus} = K^{\oplus}$ is the point closure P of Example 2.6.

(3) The hypotheses (A) and (B) are usually not satisfied in "algebraic categories". For instance, in the category **Grp** of groups (with $\mathcal{M}$ the class of monomorphisms), $\bigvee$-prime subobjects must be cyclic subgroups (since every group is the join of its cyclic subgroups). On the other hand, since $\mathbb{Z} = n\mathbb{Z} + m\mathbb{Z}$ for n, m relatively prime, the cyclic group $\mathbb{Z}$ (as a subgroup of itself) is not $\bigvee$-prime, and since every non-zero subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, this group cannot be presented as the join of its $\bigvee$-prime subgroups.

In general, the situation cannot be improved by restriction to subcategories: in every full subcategory of **Grp** which is closed under finite direct products, every non-trivial group G contains a cyclic subgroup which is not $\bigvee$-prime; just consider the cyclic subgroup generated by (a, a) in $G \times G = (G \times \{e\})(\{e\} \times G)$, with e the neutral element of G and $a \neq e$.

(4) Although conditions (A) and (B) do not hold in $\mathbf{Mod}_R$ either, it is easy to "compute" the fully additive core of a closure operator in this category: since here every additive closure operator is minimal (see Exercise 3.M), both its additive and its fully additive core are given by its (easily computed) minimal core, as discussed at the beginning of the next section. See also Example 4.8(2).

REMARKS

(1) Conditions (A) & (B) imply (a) & (b) of 4.8. Indeed, the more general statement of ((A) & (B) $\Rightarrow$ (a)) follows from the Lemma, and for ((B) $\Rightarrow$ (b)) see Exercise 4.M.

(2) When trading $\bigvee$-prime elements for $\vee$-prime elements (so that $p \leq m_1 \vee m_2$ implies $p \leq m_1$ or $p \leq m_2$), one may construct the additive core C^+ analogously to the fully additive core $C^{\oplus}$.

(3) Instead of conditions (A) and (B), let us consider:

(α) *Meets distribute over arbitrary joins in each* $\mathcal{M}/X$, i.e., each $\mathcal{M}/X$ has the structure of a frame.

(β) *Every* $m \in \mathcal{M}/X$ *is the join of the compact elements below it*, i.e., each $\mathcal{M}/X$ has the structure of an algebraic dcpo.

Then, in our approach to $C^{\oplus}$, if we replace $\bigvee$-prime elements by compact elements and put

$$c_X^{\uparrow}(m) :\cong \bigvee\{c_X(k) : k \in \mathcal{M}/X \text{ compact}, \quad k \leq m\},$$

then $C^{\uparrow}$ *is a largest directedly-additive closure operator* $\leq C$ *, i.e., the directedly-additive core of* C . (Condition (α) is needed to prove the analogue of condition (A) in this context: $f(k)$ is compact for every $f : X \to Y$ in $\mathcal{E}$ and every compact $k \in \mathcal{M}/X$.)

We note that if C is additive, also $C^{\uparrow}$ is additive, in fact fully additive (by Theorem 2.6). Similarly, for C directedly additive, also the additive core C^{+} (if constructed under the hypotheses (a) and (b) of 4.8) is directedly additive and therefore fully additive. Consequently, in the presence of conditions (a), (b) of 4.8 and of (α) , (β) as above, there are two alternative ways to construct the fully additive core of an arbitrary closure operator C : as $(C^{+})^{\uparrow}$, and as $(C^{\uparrow})^{+}$.

4.10 Minimal core and hereditary hull

Clearly, for every closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ one can find a largest minimal closure operator $\leq C$. Indeed, every minimal closure operator $D \leq C$ must satisfy

$$d_X(m) \cong m \vee d_X(o_X) \leq m \vee c_X(o_X),$$

and it is elementary to show that

$$c_X^{\mathrm{mi}}(m) :\cong m \vee c_X(o_X)$$

defines in fact a minimal closure operator $C^{\mathrm{mi}} \leq C$ (for the Continuity Condition, just use preservation of joins under direct images). C^{mi} is called the *minimal core of* C . For properties of C^{mi} , see Exercise 4.N.

Clearly, for every closure operator of $\mathbf{Mod}_R$ with induced preradical $\mathbf{r}$, the minimal core is given by the minimal closure operator $C_{\mathbf{r}}$.

Since hereditariness is stable under meet, for each closure operator C of an $\mathcal{M}$-complete category one can find a least hereditary closure operator $\geq C$, the *hereditary hull* of C . As in the case of the (fully) additive core, we are able to give an explicit description of this hull only under additional hypotheses on $\mathcal{X}$ and $\mathcal{M}$.

Let $C \in CL(\mathcal{X}, \mathcal{M})$. For every hereditary closure operator $D \geq C$ and all $m : M \to X$ in $\mathcal{M}$ one has

$$\begin{aligned} c_X(m) &\leq \bigvee\{z^{-1}(c_Z(z \cdot m)) : z : X \to Z \text{ in } \mathcal{M}\} \\ &\leq \bigvee\{z^{-1}(d_Z(z \cdot m)) : z : X \to Z \text{ in } \mathcal{M}\} \end{aligned}$$

$$\cong \ d_X(m) .$$

We therefore define

$$c_X^{\mathrm{he}}(m) :\cong \bigvee\{z^{-1}(c_Z(z \cdot m)) : z : X \to Z \text{ in } \mathcal{M}\} . \tag{$*$}$$

Clearly, each function c_X^{he} is extensive and monotone. However, in order to verify the continuity condition we assume that $\mathcal{X}$ *has the* $\mathcal{M}$*-transferability property*: for every morphism $f : X \to Y$ and all $z : X \to Z$ in $\mathcal{M}$ there is a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "z"'] & Y \arrow[d, "w"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \tag{4.6}$$

with $w \in \mathcal{M}$. (If $\mathcal{X}$ has pushouts and if $\mathcal{M}$ is left-cancellable (w.r.t Mor$\mathcal{X}$) then $\mathcal{X}$ has the $\mathcal{M}$-transferability exactly when $\mathcal{M}$ is stable under pushout.) The $\mathcal{M}$-transferability property is satisfied if $\mathcal{X}$ *has enough* $\mathcal{M}$*-injectives* so that for every object Y there is a morphism $w : Y \to W$ with W $\mathcal{M}$-injective. (Recall that Y is $\mathcal{M}$*-injective* if for all $z : X \to Z$ in $\mathcal{M}$ and every morphism $h : X \to W$ there is a morphism $g : Z \to W$ with $g \cdot z = h$.) We are now ready to prove:

THEOREM *Under* each *of the following two hypotheses the hereditary hull* C^{he} *of a closure operator* $C \in CL(\mathcal{X}, \mathcal{M})$ *exists and is described by formula* $(*)$*:*

(a) $\mathcal{X}$ *is finitely* $\mathcal{M}$*-complete and has enough* $\mathcal{M}$*-injectives;*

(b) $\mathcal{X}$ *is* $\mathcal{M}$*-complete and each* $\mathcal{M}/X$ *has the structure of a frame (so that meets distribute over arbitrary joins:* $m \wedge \bigvee_{i \in I} m_i \cong \bigvee_{i \in I} m \wedge m_i$ *), and* $\mathcal{X}$ *has the* $\mathcal{M}$*-transferability property.*

Under condition (a) C^{he} *is idempotent if* C *is idempotent, and under condition* (b) C^{he} *is* (*fully*) *additive if* C *is* (*fully*) *additive.*

Proof Let us first operate under hypothesis (b). Obviously, the function c_X^{he} defined by $(*)$ is extensive and monotone. In order to verify the continuity condition, for $f : X \to Y$ and every $z : X \to Z$ in $\mathcal{M}$ we choose g_z and $w_z : Y \to W_z$ as in (4.6) and observe that

$$(w_z \cdot f)(z^{-1}(k)) \cong g_z(z(z^{-1}(k))) \leq g_z(k)$$

holds for all $k \in \mathcal{M}/Z$, hence $f(z^{-1}(k)) \leq w_z^{-1}(g_z(k))$. With $k := w_z(z \cdot m)$ (for $m \in \mathcal{M}/X$) this gives

$$f(c_X^{\mathrm{he}}(m)) \ \cong \ \bigvee\{f(z^{-1}(c_Z(z \cdot m))) : z : X \to Z \text{ in } \mathcal{M}\}$$

$$
\begin{aligned}
&\leq \bigvee\{w_z^{-1}(g_z(c_Z(z \cdot m))) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\leq \bigvee\{w_z^{-1}(c_{W_z}(g_z(z \cdot m))) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee\{w_z^{-1}(c_{W_z}(w_z \cdot f(m))) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\leq \bigvee\{w^{-1}(c_W(w \cdot f(m))) : w : Y \to W \quad \text{in} \quad \mathcal{M}\} \\
&\cong c_Y^{\mathrm{he}}(f(m)) \,.
\end{aligned}
$$

Hence, together with our initial considerations, we have that C^{he} is a closure operator $\geq C$ with $D \geq C^{\mathrm{he}}$ for every hereditary operator $D \geq C$. Under condition (b) one has that $y^{-1}(-)$ preserves joins for every $y : Y \to X$ in $\mathcal{M}$. Therefore,

$$
\begin{aligned}
y^{-1}(c_X^{\mathrm{he}}(m)) &\cong \bigvee\{y^{-1}(z^{-1}(c_Z(z \cdot m))) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee\{(z \cdot y)^{-1}(c_Z(z \cdot y) \cdot m_Y)) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee\{w^{-1}(c_Z(w \cdot m_Y)) : w : Y \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong c_Y^{\mathrm{he}}(m_Y)
\end{aligned}
$$

for every $m \in \mathcal{M}/X$ with $m \leq y$ and $y \cdot m_Y = m$. Hence C^{he} is the hereditary hull of C .

The following computation shows that (full) additivity is inherited by C^{he} from C , with $m \cong \bigvee_{i \in I} m_i$ in $\mathcal{M}/X$:

$$
\begin{aligned}
c_X^{\mathrm{he}}(m) &\cong \bigvee\{z^{-1}(c_Z(z \cdot m)) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee\{z^{-1}(c_Z(\bigvee_{i \in I} z \cdot m_i)) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee\{z^{-1}(\bigvee_{i \in I} c_Z(z \cdot m_i)) : z : X \to Z \quad \text{in} \quad \mathcal{M}\} \\
&\cong \bigvee_{i \in I}(\bigvee\{z^{-1}(c_Z(z \cdot m_i)) : z : X \to Z \quad \text{in} \quad \mathcal{M}\}) \\
&\cong \bigvee_{i \in I} c_X^{\mathrm{he}}(m_i) \,.
\end{aligned}
$$

Let us now assume that Condition (a) holds. We can then show that the join $(*)$ exists and is given by

$$
c_X^{\mathrm{he}}(m) \cong i^{-1}(c_W(i \cdot m)) \,, \qquad (**)
$$

for any $i : X \to W$ in $\mathcal{M}$ with W $\mathcal{M}$-injective. Indeed, for every $z : X \to Z$ in $\mathcal{M}$ there is a morphism $g : Z \to W$ with $g \cdot z = i$, and one has

$$
\begin{aligned}
z^{-1}(c_Z(z \cdot m)) &\leq z^{-1}(g^{-1}(g(c_Z(z \cdot m)))) \\
&\leq i^{-1}(c_W(g(z \cdot m))) \\
&\cong i^{-1}(c_W(i \cdot m)) \,.
\end{aligned}
$$

Hence, we obtain the closure operator $C^{\mathrm{he}} \geq C$ with $D \geq C^{\mathrm{he}}$ for every $D \geq C$, as under Condition (b). Under Condition (a) its hereditariness can be shown as follows (notation as above):

$$\begin{aligned} y^{-1}(c_X^{\mathrm{he}}(m)) &\cong y^{-1}(i^{-1}(c_W(i \cdot m))) \\ &\cong (i \cdot y)^{-1}(c_W((i \cdot y) \cdot m_Y)) \\ &\cong c_Y^{\mathrm{he}}(m_Y) . \end{aligned}$$

Finally, idempotency of C^{he} follows from idempotency of C :

$$\begin{aligned} c_X^{\mathrm{he}}(c_X^{\mathrm{he}}(m)) &\cong i^{-1}(c_X(i(i^{-1}(c_X(i \cdot m))))) \\ &\leq i^{-1}(c_X(c_X(i \cdot m))) \\ &\cong i^{-1}(c_X(i \cdot m) \cong c_X^{\mathrm{he}}(m)) . \end{aligned}$$

□

Corollary *If the $\mathcal{M}$-complete category $\mathcal{X}$ is $\mathcal{M}$-wellpowered and has enough $\mathcal{M}$-injectives, then $(C^{\infty})^{\mathrm{he}}$ is the least idempotent and hereditary closure operator $\geq C \in CL(\mathcal{X}, \mathcal{M})$. In the chain*

$$(C_{\infty})^{\infty} \leq (C^{\infty})_{\infty} \leq (C^{\mathrm{he}})^{\infty} \leq (C^{\infty})^{\mathrm{he}}$$

each closure operator is idempotent and weakly hereditary. The first and last inequalities collapse to isomorphisms when C is idempotent. □

Proof The first statement follows immediately from the Theorem. In Corollary 4.6 we showed $(C_{\infty})^{\infty} \leq (C^{\infty})_{\infty}$, and that "$\cong$" holds for C idempotent. The middle inequality follows trivially from

$$(C^{\infty})_{\infty} \leq C^{\infty} \leq (C^{\mathrm{he}})^{\infty} .$$

Since $C^{\mathrm{he}} \leq (C^{\infty})^{\mathrm{he}}$ with $(C^{\infty})^{\mathrm{he}}$ idempotent, one has $(C^{\mathrm{he}})^{\infty} \leq (C^{\infty})^{\mathrm{he}}$. In case $C \cong C^{\infty}$ this inequality becomes an isomorphism since then $(C^{\infty})^{\mathrm{he}} \cong C^{\mathrm{he}} \leq (C^{\mathrm{he}})^{\infty}$. □

Examples

(1) In **Top**, the hereditary hull of a non-hereditary closure operator may be quite large. In the case of the θ-closure, one obtains the indiscrete operator : $\theta^{\mathrm{he}} \cong G$. More precisely, with K^* as in Example 4.2 one has

$$K < KK^* < \theta \qquad \text{and} \qquad K = K^{\mathrm{he}} < (KK^*)^{\mathrm{he}} \cong \theta^{\mathrm{he}} \cong G .$$

In fact, in order to show $(KK^*)^{\mathrm{he}} \cong G$ it suffices to verify that

$$(KK^*)_X^{\mathrm{he}}(\{x\}) = X$$

holds for all $X \in \mathbf{Top}$ and $x \in X$. For that we take a new point y and define the space $Y := X \cup \{y\}$ in such a way that $\{y\}$ is open in Y and that every open set in Y which meets X contains y . Then $k_Y(k_Y^*(\{x\})) = Y$, hence $\{x\}$ is $(KK^*)^{\mathrm{he}}$-dense in Y . But since $(KK^*)^{\mathrm{he}}$ is hereditary, $\{x\}$ is also $(KK^*)^{\mathrm{he}}$-dense in X .

Since Y is a T_0-space if X is T_0 , this construction persists in the category $\mathbf{Top}_0$ of T_0-spaces.

(2) That $(KK^*)^{\mathrm{he}} \cong G$ holds in **Top** and in $\mathbf{Top}_0$ is quite surprising since for its restriction to the category $\mathbf{Top}_1$ of T_1-spaces one has

$$KK^*|_{\mathbf{Top}_1} = K|_{\mathbf{Top}_1} ,$$

which *is* hereditary. Nevertheless, one can show

$$(\theta|_{\mathbf{Top}_1})^{\mathrm{he}} \cong G|_{\mathbf{Top}_1} .$$

To see this, one proceeds as in (1) but considers the disjoint union $Y := X \cup \mathbb{N}$, with each point of $\mathbb{N}$ open, and with basic neighbourhoods of a point $x \in X$ having the form $U \cup C$, where U is neighbourhood of x in X and $C \subseteq \mathbb{N}$ cofinite. Now $\theta_Y(\{x\}) \supseteq X$, hence $\theta_X^{\mathrm{he}}(\{x\}) = X$ for every $x \in X$.

(3) In general, the inequality $(C^{\mathrm{he}})^\infty \leq (C^\infty)^{\mathrm{he}}$ is strict. For instance, the idempotent hull $\uparrow^\omega$ of the up-closure $\uparrow$ in **SGph** fails to be hereditary although $\uparrow$ is hereditary. (See Exercise 4.K). More generally, one can show for certain **Set**-based categories $\mathcal{X}$ (including **SGph** and **Top**) and for every hereditary closure operator C of $\mathcal{X}$ that C^∞ is hereditary if and only if C is idempotent (see 5.10 and Exercise 5.R; a more general proof is provided in 9.1).

(4) For a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$ one can find a least hereditary preradical $\mathbf{r}^{\mathrm{he}} \geq \mathbf{r}$ (with the partial order to be defined "pointwise"), namely

$$\mathbf{r}^{\mathrm{he}}(M) = \mathbf{r}(E(M)) \cap M ,$$

with $E(M)$ an injective hull of the R-module M (see Bican, Jambor, Kepka and Němec [1982]). We claim that

$$(C_{\mathbf{r}})^{\mathrm{he}} \cong C_{\mathbf{r}^{\mathrm{he}}} .$$

In fact, since $C_{\mathbf{r}^{\mathrm{he}}}$ is a hereditary closure operator $\geq C_{\mathbf{r}}$ (see Theorem 3.4(4)) one has $(C_{\mathbf{r}})^{\mathrm{he}} \leq C_{\mathbf{r}^{\mathrm{he}}}$. From the same theorem one obtains that the preradical $\mathbf{s}$ induced by $(C_{\mathbf{r}})^{\mathrm{he}}$ is a hereditary preradical $\geq \mathbf{r}$, hence $\mathbf{r}^{\mathrm{he}} \leq \mathbf{s}$ and therefore $C_{\mathbf{r}^{\mathrm{he}}} \leq C_{\mathbf{s}} \leq (C_{\mathbf{r}})^{\mathrm{he}}$.

In **AbGrp** a corresponding result holds for maximal closure operators: see Exercise 4.P.

4.11 Productivity of idempotent closure operators

All closure operators which we encountered in Section 3 are (at least) finitely productive. We also noted that in an additive category with finite products, every closure operator (with respect to any notion of subobject) is finitely productive: see Exercise 2.J. In this section we will give conditions in the general context of a category $\mathcal{X}$ with subobjects in $\mathcal{M}$ as in 2.1 which ensure that every idempotent closure operator is finitely productive, and then discuss an extension of this result to arbitrary productivity. In these cases there is therefore no reason to worry about the (finitely) productive hull of a closure operator (but see Exercise 4.Q). As in 4.9, our conditions are tailored for applications to topology and order theory, but in most cases not to algebra. However, note that the idea for the proof of the key lemma below is borrowed from the simple trick that one applies in the case of modules.

We assume that $\mathcal{X}$ has finite products and consider $m_i : M_i \to X_i$ in $\mathcal{M}$ $(i = 1, 2)$, and we first examine the behaviour of the sections for the canonical projections $p_i : X_1 \times X_2 \to X_i$. For any section $s : X_1 \to X_1 \times X_2$ for p_1 (hence $p_1 s = 1_{X_1}$) one has

$$s(1_{X_1}) \leq 1_{X_1} \times m_2$$

if and only if there is a section $s' : X_1 \to X_1 \times M_2$ for the projection $p'_1 : X_1 \times M_2 \to X_1$ with $(1_{X_1} \times m_2)s' = s$. For such a section there is a section $s'' : M_1 \to M_1 \times M_2$ for the projection $p''_1 : M_1 \times M_2 \to M_1$ with $(m_1 \times 1_{M_2})s'' = s'm_1$.

Consequently,

$$s(m_1) \leq (1_{X_1} \times m_2)(m_1 \times 1_{M_2}) = m_1 \times m_2 : M_1 \times M_2 \to X_1 \times X_2 .$$

Therefore, if we denote by

$$\mathrm{Sect}\,(p_1, m_2)$$

the class of sections s for p_1 with $s(1_{X_1}) \leq 1_{X_1} \times m_2$, one obtains

$$\bigvee \{s(m_1) : s \in \mathrm{Sect}(p_1, m_2)\} \leq m_1 \times m_2 \qquad (*)$$

We say that the *section condition* holds for m_1, m_2 if in $(*)$ one has "$\cong$", rather than "$\leq$". In greater detail, we say that *finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ are covered by their sections* if the section condition holds for all m_1 , $m_2 \in \mathcal{M}$. (Of course, since direct products commute up to isomorphism, in this case one symmetrically has

$$\bigvee \{t(m_2) : t \in \mathrm{Sect}(p_2, m_1)\} \cong m_1 \times m_2) .$$

Let now C be a closure operator with respect to $\mathcal{M}$. We shall write $c_i(m_i)$ instead of $c_{X_i}(m_i)$ and put $m = m_1 \times m_2 : M = M_1 \times M_2 \to X = X_1 \times X_2$.

LEMMA *If finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ are covered by their sections, then*

$$(c_1(m_1) \times m_2) \vee (m_1 \times c_2(m_2)) \leq c_X(m) .$$

Proof The continuity of every $s \in \mathrm{Sect}(p_1, m_2)$ yields

$$s(c_1(m_1)) \leq c_X(s(m_1)) \leq c_X(m) \, .$$

Hence the section condition for $c_1(m_1)$, m_2 gives

$$c_1(m_1) \times m_2 \leq c_X(m) \, .$$

Symmetrically one has $m_1 \times c_2(m_2) \leq c_X(m)$. □

PROPOSITION *If finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ are covered by their sections, then every idempotent closure operator of $\mathcal{X}$ with respect to $\mathcal{M}$ is finitely productive.*

Proof Since always $c_X(m) \leq c_1(m_1) \times c_2(m_2)$, it suffices to show "$\geq$". The Lemma gives

$$c_1(m_1) \times m_2 \leq c_X(m) \, ,$$

when applied to m_1 , m_2 and

$$c_1(m_1) \times c_2(m_2) \leq c_X(c_1(m_1) \times m_2)$$

when applied to $c_1(m_1)$, m_2 . But then, with the monotonicity of c_X one obtains

$$c_1(m_1) \times c_2(m_2) \leq c_X(c_X(m)) \, .$$

Hence the desired result follows since C is assumed to be idempotent. □

REMARKS

(1) The condition that finite products of subobjects be covered by their sections is certainly essential to obtain finite productivity of an idempotent closure operator. For example, if $\mathcal{X}$ and C are given by a lattice $(X, \leq)$ and a closure operator c of the lattice (see Example 4.5), then finite productivity of C amounts to the preservation of finite meets by c . But obviously, idempotency of c does not generally imply this preservation property. Indeed, the section condition fails badly here.

(2) Idempotency is certainly not a necessary condition for finite productivity: see Example (2) below.

(3) For an example of a closure operator of **Top** (in which finite products of subobjects are covered by sections, see Example (1) below) which fails to be finitely productive, see Exercise 4.U.

We now turn to (unrestricted) productivity and therefore assume the existence of (arbitrary) products in $\mathcal{X}$. For $\mathcal{M}$-subobjects $m_i : M_i \to X_i$ $(i \in I)$ and every subset $J \subseteq I$, let

$$m_J = \prod_{i \in J} m_i : M_J = \prod_{i \in J} M_i \to X_J = \prod_{i \in J} X_i \, ,$$

and let $p_J : X = X_I \to X_J$ be the canonical map given by projection. A closure operator C of $\mathcal{X}$ with respect to $\mathcal{M}$ is said to *have the finite structure property for products* if

$$x \leq c_X(m) \Leftrightarrow (\forall F \subseteq I \text{ finite})\, p_F(x) \leq c_F(p_F(m)) \tag{$*$}$$

holds for all x , $m \in \mathcal{M}/X$; equivalently, if

$$c_X(m) \cong \bigwedge_F p_F^{-1}(c_F(p_F(m)))$$

for all $m \in \mathcal{M}/X$. (Here we write c_F instead of c_{X_F} .) The case $m = m_I$ shows that *finite productivity and the finite structure property for products entail productivity, for non-trivial products, that is: for those products whose finite product projections belong to* $\mathcal{E}$. In fact, from the latter provision one has $m_F \cong p_F(m)$. Hence, with $x :\cong \prod_{i \in I} c_i(m_i)$, finite productivity of C gives

$$p_F(x) \leq \prod_{i \in F} c_i(m_i) \cong c_F(m_F) \cong c_F(p_F(m))$$

for every $F \subseteq I$ finite. Therefore, $x \leq c_X(m)$ follows from $(*)$. The following Theorem gives a much better criterion in case C is idempotent.

THEOREM *Let C be an idempotent and finitely productive closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$. If there exists a closure operator $D \leq C$ with the finite structure property for products, then C is productive for non-trivial products.*

Proof $D \leq C$ implies $C \leq DC \leq CC$, hence $C = DC$ since C is idempotent. Therefore, in order to show $x \leq c_X(m)$ for $m = m_I$ and $x \leq \prod_{i \in I} c_i(m_i)$, it suffices to show $x \leq d_X(c_X(m))$. But since D has the finite structure property for products, for that we need to show only $p_F(x) \leq d_F(p_F(c_X(m)))$ for every finite $F \subseteq I$.

In fact, since $m \cong m_F \times m_{I \setminus F}$ and since C is finitely productive, one has $c_X(m) \cong c_F(m_F) \times c_{I \setminus F}(m_{I \setminus F})$ and therefore $p_F(c_X(m)) \cong c_F(m_F)$. Furthermore, applying finite productivity again, one obtains $c_F(m_F) \cong \prod_{i \in F} c_i(m_i)$. Therefore, $x \leq \prod_{i \in I} c_i(m_i)$ implies

$$p_F(x) \leq c_F(m_F) \cong p_F(c_X(m)) \leq d_F(p_F(c_X(m))) \, ,$$

as required. □

COROLLARY *Let finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ be covered by their sections, and let $D \in CL(\mathcal{X}, \mathcal{M})$ have the finite structure property for products. Then every idempotent closure operator $C \geq D$ is productive for non-trivial products.*
□

EXAMPLES (1) In **Top**, with $\mathcal{M}$ the class of embeddings, one has the sections $s : X_1 \to X_1 \times X_2$ of the form $s(x_1) = (x_1, a_2)$ with $a_2 \in X_2$ constant, hence finite products of subspaces are covered by their sections. Furthermore, the Kuratowski closure operator K has the finite structure property for products. Consequently, *every idempotent closure operator* C *of* **Top** *is finitely productive, and it is productive in case* $C \geq K$. Here both idempotency of C and the condition $\geq K$ are essential, see (2) and (3) below.

(2) According to the Theorem, the idempotent hull θ^{∞} of the (productive) θ-closure of **Top** is productive. It will be shown in Chapter 10 that, by contrast, θ^{ω} is not productive. Since $\theta^{\omega} \geq \theta \geq K$, we therefore cannot dispense of idempotency in the Corollary.

(3) Similarly, for the (productive) sequential closure σ of **Top**, σ^{ω} is not productive, but here also $\sigma^{\infty} = \sigma^{\omega_1}$ (cf. Exercise 4.F(c)) fails to be productive (see Chapter 10). Consequently, $\sigma^{\infty} < K$.

(4) The Corollary may similarly be applied to the categories **PrTop** and **FC**, rather than to **Top**, with K now playing the roles of the Čech and Katětov closure operator, respectively.

(5) The product of a family $\{X_i\}_{i \in I}$ in **Top** is trivial if and only if there exist i and j such that X_i is empty but X_j is not. Since for trivial products (PR) trivially holds, one can drop the condition on non-trivial products in the Theorem and the Corollary in this case.

Exercises

4.A (*Computational rules for composition and cocomposition*)

(a) Prove the rules given in Lemmas 4.2 and 4.3.

(b) Show that a closure operator C is idempotent (weakly hereditary) if and only if $C \geq CC$ ($C \leq C * C$, respectively).

(c) Show for closure operators C and D

$$\begin{aligned} D * C &\cong D \wedge C && \text{if } D \text{ is hereditary,} \\ DC &\cong D \vee C && \text{if } D \text{ is minimal .} \end{aligned}$$

(d) Show the distributive laws

$$\begin{aligned} C(A \vee B) &\cong CA \vee CB && \text{if } C \text{ is additive,} \\ C * (A \wedge B) &\cong C * A \wedge C * B && \text{if } C \text{ is hereditary} \end{aligned}$$

for closure operators A , B , C .

(e) Show the inequality

$$(D * B)(C * A) \leq (DC) * (BA).$$

Using the discrete and the trivial closure operators, show that this inequality may be strict even when any two of the four closure operators A , B , C , D coincide.

(f) Show the distributive law

$$(D * D)C \cong (DC) * (DC) \quad \text{if } C \text{ is weakly hereditary.}$$

(g) Show that the inequality

$$(C \wedge D)^2 \leq C^2 \wedge D^2$$

is in general strict. *Hint*: In **SGph** (see 3.6), consider $\downarrow$ and $\uparrow$.

4.B (*Bad properties of* σ^2) Let σ be the sequential closure operator of **Top** (see 3.3). Show:

(a) σ^2 is not hereditary. *Hint*: Consider the Arhangel'skii - Franklin space X as described in the proof of Theorem 3.3. For $M = \{(x,y) : xy > 0\}$ and $Y = M \cup \{(0,0)\}$ one has $\sigma_X(M) = X \setminus \{(0,0)\}$, $\sigma_X^2(M) = X$, but $\sigma_Y^2(M) = M$. (Cf. Arhangel'skii and Franklin [1968].)

(b) Although σ^2 is weakly hereditary, there are subobjects $M \subseteq Y \subseteq X$ with M σ^2-dense in X but not σ^2-dense in Y (cf. Exercise 2.F; see also Exercise 4.C (a) below).

4.C (*Bad properties of* $C^{\mathbf{r}}$ *and* $C_{\mathbf{r}}$)

(a) Let $\mathbf{r}(A) = \mathbf{d}(A)$ be the maximal divisible subgroup of an abelian group A (see Example (1) of 4.3). Show that $\mathbb{Z}$ is $C^{\mathbf{r}}$-dense in $\mathbb{Q}$, but that there are intermediate groups $\mathbb{Z} \subseteq A \subseteq \mathbb{Q}$ such that $\mathbb{Z}$ is not $C^{\mathbf{r}}$-dense in A ; in fact, $\mathbb{Z} \subseteq A$ is $C^{\mathbf{r}}$-dense if and only if A is a subring of $\mathbb{Q}$.

(b) Let $\mathbf{r}(A) = \mathbf{soc}(A)$ be the socle of A (see Example (1) of 4.3). Show that $\{0\}$ is $(C^{\mathbf{r}})^2$ -dense in Z_{p^2} (the cyclic group of order p^2 , p prime), but not $C^{\mathbf{r}}$-dense.

(c) For $\mathbf{r}(A) = \mathbf{p}(A) = pA$ with a prime number p , show that $C_{\mathbf{r}} * C_{\mathbf{r}}$ is not minimal (see Example 4.6(2)).

4.D (*Cocomposites and joins of hereditary closure operators*)

(a) Complete the proof of Proposition 4.3.

(b) Show for hereditary closure operators C and D that $C \vee D$ is hereditary if each $\mathcal{M}/X$ is a distributive lattice.

4.E (*Meets of additive closure operators*)

(a) For $C = \uparrow$ and $D = \downarrow$ in **PoSet**, show that C and D are fully additive, idempotent and hereditary, but that $C \wedge D$ is not additive.

(b) Let **2Top** be the category of *bitopological spaces*: objects are triples (X, τ_1, τ_2) with τ_1, τ_2 topologies on the same set X, and maps must be continuous with respect to both topologies. Show that with the usual closure with respect to the first (second) topology, one obtains additive, idempotent and hereditary closure operators $K^{(1)}$ ($K^{(2)}$, respectively) of **2Top**, but $K^{(1)} \wedge K^{(2)}$ is not additive. *Hint*: Consider the space (X, τ_1, τ_2) with $X = \{a, b, c, d\}$ and $\{a, b\}$ the only proper τ_1 - open set in X and $\{a, c\}$ the only proper τ_2-open set in X ; examine the closure of $\{b\}$ and of $\{c\}$.

4.F *(Closure operators of order α and of co-order α)* For an ordinal number α , let $\mathcal{X}_\alpha$ be the category arising from the ordered set $\alpha + 1 = \{\beta : \beta \leq \alpha\}$ (as in Example (2) of 1.10).

(a) Let C be the closure operator of $\mathcal{X}_\alpha$ induced by the closure operator c with $c(\beta) = (\beta + 1) \wedge \alpha$ of the poset $\alpha + 1$ (see Exercise 2.E). Show that C has order α .

(b) Find a closure operator D of the category $\mathcal{X}_\alpha^{op}$ (the opposite category of $\mathcal{X}_\alpha$) with co-order α .

(c) Prove that the order of σ in **Top** (cf. 3.3) is ω_1 (the least uncountable ordinal).

4.G *(The extended ordinal chains for $C = C^{\mathbf{r}}$)*

(a) Preradicals in $\mathbf{Mod}_R$ are partially ordered by inclusion:

$$\mathbf{r} \leq \mathbf{s} \Longleftrightarrow (\forall M \in \mathbf{Mod}_R) \qquad \mathbf{r}(M) \leq \mathbf{s}(M) .$$

Show the existence of class-indexed meets and joins.

(b) Establish an extended descending chain of preradicals $\mathbf{r}^\alpha$ $(\alpha \in \mathit{Ord} \cup \{\infty\})$ for every preradical $\mathbf{r}$, with $\mathbf{r}^0(M) = M$ and $\mathbf{r}^{\alpha+1}(M) = \mathbf{r}(\mathbf{r}^\alpha(M))$ for all M .

(c) Establish an extended ascending chain of preradicals $\mathbf{r}_\alpha$ $(\alpha \in \mathit{Ord} \cup \{\infty\})$, with $\mathbf{r}_0(M) = 0$ and $\mathbf{r}_{\alpha+1}(M) = \pi^{-1}(\mathbf{r}(M/\mathbf{r}_\alpha(M)))$ for all M ; here $\pi : M \to M/\mathbf{r}_\alpha(M)$ is the projection.

(d) Prove by induction for all $\alpha \in \mathit{Ord} \cup \{\infty\}$:

$$(C^{\mathbf{r}})_\alpha \cong C^{(\mathbf{r}^\alpha)} \qquad (C^{\mathbf{r}})^\alpha \cong C^{(\mathbf{r}_\alpha)} .$$

(e) Define the *Loewy length* $l(\mathbf{r})$ (*Ulm length* $u(\mathbf{r})$) to be the least $\alpha \in$ Ord $\cup$ $\{\infty\}$ with $\mathbf{r}_{\alpha+1} = \mathbf{r}_\alpha$ ($\mathbf{r}^{\alpha+1} = \mathbf{r}^\alpha$, respectively). Sow that the maximal closure operator $C^{\mathbf{r}}$ has order $l(\mathbf{r})$ and co-order $u(\mathbf{r})$.

(f) Prove that, for $C = C^{\mathbf{r}}$ with $\mathbf{r} = \mathbf{soc\ d}$ as in Corollary 4.6, C_∞ is idempotent, but C is not; and that C^∞ is weakly hereditary, but C is not.

4.H (*Revisiting conditions* (ID), (WH), (CC), (CD) *of* 2.4)

(a) Recall that the implications

$$(ID)\&(CC) \Longrightarrow (WH) \Longrightarrow (CC) \quad \text{and} \quad (WH)\&(CD) \Longrightarrow (ID) \Longrightarrow (CD)$$

hold (see 2.4); give "direct" proofs of these implications, without referring to 1.5 or 1.8.

(b) Let C_∞ be the weakly hereditary core of C. Show that C_∞ is idempotent if and only if $\mathcal{E}^C$ is closed under composition.

(c) Let C^∞ be the idempotent hull of C. Show that C^∞ is weakly hereditary if and only if $\mathcal{M}^C$ is closed under composition.

(d) With the help of Exercise 4.G, find an example of a closure operator which simultaneously shows

$$(CC)\&(CD) \not\Longrightarrow (ID) \quad \text{and} \quad (CC)\&(CD) \not\Longrightarrow (WH)\,.$$

4.I (*The indiscrete closure operator* G)

(a) Determine which of the properties described in 2.4-2.7 are enjoyed by G.

(b) Prove for $m \in \mathcal{M}/X$: m is G-dense iff, for all $e : X \to Z$ in $\mathcal{E}$, $e(m) \cong o_Z$ implies that Z is trivial; m is G-closed iff, whenever $k \geq m$ has the property $(e(m) \cong o_Z \Rightarrow e(k) \cong o_Z)$ for all $e : X \to Z$ in $\mathcal{E}$, then $k \cong m$.

4.J (*Iterations and order of fully additive closure operators*)

(a) Verify that full additivity is stable under arbitrary join and under composition of closure operators. Conclude that every "power" C^α is fully additive if C is.

(b) Show that the order of a fully additive closure operator C is at most ω (the least infinite ordinal). *Hint*: Examine $c_X(c_X^\omega(m))$ for $m \in \mathcal{M}/X$. Conclude that $\uparrow$ and $\downarrow$ in **SGph** have order ω.

4.K (*Operations on* $\uparrow$ *and* $\downarrow$ *in* **SGph**)

(a) Show that every iteration $\uparrow^n$ $(n \geq 2)$ is weakly hereditary, grounded, fully additive and productive, but neither idempotent nor hereditary. *Hint*: For the first four properties, see Theorem 3.6 and Proposition 4.2. Then consider the graph $0 \to 1 \to \ldots \to (n+1)$ and compute $\uparrow^n \{0\}$.
For the subgraph H

$$0 \qquad 2 \to 3 \to 4 \to \ldots \to (n+1)$$

one has $n \in (\uparrow^n \{0\}) \cap H$, but $n \notin \uparrow^n_H \{0\}$.

(b) For integers m, $n \geq 1$ a subset M of a spatial graph G is (m,n)-*convex* if M contains every vertex x in G for which there is a subgraph

$$a_0 \to a_1 \to \ldots \to a_{m'} = x = b_0 \to b_1 \to \ldots \to b_{n'}$$

in G with $0 \leq m' \leq m$, $0 \leq n' < n$, and a_0, $b_{n'} \in M$.
1. Prove that M is $(\uparrow^m \wedge \downarrow^n)$ -closed if and only if M is $(m+1,\ n+1)$ -convex.
2. Show for m, $n \geq 1$

$$(\uparrow^{m+1} \wedge \downarrow^n) > (\uparrow^m \wedge \downarrow^n) < (\uparrow^m \wedge \downarrow^{n+1}).$$

Hint: Let G_s be the spatial graph $a \to 1 \to 2 \to \ldots \to s \to b$. Now evaluate the closure operators in question for the subgraph $\{a,b\}$ of G_{m+n}. (Here "$<$" means " $\leq$ and $\not\cong$ ", similarly for "$>$".)
3. Show for $mn > 1$

$$(\uparrow^m \wedge \downarrow^n) > (\uparrow \wedge \downarrow)^{\max\{m,n\}}.$$

Hint: Evaluate the closure operators for M in G_{m+n-1} (see 2.).
4. Prove that the closure operators $\uparrow^m \wedge \downarrow^n$ $(m,n \geq 1)$ are mutually non-isomorphic, i.e.

$$(\uparrow^m \wedge \downarrow^n) \cong (\uparrow^k \wedge \downarrow^l) \Longrightarrow m = k\ ,\ n = l\ .$$

5. Show that the operators $\uparrow^m \wedge \downarrow^n$ are grounded and productive, but neither idempotent nor additive nor hereditary. $\uparrow^m \wedge \downarrow^n$ is weakly hereditary iff $m = n = 1$.

(c) Show that $\uparrow^\omega$ and $\downarrow^\omega$ are grounded, fully additive, idempotent, weakly hereditary and finitely productive closure operators, but neither hereditary nor productive.
Hint: For a subgraph M of G one has $x \in \uparrow^\omega(M)$ iff there is a finite path $m \to \cdot \ldots \cdot \to x$ in G with $x \in M$. To show its non-productivity, consider the spatial graph $G = \mathbb{N}$:

$$1 \to 2 \to \ldots \to n \to \ldots$$

$\{1\}$ is $\uparrow^\omega$ -dense in G. On the other hand, $\{(1,1,1,\ldots)\}$ is not $\uparrow^\omega$-dense in $G^{\mathbb{N}}$ since there is no finite chain from $(1,1,1,\ldots)$ to $(1,2,3,\ldots)$ in $G^{\mathbb{N}}$.

4.L (*Interplay between idempotent hull and additive core*) (a) Under existence-guaranteeing assumptions on $\mathcal{X}$ and $\mathcal{M}$, show

$$(C^+)^\infty \leq (C^\infty)^+\ ;$$

both operators are isomorphic if C is idempotent or additive.

(b) Prove that for a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$ one has $(C^+)^\infty \cong (C^\infty)^+$ for every closure operator C with induced preradical $\mathbf{r}$ if and only if $\mathbf{r}$ is a radical.
Hint: First observe that $(C^+)^\infty$ is the minimal closure operator $C_{\mathbf{r}}$ (cf. Example (2) of 4.8). Then show that it suffices to consider $C = C^{\mathbf{r}}$.

4.M (*Fully additive core and generalized distributivity*) (a) Show that condition (B) of 4.9 implies the *generalized distributive law*

$$\bigvee_{i\in I}\bigwedge_{j\in J_i} m_{i,j} \cong \bigwedge_{(j_i)_{i\in I}\in\prod_{i\in I}J_i}\ \bigvee_{i\in I} m_{i,j_i}$$

in $\mathcal{M}/X$ for every $X \in \mathcal{X}$ (with $\prod_{i\in I} J_i \neq \emptyset$).

(b) Suppose that $f^{-1}(-)$ preserves arbitrary joins for every morphisms $f : X \to Y$ and that the generalized distributive law holds in $\mathcal{M}/X$, for every object X of the $\mathcal{M}$-complete category $\mathcal{X}$. Show that the fully additive core $C^{\oplus}$ of a closure operator C can be constructed by

$$c_X^{\oplus}(m) \cong \bigwedge\left\{\bigvee_{i\in I} c_X(m_i) : m_i \in \mathcal{M}/X(i\in I)\,,\ \bigvee_{i\in I} m_i \geq m\right\}.$$

4.N (*Minimal core*) Show that the minimal core C^{mi} of any $C \in CL(\mathcal{X},\mathcal{M})$ is idempotent and minimal, in particular fully additive. If C is hereditary and if each $\mathcal{M}/X$ is modular, then also C^{mi} is hereditary. (Recall that a lattice L is *modular* if for all $a,b,c \in L$ with $a \geq c$ one has $a\wedge(b\vee c) = (a\wedge b)\vee c$.)

4.O (*Hereditary hull of* $\mathfrak{k}$ *for Tychonoff spaces*) Show that the hereditary hull of the restriction of the $\mathfrak{k}$-closure to Tychonoff spaces (= completely regular T_2-spaces) is the Kuratowski closure:

$$(\mathfrak{k}|_{\mathbf{Tych}})^{\mathrm{he}} \cong K|_{\mathbf{Tych}}\ .$$

Hint: Consider any compactification Y of a Tychonoff space X . For every $M \subseteq X$ one has

$$\mathfrak{k}_X^{\mathrm{he}}(M) = \mathfrak{k}_Y^{\mathrm{he}}(M)\cap X \supseteq \mathfrak{k}_Y(M)\cap X = k_Y(M)\cap X = k_X(M)\ .$$

4.P (*Hereditary hull of maximal closure operators in* **AbGrp**) Prove that for every preradical $\mathbf{r}$ of **AbGrp** one has

$$(C^{\mathbf{r}})^{\mathrm{he}} \cong C^{\mathbf{r}^{\mathrm{he}}}$$

(see Example 4.10(4)).

4.Q (*Productive hull*) Show that every closure operator of an $\mathcal{M}$-complete category has a (finitely) productive hull. Investigate which properties of C are inherited by these hulls.

4.R (*Scott closure as idempotent hull*) Prove that the Scott closure operator of **DCPO** is the idempotent hull of the up-directed down closure: scott $= (\mathrm{dir}\downarrow)^{\infty}$.

4.S (*Finite structure property for products in* $\mathbf{Mod}_R$) Prove that if a closure operator of $\mathbf{Mod}_R$ has the finite structure property for products, then its induced preradical $\mathbf{r}$ is Jansenian and both $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ are productive.

4.T (*Properties of* Q) Check the properties of Q stated in Theorem 4.7.

4.U (*A closure operator of* **Top** *that is not finitely productive*) For a subset M of a topological space X, let $x \in j_X(M)$ iff $x \in k_X(\{a\})$ or $a \in k_X(\{x\})$ for some $a \in M$. Show that $J = (j_X)_{X \in \mathbf{Top}}$ is a hereditary and fully additive closure operator of **Top** which is neither idempotent nor finitely productive. Show also that $J = K^{\oplus} \vee K^*$, with K^* as in Example 4.2(3). *Hint*: To show that J is not finitely productive, consider $X = S \times S$, with S the 2-point 3-open-set Sierpiński space. (Cf. Dikranjan, Tholen and Watson [1995].)

4.V (*Closure operators of* **Grp** *and normality*) Show that for every closure operator C of **Grp** and for every normal subgroup $N \trianglelefteq G$ also $c_G(N)$ is normal in G. Conclude

$$\nu C \nu = C\nu,$$

and that $C\nu$ is idempotent whenever C is idempotent. In this case show that the $C\nu$-closed subgroups are exactly the C-closed normal subgroups.

Notes

The constructions for both the idempotent hull and the weakly hereditary core of a closure operator via infinite (co-)iterations can be found in Dikranjan and Giuli [1987a] who, however, do not formally introduce the binary (co-)composition of closure operators. Additive cores appear in the context of topological categories in Dikranjan [1992]. The categorical constructions for the additive core, the fully additive core and the hereditary hull of a closure operator do not seem to have been published previously, and the same is true for the general construction of the indiscrete operator and for the sufficient criteria for (finite) productivity given in 4.11.

5 Closure Operators, Functors, Factorization Systems

The functorial presentation of closure operators and their well-behavedness "along functors" are the dominating themes of this chapter. Briefly, closure operators are equivalently described by (generalized functorial) factorization systems. The interplay between closure operators and preradicals which we have seen for R-modules in 3.4 extends to arbitrary categories; it is described by adjunctions which are (largely) compatible with the compositional structure. With the notion of continuity for functors between categories that come equipped with closure operators we can define – like in topology – initial and final structures. This permits us to "transport" closure operators along functors. For adjoint functors and for $\mathcal{M}$-fibrations, these "transported" closure operators can be computed effectively.

In 5.4 we present criteria for detecting classes of dense subobjects which will be used frequently in subsequent chapters.

5.1 Pointed endofunctors and prereflections

In order to describe closure operators functorially we recall some general categorical notions which will be used in various contexts.

For an arbitrary category $\mathcal{K}$, one calls a pair (C,γ) with an endofunctor $C : \mathcal{K} \to \mathcal{K}$ and a natural transformation $\gamma : Id_{\mathcal{K}} \to C$ a *pointed endofunctor* of $\mathcal{K}$. (C,γ) is *idempotent* if

$$\gamma C = C\gamma : C \to CC$$

is an isomorphism of functors (i.e., a natural equivalence). Finally, a pointed endofunctor (C,γ) is called a *prereflection* if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\gamma_A} & CA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{\gamma_B} & CB \end{array} \tag{5.1}$$

in $\mathcal{K}$ one has $h = Cf$.

For a given class $\mathcal{E}$ of morphisms in $\mathcal{K}$, a pointed endofunctor (a prereflection) (C,γ) is *$\mathcal{E}$-pointed* (an *$\mathcal{E}$-prereflection*, respectively) if $\gamma_A \in \mathcal{E}$ for all $A \in \mathcal{K}$; in case $\mathcal{E}$ is the class of all epimorphisms in $\mathcal{K}$, we shall speak of *epipointed endofunctors* and *epiprereflections*.

With each pointed endofunctor (C,γ) of $\mathcal{K}$ one associates the class

$$\text{Fix}\,(C,\gamma) = \{A : \gamma_A \text{ iso}\}\ ,$$

considered as a full subcategory of $\mathcal{K}$.

EXAMPLES

(1) Let $\mathcal{A}$ be a *full reflective subcategory* of a category $\mathcal{X}$. Hence for every $X \in \mathcal{X}$ one has an object $RX \in \mathcal{A}$ and an $\mathcal{X}$-morphism $\rho_X : X \to RX$, the reflection of X into $\mathcal{A}$, such that every $\mathcal{X}$-morphism $f : X \to A$ with $A \in \mathcal{A}$ factors uniquely as

$$f = h \cdot \rho_X$$

with $h : RX \to A$. These data define uniquely an idempotent prereflection (R, ρ) of X . If $\mathcal{A}$ is *replete*, i.e. if $X \cong A \in \mathcal{A}$ implies $X \in \mathcal{A}$ for every $X \in \mathcal{X}$, then $\mathcal{A} = \text{Fix}\,(R, \rho)$.

(2) For a topological space X , let $\Pi_0(X)$ be the set of arc-components of X , provided with the quotient topology with respect to the natural map

$$\pi_X : X \to \Pi_0(X)\ .$$

(Π_0, π) is an epiprereflection of **Top**. $\text{Fix}(\Pi_0, \pi)$ is the category of arcwise totally disconnected spaces (i.e., each arc-component is a singleton set). (Π_0, π) is *not* idempotent: for the Topologist's Sine Curve

$$X = \{(x, \sin \frac{1}{x}) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\} \subseteq \mathbb{R}^2\ ,$$

$\Pi_0(X)$ is the (two-element) Sierpiński space, whereas $\Pi_0(\Pi_0(X))$ is a singleton space.

(3) For a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$, the projection

$$\pi_M : M \to M/\mathbf{r}(M) = RM$$

gives a epiprereflection (R, π) ; it is idempotent if and only if $\mathbf{r}$ is a radical.

PROPOSITION *In each* (1) *and* (2) *below, the implications*

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

hold for a pointed endofunctor (C, γ) *, whereas in* (3) *all three statements are equivalent:*

(1) (i) (C, γ) *is epipointed,*
(ii) (C, γ) *is a prereflection,*
(iii) $\gamma C = C\gamma$.

(2) (i) (C, γ) *is idempotent,*
(ii) $\gamma_{CA} = C\gamma_A$ *is monic for every* $A \in \mathcal{K}$,
(iii) (C, γ) *is a prereflection.*

(3) (i) (C,γ) *is idempotent,*

(ii) (C,γ) *is a prereflection with* $CA \in \mathrm{Fix}(C,\gamma)$ *for all* $A \in \mathcal{K}$,

(iii) $\mathrm{Fix}\,(C,\gamma)$ *is a reflective subcategory with reflexions* γ_A , $A \in \mathcal{K}$.

□

Proof

(1) (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (iii) For all $A \in \mathcal{K}$ apply the prereflection property to

$$\begin{tikzcd} A \arrow[r, "\gamma_A"] \arrow[d, "\gamma_A"'] & CA \arrow[d, "\gamma_{CA}"] \\ CA \arrow[r, "\gamma_{CA}"] & CCA \end{tikzcd} \tag{5.2}$$

to obtain $\gamma_{CA} = C\gamma_A$.

(2) (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (iii) Given the commutative diagram (5.1) we obtain

$$\gamma_{CB} \cdot h = Ch \cdot \gamma_{CA} = C\gamma_B \cdot Cf = \gamma_{CB} \cdot Cf \, ,$$

hence $h = Cf$.

(3) can be left as an exercise. □

LEMMA *For the pointed endofunctor* (C,γ) *of* $\mathcal{K}$ *with* $\gamma C = C\gamma$, *and for an object* $A \in \mathcal{K}$, *let* γ_A *be a section. Then* $A \in \mathrm{Fix}(C,\gamma)$.

Proof Suppose $f \cdot \gamma_A = 1_A$. Then

$$\gamma_A \cdot f = Cf \cdot \gamma_{CA} = Cf \cdot C\gamma_A = C1_A = 1_{CA} \, .$$ □

THEOREM *For a prereflection* (C,γ) *of* $\mathcal{K}$, *the full subcategory* $\mathrm{Fix}\,(C,\gamma)$ *of* $\mathcal{K}$ *is closed under all (existing) limits of* $\mathcal{K}$. *In particular, it is replete and closed under retracts.*

Proof Consider a functor $H : \mathcal{D} \to \mathcal{K}$ with $Hd \in \mathrm{Fix}\,(C,\gamma)$ for all $d \in \mathcal{D}$ such that its limit $L = \varprojlim H$ exists in $\mathcal{K}$. We must show $L \in \mathrm{Fix}\,(C,\gamma)$. The limit property yields a morphism $f : CL \to L$ rendering the diagram

$$\begin{tikzcd} L \arrow[r, "\lambda_d"] & Hd \\ CL \arrow[u, "f"] \arrow[r, "C\lambda_d"] & CHd \arrow[u, "\gamma_{Hd}^{-1}"'] \end{tikzcd} \tag{5.3}$$

commutative (with λ_d a limit projection). Since

$$\lambda_d \cdot f \cdot \gamma_L = \gamma_{Hd}^{-1} \cdot C\lambda_d \cdot \gamma_L = \gamma_{Hd}^{-1} \cdot \gamma_{Hd} \cdot \lambda_d = \lambda_d$$

for all $d \in \mathcal{D}$, one has $f \cdot \gamma_L = 1_L$. Now the Lemma (in conjunction with Proposition (1)) yields $L \in \text{Fix}\,(C, \gamma)$.

If B is a retract of A, so that there are morphisms $s : B \to A$, $r : A \to B$ with $rs = 1_B$, then one has an equalizer diagram

$$B \xrightarrow{s} A \underset{sr}{\overset{1_A}{\rightrightarrows}} A \tag{5.4}$$

Hence $A \in \text{Fix}\,(C, \gamma)$ implies $B \in \text{Fix}\,(C, \gamma)$ by the preceding observation. In particular, $\text{Fix}\,(C, \gamma)$ is replete. □

5.2 Closure operators are prereflections

Throughout this section, let $\mathcal{M}$ be an arbitrary class of morphisms in the category $\mathcal{X}$. $\mathcal{M}$ can be considered a *category* which, by abuse of notation, is denoted by $\mathcal{M}$ again, as follows: its objects are the elements of the class $\mathcal{M}$, and a morphism

$$(u, v) : m \to n$$

in the category $\mathcal{M}$ is given by a pair of morphisms in $\mathcal{X}$ such that

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "v"] & Y \end{tikzcd} \tag{5.5}$$

commutes. There are the *projection functors*

$$\text{dom} : \mathcal{M} \to \mathcal{X} \quad \text{and} \quad \text{cod} : \mathcal{M} \to \mathcal{X}$$

which assign to each object $m \in \mathcal{M}$ its domain and its codomain, respectively, and there is the *structure transformation*

$$\sigma : \text{dom} \to \text{cod},$$

given by $\sigma_m := m$ for all $m \in \mathcal{M}$.

Let us now go back to the situation of 2.1/2.2 and consider a closure operator $C = (c_X)_{X \in \mathcal{X}}$ of $\mathcal{X}$ w.r.t. $\mathcal{M}$. Then we have in fact a functor

$$C : \mathcal{M} \to \mathcal{M}\,, \qquad (m : M \to X) \mapsto c_X(m)\,,$$

which assigns to a morphism $(u, v) : m \to n$ in the category $\mathcal{M}$ the morphism $(w, v) : c_X(m) \to c_X(n)$; here $w : c_X(M) \to c_X(N)$ is the "diagonal morphism" of diagram (2.3). Furthermore, we have a natural transformation

$$\gamma : Id_M \to C\,, \qquad \gamma_m := (j_m, 1_X) : m \to c_X(m)\,,$$

arising from the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "j_m"] \arrow[d, "m"'] & c_X(M) \arrow[d, "c_X(m)"] \\
X \arrow[r, "1_X"] & X
\end{tikzcd} \tag{5.6}$$

Naturality of γ is easily checked by chasing along the arrows in

$$\begin{tikzcd}
 & M \arrow[rr, "u"] \arrow[dl, "j_m"'] \arrow[dd, "m" near end] & & N \arrow[dl, "j_n"'] \arrow[dd, "n"] \\
c_X(M) \arrow[rr, "w" near start] \arrow[dd, "c_X(m)"'] & & c_Y(N) \arrow[dd, "c_Y(n)" near start] & \\
 & X \arrow[rr, "v" near start] \arrow[dl, "1_X"] & & Y \arrow[dl, "1_Y"] \\
X \arrow[rr, "v"'] & & Y &
\end{tikzcd} \tag{5.7}$$

We observe that each γ_m belongs to the following class of morphisms of the category $\mathcal{M}$:

$$(\mathcal{M}, 1) := \{(u, v) : u \in \mathcal{M}\,,\ v = 1\}\,.$$

(We note that for any morphism $(u, v) : m \to n$ in the category $\mathcal{M}$, $v = 1$ necessarily yields $u \in \mathcal{M}$). Hence (C, γ) is an $(\mathcal{M}, 1)$ -pointed endofunctor of $\mathcal{M}$. But when $\mathcal{M}$ is a class of monomorphisms in $\mathcal{X}$, then $(\mathcal{M}, 1)$ is a class of monomorphisms in $\mathcal{M}$, hence by (2)(ii) $\Rightarrow$ (iii) of Proposition 5.1, (C, γ) is in fact an $(\mathcal{M}, 1)$ -prereflection.

This proves most of the following Theorem; the rest of its proof is routine work.

THEOREM *For $\mathcal{X}$ and $\mathcal{M}$ as in 2.1, there is a bijective correspondence between*

- *closure operators of $\mathcal{X}$ w.r.t. $\mathcal{M}$, and*

• $(\mathcal{M},1)$*-prereflections of* $\mathcal{M}$.

Idempotent closure operators correspond exactly to idempotent $(\mathcal{M},1)$*-prereflections.* □

Under the correspondence described by the Theorem, isomorphism classes of closure operators correspond to isomorphism classes of $(\mathcal{M},1)$-prereflections. Recall that the closure operator $C = (c_X)_{X\in\mathcal{X}}$ and $D = (d_X)_{X\in\mathcal{X}}$ are isomorphic if $C \leq D$ and $D \leq C$, i.e., if $c_X(M) \cong d_X(m)$ for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$. This gives an isomorphism of functors $\alpha : C \xrightarrow{\sim} D$ for the induced $(\mathcal{M},1)$ - prereflections (C,γ) and (D,δ) , with the additional property that

$$\begin{tikzcd} & Id_{\mathcal{M}} \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ C \arrow[rr, "\alpha"] & & D \end{tikzcd} \tag{5.8}$$

commutes. This, by definition, means $(C,\gamma) \cong (D,\delta)$.

The Theorem indicates how to expand the notion of closure operator to the case that $\mathcal{M}$ is an *arbitrary* class of morphisms of $\mathcal{X}$:

DEFINITION A *closure operator of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$ is an $(\mathcal{M},1)$-prereflection of the category $\mathcal{M}$. In other words, the Theorem holds true by definition in the arbitrary case.

Also for a closure operator (C,γ) in the general context, when there is no danger of confusion, we often simply call C a closure operator. For $\mathcal{M}$ a class of monomorphisms, γ is uniquely determined by C anyway: for every $m \in \mathcal{M}$ one has $\mathrm{cod}\,\gamma_m = 1$, and $j_m = \mathrm{dom}\,\gamma_m$ is determined by $c_X(m)\cdot j_m = m$ when $c_X(m)$ is monic.

Since, *in the context of* 2.1 *each* $(\mathcal{M},1)$*-pointed endofunctor of* $\mathcal{M}$ *is a prereflection*, requiring a closure operator in the general context to be an $(\mathcal{M},1)$ - prereflection rather than just an $(\mathcal{M},1)$-pointed endofunctor seems like an unnecessary complication. However, a closer examination shows that when $\mathcal{M}$ is not necessarily a class of monomorphisms in $\mathcal{X}$, the prereflection requirement means precisely that the crucial Diagonalization Lemma remains true *verbatim*.

As in 2.3, also for a closure operator (C,γ) w.r.t. an arbitrary $\mathcal{M}$ one calls $m : M \to X$ in the class $\mathcal{M}$ to be *C-closed* if $j_m = \mathrm{dom}\,\gamma_m$ is an isomorphism; as before, let $\mathcal{M}^C$ be the class of C-closed elements of $\mathcal{M}$. We now have a common proof for Proposition 1.7 and Theorem 2.3, even in the current more abstract context; see the Remark below.

COROLLARY *For every closure operator* (C,γ) *with respect to a class* $\mathcal{M}$ *of*

$\mathcal{X}$-morphisms, which is closed under the formation of $\mathcal{D}$-limits, also the class $\mathcal{M}^C$ is closed under $\mathcal{D}$-limits, for any diagram type $\mathcal{D}$. □

Proof One considers a natural transformation $\mu : H \to K$ with $H, K : \mathcal{D} \to \mathcal{X}$ such that $m : \varprojlim H \to \varprojlim K$ exists in $\mathcal{X}$. We shall show that if $\mu_d \in \mathcal{M}^C$ for all $d \in \mathcal{D}$ and $m \in \mathcal{M}$, then $m \in \mathcal{M}^C$. Indeed, $\mu : H \to K$ yields a functor $M : \mathcal{D} \to \mathcal{M}$ with dom $M = H$, cod $M = K$, $\sigma M = \mu$. One routinely verifies that $m \cong \varprojlim M$ in the category $\mathcal{M}$ which contains $\mathcal{M}^C$ as a full subcategory. But by definition, $\mathcal{M}^C = \text{Fix}\,(C, \gamma)$, hence Theorem 5.1 yields $m \in \mathcal{M}^C$. □

REMARK Clearly the Corollary generalizes Theorem 2.3. That it also generalizes Proposition 1.7 can be seen as follows. If $\mathcal{X}$ has right $\mathcal{M}$-factorizations, then one can define a closure operator (C, γ) with respect to $\mathcal{M}_0 := \text{Mor}\,\mathcal{X}$ the class of all morphisms of $\mathcal{X}$, by putting $Cf := m$ and $\gamma_f := (e, 1)$ with a right $\mathcal{M}$-factorization $f = m \cdot e$. Now $\mathcal{M} = \mathcal{M}_0^C = \text{Fix}\,(C, \gamma)$ is closed under $\mathcal{D}$-limits in Mor $\mathcal{X}$, by the Corollary.

When considered without restrictions on the class $\mathcal{M}$, the concept of closure operator w.r.t. $\mathcal{M}$ becomes self-dual. To see this, let us re-draw diagram (5.8) as

$$\begin{tikzcd} M \arrow[r, "1_M"] \arrow[d, "j_m"'] & M \arrow[d, "m"] \\ c_X(M) \arrow[r, "c_X(m)"] & X \end{tikzcd} \tag{5.9}$$

It now represents a morphism

$$\delta_m := (1_M, c_X(m)) : j_m \to m$$

in the category $\mathcal{M}$. With $Dm := j_m$ one obtains a *copointed endofunctor* (D, δ) (so that $D : \mathcal{M} \to \mathcal{M}$ is a functor and $\delta : D \to Id_{\mathcal{M}}$ is a natural transformation), with the additional property that each δ_m belongs to the class

$$(1, \mathcal{M}) := \{(u, v) : u = 1\,,\ v \in \mathcal{M}\}$$

of morphisms in the category $\mathcal{M}$.

Dually to the notions introduced one says that a copointed endofunctor (D, δ) is *idempotent* if $\delta D = \delta$ is an isomorphism, and a *precoreflection* if $\delta_B \cdot h = f \cdot \delta_A$ for $f : A \to B$ implies $f = Dh$.

Routine work shows:

THEOREM* *For any class $\mathcal{M}$ of morphisms in $\mathcal{X}$, there is a bijective correspondence between*

- *closure operators of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$ *, and*

- $(1, \mathcal{M})$*-precoreflections of the category* $\mathcal{M}$.

For $\mathcal{X}$ *and* $\mathcal{M}$ *as in* 2.1, *weakly hereditary closure operators correspond to idempotent* $(1, \mathcal{M})$*-precoreflections.*

REMARK* Finally we ought to supply a proof of Theorem* of 2.3 which states that, in the context of 2.1, the class $\mathcal{E}^C$ of C-dense morphism is closed under $\mathcal{D}$-colimits. For that one defines an $(1, \mathcal{M}_0)$-precoreflection $(D\,,\ \delta)$ of $\mathcal{M}_0 :=$ Mor $\mathcal{X}$, by putting $Df = j_m \cdot e$ and $\delta_f := (1_X, c_Y(m))$ for every $f : X \to Y$ in $\mathcal{X}$ and $f = m \cdot e$ a right $\mathcal{M}$-factorization. Then $\mathcal{E}^C = \mathrm{Fix}\,(D, \delta)$ is closed under $\mathcal{D}$-colimits, by the dual of the Corollary above. □

5.3 Factorization systems

In 1.6 and 1.8, we introduced the notions of right $\mathcal{M}$-factorization and $(\mathcal{E}, \mathcal{M})$-factorization, respectively. For a closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$, and under the assumptions of 2.1, we considered the factorization

$$f = c_f \cdot d_f \qquad (*)$$

with $c_f := c_Y(m)$ and $d_f := j_m \cdot e$, for $f = m \cdot e$ an $(\mathcal{E}, \mathcal{M})$-factorization of $f : X \to Y$. This factorization gives a right $\mathcal{M}^C$-factorization of f provided C is idempotent, and it is an $(\mathcal{E}^C, \mathcal{M}^C)$-factorization of f if, in addition, C is weakly hereditary. In general, the essential property of $(*)$ is as follows:

LEMMA (*Diagonalization Property*) *For every commutative diagram*

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"] & \cdot \end{tikzcd} \qquad (5.10)$$

of morphisms in $\mathcal{X}$, *there is a uniquely determined morphism* w *rendering the diagram*

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "d_f"'] & \cdot \arrow[d, "d_g"] \\ \cdot \arrow[r, "w"] \arrow[d, "c_f"'] & \cdot \arrow[d, "c_g"] \\ \cdot \arrow[r, "v"] & \cdot \end{tikzcd} \tag{5.11}$$

commutative.

Proof First consider $(\mathcal{E}, \mathcal{M})$-factorizations $f = m \cdot e$ and $g = m' \cdot e'$ and apply the $(\mathcal{E}, \mathcal{M})$-diagonalization property to obtain a morphism u' with $u'e = e' \cdot u$ and $m' \cdot u' = u \cdot m$. Then apply the Diagonalization Lemma 2.4 to the latter identity to obtain the desired morphism w; it is trivially unique since $\mathcal{M}$ is a class of monomorphisms. □

The Corollary leads us to the following definition which requires no assumptions on the category $\mathcal{X}$:

DEFINITION A *factorization system* F of $\mathcal{X}$ is a map which gives, for every morphism f, a pair (d_f, c_f) of morphisms in $\mathcal{X}$ such that $(*)$ and the Diagonalization Property hold. (Note that there are no further conditions on d_f and c_f; for instance, c_f need not be monic in $\mathcal{X}$.) We call $(*)$ the *F-factorization* of f.

For every factorization system F, there are associated classes of morphisms,

$$\mathcal{D}^F = \{h : c_h \text{ is iso }\} \quad \text{and} \quad \mathcal{C}^F = \{h : d_h \text{ is iso }\},$$

the *left* and *right factorization class of* F, respectively. F is called a *left* (*right*) *factorization system* if, for every morphism f, one has $d_f \in \mathcal{D}^F$ ($c_f \in \mathcal{C}^F$, respectively). An *orthogonal* system is both, a left and right system.

Let us first note the following elementary properties:

REMARK For a factorization system F of $\mathcal{X}$ one has:

(1) The only endomorphism t with $td_f = d_f$ and $c_f t = c_f$ is $t = 1$.

(2) If in (5.10) both u and v are isomorphisms, then also the diagonal w is iso.

(3) $\mathcal{D}^F \cap \mathcal{C}^F$ is the class of all isomorphisms of $\mathcal{X}$.

(4) Each $\mathcal{D}^F$ and $\mathcal{C}^F$ is closed under composition with isomorphisms of $\mathcal{X}$.

For the proof of (1), use the uniqueness part of the Diagonalization Property in case $g = f$ and $u = 1$, $v = 1$. (2) and (3) follow from (1). For (4), consider an isomorphism v such that the composite $v \cdot f$ exists. Then, from (2), one has $f \in \mathcal{D}^F$ iff $v \cdot f \in \mathcal{D}^F$, and the same for $\mathcal{C}^F$. Analogously one treats the case of an isomorphism u such that the composite $g \cdot u$ exists. □

Two factorization systems $F = (f \mapsto (d_f, c_f))$ and $F' = (f \mapsto (d'_f, c'_f))$ of $\mathcal{X}$ are *isomorphic* (written as $F \cong F'$) if, for every morphism f, there is an isomorphism α_f with $\alpha_f d_f = d'_f$ and $c'_f \alpha_f = c_f$. From assertion (4) of the Remark one has that $F \cong F'$ implies $\mathcal{D}^F = D^{F'}$ and $\mathcal{C}^F = \mathcal{C}^{F'}$. However, the converse proposition does not hold in general; in other words: in general, the left and right factorization classes of a factorization system F do not determine the system:

EXAMPLE Consider the poset $(\mathbb{R}, \leq)$ as a small category $\mathcal{X}$ in the usual way (see Example (2) of 1.10). A factorization system on $\mathcal{X}$ chooses monotonely, for every pair (x, y) of real numbers (that is: for a morphism $x \to y$ in $\mathcal{X}$), a point z in the closed interval $[x, y]$ (that is: a factorization $x \to z \to y$ in $\mathcal{X}$). For instance, for every t with $0 \leq t \leq 1$, the assignment

$$(x, y) \mapsto tx + (1 - t)y$$

yields a factorization system F_t of $\mathcal{X}$. For $0 < t < 1$, one has

$$tx + (1 - t)y = x \Longleftrightarrow x = y \Longleftrightarrow tx + (1 - t)y = y ;$$

hence $\mathcal{D}^{F_t} = \mathcal{C}^{F_t}$ is the class of identity morphisms in $\mathcal{X}$. Nevertheless, for $t \neq s$, the factorization systems F_t and F_s are not isomorphic. We also note that the system F_t is neither a left nor a right system, unless $t = 0$ or $t = 1$, in which case F_t is orthogonal.

The notion of factorization system subsumes the notions of factorizations considered in Chapter 1: assigning to every morphism its right $\mathcal{M}$-factorization (provided it exists, with $\mathcal{M}$ closed under composition with isomorphisms) yields a right factorization system F with $\mathcal{C}^F = \mathcal{M}$. Conversely, for a right factorization system F, every morphism has a right $\mathcal{C}^F$-factorization. Unlike a general factorization system, a right system F is therefore determined (up to isomorphisms) by the class $\mathcal{C}^F$.

In summary, with the terminology introduced in 1.5 and 1.8 we have:

PROPOSITION *For every category $\mathcal{X}$ there is a bijective correspondence between*

- *classes $\mathcal{M}$ (closed under composition with isomorphisms) such that every morphism in $\mathcal{X}$ has a right $\mathcal{M}$-factorization,*
- *isomorphism classes of right factorization systems of $\mathcal{X}$.*

This correspondence has a left counter-part, and both correspondences can be restricted to a bijection between

- *pairs* $(\mathcal{E}, \mathcal{M})$ *such that* $\mathcal{X}$ *has* $(\mathcal{E}, \mathcal{M})$*-factorizations,*
- *isomorphism classes of orthogonal factorization systems of* $\mathcal{X}$*.* □

We already saw in the preamble to this section that, in the context of 2.1, a closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ yields, via $(*)$, a factorization system F of $\mathcal{X}$; one easily sees that

$$\mathcal{D}^F = \mathcal{E}^C \qquad \text{and} \qquad \mathcal{C}^F = \mathcal{M}^C \ .$$

F is a right system iff C is idempotent, and a left system iff C is weakly hereditary.

A closure operator defines a factorization system even in the more general context of 5.2, provided $\mathcal{X}$ has right $\mathcal{M}$-factorizations, with any class $\mathcal{M}$ which is closed under composition with isomorphisms. In fact, the uniqueness part of the *Diagonalization Property* still holds in this case (without the assumption that $\mathcal{M}$ is a class of monomorphisms), as one readily verifies. Hence an (idempotent; weakly hereditary) closure operator (C, γ) of $\mathcal{X}$ w.r.t. $\mathcal{M}$ defines, via $(*)$ a (right; left respectively) factorization system F of $\mathcal{X}$, with $\mathcal{C}^F \subseteq \mathcal{M}$.

Conversely, every factorization system F of $\mathcal{X}$ gives a closure operator (C, γ) of $\mathcal{X}$ w.r.t. the class $\mathcal{M}_0 = \mathrm{Mor}\,\mathcal{X}$ of all $\mathcal{X}$-morphisms, with $C(f) := c_f$, $\gamma_f := d_f$, and with $f = c_f \cdot d_f$ the F-factorization of f . If $\mathcal{X}$ has right $\mathcal{M}$-factorizations, and if $c_m \in \mathcal{M}$ for every $m \in \mathcal{M}$, then we may restrict (C, γ) to become a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$; the second condition certainly holds if F is a right system with $\mathcal{C}^F \subseteq \mathcal{M}$.

In summary we have that factorization systems provide an alternative in (fact: self-dual) description of closure operators, or vice versa. More precisely, the following holds:

THEOREM *For every category* $\mathcal{X}$*, there is a bijective correspondence between*

- *isomorphism classes of (idempotent; weakly hereditary) closure operators of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}_0 = \mathrm{Mor}\,\mathcal{X}$ *,*
- *isomorphism classes of (right; left, respectively) factorization systems of* $\mathcal{X}$*.*

For a fixed class $\mathcal{M} \subseteq \mathrm{Mor}\,\mathcal{X}$ *(closed under composition with isomorphisms) for which* $\mathcal{X}$ *has right* $\mathcal{M}$*-factorizations, this correspondence gives a bijection between*

- *isomorphism classes of idempotent closure operators of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$*,*
- *isomorphism classes of right factorization systems* F *with* $\mathcal{C}^F \subseteq \mathcal{M}$ *.* □

For C and F corresponding to each other, one has $\mathcal{M}_0^F = \mathcal{C}^F$. Hence, from Corollary 5.2 we obtain the first half of the following Corollary; its second half follows by dualization:

COROLLARY *For a factorization system F of $\mathcal{X}$, the class $\mathcal{C}^F$ is closed under $\mathcal{H}$-limits and $\mathcal{D}^F$ is closed under $\mathcal{H}$-colimits, for any diagram type $\mathcal{H}$.* □

5.4 Recognizing classes of C-dense and C-closed subobjects

We return to the standard situation of 2.1 and assume that $\mathcal{X}$ is $\mathcal{M}$-complete. Associated with every closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ are the classes $\mathcal{E}^C \cap \mathcal{M}$ and $\mathcal{M}^C$ of C-dense and C-closed $\mathcal{M}$-subobjects. Vice versa, given subclasses $\mathcal{D}$ and $\mathcal{C}$ of $\mathcal{M}$, we may ask which properties are required to recognize them as classes of C-dense and C-closed $\mathcal{M}$-subobjects, respectively, for a suitable closure operator C?

For a class $\mathcal{C} \subseteq \mathcal{M}$ to have a closure operator C w.r.t. $\mathcal{M}$ with $\mathcal{C} = \mathcal{M}^C$, necessarily $\mathcal{C}$ must be stable under pullback and multiple pullback; the latter property means that the $\mathcal{M}$-intersection of a family in $\mathcal{C}$ belongs to $\mathcal{C}$ (see Theorem 2.3). Vice versa, since $\mathcal{X}$ is $\mathcal{M}$-complete, stability of $\mathcal{C}$ under pullback and $\mathcal{M}$-intersection means that $\mathcal{X}$ is also $\mathcal{C}$-complete, hence one has right $\mathcal{C}$-factorizations (Theorem 1.9). According to Theorem 5.3, such a factorization corresponds to an (idempotent) closure operator C w.r.t. $\mathcal{M}$ with $\mathcal{M}^C = \mathcal{C}$. Hence one has:

PROPOSITION *A class $\mathcal{C} \subseteq \mathcal{M}$ is the class of C-closed $\mathcal{M}$-subobjects for some closure operator C (w.r.t. $\mathcal{M}$) if and only if $\mathcal{C}$ is stable under pullback and under $\mathcal{M}$-intersections. In this case, there is, up to isomorphism, only one idempotent closure operator C with $\mathcal{M}^C = \mathcal{C}$.* □

However, the dual of the Proposition does not give a characterization of classes $\mathcal{D}$ of the form $\mathcal{D} = \mathcal{E}^C \cap \mathcal{M}$ which, as we remarked in 2.3, need not be stable under pushout. In what follows we will show that there is a more lattice-theoretical version of the Proposition which also leads us to a characterization of classes of dense subobjects.

DEFINITION

(1) A class $\mathcal{A} \subseteq \mathcal{M}$ is called *left cancellable w.r.t.* $\mathcal{M}$ if $n \cdot m \in \mathcal{A}$ with $m, n \in \mathcal{M}$ implies $m \in \mathcal{A}$. Dually, $\mathcal{A}$ is *right cancellable w.r.t.* $\mathcal{M}$ if $n \cdot m \in \mathcal{A}$ with $m, n \in \mathcal{M}$ implies $n \in \mathcal{A}$.

(2) For every $f : X \to Y$ in $\mathcal{X}$, $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$, one has the following commutative diagram

$$\begin{tikzcd} M \wedge f^{-1}(N) \arrow[r] \arrow[d, "i"'] & N \arrow[d, "k"] \\ M \arrow[r] & f(M) \vee N \end{tikzcd} \tag{5.12}$$

Here the vertical arrows are canonical injections, and the horizontal arrows arise as restrictions of f. We say that a class $\mathcal{A} \subseteq \mathcal{M}$ has the $\wedge$-$\vee$-*reflection property* if, for all f, m, n as above, $k \in \mathcal{A}$ implies $i \in \mathcal{A}$; and $\mathcal{A}$ has the $\wedge$-$\vee$-*preservation property* if $i \in \mathcal{A}$ implies $k \in \mathcal{A}$.

REMARK In (5.12), it suffices to consider the case $m \cong 1_X$ and $f(1_X) \vee n \cong 1_Y$, i.e. pullback diagrams

$$\begin{tikzcd} f^{-1}(N) \arrow[r] \arrow[d, "f^{-1}(n)"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{5.13}$$

with $f(1_X) \vee n \cong 1_Y$. Hence $\mathcal{A}$ has the $\wedge$-$\vee$-reflection property ($\wedge$-$\vee$-preservation property) if and only if for every such pullback diagram, $n \in \mathcal{A}$ implies $f^{-1}(n) \in \mathcal{A}$ ($f^{-1}(n) \in \mathcal{A}$ implies $n \in \mathcal{A}$, respectively). Indeed, easy diagram chasing shows that diagram (5.12) is a pullback diagram of type (5.13) which, in turn, is a special case of (5.12).

LEMMA *A class $\mathcal{A} \subseteq \mathcal{M}$ is stable under pullback in $\mathcal{X}$ if and only if $\mathcal{A}$ is left cancellable w.r.t. $\mathcal{M}$ and has the $\wedge$-$\vee$-reflection property.*

Proof The Remark and the proof of Theorem 1.7 confirm that stability under pullback yield the $\wedge$-$\vee$-reflection and the left cancellation property. Conversely, for $f : X \to Y$ in $\mathcal{X}$ and $n \in \mathcal{M}/Y$ left cancellability of $\mathcal{A}$ w.r.t. $\mathcal{M}$ gives that $N \to f(X) \vee N$ belongs to $\mathcal{A}$ if $n : N \to Y$ does. Then the $\wedge$-$\vee$-reflection property implies that $i \cong (f \cdot m)^{-1}(n)$ belongs to $\mathcal{A}$. □

The Proposition, the Lemma and Theorem 5.4 give the following Theorem:

THEOREM (Tonolo [1995]) *A class $\mathcal{C} \subseteq \mathcal{M}$ in an $\mathcal{M}$-complete category $\mathcal{X}$ is the class of all C-closed $\mathcal{M}$-subobjects for some closure operator C w.r.t. $\mathcal{M}$ if and only if*

(a) *$\mathcal{C}$ is left cancellable w.r.t. $\mathcal{M}$,*

(b) $\mathcal{C}$ *has the* $\wedge$-$\vee$-*reflection property,*

(c) $\mathcal{C}$ *is stable under* $\mathcal{M}$-*intersection.*

In this case there is a uniquely determined idempotent closure operator C *(up to isomorphism) with* $\mathcal{C} = \mathcal{M}^C$ *. This closure operator is weakly hereditary if and only if* $\mathcal{C}$ *is closed under composition.* □

According to Exercise 2.D, the idempotent closure operator C with $\mathcal{C} = \mathcal{M}^C$ can be defined directly by

$$c_X(m) = \bigwedge\{k \in \mathcal{C}/X : k \geq m\}.$$

We shall use this construction in the "dual" situation to characterize classes of C-dense $\mathcal{M}$-subobjects.

We call a class $\mathcal{D} \subseteq \mathcal{M}$ *stable under* $\mathcal{M}$-*unions*, if $1_X \in \mathcal{D}$ for all $X \in \mathcal{X}$, and if n, $m_i \in \mathcal{M}/X$ with $n \leq m_i$ for all $i \in I \neq \emptyset$ (so that there are $\mathcal{M}$-morphisms $j_i : N \to M_i$ with $m_i \cdot j_i = n$), then $j_i \in \mathcal{D}$ for all $i \in I$ implies $j \in \mathcal{D}$, with $j : N \to \bigvee_{i \in I} M_i$ the $\mathcal{M}$-morphism with $\left(\bigvee_{i \in I} m_i\right) \cdot j = n$.

THEOREM* (Tonolo [1995]) *A class* $\mathcal{D} \subseteq \mathcal{M}$ *in an* $\mathcal{M}$-*complete category* $\mathcal{X}$ *is the class of all* C-*dense* $\mathcal{M}$-*subobjects for some closure operator* C *w.r.t.* $\mathcal{M}$ *if and only if*

(a) $\mathcal{D}$ *is right cancellable w.r.t.* $\mathcal{M}$,

(b) $\mathcal{D}$ *has the* $\wedge$-$\vee$-*preservation property,*

(c) $\mathcal{D}$ *is stable under* $\mathcal{M}$-*unions.*

In this case, there is a uniquely determined weakly hereditary closure operator C *(up to isomorphism) with* $\mathcal{D} = \mathcal{E}^C \cap \mathcal{M}$ *. This closure operator is idempotent if and only if the class* $\mathcal{D}$ *is closed under composition.*

Proof We first prove that (a)-(c) are necessary conditions. In fact, (a) and (c) follow from Corollary* 2.3 and Exercise 2.F(d). In order to show (b), we observe that diagram (5.13) can be decomposed as

$$\begin{tikzcd}
f^{-1}(N) \arrow[r] \arrow[d, "f^{-1}(n)"] & f(X) \wedge N \arrow[r] \arrow[d, "j"] & N \arrow[d, "n"] \\
X \arrow[r, "e"] & f(X) \arrow[r, "m"] & Y
\end{tikzcd} \tag{5.14}$$

with $e \in \mathcal{E}$. Hence $f^{-1}(n) \in \mathcal{E}^C$ implies $e \cdot f^{-1}(n) \in \mathcal{E}^C$ and therefore $j \in \mathcal{E}^C$ (see Exercise 2.F(b)). Since trivially $1_N \in \mathcal{E}^C$, we conclude (see Exercise 2.F(d)) that also

$$n : N \cong (f(X) \wedge N) \vee N \longrightarrow Y \cong f(X) \vee N$$

belongs to $\mathcal{E}^C$.

Conversely, for a class $\mathcal{D}$ with (a)-(c), we define a closure operator C by

$$c_X(m) = \bigvee \{k \in \mathcal{M}/X : k \cdot d = m \text{ for some } d \in \mathcal{D}\}$$

(cp. Exercise 2.C(b)). Obviously, since $1_X \in \mathcal{D}$, one has $m \le c_X(m)$. To show monotonicity, let $m \le n$ and consider k , $d \in \mathcal{M}$ with $k \cdot d = m$ and $d \in \mathcal{D}$. Then, with (a), we obtain that the canonical arrow $i : K \wedge N \to K$ (with $k \cdot i = k \wedge n$) belongs to $\mathcal{D}$. Now application of (b) with $f = 1_X$ gives that also $j : N \to K \vee N$ with $(k \vee n) \cdot j = n$ belongs to $\mathcal{D}$, hence $k \le k \vee n \le c_X(n)$. Therefore $c_X(m) \le c_X(n)$.

For $f : X \to Y$ and $m \in \mathcal{M}/X$, we must show $f(c_X(m)) \le c_Y(f(m))$. Since $f(-)$ preserves joins, it is sufficient to show that, for all k , d as above, there is $l \in \mathcal{D}$ with $f(k) \cdot l = f(m)$. Indeed, by (b) the right vertical arrow in

$$\begin{tikzcd}
K \wedge f^{-1}(f(M)) \arrow[r] \arrow[d] & f(M) \arrow[d, "l"] \\
K \arrow[r] & f(K) \cong f(K) \vee f(M)
\end{tikzcd} \tag{5.15}$$

belongs to $\mathcal{D}$ since the left vertical arrow does, because of (a).

Hence we have a closure operator C w.r.t. $\mathcal{M}$ which, obviously, satisfies $\mathcal{D} \subseteq \mathcal{E}^C \cap \mathcal{M}$. The proof that C is weakly hereditary, and that $\mathcal{D} = \mathcal{E}^C \cap \mathcal{M}$ if (and only if) condition (c) holds, is straight forward and can be left as an exercise: see Exercises 2.C(b) and 2.D(d).

Furthermore, by Theorem 2.4, C is idempotent if and only if $\mathcal{E}^C$ is closed under composition. But the latter condition is easily seen to be equivalent to $\mathcal{D}$ being closed under composition. □

COROLLARY *For the idempotent hull $\hat{C}$ and the weakly hereditary core $\check{C}$ of a closure operator C of an $\mathcal{M}$-complete category $\mathcal{X}$, one has*

$$\hat{c}_X(m) \cong \bigwedge \{k \in \mathcal{M}^C/X : k \ge m\},$$

$$\check{c}_X(m) \cong \bigvee \{k \in \mathcal{M}/X : k \cdot d = m \text{ for some } d \in \mathcal{E}^C\}.$$

In particular, $\mathcal{M}^{\hat{C}} = \mathcal{M}^C$ and $\mathcal{E}^{\check{C}} = \mathcal{E}^C$. If C is weakly hereditary (idempotent, resp.), then $\hat{C}$ ($\check{C}$, resp.) is both idempotent and weakly hereditary.

Proof The Theorem and Theorem* give uniquely determined idempotent and weakly hereditary closure operators $\hat{C}$ and $\check{C}$ with $\mathcal{M}^{\hat{C}} = \mathcal{M}^C$ and $\mathcal{E}^{\check{C}} \cap \mathcal{M} = \mathcal{E}^C \cap \mathcal{M}$, respectively, which are described as in the proofs of the Theorem and Theorem*. With Exercise 2.C one easily checks that $D \geq \hat{C}$ for every idempotent closure operator $D \geq C$, and that $E \leq \check{C}$ for every weakly hereditary operator $E \leq C$. Also note that by Proposition 2.4, $\mathcal{E}^C$ is closed under composition for C idempotent, and that by Proposition* of 2.4, $\mathcal{M}^C$ is closed under composition for C weakly hereditary, so that everything follows from the Theorem and Theorem*. □

EXAMPLES For a closure operator C of **Top** (w.r.t. the class of embeddings), one defines a subspace $M \subseteq X$ to be *totally* C*-dense* if $N \cap M$ is C-dense in N for every C-closed subspace $N \subseteq X$. It is easy to check that the class $\mathcal{D}^C$ of totally C-dense embeddings satisfies hypotheses (a), (b), (c) of Theorem*. Consequently, there exists a uniquely determined weakly hereditary closure operator C^{tot} such that total C-density means C^{tot}- density for subspaces. If C is hereditary and if $L_x := c_X(\{x\})$ is C-closed for all $x \in X \in \mathbf{Top}$, then C^{tot} may be described explicitly by

$$x \in c_X^{\mathrm{tot}}(M) \Leftrightarrow x \in c_{L_x}(M \cap L_x)\ . \qquad (*)$$

In fact, the right-hand side of $(*)$ describes a weakly hereditary closure operator with the characteristic properties of C^{tot} (as we shall show in greater generality in 9.5). The presentation $(*)$ shows that the passage $C \mapsto C^{\mathrm{tot}}$ preserves additivity and productivity. Now we can choose particular closure operators for C.

(1) (*Characterization of the* b*-closure*) For $C = K$, formula $(*)$ shows immediately that the b-closure of 3.3(b) is characterized as *the* weakly hereditary closure operator that describes total Kuratowski density, i.e., $b = K^{\mathrm{tot}}$. From this presentation one concludes in particular that b is additive and productive.

(2) For $C = b$, formula $(*)$ shows $b_X^{\mathrm{tot}}(M) = M$, hence $b^{\mathrm{tot}} = (K^{\mathrm{tot}})^{\mathrm{tot}} = S$ is the discrete closure operator.

(3) For $C = \sigma$ the sequential closure operator, one has

$$S < \sigma^{\mathrm{tot}} < b\ .$$

Indeed, since $\sigma_X(\{x\}) = k_X(\{x\})$ for all $x \in X$, formula $(*)$ shows $\sigma^{\mathrm{tot}} \leq K^{\mathrm{tot}} = b$. For $\sigma^{\mathrm{tot}} < b$, topologize the set $X = \mathbb{R} \cup \{\infty\}$ by taking X and every at most countable set that does not contain ∞ to be closed in X. Then $k_X(\{\infty\}) = X$, and no sequence in $\mathbb{R}$ converges to ∞, i.e., $\mathbb{R}$ is σ- closed in X. But ∞ belongs to $b_X(\mathbb{R}) = k_X^{\mathrm{tot}}(\mathbb{R})$. For the proof of $S < \sigma^{\mathrm{tot}}$, modify the topology of X by changing "at most countable" to "finite".

5.5 Closure operators versus $\mathcal{M}$-preradicals

For a closure operator C of the category $\mathbf{Mod}_R$ of R-modules, one obtains a preradical $\mathbf{r}$ by putting $\mathbf{r}(M) = c_M(0)$. Vice versa, given a preradical $\mathbf{r}$, we considered two closure operators, $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$, associated with $\mathbf{r}$ (see 3.4). In what follows we shall show that these constructions exist in the setting of 2.1, and we clarify their categorical meaning.

Hence an *$\mathcal{M}$-preradical* $(\mathbf{r}, r)$ is simply an $\mathcal{M}$-precoreflection of the finitely $\mathcal{M}$-complete category $\mathcal{X}$, that is: an endofunctor $\mathbf{r} : \mathcal{X} \to \mathcal{X}$, together with a natural transformation $r : \mathbf{r} \to Id_{\mathcal{X}}$ such that, for every $X \in \mathcal{X}$,

$$r_X : \mathbf{r}(X) \to X$$

belongs to $\mathcal{M}$. (Since we are in the setting of 2.1, so that $\mathcal{M}$ is a class of monomorphisms of $\mathcal{X}$, it follows from the dual of Proposition 5.1(1) that $(\mathbf{r}, r)$ is actually a precoreflection.) Often we shall simply refer to $\mathbf{r}$ as an $\mathcal{M}$-preradical.

The conglomerate

$$PRAD(\mathcal{X}, \mathcal{M})$$

of all $\mathcal{M}$-preradicals is preordered by

$$\mathbf{r} \leq \mathbf{s} \Longleftrightarrow r_X \leq s_X \quad \text{in} \quad \mathcal{M}/X \text{ for all } X \in \mathcal{X} .$$

Equivalently, one has a commutative diagram

$$\begin{tikzcd} \mathbf{r} \arrow[rr, "j"] \arrow[dr, "r"'] & & \mathbf{s} \arrow[dl, "s"] \\ & Id_{\mathcal{X}} & \end{tikzcd} \tag{5.16}$$

with a uniquely determined natural transformation j . Similarly to $CL(\mathcal{X}, \mathcal{M})$, if $\mathcal{X}$ is $\mathcal{M}$-complete, $PRAD(\mathcal{X}, \mathcal{M})$ inherits the structure of a large-complete lattice from $\mathcal{M}$, as follows:

$$\left(\bigvee_{i \in I} r_i\right)_X \cong \bigvee_{i \in I} (r_i)_X \quad \text{and} \quad \left(\bigwedge_{i \in I} r_i\right)_X \cong \bigwedge_{i \in I} (r_i)_X .$$

$PRAD(\mathcal{X}, \mathcal{M})$ has a largest element, $\mathbf{1}$, given by $(1_X)_{X \in \mathcal{X}}$, and a least element $\mathbf{0}$, given by $(o_X)_{X \in \mathcal{X}}$ (with each o_X the trivial $\mathcal{M}$-subobject of X).

Each closure operator C of $\mathcal{X}$ w.r.t. $\mathcal{M}$ induces an $\mathcal{M}$-preradical $\mathbf{t} = \pi(C)$ with

$$\mathbf{t}(X) = c_X(O_X) \quad \text{and} \quad t_X = c_X(o_X).$$

Vice versa, for $\mathbf{r} \in PRAD(\mathcal{X}, \mathcal{M})$, one considers $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ defined by

$$\begin{aligned} (c_{\mathbf{r}})_X(m) &= m \vee r_X , \\ c^{\mathbf{r}}_X(m) &= \bigwedge \{ e^{-1}(r_Z) : (\exists Z \in \mathcal{X})(\exists e : X \to Z \text{ in } \mathcal{E}) \ e(m) \cong o_Z \} . \end{aligned}$$

One readily checks that $C_{\mathbf{r}}$ is in fact a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$. The proof of the same fact for $C^{\mathbf{r}}$ is more difficult, but is in fact just a straightforward generalization of the proof of Proposition 4.7 (indiscrete closure operator G); in fact, we obviously have

$$G = C^{\mathbf{r}} \quad \text{for} \quad \mathbf{r} = \mathbf{0}\ .$$

Furthermore, the following rules hold:

LEMMA *For $\mathcal{X}$ $\mathcal{M}$-complete and $C \in CL(\mathcal{X},\mathcal{M})$, $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$ one has:*

(1) $\mathbf{r} \leq \pi(C) \iff C_{\mathbf{r}} \leq C$;

(2) $\pi(C) \leq \mathbf{r} \iff C \leq C^{\mathbf{r}}$;

(3) $\pi(C_{\mathbf{r}}) \cong \mathbf{r} \cong \pi(C^{\mathbf{r}})$.

Proof

(1) For $\mathbf{t} = \pi(C)$ and $\mathbf{r} \leq \mathbf{t}$ one has

$$(c_{\mathbf{r}})_X(m) = m \vee r_X \leq m \vee c_X(o_X) \leq c_X(m)$$

for all $m \in \mathcal{M}/X$. Conversely, evaluating $(c_{\mathbf{r}})_X(m) \leq c_X(m)$ for $m = o_X$ gives

$$r_X \leq c_X(o_X) = t_X\ .$$

(2) For $\mathbf{t} = \pi(C) \leq \mathbf{r}$ and every $e : X \to Z$ in $\mathcal{E}$ with $e(m) = o_Z$ one has $c_Z(o_Z) \leq r_Z$ and $m \leq e^{-1}(o_Z)$, hence

$$c_X(m) \leq e^{-1}\left(c_Z(o_Z)\right) \leq e^{-1}(r_Z)\ ;$$

$C \leq C^{\mathbf{r}}$ follows by construction of $C^{\mathbf{r}}$. Conversely, this inequality implies

$$c_X(o_X) \leq c^{\mathbf{r}}_X(o_X) \leq (1_X)^{-1}(r_X) = r_X$$

since $1_X(o_X) \cong o_X$.

(3) $\pi(C_{\mathbf{r}}) \cong \mathbf{r}$ is trivial. Furthermore, since

$$\begin{tikzcd} \mathbf{r}(X) \arrow[r, "\mathbf{r}(e)"] \arrow[d, "r_X"'] & \mathbf{r}(Z) \arrow[d, "r_Z"] \\ X \arrow[r, "e"] & Z \end{tikzcd} \tag{5.17}$$

commutes, one has $r_X \leq e^{-1}(r_Z)$ for every $e : X \to Z$, hence $r_X \leq c^{\mathbf{r}}_X(o_X)$. Hence $\mathbf{r} \leq \pi(C^{\mathbf{r}})$, whereas "$\geq$" follows from (2). □

PROPOSITION *Let $\mathcal{X}$ be $\mathcal{M}$-complete. The map*

$$\pi : CL(\mathcal{X},\mathcal{M}) \longrightarrow PRAD(\mathcal{X},\mathcal{M})$$

preserves all meets and joins. For every $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$, the fibre $\pi^{-1}(\mathbf{r})$ contains a least and a last element, $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ respectively. The assignments

$$\mathbf{r} \mapsto C_{\mathbf{r}} \quad and \quad \mathbf{r} \mapsto C^{\mathbf{r}}$$

define monotone maps which are left and right adjoint to π , respectively. The following rules hold:

(1) $\mathbf{r} \cong \bigvee_{i\in I} \mathbf{r}_i \Rightarrow C_{\mathbf{r}} \cong \bigvee_{i\in I} C_{\mathbf{r}_i}$;

(2) $\mathbf{r} \cong \bigwedge_{i\in I} \mathbf{r}_i \Rightarrow C^{\mathbf{r}} \cong \bigwedge_{i\in I} C^{\mathbf{r}_i}$.

□

Proof The statements on adjointness follow from the Lemma, together with Lemma 1.3. Therefore the assertions on preservation of meets and joins, including rules (1), (2) follow from Theorem 1.3. The Lemma also asserts that $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ play the indicated role in the fibre $\pi^{-1}\mathbf{r} = \{C : \pi(C) \cong \mathbf{r}\}$. □

REMARKS

(1) For $\mathcal{X} = \mathbf{Mod}_R$ with $\mathcal{M}$ the class of monomorphism, one easily checks that $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ as constructed above coincide (up to isomorphism) with the *minimal* and *maximal closure operator induced by* $\mathbf{r}$, respectively. We therefore keep this terminology also in the abstract setting, for any $\mathcal{M}$-preradical $\mathbf{r}$.

(2) The examples of preradicals in **AbGrp** given in 3.4 indicate that, in general, $PRAD(\mathcal{X},\mathcal{M})$ may be quite "big". Here is an opposite indicator: for $\mathcal{X} = \mathbf{Top}$ and $\mathcal{M}$ the class of embeddings, $PRAD(\mathcal{X},\mathcal{M})$ contains, up to isomorphism, only the $\mathcal{M}$-preradicals $\mathbf{0}$ and $\mathbf{1}$: see Exercise 2.H. $C_{\mathbf{1}} \cong C^{\mathbf{1}} \cong T$ is the trivial closure operator, $C_{\mathbf{0}} \cong S$ is discrete and $C^{\mathbf{0}} \cong G$ indiscrete.

(3) $\mathcal{M}$-preradicals can be composed in a natural way; the natural transformation $t : \mathbf{t} \to Id_{\mathcal{X}}$ belonging to the *composite* $\mathbf{t} = \mathbf{rs}$ of first $\mathbf{s}$ and then $\mathbf{r}$ is defined by

$$t_X = r_X \cdot \mathbf{r}(s_X) = s_X \cdot r_{\mathbf{s}(X)}$$

(see Exercise 5.A).

$$\begin{array}{ccc} \mathbf{r}(\mathbf{s}(X)) & \xrightarrow{\mathbf{r}(s_X)} & \mathbf{r}(X) \\ {\scriptstyle r_{\mathbf{s}(X)}}\downarrow & & \downarrow{\scriptstyle r_X} \\ \mathbf{s}(X) & \xrightarrow{s_X} & X \end{array} \tag{5.18}$$

It is easy to see that π transforms cocomposites of closure operators into composites of preradicals:

$$\pi(C * D) \cong \pi(C)\pi(D) .$$

Hence

$$\begin{aligned} \mathbf{rs} &\cong \pi(C^{\mathbf{r}})\pi(C^{\mathbf{s}}) \cong \pi(C^{\mathbf{r}} * C^{\mathbf{s}}) \\ &\cong \pi(C_{\mathbf{r}})\pi(C_{\mathbf{s}}) \cong \pi(C_{\mathbf{r}} * C_{\mathbf{s}}) . \end{aligned}$$

(4) One easily checks the rule

$$\mathbf{r} \wedge \mathbf{s} \cong \pi(C_{\mathbf{r}}C_{\mathbf{s}})$$

(with $C_{\mathbf{r}}C_{\mathbf{s}}$ the composite of the closure operators). This leaves us with the task of describing the $\mathcal{M}$-preradicals $\pi(C^{\mathbf{r}}C^{\mathbf{s}})$.

DEFINITION The *cocomposite* of $\mathbf{r}$, $\mathbf{s} \in PRAD(\mathcal{X},\mathcal{M})$ is defined by

$$(\mathbf{r} : \mathbf{s}) := \pi(C^{\mathbf{r}}C^{\mathbf{s}})$$

Explicitly, with $\mathbf{t} = (\mathbf{r} : \mathbf{s})$ one has

$$t_X = c^{\mathbf{r}}_X(s_X) = \bigwedge \left\{e^{-1}(r_Z) : (\exists e : X \to Z \text{ in } \mathcal{E})\ e(s_X) \cong o_Z\right\}$$

for every $X \in \mathcal{X}$. Since $s_X \leq c^{\mathbf{r}}_X(s_X)$ and $r_X = c^{\mathbf{r}}_X(o_X) \leq c^{\mathbf{r}}_X(s_X)$ one has $\mathbf{r} \vee \mathbf{s} \leq (\mathbf{r} : \mathbf{s})$, in particular $\mathbf{r} \leq (\mathbf{r} : \mathbf{r})$. We call $\mathbf{r}$ an *$\mathcal{M}$-radical* of $\mathcal{X}$ if $\mathbf{r} \cong (\mathbf{r} : \mathbf{r})$, and $\mathbf{r}$ is *idempotent* if $\mathbf{r} \cong \mathbf{rr}$.

THEOREM *Let $\mathcal{X}$ be $\mathcal{M}$-complete.*

(1) *For C , $D \in CL(\mathcal{X},\mathcal{M})$ one has*

$$\pi(CD) \leq (\pi(C) : \pi(D)) ,$$

and $\cong$ holds if C is maximal, i.e. if $C \cong C^{\mathbf{r}}$ for some $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$.

(2) *For $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$, $C_{\mathbf{r}}$ is idempotent; $C^{\mathbf{r}}$ is idempotent iff $\mathbf{r}$ is an $\mathcal{M}$-radical.*

(3) *For $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$, $C_{\mathbf{r}}$ is weakly hereditary iff $\mathbf{r}$ is an idempotent $\mathcal{M}$-preradical; both conditions hold if $C^{\mathbf{r}}$ is weakly hereditary.*

Proof

(1) Since, by adjointness, $C \leq C^{\pi(C)}$ and $D \leq C^{\pi(D)}$, and since composition of closure operators and π are monotone, one has

$$\pi(CD) \leq \pi(C^{\pi(C)}C^{\pi(D)}) \cong (\pi(C) : \pi(D)) .$$

For C maximal one has $C \cong C^{\pi(C)}$; hence, with $\mathbf{t} = (\pi(C) : \pi(D))$, we obtain for every $X \in \mathcal{X}$

$$t_X = C^{\pi(C)}_X(d_X(o_X)) \cong c_X(d_X(o_X)).$$

Therefore $\mathbf{t} \cong \pi(CD)$.

(2) The idempotency of $C_{\mathbf{r}}$ is trivial. If $C^{\mathbf{r}}$ is idempotent, one has, with (1),

$$\mathbf{r} \cong \pi(C^{\mathbf{r}}) \cong \pi(C^{\mathbf{r}}C^{\mathbf{r}}) \cong (\pi(C^{\mathbf{r}}) : \pi(C^{\mathbf{r}})) \cong (\mathbf{r} : \mathbf{r}) .$$

Furthermore, for all $\mathbf{r}$, $\mathbf{s} \in PRAD(\mathcal{X},\mathcal{M})$, since $\pi(C^{\mathbf{r}}C^{\mathbf{s}}) \cong (\mathbf{r} : \mathbf{s})$, one obtains

$$C^{\mathbf{r}}C^{\mathbf{s}} \leq C^{(\mathbf{r}:\mathbf{s})} \tag{$*$}$$

from the Lemma. Hence $\mathbf{r} \cong (\mathbf{r} : \mathbf{r})$ implies $C^{\mathbf{r}}C^{\mathbf{r}} \leq C^{(\mathbf{r}:\mathbf{r})} \cong C^{\mathbf{r}}$, i.e. idempotency of $C^{\mathbf{r}}$.

(3) In Remark (3) we noted the rules

$$\mathbf{rs} \cong \pi(C^{\mathbf{r}} * C^{\mathbf{s}}) \cong \pi(C_{\mathbf{r}} * C_{\mathbf{s}}).$$

Therefore, if $C^{\mathbf{r}}$ or $C_{\mathbf{r}}$ is weakly hereditary, i.e., if $C^{\mathbf{r}} * C^{\mathbf{r}} \cong C^{\mathbf{r}}$ or $C_{\mathbf{r}} * C_{\mathbf{r}} \cong C_{\mathbf{r}}$, then

$$\mathbf{rr} \cong \pi(C^{\mathbf{r}}) \cong \pi(C_{\mathbf{r}}) \cong \mathbf{r},$$

so that $\mathbf{r}$ is idempotent.

Furthermore, since $\mathbf{rs} \cong \pi(C_{\mathbf{r}} * C_{\mathbf{s}})$, the Lemma yields the rule

$$C_{\mathbf{rs}} \leq C_{\mathbf{r}} * C_{\mathbf{s}} \tag{$**$}$$

so that idempotency of $\mathbf{r}$ gives weak hereditariness of $C_{\mathbf{r}}$. (Under additional hypotheses one can show that also weak hereditariness of $C^{\mathbf{r}}$ is a necessary condition for $\mathbf{r}$ to be idempotent; see Corollary 5.6.) □

For an $\mathcal{M}$-preradical $\mathbf{r}$, one defines *powers* $\mathbf{r}^{\alpha}$ and *copowers* $\mathbf{r}_{\alpha}$ $(\alpha \in Ord \cup \{\infty\})$, as follows:

$$\begin{array}{rclcrcl}
\mathbf{r}^{0} & = & \mathbf{1} & \qquad & \mathbf{r}_{0} & = & \mathbf{0} \\
\mathbf{r}^{\alpha+1} & = & \mathbf{r}\mathbf{r}^{\alpha} & & \mathbf{r}_{\alpha+1} & = & (\mathbf{r} : \mathbf{r}_{\alpha}) \\
\mathbf{r}^{\beta} & = & \bigwedge\limits_{\gamma<\beta} \mathbf{r}^{\gamma} & & \mathbf{r}_{\beta} & = & \bigvee\limits_{\gamma<\beta} \mathbf{r}_{\gamma}
\end{array}$$

(for $\alpha \in Ord$ and β a limit ordinal or $\beta = \infty$).

COROLLARY *For every* $\mathbf{r} \in PRAD(\mathcal{X},\mathcal{M})$ *and all* $\alpha \in Ord \cup \{\infty\}$, *one has:*

(1) $\pi((C^{\mathbf{r}})^{\alpha}) \cong \mathbf{r}_{\alpha}$ *and* $\pi((C_{\mathbf{r}})_{\alpha}) \cong \mathbf{r}^{\alpha}$;

(2) $(C^{\mathbf{r}})^{\alpha} \leq C^{(\mathbf{r}_{\alpha})}$ *and* $C_{(\mathbf{r}^{\alpha})} \leq (C_{\mathbf{r}})_{\alpha}$. □

Proof With the help of the Lemma, (1) implies (2). We establish the first isomorphism of (1) by induction, the second isomorphism is done analogously:

$$\begin{aligned}
\pi((C^{\mathbf{r}})^0) &\cong \pi(S) \cong \mathbf{0} \cong \mathbf{r}_0 \,, \\
\pi((C^{\mathbf{r}})^{\alpha+1}) &\cong \pi(C^{\mathbf{r}}(C^{\mathbf{r}})^{\alpha}) \\
&\cong (\pi(C^{\mathbf{r}}) : \pi((C^{\mathbf{r}})^{\alpha})) \quad \text{by (1) of the Theorem} \\
&\cong (\mathbf{r} : \mathbf{r}_{\alpha}) \cong \mathbf{r}_{\alpha+1} \,, \\
\pi((C^{\mathbf{r}})^{\beta}) &\cong \pi\left(\bigvee_{\gamma<\beta} (C^{\mathbf{r}})^{\gamma}\right) \\
&\cong \bigvee_{\gamma<\beta} \pi((C^{\mathbf{r}})^{\gamma}) \cong \bigvee_{\gamma<\beta} \mathbf{r}_{\gamma} \cong \mathbf{r}_{\beta} \,.
\end{aligned}$$

□

EXAMPLE For the radical $\mathbf{p}$ of **AbGrp** given by p-multiples (for a prime number p, see Example 4.6(2)), we show that the second inequality of the Corollary may be strict. Indeed, since $C = C_{\mathbf{p}} \cong C^{\mathbf{p}}$ is (both minimal and) maximal, and since $(C^{\mathbf{p}})_{\infty} \cong C^{(\mathbf{p}^{\infty})}$ (see Exercise 4.6), we see that $C_{\infty} \cong C^{(\mathbf{p}^{\infty})}$. Hence $(C_{\mathbf{p}})_{\infty} \not\cong C_{(\mathbf{p}^{\infty})}$, as we shall show now that C_{∞} is not directedly additive, and consequently, not minimal. Let $X = \oplus_{n=1}^{\infty} X_n$, with $X = <c_n>$ a cyclic group of order p^n. Then each subgroup $M_n = <c_k - pc_{k+1} : k = 1, \ldots, n>$ of X is C_{∞}-closed, but the (directed) join $\sum_n M_n$ is a proper C_{∞}-dense subgroup of X.

5.6 $\mathcal{M}$-preradicals versus $\mathcal{E}$-prereflections

We wish to give a handier description of the closure operator $C^{\mathbf{r}}$ and the preradical $(\mathbf{r} : \mathbf{s})$, as defined in 5.5, in case our category $\mathcal{X}$ *has a zero object* 0. Furthermore, we assume that *the class* $\mathcal{E}$ (as given in 2.1) *is contained in the class of epimorphisms of* $\mathcal{X}$, and that $\mathcal{X}$ *has kernels and cokernels*. In other words, for all morphisms e and m we have the equalizer and coequalizer diagrams

$$\mathrm{Ker}(e) \xrightarrow{\ker(e)} X \underset{0}{\overset{e}{\rightrightarrows}} E \,, \qquad M \underset{0}{\overset{m}{\rightrightarrows}} X \xrightarrow{\mathrm{coker}(m)} \mathrm{Coker}(m)$$

(with 0 denoting the only morphism between two given objects which factors through the zero object 0). Equivalently, one has the pullback and pushout diagrams

$$\begin{tikzcd}
\mathrm{Ker}(e) \arrow[r] \arrow[d, "\ker(e)"'] \arrow[dr, phantom, "p.b."] & 0 \arrow[d] \\
X \arrow[r, "e"'] & E
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "p.o."] & 0 \arrow[d] \\
X \arrow[r, "\mathrm{coker}(m)"'] & \mathrm{Coker}(m)
\end{tikzcd}
\qquad (5.19)$$

Since $0 \to E$ belongs to $\mathcal{M}$ (consider its $(\mathcal{E}, \mathcal{M})$-factorization and use that the $\mathcal{E}$-part is epic; hence $0 \to E$ is isomorphic to o_E), and since $\mathcal{M}$ is stable under pullback, one has $\ker(e) \in \mathcal{M}$. Dually one obtains $\mathrm{coker}(m) \in \mathcal{E}$. Hence there are maps

$$\ker : \mathcal{E} \to \mathcal{M} , \quad \mathrm{coker} : \mathcal{M} \to \mathcal{E} .$$

If we consider $\mathcal{E}$ a category in the same way as we consider $\mathcal{M}$ a category in 5.2, then these maps become functors. Moreover:

LEMMA *The functor* ker *is right adjoint to* coker, *and the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, shift left, "\ker"] \arrow[dr, "\mathrm{dom}"'] & & \mathcal{M} \arrow[ll, shift left, "\mathrm{coker}"] \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{X} & \end{tikzcd} \tag{5.20}$$

commutes in the obvious sense.

Proof One easily shows $m \le \ker(\mathrm{coker}(m))$ for every $m \in \mathcal{M}/X$. The arising morphism

$$\begin{tikzcd} M \arrow[r] \arrow[d, "m"'] & \mathrm{Ker}(\mathrm{coker}(m)) \arrow[d, "\ker(\mathrm{coker}(m))"] \\ X \arrow[r, "1_X"] & X \end{tikzcd} \tag{5.21}$$

serves as a unit of the adjunction; the verification can be left to the reader. Trivially, $\mathrm{cod}(\ker(m)) = \mathrm{dom}(m)$ and $\mathrm{dom}(\mathrm{coker}(e)) = \mathrm{cod}(e)$ for all m and e . $\square$

An $\mathcal{M}$-preradical $(\mathbf{r}, r)$ can be described equivalently as a functor

$$r : \mathcal{X} \to \mathcal{M} , \quad X \mapsto r_X ,$$

with $\mathrm{cod}\, r = Id_X$. Dually, an $\mathcal{E}$-prereflection $(\mathbf{q}, q)$ (that is: a natural transformation $q : Id_{\mathcal{X}} \to \mathbf{q}$ pointwise in $\mathcal{E}$; remember that $\mathcal{E}$ is assumed to be a class of epimorphisms, so that the prereflection property comes for free) is described equivalently by a functor

$$q : \mathcal{X} \to \mathcal{E} , \ X \mapsto q_X ,$$

with $\mathrm{dom}\, q = Id_{\mathcal{X}}$. Composition of the functors r and coker gives an $\mathcal{E}$-prereflection since

$$\mathrm{dom}\,(\mathrm{coker}\, r) = \mathrm{cod}\, r = Id_{\mathcal{X}} ,$$

and for an $\mathcal{E}$-prereflection q one obtains the $\mathcal{M}$-preradical $\ker q$.

The conglomerate

$$PREF(\mathcal{X}\ ,\ \mathcal{E})$$

of all $\mathcal{E}$-prereflections is preordered by

$$(\mathbf{q},q) \leq (\mathbf{p},p) \Leftrightarrow (\exists j : \mathbf{q} \to \mathbf{p})\ jq = p\ .$$

With the Lemma we obtain:

PROPOSITION *The monotone functions*

$$PREF(\mathcal{X},\mathcal{E}) \underset{\text{coker}}{\overset{\text{ker}}{\rightleftarrows}} PRAD(\mathcal{X},\mathcal{M})$$

given by composition are adjoint to each other: coker $\dashv$ ker . *Hence* $\mathcal{M}$*-preradicals* $(\mathbf{r},r)$ *with* $\ker \operatorname{coker} r \cong r$ *are equivalently described by* $\mathcal{E}$*-prereflections* $(\mathbf{q},q)$ *with* $\operatorname{coker} \ker q \cong q$. □

EXAMPLES

(1) In the category $\mathbf{Mod}_R$ one has $\ker \operatorname{coker} \cong Id_{\mathcal{M}}$ and $\operatorname{coker} \ker \cong Id_{\mathcal{E}}$. Hence every preradical $(\mathbf{r},r)$ satisfies $\ker \operatorname{coker} r \cong r$, that is: $r \cong \ker q$ for an $\mathcal{E}$-prereflection $(\mathbf{q},q)$.

(2) In the category **Grp**, $\operatorname{coker} \ker \cong Id_{\mathcal{E}}$ remains valid but $\ker \operatorname{coker} \not\cong Id_{\mathcal{M}}$ (witnessed by every non-normal subgroup of a group). But since $\mathbf{r}(G)$ is normal in G for every preradical $(\mathbf{r},r)$ and every group G , we still have $\ker \operatorname{coker} r \cong r$.

(3) Let $\mathbf{Set}_*$ be the category of pointed sets: objects are pairs (X,x_0) with a set X and $x_0 \in X$, and morphisms $f : (X,x_0) \to (Y,y_0)$ are maps $f : X \to Y$ with $f(x_0) = y_0$. With $\mathcal{M}$ and $\mathcal{E}$ the injective and surjective maps, respectively, one easily checks that $\ker \operatorname{coker} \cong Id_{\mathcal{M}}$ holds, but $\operatorname{coker} \ker \not\cong Id_{\mathcal{E}}$. Nevertheless, as in the previous two examples, the operators of the Proposition produce a bijective correspondence since there are only two non-isomorphic preradicals and prereflections, respectively (see Exercise 5.H).

Coming back to the goal as stated at the beginning of this section, first we observe that for every $\mathcal{M}$-preradical $\mathbf{r}$ one obtains a closure operator C from the pullback diagrams

$$\begin{tikzcd} c_X(M) \arrow[r] \arrow[d, "c_X(m)"'] & \mathbf{r}(\mathrm{Coker}(m)) \arrow[d, "r_{\mathrm{Coker}(m)}"] \\ X \arrow[r, "\mathrm{coker}(m)"] & \mathrm{Coker}(m) \end{tikzcd} \tag{5.22}$$

Hence

$$c_X(m) \cong q_m^{-1}(r_{\mathrm{Coker}(m)}) \tag{$*$}$$

for every $m \in \mathcal{M}/X$, with $q_m := \mathrm{coker}(m)$. We claim that $C \cong C^{\mathbf{r}}$. Indeed, C is easily seen to be a closure operator of $\mathcal{X}$, and since $\mathrm{coker}(o_X)$ is iso one has $c_X(o_X) \cong r_X$; that is: $\pi(C) \cong \mathbf{r}$. For any other closure operator D of $\mathcal{X}$ with $\pi(D) \cong \mathbf{r}$ we have

$$\begin{aligned} c_X(m) &\cong q_m^{-1}(r_{\mathrm{Coker}(m)}) \\ &\cong q_m^{-1}(d_{\mathrm{Coker}(m)}(o_{\mathrm{Coker}(m)})) \\ &\geq d_X(q_m^{-1}(o_{\mathrm{Coker}(m)})) \\ &\geq d_X(m)\,. \end{aligned}$$

Therefore C is largest in the fibre $\pi^{-1}(\mathbf{r})$, hence $C \cong C^{\mathbf{r}}$ (see Prop. 5.5).

This proves the first statement of

THEOREM *The maximal closure operator $C^{\mathbf{r}}$ for an $\mathcal{M}$-preradical $\mathbf{r}$ of $\mathcal{X}$ is described by formula $(*)$. The $\mathcal{M}$-preradical $(\mathbf{r} : \mathbf{s})$ is given by the formula*

$$(r : s)_X \cong q_{s_X}^{-1}(r_{\mathrm{Coker}(s_X)})$$

with $q_{s_X} = \mathrm{coker}(s_X)$.

Proof By definition, $(\mathbf{r} : \mathbf{s}) \cong \pi(C^{\mathbf{r}}C^{\mathbf{s}})$ (see 5.5). Hence,

$$\begin{aligned} (r : s)_X &\cong c_X^{\mathbf{r}}(c_X^{\mathbf{s}}(o_X)) \\ &\cong c_X^{\mathbf{r}}(s_X) \\ &\cong q_{s_X}^{-1}(r_{\mathrm{Coker}(s_X)})\,. \end{aligned}$$

□

We may use $(*)$ to characterize hereditariness and weak hereditariness of the closure operator $C^{\mathbf{r}}$ in terms of the preradical $\mathbf{r}$. One calls $\mathbf{r} \in PRAD(\mathcal{X}, \mathcal{M})$ *hereditary* if

$$r_Y \cong y^{-1}(r_X) \qquad (**)$$

for every $y : Y \to X$ in $\mathcal{M}$. Clearly, this is a necessary condition for any closure operator C with $\pi(C) \cong \mathbf{r}$ to be hereditary. Indeed, $(**)$ follows immediately from (HE) of 2.5 applied to $m = o_X$:

$$c_Y(o_Y) \cong y^{-1}(c_X(o_X))\,.$$

In order to show that it is also sufficient in case $C = C^{\mathbf{r}}$, we need a hypothesis on the functor $\mathrm{coker} : \mathcal{M} \to \mathcal{E}$. We say that Coker *preserves $\mathcal{M}$-morphisms* if for all y, m, m_Y in $\mathcal{M}$ with $y \cdot m_Y = m$ one also has $\mathrm{Coker}(1_M, y)$ in $\mathcal{M}$.

$$\begin{tikzcd}
M \arrow[r, "1_M"] \arrow[d, "m_Y"'] & M \arrow[d, "m"] \\
Y \arrow[r, "y"] \arrow[d, "q_{m_Y}"'] & X \arrow[d, "q_m"] \\
\mathrm{Coker}(m_Y) \arrow[r, "\mathrm{Coker}(1_M, y)"] & \mathrm{Coker}(m)
\end{tikzcd} \tag{5.23}$$

Since the lower part of (5.23) is a pushout diagram, Coker preserves $\mathcal{M}$-morphisms whenever $\mathcal{M}$ is stable under pushout (along $\mathcal{E}$ morphisms).

COROLLARY *If* Coker *preserves $\mathcal{M}$-morphisms, then $C^{\mathbf{r}}$ is hereditary if and only if* $\mathbf{r}$ *is hereditary. If, in addition, $\mathcal{E}$ is stable under pullback, then $C^{\mathbf{r}}$ is weakly hereditary if and only if* $\mathbf{r}$ *is idempotent.* □

Proof Since $\bar{y} := \mathrm{Coker}(1_M, y) \in \mathcal{M}$ one has

$$r_{\mathrm{Coker}(m_Y)} \cong \bar{y}^{-1}(r_{\mathrm{Coker}(m)})$$

if $\mathbf{r}$ is hereditary. Consequently, with $(*)$ one derives

$$\begin{aligned}
c^{\mathbf{r}}_Y(m_Y) &\cong q^{-1}_{m_Y}(r_{\mathrm{Coker}(m_Y)}) \\
&\cong q^{-1}_{m_Y}(\bar{y}^{-1}(r_{\mathrm{Coker}(m)})) \\
&\cong y^{-1}(q^{-1}_m(r_{\mathrm{Coker}(m)})) \\
&\cong y^{-1}(c^{\mathbf{r}}_X(m)).
\end{aligned}$$

Hence $C^{\mathbf{r}}$ is hereditary.

Let now $\mathcal{E}$ be stable under pullback and consider $y = c_X(m)$ and let $j = m_Y$. Then the top arrow $q'_m : Y = c_X(M) \to \mathbf{r}(\mathrm{Coker}(m))$ of diagram (5.22) belongs to $\mathcal{E}$, and since $r_{\mathrm{Coker}(m)} \cdot q'_m \cdot j = q_m \cdot = 0$ with $r_{\mathrm{Coker}(m)}$ monic, q'_m factors through $q_j : Y \to \mathrm{Coker}(j)$ by a morphism $v : \mathrm{Coker}(j) \to \mathbf{r}(\mathrm{Coker}(m))$ belonging to $\mathcal{E}$ (cf. Exercise 2.F(b)). Since q_j is epic, the composite $r_{\mathrm{Coker}(m)} \cdot v$ is the bottom arrow of diagram (5.23), hence it belongs to $\mathcal{M}$ by hypothesis. But then also v belongs to $\mathcal{M}$ (cf. Theorem 1.7 (3)), so it is an isomorphism. Consequently, if $\mathbf{r}$ is idempotent, so that $\mathbf{r}(\mathbf{r}(\mathrm{Coker}(m))) \to \mathbf{r}(\mathrm{Coker}(m))$ is an isomorphism, also $r_{\mathrm{Coker}}(j)$ must be an isomorphism, and one has $c_Y(j) \cong 1_Y$ by formula (*). This shows that idempotency of $\mathbf{r}$ implies weak hereditariness of $C^{\mathbf{r}}$. The converse statement was shown in Theorem 5.5(3). □

EXAMPLES

(1) In $\mathbf{Mod}_R$, Coker preserves monomorphisms. Hence the Corollary provides a categorical proof of the result shown for modules in Theorem 3.4(4) and (6).

(2) In **Grp**, Coker does not preserve monomorphisms. The normal closure $\nu = C^{\mathbf{0}}$ is not (weakly) hereditary although the (pre)radical $\mathbf{0}$ is of course hereditary (cf. 3.5(1)).

(3) Although there are no non-trivial $\mathcal{M}$-preradicals in $\mathbf{Set}_*$ (see Exercise 5.H), we note that the hypothesis of the Corollary is satisfied in this case.

5.7 (C, D)-continuous functors

The transition from one category to another is described by functors. In this section we shall describe such transitions when the categories in question come equipped with closure operators with respect to given classes of subobjects. Hence, throughout this section, we consider a functor $F : \mathcal{X} \to \mathcal{Y}$ with $\mathcal{X}$ finitely $\mathcal{M}$-complete and $\mathcal{Y}$ finitely $\mathcal{N}$-complete, for classes $\mathcal{M}$ and $\mathcal{N}$ of monomorphisms, both closed under composition. Hence there are classes $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorizations and $\mathcal{Y}$ has $(\mathcal{F}, \mathcal{N})$-factorizations (see 1.8, 2.1). We assume throughout the section (with the exception of the final Remark) that F *preserves subobjects*, that is: $Fm \in \mathcal{N}$ for every $m \in \mathcal{M}$. Consequently, for every $X \in \mathcal{X}$, F induces a monotone function $\mathcal{M}/X \to \mathcal{N}/FX$. What is the impact of this assumption on basic constructions, like inverse image and direct image?

LEMMA *Let $f : X \to Y$ be a morphism in $\mathcal{X}$ and $m \in \mathcal{M}/X$, $n \in \mathcal{M}/Y$. Then*

(1) *$F(f^{-1}(n)) \le (Ff)^{-1}(Fn)$, and "$\cong$" holds for all f and n exactly when F preserves pullbacks along $\mathcal{M}$-morphisms; we say that F* preserves inverse images *in this case.*

(2) *$(Ff)(Fm) \le F(f(m))$, and "$\cong$" holds for all f and m exactly when $Fe \in \mathcal{F}$ for all $e \in \mathcal{E}$; we say that F* preserves (direct) images *in this case.* □

The easy proof can be left as an exercise to the reader.

Let us consider closure operators C and D of $\mathcal{X}$ and $\mathcal{Y}$ with respect to $\mathcal{M}$ and $\mathcal{N}$, respectively.

Definition F is called (C, D)-*continuous* if

$$Fc_X(m) \leq d_{FX}(Fm) \tag{$*$}$$

holds for all $m \in \mathcal{M}/X$, $X \in \mathcal{X}$. F is (C, D)-*preserving* if "$\leq$" may be replaced by "$\cong$".

As we saw in 5.2, the closure operators C and D are described as prereflections (C, γ) and (D, δ) of the *categories* $\mathcal{M}$ and $\mathcal{N}$, respectively. Since F preserves subobjects the functor F extends to a functor $\bar{F} : \mathcal{M} \to \mathcal{N}$. Now (C, D)-continuity means that there exists a (uniquely determined) natural transformation

$$\alpha : \bar{F}C \to D\bar{F} \quad \text{with} \quad \alpha \cdot \bar{F}\gamma = \delta\bar{F} \ .$$

In fact, for an object $m \in \mathcal{M}$,

$$\alpha_m : Fc_X(M) \to d_{FX}(FM) \quad \text{with} \quad d_{FX}(Fm) \cdot \alpha_m = Fc_X(m)$$

is defined by $(*)$. Naturality of α follows from the following diagram (with $(u, v) : m \to m'$ in $\mathcal{M}$): the upper parallelogram commutes since the rectangle and the lower parallelogram commute, since $d_{FX'}(Fm')$ is monic.

$$\begin{tikzcd}
Fc_X(M) \arrow[rr, "FC(u,v)"] \arrow[dr, "\alpha_m"] \arrow[dd, "F(c_X(m))"'] & & Fc_{X'}(M') \arrow[dr, "\alpha_{m'}"] \arrow[dd, "F(c_{X'}(m'))"' near end] & \\
& d_{FX}(FM) \arrow[rr, "D(Fu,Fv)" near start] \arrow[dl, "d_{FX}(Fm)"] & & d_{FX'}(FM') \arrow[dl, "d_{FX'}(Fm')"] \\
FX \arrow[rr, "Fv"] & & FX' &
\end{tikzcd} \tag{5.24}$$

(C, D)-preservation by F therefore means $\bar{F}C \cong D\bar{F}$.

It may seem strange at first sight that condition $(*)$ ignores arbitrary morphisms $f : X \to Y$ of $\mathcal{X}$. However, as the following proposition shows, one automatically obtains from $(*)$ a number of compatibility conditions.

Proposition *If F is (C, D)-continuous, the following inequalities hold for all $f : X \to Y$ in $\mathcal{X}$, $m \in \mathcal{M}/X$ and $n \in \mathcal{M}/Y$:*

$$
\begin{tikzcd}[column sep=tiny]
& (Ff)^{-1}(d_{FY}(Fn)) \arrow[dl, dash, "{(C,D)}"'] \arrow[dr, dash, "D"] & \\
(Ff)^{-1}(Fc_Y(n)) \arrow[d, dash, "\mathrm{im}^{-1}"'] & & d_{FX}((Ff)^{-1}(Fn)) \arrow[d, dash, "\mathrm{im}^{-1}"] \\
F(f^{-1}(c_Y(n))) \arrow[dr, dash, "C"'] & & d_{FX}(F(f^{-1}(n))) \arrow[dl, dash, "{(C,D)}"] \\
& Fc_X(f^{-1}(n)) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& d_{FY}(F(f(m))) \arrow[dl, dash, "{(C,D)}"'] \arrow[dr, dash, "\mathrm{im}"] & \\
F(c_Y(f(m))) \arrow[d, dash, "\mathrm{C}"'] & & d_{FY}((Ff)(Fm)) \arrow[d, dash, "D"] \\
F(f(c_X(m))) \arrow[dr, dash, "\mathrm{im}"'] & & (Ff)(d_{FX}(Fm)) \arrow[dl, dash, "{(C,D)}"] \\
& (Ff)(Fc_X(m)) &
\end{tikzcd}
\tag{5.25}
$$

(*with upwards directed lines to be read as* "$\leq$")

Proof Inequalities marked by im and im^{-1} follow from the Lemma. The others are due to the continuity of f (w.r.t. C), of Ff (w.r.t. D), and of F (w.r.t. (C,D)). □

The notion of continuity for functors is a straight generalization of the notion of continuity of morphisms (as given by the definition of closure operators). The following example will clarify this point and also illustrate the meaning of the inequalities given in the Proposition.

EXAMPLE Let $\varphi : A \to B$ be a **Set**-map between topological spaces. The usual Kuratowski closure in A defines a closure operation of the *poset* 2^A (see Exercise 2.E). When considered as a *category* $\mathcal{X}$, we obtain a closure operator C of $\mathcal{X}$ defined by

$$c_X(M) = k_A(M) \cap X = \overline{M} \cap X$$

for all $M \subseteq X \subseteq A$. In the same way one defines the closure operator D of the category $\mathcal{Y}$ given by the poset 2^B. The map φ defines a functor $F : \mathcal{X} \to \mathcal{Y}$ with $FX = \varphi(X)$. Continuity of the functor F is described by the condition

$$\varphi(\overline{M} \cap X) \subseteq \overline{\varphi(M)} \cap \varphi(X) \tag{$*$}$$

for all $M \subseteq X \subseteq A$, which holds exactly when φ is a continuous map.

For a morphism $f : X \to Y$ in $\mathcal{X}$, which is given by the inclusion map $X \hookrightarrow Y$, and for $M \subseteq X$, $N \subseteq Y$, the diagrams (5.25) can now be described as follows:

$$
\begin{tikzcd}[column sep=small]
 & \overline{\varphi(N)} \cap \varphi(X) \arrow[dl, dash] \arrow[dr, dash] & & & \overline{\varphi(M)} \cap \varphi(Y) \arrow[dl, dash] \arrow[dr, dash, double] & \\
\varphi(\overline{N} \cap Y) \cap \varphi(X) \arrow[d, dash] & & \overline{\varphi(N) \cap \varphi(X)} \cap \varphi(X) \arrow[d, dash] & \varphi(\overline{M} \cap Y) \arrow[d, dash] & & \overline{\varphi(M)} \cap \varphi(Y) \arrow[d, dash] \\
\varphi(\overline{N} \cap X) \arrow[dr, dash] & & \overline{\varphi(N \cap X)} \cap \varphi(X) \arrow[dl, dash] & \varphi(\overline{M} \cap X) \arrow[dr, dash, double] & & \overline{\varphi(M)} \cap \varphi(X) \arrow[dl, dash] \\
 & \varphi(\overline{N \cap X} \cap X) & & & \varphi(\overline{M} \cap X) &
\end{tikzcd}
\tag{5.26}
$$

Note that F trivially preserves images (since $f(M) = M$), but not in general inverse images unless φ is injective (since $f^{-1}(N) = N \cap X$).

The importance of the notion of continuity of functors, however, does not arise from the fact that it constitutes a generalization of the continuity of maps (or morphisms of a category with a closure operator). Rather, it can be used to construct new closure operators from old, in the same way as one constructs the initial (weak) topology or the final (quotient) topology with respect to a map. The sequence of easy properties below will lead us quickly to these constructions.

Properties on continuity:

(1) $Id_{\mathcal{X}}$ is (C, C)-continuous for every $C \in CL(\mathcal{X}, \mathcal{M})$.

(2) If F is (C, D)-continuous and $G : \mathcal{Y} \to \mathcal{Z}$ is (D, E)-continuous (with E a closure operator of $\mathcal{Z}$ with respect to a class $\mathcal{K}$), then GF is (C, E)-continuous.

(3) If $C' \leq C$ in $CL(\mathcal{X}, \mathcal{M})$ and $D \leq D'$ in $CL(\mathcal{Y}, \mathcal{N})$, then (C, D)-continuity of F implies (C', D')-continuity of F .

(4) If F is (C_i, D)-continuous for every $i \in I$, with any family $(C_i)_{i \in I}$ in $CL(\mathcal{X}, \mathcal{M})$, then F is (C, D)-continuous with $C :\cong \bigvee_{i \in I} C_i$ (see 4.1).

(5) If F is (C, D_i)-continuous for every $i \in I$, with any family $(D_i)_{i \in I}$ in $CL(\mathcal{Y}, \mathcal{N})$, then F is (C, D)-continuous with $D :\cong \bigwedge_{i \in I} D_i$ (see 4.1).

(6) Note that the case $I = \emptyset$ is permitted in each (4) and (5). Hence F is (S, D)-continuous and (C, T)-continuous for all $C \in CL(\mathcal{X}, \mathcal{M})$ and $D \in CL(\mathcal{Y}, \mathcal{N})$,

with S the discrete closure operator on $\mathcal{X}$ and T the trivial closure operator on $\mathcal{Y}$.

THEOREM *Let $F : \mathcal{X} \to \mathcal{Y}$ be a functor which preserves subobjects given by the classes $\mathcal{M}$ and $\mathcal{N}$. We then have:*

(1) *For every $D \in CL(\mathcal{Y}, \mathcal{N})$ there is a largest $C \in CL(\mathcal{X}, \mathcal{M})$ such that F is (C, D)-continuous. C is called* the initial closure operator induced by *D and F and is denoted by $D_{(F)}$.*

(2) *For every $C \in CL(\mathcal{X}, \mathcal{M})$ there is a least $D \in CL(\mathcal{Y}, \mathcal{N})$ such that F is (C, D)-continuous. D is called the* final closure operator induced by *C and F* and is denoted by $C^{(F)}$.

Proof (1) Necessarily, we must have

$$C \cong \bigvee \{C' \in CL(\mathcal{X}, \mathcal{M}) : F \text{ in } (C', D)\text{-continuous}\}.$$

It follows from property (4) above that with C defined this way, F is (C, D)-continuous.

(2) Dually, one considers

$$D \cong \bigwedge \{D' \in CL(\mathcal{Y}, \mathcal{N}) : F \text{ in } (C, D')\text{-continuous}\}$$

and evokes property (5). □

In Sections 5.8 and 5.13 we shall encounter important examples of closure operators of type $D_{(F)}$ and $C^{(F)}$. Here we just note:

COROLLARY *Under the assumptions of the Theorem, there are adjoint monotone maps*

$$CL(\mathcal{X}, \mathcal{M}) \underset{(-)_{(F)}}{\overset{(-)^{(F)}}{\rightleftarrows}} CL(\mathcal{Y}, \mathcal{N})$$

with $(-)^{(F)} \dashv (-)_{(F)}$. In particular, the formulas

$$\left(\bigvee_i C_i\right)^{(F)} \cong \bigvee_i C_i^{(F)} \quad \textit{and} \quad \left(\bigwedge_i D_i\right)_{(F)} \cong \bigwedge_i (D_i)_{(F)}$$

hold. □

Proof One must show $(C \le D_{(F)} \Leftrightarrow C^{(F)} \le D)$ for all $C \in CL(\mathcal{X}, \mathcal{M})$ and $D \in CL(\mathcal{Y}, \mathcal{N})$. But "$\Rightarrow$" follows from the implications (F is $(D_{(F)}, D)$-continuous $\Rightarrow F$ is (C, D)-continuous $\Rightarrow C^{(F)} \le D$), and "$\Leftarrow$" follows dually.
□

One also has

$$C \leq (C^{(F)})_{(F)} \qquad \text{and} \qquad (D_{(F)})^{(F)} \leq D\,.$$

We call $(C^{(F)})_{(F)}$ the *F-closure* of C and $(D_{(F)})^{(F)}$ the *F-interior* of D.

REMARK Although our blanket assumption that F preserve subobjects seems natural in order to consider continuity, we shall encounter situations when it is more convenient to consider the (C, D)-continuity condition

$$F(c_X(m)) \leq d_{FX}(Fm) \tag{$*$}$$

without the hypothesis $Fm \in \mathcal{N}$ for all $m \in \mathcal{M}$. The inequality $(*)$ still makes sense in this case, provided we naturally extend the $\leq$-relation and the notion of closure from subobjects to arbitrary morphisms with common codomain, as follows: for $g : K \to X$, $h : L \to X$ define

$$\begin{aligned} g \leq h \quad &:\Leftrightarrow \quad g(1_K) \leq h(1_L)\,, \\ c_X(g) \quad &:\cong \quad c_X(g(1_K))\,. \end{aligned}$$

The inequalities of the Lemma and properties (1)-(3) on continuity remain valid without the blanked assumption, but (4) and (5) may fail. Consequently, the existence of the initial closure operator $D_{(F)}$ and the final closure operator $C^{(F)}$ is no longer guaranteed. However, as we shall see in 5.10 and 5.13, often it is possible to construct $D_{(F)}$ and $C^{(F)}$ by means different from those used in the proof of the Theorem, without assuming that F preserve subobjects.

5.8 Lifting closure operators along $\mathcal{M}$-fibrations

In this section, we discuss functors which allow for an easy description of the initial closure operators induced by them (as defined by Theorem 5.7). Necessarily such functors should allow for an easy "lifting" of the notion of subobject. As a leading example, the reader should consider the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$: every subset $M \subseteq X$ of a topological space carries a natural topology which makes M a subspace of X.

Let $U : \mathcal{X} \to \mathcal{S}$ be a functor which, for convenience, is assumed to be faithful. An $\mathcal{X}$-morphism $f : X \to Y$ is called a *(U−)lifting* of an $\mathcal{S}$-morphism $\varphi : S \to UY$ if $UX = S$ and $Uf = \varphi$. By abuse of language, an $\mathcal{S}$-morphism $\varphi : UX \to UY$ is said to *be an $\mathcal{X}$-morphism* if it has a U-lifting $f : X \to Y$ (which, due to faithfulness, is uniquely determined). An $\mathcal{X}$-morphism $f : X \to Y$ is called *U-initial* (or *U-cartesian*) if, for every $Z \in \mathcal{X}$, an $\mathcal{S}$-morphism $\psi : UZ \to UX$ is an $\mathcal{X}$-morphism whenever $Uf \cdot \psi : UZ \to UY$ is an $\mathcal{X}$-morphism.

For a class $\mathcal{M}$ of morphisms in $\mathcal{S}$, the functor U is called an *$\mathcal{M}$-fibration* if every $\varphi : S \to UY$ has U-initial lifting. In case $\mathcal{M}$ is the class of all (mono-)morphisms of $\mathcal{S}$, one calls U a *(mono-)fibration.*

EXAMPLES

(1) $U : \mathbf{Top} \to \mathbf{Set}$ is a fibration. A continuous map $f : X \to Y$ is U-initial iff X carries the initial (or weak) topology w.r.t. f, i.e. the coarsest topology making f continuous. For $\mathcal{M}$ the class of embeddings in **Top** and for every full replete subcategory $\mathcal{A}$ of **Top** closed under subspaces, the forgetful $\mathcal{A} \to \mathbf{Set}$ is an $(\mathcal{M} \cap \mathrm{Mor}\, \mathcal{A})$ -fibration.

(2) The underlying **Set**-functor of **PoSet** is a monofibration, but not a fibration; $\mathbf{AbGrp} \to \mathbf{Set}$ is not even a monofibration.

(3) The inclusion functor $\mathcal{Y} \hookrightarrow \mathcal{X}$ of a full subcategory closed under $\mathcal{M}$-subobjects (cf. 2.8) is an $\mathcal{M}$-fibration.

LEMMA *The class* Init_U *of U-initial morphisms in $\mathcal{X}$ is closed under composition and under those limits in $\mathcal{X}$ which are preserved by U.*

Proof Closedness under composition is easy to check. For functors H, $K : \mathcal{D} \to \mathcal{X}$, consider a natural transformation $\alpha : H \to K$ pointwise in Init_U. We shall show that the induced $f : \varprojlim \longrightarrow H \varprojlim K$ is U-initial, provided U preserves $\varprojlim H$. Indeed, for any $\psi = UZ \to U(\varprojlim H)$ with $Uf \cdot \psi : UZ \to U(\varprojlim K)$ an $\mathcal{X}$-morphism, also every $U\kappa_d \cdot Uf \cdot \psi : UZ \to UKd$ (with $\kappa_d : \varprojlim K \to Kd$ a limit projection) is an $\mathcal{X}$-morphism. By U-initiality of α_d, this means that $U\lambda_d \cdot \psi : UZ \to UHd$ (with $\lambda_d : \varprojlim H \to Hd$ a limit projection) has a U-lifting $\beta_d : Z \to Hd$. Now $\beta = (\beta_d)_{d \in \mathcal{D}}$ induces a morphism $h : Z \to \varprojlim L$ with $\lambda_d \cdot h = \beta_d \ (d \in \mathcal{D})$, and one has

$$U\lambda_d \cdot Uh = U\beta_d = U\lambda_d \cdot \psi$$

for all $d \in \mathcal{D}$, hence $Uh = \psi$ when the family $(U\lambda_d)_{d \in \mathcal{D}}$ is monic in $\mathcal{S}$, in particular when U preserves the limit. □

For a class $\mathcal{M}$ of morphisms in $\mathcal{S}$, let

$$\mathcal{M}_U := U^{-1}\mathcal{M} \cap \mathrm{Init}_U \ .$$

For $\mathcal{M}$ the class of all monomorphisms in $\mathcal{S}$, we call $\mathcal{M}_U$ the class of *U-embeddings*. Under mild assumptions on the functor U, good subobject properties of $\mathcal{S}$ w.r.t. $\mathcal{M}$ are inherited by $\mathcal{X}$ w.r.t. $\mathcal{M}_U$. First of all, faithfulness of U guarantees that, if $\mathcal{M}$ is a class of monomorphisms in $\mathcal{S}$ containing all isomorphisms and closed under composition, then $\mathcal{M}_U$ has the same properties in $\mathcal{X}$. More importantly:

PROPOSITION *For a faithful $\mathcal{M}$-fibration $U : \mathcal{X} \to \mathcal{S}$ one has:*

(1) *If $\mathcal{S}$ has $\mathcal{M}$-pullbacks ($\mathcal{M}$-intersections), then $\mathcal{X}$ has $\mathcal{M}_U$-pullbacks ($\mathcal{M}_U$-intersections, resp.).*

(2) *If every morphism (sink) in $\mathcal{S}$ has a right $\mathcal{M}$-factorization, then every morphism (sink, resp.) in $\mathcal{X}$ has a right $\mathcal{M}_U$-factorization.*

(3) *If $\mathcal{S}$ is (finitely) $\mathcal{M}$-complete, then $\mathcal{X}$ is (finitely) $\mathcal{M}_U$-complete.*

□

Proof

(1) In order to construct the inverse image of $n \in \mathcal{M}_U/Y$ and $f : X \to Y$ in $\mathcal{X}$, construct the pullback diagram

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\varphi"'] & UN \arrow[d, "Un"] \\ UX \arrow[r, "Uf"] & UY \end{tikzcd} \tag{5.27}$$

in $\mathcal{S}$ with $\varphi \in \mathcal{M}$. Let $m : M \to X$ be a U-lifting of φ. Then $m \in \mathcal{M}_U$, and by U-initiality of n, ψ has U-lifting $f' : M \to N$. With the pullback property of (5.27) and the U-initiality of m , one easily verifies that

$$\begin{tikzcd} M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{5.28}$$

is a pullback diagram in $\mathcal{X}$. This shows that $\mathcal{M}$-pullbacks in $\mathcal{S}$ yield the existence of $\mathcal{M}_U$-pullbacks in $\mathcal{X}$; the proof for intersections is very similar.

(2) In order to establish the right $\mathcal{M}_U$-factorization of a sink $(f_i : X_i \to Y)_{i\in I}$ in $\mathcal{X}$, consider the right $\mathcal{M}$-factorization

$$Uf_i = (UX_i \xrightarrow{\eta_i} S \xrightarrow{\varphi} UY)$$

of $(Uf_i)_{i\in I}$, with $\varphi \in \mathcal{M}$. With $m : M \to Y$ a U-initial lifting of φ, one obtains morphisms $e_i : X_i \to M$ with $Ue_i = \eta_i$. Checking that

$$f_i = (X_i \xrightarrow{e_i} M \xrightarrow{m} Y)$$

satisfies the simultaneous diagonalization property (see 1.10) can be safely left to the reader. Lifting right $\mathcal{M}$-factorizations of morphisms is the special case when $|I| = 1$. □

Corollary *For a faithful $\mathcal{M}$-fibration U , $\mathcal{M}_U$-intersections and $\mathcal{M}_U$-unions are obtained by U-initially lifting the corresponding $\mathcal{M}$-constructions. The* *same holds true for images and inverse images; hence* U *preserves images and inverse images in the sense of Lemma* 5.7. □

The map

$$\gamma_X : \mathcal{M}_U/X \to \mathcal{M}/UX\,, \quad m \mapsto Um\,,$$

defines an order-equivalence for every $X \in \mathcal{X}$: choosing for every $\varphi \in \mathcal{M}/UX$ a U-initial lifting defines a map δ_X in the opposite direction with $\gamma_X\delta_X = id$ and $\delta_X\gamma_X \cong id$. (In case U is *amnestic*, the latter natural isomorphism may be replaced by a strict equality: see Exercise 5.M). Since U preserves images, for

$$f(-) : \mathcal{M}_U/X \to \mathcal{M}_U/Y \quad \text{and} \quad (Uf)(-) : \mathcal{M}/UX \to \mathcal{M}/UY$$

(with $f : X \to Y$ in $\mathcal{X}$) one has $\gamma_Y \cdot f(-) \cong (Uf)(-) \cdot \gamma_X$, hence $f(-) \cdot \delta_X \cong \delta_Y \cdot (Uf)(-)$. Similarly, $f^{-1}(-) \cdot \delta_Y \cong \delta_X \cdot (Uf)^{-1}(-)$.

THEOREM *Let* $U : \mathcal{X} \to \mathcal{S}$ *be a faithful* $\mathcal{M}$*-fibration. For every* $C \in CL(\mathcal{S}, \mathcal{M})$ *, one obtains a closure operator* $C_U \in CL(\mathcal{X}, \mathcal{M}_U)$ *by putting*

$$(c_U)_X(m) :\cong \delta_X(c_{UX}(Um))$$

for every $m \in \mathcal{M}_U/X$ *. Then* U *is* (C_U, C)*-preserving, and* C_U *is the initial closure operator induced by* C *and* U *in the sense of Theorem* 5.7, *i.e,* $C_U = C_{(U)}$ *. The following properties of* C *are inherited by* C_U *: idempotent, (weakly) hereditary, minimal, grounded, (fully) additive; also (finite) productivity is preserved, provided* U *preserves (finite) products.*

Proof C_U is obviously extensive and monotone, and the continuity condition holds since δ commutes with images. Hence C_U is in fact a closure operator *w.r.t.* $\mathcal{M}_U$. By definition of C_U , one has

$$\gamma_X((c_U)_X(m)) \cong c_{UX}(Um)$$

for all $m \in \mathcal{M}_U/X$, hence U is (C_U, C)-preserving. For every $D \in CL(\mathcal{X}, \mathcal{M}_U)$ such that U is (D, C)-continuous, $\gamma_X(d_X(m)) \le c_{UX}(Um)$ implies

$$d_X(M) \le \delta_X(c_{UX}(Um)) \cong (c_U)_X(m)\,.$$

for all $m \in \mathcal{M}_U/X$. Therefore, $C_U \cong C_{(U)}$.

For C idempotent one has

$$\begin{aligned}(c_U)_X((c_U)_X(m)) &\cong \delta_X(c_{UX}(U((c_U)_X(m))))\\ &\cong \delta_X(c_{UX}(\gamma_X(\delta_X(c_{UX}(Um)))))\\ &\cong \delta_X(c_{UX}(c_{UX}(Um)))\\ &\cong \delta_X(c_{UX}(Um))\end{aligned}$$

for all $m \in \mathcal{M}_U/X$. Hence C_U is idempotent. Using the fact that δ_X commutes with all meets, joins and direct and inverse images, one shows similarly that also the other properties mentioned in the Theorem are inherited by C_U from C. In case of (finite) productivity, one notes that δ_X commutes also with direct products, if U preserves them, according to the Lemma. □

REMARKS

(1) The closure operator C_U constructed by the Theorem is its own U-closure. In fact, from Corollary 5.7 one has

$$C_U \cong C_{(U)} \cong ((C_{(U)})^{(U)})_{(U)} \cong ((C_U)^{(U)})_{(U)} .$$

(2) If the faithful $\mathcal{M}$-fibration U is *essentially surjective on objects* so that every $S \in \mathcal{S}$ is of the form $S \cong UX$, $X \in \mathcal{X}$, then C is the final closure operator induced by C_U and U, i.e.

$$C \cong (C_U)^{(U)} .$$

This follows from the observation, that any $D \in CL(\mathcal{S}, \mathcal{M})$ such that U is (C_U, D)-continuous must satisfy

$$c_{UX}(Um) \le d_{UX}(Um)$$

for all $X \in \mathcal{X}$ and $m \in \mathcal{M}_U/X$. Under the hypotheses on U, this suffices to conclude $C \le D$.

(3) In case U is the inclusion functor $\mathcal{Y} \hookrightarrow \mathcal{X}$ of a full subcategory closed under $\mathcal{M}$-subobjects, then the closure operator C_U coincides with $C|_{\mathcal{Y}}$ as constructed in 2.8.

(4) We observe that, in order to define C_U, we do not need the existence of U-initial liftings of all subobjects in $\mathcal{M}$ but just of those which are closures of subobjects in $\mathcal{M}_U$. This observation turns out to be essential in some cases, as we will show in the next section.

5.9 Applications to topological groups

A *topological group* is a group G provided with a topology on G such that both $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ give continuous maps $G \times G \to G$ and $G \to G$, respectively. (We do not require any separation axiom for the topology.) A morphisms in the category **TopGrp** is a continuous homomorphism of topological groups. There are two forgetful functors of interest along which we wish to lift closure operators:

$$U : \mathbf{TopGrp} \to \mathbf{Grp} \quad \text{and} \quad V : \mathbf{TopGrp} \to \mathbf{Top} .$$

Since every subgroup of a topological group becomes a topological group when provided with the subspace topology, U is a monofibration. (U is in fact a topological functor, see Exercise 5.P.) Hence Theorem 5.8, with the subsequent Remarks, gives immediately:

PROPOSITION *Every closure operator D of* **Grp** *can be initially lifted to a closure operator D_U of* **TopGrp**. *This way $CL(\mathbf{Grp}, \mathcal{M})$ is reflectively embedded in $CL(\mathbf{TopGrp}, \mathcal{M}_U)$, with $\mathcal{M}$ the class of group monomorphisms.* □

The functor V behaves very differently from U. Since a subspace of a topological group is in general not a subgroup, V is not an $\mathcal{N}$-fibration, with $\mathcal{N}$ the class of embeddings in **Top**. However, it turns out that the lifting procedure of Theorem 5.8 is still applicable to V in case of finitely productive closure operators (see Remark (4) of 5.8), due to the following crucial lemma.

LEMMA *For every finitely productive closure operator C of* **Top**, *the closure $c_G(H)$ of a (normal) subgroup H of a topological group G in* **Top** *is again a (normal) subgroup of G.*

Proof Since $\iota : G \to G$, $x \mapsto x^{-1}$, is continuous, the continuity condition for C gives $\iota(c_G(H)) \subseteq c_G(\iota(H)) \subseteq c_G(H)$. Since C is finitely productive, we are able to argue similarly in case of the continuous map $\mu : G \times G \to G$, $(x,y) \mapsto xy$:

$$\mu(c_G(H) \times c_G(H)) = \mu(c_{G\times G}(H \times H)) \subseteq c_G(\mu(H \times H)) \subseteq c_G(H) .$$

Finally, exploiting the continuity condition for C in case of the continuous conjugation $x \mapsto gxg^{-1}$ (for every $g \in G$), we see that $c_G(H)$ is normal if H is. (Cf. Exercise 4.V.). □

From Proposition 4.11 and Theorem 5.8 (with Remark (4)) we now obtain:

THEOREM *Every finitely productive and, in particular, every idempotent closure operator C of* **Top** *can be initially lifted to a finitely productive closure operator C_V of* **TopGrp**. *If C is (weakly) hereditary (idempotent, productive), then C_V has the respective property.*

Let $FPCL(\mathbf{Top}, \mathcal{N})$ be the conglomerate of finitely productive closure operators of **Top** w.r.t. the class $\mathcal{N}$ of embeddings. We then have the two procedures of defining closure operators of **TopGrp** given by the Proposition and the Theorem, and every closure operator of **TopGrp** gives the induced preradical, as in 5.5.

$$\begin{tikzcd}
CL(\mathbf{Grp},\mathcal{M}) \arrow[dr, "(-)_U"] & & \\
& CL(\mathbf{TopGrp},\mathcal{M}_U) \arrow[r, "\pi"] & PRAD(\mathbf{TopGrp},\mathcal{M}_U) \\
FPCL(\mathbf{Top},\mathcal{N}) \arrow[ur, "(-)_V"] & &
\end{tikzcd} \tag{5.29}$$

The left and right adjoints of π are given by the minimal and maximal closure operators $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ belonging to a preradical $\mathbf{r}$, respectively (see Prop. 5.5). According to Theorem 5.6, for a subgroup H of a topological group G one has

$$(c_{\mathbf{r}})_G(H) = H \cdot \mathbf{r}(G) \quad \text{and} \quad c^{\mathbf{r}}_G(H) = q^{-1}(\mathbf{r}(G/\nu(H))) \, . \tag{$*$}$$

For this note that, as for abstract groups, $\mathbf{r}(G)$ must be normal in G (cf. 3.5(2)). Furthermore, $G/\nu(H)$ is provided with the quotient topology, and $q : G \to G/\nu(H)$ is the projection. Here $\nu = \nu_U$ is the lifting of the normal closure of **Grp** (cf. 3.5(1)) to **TopGrp**, given by the Proposition. It inherits the good properties of idempotency, full additivity and productivity from its parent in **Grp**, but also the fact that it is not weakly hereditary (just provide any witness in **Grp** with the discrete topology). The normal closure may be combined with other closure operators, as follows.

COROLLARY *For every closure operator* C *of* **TopGrp** *with* $\pi(C) = \mathbf{r}$ *one has closure operators*

$$C_{\mathbf{r}} \le C \le \nu \vee C \le \nu C \le C\nu \le C^{\mathbf{r}}$$

all of which induce the same preradical $\mathbf{r}$. □

Proof Trivially $C_{\mathbf{r}} \le C \le \nu \vee C \le \nu C$ (see Lemma 4.4(1)). In the Lemma we observed that $c_G(\nu(H))$ must be normal in G, for all $H \le G$; hence $\nu C \le C\nu$. Finally, $C\nu \le C^{\mathbf{r}}$ follows from formula $(*)$. That the closure operators induce the same preradical follows from the stated inequalities. □

EXAMPLES

(1) Lifting the Kuratowski closure operator of **Top** gives an idempotent, hereditary and productive closure operator $K = K_V$ of **TopGrp**. K is neither grounded nor additive in **TopGrp**. The topological groups for which K is grounded are exactly

the Hausdorff topological groups. (Recall that for topological groups the separation axioms T_0 , T_1 , T_2 , T_3 , $T_{3\frac{1}{2}}$ are equivalent, cf. Hewitt and Ross [1963].) For non-additivity, consider the closed subgroups $\mathbb{Z}$ and $\mathbb{Z}\sqrt{2}$ of $(\mathbb{R},+)$ whose join $\mathbb{Z}+\mathbb{Z}\sqrt{2}$ is in fact a dense subgroup of $\mathbb{R}$, according to *Kronecker's Theorem* (cf. Bourbaki [1961]).

(2) The Kuratowski closure $k_G(H)$ of a normal subgroup H of a topological group G may be formed by taking the closure of the neutral element e in G/H and pulling it back along $q : G \to G/H$, i.e., $k_G(H) = q^{-1}(k_{G/H}(\{e\}))$. This shows that $K\nu = C^{\pi(K)}$ is maximal. The minimal closure operator $C_{\pi(K)}$ is properly smaller than K : for the subgroup $\mathbb{Q}$ of $(\mathbb{R},+)$, one has $c_{\pi(K)}(\mathbb{Q}) = \mathbb{Q} \neq \mathbb{R} = k_{\mathbb{R}}(\mathbb{Q})$.

(3) For every non-abelian Hausdorff group G , the diagonal subgroup $\Delta_G \leq G\times G$ is (K-) closed but not normal, hence not $(\nu \vee K)$-closed (see Theorem 4.4(1)). Hence $K < \nu \vee K$.

(4) We give an example which shows $\nu K < K\nu$. Let G be the group of permutations of a discrete countable set, i.e., $G = S(\mathbb{N}) \subseteq \mathbb{N}^{\mathbb{N}}$, with the topology of pointwise convergence. The stabilizer subgroups $\mathrm{stab}(m) = \{g \in G : g(m) = m\}$, $m \in \mathbb{N}$, generate a neighbourhood base for the neutral element. Now consider

$$S_\omega = \bigcup_{n=1}^{\infty} S_n\ , \quad \text{with} \quad S_n = \bigcap_{m=n+1}^{\infty} \mathrm{stab}(m)\ .$$

S_ω is a dense normal countable subgroup of G , while G is uncountable. Any (non-identical) involution of $\mathbb{N}$ generates a closed subgroup H of order 2 in G whose normal closure can be shown to coincide with S_ω . Hence

$$\nu(k_G(H)) = \nu(H) = S_\omega < G = k_G(S_\omega) = k_G(\nu(H))\ .$$

(5) The lifted sequential closure operator σ and K induce the same preradical in **TopGrp** (since their point closures coincide; cf. Example 4.9(2)), thus $C^{\pi(\sigma)} = C^{\pi(K)}$. For any topological group G with a non-closed sequentially-closed normal subgroup N , one has $\sigma_G(\nu(N)) = N \neq k_G(N) = k_G(\nu(N)) = c_G^{\pi(K)}(N)$, hence $\sigma\nu < C^{\pi(\sigma)}$.

5.10 Closure operators and CS-valued functors

In a **Set**-based category $\mathcal{X}$ with a grounded, additive closure operator C , each c_X may be viewed as a map $2^X \to 2^X$, provided each subset of X carries a subobject structure. One would then obtain a functor $\mathbf{C} : \mathcal{X} \to \mathbf{PrTop}$ which may be used to transfer (pre)topological notions to $\mathcal{X}$. If C is not grounded and additive, **PrTop** must be replaced by the larger category **CS** of *closure spaces* which is defined as **PrTop** (see 3.1), except that for a **CS**-object (S, k_S) , the map k_S does *not* necessarily satisfy the axioms $k_S(M \cup N) = k_S(M) \cup k_S(N)$

and $k_S(\emptyset) = \emptyset$. The Čech closure operator K of **PrTop** has therefore a natural extension to **CS**.

More precisely, let $U : \mathcal{X} \to \mathbf{Set}$ be a faithful monofibration, and let C be a closure operator of $\mathcal{X}$ w.r.t. the class $\mathcal{M}_U$ of U-embeddings. For $X \in \mathcal{X}$, the inclusion map of every subset $S \subseteq UX$ has U-initial lifting and can therefore be assumed to be of the form $UM \subseteq UX$; furthermore, in this case we may assume for the closure $c_X(M)$ that $Uc_X(M) \subseteq UX$. Similarly, for $f : X \to Y$ in $\mathcal{X}$ and $UM \subseteq UX$, the $\mathcal{M}_U$ -image $f(M)$ can be chosen such that $U(f(M)) = (Uf)(UM) \subseteq UY$.

Let us now define a **CS**-structure c_{UX} on UX , as follows

$$k_X(UM) = Uc_X(M) .$$

Since c_X is extensive and monotone, also c_{UX} is extensive and monotone. Hence

$$\mathbf{C}X := (UX, k_X)$$

is a closure space. Furthermore, for $f : X \to Y$ in $\mathcal{X}$ one has

$$\begin{aligned} (Uf)(k_X(UM)) &= (Uf)(Uc_X(M)) = U(f(c_X(M))) \\ &\subseteq U(c_Y(f(M))) = k_Y(U(f(M))) = k_Y((Uf)(UM)) . \end{aligned}$$

Hence we have a functor $\mathbf{C}$ which makes

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\mathbf{C}"] \arrow[dr, "U"'] & & \mathbf{CS} \arrow[dl, "V"] \\ & \mathbf{Set} & \end{tikzcd} \tag{5.30}$$

(with V the forgetful functor) commute.

Vice versa, if we are given such a functor $\mathbf{C}$, denoting $\mathbf{C}X$ by (UX , k_X) , we may define a closure operator $C = (c_X)_{X\in\mathcal{X}}$ of $\mathcal{X}$ w.r.t. $\mathcal{M}_U$ as follows: $c_X(M)$, for $UM \subseteq UX$, is the U-initial lifting of $k_X(UM)$. Since the U-embeddings whose underlying **Set**-maps are inclusion maps, form a skeleton of $\mathcal{M}_U$, this defines C uniquely, up to isomorphism.

The completion of the proof of the following Theorem can be left to the reader:

THEOREM *For a faithful monofibration* $U : \mathcal{X} \to \mathbf{Set}$ *, there is a bijective correspondence between*

- *isomorphism classes of closure operators of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}_U$ *,*
- *functors* $\mathbf{C} : \mathcal{X} \to \mathbf{CS}$ *which make* (5.30) *commute.*

When replacing **CS** *by* **PrTop** (**Top**) *this correspondence remains valid for isomorphism classes of grounded and additive* (*and idempotent, resp.*) *closure operators.*

□

We note that the bijection of the Theorem respects the preorder of $CL(\mathcal{X},\mathcal{M})$, in the following sense: we have $C \leq D$ if and only if there is a natural transformation $\alpha : \mathbf{C} \to \mathbf{D}$ with $V\alpha = 1_U$.

The Theorem provides an easy tool for the construction of closure operators:

COROLLARY *Let* $U : \mathcal{X} \to \mathbf{Set}$ *and* $W : \mathcal{Y} \to \mathbf{Set}$ *be faithful monofibrations, and let* $F : \mathcal{X} \to \mathcal{Y}$ *be a functor with* $WF = U$. *Then, for every* (*grounded; additive; idempotent*) *closure operator* D *of* $\mathcal{Y}$, *w.r.t.* $\mathcal{M}_W$, *one may define a* (*grounded; additive; idempotent, respectively*) *closure operator* C *of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}_V$ *by taking for* $c_X(m)$ *the* U*-initial lifting of* $W(d_{FX}(Fm))$: $W(d_{FX}(FM)) \hookrightarrow UX$. *It is the initial closure operator induced by* D *and* F.

Proof C is the closure operator with $\mathbf{C} = \mathbf{D}F$. By definition of C, one has $F(c_X(m)) \cong d_{FX}(Fm)$, hence F is initial with respect to D and F. (*Note:* F does *not* in general preserve subobjects. Initiality is to be understood in the more general sense of Remark 5.7.) □

EXAMPLES

(1) There is a functor $(-)^* : \mathbf{Top} \to \mathbf{Top}$ taking a topological space X to a new space X^* with the same underlying set such that the closed sets of X form a base of open sets for X^*. The initial closure operator induced by the Kuratowski closure operator K of **Top** is the operator K^* of Example 4.2(3), i.e., $\mathbf{K}^* = \mathbf{K}(-)^*$.

(2) Define the *sequential modification functor* $(-)^s : \mathbf{Top} \to \mathbf{Top}$ as follows: provide a topological space X with a new topology such that $A \subseteq X^s$ is closed iff $A \subseteq X$ is sequentially closed (i.e., a limit of a convergent sequence in A stays in A). The closure operator given by $\mathbf{K}(-)^s$ is grounded, additive and idempotent since K has these properties. It is the idempotent hull of the sequential closure operator σ, i.e., $K_{X^s}(M) = \sigma_X^\infty(M)$ for all $M \subseteq X$ (see Exercise 5.Q).

(3) Let $\Theta : \mathbf{FC} \to \mathbf{FC}$ be the θ*-modification functor* of filter spaces, see Exercise 3.D, and let K be the Katětov closure operator of **FC**. The θ-closure of **FC** is given by the composite $\boldsymbol{\theta} = \mathbf{K}\Theta$. Since $\mathbf{K}$ takes values in **PrTop**, the same holds true for $\boldsymbol{\theta}$, hence θ is additive in **FC**.

(4) An analysis of the proof of the Corollary reveals that we do not need the full strength of the hypothesis that U be a monofibration: it suffices to guarantee the existence of the U-initial lifts needed to construct $c_X(m)$. An instance of the Corollary under this relaxed assumption is given by the functor Scott : **DCPO** $\to$ **Top** and the Kuratowski closure operator K of **Top**: the initial closure operator with respect to these data is scott, as defined in 3.7. Groundedness, additivity and

idempotency of scott follow from this presentation.

The Corollary may be applied in case $\mathcal{Y} = \mathbf{CS}$ and $W = V$. It is easy to check that $V : \mathbf{CS} \to \mathbf{Set}$ is a fibration, in fact: a topological functor (see Exercise 5.P). A morphism $\varphi : (S, k_S) \to (T, k_T)$ in $\mathbf{CS}$ is V-initial iff

$$k_S(M) = \varphi^{-1} k_T(\varphi(M))$$

for all $M \subseteq S$. In particular, (S, k_S) is a subspace of (T, k_T) iff $S \hookrightarrow T$ is V-initial, i.e.

$$k_S(M) = T \cap k_S(M)$$

for all $M \subseteq S$.

Let us now compare U-embeddings with V-embeddings in terms of the functor $\mathbf{C} : \mathcal{X} \to \mathbf{CS}$ induced by a closure operator C w.r.t. $\mathcal{M}_U$.

PROPOSITION *For a faithful monofibration* $U : \mathcal{X} \to \mathbf{Set}$, *the induced functor* $\mathbf{C} : \mathcal{X} \to \mathbf{CS}$ *preserves subobjects, (i.e.,* $\mathbf{C}(\mathcal{M}_U) \subseteq \mathcal{M}_V$ *) if and only if* C *is hereditary. In this case the functor* $\mathbf{C}$ *is* (C, K)*-preserving, and* C *is the initial closure operator induced by* K *and* V.

Proof For $\mathcal{M}_U$-subobjects $M \to Y \to X$ with $UM \subseteq UY \subseteq UX$,

$$c_{UY}(UM) = U c_Y(M)$$

by definition of $\mathbf{C}Y$. On the other hand, UY carries the $\mathcal{M}_V$-subobject structure k_{UY} which it inherits from $\mathbf{C}X$:

$$k_{UY}(UM) = UY \cap c_{UX}(UM) ;$$

since U preserves meets of subobjects (see Corollary 5.8), this means

$$k_{UY} = U(Y \wedge c_X(M)).$$

But $c_Y(M) \to Y \wedge c_X(M) \cong y^{-1}(c_X(M))$ is a U-initial morphism. Hence it is an isomorphism if and only if its underlying $\mathbf{Set}$-map is an isomorphism. Hence one has

$$c_{UY} = k_{UY} \iff c_Y(M) \cong Y \wedge c_X(M) \text{ for all } M \to X \text{ in } \mathcal{M}_U .$$

That $\mathbf{C}$ is (C, K)-preserving follows immediately from the relevant definitions.
□

REMARKS

(1) Considering $Y = c_X(M)$ in the proof of the Proposition, one proves that C *is weakly hereditary if and only if* $\mathbf{C}$ *preserves embeddings of type* $c_X(M) \to X$.

(2) Productivity of the closure C does *not* in general imply that the functor $\mathbf{C}$ preserves products: see Exercise 5.N.

(3) The assertion of the Proposition that C is the initial closure operator induced by K and V remains true even if C is not hereditary, with the understanding of Remark 5.7.

5.11 Closure-structured categories, uniform spaces

A striking feature of the Kuratowski closure operator K of **Top** is that a **Set**-map $\varphi : X \to Y$ of topological spaces is continuous (i.e., belongs to the category **Top**) if (and only if)

$$\varphi(c_X(M)) \subseteq c_Y(\varphi(M))$$

for every subspace M of X. In other words, the operator K alone determines which are the structure-preserving maps of topological spaces. We say that **Top** *is K-structured*, in accordance with the following definition.

As in 5.8, we work in the context of a faithful $\mathcal{M}$-fibration $U : \mathcal{X} \to \mathcal{S}$ with $\mathcal{S}$ finitely $\mathcal{M}$-complete, for $\mathcal{M}$ a class of monomorphisms closed under composition, and provide $\mathcal{X}$ with the subobject structure given by $\mathcal{M}_U$ (cf. 5.8).

DEFINITION (1) For a closure operator C of $\mathcal{X}$ w.r.t $\mathcal{M}_U$, a morphism $\varphi : UX \to UY$ in $\mathcal{S}$ with X, $Y \in \mathcal{X}$ is C-continuous (w.r.t. X and Y) if

$$\varphi(U(c_X(m))) \leq U(c_Y(\delta_Y(\varphi(Um))))$$

for all $m \in \mathcal{M}_U/X$. (Recall that δ_Y provides $\varphi(Um)$ with the U-initial structure; see 5.8.) The morphism φ is *$\mathcal{C}$-continuous*, for a subconglomerate $\mathcal{C} \subseteq CL(\mathcal{X}, \mathcal{M}_U)$, if it is C-continuous for every $C \in \mathcal{C}$. In case $\mathcal{C} = CL(\mathcal{X}, \mathcal{M}_U)$, φ is called *totally continuous*.

(2) The functor U or, more laxly, the category $\mathcal{X}$ is called *C-structured* (*$\mathcal{C}$-structured, closure-structured*) if every C-continuous ($\mathcal{C}$-continuous, totally continuous, resp.) morphism $\varphi : UX \to UY$ in $\mathcal{S}$ is an $\mathcal{X}$-morphism, i.e., of the form $\varphi = Uf$ with $f : X \to Y$.

REMARKS

(1) It follows from the continuity condition of a closure operator that $Uf : UX \to UY$ is totally continuous, for every $f : X \to Y$ in $\mathcal{X}$.

(2) In case $\mathcal{S} = \mathbf{Set}$, the category $\mathcal{X}$ is C-structured if and only if the induced functor $\mathbf{C} : \mathcal{X} \to \mathbf{CS}$ of 5.10 is full.

Note that $\mathbf{C}$ is always faithful since $U = V\mathbf{C}$ is faithful.

(3) With S the discrete and T the trivial closure operator of $\mathcal{X}$, every morphism $\varphi : UX \to UY$ in $\mathcal{S}$ is $\{S, T\}$-continuous. Hence $\mathcal{X}$ is $\{S, T\}$-structured if and only if the functor U if full.

EXAMPLES

(1) The categories **Top**, **PrTop**, and **FC** are K-structured, with K the Kuratowski, Čech, and Katětov closure operator, respectively.

(2) Each of the categories **SGph**, **PrSet**, and **PoSet** is both $\uparrow$- and $\downarrow$-structured, but not conv-structured.

(3) **Top** is not σ-structured, for σ the sequential closure operator: a map of topological spaces that preserves convergence of sequences if σ-continuous, but may fail to be continuous.

(4) The category **TopGrp** is not K_V-structured, with K_V the lifting of the Kuratowski closure operator K along $V : \mathbf{TopGrp} \to \mathbf{Top}$ (see 5.9). Consider a group G with two different topologies τ, τ' , both of which make G a topological group such that G has the same closed subgroups w.r.t. τ and τ' . Then both identity maps

$$(G,\tau) \to (G,\tau') \qquad \text{and} \qquad (G,\tau') \to (G,\tau)$$

are K_V-continuous, but at least one of these cannot be continuous.

In the remainder of this section we discuss a mono-fibration (in fact: a topological functor) $V : \mathcal{X} \to \mathbf{Set}$ which fails to be closure-structured, i.e., the structure of $\mathcal{X}$ cannot be described by its closure operators. The most natural example of this type is the category **Unif** of *uniform spaces* which, for the reader's convenience, we describe here explicitly.

For a set X we denote by $\mathbf{SGph}(X)$ the set of reflexive relations on X (since these are the spatial graph structures on X , see 3.6). A *uniformity* on X is a filter $\mathcal{U}$ on $\mathbf{SGph}(X)$ (ordered by "$\subseteq$") such that for every $E \in \mathcal{U}$ there exists F , $G \in \mathcal{U}$ with

$$\begin{aligned} F^{-1} &= \{(y,x) : (x,y) \in F\} \subseteq E\,, \\ G \circ G &= \{(x,z) : (\exists y \in X)(x,y) \in G \text{ and } (y,z) \in G\} \subseteq E\,. \end{aligned}$$

A *uniformly continuous* map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a map $f : X \to Y$ such that, for every $F \in \mathcal{V}$ there is $E \in \mathcal{U}$ such that $f : (X,E) \to (Y,F)$ is a map of spatial graphs (i.e, $(f \times f)(E) \subseteq F$).

For a uniformity $\mathcal{U}$ on X , each $E \in \mathcal{U}$ defines a pretopology $\downarrow_{(X,E)}$ on X (see Theorem 3.6). Thee meet $k_{(X,\mathcal{U})}$ of these pretopologies is described by

$$\begin{aligned} k_{(X,\mathcal{U})}(M) &= \{x \in X : (\forall E \in \mathcal{U})(\exists a \in M)(x,a) \in E\} \\ &= \{x \in X : (\forall E \in \mathcal{U}) E[x] \cap M \neq \emptyset\}\,, \end{aligned}$$

with $E[x] = \uparrow_{(X,E)} (\{x\}) = \{y \in X : (x,y) \in E\}$ and $M \subseteq X$. Since $k_{(X,\mathcal{U})}$ is idempotent, $(X, k_{(X,\mathcal{U})})$ is actually a topological space. For every $x \in X$, the sets $\{E[x] : E \in \mathcal{U}\}$ form a base of neighbourhoods at x . Every uniformly continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ gives a continuous map with respect to the induced topologies. Hence one has a functor W which makes the diagram

$$\begin{tikzcd} \mathbf{Unif} \arrow[rr, "W"] \arrow[dr, "V"'] & & \mathbf{Top} \arrow[dl, "U"] \\ & \mathbf{Set} & \end{tikzcd} \tag{5.31}$$

commute. The forgetful functors U and V are fibrations (in fact: topological functors), and with $\mathcal{M}$ the class of injective maps W is a $\mathcal{M}_U$-fibration with $(\mathcal{M}_U)_W = \mathcal{M}_V$.

Every closure operator C of **Top** may be initially lifted to a closure operator C_W of **Unif** (see Theorem 5.8). In terms of **CS**-valued functors, it is described by $\mathbf{C}W$ (see Theorem 5.10). In particular, the Kuratowski closure K of **Top** can be lifted to **Unif** and yields an idempotent, hereditary, grounded, additive and productive closure operator of **Unif** . However, unlike **Top**, **Unif**, is not K-structured: the map $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is $(K-)$continuous but not uniformly continuous. Actually, we shall show below that f is D-continuous with respect to *every* closure operator D of **Unif** , i.e., is totally continuous.

A **Set**-map $f : X \to Y$ of uniform spaces X and Y is called *uniformly approachable* for all $x \in X$ and $M \subseteq X$ if there is a uniformly continuous map $g : X \to Y$ with $g(x) = f(x)$ and $g(M) \subseteq f(M)$.

LEMMA *Every uniformly approachable* **Set**-*map of uniform spaces is totally continuous.*

Proof For every closure operator D of **Unif** and every uniformly approachable **Set**-map $f : X \to Y$, we must show $f(d_X(M)) \subseteq d_Y(f(M))$ for all $M \subseteq X$. But for $x \in d_X(M)$, one may choose $g : X \to Y$ in **Unif** with $g(x) = f(x)$ and $g(M) \subseteq f(M)$. Since g is D-continuous, one obtains

$$f(x) = g(x) \in g(d_X(M)) \subseteq d_Y(g(M)) \subseteq d_Y(f(M)).$$

□

THEOREM *The functors* V : **Unif** $\to$ **Set** *and* W : **Unif** $\to$ **Top** *are not closure-structured.*

Proof It suffices to show that every continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly approachable and therefore totally continuous since, as mentioned above, not every such function is uniformly continuous. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and consider $x \in \mathbb{R}$ and $M \subseteq \mathbb{R}$. Choose $a \in M \cap (-\infty, x)$ if this set is non-void, and put $a = x - 1$ otherwise; similarly, choose $b \in M \cap (x, \infty)$ or put $b = x + 1$. Now let $g : \mathbb{R} \to \mathbb{R}$ coincide with f on the compact interval $[a, b]$, and let it be constant $f(a)$ on $(\infty, a]$ and constant $f(b)$ on $[b, \infty)$. Then g is obviously uniformly continuous and satisfies $g(x) = f(x)$ and $g(M) \subseteq f(M)$, as required. □

The Theorem is especially surprising since, in addition to the closure operators obtained from initially lifting closure operators of **Top** along W , **Unif** has important closure operators which may not be obtained this way, as we shall see next. Call a subset N of a uniform space X *uniformly clopen* if the characteristic function $h : X \to D = \{0, 1\}$ with $h^{-1}(0) = N$ is uniformly continuous; here D is provided with the discrete uniformity $\mathcal{U}$ (i.e., the diagonal Δ_D belongs to $\mathcal{U}$).

For $M \subseteq X$, one puts

$$q_X^u(M) = \bigcap\{N \subseteq X : N \text{ uniformly clopen}\ ,\ M \subseteq N\}$$

and proves:

PROPOSITION $Q^u = (q_X^u)_{X \in \mathbf{Unif}}$ *is an idempotent, grounded, additive and productive closure operator of* **Unif**, *but neither weakly hereditary nor fully additive. It is the largest proper closure operator of* **Unif**, *but it cannot be obtained as an initial lifting of a closure operator of* **Top**. □

Proof Everything but the last statement can be proved analogously to Theorem 4.7 and is therefore left to the reader as Exercise 5.U. In order to show that Q^u cannot be described as C_W with a closure operator C of **Top**, first we observe that, for a uniform space X and $M \subseteq X$, $(c_W)_X(M)$ is uniquely determined by $c_{WX}(M)$. Consequently, for uniform spaces X and Y which, as topological space are homeomorphic, one has $(c_W)_X \cong (c_W)_Y$. With this in, mind, we consider two distinct uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathbb{Q}$ both of which provide $\mathbb{Q}$ with its natural topology. It then suffices to show that Q^u differs on $(\mathbb{Q},\mathcal{U}_1)$ and $(\mathbb{Q},\mathcal{U}_2)$.

For $\mathcal{U}_1$ we choose the usual metric uniformity. Then every uniformly continuous function $h : \mathbb{Q} \to D$ into the discrete space $D = \{0,1\}$ can be uniformly extended to the completion $\mathbb{R}$ of $\mathbb{Q}$. But since $\mathbb{R}$ is connected, every (uniformly) continuous function $\mathbb{R} \to D$ is constant. Consequently, $q^u_{(\mathbb{Q},\mathcal{U}_1)}(M)$ is $\mathbb{Q}$ for $\emptyset \neq M \subseteq \mathbb{Q}$ and empty otherwise, that is: on $(\mathbb{Q},\mathcal{U}_1)$, $\mathbb{Q}^u$ coincides with the largest grounded operator of **Unif**.

In order to define $\mathcal{U}_2$, we embed the zero-dimensional space $\mathbb{Q}$ *topologically* into the Tychonoff product $X = D^\omega$. The least filter in $\mathbf{SGph}(X)$ which contains the relations $E_n = \{(x,y) \in X^2 : p_n(x) = p_n(y)\}$, $n < \omega$ (with p_n the n-th projection) defines a uniformity on X (in fact, the only one which induces the Tychonoff topology). $\mathcal{U}_2$ is its restriction to $\mathbb{Q}$. Since X is compact, each projection p_n is uniformly continuous. Consequently, the uniformly clopen sets separate points in X and therefore in $(\mathbb{Q},\mathcal{U}_2)$. From this one derives that, on $(\mathbb{Q},\mathcal{U}_2)$, Q^u coincides with the Kuratowski closure operator $K = K_W$. □

5.12 Modifications of closure operators

In the setting of 2.1, we consider closure operators C and D of $\mathcal{X}$. A morphism $f : X \to Y$ of $\mathcal{X}$ is called (D,C)-*continuous* if

$$f(d_X(m)) \leq c_Y(f(m))$$

holds for all $m \in \mathcal{M}/X$. Since f is (C,C)-continuous (and (D,D)-continuous), it is certainly (D,C)-continuous in case $D \leq C$. Given C and a class $\mathcal{K}$ of morphisms in $\mathcal{X}$, one may ask whether there is largest closure operator D such that each morphisms in $\mathcal{K}$ is (D,C)-continuous. In this section we deal with this question when $\mathcal{K}$ is the class $\{\eta_X : X \in \mathcal{X}\}$, for a pointed endofunctor (T,η) of $\mathcal{X}$, and with the dual question.

LEMMA *For every commutative square*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "\psi"] \\ Y \arrow[r, "\varphi"] & W \end{tikzcd} \tag{5.32}$$

and every $n \in \mathcal{M}/Z$ *, one has* $f(g^{-1}(n)) \le \varphi^{-1}(\psi(n))$.

Proof Repeated adjointness arguments yield

$$\begin{aligned} g(g^{-1}(n)) \le n &\Rightarrow \psi(g(g^{-1}(n))) \le \psi(n) \\ &\Rightarrow \varphi(f(g^{-1}(n))) \le \psi(n) \\ &\Rightarrow f(g^{-1}(n)) \le \varphi^{-1}(\psi(n)) . \end{aligned}$$

□

THEOREM *For a pointed endofunctor* (T, η) *of* $\mathcal{X}$ *and every closure operator* C *of* $\mathcal{X}$ *there is a largest closure operator* ${}_TC$ *of* $\mathcal{X}$ *which makes every* $\eta_X : X \to TX$ $({}_TC, C)$*-continuous. The passage* $C \mapsto {}_TC$ *preserves arbitrary meets and idempotency of closure operators.*

Proof For any D such that all η_X , $X \in \mathcal{X}$, are (D, C)-continuous we must have $\eta_X(d_X(m)) \le c_{TX}(\eta_X(m))$, hence $d_X(m) \le \eta_X^{-1}(c_{TX}(\eta_X(m)))$ for all $m \in \mathcal{M}/X$. For the first part of the Theorem it therefore suffices to show that, when putting

$$({}_Tc)_X(m) :\cong \eta_X^{-1}(c_{TX}(\eta_X(m))) \tag{*}$$

we obtain indeed a closure operator ${}_TC$. Since the properties of extension and monotonicity are rather obvious, we just check the continuity condition for ${}_TC$. Indeed, for every $f : X \to Y$ in $\mathcal{X}$ one obtains with the Lemma from the continuity condition for C :

$$\begin{aligned} f(({}_Tc)_X(m)) &\le \eta_Y^{-1}((Tf)(c_{TX}(\eta_X(m)))) \\ &\le \eta_Y^{-1}(c_{TY}((Tf)(\eta_X(m)))) \\ &\cong \eta_Y^{-1}(c_{TY}(\eta_Y(f(m)))) \\ &\cong ({}_Tc)_Y(f(m)) . \end{aligned}$$

The rule

$$_T\left(\bigwedge_i C_i\right) \cong \bigwedge_i {}_T(C_i)$$

follows immediately from the fact that inverse images preserves meets. Finally, assume C to be idempotent. Then, for every $m \in \mathcal{M}/X$, $c_{TX}(\eta_X(m))$ is C-closed, hence $\eta_X(m) \leq \eta_X(({}_Tc)_X(m)) \leq c_{TX}(\eta_X(m)) \cong c_{TX}(c_{TX}(\eta_X(m)))$. This implies $c_{TX}(\eta_X(({}_Tc)_X(m))) \cong c_{TX}(\eta_X(m))$ and then, with the definition of ${}_TC$, $({}_Tc)_X(({}_Tc)_X(m)) \cong ({}_Tc)_X(m)$. □

THEOREM* *For a copointed endofunctor (S,ε) of $\mathcal{X}$ and every closure operator C of $\mathcal{X}$, there is a least closure operator ${}^SC$ of $\mathcal{X}$ which makes every $\varepsilon_X : SX \to X$ $(C, {}^SC)$-continuous. The passage $C \mapsto {}^SC$ preserves arbitrary joins.*

Proof For any D such that all ε_X, $X \in \mathcal{X}$, are (C,D)-continuous one has $\varepsilon_X(c_{SX}(n)) \leq d_X(\varepsilon_X(n))$ for all $n \in \mathcal{M}/SX$ or, equivalently, $c_{SX}(\varepsilon_X^{-1}(m)) \leq \varepsilon_X^{-1}(d_X(m))$, hence $d_X(m) \geq \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m)))$ for all $m \in \mathcal{M}/X$. Since trivially $d_X(m) \geq m$, one therefore puts

$${}^Sc_X(m) :=\cong m \vee \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m))) . \qquad (**)$$

We just check the continuity condition for ${}^SC$, using preservation of joins by direct images, the continuity condition for C, and the Lemma successively:

$$\begin{aligned} f({}^Sc_X(m)) &\cong f(m) \vee \varepsilon_Y((Sf)(c_{SX}(\varepsilon_X^{-1}(m)))) \\ &\leq f(m) \vee \varepsilon_Y(c_{SY}((Sf)(\varepsilon_X^{-1}(m)))) \\ &\leq f(m) \vee \varepsilon_Y(c_{SY}(\varepsilon_Y^{-1}(f(m)))) \\ &\cong {}^Sc_Y(f(m)) . \end{aligned}$$

for $f : X \to Y$ in $\mathcal{X}$ and $m \in \mathcal{M}/X$. Preservation of joins by direct images is also responsible for the validity of the rule

$${}^S\left(\bigvee_i C_i\right) \cong \bigvee_i {}^SC_i .$$

□

The closure operator ${}_TC$ as defined by the Theorem is called the (T,η)-*modification* (or just T-*modification*) of C, and ${}^SC$ as defined by Theorem * is the (S,ε)-*comodification* (or just S-*comodification*) of C. The following Corollary illustrates these constructions in an algebraic context. They will be used mostly for topological applications in 6.3 and 7.7.

COROLLARY *Let* $\mathbf{s}$ *be a preradical in* $\mathbf{Mod}_R$ *with corresponding prereflection* $\mathbf{q}$ *(i.e.,* $\mathbf{q}(X) = X/\mathbf{s}(X)$ *), and let* C *be a closure operator with induced preradical* $\mathbf{r}$. *Then the* $\mathbf{s}$*-comodification* ${}^{\mathbf{s}}C$ *of* C *induces the preradical* $\mathbf{r}\,\mathbf{s}$ *, and the* $\mathbf{q}$*-modification* ${}_{\mathbf{q}}C$ *of* C *induces the preradical* $\mathbf{r} : \mathbf{s}$ *. Furthermore, if* $C = C_{\mathbf{r}}$ *is minimal, then also* ${}^{\mathbf{s}}C$ *and* ${}_{\mathbf{q}}C$ *are minimal. Moreover, if* $C = C^{\mathbf{r}}$ *is maximal, also* ${}^{\mathbf{s}}C$ *and* ${}_{\mathbf{q}}C$ *are maximal, provided the functor* $\mathbf{s} : \mathbf{Mod}_R \to \mathbf{Mod}_R$ *preserves epimorphisms.* □

Proof For a submodule M of $X \in \mathbf{Mod}_R$, one has

$$\begin{aligned} {}^{\mathbf{s}}c_X(M) &= M + c_{\mathbf{s}(X)}(M \cap \mathbf{s}(X)) , \\ ({}_{\mathbf{q}}c)_X(M) &= p^{-1}(c_{\mathbf{q}(X)}(p(M))) \end{aligned}$$

with $p : X \to \mathbf{q}(X)$ the projection. Hence

$$\begin{aligned} {}^{\mathbf{s}}c_X(0) &= \mathbf{r}(\mathbf{s}(X)) , \\ ({}_{\mathbf{q}}c)_X(0) &= p^{-1}(\mathbf{r}(X/\mathbf{s}(X))) = (\mathbf{r} : \mathbf{s})(X) . \end{aligned}$$

If C is minimal, then $c_{\mathbf{s}(X)} = (M \cap \mathbf{s}(X)) + \mathbf{r}(\mathbf{s}(X))$, hence

$${}^{\mathbf{s}}c_X(M) = M + \mathbf{r}(\mathbf{s}(X)) ;$$

furthermore, since $p^{-1}(p(M) + \mathbf{r}(X/\mathbf{s}(X))) = M + p^{-1}(\mathbf{r}(X/\mathbf{s}(X)))$,

$$({}_{\mathbf{q}}c)_X(M) = M + (\mathbf{r} : \mathbf{s})(X) .$$

If $\mathbf{s}$ preserves epimorphisms, then the restriction $\mathbf{s}(X) \to \mathbf{s}(X/M)$ of the projection $\pi : X \to X/M$ is surjective. This implies

$$\begin{aligned} \mathbf{s}(X/M) &\cong \mathbf{s}(X)/M \cap \mathbf{s}(X) , \\ (X/M)/\mathbf{s}(X/M) &\cong (X/\mathbf{s}(X))/p(M) . \end{aligned}$$

Consequently, if C is maximal, one obtains

$$\begin{aligned} {}^{\mathbf{s}}c_X(M) &= M + \pi^{-1}(\mathbf{r}(\mathbf{s}(X/M))) \\ &= \pi^{-1}(\mathbf{r}(\mathbf{s}(X/M))) , \\ ({}_{\mathbf{q}}c)_X(M) &= p^{-1}(q^{-1}(\mathbf{r}((X/M)/\mathbf{s}(X/M)))) \\ &= p^{-1}((\mathbf{r} : \mathbf{s})(X/M)) , \end{aligned}$$

with $q : X/M \to \mathbf{q}(X/M)$ the projection. □

REMARK For a preradical $\mathbf{s}$ of $\mathbf{Mod}_R$, the following conditions are equivalent:

(i) $\mathbf{s} : \mathbf{Mod}_R \to \mathbf{Mod}_R$ preserves epimorphisms,

(ii) $\mathbf{s}$ is *cohereditary* (i.e., $\mathbf{s}(X/M) \cong (\mathbf{s}(X) + M)/M$ for all $M \leq X \in \mathbf{Mod}_R$),

(iii) $C_{\mathrm{s}} = C^{\mathrm{s}}$,

(iv) there exists a two-sided ideal J of the ring R such that

$$\mathrm{s}(X) = JX \text{ for all } X \in \mathbf{Mod}_R \ ;$$

if these conditions hold, s is a radical (cf. Theorem 3.4).

5.13 Closure operators and adjoint functors

Recall that a functor $U : \mathcal{A} \to \mathcal{X}$ is *right adjoint* if for all $X \in \mathcal{X}$ there is a (chosen) *U-universal arrow* for X ; this is a pair consisting of an object $FX \in \mathcal{A}$ and a morphism $\eta_X : X \to UFX$ such that for all $A \in \mathcal{A}$ and $f : X \to UA$ in $\mathcal{X}$ there is exactly one morphism $f^\# : FX \to A$ in $\mathcal{A}$ with $Uf^\# \cdot \eta_X = f$. Hence the correspondence

$$\frac{X \xrightarrow{f} UA}{FX \xrightarrow{f^\#} A}$$

is inverse to the map $\varphi_{X,A} : \mathcal{A}(FX, A) \to \mathcal{X}(X, UA)$ given by $g \mapsto Ug \cdot \eta_X$. The existence of a U-universal arrow for X therefore gives a natural isomorphism $\varphi_{X,-} : \mathcal{A}(FX, -) \to \mathcal{X}(X, U-)$ exhibiting $\mathcal{X}(X, U-) : \mathcal{A} \to \mathbf{Set}$ as a representable functor. Vice versa, the representability of $\mathcal{X}(X, U-)$ yields a U-universal arrow for X : with FX the representing object and with an isomorphism φ_X as above, put $\eta_X := \varphi_{X,FX}(1_{FX})$.

If U is right adjoint then there is a unique way of making F (as above) a functor $\mathcal{X} \to \mathcal{A}$ such that $\eta : Id_{\mathcal{X}} \to UF$ is a natural transformation, by putting $Ff = (\eta_Y \cdot f)^\#$.

$$\begin{array}{ccccc} X & \xrightarrow{\quad \eta_X \quad} & UFX & \qquad & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UFf} & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow{\quad \eta_Y \quad} & UFY & & FY \end{array} \tag{5.33}$$

Moreover there is a natural transformation $\varepsilon : FU \to Id_{\mathcal{A}}$ defined by $\varepsilon_A = (1_{UA})^\#$ which satisfies the so-called *triangular identities*

$$U\varepsilon \cdot \eta U = 1_U \qquad \text{and} \qquad \varepsilon F \cdot F\eta = 1_F \ .$$

Any pair of functors $(F : \mathcal{X} \to \mathcal{A}\ ,\ U : \mathcal{A} \to X)$ is called *adjoint* if there are natural transformations $\eta : Id_X \to UF$ and $\varepsilon : FU \to Id_{\mathcal{A}}$ satisfying the triangular identities; these are called the *unit* and *counit* of the adjunction, respectively. Usually one writes

$$F \dashv U \quad \text{or} \quad F \ \underset{\varepsilon}{\overset{\eta}{\dashv}}\ U : \mathcal{A} \to \mathcal{X}$$

in this situation and calls F *left adjoint to* U and U *right adjoint to* F. One easily derives from the triangular identities that U is in fact a right adjoint functor in the sense of the initial definition; just put $f^{\#} := \varepsilon_A \cdot Ff$ in order to show that (FX, η_X) is a U-universal arrow for X. The notion of adjointness is self dual, in the sense that

$$F \underset{\varepsilon}{\overset{\eta}{\dashv}} U : \mathcal{A} \to \mathcal{X} \Leftrightarrow U^{op} \underset{\eta^{op}}{\overset{\varepsilon^{op}}{\dashv}} F^{op} : \mathcal{X}^{op} \to \mathcal{A}^{op}$$

(with $\mathcal{A}^{op}$ denoting the opposite category of $\mathcal{A}$). This means in particular that for every $A \in \mathcal{A}$, the pair (UA, ε_A) is an F-*couniversal arrow* for A, in the sense dual to universality as defined before.

The first part of the following Proposition recalls the most prominent property of adjoint functors the proof of which is left as Exercise 5.W. In the special case of preordered classes it was shown in Proposition 1.3.

PROPOSITION *Let $F \dashv U : \mathcal{A} \to \mathcal{X}$ be adjoint functors. Then:*

(1) *U preserves all (existing) limits of $\mathcal{A}$, and F preserves all (existing) colimits of $\mathcal{X}$.*

(2) *If $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorizations and if $\mathcal{A}$ has $(\mathcal{F}, \mathcal{N})$-factorizations, then $U\mathcal{N} \subseteq \mathcal{M}$ if and only if $F\mathcal{E} \subseteq \mathcal{F}$.*

□

Proof (2) In the notation of 1.8 one has $\mathcal{E} = \mathcal{M}^{\perp}$ and $\mathcal{F} = \mathcal{N}^{\perp}$. Hence, in order to show $F\mathcal{E} \subseteq \mathcal{F}$ if $U\mathcal{N} \subseteq \mathcal{M}$, it suffices to show that $Fe \perp n$ holds for all $n \in \mathcal{N}$. But with the adjointness property one easily shows that this is equivalent to $e \perp Un$ for all $n \in \mathcal{N}$, which holds by hypothesis.

That $U\mathcal{N} \subseteq \mathcal{M}$ is also a necessary condition for $F\mathcal{E} \subseteq \mathcal{F}$ follows dually since $\mathcal{M} = \mathcal{E}_{\perp}$ and $\mathcal{N} = F_{\perp}$ (see 1.8). □

For the remainder of this section we keep the hypotheses and notations of Proposition (2) and assume $\mathcal{X}$ to be finitely $\mathcal{M}$-complete and $\mathcal{A}$ finitely $\mathcal{N}$-complete. Hence U is required to preserve subobjects but F is not. We adopt the conventions of Remark 5.7 regarding the application of a closure operator to morphisms not necessarily in $\mathcal{M}$ or $\mathcal{N}$.

Given a closure operator D of $\mathcal{A}$ w.r.t. $\mathcal{N}$, we wish to give an explicit description of the initial closure operator $D_{(F)}$ induced by D and F. For $m : M \to X$ in $\mathcal{M}$, consider

$$d^{\eta}_X(m) := \eta_X^{-1}(Ud_{FX}(Fm)) , \qquad (*)$$

with $d_{FX}(Fm) = d_{FX}((Fm)(1_{FM}))$; see Remark 5.7.

$$\begin{tikzcd}
M \arrow[rr, "\eta_M"] \arrow[dd, "m"'] \arrow[dr] & & UFM \arrow[dd, "UFm"'] \arrow[dr] & \\
& \cdot \arrow[rr, "t"] \arrow[dl, "d^\eta_X(m)"] & & \cdot \arrow[dl, "Ud_{FX}(Fm)"] \\
X \arrow[rr, "\eta_X"] & & UFX &
\end{tikzcd} \tag{5.34}$$

LEMMA *Formula* $(*)$ *defines a closure operator* D^η *of* $\mathcal{X}$ *; it is the initial closure operator induced by* D *and* F *:* $D^\eta \cong D_{(F)}$.

Proof Extension and monotonicity are easily checked for D^η . For every $f : X \to Y$ in $\mathcal{X}$ and $m \in \mathcal{M}/X$ one obtains with Lemma 5.12, Lemma 5.7, and with the continuity condition for D :

$$\begin{aligned}
f(d^\eta_X(m)) &\leq \eta_Y^{-1}((UFf)(Ud_{FX}(Fm))) \\
&\leq \eta_Y^{-1}(U((Ff)(d_{FX}(Fm)))) \\
&\leq \eta_Y^{-1}(Ud_{FY}((Ff)(Fm))) \\
&\cong \eta_Y^{-1}(Ud_{FY}(F(f(m)))) \\
&\cong d^\eta_Y(f(m)) .
\end{aligned}$$

Hence D^η is a closure operator. Next we show that F is (D^η, D)-continuous, i.e., $Fd^\eta_X(m) \leq d_{FX}(Fm)$ for all $m \in \mathcal{M}/X$. For this one simply observes that the horizontal morphism t of (5.34) corresponds by adjunction to a morphism $t^\#$ with $d_{FX}(Fm) \cdot t^\# = Fd^\eta_X(m)$. For any other closure operator $C \in CL(\mathcal{X}, \mathcal{M}))$ such that F is (C, D)-continuous, one has $Fc_X(m) \leq d_{FX}(Fm)$; since $\eta_X(c_X(m)) \leq UFc_X(m)$, this implies

$$c_X(m) \leq \eta_X^{-1}(UFc_X(m)) \leq \eta_X^{-1}(Ud_{FX}(Fm)) \cong d^\eta_X(m) .$$

Hence we have $D^\eta \cong D_{(F)}$. □

D^η was obtained by pulling back D along the unit η . We now investigate the "dual" procedure: mapping a closure operator C of $\mathcal{X}$ along the counit ε to obtain a closure operator ${}^\varepsilon C$ of $\mathcal{A}$. For $n : N \to A$ in $\mathcal{N}$ we put

$${}^\varepsilon c_A(n) = n \vee \varepsilon_A(Fc_{UA}(Un)) , \tag{**}$$

with $\varepsilon_A(Fc_{UA}(Un)) = \varepsilon_A((Fc_{UA}(Un))(1))$; see Remark 5.7.

$$\begin{tikzcd}
FUN \arrow[rr, "\varepsilon_N"] \arrow[dd, "FUn"'] \arrow[dr] & & N \arrow[dd, "n"'] \arrow[dr] & \\
& \cdot \arrow[rr] \arrow[dl, "Fc_{UA}(Un)"] & & \cdot \arrow[dl, "{}^{\varepsilon}c_A(n)"] \\
FUA \arrow[rr, "\varepsilon_A"] & & A &
\end{tikzcd} \tag{5,34'}$$

Note that in case ε_N belongs to $\mathcal{E}$, one has $n \leq \varepsilon_A(Fc_{UA}(Un))$, which simplifies formula $(**)$.

LEMMA* *Formula* $(**)$ *defines a closure operator* ${}^{\varepsilon}C$ *of* $\mathcal{A}$; *it is the final closure operator induced by* C *and* F : ${}^{\varepsilon}C \cong C^{(F)}$.

Proof We check only the continuity condition for ${}^{\varepsilon}C$, using the same tools as in the proof of the Lemma, plus preservation of joins by the direct image along $f : A \to B$ in $\mathcal{A}$:

$$\begin{aligned}
f({}^{\varepsilon}c_A(n)) &\cong f(n) \vee (f \cdot \varepsilon_A \cdot Fc_{UA}(Un))(1) \\
&\cong f(n) \vee (\varepsilon_B \cdot FUf \cdot Fc_{UA}(Un))(1) \\
&\cong f(n) \vee \varepsilon_B(F((Uf)(c_{UA}(Un)))) \\
&\leq f(n) \vee \varepsilon_B(Fc_{UB}((Uf)(Un))) \\
&\leq f(n) \vee \varepsilon_B(Fc_{UB}(U(f(n)))) \\
&\cong {}^{\varepsilon}c_B(f(n)) .
\end{aligned}$$

In order to show ${}^{\varepsilon}C \cong C^{(F)}$, it suffices to show that $(C \mapsto {}^{\varepsilon}C)$ is left adjoint to $(D \mapsto D^{\eta} \cong D_{(F)})$ since we already know that $(C \mapsto C^{(F)})$ is left adjoint to $(D \mapsto D_{(F)})$, see Corollary 5.7. Hence it is sufficient to show

$$C \leq D^{\eta} \Leftrightarrow {}^{\varepsilon}C \leq D$$

for all $C \in CL(\mathcal{X}, \mathcal{M})$ and $D \in CL(\mathcal{A}, \mathcal{N})$.

"$\Rightarrow$" For all $n : N \to A$ in $\mathcal{N}$, $c_{UA}(Un) \leq d^{\eta}_{UA}(Un)$ by hypothesis, and $Fd^{\eta}_{UA}(Un) \leq d_{FUA}(FUn)$ since F is (D^{η}, D)-continuous (according to the Lemma). Hence

$$\begin{aligned}
{}^{\varepsilon}c_A(n) &\leq n \vee \varepsilon_A(Fd^{\eta}_{UA}(Un)) \leq n \vee \varepsilon_A(d_{FUA}(FUn)) \\
&\leq n \vee d_A(\varepsilon_A(FUn)) \leq d_A(n) .
\end{aligned}$$

"$\Leftarrow$" Let ${}^{\varepsilon}C \leq D$ and consider $m : M \to X$ in $\mathcal{M}$. Then

$$\eta_X(c_X(m)) \leq c_{UFX}(\eta_X(m)) \leq c_{UFX}(UFm) =: m' ,$$

and with $A = FX$ and the triangular identity one has

$$m' \cong (U\varepsilon_A)((\eta_{UA}(m'))) \leq (U\varepsilon_A)(UFm') \leq U(\varepsilon_A(Fm')) .$$

Consequently, by hypothesis,

$$\eta_X(c_X(m)) \leq U(\varepsilon_{FX}(Fc_{UFX}(UFm))) \leq Ud_{FX}(Fm) ,$$

and this implies

$$c_X(m) \leq \eta_X^{-1}(Ud_{FX}(Fm)) \cong d_X^{\eta}(m) .$$

$\square$

The Lemma and Lemma* prove the first part of the following Theorem.

Theorem *Let* $F \underset{\varepsilon}{\overset{\eta}{\dashv}} U : \mathcal{A} \to \mathcal{X}$ *be adjoint functors with* $U\mathcal{N} \subseteq \mathcal{M}$.

(1) *Then the assignments* $C \mapsto {}^{\varepsilon}C$ *and* $D \mapsto D^{\eta}$ *define adjoint maps*

$$CL(\mathcal{X},\mathcal{M}) \underset{(-)^{\eta}}{\overset{{}^{\varepsilon}(-)}{\rightleftarrows}} CL(\mathcal{A},\mathcal{N}) ,$$

with ${}^{\varepsilon}(-) \cong (-)^{(F)} \dashv (-)_{(F)} \cong (-)^{\eta}$. *In particular,* F *is* $(C, {}^{\varepsilon}C)$-*continuous and* (D^{η}, D)-*continuous for all* $C \in CL(\mathcal{X},\mathcal{M})$ *and* $D \in CL(\mathcal{A},\mathcal{N})$.

(2) *If* $\mathcal{N} \subseteq \mathrm{Init}_U$, *then* $(D_{(F)})_{(U)} \leq D$ *for all* $D \in CL(\mathcal{A},\mathcal{N})$, *and if* U *is a faithful fibration, then* $(C_{(U)})_{(F)} \cong C$ *for all* $C \in CL(\mathcal{X},\mathcal{M})$. *In the latter case one has*

$$(-)^{(U)} \dashv (-)_{(U)} \cong (-)^{(F)} \dashv (-)_{(F)} ;$$

in particular, U *is* $({}^{\varepsilon}C, C)$-*continuous for all* $C \in CL(\mathcal{X},\mathcal{M})$.

Proof (2) For D, $E \in CL(\mathcal{A},\mathcal{N})$, assume U to be (E, D^{η})-continuous. Then $Ue_A(n) \leq d^{\eta}_{UA}(n)$ for every $n : N \to A$ in $\mathcal{N}$, hence $\eta_{UA}(Ue_A(n)) \leq Ud_{FUA}(FUn)$. Since $U\varepsilon_A \cdot \eta_{UA} = 1_{UA}$, this implies

$$\begin{aligned} Ue_A(n) &\leq (U\varepsilon_A)(Ud_{FUA}(FUn)) \\ &\leq U(\varepsilon_A(d_{FUA}(FUn))) \\ &\leq Ud_A(\varepsilon_A(FUn)) \\ &\leq Ud_A(n) , \end{aligned}$$

hence $e_A(n) \le d_A(n)$ if $\mathcal{N} \subseteq \mathrm{Init}_U$. This proves $E \le D$ whenever $E \le (D^\eta)_{(U)}$, hence $(D_{(F)})_{(U)} \le D$ with (1).

Let now U be a faithful $\mathcal{M}$-fibration. First we show that the units η_X must be isomorphisms in this case. In fact, let $e : A \to FX$ be a U-initial lifting of $\eta_X : X \to UFX$, and consider $f = (1_X)^\# : FX \to A$. Then $Uf \cdot Ue = 1_{UA}$, hence $f \cdot e = 1_A$. Since $U(e \cdot f) \cdot \eta_X = \eta_X$ one also has $e \cdot f = 1_{FX}$, hence e and therefore η_X is an isomorphism. With $C \in CL(\mathcal{X}, \mathcal{M})$ and $D = C_U = C_{(U)}$ its U-initial lifting, one therefore has for all $m : M \to X$ in $\mathcal{M}$

$$d^\eta_X(m) \cong \eta_X^{-1}(Ud_{FX}(Fm)) \cong \eta_X^{-1}(c_{UFX}(UFm)) \cong c_X(m) \ .$$

With (1), this proves $(C_{(U)})_{(F)} \cong C$. Together with $(D_{(F)})_{(U)} \le D$, one derives $(-)_{(U)} \dashv (-)_{(F)}$. Since also $(-)^{(F)} \dashv (-)_{(F)}$, we must have $(-)_{(U)} \cong (-)^{(F)}$. Finally, since $(-)^{(F)} \cong {}^\varepsilon(-)$, one concludes $C_{(U)} \cong {}^\varepsilon C$, hence U is $({}^\varepsilon C, C)$-continuous for all C . □

COROLLARY *Let $U : \mathcal{A} \to \mathcal{X}$ be a faithful fibration with left adjoint F . Then $C \mapsto C_U$ embeds $CL(\mathcal{X}, \mathcal{M})$ reflectively and coreflectively into $CL(\mathcal{A}, \mathcal{N})$. Every $C \in CL(\mathcal{X}, \mathcal{M})$ is its own U-interior and F-closure in the sense of* Corollary 5.7. *Finally, $(-)_{(U)} \cong (-)^{(F)}$ preserves all meets and joins in $CL(\mathcal{X}, \mathcal{M})$.* □

Proof The map $C \mapsto C_U \cong C_{(U)} \cong C^{(F)}$ has both a left and a right adjoint, $(-)^{(U)}$ and $(-)_{(F)}$ respectively. Since $(C_{(U)})_{(F)} \cong C$, the map provides an embedding, and each C in its own F-closure. Since $((C_{(U)})^{(U)})_{(U)} \cong C_{(U)}$ (see Remark 5.8(1)), an application of $(-)_{(F)}$ gives $(C_{(U)})^{(U)} \cong C$. Hence C is its own U-interior. □

EXAMPLES

(1) The forgetful functor U : **Top** $\to$ **Set** is a fibration and has a left adjoint (discrete topology). If D is the Kuratowski closure operator (or any other proper closure operator of **Top** w.r.t. the class of embeddings), then D^η is the discrete closure operator of **Set** (see Lemma 4.7). In particular, U is *not* (D, D^η)-continuous.

(2) The forgetful functor U : **TopGrp** $\to$ **Top** has a left adjoint F (free topological group). For every finitely productive closure operator of C of **Top** (w.r.t. the class of embeddings), U is $({}^\varepsilon C, C)$-preserving (see Exercise 5.X). Hence ${}^\varepsilon C$ is both the initial closure operator $C_{(U)}$ induced by C and U and final closure operator $C^{(F)}$ induced by C and F (see Lemma*).

(3) Let $U : \mathcal{A} \hookrightarrow \mathcal{X}$ be the embedding of a full $\mathcal{E}$-reflective subcategory. Since the counit is an isomorphism, for every $C \in CL(\mathcal{X}, \mathcal{M})$, ${}^\varepsilon C$ is (isomorphic to) the restriction $C|_\mathcal{A}$ (cf. 2.9), hence ${}^\varepsilon C$ is again the initial closure operator w.r.t. C and U . *Warning:* In general $CL(\mathcal{X}, \mathcal{M})$ cannot be embedded into $CL(\mathcal{A}, \mathcal{M})$; for instance, in **Top** consider the full subcategory of spaces with at

most one element. This shows that in part (2) of the Theorem, and in the Corollary, it is essential to assume U to be a fibration, not just an $\mathcal{M}$-fibration.

REMARK In the setting of the Theorem (1), let $T = UF$ and $S = FU$, so that one has pointed and copointed endofunctors (T, η) and (S, ε). The constructions of this section may then be compared with the modification and comodification as given in 5.12, as follows:

(1) $(D_{(F)})^{(F)} \cong {}^{\varepsilon}(D^{\eta}) \leq {}^{S}\tilde{D} \leq {}^{S}D \leq D$,

(2) $C \leq {}_{T}C \leq {}_{T}\tilde{C} \leq ({}^{\varepsilon}C)^{\eta} \cong (C^{(F)})_{(F)}$

holds for all $C \in CL(\mathcal{X}, \mathcal{M})$, $D \in CL(\mathcal{A}, \mathcal{N})$, with ${}_{T}\tilde{C}$ and ${}^{S}\tilde{D}$ defined as in Exercise 5.V. In fact, for $m : M \to X$ in $\mathcal{M}$ and with $m' = c_{UFX}(UFm)$ and $A = FX$, we showed $m' \leq U(\varepsilon_A(Fm'))$ in the proof of Lemma*. Pulling back the last inequality along η_X gives $({}_{T}\tilde{c})_X(m) \leq ({}^{\varepsilon}c)^{\eta}_X(m)$. This proves the only critical part of (1). Likewise, the only critical part of (2) is to show ${}^{\varepsilon}(d^{\eta})_B(n) \leq {}^{S}\tilde{d}_B(n)$ for all $n : N \to B$ in $\mathcal{N}$. But since F is (D^{η}, D)-continuous, one has $Fd^{\eta}_{UB}(Un) \leq d_{FUB}(FUn)$, which implies the desired inequality when one takes the respective direct images under ε_B.

5.14 External closure operators

In algebra one obtains the least subgroup generated by a subset M of a group G by

$$\langle M \rangle = \bigcap \{H \leq G : M \subseteq H\},$$

and similarly for any other algebraic structure. Formally this closure process is not covered by the notion of closure operator of a category. In this section we briefly mention the more general notion of external closure operator and how it relates to the internal notion as given in 2.2.

We consider a functor $U : \mathcal{A} \to \mathcal{X}$ and classes $\mathcal{N}$, $\mathcal{M}$ of monomorphisms in $\mathcal{A}$, $\mathcal{X}$, respectively, both closed under composition, such that (for simplicity) $\mathcal{A}$ is $\mathcal{N}$-complete, $\mathcal{X}$ is $\mathcal{M}$-complete, and $U\mathcal{N} \subseteq \mathcal{M}$. An *external closure operator* D of $\mathcal{A}$ (with respect to U, $\mathcal{N}$, $\mathcal{M}$) is given by a family of maps $d_A : \mathcal{M}/UA \to \mathcal{N}/A$, $A \in \mathcal{A}$, such that

(i) (*Extension*) $m \leq Ud_A(m)$,

(ii) (*Monotonicity*) $m \leq m'$ implies $d_A(m) \leq d_A(m')$,

(iii) (*Continuity*) $f(d_A(m)) \leq d_B((Uf)(m))$,

for all $f : A \to B$ in $\mathcal{A}$ and m, $m' \in \mathcal{M}/UA$.

Clearly, a closure operator (in the sense of 2.2) is an external closure operator with respect to $U = Id_{\mathcal{X}}$, $\mathcal{N} = \mathcal{M}$. The question is to which extent the two notions differ in general. In what follows we compare the conglomerate

$$CL^{\text{ext}}(U, \mathcal{N}, \mathcal{M})$$

of all external closure operators of $\mathcal{A}$ w.r.t. U , $\mathcal{N}$, $\mathcal{M}$ with the conglomerate $CL(\mathcal{A},\mathcal{N})$. The former conglomerate is, like the latter, preordered, and joins of non-empty families of external closure operators exist and are formed "pointwise". The largest (=*trivial*) external closure operator T^{ext} is defined by $t_A^{\text{ext}}(m) = 1_A$ for all $m \in \mathcal{M}/UA$, $A \in \mathcal{A}$. With regard to the least external closure operator and the formation of meets one has:

LEMMA *Let U transform $\mathcal{N}$-intersections into $\mathcal{M}$-intersections. Then there is a least (=discrete) external closure operator S^{ext} of $\mathcal{A}$ w.r.t. U , $\mathcal{N}$, $\mathcal{M}$, given by*

$$s_A^{\text{ext}}(m) \cong \bigwedge\{n \in \mathcal{N}/A : m \leq Un\}$$

for all $m \in \mathcal{M}/UA$, $A \in \mathcal{A}$. All meets in $CL^{\text{ext}}(U,\mathcal{M},\mathcal{N})$ exist and are formed "pointwise".

Proof Every external closure operator D satisfies $m \leq Ud_A(m)$, hence $s_A^{\text{ext}}(m) \leq d_A(m)$ for all $m \in \mathcal{M}/UA$. Monotonicity for S^{ext} is trivial, and extension follows from the hypothesis on U . For the continuity condition, one observes that $m \leq Un$ implies $(Uf)(m) \leq (Uf)(Un) \leq U(f(n))$ with Lemma 5.7, hence

$$\begin{aligned} f(s_A^{\text{ext}}(m)) &\leq \bigwedge\{f(n) : n \in \mathcal{N}/A\,,\ m \leq Un\} \\ &\leq \bigwedge\{k \in \mathcal{N}/B : (Uf)(m) \leq Uk\} \\ &\cong s_B^{\text{ext}}((Uf)(m))\,. \end{aligned}$$

Checking that meets can be formed pointwise requires a similar argumentation. □

REMARK If U has a left adjoint F (so that U preserves limits and therefore satisfies the hypothesis of the Lemma), then

$$s_A^{\text{ext}}(m) \cong m^{\#}(1_{FM})\,,$$

with $m^{\#} : FM \to A$ corresponding to $m : M \to UA$ in $\mathcal{M}$ by adjunction. In fact, $m \leq U(m^{\#}(1_{FM}))$, and if $m \leq Un$ with $n : N \to A$ in $\mathcal{N}$, so that $Un \cdot j = m$ for a morphism $j : M \to UN$, then $n \cdot j^{\#} = m^{\#}$ and therefore $m^{\#}(1_{FM}) \leq n$.

The external closure operator S^{ext} can now be used to "externalize" every (internal) closure operator C of $\mathcal{A}$ w.r.t. $\mathcal{N}$, as follows:

$$c_A^{\text{ext}}(m) :\cong c_A(s_A^{\text{ext}}(m))$$

for all $m \in \mathcal{M}/UA$, $A \in \mathcal{A}$. In fact, since

$$m \leq Us_A^{\text{ext}}(m) \leq Uc_A(s_A^{\text{ext}}(m)) \cong Uc_A^{\text{ext}}(m)\,,$$

$$\begin{aligned} f(c_A^{\text{ext}}(m)) &\cong f(c_A(s_A^{\text{ext}}(m))) \\ &\leq c_B(f(s_A^{\text{ext}}(m))) \\ &\leq c_B(s_B^{\text{ext}}((Uf)(m))) \cong c_B^{\text{ext}}((Uf)(m)) \, , \end{aligned}$$

extension and continuity hold for C^{ext}, and similarly for monotonicity. Note that for $C = S$ the discrete closure operator, C^{ext} is in fact the external closure operator S^{ext} of the Lemma; similarly in case $C = T$. Since joins and meets are formed pointwise, these are preserved when passing from C to C^{ext}. This proves the first part of:

PROPOSITION *If U transforms $\mathcal{N}$-intersections into $\mathcal{M}$-intersections, then the map $(-)^{\text{ext}} : CL(\mathcal{A},\mathcal{N}) \to CL^{\text{ext}}(U,\mathcal{N},\mathcal{M})$ preserves all meets and joins. The map is an order-embedding if $\mathcal{N} \subseteq \text{Init}_U$.* □

Proof In case $\mathcal{N} \subseteq \text{Init}_U$ one has $(n \leq k \Leftrightarrow Un \leq Uk)$ for all n, $k \in \mathcal{N}/A$, $A \in \mathcal{A}$, and therefore $s_A^{\text{ext}}(Un) \cong n$. This implies $C \leq D$ whenever $C^{\text{ext}} \leq D^{\text{ext}}$, for all $C, D \in CL(\mathcal{A},\mathcal{N})$. □

Under the hypothesis $\mathcal{N} \subseteq \text{Init}_U$ we can describe the reflector of $(-)^{\text{ext}}$ quite easily, i.e., every external closure operator D of $\mathcal{A}$ w.r.t. U, $\mathcal{N}$, $\mathcal{M}$ can be "internalized", by

$$d_A^{\text{int}}(n) := d_A(Un)$$

for all $n \in \mathcal{N}/A$. Extension for D^{int} follows from $\mathcal{N} \subseteq \text{Init}_U$, monotonicity is trivial, and

$$f(d_A^{\text{int}}(n)) \leq d_B((Uf)(Un)) \leq d_B(U(f(n))) \cong d_B^{\text{int}}(f(n))$$

shows that the continuity condition is satisfied.

THEOREM *Let $\mathcal{N} \subseteq \text{Init}_U$ and let U transform $\mathcal{N}$-intersections into $\mathcal{M}$-intersections. Then the maps*

$$(-)^{\text{int}} \dashv (-)^{\text{ext}} : CL(\mathcal{A},\mathcal{N}) \to CL^{\text{ext}}(U,\mathcal{N},\mathcal{M})$$

are adjoint, with both maps preserving all meets and joins, and with $(C^{\text{ext}})^{\text{int}} \cong C$ for all $C \in CL(\mathcal{A},\mathcal{N})$. Moreover, if U is a faithful $\mathcal{M}$-fibration, with $\mathcal{N} = \mathcal{M}_U (= U^{-1}\mathcal{M} \cap \text{Init}_U)$, then both maps are order-equivalences.

Proof Preservation of meets and joins of non-empty families by $(-)^{\text{int}}$ is clear since they are formed pointwise. Also, $(T^{\text{ext}})^{\text{int}} \cong T$ and $(S^{\text{ext}})^{\text{int}} \cong S$ (see the proof of the Proposition). The latter isomorphism gives

$$(c^{\text{ext}})_A^{\text{int}}(n) \cong c_A(s_A^{\text{ext}}(Un)) \cong c_A(n)$$

for all $n \in \mathcal{N}/A$, $A \in \mathcal{A}$, hence $(C^{\text{ext}})^{\text{int}} \cong C$ for all $C \in CL(\mathcal{A},\mathcal{N})$.

Furthermore,

$$d_A(m) \leq d_A(U s_A^{\text{ext}}(m)) \cong d_A^{\text{int}}(s_A^{\text{ext}}(m)) \cong (d^{\text{int}})_A^{\text{ext}}(m)$$

for all $m \in \mathcal{M}/UA$, $A \in \mathcal{A}$, hence $D \leq (D^{\text{int}})^{\text{ext}}$ for all $D \in CL^{\text{ext}}(U,\mathcal{N},\mathcal{M})$.

Finally, if U is an $\mathcal{M}$-fibration and $\mathcal{N} = \mathcal{M}_U$, one has an order-equivalence

$$\gamma_A : \mathcal{N}/A \to \mathcal{M}/UA\,, \qquad n \mapsto Un\,,$$

with right adjoint δ_A , $m \mapsto$ U-initial lifting of m ; see 5.8. It is easy to see that $\delta_A(m) = s_A^{\text{ext}}(m)$. Hence

$$c_A^{\text{ext}}(m) \cong c_A(\delta_A(m)) \quad \text{and} \quad d_A^{\text{int}}(n) \cong d_A(\gamma_a(n))$$

for all $m \in \mathcal{M}/UA$, $n \in \mathcal{N}/A$, $A \in \mathcal{A}$, and $C \in CL(\mathcal{N},\mathcal{A})$, $D \in CL^{\text{ext}}(U,\mathcal{N},\mathcal{M})$. Hence the order-equivalence $\gamma_A \dashv \delta_A$ induces the order-equivalence $(-)^{\text{int}} \dashv (-)^{\text{ext}}$. □

The assignment $D \mapsto D^{\text{int}}$ may be one-to-one without assuming U to be an $\mathcal{M}$-fibration, provided we restrict ourselves to special types of external closure operators. $D \in CL^{\text{ext}}(U,\mathcal{N},\mathcal{M})$ is called *idempotent* if $d_A(Ud_A(m)) \cong d_A(m)$ for all $m \in \mathcal{M}/A$, $A \in \mathcal{A}$. If $\mathcal{N} \subseteq \text{Init}_U$ and if D is idempotent, then also D^{int} is idempotent; moreover, $C^{\text{int}} \leq D^{\text{int}}$ implies

$$c_A(m) \leq c_A(Ud_A(m)) \cong c_A^{\text{int}}(d_A(m)) \leq d_A^{\text{int}}(d_A(m)) \cong d_A(m)$$

for all $m \in \mathcal{M}/A$, $A \in \mathcal{A}$, hence $C \leq D$. Consequently, one gets an order-embedding

$$(-)^{\text{int}} : IDCL^{\text{ext}}(U,\mathcal{N},\mathcal{M}) \to IDCL(\mathcal{A},\mathcal{N})$$

by restriction to the conglomerates of idempotent external and internal closure operators.

Corollary *If $\mathcal{N} \subseteq \text{Init}_U$ and if U transforms $\mathcal{N}$-intersections into $\mathcal{M}$-intersections, then $(-)^{\text{int}} : IDCL^{\text{ext}}(U,\mathcal{N},\mathcal{M}) \to IDCL(\mathcal{A},\mathcal{N})$ is an order-equivalence.* □

Examples

(1) The least external closure operator S^{ext} w.r.t. the forgetful functor U : **Grp** $\to$ **Set** assigns to a subset M of a group G the least subgroup $\langle M \rangle$ generated by M . (Similarly for other algebraic structures.) For U the underlying **Set**-functor of **DCPO** , one has

$$s_X^{\text{ext}}(Z) = \text{dir}_X Z$$

as defined in 3.7.

(2) We give an example of an external closure operator D w.r.t. the forgetful functor $U : \mathbf{DCPO} \to \mathbf{Set}$ which cannot be presented as the externalization of an internal closure operator of **DCPO**. For a subset Z of a dcpo X, define D by

$$d_X(Z) := \mathrm{dir}_X(\mathrm{conv}_X(Z)) \ ,$$

cf. 3.6. Then D is not of the form C^{ext} for some C since $D < (D^{\mathrm{int}})^{\mathrm{ext}}$. Indeed, for X given by

$$\begin{tikzcd} & & \mathrm{u} \arrow[dl, dash] \arrow[ddr, dash] & \\ & \mathrm{v} \arrow[dl, dash] & & \\ \mathrm{w} & & & \mathrm{z} \end{tikzcd} \tag{5.35}$$

and $Z := \{z, w\}$, one has

$$d_X(Z) = \{z, w, u\} \neq X = \mathrm{dir}_X(\mathrm{conv}_X(\mathrm{dir}_X Z)) = (d^{\mathrm{int}})^{\mathrm{ext}}_X(Z) \ .$$

Exercises

5.A (*Composition of (co-)pointed endofunctors*) A pointed endofunctor (C, γ) of a category $\mathcal{K}$ with $\gamma C = C\gamma$ is also called *wellpointed*.

(a) Show that for wellpointed endofunctors (C, γ), (D, δ) of $\mathcal{K}$ also their *composite*

$$(D, \delta)(C, \gamma) = (DC, \delta\gamma)$$

with DC the composite of functors and $\delta\gamma := \delta C \cdot \gamma = D\gamma \cdot \delta$ is wellpointed, and

$$\mathrm{Fix}(DC \ , \ \delta\gamma) = \mathrm{Fix}(D, \delta) \cap \mathrm{Fix}(C, \gamma)$$

(b) Suppose that for the pointed endofunctors (C_i, γ_i), $i \in I$, the multiple pushout

$$\begin{tikzcd} & C_iA \arrow[dr] & \\ A \arrow[ur, "\gamma_{iA}"] \arrow[rr, "\gamma_A"] & & CA \end{tikzcd} \tag{5.36}$$

exists for all $A \in \mathcal{K}$. Show that (C, γ) is a wellpointed endofunctor if every $(C_i \ , \ \gamma_i)$ is wellpointed, and that

$$\mathrm{Fix}(C \ , \ \gamma) = \bigcap_{i \in I} \mathrm{Fix}(C_i, \gamma_i) \ .$$

(c) Verify that the correspondence of Theorem 5.2 transforms composites of closure operators in the sense of 4.2 into composites of wellpointed endofunctors.

(d) Define the composite of copointed endofunctors dually to (a) and dualize statements (a) and (b). Show that the correspondence of the dual of Theorem 5.2 transforms cocomposites of closure operators in the sense of 4.3 into composites of copointed endofunctors.

5.B (*Orthogonal subcategories*) A morphism $f : X \to Y$ in a category $\mathcal{X}$ is *orthogonal* to an object $Z \in \mathcal{X}$, written as $f \perp Z$, if the map $\mathcal{X}(f,Z) : \mathcal{X}(Y,Z) \to \mathcal{X}(X,Z)$ is bijective (i.e., every $g : X \to Z$ factors uniquely as $g = h \cdot f$). For a class $\mathcal{H}$ of morphism in $\mathcal{X}$, let

$$\mathcal{H}^{\perp} = \{Z : (\forall f \in \mathcal{H})\ f \perp Z\} .$$

A full subcategory of $\mathcal{X}$ is *orthogonal* if its class objects is $\mathcal{H}^{\perp}$ for some class $\mathcal{H}$.

(a) For a full subcategory $\mathcal{A}$ of $\mathcal{X}$, prove the implications (i) $\to$ (ii) $\to$ (iii) $\to$ (iv), with

(i) $\mathcal{A}$ is reflective,

(ii) $\mathcal{A} = \mathrm{Fix}(C,\gamma)$ for a wellpointed (cf. 5.A) endofunctor (C,γ) of $\mathcal{X}$,

(iii) $\mathcal{A}$ is orthogonal,

(iv) $\mathcal{A}$ is closed under all (existing) limits in $\mathcal{X}$.

Hint for (ii) $\to$ (iii): consider $\mathcal{H} = \{\gamma_X : X \in \mathcal{X}\}$.

(b) Show that the intersection of any collection of orthogonal subcategories is orthogonal.

(c) Show that the category **CBoo** of complete Boolean algebras as a full subcategory of the category **Frm** of frames (=complete lattices with $a \wedge \bigvee b_i = \bigvee a \wedge b_i$, with morphisms preserving arbitrary joins and finite meets) is orthogonal. *Hint*: Show that complements exist in a frame B if and only if it is orthogonal to the embedding

$$\begin{tikzcd}
1 \arrow[d, no head] & & & 1 \arrow[dl, no head] \arrow[dr, no head] & \\
a \arrow[d, no head] \arrow[rr] & & a \arrow[dr, no head] & & a' \arrow[dl, no head] \\
0 & & & 0 &
\end{tikzcd} \tag{5.37}$$

5.C (*Factorization systems as functors*)

(a) Consider the class $\mathcal{M}_0 = \mathrm{Mor}\,\mathcal{X}$ as a category (as in 5.2) and show that there is a full embedding

$$E : \mathcal{X} \to \mathrm{Mor}\,\mathcal{X}$$

which sends $f : X \to Y$ in $\mathcal{X}$ to $(f, f) : 1_X \to 1_Y$ in $\mathrm{Mor}\,\mathcal{X}$.

(b) Show that every factorization system F of $\mathcal{X}$ defines a functor $\mathbf{F} : \mathrm{Mor}\,\mathcal{X} \to \mathcal{X}$ with $\mathbf{F}E \cong Id_{\mathcal{X}}$, with

$$\mathbf{F}f = \mathrm{cod}(d_f) = \mathrm{dom}(c_f)$$

for all $f \in \mathrm{Mor}\,\mathcal{X}$.

(c) Let $\mathbf{F} : \mathrm{Mor}\,\mathcal{X} \to \mathcal{X}$ be any functor with $\mathbf{F}E \cong Id_{\mathcal{X}}$. Find a functor $\mathbf{F}'$ with $\mathbf{F}'E = Id_{\mathcal{X}}$ and $\mathbf{F} \cong \mathbf{F}'$.

(d) Let $\mathbf{F} : \mathrm{Mor}\,\mathcal{X} \to \mathcal{X}$ be a functor with $\mathbf{F}E = Id_{\mathcal{X}}$, and for every $f : X \to Y$, define

$$\begin{aligned} d_f &:= \mathbf{F}(1, f) \quad (\text{with } (1, f) : 1 \to f \text{ in } \mathrm{Mor}\,\mathcal{X}), \\ c_f &:= \mathbf{F}(f, 1) \quad (\text{with } (f, 1) : f \to 1 \text{ in } \mathrm{Mor}\,\mathcal{X}). \end{aligned}$$

Show that $f = c_f \cdot d_f$ holds, and that the Diagonalization Property 5.2 holds "weakly", that is: the diagonal w of (5.11) exists but is not required to be unique.

(e) Concluded that a factorization system F is fully determined by the assignment

$$f \mapsto \mathrm{cod}(d_f) = \mathrm{dom}(c_f) .$$

5.D (*Comparing factorization systems*) Prove:

(a) If $\mathcal{X}$ has both $(\mathcal{E}, \mathcal{M})$- and $(\mathcal{E}', \mathcal{M}')$-factorizations, then $\mathcal{E} \subseteq \mathcal{E}'$ if and only if $\mathcal{M}' \subseteq \mathcal{M}$.

(b) Let F and F' be factorization systems of $\mathcal{X}$ with induced functors $\mathbf{F}$, $\mathbf{F}' : \mathrm{Mor}\,\mathcal{X} \to \mathcal{X}$ (see 5.C). Then $\mathcal{D}^F \subseteq \mathcal{D}^{F'}$ implies the existence of a natural transformation $\alpha : \mathbf{F} \to \mathbf{F}'$ with αE iso, provided F' is a left factorization system.

(c) Dualize (b).

(d) For factorization systems F , F' of $\mathcal{X}$, assume the existence of a natural transformation $\alpha : \mathbf{F} \to \mathbf{F}'$ with αE iso. Show $\mathcal{D}^F \subseteq \mathcal{D}^{F'}$ and $\mathcal{C}^{F'} \subseteq \mathcal{C}^F$.

(e) Does the converse proposition of (d) hold true?

5.E (*Characterizing orthogonal factorization systems*)

(a) Prove that the following statements for a factorization system F of $\mathcal{X}$ are equivalent:

(i) F is orthogonal;

(ii) F is a left system, and $\mathcal{D}^F$ is closed under composition ;

(iii) F is a right system, and $\mathcal{C}^F$ is closed under composition.

Compare this result with Theorems 1.8 and 2.4.

(b) (Cf. Korostenski and Tholen [1993]) Let $\mathbf{F} : \mathrm{Mor}\mathcal{X} \to \mathcal{X}$ be a functor with $\mathbf{F}E = Id_{\mathcal{X}}$ and $d_f \in \{h : c_h \text{ iso}\}$ and $c_f \in \{h : d_h \text{ iso}\}$ for every morphism f (cf. Exercise 5.C(d)). Prove that $f \mapsto (d_f, c_f)$ defines an orthogonal factorization of $\mathcal{X}$.

Hint: In order to show uniqueness of diagonals, consider any to diagonals w_1, w_2 in (5.11) and apply $\mathbf{F}$ to

$$d_f \xrightarrow{(d_f,1)} 1 \xrightarrow{(w_i,w_i)} 1 \xrightarrow{(1,c_g)} c_g \, .$$

5.F (*Factorization systems of* **Set**) Every category admits the factorization systems $f \mapsto (f, 1_Y)$ and $f \mapsto (1_X, f)$ for $f : X \to Y$. In **Set** one has, in addition, the usual (Epi, Mono)-factorization system, and the system induced by the indiscrete operator (w.r.t. the class of all monomorphisms), see Example (1) of 4.7. Prove that these are, up to isomorphisms, the only factorization systems of **Set**, and that each of them is orthogonal.

Hint: First investigate the maps $s_X : \emptyset \hookrightarrow X$ and $t_X : X \to 1$ (with 1 a singleton set) and prove for every factorization system F of **Set**: 1. If $s_1 \in \mathcal{C}^F$, then $s_X \in \mathcal{C}^F$ for all X ; 2. If $s_1 \notin \mathcal{C}^F$, then F is isomorphic to $(f \mapsto (f, 1_Y))$; 3. for $X \neq \emptyset$ one has $t_X \in \mathcal{D}^F \cup \mathcal{C}^F$; 4. if $t_X \in \mathcal{C}^F$ for $|X| > 1$, then F is isomorphic to ($f \mapsto (1_X, f)$); 5. for all f, d_f is epic or c_f is monic. Finally apply 5. to $f = t_X$ to prove that F must be isomorphic to one of the four factorization systems. (See also Lemma 4.7.)

5.G (*Rules for maximal closure operators induced by preradicals*) Show that, for preradicals $\mathbf{r}$ and $\mathbf{s}$ of R-modules, their cocomposite $(\mathbf{r} : \mathbf{s})$ as defined in Definition 5.5 is described by

$$(\mathbf{r} : \mathbf{s})(M) = p^{-1}(\mathbf{r}(M/\mathbf{s}(M)))$$

for all $M \in \mathbf{Mod}_R$, with $p : M \to M/\mathbf{s}(M)$ the projection. Verify (or revisit) the following rules (see Exercise 4.G):

(a) $\mathbf{r} = \bigvee_{i \in I} \mathbf{r}_i \Rightarrow C^{\mathbf{r}} \cong \bigvee_{i \in I} C^{\mathbf{r}_i}$; same for $\bigwedge$;

(b) $C^{\mathbf{r}} C^{\mathbf{s}} \cong C^{(\mathbf{r}:\mathbf{s})}$ and $C^{\mathbf{r}} * C^{\mathbf{s}} \cong C^{\mathbf{rs}}$;

(c) $(C^{\mathbf{r}})_\alpha \cong C^{(\mathbf{r}^\alpha)}$ and $(C^{\mathbf{r}})^\alpha \cong C^{(\mathbf{r}_\alpha)}$ for all $\alpha \in Ord \cup \{\infty\}$.

(d) Show that the second isomorphism of (c) may be verified in the general context of 5.5, provided inverse images commute with directed joins.

5.H (*Scarcity of preradicals in* $\mathbf{Set}_*$) Prove that in the category $\mathbf{Set}_*$ of pointed sets (see Example(3) of 5.6), there are only two non-isomorphic $\mathcal{M}$-preradicals and only two non-isomorphic $\mathcal{E}$-prereflections.

5.I (*Hereditariness of minimal closure operators*) Recall that a lattice is *modular* if the modular law

$$a \geq c \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$$

holds. Assuming that $\mathcal{M}/X$ is modular for all $X \in \mathcal{X}$, prove that $C_{\mathbf{r}}$ is hereditary if and only if $\mathbf{r}$ is hereditary, for every $\mathbf{r} \in PRAD(\mathcal{X}, \mathcal{M})$.

5.J (*Wielandt's radical,* cf. Robinson [1976]) A subgroup S of a group G is called *subnormal* if there exists a finite chain

$$S = N_1 \leq N_2 \leq \cdots \leq N_n = G$$

such that each N_k is normal in N_{k+1} $k = 1, 2, \ldots, n-1$. Wielandt's subgroup is defined by

$$\omega(G) := \bigcap \{N_G(S) : S \leq G \text{ subnormal}\} ,$$

with $N_G(S)$ denoting the normalizer of S in G. Show that ω is a radical of the category **Grp** (and its monomorphisms). Investigate the induced minimal and maximal closure operators and their properties.

5.K (*Idempotent $\mathcal{M}$-preradicals and $\mathcal{M}$-coreflective subcategories*) Show that the coreflector of a full $\mathcal{M}$-coreflective subcategory defines an idempotent $\mathcal{M}$-preradical of the category $\mathcal{X}$ (with $\mathcal{M}$ a class of monomorphisms of $\mathcal{X}$). Prove that this way one obtains a categorical equivalence of $PRAD(\mathcal{X}, \mathcal{M})$ and the partially ordered conglomerate of all full and replete $\mathcal{M}$-coreflective subcategories of $\mathcal{X}$.

5.L (*Comparing initial topologies and initial closure operators*) Let the functor $F : \mathcal{X} \to \mathcal{Y}$ be given by a **Set**-map $\varphi : A \to B$ as in Example 5.7.

(a) For a topology on B, consider its closure operator D of $\mathcal{Y}$ (as defined in 5.7) and the closure operator C of $\mathcal{X}$ given by the initial (=weak) topology on A w.r.t. φ. Show that, in general, one has $C < D_{(F)}$.

(b) Establish an analogous result regarding final structures.

5.M (*Uniqueness of U-initial liftings, isomorphisms*) Show for $U : \mathcal{X} \to \mathcal{S}$ faithful:

(a) An $\mathcal{X}$-morphism f is an isomorphism if and only if f is U-initial and Uf is an isomorphism in $\mathcal{S}$.

(b) Show that U-initial liftings of a morphism $S \to UY$ in $\mathcal{S}$ with $Y \in \mathcal{X}$ are "unique up to isomorphisms in the U-fibre of S"; they are uniquely determined if U is *amnestic*, i.e. any isomorphism i in $\mathcal{X}$ with Ui an identity morphism in $\mathcal{S}$ must an identity morphism in $\mathcal{X}$.

5.N (*Products in* **CS**, *non-preservation of products by* **C**)

(a) Prove that direct products in the category **CS** are formed as follows: for closure spaces (S_i, k_i), provide the set $S = \prod_{i \in I} S_i$ with a **CS**-structure by putting

$$k_S(M) = \prod_{i \in I} k_i(p_i(M))$$

for all $M \subseteq S$ (p_i is the i-th projection).

(b) Show that **PrTop** is a full coreflective subcategory of **CS**, but that direct products are not preserved by the embedding. *Hint*: For the coreflector, find the "additive core" of a **CS**-structure.

(c) For a faithful monofibration $U : \mathcal{X} \to \mathbf{Set}$ and a grounded, productive closure operator C on $\mathcal{X}$, $\mathbf{C} : \mathcal{X} \to \mathbf{CS}$ may not preserve products.

5.O (*The non-hereditariness syndrom*) Use Example(1) of 4.4 to illustrate that the functor $\boldsymbol{\theta} : \mathbf{Top} \to \mathbf{CS}$ induced by the θ-closure of **Top** does not preserve subspace embeddings. *Hint:* $\boldsymbol{\theta}(F \cup \{0\}) \to \boldsymbol{\theta}(X)$ is not an embedding in **CS**.

5.P (*Topological functors*) Let $U : \mathcal{X} \to \mathcal{S}$ be a faithful functor. A *source* (= family of morphisms with common domain) $\sigma = (f_i : X \to Y_i)_{i \in I}$ in $\mathcal{X}$ is called *U-initial* if, for every $Z \in \mathcal{X}$, an $\mathcal{S}$-morphisms $\psi : UZ \to UX$ is an $\mathcal{X}$-morphism whenever each $Uf_i \cdot \psi : UZ \to UY$ is an $\mathcal{X}$-morphism. σ is a *(U-)lifting* of a source $\tau = (\varphi_i : S \to UY_i)_{i \in I}$ in $\mathcal{X}$ (more precisely: of a *U-source* $(Y_i\ ,\ \varphi_i \cdot S \to UY_i)_{i \in I}$) if $Uf_i = \varphi_i$ for all $i \in I$. The indexing system is allowed to be a proper class, or it may be empty, in which case σ and τ have to be identified with the objects X and S, respectively. In case I is a singleton set, (U-initial) sources are just (U-initial) morphisms. Trading U for $U^{op} : \mathcal{X}^{op} \to \mathcal{S}^{op}$ one obtains the notion dual to (U-initial) source, namely (*U-final*) *sink*. U is called a *cofibration* if U^{op} is a fibration (see 5.8).

(a) Prove that the following statements are logically equivalent:

(i) every U-source in $\mathcal{S}$ has a U-initial lifting in $\mathcal{X}$,

(ii) every U-sink in $\mathcal{S}$ has a U-final lifting in $\mathcal{X}$,

(iii) U is both a fibration and a cofibration, and for every object S in $\mathcal{S}$, the fibre $U^{-1}S$ has the structure of a large-complete lattice (i.e., when preordered by $X \leq X'$ iff 1_S can be lifted to an $\mathcal{X}$-morphism $X \to X'$, then class-indexed infima and suprema exist in $U^{-1}S$).

U is called *topological* (also: $\mathcal{X}$ is a *topological category* over $\mathcal{S}$), if the equivalent conditions i.-iii. hold. Note that U is topological if and only if U^{op} is topological.

(b) Show that the U-initial lifting of a limit cone in $\mathcal{S}$ is a limit cone in $\mathcal{X}$. For U topological, conclude that if $\mathcal{S}$ has all (co)limits of a fixed diagram type, then also $\mathcal{X}$ has all (co)limits of that type, and U preserves them.

(c) Prove that a topological functor has both a left and a right adjoint (given by "discrete" and "indiscrete structures").

(d) Prove that the following categories are topological over **Set** : **Top**, **PrTop**, **FC**, **SGph**, **PrSet**, **CS**, **Unif** (cf. Exercises 3.A, 3.C).

(e) Prove that the forgetful functor **TopGrp** $\to$ **Grp** is topological.

5.Q (*Idempotent hull of* σ) Prove that the idempotent hull of the sequential closure operator in **Top** may be obtained as described in Example 5.10 (2). (See also Exercise 4.F (c).)

5.R (*Multiplicative structure for closure operators of* **Top**) According to Theorem 5.10, idempotent, grounded and additive closure operators of **Top** correspond bijectively to endofunctors of **Top** that commute with the underlying **Set**-functor. For such closure operators define a product $C\Box D$ that corresponds to the composition of endofunctors. Show that this product is associative, and that the Kuratowski closure operator is neutral w.r.t. $\Box$. Is $C\Box D$ hereditary if C and D are?

5.S (*Lifting of* K^* *to* **TopGrp**) According to Theorem 5.9, the closure operator K^* of **Top** of Example 4.2(3) can be lifted to **TopGrp**.

(a) Show that K and K^* induce the same preradical $\pi(K)$ of **TopGrp**, and that K^* is the minimal closure operator belonging to $\pi(K)$.

(b) With ν the normal closure in **TopGrp**, show

$$C_{\pi(K)} = K^* < \nu \vee K^* = \nu K^* = K^*\nu < C^{\pi(K)} .$$

5.T (*More on* ν *in* **TopGrp**) Prove that there is no finitely productive closure operator C of **Top** such that $C_V = \nu_U$ (terminology of 5.9). *Hint:* Assume that C exists and consider a subgroup H of S_ω as in Example 5.9 (4). Then $C \leq Q$ by Theorem 4.7, and H is both ν-dense and ν-closed in S_ω .

5.U (*The largest proper closure operator of* **Unif**) Prove Proposition 5.11.

5.V (*Modified modifications*) Let (T,η) and (S,ε) be pointed and copointed endofunctors of $\mathcal{X}$, respectively. For a closure operator $C \in CL(\mathcal{X},\mathcal{M})$ as in 2.1, show that the formulas

$$({}_T\tilde{c})_X(m) = \eta_X^{-1}(c_{TX}(Tm)) \qquad \text{and} \qquad {}^S\tilde{c}_X(m) = m \vee \varepsilon_X(c_{SX}(Sm))$$

define closure operators ${}_T\tilde{C}$ and ${}^S\tilde{C}$, respectively (cf. Remark 5.7). Furthermore, prove:

(1) $C \leq {}_TC \leq {}_T\tilde{C}$ and ${}^S\tilde{C} \leq {}^SC \leq C$.

(2) $\quad {}_TC \cong {}_T\tilde{C}$ if $\eta_X \in \mathcal{E}$ for all $X \in \mathcal{X}$.

5.W (*Recap of adjoint functors*) Prove Proposition 5.13.

5.X (*Initial closure operator w.r.t.* $U : \mathbf{TopGrp} \to \mathbf{Top}$) . Prove that the following statements are equivalent for every closure operator C of **Top** w.r.t. the class of embeddings:

(i) U is $({}^{\varepsilon}C, C)$-continuous (cf. Example 5.13(2)),

(ii) U is $({}^{\varepsilon}C, C)$-preserving,

(iii) for every subgroup N of a topological group A , $c_{UA}(UN)$ is a subgroup of A .

Conclude that for C finitely productive, ${}^{\varepsilon}C$ is the initial closure operator $C_{(U)}$.

5.Y (*Idempotent hull and additive core of modified closure operators*) Let $\mathcal{A}$ be an $\mathcal{E}$-reflective subcategory of $\mathcal{X}$ with reflector R. Prove the formulas

$$({}_RC)^{\infty} = {}_R(C^{\infty}) \text{ and } ({}_RC)^{+} = {}_R(C^{+})$$

for any closure operator C of $\mathcal{X}$ in case $\mathcal{X} = \mathbf{Top}$ and $\mathcal{X} = \mathbf{Mod}_R$. Find conditions on the subobject lattices $\mathcal{M}/X$ under which these formulas remain valid for arbitrary $\mathcal{X}$. *Hint:* Consult Exercise 3.M and Section 4.8.

5.Z (*Approach spaces;* cf. Lowen [1993]) An approach space is a set X provided with a family $\{t_{\varepsilon,X}\}_{\varepsilon\in[0,+\infty)}$ of extensive maps $t_{\varepsilon,X} : 2^X \to 2^X$ (where 2^X is the power set of X), such that:

1. $t_{\varepsilon,X}(\emptyset) = \emptyset$;
2. $t_{\varepsilon,X}(A \cup B) = t_{\varepsilon,X}(A) \cup t_{\varepsilon,X}(B)$ for all $A, B \subseteq X$;
3. $t_{\varepsilon,X}(t_{\gamma,X}(A)) \leq t_{\varepsilon+\gamma,X}(A)$ for every $\gamma \in [0, +\infty)$; and
4. $t_{\varepsilon,X}(A) = \bigcap\{t_{\gamma,X}(A) : \gamma > \varepsilon\}$ for every $A \subseteq X$.

A morphism between two approach spaces $(X, \{t_{\varepsilon,X}\})$ and $(Y, \{t_{\varepsilon,Y}\})$ is a set-map $f : X \to Y$ such that $f(t_{\varepsilon,X}(A)) \subseteq t_{\varepsilon,Y}(f(A))$ for each subset A of X and for each $\varepsilon \in [0, +\infty)$. Denote by **AS** the category of approach spaces and by $\mathcal{M}$ the class of all embeddings in **AS**. Show:

(a) **AS** is an $\mathcal{M}$-complete category;

(b) taking for each $\varepsilon \in [0, +\infty)$ $T_\varepsilon = (t_{\varepsilon,X})_{X\in\mathbf{AS}}$ one obtains a grounded and additive closure operator of **AS**;

(c) $T_\varepsilon T_\gamma \leq T_{\varepsilon+\gamma}$; in particular $T_\varepsilon \leq T_\gamma$ when $\varepsilon \leq \gamma$;

(d) the forgetful functor $U : \mathbf{AS} \to \mathbf{Set}$ is $\{T_\varepsilon\}_{\varepsilon\in[0,+\infty)}$-structured;

(e) there is a functor O : **Met** $\to$ **AS** such that with the forgetful functor V : **Met** $\to$ **Set** one has $V = U \circ O$. *Hint*: For a metric space (X, d) define $O(X, d)$ to be the approach space $(X, \{t_{\varepsilon,X}\})$ with $t_{\varepsilon,X}(A) := \{x \in X : \mathrm{dist}(x, A) \leq \varepsilon\}$ for each $\varepsilon \in [0, +\infty)$.

Notes

A comprehensive study of pointed endofunctors and their applications was given by Kelly [1980]. Prereflections were introduced by Börger [1981] and studied by Tholen [1987] and by Rosický and Tholen [1988]. The functorial definition of closure operator appears already in the paper by Dikranjan and Giuli [1987a], with the case of no mono-assumption and the connection with generalized factorization systems being presented by Dikranjan, Giuli and Tholen [1989]. The functorial views of factorization systems can be traced back (at least) to Linton [1969] and culminates in their presentation as Eilenberg-Moore algebras (see Korostenski and Tholen [1993]). While maximal closure operators defined by preradicals of modules have been present at least implicitly in the literature on torsion theories for some time, they appeared in the formal setting of closure operators not before Dikranjan and Giuli [1987a], and their study in the context of arbitrary categories is certainly new. Likewise, the concept of continuity of functors between categories equipped with closure operators, and the notion of mixed continuity of morphisms with respect to two given closure operators were developed in the course of writing this book, in order to interpret abstractly the various constructions for transporting closure operators along functors. Of these, only the special, but fundamental case of a modification along a reflexion appears in the literature (see the Notes of Chapter 6). External closure operators were studied by Castellini [1986a]. The characterization of density classes with respect to a closure operator was given and communicated to the authors by Tonolo in 1992 and is due to appear in Tonolo [1995].

6 Regular Closure Operators

Regular closure operators provide the key instrument for attacking the epimorphism problem in a subcategory $\mathcal{A}$ of the given (and, in general, better behaved) category $\mathcal{X}$. Depending on $\mathcal{A}$, one defines the $\mathcal{A}$-regular closure operator of $\mathcal{X}$ in such a way that its dense morphisms in $\mathcal{A}$ are exactly the epimorphisms of $\mathcal{A}$. Now everything depends on being able to "compute" the $\mathcal{A}$-regular closure effectively. The strong modification of a closure operator as introduced in 6.6 turns out to play a major role in this, as we shall see in the following two chapters. It arises quite naturally after we have provided two powerful criteria for closedness with respect to the $\mathcal{A}$-regular closure (6.4, 6.5). At least for additive categories this leads to a complete characterization of regular closure operators as maximal closure operators. The rest of the chapter is devoted to the (quite particular) case of weakly hereditary closure operators which, roughly, characterize torsionfree classes in algebraic contexts and give a general notion of disconnectedness in topological contexts.

6.1 $\mathcal{A}$-epimorphisms and $\mathcal{A}$-regular monomorphisms

One of the the main goals of this book is to develop techniques that allow for easy characterizations of the class of $\mathcal{A}$-epimorphisms in a category $\mathcal{X}$, with $\mathcal{A}$ a full and replete subcategory of $\mathcal{X}$. First we give the most relevant definitions.

(1) A morphism $f : X \to Y$ of $\mathcal{X}$ is an $\mathcal{A}$-*epimorphism* if for all $u, v : Y \to A$ with $A \in \mathcal{A}$, one has the implication $(u \cdot f = v \cdot f \Rightarrow u = v)$. The class of all $\mathcal{A}$-epimorphisms in $\mathcal{X}$ is denoted by $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$.

(2) f is an $\mathcal{A}$-*regular monomorphism* if every morphism $g : Z \to Y$ with $f \overset{\mathcal{A}}{\Rightarrow} g$ factors through f by a unique morphism $h : Z \to X$; here $f \overset{\mathcal{A}}{\Rightarrow} g$ means that for all $u, v : Y \to A$ with $A \in \mathcal{A}$, one has the implication $(u \cdot f = v \cdot f \Rightarrow u \cdot g = v \cdot g)$. The class of all $\mathcal{A}$-regular monomorphism in $\mathcal{X}$ is denoted by $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \\ Z \arrow[u, "h"] \arrow[ur, "g"] & & \end{tikzcd} \tag{6.1}$$

REMARKS

(1) In case $\mathcal{A} = \mathcal{X}$, an $\mathcal{A}$-epimorphism ($\mathcal{A}$-regular monomorphism) of $\mathcal{X}$ is simply an *epimorphism* (*regular monomorphism*, respectively) of $\mathcal{X}$. We write

$\mathrm{Epi}(\mathcal{X}) = \mathrm{Epi}_{\mathcal{X}}(\mathcal{X})$, $\mathrm{Reg}(\mathcal{X}) = \mathrm{Reg}_{\mathcal{X}}(\mathcal{X})$. For full subcategories $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{X}$, one has the inclusions

$$\mathrm{Epi}(\mathcal{X}) \subseteq \mathrm{Epi}_{\mathcal{X}}(\mathcal{B}) \subseteq \mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \text{ and } \mathrm{Reg}_{\mathcal{X}}(\mathcal{A}) \subseteq \mathrm{Reg}_{\mathcal{X}}(\mathcal{B}) \subseteq \mathrm{Reg}(\mathcal{X}).$$

In particular, every epimorphism of $\mathcal{X}$ is $\mathcal{A}$-epic, and every $\mathcal{A}$-regular monomorphism is a regular monomorphism of $\mathcal{X}$.

(2) An $\mathcal{A}$-epimorphism is orthogonal to every $\mathcal{A}$-regular monomorphism (cf. Theorem 1.8). In particular, every regular monomorphism of $\mathcal{X}$ is a strong monomorphism of $\mathcal{X}$ (see Exercise 1.E), but not conversely (see Exercise 6.C).

(3) A morphism f is an $\mathcal{A}$-epimorphism of $\mathcal{X}$ iff $f \overset{\mathcal{A}}{\Rightarrow} 1$. Consequently, f is both an $\mathcal{A}$-epimorphism and an $\mathcal{A}$-regular monomorphism iff f is an isomorphism of $\mathcal{X}$.

PROPOSITION *For any diagram type $\mathcal{D}$, the class $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ is closed under $\mathcal{D}$-limits in $\mathcal{X}$, and the class $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$ is closed under $\mathcal{D}$-colimits in $\mathcal{X}$.*

Proof Let $\mu : H \to K$ be a natural transformation with $H, K : \mathcal{D} \to \mathcal{X}$ and $\mu_d \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ for all $d \in \mathcal{D}$, such that $m : \varprojlim H \to \varprojlim K$ exists in $\mathcal{X}$. Indeed, for every $g : Z \to \varprojlim K$ with $m \overset{\mathcal{A}}{\Rightarrow} g$ one has $\mu_d \overset{\mathcal{A}}{\Rightarrow} \kappa_d \cdot g$ for all d (with κ_d the limit projection) and therefore a unique factorization $\alpha_d : Z \to H_d$ which must be natural in d . This then gives the unique morphism $a : Z \to \varprojlim H$ with $m \cdot a = g$. Hence $m \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ follows.

Closedness of $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$ under colimits is even easier to establish. □

The following lemma is useful when testing $\mathcal{A}$-regularity:

LEMMA *Let the morphism $m : M \to X$ of $\mathcal{X}$ have a cokernelpair in $\mathcal{X}$ (so that the pushout diagram*

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "m"'] & X \arrow[d, "j"] \\ X \arrow[r, "i"] & K \end{tikzcd} \tag{6.2}$$

exists in $\mathcal{X}$), and let $K = X +_M X$ admit an $\mathcal{A}$-reflexion (so that there is a morphism $\rho_K : K \to RK$ with $RK \in \mathcal{A}$ and the usual universal property; see 2.8). Then the following statements are equivalent:

(i) *m is an $\mathcal{A}$-regular monomorphism,*

(ii) *m is the equalizer of $\rho_K \cdot i$ and $\rho_K \cdot j$,*

(iii) m *is the equalizer of some pair* $u, v : X \to A$ *with* $A \in \mathcal{A}$ *.*

Proof (i) $\Rightarrow$ (ii) One uses the universal property of K and of RK to show that any $g : Z \to X$ with $\rho_K \cdot i \cdot g = \rho_K \cdot j \cdot g$ must satisfy $m \overset{\mathcal{A}}{\Rightarrow} g$ and therefore factors through m by hypothesis. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) is trivial. □

We call $m : M \to X$ in $\mathcal{X}$ an *$\mathcal{A}$-section* if $X \in \mathcal{A}$ and there is a morphism $t : X \to M$ with $t \cdot m = 1_M$. Then m is the equalizer of $m \cdot t$ and 1_X and therefore an $\mathcal{A}$-regular monomorphism. (Note that the implication (iii) $\Rightarrow$ (i) of the Lemma does not require any of its hypotheses.)

THEOREM *Let* $\mathcal{X}$ *have* $(\mathcal{E}, \mathcal{M})$*-factorizations* (*see* 1.8). *Then the statements*

(i) $\mathcal{E} \subseteq \mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$,

(ii) $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A}) \subseteq \mathcal{M}$

are equivalent if $\mathcal{X}$ *has equalizers of pairs of morphisms with codomain in* $\mathcal{A}$ *, or if direct products of type* $A \times A$ *with* $A \in \mathcal{A}$ *exist in* $\mathcal{X}$ *. In the latter case,* (i) *and* (ii) *are equivalent to* (iii), *and also to* (iv) *and* (v) *if the products* $A \times A$ *belong to* $\mathcal{A}$ *:*

(iii) *the "diagonal"* $\delta_A : A \to A \times A$ *belongs to* $\mathcal{M}$ *for every* $A \in \mathcal{A}$ *,*

(iv) *every* $\mathcal{A}$*-section belongs to* $\mathcal{M}$ *,*

(v) *for all morphisms* $f : X \to A$ *and* $g : A \to Y$ *with* $A \in \mathcal{A}$ *,* $g \cdot f \in \mathcal{M}$ *implies* $f \in \mathcal{M}$ *.*

Proof The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) hold without the assumptions on limits in $\mathcal{X}$. For (i) $\Rightarrow$ (ii) one forms the $(\mathcal{E}, \mathcal{M})$-factorization $m \cdot e = f$ of an $\mathcal{A}$-regular monomorphism f and has $f \overset{\mathcal{A}}{\Rightarrow} m$ since e is $\mathcal{A}$-epic. Then m factors as $m = f \cdot t$, and it is easy to see that t is inverse to e (even without assuming $\mathcal{M}$ to be a class of monomorphisms, just using the orthogonality relation $e \perp m$). Hence $f \cong m \in \mathcal{M}$. The implication (ii) $\Rightarrow$ (iv) is trivial. For (iv) $\Rightarrow$ (v) one $(\mathcal{E}, \mathcal{M})$-factors f as $f = m \cdot e$ and uses the $(\mathcal{E}, \mathcal{M})$-diagonalization property to obtain from $e \perp g \cdot f$ a morphism t with $t \cdot e = 1$. Since e is $\mathcal{A}$-epic, from $m \cdot e = f \cdot t \cdot e$ one concludes $m = f \cdot t = m \cdot e \cdot t$. Another application of the $(\mathcal{E}, \mathcal{M})$-diagonalization property gives $e \cdot t = 1$ since $e \perp m$. Consequently, e is iso and therefore $f \in \mathcal{M}$. For (v) $\Rightarrow$ (iv) one uses the fact that $\mathcal{M}$ contains all isomorphisms, so that $t \cdot m = 1 \in \mathcal{M}$ (with m the given $\mathcal{A}$-section) implies $m \in \mathcal{M}$.

If $\mathcal{X}$ has equalizers as stated, then one shows (ii) $\Rightarrow$ (i) by factoring every $e : X \to Y$ in $\mathcal{E}$ through the equalizer m of any given pair of morphisms $u, v : Y \to A$ with $A \in \mathcal{A}$ and $u \cdot e = v \cdot e$, as $e = m \cdot s$. Now $e \perp m$ forces m iso, hence $u = v$.

If $\mathcal{X}$ has products as stated, then (ii) $\Rightarrow$ (iii) holds since δ_A is the equalizer of the projections $p_1, p_2 : A \times A \to A$. For (iii) $\Rightarrow$ (i), one again considers $e : X \to Y$ in $\mathcal{E}$ and $u, v : Y \to A$ with $A \in \mathcal{A}$ and $u \cdot e = v \cdot e$. Then $< u, v > \cdot e =$

$\delta_A \cdot u \cdot e$, so that $e \perp \delta_A$ yields a morphism $d : Y \to A$ with $\delta_A \cdot d = < u, v >$ and therefore $u = p_1 \cdot \delta_A \cdot d = p_2 \cdot \delta_A \cdot d = v$.

Finally, if the products $A \times A$ belong to $\mathcal{A}$ for every $A \in \mathcal{A}$, then δ_A is an $\mathcal{A}$-section and the implication (iv) $\Rightarrow$ (iii) follows trivially. □

COROLLARY *Let $\mathcal{X}$ have equalizers (or finite products) and $(\mathcal{E}, \mathcal{M})$-factorizations with $\mathcal{M} = \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$. Then $\mathcal{E} = \mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$.*

Proof If one $(\mathcal{E}, \mathcal{M})$-factors an $\mathcal{A}$-epimorphism, then the $\mathcal{M}$-part is $\mathcal{A}$-epic as well, hence an isomorphism when the $\mathcal{M}$-part is $\mathcal{A}$-regular. Hence $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \subseteq \mathcal{E}$, and the reverse inclusion follows from the Theorem. □

EXAMPLES (Only for $\mathcal{A} = \mathcal{X}$, others follow in subsequent sections.)

(1) In the category $\mathbf{Mod}_R$ of R-modules, every submodule is a kernel. Hence $\mathrm{Reg}(\mathbf{Mod}_R)$ is exactly the class of monomorphisms and $\mathrm{Epi}(\mathbf{Mod}_R)$ is the class of surjective homomorphisms.

(2) Schreier's [1927] Amalgamation Theorem shows that when forming the free product $G *_U G$ with amalgamated subgroup U , then U is the equalizer of the two injections $i, j : G \to G *_U G$ (as in the Lemma). Hence $\mathrm{Reg}(\mathbf{Grp})$ is again the class of all monomorphisms. An analogous result holds when $\mathbf{Grp}$ is traded for the category $\mathbf{CompGrp}$ of all compact (Hausdorff) groups : see Poguntke [1973]. As a consequence, epimorphisms are surjective in both categories. For a direct proof of the latter result in the case of groups, see Exercise 6.B.

(3) Let us consider the class $\mathcal{M}$ of all (subspace-) embeddings in each of the categories $\mathbf{Top}, \mathbf{Top}_0, \mathbf{Top}_1,$ and $\mathbf{Haus}$: in $\mathbf{Top}$ and $\mathbf{Top}_1$ it coincides with the class of all regular monomorphisms, but in $\mathbf{Top}_0$ and $\mathbf{Haus}$ it does not. For the positive result in $\mathbf{Top}_1$ one simply shows that the amalgamated sum $X +_M X$ formed in $\mathbf{Top}$ belongs to $\mathbf{Top}_1$ if X does. For the negative result in $\mathbf{Top}_0$, Baron [1968] gave an example of a proper epimorphic embedding in the category $\mathbf{Top}_0$. These categories will be considered in greater detail in 6.5 below.

6.2 $\mathcal{A}$-epi closure and $\mathcal{A}$-regular closure

We return to the standard setting of 2.1 and consider an $\mathcal{M}$-complete category $\mathcal{X}$ such that $\mathcal{M}$ is closed under composition. We say that *$\mathcal{M}$-unions are epic* if for $m \cong \bigvee_{i \in I} m_i$ in $\mathcal{M}/X$, the sink $(j_i : M_i \to M)_{i \in I}$ with $m \cdot j_i = m_i$ for all $i \in I$ is epic, so that $u \cdot j_i = v \cdot j_i$ for all $i \in I$ (with $u, v : M \to Y$ in $\mathcal{X}$) implies $u = v$. Similarly to (ii) $\Rightarrow$ (i) of Theorem 6.1 one proves:

LEMMA *If $\mathcal{X}$ has equalizers and if $\mathcal{M}$ contains all regular monomorphisms, then $\mathcal{M}$-unions are epic in $\mathcal{X}$.*

Proof If $u \cdot j_i = v \cdot j_i$ holds for all $i \in I$, then every j_i factors through the equalizer k of u, v, hence $m_i \leq k \cdot m \leq m$ in $\mathcal{M}/X$ for all $i \in I$. This forces k to be an isomorphism. □

THEOREM *Assume* $\mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M}$. *Then for every full subcategory* $\mathcal{A}$ *of* $\mathcal{X}$,

(1) *there is a uniquely determined idempotent closure operator* $\mathrm{reg}^{\mathcal{A}}$ *of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$ *such that* $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ *is the class of* $\mathrm{reg}^{\mathcal{A}}$*-closed* $\mathcal{M}$*-subobjects; furthermore,* $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$ *is the class of* $\mathrm{reg}^{\mathcal{A}}$*-dense morphisms, provided* $\mathcal{X}$ *has equalizers. Under this provision*

(2) *there is a uniquely determined weakly hereditary closure operator* $\mathrm{epi}^{\mathcal{A}}$ *of* $\mathcal{X}$ *w.r.t.* $\mathcal{M}$ *such that* $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$ *is the class of* $\mathrm{epi}^{\mathcal{A}}$*-dense morphisms. The* $\mathrm{epi}^{\mathcal{A}}$*-closed* $\mathcal{M}$*-subobjects are exactly the* $\mathcal{A}$-extremal monomorphisms *in* $\mathcal{M}$, *i. e., the morphisms* $m \in \mathcal{M}$ *such that* $m = k \cdot d$ *with* $k \in \mathcal{M}$ *and an* $\mathcal{A}$*-epimorphism* d *is possible only for* d *an isomorphism.*

(3) $\mathrm{epi}^{\mathcal{A}}$ *is idempotent and weakly hereditary, in fact the weakly hereditary core of* $\mathrm{reg}^{\mathcal{A}}$.

Proof (1) Existence and uniqueness of $\mathrm{reg}^{\mathcal{A}}$ follow from Propositions 5.4 and 6.1, in conjunction with Theorem 1.7. Explicitly, for $m \in \mathcal{M}/X$, one has

$$\mathrm{reg}_X^{\mathcal{A}}(m) = \bigwedge\{k \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})/X : k \geq m\}. \qquad (*)$$

If $u \cdot m = v \cdot m$ holds for $u, v : X \to A$ with $A \in \mathcal{A}$, then m factors through the equalizer k of u and v (if it exists); hence $\mathrm{reg}_X^{\mathcal{A}}(m)$ factors through k, and $u \cdot \mathrm{reg}_X^{\mathcal{A}}(m) = v \cdot \mathrm{reg}_X^{\mathcal{A}}(m)$ follows. Consequently, if m is $\mathrm{reg}^{\mathcal{A}}$-dense (so that $\mathrm{reg}_X^{\mathcal{A}}(m)$ is iso), then m is $\mathcal{A}$-epic. Conversely, if m is $\mathcal{A}$-epic, also $\mathrm{reg}_X^{\mathcal{A}}(m)$ is $\mathcal{A}$-epic and therefore an isomorphism.

(2) We apply Theorem* of 5.4 to the class $\mathcal{D} = \mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{M}$. Indeed, $\mathcal{D}$ is right cancellable w.r.t. $\mathcal{M}$ and satisfies also conditions (b) and (c) of that Theorem since, according to the Lemma, $\mathcal{M}$-unions are epic in $\mathcal{X}$. Explicitly, in order to check (c) stability of $\mathcal{D}$ under $\mathcal{M}$-unions, consider $n \leq m_i$ in $\mathcal{M}/X$ with $i \in I \neq \emptyset$ such that each j_i with $m_i.j_i = n$ belongs to $\mathcal{D}$, and for j with $m.j = n$ and $m \cong \bigvee_{i \in I} n_i$ assume $u.j = v.j$ for u, v with codomain in $\mathcal{A}$. Since each j_i is $\mathcal{A}$-epic one has $u.k_i = v.k_i$, for k_i such that $m.k_i = m_i$. Since $(k_i)_{i \in I}$ is epic, $u = v$ follows. When checking that (b) $\mathcal{D}$ has the $\wedge$-$\vee$-preservation property one relies on the same arguments.

Hence Theorem* of 5.4 gives a uniquely determined weakly hereditary closure operator $\mathrm{epi}^{\mathcal{A}}$ such that $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathcal{M}$ is the class of $\mathrm{epi}^{\mathcal{A}}$-dense $\mathcal{M}$-subobjects. But the latter condition is easily seen to be equivalent to saying that $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$ is the class of $\mathrm{epi}^{\mathcal{A}}$-dense morphisms (see Corollary* of 2.3). Explicitly, for $m \in \mathcal{M}/X$ one has

$$\mathrm{epi}_X^{\mathcal{A}}(m) = \bigvee\{k \in \mathcal{M}/X : k \cdot d = m \text{ for some } d \in \mathrm{Epi}_{\mathcal{X}}(\mathcal{A})\}. \qquad (**)$$

If $\mathrm{epi}_X^{\mathcal{A}}(m)$ is factored as $\mathrm{epi}_X^{\mathcal{A}}(m) = n \cdot e$ with $n \in \mathcal{M}$ and $e \in \mathcal{E}$, then $m = n \cdot e \cdot j_m$ with $e \cdot j_m \in \mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$. Hence (**) gives that e must be an isomorphism. Consequently, every $\mathrm{epi}^{\mathcal{A}}$-closed $\mathcal{M}$-subobject is $\mathcal{A}$-extremal. The converse holds trivially since $j_m \in \mathrm{Epi}_{\mathcal{X}}(\mathcal{A})$.

(3) follows from (1) and (2) with Theorem* 5.4 and Corollary 5.4. □

DEFINITION Under the hypotheses of the Theorem, we call $\mathrm{reg}^{\mathcal{A}}$ and $\mathrm{epi}^{\mathcal{A}}$ the *$\mathcal{A}$-regular* and the *$\mathcal{A}$-epi closure operator of* $\mathcal{X}$, respectively. Instead of $\mathrm{reg}^{\mathcal{A}}$-closed (= $\mathcal{A}$-regular) $\mathcal{M}$-subobjects we often speak of *$\mathcal{A}$-closed* $\mathcal{M}$-subobjects, and $\mathrm{reg}^{\mathcal{A}}$-dense (= $\mathcal{A}$-epic) morphisms are often called *$\mathcal{A}$-dense*. Any closure operator C of $\mathcal{X}$ is called *regular* if it is the $\mathcal{A}$-regular closure operator for some full subcategory $\mathcal{A}$ of $\mathcal{X}$.

COROLLARY *Under the hypotheses of the Theorem, the following statements are equivalent:*

(i) $\mathrm{reg}^{\mathcal{A}} = \mathrm{epi}^{\mathcal{A}}$,

(ii) $\mathrm{reg}^{\mathcal{A}}$ *is weakly hereditary,*

(iii) $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ *is closed under composition,*

(iv) *every $\mathcal{A}$-extremal monomorphism in $\mathcal{M}$ is $\mathcal{A}$-closed.*

Proof From the Theorem, in conjunction with Theorem 2.4. □

REMARKS We continue to operate under the hypotheses of the Theorem.

(1) One has $m \le \mathrm{epi}_X^{\mathcal{A}}(m) \le \mathrm{reg}_X^{\mathcal{A}}(m)$ for all $m \in \mathcal{M}/\mathcal{X}$.

(2) If $m : M \to X$ in $\mathcal{M}$ has a cokernelpair $i, j : X \to K$ in $\mathcal{X}$ and if K has an $\mathcal{A}$-reflexion ρ_K (see Lemma 6.1), then the $\mathcal{A}$-regular closure of m can be computed as

$$\mathrm{reg}_X^{\mathcal{A}}(m) \cong \mathrm{equalizer}(\rho_K \cdot i, \rho_K \cdot j).$$

Indeed, this equalizer is $\mathcal{A}$-regular and factors through every $k \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})/X$ with $k \ge m$.

(3) We showed in the proof of the Theorem that $m \overset{\mathcal{A}}{\Rightarrow} \mathrm{reg}_X^{\mathcal{A}}(m)$ holds if $\mathcal{X}$ has equalizers. Conversely, for any $k \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})/X$ with $k \ge m$ and $m \overset{\mathcal{A}}{\Rightarrow} k$, one necessarily has $k \cong \mathrm{reg}_X^{\mathcal{A}}(m)$.

(4) $\mathcal{A}$ -extremal morphisms m in $\mathcal{M}$ are characterized by the property that $m = f \cdot d$ with an $\mathcal{A}$-epimorphism d is possible only if there is a morphism t with $m \cdot t = f$.

(5) By Remark 6.1 (1), $\mathrm{reg}^{\mathcal{B}} \leq \mathrm{reg}^{\mathcal{A}}$ if $\mathcal{A} \subseteq \mathcal{B}$.

(6) The trivial operator is regular: $T \cong \mathrm{reg}^{\emptyset}$ (since $\mathrm{Reg}_{\mathcal{X}}(\emptyset) = \mathrm{Iso}(\mathcal{X})$). The discrete operator S is regular if and only if $\mathcal{M} = \mathrm{Reg}(\mathcal{X})$; in this case, $S \cong \mathrm{reg}^{\mathcal{X}}$. Every non-idempotent closure operator is non-regular. But also idempotent operators may fail to be regular, the most famous example being K in **Top**, as we shall see in Example 6.9(3).

When trying to characterize the epimorphisms of the subcategory $\mathcal{A}$ of a category $\mathcal{X}$, usually we shall choose $\mathcal{M}$ such that morphisms of $\mathcal{E}$ are easily described, as the surjective morphisms of $\mathcal{X}$, say. Since the $\mathcal{A}$-epimorphisms of $\mathcal{X}$ which belong to $\mathcal{A}$ are precisely the epimorphisms of the category $\mathcal{A}$, in order to recognize them as morphisms belonging to $\mathcal{E}$ one may just check condition (iv) of the following proposition. Sufficient conditions for (iv) are given in Section 6.4.

PROPOSITION *If $\mathcal{X}$ has equalizers with* $\mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M}$ *, and if $\mathcal{A}$ is a full replete subcategory of $\mathcal{X}$, then for the conditions* (i)-(iv) *below one has* (i) $\Leftrightarrow$ (ii) *and* (i) & (iii) $\Leftrightarrow$ (iv)*:*

(i) *every $\mathcal{A}$-epimorphism of $\mathcal{X}$ with codomain in $\mathcal{A}$ belongs to $\mathcal{E}$,*

(ii) $m \cong \mathrm{epi}_X^{\mathcal{A}}(m)$ *for all* $m \in \mathcal{M}/X$, $X \in \mathcal{A}$,

(iii) $\mathrm{epi}_X^{\mathcal{A}}(m) \cong \mathrm{reg}_X^{\mathcal{A}}(m)$ *for all* $m \in \mathcal{M}/X$, $X \in \mathcal{A}$,

(iv) *every morphism in $\mathcal{M}$ with codomain in $\mathcal{A}$ is $\mathcal{A}$-closed.*

Proof (i) & (iii) $\Leftrightarrow$ (iv) is trivial (see Remark (1)). For (i) $\Rightarrow$ (ii), since $\mathrm{epi}^{\mathcal{A}}$ is weakly hereditary, the morphism $M \to \mathrm{epi}_X^{\mathcal{A}}(M)$ is $\mathcal{A}$-epic and therefore in $\mathcal{E} \cap \mathcal{M}$ under hypothesis (i), hence iso. Conversely, in the $(\mathcal{E},\mathcal{M})$-factorization $q = m \cdot e$ of an $\mathcal{A}$-epimorphism q with codomain in $\mathcal{A}$ also $m \cong \mathrm{epi}_X^{\mathcal{A}}(m)$ is $\mathcal{A}$-epic, hence iso and therefore $q \in \mathcal{E}$. □

EXAMPLES

(1) Let $\mathcal{X} = \mathbf{AbGrp}$ be the category of abelian groups, with the usual subobject structure. For a subgroup $M \leq X$, the cokernelpair $i, j : X \to X +_M X$ may be constructed as

$$<1_X, p>, <1_X, 0>: X \to X \times (X/M),$$

with $p : X \to X/M$ the canonical projection, hence

$$X +_M X \cong X \times (X/M).$$

(This remains true in any additive category, see Exercise 6.O.) For $\mathcal{A}$ the subcategory of torsion-free abelian groups, the $\mathcal{A}$-reflexion of X is the projection

$$\rho_X : X \to X/\mathbf{t}(X),$$

with $\mathbf{t}(X)$ the torsion subgroup of X (cf. Example 3.4(1)). With the formula given in Remark (2) one obtains

$$\mathrm{reg}_X^{\mathcal{A}}(M) = \{x \in X : (\exists n \geq 1)\ nx \in M\},$$

hence $\mathrm{reg}^{\mathcal{A}}$ *is the maximal closure operator* $C^{\mathbf{t}}$. As a hereditary closure operator it is in particular weakly hereditary, hence $\mathrm{reg}^{\mathcal{A}} = \mathrm{epi}^{\mathcal{A}}$. Note, for example that $\mathbb{Q} \leq \mathbb{R}$ is $\mathcal{A}$-closed, while $\mathbb{Z} \leq \mathbb{Q}$ is $\mathcal{A}$-dense. Hence $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category $\mathcal{A}$. We note furthermore that for every $n \geq 1$ also $n\mathbb{Z} \to \mathbb{Z}$ is $\mathcal{A}$-dense while $0 \leq \mathbb{Z}$ is $\mathcal{A}$-closed, so that $\mathbb{Z}$ has no proper non-trivial $\mathcal{A}$-closed subgroup. In general, $f : X \to Y$ is an epimorphism of $\mathcal{A}$ if and only if for every $y \in Y$ there is $n \geq 1$ and $x \in X$ such that $ny = f(x)$.

(2) For a class $\mathcal{A}$ of universal algebras and for $C \leq B \in \mathcal{A}$, Isbell [1966] defined the *dominion* of C in B as the subalgebra

$$\{x \in C : (\forall f, g : B \to A \in \mathcal{A} \text{ homom.})\ f|_C = g_C \Rightarrow f(x) = g(x)\}$$

of B; this is simply the $\mathcal{A}$-regular closure of C in B. He also considered the weakly hereditary core $(\mathrm{reg}^{\mathcal{A}})_\infty$ of the $\mathcal{A}$-regular closure for universal algebras (using the ∞-subscript notation !), i.e., the closure operator $\mathrm{epi}^{\mathcal{A}}$, in order to describe the epimorphisms of $\mathcal{A}$ (cf. 8.9).

(3) For any class $\mathcal{A}$ of topological spaces (which contain a space with at least two points) and any space X, Salbany [1976] considered the coarsest topology on X in which $\mathcal{A}$-closed sets are closed. The space obtained in this way, when presented as a pretopological space, is simply $(X, (\mathrm{reg}^{\mathcal{A}})^+)$. In fact, $(\mathrm{reg}^{\mathcal{A}})^+$ is the coarsest closure operator C such that $\mathcal{A}$-closed sets are C-closed (i.e., $C \leq \mathrm{reg}^{\mathcal{A}}$) and C-closed sets are closed under binary union (i.e., C is additive). (Note that due to the existence of a non-trivial space in $\mathcal{A}$, $\mathrm{reg}^{\mathcal{A}}$ is grounded; cf. Example 6.9(1) below.)

6.3 Computing the $\mathcal{A}$-regular closure for reflective $\mathcal{A}$

In an $\mathcal{M}$-complete category $\mathcal{X}$ with equalizers, with $\mathcal{M}$ closed under composition and containing all regular monomorphisms, we consider a full and replete *reflective* subcategory $\mathcal{A}$. In what follows we shall show that the $\mathcal{A}$-regular closure operator of $\mathcal{X}$ is completely determined by its behavior on $\mathcal{A}$.

Hence for every $X \in \mathcal{X}$ one has a universal map $\rho_X : X \to RX$ with $RX \in \mathcal{A}$, which gives rise to a pointed endofunctor (R, ρ) as in Example (1) of 5.1. As in 5.12, we can then form the *R-modification* ${}_R(\mathrm{reg}^{\mathcal{A}})$ of the regular closure operator $\mathrm{reg}^{\mathcal{A}}$ of $\mathcal{X}$, by

$$_R(\mathrm{reg}^{\mathcal{A}})_X(m) \cong \rho_X^{-1}(\mathrm{reg}_{RX}^{\mathcal{A}}(\rho_X(m))). \tag{$*$}$$

Furthermore, if $\mathcal{A}$ is $\mathcal{E}$-reflective, with $\mathcal{E}$ determined by $\mathcal{M}$ as in 2.1, so that $\mathcal{A}$ is closed under $\mathcal{M}$-subobjects, then $\mathrm{reg}^{\mathcal{A}}$ can be restricted to a closure operator of $\mathcal{A}$ (cf. 2.8). Since R may be considered the left adjoint of the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{X}$, with unit ρ, we can also form the *initial closure operator* induced by $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ and R (cf. 5.7), which may be computed by

$$(\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}})^{\rho}_{X}(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(Rm)). \tag{**}$$

(see Lemma 5.13).

THEOREM

(1) *The regular closure operator* $\mathrm{reg}^{\mathcal{A}}$ *is its own R-modification. It is therefore the largest closure operator* D *of* $\mathcal{X}$ *for which every* $\mathcal{A}$*-reflexion* ρ_X *is* $(D, \mathrm{reg}^{\mathcal{A}})$*-continuous.*

(2) *If* $\mathcal{A}$ *is* $\mathcal{E}$*-reflective, then* $\mathrm{reg}^{\mathcal{A}}$ *is the initial closure operator induced by its restriction* $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ *and* R *. It is therefore the largest closure operator* D *of* $\mathcal{X}$ *for which the* $\mathcal{A}$*-reflector* $R : \mathcal{X} \to \mathcal{A}$ *is* $(D, \mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}})$*-continuous.*

Proof In view of Theorems 5.12 and 5.7, it suffices to show formulas (*) and (**) with the left-hand sides replaced by $\mathrm{reg}^{\mathcal{A}}_X(m)$ in each case. For $m : M \to X$ in $\mathcal{M}$, we form the $(\mathcal{E}, \mathcal{M})$-factorization $n \cdot e = \rho_X \cdot m$ of $\rho_X \cdot m$, so that $n \cong \rho_X(m) : N \to RX$, and apply the Diagonalization Lemma 2.4 to obtain the following commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\quad e \quad} & N \\
\downarrow & & \downarrow \\
\mathrm{reg}^{\mathcal{A}}_X(M) & \xrightarrow{\quad w \quad} & \mathrm{reg}^{\mathcal{A}}_{RX}(N) \\
{\scriptstyle \mathrm{reg}^{\mathcal{A}}_X(m)}\downarrow & & \downarrow{\scriptstyle \mathrm{reg}^{\mathcal{A}}_{RX}(n)} \\
X & \xrightarrow{\quad \rho_X \quad} & RX
\end{array} \tag{6.3}$$

We must now show that the lower part of (6.3) forms a pullback diagram. Hence we consider x, y with $\rho_X \cdot x = \mathrm{reg}^{\mathcal{A}}_{RX}(n) \cdot y$. Since $\mathrm{reg}^{\mathcal{A}}_{RX}(n)$ is monic, it suffices to show that x factors through $\mathrm{reg}^{\mathcal{A}}_X(m)$, and for that we just need to show $\mathrm{reg}^{\mathcal{A}}_X(m) \overset{\mathcal{A}}{\Rightarrow} x$. We therefore consider $u, v : X \to A$ with $u \cdot \mathrm{reg}^{\mathcal{A}}_X(m) = v \cdot \mathrm{reg}^{\mathcal{A}}_X(m)$ and $A \in \mathcal{A}$. Since u, v factor as $u' \cdot \rho_X = u$ and $v' \cdot \rho_X = v$, and since $e \in \mathcal{E}$ is epic (due to the existence of equalizers, see Theorem 6.1), one derives $u' \cdot n = v' \cdot n$ and then $u' \cdot \mathrm{reg}^{\mathcal{A}}_{RX}(n) = v' \cdot \mathrm{reg}^{\mathcal{A}}_{RX}(n)$ (with Remark (3) of 6.3). This implies

$$u \cdot x = u' \cdot \rho_X \cdot x = u' \cdot \mathrm{reg}^{\mathcal{A}}_{RX}(n) \cdot y = v' \cdot \mathrm{reg}^{\mathcal{A}}_{RX}(n) \cdot y = v' \cdot \rho_X \cdot x = v \cdot x,$$

as desired. This completes the proof of (1).

For (2) one considers an $(\mathcal{E},\mathcal{M})$-factorization $d\cdot n = Rm$ of $Rm : RM \to RX$. With $e := d\cdot\rho_M$ one obtains again the commutative diagram (6.3) and shows that its lower part is a pullback diagram as before. □

REMARK We note that $\mathcal{E}$-reflectivity of $\mathcal{A}$ was assumed in (2) of the Theorem only in order to be able to restrict $\mathrm{reg}^{\mathcal{A}}$ to a closure operator of $\mathcal{A}$. But the formulas

$$\mathrm{reg}^{\mathcal{A}}_X(m) \cong \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(\rho_X(m))) \cong \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(Rm))$$

actually hold for every reflective subcategory $\mathcal{A}$.

A particularly useful consequence of formula (*) , of Proposition 6.1 and of Exercise 1.K(d) is:

COROLLARY *An $\mathcal{M}$-subobject m of X is $\mathcal{A}$-closed if and only if there is an $\mathcal{A}$-closed $\mathcal{M}$-subobject n of RX with $m = \rho_X^{-1}(n)$. In this case necessarily $n \cong \rho_X(m)$, provided that $\mathcal{A}$ is $\mathcal{E}$-reflective and $\mathcal{E}$ is stable under pullback.* □

Of particular interest is the question whether the least subobject $o_X : O_X \to X$ of an object X is $\mathcal{A}$-closed. Formula (*) allows us to reduce this problem to the case $X \in \mathcal{A}$:

PROPOSITION *For an object $X \in \mathcal{X}$, o_X is $\mathcal{A}$-closed if o_{RX} is $\mathcal{A}$-closed and $\rho_X^{-1}(o_{RX}) \cong o_X$. Conversely, if o_X is $\mathcal{A}$-closed, then $\rho_X^{-1}(o_{RX}) \cong o_X$, and also o_{RX} is $\mathcal{A}$-closed provided that $\mathcal{A}$ is $\mathcal{E}$-reflective and $\mathcal{E}$ is stable under pullback.*

Proof The first assertion follows from Proposition 6.1. If $\mathrm{reg}^{\mathcal{A}}_X(o_X) \cong o_X$, then $\rho_X(o_X) \cong o_{RX}$ (cf. 1.11) and (*) gives:

$$\rho_X^{-1}(o_{RX}) \le \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(o_{RX})) \cong \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(\rho_X(o_X))) \cong \mathrm{reg}^{\mathcal{A}}_X(o_X) \cong o_X,$$

hence $\rho_X^{-1}(o_{RX}) \cong o_X$. Furthermore, if $\rho_X \in \mathcal{E}$ with $\mathcal{E}$ is stable under pullback, with Exercise 1.K (d) and (*) one concludes:

$$\mathrm{reg}^{\mathcal{A}}_{RX}(o_{RX}) \cong \rho_X(\rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(o_{RX}))) \cong \rho_X(\mathrm{reg}^{\mathcal{A}}_X(o_X)) \cong \rho_X(o_X) \cong o_{RX}.$$

□

Conditions under which o_X with $X \in \mathcal{A}$ is $\mathcal{A}$-closed will be described in 6.5. The conditions in the above Proposition are not not necessary in general.

EXAMPLE In the category **Fld** of fields with $\mathcal{M}$ the class of all morphisms (=monomorphisms), $\mathcal{E}$ is the class of all isomorphisms, so that $\mathcal{E}$ is stable under pullback, and the only full replete $\mathcal{E}$-reflective subcategory $\mathcal{A}$ is the whole category **Fld** . However, **Fld** has many (non- $\mathcal{E}$-)reflective subcategories, each of which is necessarily bireflective (cf. Exercise 3.L and 6.H). We claim that for every reflective subcategory $\mathcal{A}$, o_X (=the embedding of the prime field of X into X) is $\mathcal{A}$-closed. In fact, since **Fld** has a least reflective subcategory, given by the subcategory **PerFld** of perfect fields (cf. Proposition 8.10), it suffices to show that o_X is **PerFld**-closed (see Remark 6.1(1)). But it turns out that a perfect subfield is always **PerFld**-closed (again, by Proposition 8.10), so it is enough to note that the prime field of X is perfect.

6.4 The magic cube

Let $\mathcal{X}$ be $\mathcal{M}$-complete, with $\mathcal{M}$ *containing all regular monomorphisms* and being closed under composition. We assume $\mathcal{X}$ to be finitely complete and finitely co-complete, and consider a full reflective and replete subcategory $\mathcal{A}$ of $\mathcal{X}$. Our goal is to *find sufficient conditions for a given regular monomorphism* $m : M \to X$ *in* $\mathcal{X}$ *with* $X \in \mathcal{A}$ *to be* $\mathcal{A}$*-closed*. As in Lemma 6.1, we form the cokernelpair $i, j : X \to K = X +_M X$ of m in $\mathcal{X}$, so that m is the equalizer of $(i \cdot m, j \cdot m)$; equivalently, diagram (6.2) is a pullback (since $i \cdot a = j \cdot b$ implies $a = \varepsilon \cdot i \cdot a = \varepsilon \cdot j \cdot b = b$, with $\varepsilon : K \to X$ the common retraction of i and j).

Let now $\rho_K : K \to RK$ be the $\mathcal{A}$-reflexion of K and consider the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "m"] & X \arrow[r, "1_X"] \arrow[d, "i"] & X \arrow[d, "\rho_K \cdot i"] \\
X \arrow[r, "j"] & K \arrow[r, "\rho_K"] & RK
\end{tikzcd} \tag{6.4}$$

Calling m *strongly* $\mathcal{A}$*-closed* if i is $\mathcal{A}$-closed, we have :

PROPOSITION *m is strongly $\mathcal{A}$-closed if and only if the right square in* (6.4) *is a pullback diagram, and then m is $\mathcal{A}$-closed.*

Proof Since the left square in (6.4) is a pullback diagram and since $\mathcal{A}$-closedness is stable under pullback (see Prop. 6.1), every strongly $\mathcal{A}$-closed morphism is $\mathcal{A}$-closed. Since ε factors as $\varepsilon' \cdot \rho_K = \varepsilon$ (since $X \in \mathcal{A}$), $\rho_K \cdot i$ is an $\mathcal{A}$-section, so that i must be $\mathcal{A}$-closed if the right square in (6.4) is a pullback diagram. Conversely, if i is $\mathcal{A}$-closed, then i is the equalizer of a pair of morphisms $u, v : K \to A$ with $A \in \mathcal{A}$, which factor through ρ_K as $u' \cdot \rho_K = u$ and $v' \cdot \rho_K = v$. Now we can check the pullback property, as follows: from $\rho_K \cdot a = \rho_K \cdot i \cdot b$ one has

$$u \cdot a = u' \cdot \rho_K \cdot i \cdot b = u \cdot i \cdot b = v \cdot i \cdot b = v' \cdot \rho_K \cdot i \cdot b = v' \cdot \rho_K \cdot a = v \cdot a,$$

so that the equalizer property gives c with $i \cdot c = a$, but when applying ε' to $\rho_K \cdot i \cdot c = \rho_K \cdot i \cdot b$ one gets $c = b$, hence $i \cdot b = a$, as desired. □

In what follows we describe a different method of identifying the right square of

(6.4) as a pullback, by forming the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "i"] & X \times X \arrow[r, "\pi_2"] \arrow[d, "1 \times (\rho_K \cdot i)"] & X \arrow[d, "\rho_K \cdot i"] \\
K \arrow[r, "w"] & X \times RK \arrow[r, "\pi_2"] & RK
\end{tikzcd} \tag{6.5}$$

Here (each) π_2 is a product projection, and $\delta = <1_X, 1_X>$ and $w = <\varepsilon, \rho_K>$ are given by the product property. Note that the right square of (6.5) is always a pullback diagram. Hence the left square of (6.5) is a pullback diagram if and only if the whole diagram (6.5) is a pullback diagram, which is exactly the right square of (6.4). With the Proposition, we have therefore proved

LEMMA $\quad m$ *is strongly* $\mathcal{A}$*-closed if and only if the left square in* (6.5) *is a pullback diagram.* □

Recall that the cokernelpair $i, j : X \to K$ of m can be constructed in two stages, by first forming the coproduct $k, l : X \to X + X$ in $\mathcal{X}$ and then the coequalizer $c : X + X \to K$ of $(k \cdot m, l \cdot m)$ in $\mathcal{X}$. Hence we may assume $i = c \cdot k, j = c \cdot l$. Furthermore, let

- $q : R(X+X) \to L$ be the coequalizer of $(\rho \cdot k \cdot m, \rho \cdot l \cdot m)$, with $\rho = \rho_{X+X}$ the $\mathcal{A}$-reflexion of $X + X$,
- $u : K \to L$ the morphism with $u \cdot c = q \cdot \rho$,
- $v = <\varepsilon'', \rho_L>: L \to X \times RL$ the induced morphism with ρ_L the $\mathcal{A}$-reflexion of L and $\varepsilon'' : L \to X$ the morphism with $\varepsilon'' \cdot q \cdot \rho \cdot k = \varepsilon'' \cdot q \cdot \rho \cdot l$.

We let $a := 1 \times (\rho_K \cdot i) : X \times X \to X \times RK$ and $b := 1 \times (Ru \cdot \rho_K \cdot i) : X \times X \to X \times RL$.

$$\begin{tikzcd}
& X \arrow[rr, "\delta"] \arrow[dd, "u \cdot i"] & & X \times X \arrow[dd, "b"] \\
X \arrow[ur, "1"] \arrow[rr, "\delta"] \arrow[dd, "i"] & & X \times X \arrow[ur, "1"] \arrow[dd, "a"] & \\
& L \arrow[rr, "v"] & & X \times RL \\
K \arrow[ur, "u"] \arrow[rr, "w"] & & X \times RK \arrow[ur, "1 \times Ru"] &
\end{tikzcd} \tag{6.6}$$

It is a routine exercise to check that the "magic cube" (6.6) commutes. Furthermore, let us note that $Ru : RK \to RL$ is an isomorphism. In fact, Rc :

$R(X+X) \to K$ factors as $x \cdot q = Rc$, and $x : L \to RK$ factors as $y \cdot \rho_L = x$. From $Ru \cdot Rc \cdot \rho = Ru \cdot \rho_K \cdot c = \rho_L \cdot u \cdot c = \rho_L \cdot q \cdot \rho$ one obtains $Ru \cdot Rc = \rho_L \cdot q$ so that, in particular, Ru is $\mathcal{A}$-epic. But Ru is also an $\mathcal{A}$-section, since $y \cdot Ru \cdot Rc = y \cdot \rho_L \cdot q = x \cdot q = Rc$, with Rc $\mathcal{A}$-epic (since the left adjoint $R : \mathcal{X} \to \mathcal{A}$ preserves epimorphisms). We therefore have that *the right face of* (6.6) *is a pullback diagram, and so is the top face.* Furthermore we note that *all vertical arrows are sections and therefore belong to* $\mathcal{M}$. We are now in a position to prove:

THEOREM (Magic Cube Theorem) *For the statements* (i)-(xi) *below one has the following implications:*

$$\begin{tikzcd}
\text{(iii)} \arrow[rr, Rightarrow] \arrow[dd, Rightarrow] & & \text{(vii)} \arrow[rr, Rightarrow] & & \text{(viii)} \arrow[dd, Rightarrow] \\
& \text{(ii)} \arrow[ul, Rightarrow] \arrow[dr, Rightarrow] & & & \\
\text{(iv)} \arrow[dd, Rightarrow] & & \text{(x)} \arrow[ddrr, Rightarrow] & & \text{(ix)} \arrow[dd, Rightarrow] \\
& \text{(i)} \arrow[ur, Rightarrow] \arrow[dl, Rightarrow] & & & \\
\text{(v)} \arrow[rr, Leftrightarrow] & & \text{(vi)} \arrow[rr, Rightarrow] & & \text{(xi)}
\end{tikzcd}$$

In addition, (i) $\Rightarrow$ (ii) *holds when* $\mathcal{A}$ *is epireflective.*

(i) $K \in \mathcal{A}$,

(ii) $L \in \mathcal{A}$ *and* w *is monic,*

(iii) $L \in \mathcal{A}$ *and* u *is monic,*

(iv) $w^{-1}(a) \cong i$ *and* $v^{-1}(b) \cong u \cdot i$,

(v) $\rho_K^{-1}(\rho_K \cdot i) \cong i$,

(vi) m *is strongly* $\mathcal{A}$*-closed,*

(vii) $L \in \mathcal{A}$ *and* $u^{-1}(u \cdot i) \cong i$,

(viii) v *is monic and* $\rho_K^{-1}(\rho_K \cdot i) \cong i$,

(ix) v *is monic and* m *is an equalizer of* $(q \cdot \rho_K \cdot k, q \cdot \rho_K \cdot l)$,

(x) w *is monic,*

(xi) m *is* $\mathcal{A}$*-closed.*

Proof (ii) $\Rightarrow$ (iii) Since the bottom face of (6.6) commutes, with $1 \times Ru$ iso, u is a monomorphism if w is one.

(iii) $\Rightarrow$ (iv) First we show that $L \in \mathcal{A}$ implies that the back face of (6.6) is a pullback diagram. Consider x, y with $v \cdot x = b \cdot y$. An application of the second projection of $X \times RL$ gives

$$\rho_L \cdot x = Ru \cdot \rho_K \cdot i \cdot \pi_2 \cdot y = \rho_L \cdot u \cdot i \cdot \pi_2 \cdot y,$$

with π_2 the second projection of $X \times X$, hence $x = u \cdot i \cdot \pi_2 \cdot y$ under the assumption $L \in \mathcal{A}$. Furthermore, from $b \cdot y = v \cdot x = v \cdot u \cdot i \cdot \pi_2 \cdot y = b \cdot \delta \cdot \pi_2 \cdot y$ with b monic one has $y = \delta \cdot \pi_2 \cdot y$, so that $\pi_2 \cdot y (= \pi_1 \cdot y)$ gives the desired factorization. Now we have that the concatenation (top& back)=(front& bottom) of (6.6) is a pullback, so that the front face must be a pullback diagram since u is monic, by hypothesis.

(iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Rightarrow$ (xi) follow from the Proposition and the Lemma.

(iii) $\Rightarrow$ (vii) We clearly showed that (iii) implies that the front face and therefore the concatenation (front& right)=(left&back) of (6.6) is a pullback. Since δ is monic, this implies that the left face is a pullback diagram.

(vii) $\Rightarrow$ (viii) Since $\pi_2 \cdot v = \rho_L$ (with $\pi_2 : X \times RL \to RL$), ρ_L monic implies v monic. Furthermore, $L \in \mathcal{A}$ also implies that the back face of (6.6) is a pullback (as shown above). Hence if the left face is a pullback, also the front face must be one, and this means $\rho_K^{-1}(\rho_K \cdot i) = i$ by the Lemma and the Proposition.

(viii) $\Rightarrow$ (ix) We show that if the right face of (6.6) is a pullback then m is an equalizer of $(q \cdot \rho \cdot k, q \cdot \rho \cdot l)$. In fact, if any z satisfies $q \cdot \rho \cdot k \cdot z = q \cdot \rho \cdot l \cdot z$, since $Ru \cdot \rho_K \cdot c = \rho_L \cdot q \cdot \rho$, then

$$(\rho_K \cdot i) \cdot z = \rho_K \cdot c \cdot k \cdot z = (Ru)^{-1} \cdot \rho_L \cdot q \cdot \rho \cdot k \cdot z = (Ru)^{-1} \cdot \rho_L \cdot q \cdot \rho \cdot l \cdot z = \rho_K \cdot (c \cdot l \cdot z).$$

Hence the pullback property shows that we must have $c \cdot k \cdot z = c \cdot l \cdot z$. But since m is the equalizer of (i, j) , the equation $i \cdot z = j \cdot z$ makes z factor through m , as desired.

(ix) $\Rightarrow$ (xi) Under the assumption (ix), m is the equalizer of $(v \cdot q \cdot \rho \cdot k, v \cdot q \cdot \rho \cdot l)$, with the codomain $X \times RL$ of v belonging to the reflective subcategory $\mathcal{A}$. Hence m is $\mathcal{A}$-closed by Lemma 6.1.

(ii) $\Rightarrow$ (x) is trivial, and for (x) $\Rightarrow$ (xi) observe that since m is the equalizer of (i, j) it is also the equalizer of $(w \cdot i, w \cdot j)$ when w is monic, with the codomain $X \times RK$ of w belonging to $\mathcal{A}$.

(i) $\Rightarrow$ (v) is trivial since ρ_K is iso if $K \in \mathcal{A}$, and for (i) $\Rightarrow$ (x) note that w is monic if ρ_K is monic, since $\pi_2 \cdot w = \rho_K$ (with $\pi_2 : X \times RK \to RK$).

Finally we show (i) $\Rightarrow$ (ii) under the condition that $\mathcal{A}$ is epireflective. First we note that $\pi_2 \cdot w = \rho_K$ shows that w is a section if $K \in \mathcal{A}$. But then also u is a section, since $v \cdot u = (1 \times Ru) \cdot w$, with $1 \times Ru$ iso. On the other hand, u is epic since $u \cdot c = q \cdot \rho$ is epic by hypothesis. Hence u is iso and v must be a section. But this means that L is a retract of the $\mathcal{A}$-object $X \times RL$ and therefore in $\mathcal{A}$ itself. □

COROLLARY *If $\mathcal{A}$ is strongly epireflective in $\mathcal{X}$, then conditions* (i), (ii) *and* (x) *of the Theorem are equivalent and imply all other conditions of the Theorem.*

Proof We need to show only (x) $\Rightarrow$ (i), assuming that $\mathcal{A}$ is strongly epireflective. But in this case $\mathcal{A}$ is closed under monomorphisms, so that $w : K \to X \times RK$ monic implies $K \in \mathcal{A}$ since $X \times RK \in \mathcal{A}$. □

REMARKS

(1) Both the Theorem and Corollary can be simplified in case $\mathcal{A}$ satisfies the condition

$$Y \in \mathcal{A} \;\Rightarrow\; Y + Y \in \mathcal{A} \tag{$*$}$$

for all objects $Y \in \mathcal{X}$. One then has $X + X \cong R(X + X)$, hence $u : K \to L$ is an isomorphism, and the Magic Cube collapses to its front face, say. Only the lower triangle in the scheme of implications in the Theorem does matter now, all other conditions appearing in the Theorem are trivially equivalent to at least one appearing in that triangle, provided that $\mathcal{A}$ is epireflective.

(2) Note that condition (*) follows necessarily from condition (i) of the Theorem, provided that the least $\mathcal{M}$-subobject o_Y of Y is $\mathcal{A}$-closed (see Proposition 6.3 and Exercise 6.I).

EXAMPLES

(1) Let $\mathcal{X}$ be the category $\mathbf{Mod}_R$, with its usual subobject structure. Since we are in an additive category, we have $Y + Y \cong Y \times Y$, so that condition (*) of Remark (1) is satisfied. Furthermore, one has $X +_M X \cong X \times (X/M)$ for every submodule $M \leq X$ (see Example 6.2 and Exercise 6.O). Since there is a trivial equalizer diagram

$$X/M \to X \times (X/M) \overset{1\times 1}{\underset{0\times 1}{\rightrightarrows}} X \times (X/M)$$

one concludes for every (full replete) reflective subcategory $\mathcal{A}$ and for every submodule $M \leq X \in \mathcal{A}$:

$$X +_M X \in \mathcal{A} \Leftrightarrow X/M \in \mathcal{A}.$$

Hence the essential implications arising from the Magic Cube Theorem are as follows:

$$X/M \in \mathcal{A} \Rightarrow M \text{ strongly } \mathcal{A}\text{-closed in } X \Rightarrow M \ \ \mathcal{A}\text{-closed in } X.$$

Moreover, for $\mathcal{A}$ epireflective we have

$$X/M \in \mathcal{A} \Leftrightarrow M \text{ strongly } \mathcal{A}\text{-closed in } X.$$

In fact, if $M \leq X$ is strongly $\mathcal{A}$-closed, then $X \times 0 \leq X \times (X/M)$ is $\mathcal{A}$-closed (see the construction of $i : X \to X +_M X$ in Example 6.2), hence its pullback along $<0, 1>: X \times (X/M)$ is $\mathcal{A}$-closed, which is $0 \leq X/M$. Proposition 6.3 gives $\rho_{X/M}^{-1}(0) \cong 0$, i. e., X/M has an injective and therefore bijective $\mathcal{A}$-reflexion $\rho_{X/M}$, and this shows $X/M \in \mathcal{A}$.

We shall show in 6.7 that in fact $\mathcal{A}$-closedness implies strong $\mathcal{A}$-closedness whenever $\mathcal{A}$ is epireflective; in other words : *all statements of the Magic Cube Theorem are equivalent*, even without assuming $X \in \mathcal{A}$.

(2) Let $\mathcal{A} = \mathbf{AbGrp}$ be the (strongly) epireflective subcategory of Abelian groups in $\mathcal{X} = \mathbf{Grp}$, with its usual subobject structure. A subgroup M of an abelian group X is the kernel of the projection $X \to X/M$ and therefore $\mathcal{A}$-closed. The sum $X+X$ in $\mathbf{Grp}$ is the free product $X*X$, and $K = X*_M X$ is the free product with amalgamated subgroup M; clearly, K is not abelian unless $M = X$. Under additive notation, the abelianization $R(X*X)$ is the direct sum $X \times X$ in $\mathbf{AbGrp}$, hence $L = X \times X/\overline{M}$ with $\overline{M} = \{(x,-x)|x \in M\}$, so that we can take $RK = RL = L$. It is now easy to see that the right part of (6.4) fails to be a pullback diagram, unless $M = X$. Hence, for $M \neq X$, *M is $\mathcal{A}$-closed in $\mathcal{X}$ but not strongly $\mathcal{A}$-closed.* In terms of the conditions of the Magic Cube Theorem, one has for $M \neq X$: none of the equivalent conditions (i), (ii), (x) holds, and although $L \in \mathcal{A}$ and the canonical map $v : L \to A \times L$ is monic, conditions (iii)-(viii) do not hold either; however, (ix) and (xi) are satisfied.

6.5 Frolík's Lemma

In this section we provide a simple criterion for $\mathcal{A}$-closedness under **Set**-like conditions. These conditions are similar to those used in Section 4.9 where "points" were simulated by $\bigvee$-prime subobjects. Here we work with $\vee$-prime subobjects instead, calling $p \in \mathcal{M}/X$ *$\vee$-prime* if $p \neq o_X$ and if $p \leq m \vee n$ implies $p \leq m$ or $p \leq n$. We say that *$\mathcal{M}$ is generated by its $\vee$-prime elements* if the class $\mathcal{P}$ of $\vee$-prime elements in $\mathcal{M}$ satisfies the following two conditions

(A) $f(p) \in \mathcal{P}/Y$ for every $f : X \to Y$ and $p \in \mathcal{P}/X$,

(B) $m \cong \bigvee\{p \in \mathcal{P}/X : p \leq m\}$ for every $m \in \mathcal{M}/X$.

As in 6.4, we assume our $\mathcal{M}$-complete category $\mathcal{X}$ (with $\mathcal{M}$ containing all regular monomorphisms and being closed under composition) to have binary products and equalizers as well as binary coproducts and coequalizers, and consider a reflective and replete subcategory $\mathcal{A}$ of $\mathcal{X}$.

We keep the notations of the Magic Cube Theorem and prove :

THEOREM (Frolík's Lemma) *Let $\mathcal{M}$ be generated by its $\vee$-prime elements. Then for a regular monomorphism $m : M \to X$ with $X \in \mathcal{A}$ to be $\mathcal{A}$-closed it is necessary that the morphism $<\varepsilon, \rho_K>: K \to X \times RK$ be a monomorphism. Hence, in case $\mathcal{A}$ is strongly epireflective, all conditions of the Magic Cube Theorem are equivalent. In particular, m is $\mathcal{A}$-closed if and only if $K = X +_M X$ belongs to $\mathcal{A}$.*

Proof We must show only that when m is $\mathcal{A}$-closed then necessarily $w = <\varepsilon, \rho_K>$ is monic. Hence we consider morphisms $s,t : Z \to K$ with $w \cdot s = w \cdot t$, and since the union $1_Z \cong \bigvee \mathcal{P}/Z$ is epic (see condition (B) and Lemma 6.2), it suffices to show $s \cdot p = t \cdot p$ for all $p \in \mathcal{P}/Z$. Both $s \cdot p$ and $t \cdot p$ can be $(\mathcal{E},\mathcal{M})$-factored as $s \cdot p = p_s \cdot e_s$ and $t \cdot p = p_t \cdot e_t$, respectively, with $p_s \cong p(s) \in \mathcal{P}$ and $p_t \cong p(t) \in \mathcal{P}$ (see condition (A)). As injections of the cokernelpair of m, both i and j are sections and therefore belong to $\mathcal{M}$, and since the regular epimorphism

c belongs to $\mathcal{E}$ (Theorem 6.1, dual), one has $1_K \cong i \vee j$. Now $\vee$-primeness of $p_s : P_s \to K$ and $p_t : P_t \to K$ yields morphisms $x : P_s \to X$ and $y : P_t \to X$ with $p_s \in \{i \cdot x, j \cdot x\}$ and $p_t \in \{i \cdot y, j \cdot y\}$, hence $p_z = g \cdot x$ and $p_w = h \cdot y$ with $g, h \in \{i, j\}$. With $\pi_1 : X \times RK \to X$ the projection, one obtains

$$\pi_1 \cdot w \cdot s \cdot p = \varepsilon \cdot p_s \cdot e_s = \varepsilon \cdot g \cdot x \cdot e_s = x \cdot e_s$$

and symmetrically, $\pi_1 \cdot w \cdot t \cdot p = y \cdot e_t$, so that $w \cdot s = w \cdot t$ implies $x \cdot e_s = y \cdot e_t$. In case $g = h$, this gives immediately

$$s \cdot p = p_s \cdot e_s = g \cdot x \cdot e_s = g \cdot y \cdot = h \cdot y \cdot e_t = p_t \cdot e_t = t \cdot p.$$

Otherwise one has to invoke the fact that the $\mathcal{A}$-closed morphism m is an equalizer of $(\rho_K \cdot i, \rho_K \cdot j)$, as follows. Since

$$\rho_K \cdot g \cdot x \cdot e_s = \rho_K \cdot p_s \cdot e_s = \pi_2 \cdot w \cdot s \cdot p = \pi_2 \cdot w \cdot t \cdot p = \rho_K \cdot p_t \cdot e_t = \rho_K \cdot h \cdot y \cdot e_t$$

the morphism $x \cdot e_s = y \cdot e_t$ factors as $x \cdot e_s = m \cdot z$. Therefore,

$$s \cdot p = p_s \cdot e_s = g \cdot x \cdot e_s = g \cdot m \cdot z = h \cdot m \cdot z = h \cdot y \cdot e_t = p_t \cdot e_t = t \cdot p,$$

which completes the proof. □

REMARK For every topological category $\mathcal{X}$ over **Set** with its usual subobject structure, given by the class $\mathcal{M}$ of all regular monomorphisms, the hypothesis that $\mathcal{M}$ be generated by its $\vee$-prime elements (=actual points) trivially holds. This is the context (actually for $\mathcal{X} = \mathbf{Top}$) in which Frolík proved the Theorem. In the examples below we give a number of applications in this context.

EXAMPLES Let $\mathcal{X} = \mathbf{Top}$ and $\mathcal{M} = \mathrm{Reg}(\mathcal{X})$. For a subspace M of a space X , the space $K = X +_M X$ has as its underlying set $X \times \{1,2\}/\sim$, with

$$(x, \nu) \sim (y, \mu) \Leftrightarrow x = y \in M \text{ or } (x, \nu) = (y, \mu).$$

With the canonical injections $i : X \to K$, $x \mapsto c(x, 1)$, and $j : X \to K$, $x \mapsto c(x, 2)$, a subset $U \subseteq K$ is open iff

$$i^{-1}(U) = \{x \in X : c(x,1) \in U\},\ j^{-1}(U) = \{x \in X : c(x,2) \in U\}$$

are open in X . Note that for $V = i^{-1}(U)$ and $W = j^{-1}(U)$, $V \cap M = W \cap M$ holds. Conversely, for every pair V, W of open subsets of X with $V \cap M = W \cap M$, the subset $U = i(V) \cup j(W)$ of K is open. For some familiar reflective subcategories $\mathcal{A}$ we check whether $K \in \mathcal{A}$ whenever $X \in \mathcal{A}$:

(1) $\mathcal{A} = \mathbf{Top}_1$, the category of T_1-spaces. In this case we have $K \in \mathcal{A}$. Indeed, the critical pairs of points to be T_1-separated are of the form $c(x,1)$, $c(x,2)$, with $x \notin M$. But in this case $W = K \setminus c(x,2)$ is an open neighbourhood of $c(x,1)$ since $i^{-1}(W) = X$ and $j^{-1}(W) = X \setminus \{x\}$ are both open in X . Consequently, the *regular monomorphisms in* $\mathbf{Top}_1$ *are exactly the subspace embeddings, and the epimorphisms in* $\mathbf{Top}_1$ *are surjective.*

(2) $\mathcal{A} = \mathbf{Top}_2 = \mathbf{Haus}$, the category of Hausdorff spaces. For $X = \mathbb{R}$ and $M = \mathbb{Q}$, the space K fails to be Hausdorff since $c(x,1)$ and $c(x,2)$ with $x \notin \mathbb{Q}$ cannot be T_2-separated, since any two open neighbourhoods of $c(x,1)$, $c(x,2)$ contain common rational points. However, in general, for $M \subseteq X$ (Kuratowski-) closed, one has $K \in \mathbf{Haus}$, and conversely. Consequently, the regular monomorphisms in **Haus** are exactly the (Kuratowski-) closed subspace embeddings, and the *epimorphisms in* **Haus** *are exactly the (Kuratowski-) dense maps.*

(3) $\mathcal{A} = \mathbf{Top}_0$, the category of T_0-spaces. In this case we have $K \in \mathcal{A}$ if and only if the subspace M of X is b-closed. Indeed, assume M to be b-closed. Again the critical pairs of points to be T_0-separated are of the form $c(x,1)$, $c(x,2)$, with $x \notin M$. Since M is b-closed, there exists an open neighborhood V of x such that $k_X(\{x\})$ does not meet $V \cap M$. Hence the open set $W = V \setminus k_X(\{x\})$ satisfies $V \cap M = W \cap M$. Thus $U = i(V) \cup j(W)$ is an open neighbourhood of $c(x,1)$ (since $i^{-1}(U) = V$ and $j^{-1}(W) = W$ are both open in X) which misses $c(x,2)$. On the other hand, if $K \in \mathcal{A}$, then for $x \notin M$ one can find an open neighbourhood of $c(x,1)$ missing $c(x,2)$. Then U misses also $k_K(\{c(x,2)\})$. Hence, from $j(k_X(\{x\})) \subseteq k_K(\{c(x,2)\})$, V is an open neighbourhood of x such that $V \cap M \cap k_X(\{x\}) = \emptyset$. This yields $x \notin b_X(M)$. Consequently, the regular monomorphisms in $\mathbf{Top}_0$ are precisely the b-closed subspace embeddings, and the *epimorphisms in* $\mathbf{Top}_0$ *are precisely the b-dense maps*, hence not surjective in general.

Without assuming that $\mathcal{X}$ satisfies the hypothesis of Frolík's Lemma, we say that a category $\mathcal{X}$ *satisfies Frolík's Lemma* if for every $\mathcal{A}$-closed $\mathcal{M}$-subobject $m \in \mathcal{M}/X$ with $X \in \mathcal{A}$ also $X +_M X$ belongs to $\mathcal{A}$, i.e., if conditions (i) and (xi) of the Magic Cube Theorem are equivalent. In this case every $\mathcal{A}$-closed $m \in \mathcal{M}/X$ with $X \in \mathcal{A}$ is strongly $\mathcal{A}$-closed. We shall see next that the restriction to subobjects in $\mathcal{A}$ can be avoided and first show the following useful formula:

LEMMA *For every $m : M \to X$ in $\mathcal{M}$, there is a canonical isomorphism*

$$R(X +_M X) \cong RX +_N RX$$

with $n \cong \mathrm{reg}^{\mathcal{A}}_{RX}(\rho_X(m)) : N \to X$.

Proof As usual, let $i, j : X \to K = X +_M X$ be the cokernelpair of m in $\mathcal{X}$, and let $i', j' : RX \to K' = RX +_N RX$ be the cokernelpair of n in $\mathcal{X}$. Since n is $\mathcal{A}$-closed, $K' \in \mathcal{A}$ by assumption on $\mathcal{X}$. Hence the unique morphism $s : K \to K'$ with $s \cdot i = i' \cdot \rho_X$ and $s \cdot j = j' \cdot \rho_X$ factors through ρ_K by a morphism $u : RK \to K'$. Since $\rho_X \cdot \overset{\mathcal{A}}{\Rightarrow} n$ by definition of n , and since

$$Ri \cdot \rho_X \cdot m = \rho_K \cdot i \cdot m = \rho_K \cdot j \cdot m = Rj \cdot \rho_X \cdot m,$$

one has $Ri \cdot n = Rj \cdot n$. Now the cokernel property of K' gives a morphism $v : K' \to RK$ with $v \cdot i' = Ri$ and $v \cdot j' = Rj$. It is easy to check that v is inverse to u . □

COROLLARY *If $\mathcal{X}$ satisfies Frolík's Lemma and if $\mathcal{A}$ is strongly epireflective in $\mathcal{X}$, then every $\mathcal{A}$-closed $\mathcal{M}$-subobject in $\mathcal{X}$ is strongly $\mathcal{A}$-closed.*

□

Proof With the notation of the Lemma, we have a commutative diagram

$$\begin{tikzcd}
 & N \arrow[rr, "n"] \arrow[dd, "n" near start] & & RX \arrow[dd, "Ri"] \\
M \arrow[ur, "e"] \arrow[rr, "m" near start, crossing over] \arrow[dd, "m"'] & & X \arrow[ur, "\rho_X"] & \\
 & RX \arrow[rr, "Rj" near start] & & RK \\
X \arrow[ur, "\rho_X"] \arrow[rr, "j"'] & & K \arrow[ur, "\rho_K"'] \arrow[from=uu, "i" near start, crossing over] &
\end{tikzcd} \tag{6.7}$$

The Lemma gives that the back face is a pullback diagram since n is $\mathcal{A}$-closed. With the $\mathcal{A}$-closedness of m and the definition of n one checks very similarly to the proof given for Proposition 6.4, that the left face is a pullback diagram. Hence the concatenation of the front and the right face is a pullback diagram. Using the common retraction ε of i and j , one easily checks that the right face must be a pullback diagram. But since n is strongly $\mathcal{A}$-closed, so that Ri is $\mathcal{A}$-closed (by the Lemma), also its pullback i along ρ_K is $\mathcal{A}$-closed. Hence m is strongly $\mathcal{A}$-closed. □

We have seen that every topological category over **Set** satisfies Frolík's Lemma, and we shall show the same for $\mathbf{Mod}_R$ in Theorem 6.7. Example 6.4 (2) shows that **Grp** does not satisfy Frolík's Lemma. A useful application of the Lemma is given in Exercise 6.U.

6.6 The strong modification of a closure operator

We have seen in the preceding sections that the notion of strong $\mathcal{A}$-closedness is a useful tool in order to recognize a given regular monomorphism as being $\mathcal{A}$-closed. Here we revisit this theme from a different perspective. Starting with an arbitrary closure operator C , we modify the C-closure of each subobject "along its cokernelpair", similarly to the modification procedure used in Section 5.12. The closure operator $\tilde{C}$ obtained this way will lead not only to a characterization of additive regular closure operators in 7.5 but turns out to be useful also for various examples (see below).

Let $\mathcal{X}$ have cokernelpairs and be $\mathcal{M}$-complete, with $\mathcal{M} \subseteq \mathrm{Reg}(\mathcal{X})$ closed under composition, and let C be a closure operator w.r.t. $\mathcal{M}$.

DEFINITION

(1) An $\mathcal{M}$-subobject $m : M \to X$ in $\mathcal{X}$ is called *strongly C-closed* if (at least one of) the canonical injections $i, j : X \to K = X +_M X$ of the cokernelpair of m are C-closed. (Note that, as sections, i and j belong to $\mathcal{M}$.)

(2) C is called *strong* if every C-closed $\mathcal{M}$-subobject is strongly C-closed.

(3) The *strong modification* $\tilde{C}$ of C is defined by

$$\tilde{c}_X(m) := j^{-1}(c_K(i)).$$

PROPOSITION *$\tilde{C}$ is a closure operator of $\mathcal{X}$ with $C \leq \tilde{C}$, and $C \mapsto \tilde{C}$ defines an endofunctor of $CL(\mathcal{X}, \mathcal{M})$.*

Proof Since $j \cdot m \leq i$, the C-continuity of j gives $j(c_X(m)) \leq c_K(j \cdot m) \leq c_K(i)$ and therefore $c_X(m) \leq j^{-1}(c_K(i)) = \tilde{c}_X(m)$; in particular, $m \leq \tilde{c}_X(m)$. If $m_1 \leq m_2 \in \mathcal{M}/X$, in self-explaining notation one considers the unique morphism $t : K_1 \to K_2$ with $t \cdot i_1 = i_2$ and $t \cdot j_1 = j_2$ and obtains

$$\begin{aligned} j_2(\tilde{c}_X(m_1)) &\cong t(j_1(j_1^{-1}(c_{K_1}(i_1)))) \\ &\leq t(c_{K_1}(i_1)) \\ &\leq c_{K_2}(t(i_1)) \\ &\leq c_{K_2}(i_2) , \end{aligned}$$

hence $\tilde{c}_X(m_1) \leq j_2^{-1}(c_{K_2}(i_2)) \cong \tilde{c}_X(m_2)$.

For the continuity condition for $\tilde{C}$, consider $f : X \to Y$ in $\mathcal{X}$ and $m_1 \in \mathcal{M}/X$, and let $m_2 \cong f(m_1)$. Again, in self-explanatory notation, one now has a unique morphism $g : K_1 \to K_2$ with $g \cdot i_1 = i_2 \cdot f$ and $g \cdot j_1 = j_2 \cdot f$ and obtains

$$\begin{aligned} j_2(f(\tilde{c}_X(m_1))) &\cong g(j_1(j_1^{-1}(c_{K_1}(i_1)))) \\ &\leq g(c_{K_1}(i_1)) \\ &\leq c_{K_2}(g(i_1)) \\ &\leq c_{K_2}(i_2) , \end{aligned}$$

hence $f(\tilde{c}_X(m_1)) \leq j_2^{-1}(c_{K_2}(i_2)) \cong \tilde{c}_Y(f(m_1))$.

If $C \leq D$ in $CL(\mathcal{X}, \mathcal{M})$, then $c_X(m) \leq d_X(m)$ trivially implies $\tilde{c}_X(m) \leq \tilde{d}_X(m)$. □

REMARKS

(1) The definition of $\tilde{c}_X(m)$ does not depend on the order in which the injections of the cokernelpair of m are used, i. e.

$$\tilde{c}_X(m) \cong i^{-1}(c_K(j)).$$

Indeed, with $s : K \to K$ the isomorphism such that $s \cdot i = j$ and $s \cdot j = i$ one has $c_K(j) \cong c_K(s \cdot i) \cong s(c_K(i))$, hence $i^{-1}(c_K(j)) \cong j^{-1}(c_K(i))$.

(2) The hypothesis $\mathcal{M} \subseteq \mathrm{Reg}(\mathcal{X})$ guarantees that for every $m \in \mathcal{M}$ diagram (6.2) is a pullback. Hence every strongly C-closed $\mathcal{M}$-closed subobject is C-closed.

We now use the hypotheses used in Frolík's Lemma in order to describe the $\tilde{C}$-closed subobjects. First we prove:

LEMMA *Under condition (B) of* 6.5, *a morphism* $f : X \to Y$ *in* $\mathcal{X}$ *with the property that every* $y \in \mathcal{P}/Y$ *factors as* $y = f \cdot x$ *must belong to* $\mathcal{E}$.

Proof Consider the $(\mathcal{E}, \mathcal{M})$-factorization $f = m \cdot e$. By hypothesis, the sink $\mathcal{P}/Y$ factors through m . From the defining property of the $\mathcal{M}$-union $1_Y = \bigvee(\mathcal{P}/Y)$ one obtains a morphism w with $m \cdot w = 1_Y$, so that m must be an isomorphism and f must belong to $\mathcal{E}$. □

THEOREM *Let* $\mathcal{M}$ *be generated by its* $\vee$*-prime elements (see* 6.5*). Then* $m \in \mathcal{M}/X$ *is* $\tilde{C}$*-closed if and only if* m *is strongly* C*-closed.*

Proof Trivially, strong C-closedness of m implies its $\tilde{C}$-closedness: $\tilde{c}_X(m) \cong j^{-1}(c_K(i)) \cong j^{-1}(i) \cong m$. Conversely, if $m \cong j^{-1}(c_K(i))$, we must show that the morphism $k : X \to c_K(X)$ with $c_K(i) \cdot k = i$ belongs to $\mathcal{E}$ and is therefore an isomorphism. For this it suffices to show that every $\vee$-prime $\mathcal{M}$-subobject y of $c_K(X)$ factors through X . But since $1_K = i \vee j$, the $\vee$-prime $\mathcal{M}$-subobject $c_K(i) \cdot y$ factors through i or j . In the latter case, one has z such that $j \cdot z = c_K(i) \cdot y$, so that the pullback property of $m \cong j^{-1}(c_K(i))$ gives an x with $k \cdot m \cdot x = y$, while in the former case y trivially factors through k . □

COROLLARY *Let* $\mathcal{M}$ *be generated by its* $\vee$*-prime elements. Then* C *is strong if and only if* C *and* $\tilde{C}$ *have isomorphic idempotent hulls. If* C *is the regular closure operator of a strongly epireflective subcategory, then* C *is strong.*

Proof According to the Theorem, every C-closed $\mathcal{M}$-subobject is $\tilde{C}$-closed. Since $C \leq \tilde{C}$, this means that C and $\tilde{C}$ have the same closed $\mathcal{M}$-subobjects; equivalently, C and $\tilde{C}$ have isomorphic idempotent hulls (see Corollary 5.4). The regular closure operator of a strongly epireflective subcategory is strong, by Corollary 6.5. □

EXAMPLES

(1) Using the explicit description of the adjunction space $X +_M X$ given in Example 6.5, one easily verifies that the Kuratowski closure operator coincides with its strong modification. It is therefore strong. (A more general result is proved in 8.4.)

(2) The strongly epireflective subcategory **AbGrp** of **Grp** provides a regular

closure operator which is not strong (see Example 6.4(2)). Hence the assumption that $\mathcal{M}$ be generated by its $\vee$-prime elements is essential for the validity of the Theorem and its Corollary. On the other hand, the assumption is by no means a necessary condition, as we shall see next.

(3) *For every closure operator* C *of* $\mathbf{Mod}_R$ *and every submodule* $M \leq X$, one has

$$M \leq X \textit{ is } \tilde{C}\textit{-closed} \Leftrightarrow \mathbf{r}(X/M) = 0 \Leftrightarrow M \leq X \textit{ is strongly } C\textit{-closed},$$

with $\mathbf{r} = \pi(C)$ *the preradical induced by* C (cf. 5.5). In fact, these equivalences follow easily once we have shown: $\tilde{C}$ *is the maximal closure operator* $C^{\mathbf{r}}$ *, and maximal closure operators are strong.* In fact, if we construct $K = X +_M X = X + (X/M)$ as in 6.2 and 6.4, with canonical injections $i = <1,0>$, $j = <1,p>$ and $p : X \to X/M$ the projection, then

$$c_K(i(X)) = c_K(X \times 0) = c_X(X) \times c_{X/M}(0) = X \times \mathbf{r}(X/M)$$

since C is finitely productive (cf. Exercise 2.J). Therefore

$$\tilde{c}_X(M) = j^{-1}(X \times \mathbf{r}(X/M)) = p^{-1}(\mathbf{r}(X/M)),$$

and for $M \leq X$ to be $C^{\mathbf{r}}$-closed means exactly $\mathbf{r}(X/M) = 0$ (cf. Prop. 3.4). Hence, if $M \leq X$ is $C^{\mathbf{r}}$-closed, one has

$$c_K(i(X)) = X \times 0 = i(X).$$

(4) We are now able to name a closure operator of $\mathbf{Mod}_R$ that fails to be strong. Simply consider any preradical $\mathbf{r}$ which fails to be a radical. (For instance, consider $\mathbf{soc}$; cf. Example 4.3(1).) Then the minimal closure operator $C = C_{\mathbf{r}}$ is not strong: although $\mathbf{r}(X)$ is C-closed in X , $i(\mathbf{r}(X))$ fails to be C-closed in K whenever $\mathbf{r}(X/\mathbf{r}(X)) \neq 0$.

6.7 Regular closure in pointed and in additive categories

In this section we provide a simple formula for the computation of the $\mathcal{A}$-regular closure in an additive category when $\mathcal{A}$ is $\mathcal{E}$-reflective. This will then lead us to a characterization of the idempotent maximal closure operators.

First we consider an arbitrary $\mathcal{M}$-complete category $\mathcal{X}$ with $\mathcal{M}$ closed under composition and revisit the adjunctions

$$C_{(-)} \dashv \pi \dashv C^{(-)} : PRAD(\mathcal{X},\mathcal{M}) \to CL(\mathcal{X},\mathcal{M})$$

of 5.5, which assign to every preradical $\mathbf{r}$ the minimal and maximal closure operators $C_{\mathbf{r}}$ and $C^{\mathbf{r}}$ and to every closure operator C the induced $\mathcal{M}$-preradical $\pi(C) = \mathbf{r}$, with $\mathbf{r}_X = c_X(o_X)$. Recall that $\mathbf{r}$ is an $\mathcal{M}$-radical iff $(\mathbf{r} : \mathbf{r}) \cong \mathbf{r}$; according to Theorem 5.5(2), this means exactly that $C^{\mathbf{r}}$ is idempotent. We call a closure operator $C \in CL(\mathcal{X},\mathcal{M})$ *radical* if $\pi(C)$ is a radical, that is, if its *maximal hull* $C^{\pi(C)}$ is idempotent. By restriction we then have adjunctions

$$C_{(-)} \dashv \pi \dashv C^{(-)} : RAD(\mathcal{X}, \mathcal{M}) \to RCL(\mathcal{X}, \mathcal{M}) \qquad (*)$$

with $RAD(\mathcal{X}, \mathcal{M})$ denoting the conglomerate of $\mathcal{M}$-radicals in $\mathcal{X}$ and $RCL(\mathcal{X}, \mathcal{M})$ the conglomerate of radical closure operators of $\mathcal{X}$.

Let us now move to the setting of 5.6 where $\mathcal{X}$ is a pointed category (so that $\mathcal{X}$ has a zero object) with kernels and cokernels; also, $\mathcal{E}$ is assumed to be a class of epimorphisms. Then we have another adjunction

$$\mathrm{coker} \dashv \ker : PREF(\mathcal{X}, \mathcal{E}) \to PRAD(\mathcal{X}, \mathcal{M}).$$

We denote by

$$REF(\mathcal{X}, \mathcal{E})$$

the subconglomerate of idempotent $\mathcal{E}$-prereflections, which we also call *$\mathcal{E}$-reflections*. In fact, by Proposition 5.1(3), every $\mathcal{E}$-reflection (R, ρ) induces the $\mathcal{E}$-reflective subcategory $\mathrm{Fix}(R, \rho)$, and trivially, every (full and replete) $\mathcal{E}$-reflective subcategory of $\mathcal{X}$ arises in this way. Hence we can think of $REF(\mathcal{X}, \mathcal{E})$ as of the conglomerate of all $\mathcal{E}$-reflective subcategories of $\mathcal{X}$, but since

$$(R, \rho) \leq (R', \rho') \Leftrightarrow \mathrm{Fix}(R', \rho') \subseteq \mathrm{Fix}\ (R, \rho)$$

we must remember that the preorder of $REF(\mathcal{X}, \mathcal{E})$ is opposite to "$\subseteq$".

For an $\mathcal{E}$-reflective subcategory $\mathcal{A} = \mathrm{Fix}(R, \rho)$ of $\mathcal{X}$, we call

$$\mathbf{r}^{\mathcal{A}} := \ker(R, \rho)$$

the *$\mathcal{A}$-regular $\mathcal{M}$-preradical* of $\mathcal{X}$. We want to show that $\mathbf{r}^{\mathcal{A}}$ is actually a radical. For that we first state:

Lemma *An $\mathcal{M}$-preradical $\mathbf{r}$ is a radical if and only if*

$$\begin{tikzcd} \mathbf{r}(X) \arrow[r, "\mathbf{r}(\rho_X)"] \arrow[d, "r_X"'] & \mathbf{r}(RX) \arrow[d, "r_{RX}"] \\ X \arrow[r, "\rho_X"] & RX \end{tikzcd} \qquad (6.8)$$

is a pullback diagram for every $X \in \mathcal{X}$ *(with* $\rho_X = \mathrm{coker}(\mathbf{r}_X)$ *). If* $\mathcal{E}$ *is stable under pullback, then* $\mathbf{r}(RX) \cong 0$ *is a necessary condition for* $\mathbf{r}$ *to be a radical; it is also sufficient whenever* $\mathbf{r} \cong \ker(\mathrm{coker}\, \mathbf{r})$.

Proof The first assertion follows immediately from the definition of radical and from Theorem 5.6. For the second assertion, if $\mathbf{r}$ is a radical, then in the pullback diagram (6.8) $\mathbf{r}(\rho_X)$ belongs to $\mathcal{E}$, hence

$$r_{RX} \cong \rho_X(r_X) \cong \rho_X \cdot r_X(1_{\mathbf{r}(X)}) \cong 0(1_{\mathbf{r}(X)}) \cong o_{RX}.$$

Viceversa, if $r_{RX} \cong o_{RX}$ and $\ker(\operatorname{coker} r_X)$, then

$$r_X \cong {\rho_X}^{-1}(o_{RX}) \cong {\rho_X}^{-1}(r_{RX})$$

(cf. diagram (5.19)). $\square$

PROPOSITION

(1) *For a full and replete $\mathcal{E}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, the $\mathcal{A}$-regular $\mathcal{M}$-preradical $\mathbf{r}^{\mathcal{A}}$ is a radical. If each $\mathcal{A}$-reflexion ρ_X satisfies* $\operatorname{coker}(\ker\rho_X) \cong \rho_X$ *, then $\mathcal{A} = \{X \in \mathcal{X} : \mathbf{r}^{\mathcal{A}}(X) \cong 0\}$.*

(2) *For an $\mathcal{M}$-radical $\mathbf{r}$ of $\mathcal{X}$* $\operatorname{coker}\mathbf{r} = (R, \rho)$ *is an $\mathcal{E}$-reflection, provided $\mathcal{E}$ is stable under pullback.*

Proof

(1) We write $r_X = \ker(\rho_X)$ and $\tilde{\rho}_X = \operatorname{coker}(r_X) : X \to \tilde{R}X$. There is then a unique morphism $p : \tilde{R}X \to RX$ with $p \cdot \tilde{\rho}_X = \rho_X$, and p factors as $p = t \cdot \rho_{\tilde{R}X}$ with $t : R(\tilde{R}X) \to RX$. Furthermore, there is $s : RX \to R(\tilde{R}X)$ with $s \cdot \rho_X = \rho_{\tilde{R}X} \cdot \tilde{\rho}_X$. One easily checks that s and t are inverse to each other, hence $\ker(\rho_{\tilde{R}X}) \cong \ker(p)$. Therefore

$$r_X \cong \ker(p \cdot \tilde{\rho}_X) \cong (\tilde{\rho}_X)^{-1}(\ker(p)) \cong (\tilde{\rho}_X)^{-1}(\ker(\rho_{\tilde{R}X})) \cong (\tilde{\rho}_X)^{-1}(r_{\tilde{R}X}),$$

so that $\mathbf{r} = \mathbf{r}^{\mathcal{A}}$ is a radical by the Lemma. Furthermore, for every $X \in \mathcal{A}$ one has ρ_X iso and therefore $\mathbf{r}^{\mathcal{A}} \cong \ker(\rho_X) \cong o_X$. Conversely, $\ker(\rho_X) \cong o_X$ implies $\operatorname{coker}(\ker(\rho_X)) \cong 1_X$, hence $\rho_X \cong 1_X$ under the given hypothesis; consequently, $X \in \mathcal{A}$.

(2) By the Lemma, for a radical $\mathbf{r}$ one has $\ker(r_{RX}) \cong o_X$, hence

$$\rho_{RX} = \operatorname{coker}(r_{RX}) \cong 1_X$$

is an isomorphism for every $X \in \mathcal{X}$.

$\square$

For $\mathcal{E}$ stable under pullback, the Proposition gives an adjunction

$$\operatorname{coker} \dashv \ker \;: REF(\mathcal{X}, \mathcal{E}) \to RAD(\mathcal{X}, \mathcal{M}) \tag{**}$$

The $\mathcal{E}$-reflective subcategories closed under this correspondence are exactly the subcategories $\mathcal{A}$ of the form

$$\mathcal{A} = \{X \in \mathcal{X} : \mathbf{r}(X) \cong 0\}$$

for some radical $\mathbf{r}$ with $\ker(\operatorname{coker}\mathbf{r}) \cong \mathbf{r}$, i. e. the *torsionfree classes* of radicals closed under the correspondence.

Getting back to our original goal of describing the regular closure, we consider the monotone function that assigns to every $\mathcal{E}$-reflective subcategory $\mathcal{A}$ its regular closure operator $\mathrm{reg}^{\mathcal{A}}$ and show for additive $\mathcal{X}$ that $\mathrm{reg}^{(-)}$ is (up to isomorphism) the composite of (*), (**).

$$\begin{tikzcd} REF(\mathcal{X},\mathcal{E}) \arrow[rr, "\mathrm{reg}^{(-)}"] \arrow[dr, "\ker"'] & & RCL(\mathcal{X},\mathcal{M}) \\ & RAD(\mathcal{X},\mathcal{M}) \arrow[ur, "C^{(-)}"'] & \end{tikzcd} \tag{6.9}$$

Since $\pi C^{(-)} = Id$, this implies $\ker \cong \pi\mathrm{reg}^{(-)}$, and the latter fact remains true even without the assumption of additivity:

THEOREM *Let $\mathcal{A}$ be a full and replete $\mathcal{E}$-reflective subcategory of an $\mathcal{M}$-complete category $\mathcal{X}$ with kernels and cokernels. Then the $\mathcal{A}$-regular closure operator $\mathrm{reg}^{\mathcal{A}}$ induces the $\mathcal{A}$-regular radical $\mathbf{r}^{\mathcal{A}} \cong \pi(\mathrm{reg}^{\mathcal{A}})$, hence $\mathrm{reg}^{\mathcal{A}} \leq C^{\mathbf{r}^{\mathcal{A}}}$. If $\mathcal{X}$ is additive, then $\mathrm{reg}^{\mathcal{A}}$ is isomorphic to the maximal closure operator of $\mathbf{r}^{\mathcal{A}}$; in particular, if the $\mathcal{A}$-reflexion $\rho : 1 \to R$ satisfies $\mathrm{coker}(\ker\rho) \cong \rho$, then $m : M \to X$ is $\mathcal{A}$-closed if and only if $X/M \cong \mathrm{Coker}(m) \in \mathcal{A}$, and m is $\mathcal{A}$-dense if and only if $R(X/M) = 0$.*

Proof In order to verify the first statement, we must show that $\ker(\rho_X)$ is the $\mathcal{A}$-regular closure of o_X for every $X \in \mathcal{X}$, with ρ_X the $\mathcal{A}$-reflexion of X . As an equalizer of $\rho_X, 0 : X \to RX$, $\ker(\rho_X)$ is $\mathcal{A}$-closed. Furthermore, every pair of morphisms $u, v : X \to A \in \mathcal{A}$ factors through ρ_X as $u = u^* \cdot \rho_X, v = v^* \cdot \rho_X$, hence $u \cdot \ker(\rho_X) = 0 = v \cdot \ker(\rho_X)$. This shows $o_X \overset{\mathcal{A}}{\Rightarrow} \ker(\rho_X)$, hence $\mathrm{reg}^X_{\mathcal{A}}(m) \cong \ker(\rho_X)$ (see Exercise 6.A(1)).

Let $\mathcal{X}$ be *additive*. Hence each hom-set has the structure of an abelian group, with the zero-morphism being its neutral element and with the composition of the category distributing over addition from either side. According to Theorem 5.6 we must show

$$\mathrm{reg}^{\mathcal{A}}_X(m) \cong \ker(\rho_{X/M} \cdot q_m) \cong {q_m}^{-1}(\ker(\rho_{X/M})),$$

with $q_m = \mathrm{coker}(m) : X \to X/M := \mathrm{Coker}(m)$, $m : M \to X$ in $\mathcal{M}$. Again as an equalizer of a pair of morphisms with codomain in $\mathcal{A}$, $k := \ker(\rho_{X/M} \cdot q_m)$ is $\mathcal{A}$-closed, with $k \geq m$. It therefore suffices to show $m \overset{\mathcal{A}}{\Rightarrow} k$ (again, see Exercise 6.A(1)). Indeed, for $u, v : X \to A$ with $A \in \mathcal{A}$ and $u \cdot m = v \cdot m$, one has $d \cdot m = 0$ for $d := u - v$, hence d factors as $t \cdot q_m = d$. But then $t : X/M \to A$ factors as $s \cdot \rho_{X/M} = t$, hence $d \cdot k = s \cdot \rho_{X/M} \cdot q_m \cdot k = s \cdot 0 = 0$, and this means $u \cdot k = v \cdot k$.

Finally, m is $\mathcal{A}$-closed if and only if m is $C^{\mathbf{r}^{\mathcal{A}}}$-closed, that is:

$$m \cong q_m^{-1}(r_{X/M}),\ \ o_{X/M} \cong q_m(m) \cong q_m(q_m^{-1}(r_{X/M})) \cong r_{X/M}.$$

According to the Proposition, this means $X/M \in \mathcal{A}$ under the hypothesis coker(ker ρ) $\cong \rho$. The argumentation for $\mathcal{A}$-denseness is similar. □

Since both $C^{(-)}$ and ker have adjoints, also their composite has a left adjoint. For additive $\mathcal{X}$ this means:

$$\mathrm{coker}\pi \dashv \mathrm{reg}^{(-)} \ : REF(\mathcal{X},\mathcal{E}) \to RCL(\mathcal{X},\mathcal{M}) \qquad (***)$$

If $\mathrm{coker} \cdot \mathrm{ker} = Id_{REF(\mathcal{X},\mathcal{E})}$, then every $\mathcal{E}$-reflection is closed under this correspondence. Furthermore,

COROLLARY *If* $\mathrm{ker} \cdot \mathrm{coker} \cong Id_{RAD(\mathcal{X},\mathcal{M})}$ *, then the regular closure operators induced by the* $\mathcal{E}$*-reflective subcategories are exactly the idempotent maximal closure operators of the additive category* $\mathcal{X}$ *.* □

REMARK In an abelian category, every monomorphism is a kernel and every epimorphism is a cokernel, hence $\mathrm{coker} \cdot \mathrm{ker} \cong Id_{REF(\mathcal{X},\mathcal{E})}$ and $\mathrm{ker} \cdot \mathrm{coker} \cong Id_{RAD(\mathcal{X},\mathcal{M})}$ holds true trivially. However, we note that these equalities are also available in the pointed but non-additive category **Grp** (cf. Example 5.6(2)). We note further that we have used additivity for the sole purpose of making sure that every $\mathcal{A}$-regular monomorphism is $\mathcal{A}$-*normal* (i. e., the kernel of a morphism with codomain in $\mathcal{A}$). Hence the assumption of additivity could be avoided, by working with $\mathcal{A}$-normal monomorphisms instead of $\mathcal{A}$-regular ones and with the $\mathcal{A}$-normal closure instead of the $\mathcal{A}$-regular one. We prefer not to do so in order not to lose the immediate contact with the epimorphism problem.

EXAMPLES

(1) The abelian category $\mathbf{Mod}_R$ satisfies all hypotheses appearing in this section. The adjunction (**) gives a bijective correspondence between epireflective subcategories of $\mathbf{Mod}_R$ and radicals of $\mathbf{Mod}_R$, representing every epireflective subcategory $\mathcal{A}$ as the torsionfree class of the $\mathcal{A}$-regular closure $\mathbf{r}^{\mathcal{A}}$. The $\mathcal{A}$-regular radical $\mathrm{reg}^{\mathcal{A}}$ is the maximal closure operator given by $\mathbf{r}^{\mathcal{A}}$.

Here is a list of radicals of **AbGrp** (= $\mathbf{Mod}_{\mathbb{Z}}$) and their corresponding torsionfree classes (cf. Examples 3.4 and 4.6).

- $\mathbf{t}$ (torsion subgroup) : torsion-free abelian groups
- $\mathbf{t}_p$ (p-torsion subgroup) : p-torsion-free abelian groups
- $\mathbf{d}$ (maximal divisible subgroup) : reduced abelian groups
- $\mathbf{d}_p$ (maximal p-divisible subgroup) : abelian groups without p-divisible subgroups except 0
- $\mathbf{f}$ (Frattini subgroup) : the subcategory cogenerated by simple cyclic abelian groups
- $\mathbf{p}$ (p-radical) : abelian groups of exponent p .

(2) The bijective correspondence between epireflective subcategories (=quasivarieties) and radicals remains true in the pointed category **Grp** , for the reason given in the Remark. (Recall that a quasivariety of groups is a full and replete subcategory of **Grp** which is closed under subobjects and direct products; it is a variety if it is also closed under homomorphic images.) The preradical $\mathbf{k}$ given by the commutator subgroup is (according to the Lemma) actually a radical, with corresponding variety **AbGrp** . Descending further along the derived series of a group, the powers $\mathbf{k}^n$ $(n \geq 1)$ give the varieties $\mathbf{S}_n\mathbf{Grp}$ of *soluble* groups of class n . With the lower central series (instead of the derived series) one produces in a similar fashion the varieties $\mathbf{N}_n\mathbf{Grp}$ of *nilpotent* groups of class n . Finally, for the *radical* $\mathbf{n}$ (for any $n \geq 1$)

$$\mathbf{n}(G) := < g^n : g \in G >$$

one obtains the *Burnside variety* $\mathbf{B}_n\mathbf{Grp}$ of groups of exponent n . (Note that the question of whether the finitely generated objects of $\mathbf{B}_n\mathbf{Grp}$ are finite is known as the Burnside Problem; it is still open for small n , i. e., for $5 \leq n \leq 665$.)

(3) If $\mathcal{X}$ is not additive, then the regular closure operator $\mathrm{reg}^{\mathcal{A}}$ of an $\mathcal{E}$-reflective subcategory $\mathcal{A}$ may be strictly smaller than the maximal closure operator of $\mathbf{r}^{\mathcal{A}}$. Again, consider $\mathcal{X} = \mathbf{Grp}$ and let $\mathcal{A} = \mathcal{X}$. Then $\mathbf{r}^{\mathcal{A}} = \mathbf{0}$, with corresponding maximal closure operator ν (cf. Example 5.6(2)). However, since every monomorphism in **Grp** is regular (cf. Example 6.1(2) and Exercise 6.B), the $\mathcal{A}$-regular closure is the discrete operator $S < \nu$.

In general, with an arbitrary epireflective category $\mathcal{A}$ of **Grp** , for the formula $\mathrm{reg}^{\mathcal{A}} = C^{\mathbf{r}^{\mathcal{A}}}$ to hold it is necessary that

$$\nu = C^{\mathbf{0}} \leq C^{\mathbf{r}^{\mathcal{A}}} = \mathrm{reg}^{\mathcal{A}};$$

equivalently, that every $\mathcal{A}$-closed subgroup of a group is normal. In fact, an easy adoption of the proof of the Theorem shows, that this condition is also sufficient. In particular, if $\mathcal{A}$ is contained in the variety **AbGrp** , so that $\mathcal{A}$-closed subgroups are necessarily normal (just check that the equalizer of two homomorphisms into an abelian group is normal in their domain), then the *$\mathcal{A}$-regular closure operator is maximal.* For example, the regular closure operator w.r.t. **AbGrp** is the maximal closure operator of the radical $\mathbf{k}$.

6.8 Clementino's Theorem

In this section we provide necessary and sufficient conditions for the $\mathcal{A}$-regular closure operator to be weakly hereditary; equivalently, for $\mathcal{A}$-closed subobjects to be closed under composition. Our category $\mathcal{X}$ is assumed to be finitely complete and $\mathcal{M}$-complete, with $\mathcal{M}$ closed under composition and containing all regular monomorphisms. Consequently, the class $\mathcal{E}$ such that $\mathcal{X}$ has $(\mathcal{E},\mathcal{M})$-factorizations is a class of epimorphisms in $\mathcal{X}$, and our crucial additional hypothesis throughout this section is that $\mathcal{E}$ is a *surjectivity class in* $\mathcal{X}$. This means that there is a class $\mathcal{P}$ of objects in $\mathcal{X}$ such that for every morphism $f : X \to Y$ in $\mathcal{X}$ one has $f \in \mathcal{E}$ if and only if the following condition holds :

$$\text{for every } y : P \to Y \text{ in } \mathcal{X} \text{ with } P \in \mathcal{P}$$

$$\text{there is an } x : P \to X \text{ in } \mathcal{X} \text{ with } f \cdot x = y\,.$$

In other words, $\mathcal{E}$ contains exactly those morphisms f for which every $P \in \mathcal{P}$ is projective with respect to f . It is easy to check that the surjectivity class $\mathcal{E}$ *is necessarily stable under pullback* (cf. Exercise 1.L, where the case of a single-object class $\mathcal{P}$ is considered).

The full replete subcategory $\mathcal{A}$ of $\mathcal{X}$ is assumed to be $\mathcal{E}$-reflective, hence closed under $\mathcal{M}$-subobjects. Recall that the $\mathcal{A}$-regular closure of $m : M \to X$ in $\mathcal{M}$ may be computed as

$$\mathrm{reg}_X^{\mathcal{A}}(m) \cong \rho_X^{-1}(\mathrm{reg}_{RX}^{\mathcal{A}}(\rho_X(m))), \tag{$*$}$$

with $\rho_X : X \to RX$ denoting the $\mathcal{A}$-reflexion of X (see Theorem 6.3). Consequently, an $\mathcal{M}$-subobject m of X is $\mathcal{A}$-closed if and only if there is an $\mathcal{A}$-closed $\mathcal{M}$-subobject n of RX with $m \cong \rho_X^{-1}(n)$ (see Corollary 6.3).

For such $m : M \to X$ and $n : N \to RX$ the pullback projection $M \to N$ factors through ρ_M by a unique $\mathcal{E}$-morphism m_ρ since N belongs to $\mathcal{A}$. Hence we have a pullback diagram

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "m"'] & RM \arrow[r, "m_\rho"] & N \arrow[d, "n"] \\
X \arrow[rr, "\rho_X"] & & RX
\end{tikzcd} \tag{6.10}$$

We call m_ρ the *ρ-defect of* m and note that m_ρ belongs to $\mathcal{E}$ (since $M \to N$ belongs to $\mathcal{E}$, as a pullback of ρ_X).

A morphism $f : X \to Y$ is said to *preserve* $\mathcal{A}$*-closedness* if for every $\mathcal{A}$-closed $\mathcal{M}$-subobject m of X also $f(m)$ is $\mathcal{A}$-closed. Corollary 6.3 implies that *each* $\mathcal{A}$*-reflexion preserves* $\mathcal{A}$*-closedness.* It turns out that preservation of $\mathcal{A}$-closedness by m_ρ is a crucial condition for $\mathrm{reg}^{\mathcal{A}}$ to be weakly hereditary.

PROPOSITION *Let the ρ-defects of $\mathcal{A}$-closed $\mathcal{M}$-subobjects be monic. Then* $\mathrm{reg}^{\mathcal{A}}$ *is weakly hereditary in* $\mathcal{X}$ *if and only if its restriction to* $\mathcal{A}$ *is weakly hereditary and the ρ-defects of $\mathcal{A}$-closed $\mathcal{M}$-subobjects preserve $\mathcal{A}$-closedness.*

Proof First we show that the given conditions are sufficient for $\mathrm{reg}^{\mathcal{A}}$ to be weakly hereditary. Hence we must show that $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A})$ is closed under composition and consider $\mathcal{A}$-closed $\mathcal{M}$-subobjects $m : M \to X$ and $m' : M' \to M$. According to Corollary 6.3, let $n : N \to RX$ be $\mathcal{A}$-closed such that (6.10) is a pullback diagram, and let $k : K \to RM$ be $\mathcal{A}$-closed such that the left square in

$$\begin{array}{ccccc} M' & \xrightarrow{\quad e \quad} & K & \xrightarrow{\quad d \quad} & N' \\ \Big\downarrow{\scriptstyle m'} & & \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle n'} \\ M & \xrightarrow{\quad \rho_M \quad} & RM & \xrightarrow{\quad m_\rho \quad} & N \end{array} \tag{6.11}$$

is a pullback diagram. In the right square, we let $n' \cong m_\rho(k)$, hence $d \in \mathcal{E}$. By hypothesis, $n \cdot n'$ is $\mathcal{A}$-closed, hence also $\rho_X^{-1}(n \cdot n')$ is $\mathcal{A}$-closed. Now it suffices to show that the unique morphism $j : M' \to L = \rho_X^{-1}(N')$ with $\rho_X^{-1}(n \cdot n') \cdot j = m \cdot m'$ and $p \cdot j = d \cdot e$ (with $p : L \to N'$ the pullback projection) belongs to $\mathcal{E}$, in order to conclude that $m \cdot m' \cong \rho_X^{-1}(n \cdot n')$ is $\mathcal{A}$-closed.

Hence we must show that every $P \in \mathcal{P}$ is projective with respect to j , and we consider a morphism $y : P \to L$. Since $d \cdot e \in \mathcal{E}$, the morphism $w = p \cdot y$ factors as $d \cdot e \cdot x = w$. The pullback property of (6.10) gives a morphism $v : L \to M$ with $m \cdot v = \rho_X^{-1}(n \cdot n')$ and $m_\rho \cdot \rho_M \cdot v = n' \cdot p$, hence

$$m_\rho \cdot \rho_M \cdot v \cdot y = n' \cdot p \cdot y = n' \cdot d \cdot e \cdot x = m_\rho \cdot k \cdot e \cdot x$$

and therefore $\rho_M \cdot v \cdot y = k \cdot e \cdot x$, by hypothesis on m_ρ . The pullback diagram

$$\begin{array}{ccc} Q & \xrightarrow{\quad c \quad} & P \\ {\scriptstyle a}\Big\downarrow & & \Big\downarrow{\scriptstyle e \cdot x} \\ M' & \xrightarrow{\quad e \quad} & K \end{array} \tag{6.12}$$

can be glued to the left part of (6.11), and from the composite pullback diagram one obtains a morphism $b : P \to Q$ with $c \cdot b = 1$ and $m' \cdot a \cdot b = v \cdot y$. Now

$$\rho_X^{-1}(n \cdot n') \cdot j \cdot a \cdot b = m \cdot m' \cdot a \cdot b = m \cdot v \cdot y = \rho_X^{-1}(n \cdot n') \cdot y,$$

hence $j \cdot (a \cdot b) = y$, as desired.

Conversely, we must prove that the given conditions are necessary. This is trivially true for the restriction of $\mathrm{reg}^{\mathcal{A}}$ to $\mathcal{A}$ to be weakly hereditary. In order to show that the defect m_ρ of $m \in \mathrm{Reg}_{\mathcal{X}}(\mathcal{A})/X$ preserves $\mathcal{A}$-closedness, we consider an $\mathcal{A}$-closed subobject $k : K \to RM$ and let $m' \cong \rho_M^{-1}(k)$ and $n' \cong m_\rho(k)$. Then m' is $\mathcal{A}$-closed, and in order to show that n' is $\mathcal{A}$-closed, it suffices to verify that $n \cdot n'$ $\mathcal{A}$-closed (see Corollary 2.3). But since $\rho_X(m \cdot m') \cong n \cdot n'$, with $m \cdot m'$ $\mathcal{A}$-closed by hypothesis, this follows since ρ_X preserves $\mathcal{A}$-closedness, as remarked before the Proposition. □

Clementino's Theorem deals with the case that $\mathcal{E}$ is the surjectivity class belonging to $\mathcal{P} = \{T\}$, with T the terminal object of $\mathcal{A}$ (so that $f \in \mathcal{E}$ holds iff

T is projective with respect to f). Then the monic condition of the Proposition turns out to be necessary as well:

THEOREM *Suppose that $\mathcal{E}$ is the surjectivity class belonging to the terminal object. Then the $\mathcal{A}$-regular closure operator of $\mathcal{X}$ is weakly hereditary if and only if its restriction to $\mathcal{A}$ is weakly hereditary and the ρ-defects of $\mathcal{A}$-closed $\mathcal{M}$-subobjects are monic and preserve $\mathcal{A}$-closedness.*

Proof We are left with having to show that the ρ-defects of $\mathcal{A}$-closed $\mathcal{M}$-subobjects are monic if $\mathrm{reg}^{\mathcal{A}}$ is weakly hereditary in $\mathcal{X}$. First we prove two auxiliary claims.

CLAIM 1 *For every pullback diagram* (6.10) *with $N = T$, m_ρ is an isomorphism.*

Proof 1 First we observe that any morphism $n : T \to RX$ is an $\mathcal{A}$-section and therefore an $\mathcal{A}$-closed $\mathcal{M}$-subobject (see the remark before Theorem 6.1), hence also m is $\mathcal{A}$-closed. Since $m_\rho \in \mathcal{E}$, the assumption on $\mathcal{E}$ gives a morphism $k : T \to RM$ with $m_\rho \cdot k = 1$. As before, k is $\mathcal{A}$-closed, and so is its pullback $m' \cong \rho_M^{-1}(k)$, hence $m \cdot m'$ is $\mathcal{A}$-closed by hypothesis. Since $\rho_X(m \cdot m') \cong n$, with Corollary 6.3 one has

$$m \cdot m' \cong \mathrm{reg}_X^{\mathcal{A}}(m \cdot m') \cong \rho_X^{-1}(n) \cong m,$$

hence m' is iso. But then k must be in $\mathcal{E}$ and therefore be iso as well, and this finally means that m_ρ must be an isomorphism.

CLAIM 2 *For $m : M \to X$ $\mathcal{A}$-closed, $m_\rho \cdot s = m_\rho \cdot t$ with $s, t : T \to RM$ implies $s = t$.*

Proof 2 According to Claim 1, the ρ-defect of $\tilde{m} \cong \rho_X^{-1}(\tilde{n})$ with $\tilde{n} := n \cdot m_\rho \cdot s = n \cdot m_\rho \cdot t$ is an isomorphism. With $j : \tilde{M} \to M$ arising from $\tilde{m} \leq m$, the morphism $\rho_M \cdot j : \tilde{M} \to RM$ therefore factors through $\rho_{\tilde{M}}$, as $a \cdot \rho_{\tilde{M}} = \rho_M \cdot j$. One now concludes

$$s \cong \rho_M((\rho_M^{-1}(s)) \leq \rho_M(j) \cong a,$$

$$t \cong \rho_M((\rho_M^{-1}(t)) \leq \rho_M(j) \cong a,$$

which implies $s = a = t$ as claimed, since T is a terminal object.

If we finally consider arbitrary morphisms $u, v : U \to RM$ with $m_\rho \cdot u = m_\rho \cdot v$, then the equalizer $w : W \to U$ of (u, v) must belong to $\mathcal{E}$ and therefore be an isomorphism: indeed, for every $z : T \to U$ one has $uz = vz$ according to Claim 2, so that z factors through w . □

REMARKS

(1) The assumption of finite completeness of $\mathcal{X}$, in addition to $\mathcal{M}$-completeness, is not used to its fullest, neither in the Proposition nor in the Theorem. In the

Proposition we used it only when forming the pullback (6.11), and even that can be avoided when the arrow $e \cdot x$ is in $\mathcal{M}$, as is the case when $\mathcal{P}$ contains the terminal object only, i. e., under the hypothesis of the Theorem. In the Theorem, we use the terminal object and equalizers.

(2) Note that in Claim 1 of the Proof of the Theorem, we have shown (under the assumption that $\mathcal{E}$ be the surjectivity class of $\{T\}$ and that $\mathrm{reg}^{\mathcal{A}}$ be weakly hereditary) that for the pullback $m = \rho_X^{-1}(n) : M \to X$ of any $m : T \to RX$, the $\mathcal{A}$-reflexion of M is (isomorphic) to $M \to T$. We say that *fibres of $\mathcal{A}$-reflexions have trivial $\mathcal{A}$-reflexions* in this case. Since in the proof of the Theorem we showed that ρ-defects are monic based solely on the validity of Claim 1, we finally obtain Clementino's Theorem [1993] as originally stated.

COROLLARY *Under the assumptions of the Theorem, the $\mathcal{A}$-regular closure operator of $\mathcal{X}$ is weakly hereditary if and only if*

(1) *its restriction to $\mathcal{A}$ is weakly hereditary,*

(2) *fibres of $\mathcal{A}$-reflexions have trivial $\mathcal{A}$-reflexions,*

(3) *ρ-defects of $\mathcal{A}$-closed subobjects preserve $\mathcal{A}$-closedness.*

□

The hypotheses of the Theorem are tailored for applications in Topology which we discuss in the next section. However, the Proposition as well as parts of the proof of the Theorem remain valid in algebraic categories, as we shall see next.

EXAMPLES

(1) We have shown in 6.7 that, for $\mathcal{X} = \mathbf{Mod}_S$, one has a bijective correspondence between epireflective subcategories and radicals, given by $\mathcal{A} \mapsto \mathbf{r}^{\mathcal{A}}$, and that the $\mathcal{A}$-regular closure is just the maximal closure operator of $\mathbf{r}^{\mathcal{A}}$. According to Theorem 3.4 then, $\mathrm{reg}^{\mathcal{A}}$ is weakly hereditary if and only if $\mathbf{r}^{\mathcal{A}}$ is idempotent. Hence full and replete *epireflective subcategories with weakly hereditary regular closure operator correspond bijectively to idempotent radicals*. Such radicals are also called *torsion theories*, and the corresponding epireflective subcategories are precisely the torsionfree classes. For example, the radicals $\mathbf{t}$ and $\mathbf{d}$ are torsion theories of $\mathbf{AbGrp} = \mathbf{Mod}_{\mathbb{Z}}$, but $\mathbf{f}$ is not (see Examples 3.4).

Let us now compare these consequences of 6.7 with the results presented in this section. First of all, $\mathcal{E} = \mathrm{Epi}(\mathbf{Mod}_S)$ is in fact a surjectivity class, since S is (free and therefore) projective in $\mathbf{Mod}_S$, and $\mathcal{E}$ is simply the surjectivity class belonging to $\mathcal{P} = \{S\}$. Hence the general assumption of this section is satisfied, although the hypothesis of the Theorem is not. However, an adaption of its proof allows us to show that *the following conditions are equivalent for a full and replete epireflective subcategory $\mathcal{A}$ with $\mathcal{A}$-reflexions $\rho_X : X \to RX$ and induced radical* $\mathbf{r} = \mathbf{r}^{\mathcal{A}}$:

(i) *the regular closure operator* $\mathrm{reg}^{\mathcal{A}}$ *is weakly hereditary,*

(ii) $\mathbf{r}$ *is idempotent,*

(iii) $R(\mathbf{r}(X)) = 0$ *for every* S*-module* X *,*

(iv) *the restriction* $\rho_X^{-1}(N) \to N$ *of* ρ_X *is an* $\mathcal{A}$*-reflexion of* $\rho_X^{-1}(N)$ *for every submodule* $N \leq RX$ *,*

(v) $\mathcal{A}$ *is closed under extensions, that is: for every* $M \leq X$ *, if* M *and* $X/M \in \mathcal{A}$ *, then* $X \in \mathcal{A}$ *.*

In fact, we already know that (i) is equivalent to (ii), and (iii) is simply a reformulation of (ii). For (iii) $\Leftrightarrow$ (iv), first note that (iii) means that the ρ-defect of $\mathbf{r}(X) \to X$ is the map $0 \to 0$, while (iv) means that the ρ-defect of $M := \rho_X^{-1}(N)$ is an isomorphism for every $N \leq RX$ (not only when N is $\mathcal{A}$-closed !). Hence (iv) trivially implies (iii) while (iii) $\Rightarrow$ (iv) can be shown analogously to Claim 2 in the proof of the Theorem (although the hypotheses of the Theorem are not satisfied here), as follows. In the notation of diagram (6.10), since $m_\rho \in \mathcal{E}$, it suffices to show that $\ker(m_\rho) = 0$. But this is obvious in the presence of (iii): since (6.10) is a pullback, we have $\mathbf{r}(X) \leq M$, and since $\mathbf{r}(X) \to 0$ is an $\mathcal{A}$-reflexion, $\rho_M|_{\mathbf{r}(X)} = 0$, hence $\ker(\rho_X) = \mathbf{r}(X) \leq \ker(\rho_M) = \mathbf{r}(M)$; therefore, $\ker(m_\rho \cdot \rho_M) = \ker(\rho_X) = \ker(\rho_M)$, which implies $\ker(m_\rho) = 0$. To prove the equivalence of (v) with the other conditions is left as Exercise 6.P.

In summary we see that for $\mathcal{X} = \mathbf{Mod}_S$, the Corollary remains valid, but that of the necessary and sufficient conditions (1)-(3) given, (2) is the same as (iii) above while (1) and (3) have become redundant.

(2) Quite surprisingly, conditions (i)-(v) of (1) remain equivalent in case $\mathcal{X} = \mathbf{Grp}$, for any full and replete epireflective subcategory $\mathcal{A}$. In addition, they imply

(vi) *the restriction of* $\mathrm{reg}^{\mathcal{A}}$ *to* $\mathcal{A}$ *is discrete.*

The proof of (v) $\Rightarrow$ (vi) uses the powerful tool of *Kurosh's Subgroup Theorem*; for a complete proof we must refer the reader to Fay [1995]. The proof that conditions (ii)-(v) of (1) remain equivalent can proceed as in the module case, as well as implication (i) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (i) one argues as follows. Since (ii) implies (iv), so that ρ-defects of $\mathcal{A}$-closed subobjects are monic, the Proposition gives (i) since the discrete (!) restriction of $\mathrm{reg}^{\mathcal{A}}$ to $\mathcal{A}$ is weakly hereditary, and since ρ-defects of $\mathcal{A}$-closed subobjects preserve $\mathcal{A}$-closedness, by (iv).

As in the case of modules, the Corollary remains valid for $\mathcal{X} = \mathbf{Grp}$, but with conditions (1) and (3) having become redundant. However, even for "good" subcategories of $\mathcal{A}$, the $\mathcal{A}$-regular closure operator may fail to be weakly hereditary . For example, for $\mathcal{A} = \mathbf{AbGrp}$ one has $\mathrm{reg}^{\mathcal{A}} = C^{\mathbf{k}}$ (see Example 6.7 (3)). Then every subgroup of an abelian group is $\mathcal{A}$-closed, hence the restriction of $\mathrm{reg}^{\mathcal{A}}$ to $\mathcal{A}$ is discrete and therefore trivially weakly hereditary (more generally, see Exercise 6.Q). However, since $\mathbf{k}$ is not idempotent, $\mathrm{reg}^{\mathcal{A}}$ is not weakly hereditary in $\mathbf{Grp}$.

6.9 Regular closure for topological spaces

In this section we wish to apply Clementino's Theorem to the category **Top** and determine which strongly epireflective subcategories $\mathcal{A}$ have the property that every subobject of an object in $\mathcal{A}$ is $\mathcal{A}$-closed, so that then every epimorphism in $\mathcal{A}$ must be surjective. We show that this never happens for $\mathcal{A} \subseteq$ **Haus** (unless every space in $\mathcal{A}$ has at most one point). Nevertheless, "outside **Haus** " this property is quite frequent (with the prominent exception of $\mathcal{A} = \mathbf{Top}_0$, see Example 6.5(3)), and it allows for a perfect characterization in terms of weak hereditariness and of full additivity of the $\mathcal{A}$-regular closure operator.

First let $\mathcal{A}$ be any full replete subcategory of **Top** . If $\mathcal{A}$ is *non-trivial*, that is, if $\mathcal{A}$ contains a space A with at least two points, then $\mathrm{reg}^{\mathcal{A}}$ is grounded. (In fact, any two (constant) maps $X \to A$ must agree on $\mathrm{reg}^{\mathcal{A}}_X(\emptyset)$ since they agree on $\emptyset$, which is possible only if $\mathrm{reg}^{\mathcal{A}}_X(\emptyset) = \emptyset$.) On the other hand, if $\mathcal{A}$ does not contain a two-point space (including the case $\mathcal{A} = \emptyset$), then its regular closure is the only non-grounded closure operator of **Top** , namely the trivial operator T (cf. Exercise 2.H). Hence, $\mathcal{A}$ *is non-trivial if and only if its regular closure operator is non-trivial.* In what follows we always assume $\mathcal{A}$ to be non-trivial.

LEMMA *For any space X and the conditions below, one has (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), while all are equivalent if $\mathcal{A}$ is strongly epireflective.*

(i) $X \in \mathcal{A}$;

(ii) *for all* $x \in X$, $\{x\}$ *is $\mathcal{A}$-closed in* X ;

(iii) *the fully additive core of* $\mathrm{reg}^{\mathcal{A}}$ *is discrete on* X , *that is:* $(\mathrm{reg}^{\mathcal{A}})^{\oplus}_X = s_X$;

(iv) *(if $\mathcal{A}$ is reflective) the $\mathcal{A}$-reflexion of X is monic.*

Proof (i) $\Rightarrow$ (ii) For every $x \in X$, $\{x\}$ is the equalizer of the identity map on X and the map constant x . (ii) $\Rightarrow$ (iii) For $\emptyset \neq M \subseteq X$, one has

$$(\mathrm{reg}^{\mathcal{A}})^{\oplus}_X(M) = \bigcup_{x \in X} \mathrm{reg}^{\mathcal{A}}_X(\{x\}) = M$$

(see 4.9), hence (iii) holds since $\mathrm{reg}^{\mathcal{A}}$ is non-trivial. (iii) $\Rightarrow$ (iv) From Theorem 6.3 one has the formula

$$\mathrm{reg}^{\mathcal{A}}_X(\{x\}) = \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(\rho_X(\{x\}))),$$

and (iii) implies

$$\mathrm{reg}^{\mathcal{A}}_X(\{x\}) = (\mathrm{reg}^{\mathcal{A}})^{\oplus}_X(\{x\}) = \{x\},$$

which then gives the injectivity of the map ρ_X . (iv) $\Rightarrow$ (i) As a monic strong epimorphism, ρ_X is an isomorphism. $\square$

PROPOSITION *Every subspace embedding in an epireflective subcategory $\mathcal{A}$ of* **Top** *is a regular monomorphism of the category $\mathcal{A}$ if and only if the $\mathcal{A}$-*

regular closure operator of **Top** *is fully additive. In this case, epimorphisms of the category* $\mathcal{A}$ *are surjective.*

Proof To say that all subobjects of objects in $\mathcal{A}$ are regular monomorphisms of $\mathcal{A}$, hence $\mathcal{A}$-closed, is the same as to say that $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ is discrete. In this case, formula (*) of 6.3 gives

$$\mathrm{reg}^{\mathcal{A}}_X(M) = \rho_X^{-1}(\mathrm{reg}^{\mathcal{A}}_{RX}(\rho_X(M))) = \rho_X^{-1}(\rho_X(M)),$$

which shows that $\mathrm{reg}^{\mathcal{A}}$ must be fully additive. Conversely, if $\mathrm{reg}^{\mathcal{A}} = (\mathrm{reg}^{\mathcal{A}})^{\oplus}$, implication (i) $\Rightarrow$ (iii) of the Lemma gives that $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ is discrete. For the additional statement see Corollary 6.1. □

We remark that the Lemma and the Proposition remain valid for any topological category over **Set** (see Exercise 5.P) such that constant maps are morphisms in the category. Hence **Top** may be traded for **PrSet**, **FC**, **Gph**, **SGph**, **Unif**, etc. The following Theorem, however, makes essential use of the specific structure of **Top**.

THEOREM *The regular closure operator of a strongly epireflective subcategory* $\mathcal{A}$ *of* **Top** *with* $\mathcal{A} \subseteq \mathbf{Top}_1$ *is weakly hereditary if and only if every subspace embedding in* $\mathcal{A}$ *is* $\mathcal{A}$*-closed and fibres of* $\mathcal{A}$*-reflexions have trivial* $\mathcal{A}$*-reflexions. In this case the* $\mathcal{A}$*-regular closure operator is fully additive.*

Proof *"if"* If $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ is discrete, then it is trivially weakly hereditary. Furthermore, the ρ-defects, as maps in $\mathcal{A}$, trivially preserve $\mathcal{A}$-closedness in this case. Hence, if fibres of $\mathcal{A}$-reflexions have trivial $\mathcal{A}$-reflexions, Corollary 6.8 gives weak hereditariness of $\mathrm{reg}^{\mathcal{A}}$.

"only if" Let $\mathrm{reg}^{\mathcal{A}}$ be weakly hereditary. By Corollary 6.8, fibres of $\mathcal{A}$-reflexions have trivial $\mathcal{A}$-reflexions. We must show that $\mathrm{reg}^{\mathcal{A}}$ is discrete and consider $\emptyset \neq M \subseteq X \in \mathcal{A}$. In order to see that M is $\mathcal{A}$-closed in X, by Frolík's Lemma we must actually show that $X +_M X$ belongs to $\mathcal{A}$, and for that it suffices to show that every point in $X +_M X$ is $\mathcal{A}$-closed, according to the Lemma.

Indeed, for every $x \in X$, $\{x\}$ is $\mathcal{A}$-closed in X, hence also its fibre $\varepsilon^{-1}x$ (along the canonical map $\varepsilon : X +_M X \to X$) in $X +_M X$ is $\mathcal{A}$-closed. But since $\mathcal{A} \subseteq \mathbf{Top}_1$, the subspace $\varepsilon^{-1}x = \{c(x,1), c(x,2)\}$ of the T_1-space $X +_M X$ is discrete and therefore belongs to $\mathcal{A}$. (Note that the non-trivial subcategory $\mathcal{A}$ contains a space A with at least two points; hence the 2-point discrete space is a subspace of A and belongs to $\mathcal{A}$.) By the Lemma, $\{c(x,1)\}$ is $\mathcal{A}$-closed in $\varepsilon^{-1}x$, hence $\mathcal{A}$-closed in $X +_M X$, since $\mathcal{A}$-closedness is transitive when $\mathrm{reg}^{\mathcal{A}}$ is weakly hereditary.

By the Proposition, the condition that subspace-embeddings be $\mathcal{A}$-closed may equivalently be replaced by the condition that $\mathrm{reg}^{\mathcal{A}}$ be fully additive. □

COROLLARY *For a (non-trivial) strongly epireflective subcategory* $\mathcal{A}$ *of* **Top**, *one has* $\mathcal{A} \subseteq$ **Haus** *if and only if every* $\mathcal{A}$*-closed embedding in* $\mathcal{A}$ *is Kuratowski-closed. In this case the* $\mathcal{A}$*-regular closure operator fails to be weakly hereditary.*

Proof To say that every $\mathcal{A}$-closed subobject embedding in $\mathcal{A}$ is Kuratowski-closed means equivalently $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} \geq K|_{\mathcal{A}}$ (which is equivalent to $\mathrm{reg}^{\mathcal{A}} \geq K$, by Theorem 6.3; see Exercise 6.G). Under this condition, for every $X \in \mathcal{A}$, the diagonal $\delta_X : X \to X \times X$ is $\mathcal{A}$-closed (see Theorem 6.1(iii)), hence (Kuratowski-) closed, so that X must be Hausdorff. Conversely, $\mathcal{A} \subseteq$ **Haus** implies $\mathrm{reg}^{\mathcal{A}} \geq \mathrm{reg}^{\mathbf{Haus}} \geq \mathrm{epi}^{\mathbf{Haus}} = K$ on $\mathcal{A}$ (see Example 6.5(2)). If we would ask $\mathrm{reg}^{\mathcal{A}}$ to be weakly hereditary, by the Theorem then every space $X \in \mathcal{A}$ would have to be discrete. But since $\mathcal{A}$ contains a space with at least two points and all its infinite products, this would lead to a contradiction. □

Remarks

(1) The assumption $\mathcal{A} \subseteq \mathbf{Top}_1$ in the Theorem can be relaxed to $\mathcal{A} \neq \mathbf{Top}_0$. In other words, necessity and sufficiency of the given conditions for weak hereditariness of $\mathrm{reg}^{\mathcal{A}}$ remain valid for every strongly epireflective $\mathcal{A}$, with the only exception of $\mathcal{A} = \mathbf{Top}_0$ (see Example (2) below). In fact, the statement of the Theorem is trivially true for $\mathcal{A} = \mathbf{Top}$ (in which case $\mathrm{reg}^{\mathcal{A}} = S$) and also for $\mathcal{A} = \mathcal{T} = \{X : |X| \leq 1\}$ (see Example (1) below). Hence we only need to show:

Every strongly epireflective proper subcategory $\mathcal{A}$ *of* **Top** *different from* $\mathbf{Top}_0$ *is contained in* $\mathbf{Top}_1$.

Proof If $\mathcal{A}$ contains a non-T_0 space, then that space has an indiscrete 2-element subspace I_2 in $\mathcal{A}$, and $\mathcal{A}$ must be **Top** (since every space allows for a continuous injective map to a power of I_2). Hence we can assume that we have proper inclusions $\mathcal{T} \subset \mathcal{A} \subset \mathbf{Top}_0$. Since every T_0-space is homeomorphic to a subspace of a power of the Sierpiński dyad, we conclude that 2-element spaces in $\mathcal{A}$ must be discrete. But then every space in $\mathcal{A}$ must be T_1 (since T_1-separation can be detected on 2-element subspaces). □

(2) The Corollary remains valid for every (not necessarily strongly) epireflective subcategory $\mathcal{A}$ of **Top**. In fact, in 7.1 we shall state (the easy fact) that $\mathcal{A}$and its strongly epireflective hull $\mathcal{B}$ in **Top** have the same regular closure. Since **Haus** is strongly epireflective, one may therefore just apply the Corollary to $\mathcal{B}$ and obtain it for $\mathcal{A}$ as well.

(3) The Corollary indicates a certain scarcity of weakly hereditary regular closure operators. Nevertheless, it is quite easy to produce for every cardinal number α a distinct strongly epireflective subcategory $\mathcal{H}_\alpha$ with $\mathbf{Haus} \subseteq \mathcal{H}_\alpha \subseteq \mathbf{Top}_1$, and with weakly hereditary regular closure operator: see Example (5) below.

Examples

(1) As mentioned at the beginning of this section, the trivial closure operator T is the regular closure operator of the strongly epireflective subcategory $\mathcal{T} = \{X :$

$|X| \leq 1\}$ of **Top** . For any other strongly epireflective subcategory $\mathcal{A}$ we have $\mathrm{reg}^{\mathcal{A}} \leq Q$ (the largest proper closure operator of **Top** ; see Theorem 4.7). Indeed, if we had $\mathrm{reg}^{\mathcal{A}} = G$ (the indiscrete closure operator), since G is fully additive, from the Lemma we would have $\mathcal{A} = \{X : g_X = s_X\} \subseteq \mathcal{T}$. The operator Q is the regular closure operator of the least strongly epireflective subcategory which contains a two-point space (this is the category of spaces with trivial quasi-components), as well as of the least epireflective subcategory which contains a two-point discrete space (this is the smaller category **0-Top** of zero-dimensional spaces, see Exercise 6.R).

(2) The regular closure operator of $\mathbf{Top}_0$ in **Top** is the b-closure. We already showed in Example 6.5 (3) that its restriction to $\mathbf{Top}_0$ coincides with b . In light of Theorem 6.3 it suffices to show the formula

$$b_X(M) = \rho_X^{-1}(b_{RX}(\rho_X(M))), \tag{*}$$

in order to conclude the validity of $b = \mathrm{reg}^{\mathbf{Top}_0}$ for the *whole* category **Top** . For this recall that the $\mathbf{Top}_0$-reflexion $\rho_X : X \to RX = X/\sim$ of a space X is given by the equivalence relation ($x \sim y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}$) on X . Hence both closed sets and open sets are $\sim$-saturated. Consequently, the map ρ_X is both open and closed. Assume that $x \in X \setminus b_X(M)$. Then there exists an open neighbourhood U of x such that for the closed set $A = \overline{\{x\}}$, $U \cap A \cap M = \emptyset$. By the above-mentioned properties of ρ_X , the set $\rho_X(U)$ is an open neighbourhood of $y = \rho_X(x)$, and $\rho_X(A)$ is closed in RX , hence it is the closure of $\{y\}$. Moreover, by the $\sim$-saturatedness of U and A we have $\rho_X(U) \cap \rho_X(A) \cap \rho_X(M) = \emptyset$. This shows that $y \notin b_{RX}(\rho_X(M))$ and consequently the inclusion "$\supseteq$" in (*). The other inclusion follows from the continuity property applied to b.

As a hereditary closure operator, b is in particular weakly hereditary. Also, fibres of $\mathbf{Top}_0$-reflexions have trivial $\mathbf{Top}_0$-reflexion, but not every subspace of a T_0-space is b-closed. Hence the assertion of the Theorem is not valid for $\mathcal{A} = \mathbf{Top}_0$.

(3) The regular closure operator of **Haus** in **Top** coincides with the Kuratowski closure K on **Haus** (cf. Example 6.5 (2)). But since K is (weakly) hereditary, according to the Corollary, the **Haus**-regular closure cannot coincide with K on the whole category **Top** . In fact, K is not *regular*: if we had $K = \mathrm{reg}^{\mathcal{A}}$, we could assume $\mathcal{A}$ to be strongly epireflective, with $\mathcal{A} \subseteq \mathbf{Top}_1$ (by Remark (1), since $\mathrm{reg}^{\mathbf{Top}_0} = b$ and $\mathrm{reg}^{\mathbf{Top}} = S$); but then K would have to be fully additive, by the Theorem.

(4) The regular closure operator of the category **Tych** of Tychonoff spaces in **Top** and the regular closure operator of the category **FHaus** of functionally Hausdorff spaces in **Top** (spaces in which every pair of distinct points can be separated by real-valued continuous functions) coincide with the *zero operator* Z defined by

$$z_X(M) = \bigcap\{f^{-1}(0) : f : X \to [0,1] \text{ continuous }, f|_M = 0\}.$$

To prove $\mathrm{reg}^{\mathbf{Tych}} = Z$, it suffices, as in (2), to show that $\mathrm{reg}^{\mathbf{Tych}}$ coincides with Z on **Tych** and the counterpart of (*) for Z and the **Tych**-reflexion

$\rho_X : X \to RX$. For the first property note that both Z and $\mathrm{reg}^{\mathbf{Tych}}$ coincide with K on **Tych** (see Exercise 6.T for the latter coincidence). A proof for $\mathrm{reg}^{\mathbf{FHaus}} = Z$ is given in Example 7.5(1).

(5) (Hoffmann's [1979] categories $\mathcal{H}_\alpha$) For an infinite cardinal α denote by X_α the (unique up to homeomorphism) space of cardinality α having the cofinite topology. Then the full subcategory $\mathcal{H}_\alpha$ of **Top** consisting of spaces Y such that every continuous map $f : X_\alpha \to Y$ is constant, is strongly epireflective and satisfies $\mathbf{Haus} \subseteq \mathcal{H}_\alpha \subseteq \mathbf{Top}_1$. Moreover, for $\alpha < \beta$, $\mathcal{H}_\alpha$ is properly contained in $\mathcal{H}_\beta$ since $X_\alpha \in \mathcal{H}_\beta$ but $X_\alpha \notin \mathcal{H}_\alpha$. Since fibres of $\mathcal{H}_\alpha$-reflexions have trivial $\mathcal{H}_\alpha$-reflexion, and since every subspace embedding in $\mathcal{H}_\alpha$ is $\mathcal{H}_\alpha$-closed (see Exercise 6.T), the Theorem yields that $\mathrm{reg}^{\mathcal{H}_\alpha}$ is weakly hereditary.

6.10 Pointed topological spaces

In this section we show how the results of 6.7 when applied to the pointed category $\mathbf{Top}_*$ yield useful results also for the non-pointed category **Top** . In particular, we obtain a characterization of epiprereflections of **Top** in terms of closure operators.

The category $\mathbf{Top}_*$ of pointed topological spaces is provided with its natural embedding-subobject structure $\mathcal{M}_*$, hence $\mathcal{M}_* = U^{-1}\mathcal{M}$, with the forgetful functor $U : \mathbf{Top}_* \to \mathbf{Top}$, $(X,x) \mapsto X$, and with $\mathcal{M}$ the class of embeddings in **Top** . The functor U has a left adjoint F which adds to a space X a new discrete point; we shall write

$$|(X,x)| = U(X,x) \text{ and } X_* = FX = (X + \{*\}, *).$$

According to Theorem 5.13, this leads to an adjunction

$$|-| \dashv (-)^* : CL(\mathbf{Top}_*, \mathcal{M}_*) \to CL(\mathbf{Top}, \mathcal{M}),$$

which may be described as follows: for $C \in CL(\mathbf{Top}, \mathcal{M})$, $|C|$ is defined by

$$|c|_{(X,x)}(M,x) := (c_X(M), x), \tag{$*$}$$

and for $D \in CL(\mathbf{Top}_*, \mathcal{M}_*)$, D^* is defined by

$$d^*_X(M) := |d_{X_*}(M_*)| \cap X. \tag{$**$}$$

(In the terminology of 5.13, one has $|C| = {}^{\varepsilon}C$ and $D^* = D^{\eta}$.) Somewhat surprisingly, the map $|-|$ not only has a right but also a left adjoint; for $D \in CL(\mathbf{Top}_*, \mathcal{M}_*)$, define $\overline{D}$ by

$$\overline{d}_X(M) := \bigcup_{x \in M} |d_{(X,x)}(M,x)|.$$

PROPOSITION *For $C \in CL(\mathbf{Top}, \mathcal{M})$ and $D \in CL(\mathbf{Top}_*, \mathcal{M}_*)$, the formulas above define closure operators $|C| \in CL(\mathbf{Top}_*, \mathcal{M}_*)$ and $D^*, \overline{D} \in CL(\mathbf{Top}, \mathcal{M})$, respectively, with*

$$\overline{|C|} \le C \le |C|^* \text{ and } |D^*| \le D \le |\overline{D}|;$$

in fact, $\overline{|C|}$, C , $|C|^*$ *coincide on non-empty subspaces.*

Proof The inequalities not involving $\overline{(\ )}$ follow from Theorem 5.13. To show that $\overline{D}$ as defined above is a closure operator of **Top** is an easy exercise. For every $M \subseteq X \in \mathbf{Top}$ and $C \in CL(\mathbf{Top}, \mathcal{M})$ one has

$$\overline{|c|}_X(M) = \bigcup_{x \in M} |(c_X(M), x)| \subseteq c_X(M),$$

with the inclusion being proper only for $C = T$ and $M = \emptyset$. Furthermore, for $M \neq \emptyset$, one has

$$c_{|X_*|}(|M_*|) = c_X(M) \cup \{*\}$$

(just exploit the continuity condition for $f : X + \{*\} \to X$ with $f|_X = id_X$ and $f(*) = a$, for some $a \in M$). This gives

$$|c|^*_X(M) = c_{|X_*|}(|M_*|) \cap X = c_X(M),$$

while the last equality sign must be relaxed to "$\supseteq$" in case $M = \emptyset$. Similarly one shows $|D^*| \leq D$ for $D \in CL(\mathbf{Top}_*, \mathcal{M}_*)$. Finally, for $(M, x_0) \subseteq (X, x_0) \in \mathbf{Top}_*$ one has

$$|\overline{d}|_{(X,x_0)}(M, x_0) = (\overline{d}_X(M), x_0) = \bigcup_{x \in M} (|d_{(X,x)}(M, x)|, x_0),$$

which contains the pointed set $d_{(X,x_0)}(M, x_0)$ since $x_0 \in M$. □

In complete analogy to the adjunctions

(i) $\overline{(\ )} \dashv |-| \dashv (-)^* \quad : CL(\mathbf{Top}_*, \mathcal{M}_*) \longrightarrow CL(\mathbf{Top}, \mathcal{M})$

we can define adjunctions

(ii) $\overline{(\ )} \dashv |-| \dashv (-)^* \ : \ PREF(\mathbf{Top}_*, \overline{\mathcal{E}}_*) \longrightarrow PREF(\mathbf{Top}, \overline{\mathcal{E}}),$

with $\overline{\mathcal{E}}$, $\overline{\mathcal{E}}_*$ denoting the class of quotient maps (=strong epimorphisms) in the categories **Top**, $\mathbf{Top}_*$ respectively. For $(R, \rho) \in PREF(\mathbf{Top}, \overline{\mathcal{E}})$ one defines $(|R|, |\rho|) \in PREF(\mathbf{Top}_*, \overline{\mathcal{E}}_*)$ by

$$|\rho|_{(X,x_0)} := \rho_X : (X, x_0) \to |R|(X, x_0) := (RX, \rho_X(x_0)).$$

For $(S, \sigma) \in PREF(\mathbf{Top}_*, \overline{\mathcal{E}}_*)$, one defines $(S^*, \sigma^*) \in PREF(\mathbf{Top}, \mathcal{E})$ as a family of quotient maps

$$\sigma^*_X := \sigma_{X_*}|_X : X \to S^*X := \sigma_{X_*}(X) \subseteq SX_*;$$

note that S^*X is not required to be a subspace of SX_* . Finally, $(\overline{S}, \overline{\sigma})$ is, for every $X \in \mathbf{Top}$, defined by the *multiple pushout* diagram

$$\begin{tikzcd} X \arrow[rr, "\overline{\sigma}X"] \arrow[dr, "\sigma_{(X,x)}"'] & & \overline{S}X \\ & |S(X,x)| \arrow[ur, "p_x\ (x \in X)"'] & \end{tikzcd} \tag{6.13}$$

Hence $\overline{S}X \cong X/\sim$, with $\sim$ the equivalence relation such that $a \sim b$ for every pair $a, b \in X$ for which $\sigma_{(X,x)}(a) = \sigma_{(X,x)}(b)$ for some $x \in X$. The proof of the following Proposition* must be left as an exercise:

PROPOSITION* *For* $(R,\rho) \in PREF(\mathbf{Top}, \overline{\mathcal{E}})$ *and* $(S,\sigma) \in PREF(\mathbf{Top}_*, \overline{\mathcal{E}}_*)$ *one has*

$$(\overline{|R|}, \overline{|\rho|}) \cong (R,\rho) \leq (|R|^*, |\rho|^*) \text{ and } (|S|^*, |\sigma|^*) \leq (S,\sigma) \leq (\overline{|S|}, \overline{|\sigma|}).$$

Furthermore,

$$\mathrm{Fix}(|R|, |\rho|) = \{(X, x_0) \in \mathbf{Top}_* : X \in \mathrm{Fix}(R,\rho)\},$$

$$\mathrm{Fix}(\overline{S}, \overline{\sigma}) = \{X \in \mathbf{Top} : (\forall x \in X)(X,x) \in \mathrm{Fix}(S,\sigma)\},$$

and $(|R|, |\rho|)$ *is an* $\overline{\mathcal{E}}_*$*-reflection if* (R,ρ) *is an* $\overline{\mathcal{E}}$*-reflection.*

Consequently, for an $\overline{\mathcal{E}}$-(pre)reflection (R,ρ) and $\mathcal{A} := \mathrm{Fix}(R,\rho)$, $|\mathcal{A}| = \{(X, x_0) \in \mathbf{Top}_* : X \in \mathcal{A}\}$ is the subcategory belonging to the $\overline{\mathcal{E}}_*$-(pre)reflection $(|R|, |\rho|)$. For the regular closures belonging to $\mathcal{A}$ and $|\mathcal{A}|$ one can prove:

LEMMA $\mathrm{reg}^{|\mathcal{A}|}_{(X,x_0)}(M, x_0) = (\mathrm{reg}^{\mathcal{A}}_X(M), x_0)$.

Proof Since $x_0 \in M$, with $i, j : X \to X +_M X$ the cokernel pair of $M \subseteq X$ in $\mathbf{Top}$ and $\overline{x}_0 = i(x_0) = j(x_0)$,

$$i, j : (X, x_0) \to (X +_M X, \overline{x}_0)$$

is the cokernel pair of $(M, x_0) \subseteq (X, x_0)$ in $\mathbf{Top}_*$. Hence the formula of the Lemma follows from the formula given in Remark 6.2(2). □

We call a closure operator $C \in CL(\mathbf{Top}, \mathcal{M})$ *pointedly radical* if $|C| \in CL(\mathbf{Top}_*, \mathcal{M}_*)$ is radical and denote by $R^*CL(\mathbf{Top}, \mathcal{M})$ the conglomerate of all pointedly radical closure operators of $\mathbf{Top}$. Now the Lemma gives the following commutative diagram:

$$\begin{tikzcd}
(P)REF(\mathbf{Top}_*,\overline{\mathcal{E}}) \arrow[r, "\mathrm{reg}_*^{(-)}"] & (R)CL(\mathbf{Top}_*,\mathcal{M}_*) \\
(P)REF(\mathbf{Top},\overline{\mathcal{E}}) \arrow[u, "|-|"] \arrow[r, "\mathrm{reg}^{(-)}"] & (R^*)CL(\mathbf{Top},\mathcal{M}) \arrow[u, "|-|"']
\end{tikzcd} \tag{6.14}$$

As in 6.7 one has that $\overline{\mathcal{E}}_*$-(pre)reflections and radical closure operators of $\mathbf{Top}_*$ are related to (pre)radicals of $\mathbf{Top}_*$ via the adjunctions

(iii) $C_{(-)} \dashv \pi \dashv C^{(-)} \,:\, (P)RAD(\mathbf{Top}_*,\mathcal{M}_*) \longrightarrow (R)CL(\mathbf{Top}_*,\mathcal{M}_*)$

(iv) coker $\dashv$ ker $:\, (P)REF(\mathbf{Top}_*,\overline{\mathcal{E}}_*) \longrightarrow (P)RAD(\mathbf{Top}_*,\mathcal{M}_*)$.

(Note that, in general, it is not guaranteed that coker maps a radical to a reflection since $\overline{\mathcal{E}}_*$ is not stable under pullback; see Proposition 6.7(2).) Composing of π and ker with the functors $|-|$ of (i) and (ii), respectively, leads to functors

$$|\pi| : (R^*)CL(\mathbf{Top},\mathcal{M}) \to (P)RAD(\mathbf{Top}_*,\mathcal{M}_*),$$

$$|\mathrm{ker}| : (P)REF(\mathbf{Top},\overline{\mathcal{E}}) \to (P)RAD(\mathbf{Top}_*,\mathcal{M}_*),$$

which are described explicitly by

$$|\pi|(C)(X,x_0) = (c_X(\{x_0\}),x_0),$$

$$|\mathrm{ker}|(R,\rho)(X,x_0) = (\rho^{-1}(\rho(x_0)),x_0).$$

As in (the non-additive part of) Theorem 6.7 one obtains with the Lemma

$$|\pi|\mathrm{reg}^{(-)} = |\mathrm{ker}|,$$

for $\overline{\mathcal{E}}$-reflections, hence

$$\mathrm{reg}_X^{\mathcal{A}}(\{x_0\}) = \rho^{-1}(\rho(x_0))$$

for $\mathcal{A}$ $\overline{\mathcal{E}}$-reflective and $x_0 \in X \in \mathbf{Top}$. This suggests to consider, for every $(R,\rho) \in PREF(\mathbf{Top},\overline{\mathcal{E}})$, the *ρ-saturation operator* $\mathrm{sat}^{(R,\rho)} \in CL(\mathbf{Top},\mathcal{M})$ defined by

$$\mathrm{sat}_X^{(R,\rho)}(M) := \rho^{-1}(\rho(M)).$$

THEOREM

(1) *$|\pi|$ establishes a bijection between fully additive (and weakly hereditary) closure operators of* $\mathbf{Top}$ *and (idempotent) preradicals of* $\mathbf{Top}_*$. *Its inverse assigns to a preradical* $\mathbf{r}$ *the closure operator* $\overline{C}_{\mathbf{r}}$ *with*

$$(\overline{c}_{\mathbf{r}})_X(M) = \bigcup_{x\in M} |\mathbf{r}(X,x)|.$$

(2) $|\ker|$ *embeds* $PREF(\mathbf{Top},\overline{\mathcal{E}})/\cong$ *reflectively into* $PRAD(\mathbf{Top}_*,\mathcal{M}_*)$. *Its reflector* $\overline{\mathrm{coker}}$ *assigns to a preradical* $\mathbf{r}$ *the prereflection* (R,ρ) , *with* $\rho_X : X \to RX = X/\sim_{\mathbf{r}}$ *the canonical projection with respect to the equivalence relation*

$$a \sim_{\mathbf{r}} b \Leftrightarrow (\exists x_1,\ldots,x_n \in X)(a \in |\mathbf{r}(X,x_1)|, b \in |\mathbf{r}(X,x_n)|, \textit{ and}$$

$$|\mathbf{r}(X,x_i)| \cap |\mathbf{r}(X,x_{i+1})| \neq \emptyset \textit{ for } i = 1,\ldots,n-1).$$

Hence $|\ker|$ *establishes a bijection between isomorphism classes of* $\overline{\mathcal{E}}$*-prereflections of* $\mathbf{Top}$ *and those preradicals* $\mathbf{r}$ *of* $\mathbf{Top}_*$ *for which* $\{\mathbf{r}(X,x) : x \in X\}$ *gives a partition for every* $X \in \mathbf{Top}$.

(3) *For every* $\overline{\mathcal{E}}$*-reflection* (R,ρ) *and* $\mathbf{r} = |\ker|(R,\rho)$,

$$\mathrm{sat}^{(R,\rho)} = \overline{C_{\mathbf{r}}}.$$

If (R,ρ) *is an* $\overline{\mathcal{E}}$*-reflection with* $\mathcal{A} = \mathrm{Fix}(R,\rho)$ *non-trivial, then* $\mathrm{reg}^{\mathcal{A}} \leq (C^{\mathbf{r}})^*$, *and*

$$\mathrm{sat}^{(R,\rho)} = (\mathrm{reg}^{\mathcal{A}})^{\oplus} = ((C^{\mathbf{r}})^*)^{\oplus}.$$

(4) $\mathrm{sat}^{(-)}$ *establishes a bijection between isomorphisms classes of* $\overline{\mathcal{E}}$*-prereflections and closure operators* $C \in CL(\mathbf{Top},\mathcal{M})$ *which are fully additive, idempotent and* symmetric, *so that* $(y \in c_X(\{x\}) \Leftrightarrow x \in c_X(\{y\}))$ *for all* $x, y \in X \in \mathbf{Top}$.

$$\begin{tikzcd}[column sep=large, row sep=large]
PRAD(\mathbf{Top}_*,\mathcal{M}_*) \arrow[r, shift left, leftarrow, "|\pi|"] \arrow[r, shift right, swap, "(C^{(-)})^*"] \arrow[d, shift right, swap, "\overline{\mathrm{coker}}"] & CL(\mathbf{Top},\mathcal{M}) \arrow[d, shift right, swap, "(-)^{\oplus}"] \\
PREF(\mathbf{Top},\overline{\mathcal{E}}) \arrow[u, shift right, swap, "|\ker|"] \arrow[r, "\mathrm{sat}"] & FACL(\mathbf{Top},\mathcal{M}) \arrow[u, shift right, swap, "\mathrm{incl.}"]
\end{tikzcd} \qquad (6.15)$$

Diagram (6.15) illustrates the situation, its inner arrows commute, and the outer arrows denote the left adjoints.

Proof (1) The adjunctions (i) and (iii) give the adjunction $\overline{C_{(-)}} \dashv |\pi|$, and for all $\mathbf{r} \in PRAD(\mathbf{Top}_*,\mathcal{M}_*)$, $C \in CL(\mathbf{Top},\mathcal{M})$, and $x_0 \in X \in \mathbf{Top}$ one has

$$|\pi|(\overline{C_{\mathbf{r}}})(X,x_0) = ((\overline{c_{\mathbf{r}}})_X(\{x_0\}),x_0) = \mathbf{r}(X,x_0),$$

$$(\overline{c_{|\pi|(C)}})_X(\{x_0\}) = |(c_X(\{x_0\}),x_0)| = c_X(\{x_0\}).$$

Hence $|\pi|(\overline{C_{\mathbf{r}}}) = \mathbf{r}$ and $\overline{C}_{|\pi|(C)} = C^{\oplus}$, which proves the claim.

(2) The adjunctions (ii) and (iv) give the adjunction $\overline{\mathrm{coker}} \dashv |\ker|$. The given description for $(R,\rho) = \overline{\mathrm{coker}(\mathbf{r})}$ arises from an explicit construction of the multiple pushout (6.12) in case $(S,\sigma) = \mathrm{coker}(\mathbf{r})$: the equivalence relation induced by $\overline{\sigma}_X$

is simply the transitive hull of the union of the equivalence relations induced by the maps $\sigma_{(X,x)}$, $x \in X$. We are left with having to show

$$\overline{\text{coker}}(|\text{ker}|(R,\rho)) \cong (R,\rho).$$

But with $\mathbf{r} = |\text{ker}|(R,\rho)$ one has $|\mathbf{r}(X,x_0)| = \rho_X^{-1}(\rho(x_0))$ for all $x_0 \in X \in \mathbf{Top}$, hence the explicit description of $\sim_{\mathbf{r}}$ shows immediately that this is exactly the equivalence relation induced by ρ_X , and this proves the claim.

(3) Clearly,

$$(\overline{c_{\mathbf{r}}})_X(M) = \bigcup_{x \in M} |\mathbf{r}(X,x)| = \bigcup_{x \in M} \rho_X^{-1}(\rho_X(x)) = \text{sat}_X^{(R,\rho)}(M)$$

for all $M \subseteq X \in \mathbf{Top}$. Furthermore, with $q_{M_*} = \text{coker}(M^*) : X_* \to X_*/M_*$,

$$(c^{\mathbf{r}})^*_X(M) = |q_{M_*}{}^{-1}(\mathbf{r}(X_*/M_*))| \cap X = \{x \in X : \rho_{X_*/M_*}(q_{M_*}(x)) = \rho_{X_*/M_*}(q_{M_*}(*))\}$$

is the equalizer of $(\rho_{X_*/M_*} \cdot q_{M_*})|_X : X \to R(X_*/M_*)$ and a constant map. Hence, when (R,ρ) is a reflection with $\mathcal{A} = \text{Fix}(R,\rho)$, then $(c^{\mathbf{r}})^*_X(M)$ is $\mathcal{A}$-closed, hence $\text{reg}_X^{\mathcal{A}}(M) \subseteq (c^{\mathbf{r}})^*_X(M)$.

If $M = \{x_0\}$ is a singleton set, and if we consider any continuous maps $f, g : X \to A \in \mathcal{A}$ with $f(x_0) = g(x_0) = a_0$, then there are extensions $f_*, g_* : |X_*| \to A$ with $f_*(*) = g_*(*) = a_0$. Since f_* and g_* coincide on $\{*, x_0\} = |M_*|$, they both factor through q_{M_*} and then (since $A \in \mathcal{A}$) through ρ_{X_*/M_*} . Hence there are continuous maps $f^\#, g^\# : R(X_*/M_*) \to A$ with

$$f_* = f^\# \cdot \rho_{X_*/M_*} \cdot q_{M_*}, \quad g_* = g^\# \cdot \rho_{X_*/M_*} \cdot q_{M_*}.$$

Therefore f and g coincide on $(c^{\mathbf{r}})^*_X(\{x_0\})$ which then must coincide with $\text{reg}_X^{\mathcal{A}}(\{x_0\}) = \rho_X^{-1}(\rho_X(x_0)) = \text{sat}_X^{(R,\rho)}(\{x_0\})$. Since obviously $\text{sat}^{(R,\rho)}$ is fully additive, this shows that sat is the fully additive core of both $\text{reg}^{\mathcal{A}}$ and $(C^{\mathbf{r}})^*$.

(4) From identities established above we obtain

$$\overline{\text{coker}} \cdot |\pi| \cdot \text{incl} \cdot \text{sat}^{(-)} \cong \overline{\text{coker}} \cdot |\pi| \cdot \overline{C_{(-)}} \cdot |\text{ker}| \cong \text{id}.$$

We are left with having to show

$$\text{sat}^{(-)} \cdot \overline{\text{coker}} \cdot |\pi|(C) = C$$

for every fully additive, idempotent and symmetric closure operator C of $\mathbf{Top}$. But with $\mathbf{r} = |\pi|(C)$ and $(R,\rho) = \overline{\text{coker}}(\mathbf{r})$ one has

$$\text{sat}_X^{(R,\rho)}(\{x_0\}) = \rho_X^{-1}(\rho_X(x_0)) = \{a \in X : a \sim_{\mathbf{r}} x_0\}$$

for all $x_0 \in X \in \mathbf{Top}$. The description of $\sim_{\mathbf{r}}$ given in (2) shows

$$a \sim_{\mathbf{r}} x_0 \Leftrightarrow c_X(\{a\}) = c_X(\{x_0\})$$

whenever C is idempotent and symmetric. Hence $\mathrm{sat}^{(R,\rho)}$ and C coincide on singletons, hence are equal iff C is fully additive. □

COROLLARY *A closure operator C of* **Top** *is the fully additive core of the regular closure of a strongly epireflective subcategory $\mathcal{A}$ of* **Top** *if and only if*

(1) *C is fully additive, idempotent and symmetric,*

(2) *for every $X \in$* **Top** *, the projection $X \to X/\sim_C$ with ($x \sim_C y \Leftrightarrow c_X(\{x\}) = c_X(\{y\})$) preserves C-closedness.*

In this case, $\mathcal{A}$ contains exactly those spaces in which every point is C-closed. □

Proof If $C = (\mathrm{reg}^{\mathcal{A}})^{\oplus}$ for an $\overline{\mathcal{E}}$-reflective subcategory $\mathcal{A}$ of **Top** , then $C = \mathrm{sat}^{(R,\rho)}$ according to Theorem (3), which satisfies condition (1). Up to isomorphism $X \to X/\sim_C$ coincides with $\rho_X : X \to RX$, which preserves $\mathrm{sat}^{(R,\rho)}$-closedness.

Conversely, for C with (1) and (2), let $(R,\rho) = \overline{\mathrm{coker}}(|\pi|(C)\ )$, so that $\rho_X : X \to RX = X/\sim_C$ is the projection. Then

$$\mathcal{A} = \mathrm{Fix}(R,\rho) = \{X : \rho_X \text{ is monic}\} = \{X : (\forall x \in X)\{x\} \text{ is } C\text{-closed in } X\}.$$

Now it suffices to show that (R,ρ) is a reflection, i.e., that $RX \in \mathcal{A}$ for all X . But (2) implies

$$c_{RX}(\{\rho_X(x)\}) = \rho_X(c_X(\{x\})) = \rho_X(\{x\})$$

for all $x \in X$, so that points in RX are C-closed. □

EXAMPLE The following $\mathcal{M}_*$-radicals of $\mathbf{Top}_*$ give partitions for every space X and therefore correspond to $\overline{\mathcal{E}}$ -prereflections of **Top** (see Theorem (2)):

- $\mathbf{c}(X,x) =$ connected component of x in X ,
- $\mathbf{q}(X,x) =$ quasi-component of x in X $= |\pi(Q)|$ (cf. 4.7),
- $\alpha(X,x) =$ arccomponent of x in X ,

Of these prereflections, **c** and **q** lead to reflections, but α does not. (Consider the Topologist's Sine Curve (cf. Example 5.1 (2)).)

Exercises

6.A (*Characterizing $\mathcal{A}$-regular closure and $\mathcal{A}$-epimorphisms*)

(a) Under the hypotheses of Theorem 6.2 show that n is (isomorphic to) the $\mathcal{A}$-regular closure of $m \in \mathcal{M}$ if and only if $n' \geq m$ is $\mathcal{A}$-regular and $m \overset{\mathcal{A}}{\Rightarrow} n$.

(b) Under the hypothesis of Lemma 6.1 show that a morphism $m : M \to X$ in $\mathcal{X}$ is $\mathcal{A}$-epic if and only if $\rho_K \cdot i = \rho_K \cdot j$.

6.B (*Surjectivity of epimorphisms in* **Grp**) Show that all monomorphisms in **Grp** and in the category $\mathbf{Grp}_{fin}$ of finite groups are regular, and conclude that the epimorphisms in **Grp** and $\mathbf{Grp}_{fin}$ are precisely the surjective homomorphisms. *Hint* (cf. Adámek-Herrlich-Strecker [1990], p. 117): Let M be a subgroup of a (finite) group G. Consider the (finite) permutation group S of the (finite) set X obtained from the set of all left M-cosets of G by adjoining a single new element $\tilde{M}$, and denote by $\rho \in S$ the transposition of the elements M (which is the coset eM) and $\tilde{M}$ of X . Now the inclusion $M \to G$ is the equalizer of the homomorphisms $f_1, f_2 : G \to S$ defined as follows: $f_1(g)(g'M) = gg'M$, $f_1(g)(\tilde{M}) = \tilde{M}$, $f_2(g) = \rho \cdot f_1(g) \cdot \rho$.

6.C (*Strong monomorphisms need not be regular*)

(a) Prove that if the composite $j \cdot i$ is a regular monomorphism with j monic, then also i is a regular monomorphism.

(b) Let S be a semigroup with 0 (hence $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$), and let I be a two-sided ideal of S (hence $IS \cup SI \subseteq I$). Then the inclusion $I \to S$ is a regular monomorphism in the category **SGrp** of semigroups. If S is commutative and the subset $B = \{0\} \cup S \setminus I$ is a subsemigroup of S with $BBI = \{0\}$ then also the inclusion $B \to S$ is a regular monomorphism. *Hint*: Let the "quotient" S/I have underlying set B and a multiplication $*$ defined by $a * b = ab$ if $ab \in S \setminus I$ and $a * b = 0$ otherwise.

(c) (Cf. Adámek-Herrlich-Strecker [1990], p. 117) Define a multiplication on $S = \{0, a, b, c, d, e\}$ such that $B = \{0, a, b, c\}$ is a subsemigroup with zero multiplication, $d^2 = e$, $da = ad = b$, $db = bd = c$ and $dc = cd = 0$. Consider the subsemigroups $A = \{0, a, b\}$, $D = \{0, d, e\}$ and $E = \{0, b, c\}$. Then the inclusions $m : A \to B$ and $n : B \to S$ are regular monos, while $n \cdot m : A \to S$ is a strong monomorphism, but not regular. Check also the inclusion maps $i : D \to S$, $i : E \to B$ and $j : B \to S$ and $j : E \to S$ for regularity. *Hint*: Use the equalities $b = da$ and $c = bd$ to conclude that for each pair of morphisms $f, g : S \to S'$ in **SGrp** coinciding on A also $f(c) = g(c)$ holds.

6.D (*$\mathcal{A}$-epi closure versus $\mathcal{A}$-regular closure*) Find examples such that (1) $\mathrm{epi}^{\mathcal{A}} = \mathrm{reg}^{\mathcal{A}}$ is not discrete on $\mathcal{X}$ and (2) $\mathrm{epi}^{\mathcal{A}}$ is discrete but $\mathrm{reg}^{\mathcal{A}}$ is not.

6.E (*Productivity of regular closure operators*) Let $\mathcal{A}$ be a reflective subcategory of a complete and $\mathcal{M}$-complete category $\mathcal{X}$ with cokernelpairs. Then the $\mathcal{A}$-regular closure operator is productive. *Hint*: If $m_i = \mathrm{equalizer}(f_i, g_i)$ for every $i \in I$, then $\prod_{i \in I} m_i \cong \mathrm{equalizer}(\prod_{i \in I} f_i, \prod_{i \in I} g_i)$.

6.F (*When $\mathcal{A}$-regular monos are iso*) An object T of $\mathcal{X}$ is *preterminal* if every hom-set $\mathcal{X}(X,T)$ has at most one element. Show that for the statements below, one has (a) $\Rightarrow$ (b) $\Rightarrow$ (c), while (b) $\Rightarrow$ (a) holds when $\mathcal{A}$ has an initial object, and (c) $\Rightarrow$ (a) holds when $\mathcal{A}$ has equalizers.

(a) Every object in $\mathcal{A}$ is preterminal,

(b) every morphism in $\mathcal{A}$ is $\mathcal{A}$-epic.

(c) every $\mathcal{A}$-regular morphism in $\mathcal{X}$ is iso.

6.G (*Restricting the $\mathcal{A}$-regular closure to $\mathcal{A}$*) In the context of 6.3, let $\mathcal{A}$ be $\mathcal{E}$-reflective in $\mathcal{X}$. Show that the restriction of $\mathrm{reg}^{\mathcal{A}}$ to $\mathcal{A}$ is weakly hereditary if and only if the class of regular monomorphisms in the category $\mathcal{A}$ is closed under composition. Furthermore, show $\mathrm{reg}^{\mathcal{A}} \geq C$ whenever $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} \geq C|_{\mathcal{A}}$, for any $C \in CL(\mathcal{X}, \mathcal{M})$.

6.H (*Bireflective subcategories*) Let $\mathcal{A}$ be a full replete reflective subcategory of a category $\mathcal{X}$, such that every reflexion morphism is monic. Show that then each reflexion is also epic, hence a *bimorphism*. Prove that every regular monomorphism of $\mathcal{X}$ is $\mathcal{A}$-closed and every $\mathcal{A}$-epimorphism is epic in $\mathcal{X}$. Furthermore, in the context and in the notation of the Magic Cube Theorem, show that each of u, v, w is a monomorphism. Simplify the Magic Cube Theorem accordingly.

6.I (*$\mathcal{A}$-closedness of least subobjects*) Prove in the context of Frolík's Lemma for every object $X \in \mathcal{X}$: the least $\mathcal{M}$-subobject o_X is $\mathcal{A}$-closed if and only if the sum $X + X$ belongs to $\mathcal{A}$.

6.J (*Strong modification*) Prove under the hypothesis of Theorem 6.6 that $C \leq D$ with D idempotent and strong implies $\tilde{C} \leq D$.

6.K (*ρ-defects and preservation of $\mathcal{A}$-closedness*) Prove in the setting of 6.7 that ρ-defects of $\mathcal{A}$-closed $\mathcal{M}$-subobjects preserve $\mathcal{A}$-closedness if and only if for every $\mathcal{A}$-closed $\mathcal{M}$-subobject $n : N \to RX$, the "restriction" $\rho_X^{-1}(N) \to N$ of ρ_X preserves $\mathcal{A}$-closedness.

6.L (*Computing the $\mathcal{A}$-regular closure in $\mathbf{Mod}_R$*) Show that for every full subcategory $\mathcal{A}$ of $\mathbf{Mod}_R$ the $\mathcal{A}$-regular closure of $M \leq X \in \mathbf{Mod}_R$ can be computed as $\mathrm{reg}_X^{\mathcal{A}}(M) = \bigcap\{\ker f \,|\, f : M \to A \in \mathcal{A}\ R\text{-linear}\ \&\ f(M) = 0\}$. Does a similar formula hold true in $\mathbf{Grp}$?

6.M (*Additivity of regular closure operators in $\mathbf{Mod}_R$*) Prove that the regular closure operator of an epireflective subcategory $\mathcal{A}$ of $\mathbf{Mod}_R$ is additive if and only if the $\mathcal{A}$-regular radical is cohereditary.

6.N (*Epimorphisms in subcategories of R-modules*) Prove that the $\mathcal{A}$-epiclosure of an epireflective subcategory $\mathcal{A}$ of $\mathbf{Mod}_R$ is maximal; more precisely, it is the maximal closure operator of the idempotent core $\mathbf{s} = (\mathbf{r}^{\mathcal{A}})^{\infty}$ of the $\mathcal{A}$-regular radical if and only if $C^{\mathbf{s}}|_{\mathcal{A}}$ is discrete; that is, if $\mathbf{s}(X/M) = 0$ for all $M \leq X \in \mathcal{A}$.

6.O (*Cokernelpairs in additive categories*) Extend the formula

$$X +_M X \cong X \times X/M$$

from modules to additive categories. More precisely, show that in an additive category with cokernelpairs and binary (co)products, the cokernelpair of $m : M \to X$ can be constructed as

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "m"'] & X \arrow[d, "<1, \mathrm{coker}(m)>"] \\ X \arrow[r, "<1,0>"] & X \times X/M \end{tikzcd} \tag{6.16}$$

6.P (*Torsion-free classes of R-modules and groups*)

(a) Prove that a full and replete epireflective subcategory $\mathcal{A}$ of $\mathbf{Mod}_R$ has a weakly hereditary regular closure operator if and only if $\mathcal{A}$ is closed under extensions (see condition (v) of Example 6.8(1)).

(b) Prove the same statement for the category **Grp** in lieu of $\mathbf{Mod}_R$.

6.Q (*Regular closure of varieties of abelian groups in* **Grp**) For a variety $\mathcal{A}$ in **Grp** with $\mathcal{A} \subseteq$ **AbGrp** , prove that the restriction of $\mathrm{reg}^{\mathcal{A}}$ to $\mathcal{A}$ is discrete.

6.R (Q *as a regular closure operator*) Confirm the claims on the largest proper closure operator Q of **Top** (see 4.7) made in Example 6.9 (1). *Hint:* Let $\mathcal{A}$ denote the category of spaces with trivial quasicomponent. As noted in Example 6.9 (1), $\mathrm{reg}^{\mathcal{A}} \leq \mathrm{reg}^{\mathbf{0\text{-}Top}} \leq Q$. To prove $\mathrm{reg}^{\mathcal{A}} \geq Q$, note that for $M \subseteq X \in$ **Top** $\mathrm{reg}^{\mathcal{A}}{}_X(M)$ is the intersection of equalizers of pairs $u, v : X \to A \in \mathcal{A}$, coinciding on M (cf. Theorem 6.2 and Lemma 6.1). Since each $A \in \mathcal{A}$ admits a continuous injection into a power of the discrete dyad $D = \{0,1\}$, actually only pairs $u, v : X \to D$ suffice. It remains to note that such equalizers are clopen.

6.S (*When regular monomorphisms in* $\mathcal{A}$ *coincide with the closed embeddings*)

Show that the regular monomorphisms in an epireflective subcategory $\mathcal{A}$ of **Top** are exactly the (Kuratowski-) closed subspace embeddings, when $\mathcal{A}$ is one of the following subcategories: **Tych** , **0-Top** (0-dimensional spaces), **Reg** (regular spaces), **DHaus** (totally disconnected Hausdorff spaces), **SHaus** (*strongly Hausdorff spaces*, i.e., spaces X such that for every infinite subset $M \subseteq X$ there is a sequence $\{U_n : n \in \mathbb{N}\}$ of pointwise disjoint open sets such that $M \cap U_n \neq \emptyset$ for each $n \in \mathbb{N}$, cf. A. Hajnal and I. Juhász [1969]) . Conclude that in these categories epimorphisms are dense maps. *Hint:* As in Corollary 6.9, the $\mathcal{A}$-closed embeddings are (Kuratowski-) closed. To prove the converse, apply Frolík's Lemma.

6.T (*Disconnectedness*) (Cf. Arhangel'skii and Wiegandt [1975])

(a) Let $\mathcal{A}$ be a strongly epireflective subcategory of **Top** with weakly hereditary regular closure operator. Show that, with $\mathcal{C}$ the full subcategory of topological spaces with trivial $\mathcal{A}$-reflexion, a space $X \in \mathbf{Top}$ belongs to $\mathcal{A}$ if and only if every map $C \to X$ with $C \in \mathcal{C}$, is constant.

(b) A *disconnectedness* of **Top** is a full subcategory $\mathcal{A}$ with the property described in (a), i.e., for some full subcategory of topological spaces $\mathcal{C}$, a space $X \in \mathbf{Top}$ belongs to $\mathcal{A}$ if and only if every map $C \to X$, with $C \in \mathcal{C}$, is constant. Show that every disconnectedness of **Top** is a strongly epireflective subcategory of **Top** with weakly hereditary regular closure operator.

(c) Show that $\mathrm{reg}^{\mathbf{Top}_1}$ is weakly hereditary. *Hint:* $\mathbf{Top}_1$ is a disconnectedness of **Top** w.r.t. the class $\mathcal{C}$ formed by Sierpiński dyad as its only member.

6.U (*Recovering* $\mathrm{reg}^{\mathcal{A}}$ *from* $(\mathrm{reg}^{\mathcal{A}})^{\oplus}$ *in* **Top**) For $\mathcal{A}$ strongly epireflective in **Top**, show

$$((\mathrm{reg}^{\mathcal{A}})^{\oplus})^{\check{}} = \mathrm{reg}^{\mathcal{A}}.$$

Hint: In the notation of Lemma 6.5, first show $(\mathrm{reg}^{\mathcal{A}})^{\oplus}_{K}(j(X)) = j(X) \cup i(\mathrm{reg}^{\mathcal{A}}_{X}(M))$, using the discreteness of $(\mathrm{reg}^{\mathcal{A}})^{\oplus}$ on $\mathcal{A}$ (see Lemma 6.9).

6.V (*Recovering* $\mathrm{reg}^{\mathcal{A}}$ *from* $(\mathrm{reg}^{\mathcal{A}})^{\oplus}$ *in* $\mathbf{Mod}_R$)

Hint: Use Exercise 3.M (b) to show $(\mathrm{reg}^{\mathcal{A}})^{\oplus} = C_{\mathbf{r}}$ with $\mathbf{r}$ the $\mathcal{A}$-regular radical. Then use Example 6.6 (2) and Theorem 6.7.

6.W (*Regular closure for non-abelian torsionfree groups*) (Cf. Fay and Walls [1994])

(a) Show that the full subcategory $\mathcal{A}$ of **Grp** having as objects all torsionfree groups, is closed under extension (that is: if $N \trianglelefteq G$ with $N \in \mathcal{A}, G/N \in \mathcal{A}$, then $G \in \mathcal{A}$). Conclude that $\mathrm{reg}^{\mathcal{A}}$ is discrete on $\mathcal{A}$.

(b) For a subgroup H of a group G one says that H is *isolated* if for all $x \in G$ and $n > 0$, $x^n \in H$ yields $x \in H$. Show that the class of isolated subgroup embeddings is closed under intersection and pullback, so that it determines an idempotent closure operator $I = \{i_G\}_{G \in \mathbf{Grp}}$; explicitly, for $H \leq G$, $i_G(H) = \bigcap\{N : H \leq N \leq G \text{ isolated}\}$. Prove that the preradical $\pi(I)$ corresponds to the reflection onto torsionfree groups. Check that $\pi(I)$ is a hereditary radical. Then compute $\mathrm{reg}^{\mathcal{A}}$ for $\mathcal{A}$ as in (a).

(c) Let $\mathcal{R}$ denote the full subcategory $\mathcal{A}$ of **Grp** having as objects all *R-groups*, i.e., groups G such that $x^n = y^n$ with $n > 0$ always implies $x = y$ in G. Show that for $H \leq G \in \mathcal{R}$, $\mathrm{reg}^{\mathcal{R}}_{G}(H) = i_G(H)$ and conclude that $\mathcal{R}$ *is not closed under extension*. Give an example of a torsionfree group which is not an R-group.

Hints: (a) Apply Example 6.8 (2). (b) For $H \leq G \in \mathbf{Grp}$, $\mathrm{reg}^{\mathcal{A}}_{G}(H) = H \cdot i_G(\{1\})$. (c) Show that for $H \leq G \in \mathcal{R}$, the cokernelpair (=amalgamated free product) $G *_H G$ belongs to $\mathcal{R}$ if H is isolated. Moreover, if $x^n \in H$ for some $n > 0$, then the $\mathcal{R}$-reflexion $\rho : G *_H G \to R(G *_H G)$ sends $i(x)$ and $j(x)$ to the same element since $i(x)^n = j(y)^n$. This proves $\mathrm{reg}^{\mathcal{R}} = I$ when restricted to $\mathcal{R}$. Finally, consider $\mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}$.

Notes

The regular closure and epi-closure have been defined for categories of algebras by Isbell [1966] and for categories of topological spaces by Salbany [1976] and Cagliari and Cicchese [1983]. The modification formula 6.3 was given by Dikranjan and Giuli [1984] in topological contexts and by Dikranjan, Giuli and Tholen [1989] for general categories. Frolík communicated a proof of Theorem 6.5 in the context of subcategories of **Top** to Dikranjan and Giuli in 1983 (see Dikranjan and Giuli [1983]). A categorical version (but more restrictive than the one given in 6.5) was given by Dikranjan, Giuli and Tholen [1989]. Strong modifications were defined by Dikranjan [1992], and the characterization of weakly hereditary regular closure operators (see 6.8) was given by Clementino [1992], [1993]. The correspondences of 6.10 concerning pointed topological spaces are new.

7 Subcategories Defined by Closure Operators

A Hausdorff space X is characterized by the property that its diagonal $\Delta_X \subseteq X \times X$ is (Kuratowski-) closed. In this way, every closure operator C of a category $\mathcal{X}$ defines the Delta-subcategory $\Delta(C)$ of objects with C-closed diagonal, and subcategories appearing as Delta-subcategories are in any "good" category $\mathcal{X}$ characterized as the strongly epireflective ones. What then is the regular closure operator induced by $\Delta(C)$? Under quite "topological" conditions on $\mathcal{X}$, we show for additive C that this closure can be computed as the idempotent hull of the strong modification of C, at least where it matters: for subobjects in $\Delta(C)$ (see Theorem 7.4). This leads to a complete characterization of additive regular closure operators in the given context.

Another approach of learning about the epimorphisms of a subcategory $\mathcal{A}$ of $\mathcal{X}$ is to embed $\mathcal{A}$ into larger subcategories $\mathcal{B}$ with potentially better epi-behaviour, such that epimorphisms of $\mathcal{A}$ are still epimorphisms of $\mathcal{B}$. Two such extensions are discussed in this chapter, the epi-closure $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$ and the maximal epi-preserving extension $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$. Both are described in terms of closure operators, with epi-closures being characterized as those Delta-subcategories induced by weakly hereditary closure operators.

7.1 The Salbany correspondence

Our first goal in this chapter is to solve the problem whether a subcategory can be recovered from its regular closure operator. As in 6.2, we work with a full subcategory $\mathcal{A}$ of an $\mathcal{M}$-complete category $\mathcal{X}$ with equalizers such that $\mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M}$ and $\mathcal{M}$ is closed under composition. We recall that the $\mathrm{reg}^{\mathcal{A}}$-closed $\mathcal{M}$-subobjects are then given by the $\mathcal{A}$-regular monomorphisms, and the $\mathrm{reg}^{\mathcal{A}}$-dense morphisms are exactly the $\mathcal{A}$-epimorphisms.

We observe first that in order to recover $\mathcal{A}$ from $\mathrm{reg}^{\mathcal{A}}$, the subcategory $\mathcal{A}$ should be closed under mono-sources. Recall that a *source* is simply a (possibly large) family $(p_i : B \to A_i)_{i \in I}$ of morphisms in $\mathcal{X}$ with common domain B; in case $I = \emptyset$, the source is identified with B. The source is *monic* if for all $x, y : T \to B$ with $p_i \cdot x = p_i \cdot y$ for all $i \in I$ one has $x = y$. For every full subcategory $\mathcal{A}$ of $\mathcal{X}$,

$$S(\mathcal{A}) := S_{\mathcal{X}}(\mathcal{A}) = \{B \in \mathcal{X} : B \text{ is the domain of some mono-source with codomain in } \mathcal{A}\}$$

is the *closure of $\mathcal{A}$ under mono-sources in $\mathcal{X}$*. We note that, if $\mathcal{A}$ is reflective in $\mathcal{X}$ (with $\mathcal{A}$-reflexion $\rho : 1 \to R$), then $S_{\mathcal{X}}(\mathcal{A})$ has a simplified description as

$$S_{\mathcal{X}}(\mathcal{A}) = \{B \in \mathcal{X} : \rho_B \text{ is monic}\}.$$

LEMMA *For every full subcategory $\mathcal{A}$ of $\mathcal{X}$, $\mathcal{A}$ and $S_{\mathcal{X}}(\mathcal{A})$ give the same regular closure in $\mathcal{X}$.*

Proof Since regular closure operators are idempotent, it suffices to show $\mathrm{Reg}_{\mathcal{X}}(\mathcal{A}) = \mathrm{Reg}_{\mathcal{X}}(S(\mathcal{A}))$, and for that it suffices to show that $f \overset{\mathcal{A}}{\Rightarrow} g$ holds if and only if $f \overset{S(\mathcal{A})}{\Rightarrow} g$ (see 6.1). But this follows immediately from the definition of $S(\mathcal{A})$. □

Each object $A \in \mathcal{A}$ satisfies $m \overset{A}{\Rightarrow} \mathrm{reg}_X^{\mathcal{A}}(m)$ for every $m \in \mathcal{M}/X$, $X \in \mathcal{X}$ (see Remark 6.2 (3)); here $m \overset{A}{\Rightarrow} k$ means $m \overset{\{A\}}{\Rightarrow} k$ as defined in 6.1, that is : $u \cdot m = v \cdot m$ with $u, v : X \to A$ always implies $u \cdot k = v \cdot k$. Replacing $\mathrm{reg}^{\mathcal{A}}$ by an arbitrary closure operator we arrive at the following general definition.

Definition For every closure operator C of $\mathcal{X}$, the *Delta-subcategory* $\Delta(C)$ induced by C is the full subcategory of $\mathcal{X}$ with object class

$$\{A \in \mathcal{X} : (\forall m \in \mathcal{M})\, m \overset{A}{\Rightarrow} c(m)\}$$

Before comparing the subcategories $\mathcal{A}$ and $\Delta(\mathrm{reg}^{\mathcal{A}})$ in detail, we must justify the notation $\Delta(C)$.

Proposition *For every closure operator C of $\mathcal{X}$, $\Delta(C)$ is closed under mono-sources. If $\mathcal{X}$ has finite products, then the object class of $\Delta(C)$ is given by*

$$\{A \in \mathcal{X} : \delta_A =< 1_A, 1_A >: A \to A \times A \text{ is } C\text{-closed}\}.$$

Proof Let $(p_i : B \to A_i)_{i \in I}$ be a mono-source with all $A_i \in \Delta(C)$, assume $u \cdot m = v \cdot m$ for $m \in \mathcal{M}$ and $u, v : X \to B$. Then $p_i \cdot u \cdot m = p_i \cdot v \cdot m$, hence $p_i \cdot u \cdot c_X(m) = p_i \cdot v \cdot c_X(m)$ since $A_i \in \Delta(C)$, for all $i \in I$. This implies $u \cdot c_X(m) = v \cdot c_X(m)$.

For every object A, δ_A is the equalizer of the projections $p_1, p_2 : A \times A \to A$, hence $\delta_A \in \mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M}$. For $A \in \Delta(C)$ one therefore has $p_1 \cdot c_{A \times A}(\delta_A) = p_2 \cdot c_{A \times A}(\delta_A)$, hence $c_{A \times A}(\delta_A) \leq \delta_A$ by the equalizer property. Conversely, if δ_A is C-closed, $u \cdot m = v \cdot m$ with $m \in \mathcal{M}$ and $u, v : X \to A$ gives a commutative diagram

$$\begin{tikzcd} M \arrow[r, "u \cdot m"] \arrow[d, "m"'] & A \arrow[d, "\delta_A"] \\ X \arrow[r, "< u, v >"] & A \times A \end{tikzcd} \tag{7.1}$$

and therefore a morphism $t : c_X(M) \to A$ with

$$\delta_A \cdot t =< u, v > \cdot c_X(m),$$

by the Diagonalization Lemma 2.4. Hence

$$u \cdot c_X(m) = p_1 \cdot \delta_A \cdot t = t = p_2 \cdot \delta_A \cdot t = v \cdot c_X(m). \qquad \square$$

EXAMPLES

(1) For any closure operator C of $\mathbf{Mod}_R$, $\Delta(C)$ contains precisely those R-modules A in which the trivial submodule 0 is C-closed. In fact, the definition of $\Delta(C)$ shows immediately $c_A(0) = 0$ for $A \in \Delta(C)$ (just consider the zero endomorphism of X and id_A for u and v). On the other hand, there is a pullback diagram

$$\begin{tikzcd} A \arrow[r, "u \cdot m"] \arrow[d, "\delta_A"'] & 0 \arrow[d] \\ A \times A \arrow[r, "d"] & A \end{tikzcd} \tag{7.2}$$

for the map $d(x, y) = x - y$, which shows that C-closedness of 0 in A implies C-closedness of δ_A , hence $A \in \Delta(C)$.

Obviously the statement remains true in any additive category with finite products.

(2) A group G is abelian if and only if (the image of) the diagonal δ_G is normal in $G \times G$, i.e. closed w.r.t. the normal closure ν in $\mathbf{Grp}$. Hence $\Delta(\nu) = \mathbf{AbGrp}$.

(3) A topological space X satisfies the Hausdorff separation axiom if and only if its diagonal δ_X is (K-) closed in $X \times X$, hence $\Delta(K) = \mathbf{Haus}$, with K the Kuratowski closure operator of $\mathbf{Top}$. For the fully additive core $K^{\oplus}$ of K and for the inverse Kuratowski closure operator K^* (cf. Example 4.2 (3)), one easily shows $\Delta(K^{\oplus}) = \Delta(K^*) = \mathbf{Top}_1$. Note that nevertheless K and K^* are not comparable by the preorder of closure operators.

(4) For the b-closure of $\mathbf{Top}$ one has $\Delta(b) = \mathbf{Top}_0$. Recall that b is the additive core of $K \wedge K^*$ (cf. Example 4.8 (1)). Although the additive core of $K^{\oplus} \wedge K^*$ is strictly smaller than b , its Delta-subcategory is also $\mathbf{Top}_0$.

(5) $\Delta(\theta)$ is the category $\mathbf{Ury}$ of *Urysohn spaces*, i.e. the full subcategory of $\mathbf{Top}$ containing those spaces X in which any two distinct points can be separated by disjoint *closed* neighbourhoods.

(6) In the category $\mathbf{PoSet}$, the Delta-subcategory induced by each $\uparrow, \downarrow$ and conv is the category of (discrete partially ordered) sets.

The assignment $C \mapsto \Delta(C)$ is the counterpart of the assignment $\mathcal{A} \mapsto \mathrm{reg}^{\mathcal{A}}$, in the following sense:

THEOREM *There is an adjunction*

$$\Delta \dashv \mathrm{reg}^{(-)} : SUB(\mathcal{X})^{op} \to CL(\mathcal{X},\mathcal{M}),$$

called the Salbany correspondence, *with* $SUB(\mathcal{X})$ *denoting the conglomerate of all full subcategories of* $\mathcal{X}$ *, ordered by inclusion. Hence a subcategory* $\mathcal{A}$ *is the Delta-subcategory of some closure operator if and only if* $\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}})$ *, and a closure operator* C *is the regular closure operator of some subcategory if and only if* $C \cong \mathrm{reg}^{\Delta(C)}$ *.*

Proof The inclusion $\mathcal{A} \subseteq \Delta(\mathrm{reg}^{\mathcal{A}})$ for every subcategory $\mathcal{A}$ follows from the relation $m \overset{\mathcal{A}}{\Rightarrow} \mathrm{reg}^{\mathcal{A}}(m)$ for all $m \in \mathcal{M}$ (cf. Remark 6.2 (3)). Furthermore, Δ is like $\mathrm{reg}^{(-)}$ order-preserving. Hence we need to show only $C \leq \mathrm{reg}^{\Delta(C)}$ for all $C \in CL(\mathcal{X},\mathcal{M})$. But for all $k \geq m \in \mathcal{M}$ one has $k \overset{\Delta(C)}{\Rightarrow} m$ by definition of $\Delta(C)$, hence $c(m) \leq k$ whenever k is $\Delta(C)$-closed; consequently, $c(m) \leq \mathrm{reg}^{\Delta(C)}(m)$. □

COROLLARY *For any families of subcategories and of closure operators, one has the rules*

$$\mathrm{reg}^{\bigcup \mathcal{A}_i} = \bigwedge \mathrm{reg}^{\mathcal{A}_i} \quad \textit{and} \quad \Delta(\bigvee C_i) = \bigcap \Delta(C_i).$$

□

In the next section we shall give sufficient conditions for $\mathcal{A}$ to be a Delta-category and thereby solve the problem of recovering $\mathcal{A}$ from its regular closure.

7.2 Two diagonal theorems

Our general hypotheses are as in 7.1. We shall give two sets of conditions leading to a characterization of Delta-subcategories, one suitable for applications in algebra and the other applicable also in topology.

We first assume $\mathcal{X}$ to be pointed with kernels and cokernels.

PROPOSITION *(Pointed Diagonal Theorem) Let* $\mathcal{A}$ *be a full and replete* $\mathcal{E}$*-reflective subcategory of* $\mathcal{X}$ *such that each reflexion* ρ_X *satisfies* $\mathrm{coker}(\ker\rho_X) \cong \rho_X$*. Then* $\mathcal{A}$ *is a Delta-subcategory of* $\mathcal{X}$ *.*

Proof With $r_B = \ker\rho_B : \mathbf{r}(B) \to B$ the $\mathcal{A}$-regular radical of $B = \Delta(\mathrm{reg}^{\mathcal{A}})$, according to Proposition 6.7 it suffices to show $\mathbf{r}(B) \cong 0$ in order to obtain $B \in \mathcal{A}$. From Theorem 6.7 we know that r_B is the $\mathcal{A}$-regular closure of $o_B : 0 \to B$. With t_B the zero endomorphism of B (which factors through 0), from $B \in \Delta(\mathrm{reg}^{\mathcal{A}})$ and $1_B \cdot o_B = t_B \cdot o_B$ one derives $1_B \cdot r_B = t_B \cdot r_B$, i.e. r_B is a zero morphism, which is possible only if $r_B \cong o_B$. □

If $\mathcal{X}$ is not pointed, we have to mimic "points" in a different manner. Recall that a class $\mathcal{P}$ of objects of $\mathcal{X}$ is *generating* if the sink

$$\mathcal{X}(\mathcal{P},X) := \bigcup_{P\in\mathcal{P}} \mathcal{X}(P,X)$$

is epic for every $X \in \mathcal{X}$. If $\mathcal{X}$ has finite products, we may consider, for every $P \in \mathcal{P}$ and $B \in \mathcal{X}$, the canonical morphism

$$k_{P,B} = < Rq_1, Rq_2 >: R(P \times B) \to RP \times RB$$

with q_1, q_2 the projections of $P \times B$. To say that $k_{P,B}$ is monic is the same as to say that the pair (Rq_1, Rq_2) is monic as a source.

THEOREM *(Generating Diagonal Theorem) Let the category $\mathcal{X}$ with finite products contain a generating class $\mathcal{P}$ such that each morphism $k_{P,B}$ ($P \in \mathcal{P}$, $B \in \mathcal{X}$) is monic. Then*

$$\Delta(\mathrm{reg}^{\mathcal{A}}) = S(\mathcal{A})$$

holds for every full reflective subcategory $\mathcal{A}$ of $\mathcal{X}$. In particular, if $\mathcal{A}$ is strongly epireflective, then $\mathcal{A}$ is a Delta-subcategory.

Proof Since $\Delta(\mathrm{reg}^{\mathcal{A}})$ is closed under mono-sources, from $\mathcal{A} \subseteq \Delta(\mathrm{reg}^{\mathcal{A}})$ one actually has $S(\mathcal{A}) \subseteq \Delta(\mathrm{reg}^{\mathcal{A}})$. We must now show that for every $B \in \Delta(\mathrm{reg}^{\mathcal{A}})$, the $\mathcal{A}$-reflexion ρ_B is monic (cf. Lemma 7.1). Since $\mathcal{P}$ is a generating class, it suffices to verify that $\rho_B \cdot x = \rho_B \cdot y$ with $x, y : P \to B$ and $P \in \mathcal{P}$ implies $x = y$. To this end we first show $\delta_A \overset{\mathcal{A}}{\Rightarrow} < x, y >: P \to B \times B$. In fact, if $u \cdot \delta_B = v \cdot \delta_B$ with $u, v : B \times B \to A \in \mathcal{A}$, then we may form the induced morphisms $u_x, v_x : R(P \times B) \to A$ with

$$u_x \cdot \rho_{P\times B} = u \cdot (x \times 1) \text{ and } v_x \cdot \rho_{P\times B} = v \cdot (x \times 1)$$

and first obtain

$$Rq_1 \cdot \rho_{P\times B} \cdot < 1, x >= \rho_P \cdot q_1 \cdot < 1, x >= \rho_P = Rq_1 \cdot \rho_{P\times B} \cdot < 1, y >,$$

$$Rq_2 \cdot \rho_{P\times B} \cdot < 1, x >= \rho_B \cdot q_2 \cdot < 1, x >= \rho_B \cdot x = \rho_B \cdot y = Rq_2 \cdot \rho_{P\times B} \cdot < 1, y >,$$

hence $\rho_{P\times B} \cdot < 1, x >= \rho_{P\times B} \cdot < 1, y >$ under the assumption on $\mathcal{P}$. This gives

$$u \cdot < x, y >= u \cdot (x \times 1) \cdot < 1, y >= u_x \cdot \rho_{P\times B} \cdot < 1, y >=$$

$$u_x \cdot \rho_{P\times B} \cdot < 1, x >= u \cdot < x, x >= u \cdot \delta_B \cdot x = v \cdot \delta_B \cdot x = v \cdot < x, y >,$$

with the last identity to be obtained as in the previous steps. This completes the proof of $\delta_B \overset{\mathcal{A}}{\Rightarrow} < x, y >$ which, by hypothesis on B, yields a morphism $z : P \to B$ with $\delta_B \cdot z =< x, y >$. When applying the product projections to this identity we obtain $z = x = y$, as desired. □

COROLLARY *Let $\mathcal{A}$ be a full and replete strongly epireflective subcategory of the category $\mathcal{X}$ with finite products. Then $\mathcal{A}$ is a Delta-subcategory of $\mathcal{X}$ under* each *of the following two conditions:*

(a) *the $\mathcal{A}$-reflector $R : \mathcal{X} \to \mathcal{X}$ preserves finite products, or*

(b) *the terminal object of $\mathcal{X}$ forms a (single-object) generating class of $\mathcal{X}$.*

Proof First note that strong epireflectivity makes $\mathcal{A}$ closed under mono-sources, hence $\mathcal{A} = S(\mathcal{A})$. Under condition (a), in the Theorem, simply choose $\mathcal{P} = \mathcal{X}$. Under condition (b), note that for the terminal object P of $\mathcal{X}$ and for every object $B \in \mathcal{X}$, the product projection $q_2 : P \times B \to B$ is an isomorphism, hence (Rq_1, Rq_2) is trivially monic. □

REMARK The hypothesis of the Theorem and the Corollary that $\mathcal{A}$ be reflective can be avoided, provided that every source in $\mathcal{X}$ has a (strong epi, mono-source)-factorization (cf. Exercise 7.A). In this case $S(\mathcal{A})$ is always (strongly epi-) reflective, so one can apply the Theorem to $S(\mathcal{A})$ rather than to $\mathcal{A}$ itself and obtains with the Lemma the same result for any full subcategory $\mathcal{A}$ of $\mathcal{X}$:

$$\Delta(\mathrm{reg}^{\mathcal{A}}) = \Delta(\mathrm{reg}^{S(\mathcal{A})}) = S(S(\mathcal{A})) = S(\mathcal{A}).$$

EXAMPLES

(1) The Proposition (as well as the Theorem) applies to the category **Grp** of groups. From Example 6.7 (3) we know that for $\mathcal{A} = \mathbf{AbGrp}$, $\mathrm{reg}^{\mathcal{A}} = \mathbf{k}$, hence $\Delta(C^{\mathbf{k}}) = \mathcal{A}$. Since $\nu = C^{\mathbf{0}}$, we also have $\Delta(C^{\mathbf{0}}) = \mathcal{A}$ (cf. Example 7.1(1)). Hence the Delta-subcategories of distinct maximal closure operators may coincide.

(2) The Theorem (more precisely: its Corollary) is applicable to the category **Top** and its strongly strongly epireflective subcategories, for instance to $\mathbf{Top}_0$, $\mathbf{Top}_1$, **Haus** and **Ury** . Since the regular closure of $\mathbf{Top}_0$ is b , the Theorem reproduces the equality $\Delta(b) = \mathbf{Top}_0$ (cf. Example 7.1(2)).

(3) The following example shows that the hypothesis of the existence of a generating class $\mathcal{P}$ in $\mathcal{X}$ such that each $k_{P,B}$ is monic $(P \in \mathcal{P}, B \in \mathcal{X})$, is by no means a necessary condition for $\Delta(\mathrm{reg}^{\mathcal{A}}) = S(\mathcal{A})$. Let $\mathcal{X}$ be the category $(\mathbf{Set} \times \mathbf{Set})^{op}$ with the subobject structure given by monomorphisms (i.e., pairs of surjective maps in **Set**), and consider the (strongly) epireflective subcategory $\mathcal{A}$ of objects (A, B) with $(A = \emptyset \Leftrightarrow B = \emptyset)$. It is easy to check that the $\mathcal{A}$-regular closure is discrete on $\mathcal{A}$ but trivial outside $\mathcal{A}$, which implies $\Delta(\mathrm{reg}^{\mathcal{A}}) = \mathcal{A}$. On the other hand, each generating class $\mathcal{P}$ of $\mathcal{X}$ must contain an object (P, Q) with Q not empty and P having at least two elements, but $k_{(P,Q),(P,\emptyset)}$ is not monic in in $\mathcal{X}$. that

PROBLEM *Is every full and replete strongly epireflective subcategory of a category satisfying the general hypotheses of this section a Delta-subcategory ?*

7.3 Essentially equivalent closure operators

When trying to characterize the epimorphisms of a strongly epireflective subcategory $\mathcal{A}$ of $\mathcal{X}$, it should be helpful to find an easily described closure operator C of $\mathcal{X}$ whose idempotent hull coincides with $\mathrm{reg}^{\mathcal{A}}$ on $\mathcal{A}$. The following definition and theorem describe this situation more precisely. We continue to work under the hypotheses of 7.1.

DEFINITON Two closure operators are called *essentially equivalent* if and only if they define the same Delta-subcategory $\mathcal{A} = \Delta(C) = \Delta(D)$ and they are isomorphic on $\mathcal{A}$, i.e., $C|_{\mathcal{A}} \cong D|_{\mathcal{A}}$.

A closure operator C and its idempotent hull $\hat{C}$ always induce the same Delta-subcategory (cf. Exercise 7.D), but may fail to be essentially equivalent (see Example 7.1 (4)).

THEOREM *Let $\mathcal{A}$ be a Delta-subcategory of $\mathcal{X}$, and let C be a closure operator of $\mathcal{X}$. Then the conditions* (i), (ii), (iii) *below are equivalent and imply* (iv)*:*

(i) $\mathrm{reg}^{\mathcal{A}}$ *and* $\hat{C}$ *are essentially equivalent,*

(ii) $\mathcal{A} = \Delta(C)$, *and an $\mathcal{M}$-subobject in $\mathcal{A}$ is $\mathcal{A}$-regular if and only if it is C-closed,*

(iii) $\mathcal{A} = \Delta(C)$, *and every C-closed $\mathcal{M}$-subobject in $\mathcal{A}$ is $\mathcal{A}$-regular,*

(iv) $\mathcal{A} = \Delta(C)$, *and a morphism in $\mathcal{A}$ is $\mathcal{A}$-epic if and only if it is $\hat{C}$-dense.*

All four conditions are equivalent if $\mathrm{reg}^{\mathcal{A}}$ *is weakly hereditary on* $\mathcal{A}$.

Proof (i) $\Rightarrow$ (ii) By hypothesis one has

$$\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}}) = \Delta(\hat{C}) = \Delta(C).$$

Furthermore, since $\hat{C}|_{\mathcal{A}} = \mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ and $\mathcal{M}^{\hat{C}} = \mathcal{M}^{C}$ (cf. Corollary 5.4), $\mathcal{M}$-subobjects in $\mathcal{A}$ are $\mathcal{A}$-regular if and only if they are C-closed.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) From $\mathcal{A} = \Delta(C)$ one has $C \le \mathrm{reg}^{\Delta(C)} = \mathrm{reg}^{\mathcal{A}}$ from Theorem 7.1, hence $\mathcal{A}$-regular $\mathcal{M}$-subobjects are C-closed. With the hypothesis of (iii), this means $\mathrm{Reg}(\mathcal{A}) = \mathcal{M}^{C} \cap \mathrm{Mor}\,\mathcal{A} = \mathcal{M}^{\hat{C}} \cap \mathrm{Mor}\,\mathcal{A}$, which implies $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = \hat{C}|_{\mathcal{A}}$ with Proposition 5.4 since both operators are idempotent.

(i) $\Rightarrow$ (iv) is shown as (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (iii) If epimorphisms of the category $\mathcal{A}$ are $\hat{C}$-dense, then every C-closed $\mathcal{M}$-subobject in $\mathcal{A}$ is an extremal monomorphism of $\mathcal{A}$, i.e., $\mathrm{epi}^{\mathcal{A}}$-closed (cf. Theorem 6.2(2)). But under the additional hypothesis, $\mathrm{epi}^{\mathcal{A}}|_{\mathcal{A}} = \mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ (cf. Corollary 6.2). $\square$

EXAMPLE (*Epimorphisms of* **Ury**) We want to show that the epimorphisms of **Ury** are exactly the θ^{∞}-dense maps. From Example 7.1 (4) we have $\mathbf{Ury} = \Delta(\theta)$, so that according to the Theorem, it suffices to show that every θ-closed subset $M \subseteq X \in \mathbf{Ury}$ is **Ury**-regular. We prove $K = X +_M X \in \mathbf{Ury}$ and consider $x, y \in K$. If $\varepsilon(x) = \varepsilon(y)$ with the common retraction ε of the canonical injections i, j , then $\varepsilon(x), \varepsilon(y)$ can be separated in X by disjoint closed neighbourhoods U, V , respectively, and $\varepsilon^{-1}(U) = \varepsilon^{-1}(V)$ are disjoint closed neighbourhoods of x, y in K , respectively. If $\varepsilon(x) = \varepsilon(y)$, we may assume $x = i(a), y = j(a)$ and $a \in M$. The θ-closedness of M produces an open neighbourhood W of a in X with $\overline{W} \cap M = \emptyset$. Now $i(W)$ is an open neighbourhood of x in K with $\overline{i(W)} \subseteq i(X \setminus M)$ since for every $z \in X$, there is a neighbourhood Z of $j(z)$ which misses $i(W)$: for $z \in M$ one has a neighbourhood N of z with $N \cap M = \emptyset$, and one can take $Z := i(N) \cup j(N)$; for $z \in X \setminus M$, one takes $Z = j(X)$. Similarly $\overline{j(W)} \subseteq j(X \setminus M)$. Hence we have produced disjoint closed neighbourhoods of x and y in K .

7.4 Regular hull and essentially strong closure operators

We continue to work under the hypotheses of 7.1 on $\mathcal{X}$. Then Theorem 7.1 tells us that every closure operator C of $\mathcal{X}$ has a *regular hull*

$$C^{\text{reg}} = \text{reg}^{\Delta(C)}.$$

In fact, for every full subcategory $\mathcal{A}$ of $\mathcal{X}$ with $C \leq \text{reg}^{\mathcal{A}}$ one has (by adjunction) $\mathcal{A} \subseteq \Delta(C)$, hence $\text{reg}^{\Delta(C)} \leq \text{reg}^{\mathcal{A}}$. The gap between C and its regular hull may be substantial, even for "good" operators C, as the following example shows.

Example In the category **Grp**, the regular hull of the normal closure ν (that is the maximal closure operator $C^{\mathbf{0}}$), is the maximal closure operator $C^{\mathbf{k}}$. Indeed, with Example 7.1 (1) and 6.7(3) one has

$$\nu^{\text{reg}} = \text{reg}^{\Delta(\nu)} = \text{reg}^{\mathcal{A}} = C^{\mathbf{k}},$$

with $\mathcal{A} = \mathbf{AbGrp}$. Even when we replace ν by its idempotent hull ν^∞ , these equalities remain valid, although $\nu^\infty < C^{\mathbf{k}}$ (as evaluation on the trivial subgroup shows).

Essentially strong closure operators as defined below are designed in order to gain further insight in the gap between C and its regular hull. Recall first that a closure operator C is strong if every C-closed $\mathcal{M}$-subobject $m : M \to X$ is strongly C-closed, i.e., if the canonical injections $i, j : X \to K = X +_M X$ of its cokernelpair are C-closed. Obviously, in order to determine the epimorphisms of $\mathcal{A} = \Delta(C)$, it is sufficient to have this property whenever $X \in \mathcal{A}$. Assuming that $\mathcal{X}$ has cokernelpairs, we therefore define:

Definition A closure operator is *essentially strong* if every C-closed subobject $m : M \to X$, with $X \in \Delta(C)$, is strongly C-closed.

Clearly, every strong closure operator is essentially strong. Hence, whenever regular closure operators are strong (in particular, if $\mathcal{M}$ is generated by its $\vee$-prime elements, see Frolík's Lemma 6.5), then they are *a fortiori* essentially strong.

On the other hand, we have seen that the regular closure operator of $\mathcal{A} =$ **AbGrp** in **Grp** is not strong (cf. Example 6.6(1)); however, since $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}}$ is discrete and $\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}})$, $\mathrm{reg}^{\mathcal{A}}$ is still essentially strong.

We are now able to take advantage of the strong modification $\tilde{C}$ of C as constructed in 6.6, as follows.

PROPOSITION *Let $\mathcal{M}$ be generated by its $\vee$-prime elements. Then every closure operator C of the category $\mathcal{X}$ with cokernelpairs such that $\Delta(C)$ is strongly epireflective in $\mathcal{X}$, satisfies the inequalities*

$$C \leq \tilde{C} \leq C^{\mathrm{reg}},$$

and C and $\tilde{C}$ induce the same Delta-subcategory of $\mathcal{X}$.

Proof $C \leq \tilde{C}$ was shown in Proposition 6.6. For $\tilde{C} \leq C^{\mathrm{reg}} = \mathrm{reg}^{\Delta(C)}$ it suffices to show that every $\Delta(C)$-closed $\mathcal{M}$-subobject m is $\tilde{C}$-closed. But by Corollary 6.5, every $\Delta(C)$-closed subobject m is even strongly $\Delta(C)$-closed, i.e., its cokernelpair injections i, j are $\Delta(C)$-closed, hence C-closed (since $C \leq \mathrm{reg}^{\Delta(C)}$), and this means that m is strongly C-closed, which trivially implies that m is $\tilde{C}$-closed.

Since $\Delta(C) = \Delta(\mathrm{reg}^{\Delta(C)})$, the equality $\Delta(C) = \Delta(\tilde{C})$ follows trivially. □

Recall from Corollary 6.6 that if $\mathcal{M}$ is generated by its $\vee$-prime elements, then C is strong if and only if C and $\tilde{C}$ have isomorphic idempotent hulls. Here is an "essential version" of this fact:

COROLLARY *Under the assumption of the Proposition, a closure operator C is essentially strong if and only if C and $\tilde{C}$ have essentially equivalent idempotent hulls. In particular, if $C \cong \tilde{C}$ then C is essentially strong.*

Proof From the Proposition we obtain (with Exercise 7.D)

$$\Delta(\hat{C}) = \Delta(C) = \Delta(\tilde{C}) = \Delta(\hat{\tilde{C}}).$$

Hence C and $\tilde{C}$ have essentially equivalent idempotent hulls if and only if they coincide on $\Delta(C)$. But the latter property means equivalently that in $\Delta(C)$, C and $\tilde{C}$ give the same notion of closedness. However, by Theorem 6.6, $\tilde{C}$-closedness means strong C-closedness. □

In Theorem 7.3 we described the situation when C^{∞} and C^{reg} are essentially equivalent (consider $\mathcal{A} = \Delta(C)$ in the Theorem). With the Proposition we see that a finer approach would be to compare $\hat{\tilde{C}}$ and C^{reg}. In fact, we shall show below that for *additive* C, the latter two closure operators are already essentially equivalent, under the following hypotheses on $\mathcal{X}$:

- $\mathcal{X}$ is $\mathcal{M}$-complete, with $\mathcal{M}$ closed under composition and containing all regular monomorphisms of $\mathcal{X}$,
- $\mathcal{X}$ is finitely $\mathcal{M}$-complete and has cokernelpairs,
- $\mathcal{M}$ is generated by its $\vee$-prime elements, and each subobject lattice $\mathcal{M}/X$ is distributive.

THEOREM *Let C be an additive closure operator of a category $\mathcal{X}$ satisfying the hypotheses listed above, such that $\Delta(C)$ is strongly epireflective in $\mathcal{X}$. Then:*

(1) *C^{reg} and $\hat{\hat{C}}$ are essentially equivalent.*

(2) *$\tilde{C}$ is essentially strong.*

(3) *C is essentially strong if and only if C^{reg} and $\hat{C}$ are essentially equivalent; in this case, the epimorphisms of $\Delta(C)$ are precisely the $\hat{C}$-dense morphisms in $\Delta(C)$.*

Proof Once we have shown (1), then (2) and (3) follow with the Corollary. Hence we have to prove only (1). But since C^{reg} and $\hat{\hat{C}}$ induce the same Delta-subcategory and since always $\hat{\hat{C}} \leq C^{\text{reg}}$ by the Proposition (note that C^{reg} is idempotent), it suffices to show that every $\tilde{C}$-closed $\mathcal{M}$-subobject $m : M \to X$ with $X \in \Delta(C)$ is $\Delta(C)$-closed, and according to Frolík's Lemma 6.5 this means that we must show $K \in \Delta(C)$, with $i, j : X \to K = X +_M X$ the cokernelpair of m. But since m is $\tilde{C}$-closed, i and j are C-closed (cf. Theorem 6.6) and then also $z := i^2 \vee j^2$, since C is additive (cf. Proposition 2.6). Since $i \vee j \cong 1_K$, we obtain for the diagonal $\delta_K : K \to K \times K$

$$\delta_K \cong (\delta_K \cdot i) \vee (\delta_K \cdot j) \cong (i^2 \cdot \delta_X) \vee (j^2 \cdot \delta_X) \leq i^2 \vee j^2 = z,$$

hence $d := c_{K \times K}(\delta_K) \leq z$.

For the common retraction $\varepsilon : K \to X$ of i and j, one has $\varepsilon^2(\delta_K) \cong \delta_X$. Since $c_{X \times X}(\delta_X) \cong \delta_X$ by hypothesis on X, C-continuity of ε^2 gives

$$\varepsilon^2(d) \leq c(\varepsilon^2(\delta_K)) \cong c(\delta_X) \cong \delta_X,$$

hence $d \leq (\varepsilon^2)^{-1}(\delta_X)$. In addition one easily establishes the inequalities

$$i^2 \wedge (\varepsilon^2)^{-1}(\delta_X) \leq \delta_K \text{ and } j^2 \wedge (\varepsilon^2)^{-1}(\delta_X) \leq \delta_K.$$

(With $u := (\varepsilon^2)^{-1}(\delta_X) : U \to K \times K$ and $v : U \to X$ the morphism with $\delta_X \cdot v = \varepsilon^2 \cdot u$, just confirm the equation $\delta_K \cdot i \cdot v = u$.) Now the distributivity of $\mathcal{M}/K \times K$ allows to conclude $d \leq z \wedge (\varepsilon^2)^{-1}(\delta_X) \leq (i^2 \wedge (\varepsilon^2)^{-1}(\delta_X)) \vee (j^2 \wedge (\varepsilon^2)^{-1}(\delta_X)) \leq \delta_K$, which means that $\delta_K \cong c(\delta_K)$ must be C-closed, hence $K \in \Delta(C)$ by Proposition 7.1. □

REMARKS

(1) If $\mathcal{X}$ permits the construction of the additive core C^+ as in 4.8, then statement (1) of the Theorem can be slightly generalized , as follows:

$$(C^+)^{\mathrm{reg}} \leq \hat{\hat{C}}|_{\Delta(C^+)} \leq C^{\mathrm{reg}}|_{\Delta(C^+)}.$$

The proof can remain essentially unchanged; just observe that the morphism z is C^+-closed, and it suffices to consider $X \in \Delta(C^+)$.

(2) In Exercise 7.H it is shown that the sequential closure σ of **Top** is not essentially strong. With the Theorem, this shows that there is an additive, hereditary closure operator which does not coincide with its strong modification.

(3) The Theorem remains valid in the category $\mathbf{Mod}_R$ (more generally in suitable additive categories), although in $\mathbf{Mod}_R$ the subobject lattices fail to be distributive: see Exercise 7.I.

7.5 Characterization of additive regular closure operators

For simplicity, we assume the hypotheses of Theorem 7.4 throughout this section. In addition we need that $\mathcal{E}$ is stable under pullback and that f^{-1} (-) preserves arbitrary joins for every morphism f, which, in particular gives every subobject lattice the structure of a frame. A regular closure operator C of $\mathcal{X}$ (one that is of the form $C \cong \mathrm{reg}^{\mathcal{A}}$ or, equivalently, $C \cong C^{\mathrm{reg}}$) is certainly idempotent (Theorem 6.2) and essentially strong (as remarked after Definition 7.4). Furthermore, C is determined by its values in $\Delta(C)$, according to the formula

$$c_X(m) \cong \rho_X^{-1}(c_{RX}(\rho_X(m)))$$

established in 6.3, with $\rho_X : X \to RX$ the $\Delta(C)$-reflexion of $X \in \mathcal{X}$. For convenience, we call *initial* any closure operator C satisfying this formula for all $X \in \mathcal{X}$ (cf. Exercise 7.J). Idempotency, essential strength and initiality are therefore necessary conditions for a closure operator to be regular. We shall show that they are also sufficient, provided that C is additive. First we must prove:

LEMMA *The idempotent hull of an initial closure operator is initial.*

Proof Since $\hat{C} \cong C^{\infty}$ is constructed as in 4.6, it suffices to show that

- the composite DC of two initial closure operators with the same Δ-subcategory is initial,
- the join $\bigvee_{i \in I} C_i$ of any non-empty family of initial closure operators with the same Delta-subcategory is initial.

For the first assertion we observe that the strong epimorphism $\rho_X : X \to RX$ (the reflexion of the common Delta-subcategory) belongs to $\mathcal{E}$ (since every $\mathcal{M}$-morphism is monic), so that pullback stability of $\mathcal{E}$ gives $\rho_X(\rho_X^{-1}(k)) \cong k$ for all $k \in \mathcal{M}/RX$. One therefore has

$$d(c(m)) \cong \rho_X^{-1}(d(\rho_X(c(m)))) \cong \rho_X^{-1}((d(c(\rho_X(m))))$$

for all $m \in \mathcal{M}/RX$, so that DC is initial.

For the second assertion, simply use the fact that $\rho_X(\text{-})$ preserves arbitrary joins as a left adjoint and that $\rho_X^{-1}(\text{-})$ does so by hypothesis. □

THEOREM *Let C be an additive closure operator of $\mathcal{X}$. Then:*

(1) *If C is initial and essentially strong, then $\hat{C}$ is regular.*

(2) *C is regular if and only if C is idempotent, initial and essentially strong.*

Proof (1) By Theorem 7.4 (3), $\hat{C}$ and C^{reg} are essentially equivalent. But according to the Lemma and to Theorem 6.3, both operators are initial. Hence coincidence on the common Delta-subcategory implies global coincidence (up to isomorphism). (2) follows from (1) and the initial remark of this section. □

PROBLEM *Does the Theorem remain true without the assumption of additivity?*

In trying not to use additivity a priori, one is tempted to replace a potentially non-additive regular closure operator by its additive core. We shall show next in the context of a topological category $\mathcal{X}$ over **Set** with its usual subobject structure (which automatically satisfies all hypotheses of this section !), that the passage from $\mathrm{reg}^{\mathcal{A}}$ to $(\mathrm{reg}^{\mathcal{A}})^+$ does not affect the validity of the Generating Diagonal Theorem 7.2, but that the important property of essential strength will be lost, unless $\mathrm{reg}^{\mathcal{A}}$ was already additive. (We assume the topological category to have the property that constant **Set**-maps between $\mathcal{X}$-objects lift to $\mathcal{X}$-morphisms.)

PROPOSITION *(Additive Diagonal Theorem) Every strongly epireflective subcategory $\mathcal{A}$ of a topological category $\mathcal{X}$ over* **Set** *is the Delta-subcategory of an additive closure operator of $\mathcal{X}$; more precisely,*

$$\mathcal{A} = \Delta((\mathrm{reg}^{\mathcal{A}})^+).$$

Proof Since $(\mathrm{reg}^{\mathcal{A}})^+ \leq \mathrm{reg}^{\mathcal{A}}$, trivially $\mathcal{A} \subseteq \Delta((\mathrm{reg}^{\mathcal{A}})^+)$. Conversely, let $X \in \Delta(C)$ with $C = (\mathrm{reg}^{\mathcal{A}})^+$. We denote the underlying set of the object X again by X and prove that for every $x \in X$, the set $\{x\}$ is C-closed. With Lemma 4.11, for every $y \in c_X(\{x\})$ we have

$$(x, y) \in \{x\} \times c_X(\{x\}) \subseteq c_{X\times X}(\{(x, x)\}) \subseteq c_{X\times X}(\Delta_X) = \Delta_X,$$

with $\Delta_X := \delta_X(X)$, hence $x = y$. According to the construction of the additive core, the C-closed set $\{x\}$ is an intersection of sets F_i each of which is a non-empty finite union of $\mathcal{A}$-closed sets. Since x must belong to at least one member of this union, one may assume F_i to be $\mathcal{A}$-closed, so that also $\{x\}$ is $\mathcal{A}$-closed. Now we can apply Lemma 6.9 (which holds true not only for **Top**, but for every topological category over **Set** – see the remark after Proposition 6.9) and conclude $X \in \mathcal{A}$. □

COROLLARY *For a regular closure operator C of a topological category $\mathcal{X}$ over* **Set**, *the following are equivalent:*

(i) C^+ *is essentially strong,*

(ii) C^+ *is regular,*

(iii) C *is additive.*

Proof (i) $\Rightarrow$ (ii) With C also C^+ is idempotent. Likewise, initiality is inherited from C by C^+ (see Exercise 7.J). Hence one may apply the Theorem to conclude that C^+ must be regular.

(ii) $\Rightarrow$ (iii) With $\mathcal{A} = \Delta(C)$ one obtains from the Proposition

$$C = \mathrm{reg}^{\mathcal{A}} = \mathrm{reg}^{\Delta(C^+)} = (C^+)^{\mathrm{reg}},$$

hence (ii) implies that $C = C^+$ must be additive.

(iii) $\Rightarrow$ (i) is trivial. □

The Corollary provides a device for constructing an additive, idempotent and initial closure operator C which fails to be essentially strong: simply take the additive core of a non-additive regular closure operator. That such closure operators exist is shown in Example (2) below.

EXAMPLES

(1) Let Z be the closure operator in **Top** which to every subset M of a topological space X assigns the intersection of all zero-sets containing M (cf. Example 6.9(4)). It is easy to see that Z is an idempotent, additive and initial closure operator with $\Delta(Z)$ = **FHaus**, the category of *functionally Hausdorff spaces*, i.e., of spaces X in which distinct points may be separated by real-valued continuous functions on X. Furthermore, it is not difficult to show $\tilde{Z} = Z$, hence Z is essentially strong by Corollary 7.4. Consequently, by the Theorem, Z is the regular closure operator of **FHaus**.

(2) (Cagliari and Cicchese [1982]) Let $\mathcal{A}$ be a proper rigid class of topological spaces, i. e., a large full subcategory of **Top** in which all morphisms are constant or identity morphisms. (For existence of $\mathcal{A}$, see Kannan and Rajagopalan [1978].) According to Lemma 6.9, singleton subsets of spaces in $\mathcal{A}$ are $\mathcal{A}$-closed. However, any two-element subset $\{x, y\}$ of a space $X \in \mathcal{A}$ is $\mathcal{A}$-dense in X. In fact, its $\mathcal{A}$-closure is an intersection of equalizers of pairs $f_i, g_i : X \to Y_i \in \mathcal{A}$ with $f_i(x) = g_i(x)$, $f_i(y) = g_i(y)$; if $Y_i = X$, then $f_i = g_i = 1$, and if $Y_i \neq X$, then $f_i = g_i$ is constant, so that each equalizer gives X. Hence $\mathrm{reg}^{\mathcal{A}}$ is not additive.

7.6 The Pumplün-Röhrl correspondence

We return to the setting of 6.2/7.1 and consider an $\mathcal{M}$-complete category $\mathcal{X}$ with equalizers which belong to $\mathcal{M}$, with $\mathcal{M}$ closed under composition.

Delta-subcategories may be thought of as categories of separated objects. A slightly more special notion of separation arises when one considers the relation

$$d \overset{A}{\Rightarrow} 1$$

between morphisms d and objects A in $\mathcal{X}$, i.e., for all morphisms u, v with codomain A and composable with d from the right one has

$$d \cdot u = d \cdot v \Rightarrow u = v.$$

For a full subcategory $\mathcal{A}$ of $\mathcal{X}$ we defined in 6.1 the morphism class

$$\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) = \{d : (\forall A \in \mathcal{A})\ d \overset{A}{\Rightarrow} 1\}.$$

"Conversely", for a class $\mathcal{D}$ of morphisms in $\mathcal{X}$, let

$$\mathrm{Sep}_{\mathcal{X}}(\mathcal{D}) := \{A : (\forall d \in \mathcal{D})\ d \overset{A}{\Rightarrow} 1\}$$

be the class of *$\mathcal{D}$-separated objects* in $\mathcal{X}$, considered as a full subcategory of $\mathcal{X}$. It is elementary to verify that for every $\mathcal{A}$ and $\mathcal{D}$ one has

$$\mathcal{D} \subseteq \mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \Rightarrow \mathcal{A} \subseteq \mathrm{Sep}_{\mathcal{X}}(\mathcal{D}).$$

Since $\mathrm{Epi}_{\mathcal{X}}(\text{-})$ is order preserving, this proves the first part of:

PROPOSITION *There is an adjunction*

$$\mathrm{Sep}_{\mathcal{X}} \dashv \mathrm{Epi}_{\mathcal{X}} : SUB(\mathcal{X})^{op} \to MOR(\mathcal{X}),$$

called the PR-correspondence, *with* $MOR(\mathcal{X})$ *denoting the conglomerate of all subclasses of morphisms of* $\mathcal{X}$ *. If* $\mathcal{X}$ *has finite products, the following assertions are equivalent for every morphism class* $\mathcal{D}$ *and every object* A *in* $\mathcal{X}$*:*

(i) $A \in \mathrm{Sep}_{\mathcal{X}}(\mathcal{D})$;

(ii) $\delta_A : A \to A \times A$ *belongs to* $\mathcal{D}_\perp$*;*

(iii) *for all* $f : X \to A$ *in* $\mathcal{X}$*,* $<1, f>: X \to X \times A$ *belongs to* $\mathcal{D}_\perp$*;*

(iv) *for all* $f, g : X \to A$ *in* $\mathcal{X}$*, the equalizer of* f, g *belongs to* $\mathcal{D}_\perp$*,*

with $\mathcal{D}_\perp = \{n : (\forall d \in \mathcal{D})\ d \perp n\}$ *defined as in Theorem* 1.8.

Proof We sketch the proof of (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). In (ii), $<1, f>$ is the equalizer of $p_1 f$ and p_2, with p_1, p_2 the product projections, and in (ii) $\delta_A = <1_A, 1_A>$, so that the first two implications are clear. For (ii) $\Rightarrow$ (i), consider

$m : M \to X$ in $\mathcal{D}$ and $u, v : X \to A$ with $u \cdot m = v \cdot v$. Then the commutative diagram (7.1) gives a morphism s with $\delta_A \cdot s =< U, v >$, hence $u = v$. Finally, for (i) $\Rightarrow$ (iv), whenever one has $t \cdot a = b \cdot m$ with t the equalizer of f, g and with $m \in \mathcal{D}$, then $f \cdot b = g \cdot b$ by hypothesis on A. Hence b factors through the equalizer t, which gives the desired diagonal. □

Remarks

(1) The equivalence of conditions (i) and (iv) remains valid without the existence of finite products in $\mathcal{X}$, since the given proof does not use them, and also (i) $\Rightarrow$ (iv) can be shown directly without the use of products.

(2) For every class $\mathcal{D}$, $\mathrm{Sep}_{\mathcal{X}}(\mathcal{D})$ is closed under mono-sources. Since also $\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) = \mathrm{Epi}_{\mathcal{X}}(S(\mathcal{A}))$, with $S(\mathcal{A})$ the closure of $\mathcal{A}$ under mono-sources in $\mathcal{X}$ (see 7.1), no generality is lost when restricting the PR-correspondence to subclasses $\mathcal{A}$ closed under mono-sources.

(3) For every class $\mathcal{D}$, we may assume $\mathcal{D} \subseteq \mathcal{M}$ when determining the $\mathcal{D}$-separated objects. Indeed, since $\mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M}$ and therefore $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{X})$, one easily shows $\mathrm{Sep}_{\mathcal{X}}(\mathcal{D}) = \mathrm{Sep}_{\mathcal{X}}(\mathcal{D}_{\mathcal{M}})$, with

$$\mathcal{D}_{\mathcal{M}} := \{d(1_X) : d : X \to Y \text{ in } \mathcal{D}\} \subseteq \mathcal{M}.$$

Note that $\mathcal{D}_{\mathcal{M}}$ is a subclass of $\mathcal{D}$ whenever $\mathcal{D}$ is right cancellable.

(4) For every class $\mathcal{D}$, the class $\mathcal{D}_{\perp} \cap \mathcal{M}$ is stable under pullback and under $\mathcal{M}$-intersections. According to Proposition 5.4 it is therefore the class of C-closed subobjects for a uniquely determined idempotent closure operator $\hat{\tau}^{\mathcal{D}}$. Hence the Proposition together with Proposition 7.1 gives that

$$\mathrm{Sep}_{\mathcal{X}}(\mathcal{D}) = \Delta(\hat{\tau}^{\mathcal{D}})$$

is a particular Delta-subcategory, in case $\mathcal{X}$ has finite products.

(5) In the category $\mathcal{X} = \mathbf{Top}$ one may in fact find Delta-subcategories $\mathcal{A}$ which cannot be presented in the form $\mathrm{Sep}_{\mathcal{X}}(\mathcal{D})$. Cagliari and Cicchese [1982] showed that the category $\mathcal{A} = S(\{D_2\})$ with D_2 the two-point discrete space is such a subcategory, by confirming that the *Tychonoff corkscrew* (see Steen and Seebach [1978], #90) belongs to

$$\mathrm{E}(\mathcal{A}) = \mathrm{E}_{\mathcal{X}}(\mathcal{A}) := \mathrm{Sep}_{\mathcal{X}}(\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}),$$

but not to $\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}})$ itself.

In general, from reasons which become clear by the Corollary below, we call $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$ the *epi-closure* of $\mathcal{A}$ in $\mathcal{X}$. The true reason for the statement of Remark (4) is given by:

THEOREM *The PR-correspondence factors through the Salbany-correspondence:*

$$\begin{tikzcd} SUB(\mathcal{X})^{op} \arrow[rr, shift left, "\mathrm{Epi}_{\mathcal{X}}"] \arrow[dr, shift left, "\Delta"] & & MOR(\mathcal{X}) \arrow[ll, shift left, "\mathrm{Sep}_{\mathcal{X}}"] \arrow[dl, shift left, "\tau^{(-)}"'] \\ & CL(\mathcal{X},\mathcal{M}) \arrow[ul, shift left, "\mathrm{reg}^{(-)}"] \arrow[ur, shift left, "\mathcal{E}^{(-)}"'] & \end{tikzcd} \tag{7.3}$$

Here the outer arrows denote the right adjoints, with $\mathcal{E}^{(-)}$ assigning to each closure operator the class $\mathcal{E}^C$ of C-dense maps and $\tau^{(-)}$ denoting its left adjoint.

Proof After the Proposition and Theorem 7.1, we just need to show that $\mathcal{E}^{(-)}$ does in fact have a left adjoint $\tau^{(-)}$ and that the right adjoints (and therefore the left adjoints) in (7.3) commute. But the latter statement follows from Theorem 6.2(1), while the former may be concluded immediately with Theorem 1.3(1) from the preservation of meets by $\mathcal{E}^{(-)}$ (see Proposition 4.4(3)). But we may also give a somewhat more concrete description of the operator $\tau^{\mathcal{D}}$ (for any morphism class $\mathcal{D}$), as follows: given $\mathcal{D} \subseteq \mathrm{Mor}\,\mathcal{X}$, pass to $\mathcal{D}_{\mathcal{M}}$ as in Remark (3), consider the least subclass $\overline{\mathcal{D}_{\mathcal{M}}}$ of $\mathcal{M}$ containing $\mathcal{D}_{\mathcal{M}}$ satisfying properties (a), (b), (c) of Theorem* of 5.4, and then let $\tau^{\mathcal{D}}$ be the weakly hereditary closure operator whose dense maps are given by $\overline{\mathcal{D}_{\mathcal{M}}}$. In fact, in case $\mathcal{D} = \mathcal{E}^C$ for a closure operator C of $\mathcal{X}$, we already know that $\mathcal{D}_{\mathcal{M}} = \mathcal{D} \cap \mathcal{M}$ satisfies properties (a), (b), (c), hence $\overline{\mathcal{D}_{\mathcal{M}}} = \mathcal{E}^C$, and $\tau^{\mathcal{D}}$ is the weakly hereditary core of C (cf. Corollary 5.4). Consequently, one has $\tau^{\mathcal{E}^C} \leq C$. For arbitrary $\mathcal{D}$, we must show $\mathcal{D} \subseteq \mathcal{E}^{\tau^{\mathcal{D}}}$; but this is clear since $\mathcal{E}^{\tau^{\mathcal{D}}} = \overline{\mathcal{D}_{\mathcal{M}}}$, so that

$$\mathcal{D} \subseteq \mathcal{E} \cdot \mathcal{D}_{\mathcal{M}} \subseteq \mathcal{E} \cdot \mathcal{E}^{\tau^{\mathcal{D}}} \subseteq \mathcal{E}^{\tau^{\mathcal{D}}}.$$

$\square$

We remark that since regular closure operators are idempotent, the left diagonal adjunction of (7.3) factors through the conglomerate $IDCL(\mathcal{X},\mathcal{M})$ of idempotent operators. Note that the idempotent hull $\hat{\tau}^{\mathcal{D}}$ of $\tau^{\mathcal{D}}$ for any $\mathcal{D}$ induces the same Delta-subcategory as $\tau^{\mathcal{D}}$ and has the simple description given by Remark (4) in case $\mathcal{X}$ has finite products. We remark further that $\tau^{\mathcal{D}}$ is always weakly hereditary, so that the right diagonal adjunction of (7.3) factors through the conglomerate $WHCL(\mathcal{X},\mathcal{M})$ of weakly hereditary operators. Its idempotent hull $\hat{\tau}^{\mathcal{D}}$ is therefore both, idempotent and weakly hereditary (see Corollary 5.4).

This leads to the following refinement of diagram (7.3), and to a complete characterization of categories of separated objects as Delta-subcategories, as given by the Corollary below.

$$\begin{tikzcd}[column sep=large] SUB(\mathcal{X})^{op} \arrow[rr, shift left, "\mathrm{Epi}_{\mathcal{X}}"] \arrow[d, shift right, "\mathrm{reg}^{(-)}"'] & & \mathrm{MOR}(\mathcal{X}) \arrow[ll, shift left, "\mathrm{Sep}_{\mathcal{X}}"] \arrow[d, shift right, "\tau^{(-)}"'] \\ IDCL(\mathcal{X},\mathcal{M}) \arrow[u, shift right, "\Delta"'] \arrow[r, shift right, "\mathrm{incl}"'] & CL(\mathcal{X},\mathcal{M}) \arrow[l, shift right, "\hat{(-)}"'] \arrow[r, shift right, "\check{(-)}"'] & WHCL(\mathcal{X},\mathcal{M}) \arrow[u, shift right, "\mathcal{E}^{(-)}"'] \arrow[l, shift right, "\mathrm{incl}"'] \end{tikzcd} \tag{7.4}$$

Here the inner arrows are left adjoints to the outer arrows. For the epi-closure of a subcategory (as defined after Remark (5)), we conclude:

COROLLARY *For any full subcategory $\mathcal{A}$ of $\mathcal{X}$, $\mathrm{E}_{\mathcal{X}}(\mathcal{A}) = \Delta(\mathrm{epi}^{\mathcal{A}})$, and the following assertions are equivalent:*

(i) $\mathcal{A} = \mathrm{E}_{\mathcal{X}}(\mathcal{A})$*;*

(ii) $\mathcal{A} = \mathrm{Sep}_{\mathcal{X}}(\mathcal{D})$ *for some class $\mathcal{D}$ of morphisms;*

(iii) $\mathcal{A} = \Delta(\mathrm{epi}^{\mathcal{A}})$*;*

(iv) $\mathcal{A} = \Delta(C)$ *for some idempotent and weakly hereditary closure operator C of $\mathcal{X}$;*

(v) $\mathcal{A} = \Delta(C)$ *for some weakly hereditary closure operator C of $\mathcal{X}$.*

Proof First we note that for a weakly hereditary closure operator C one has $C \cong \tau^{\mathcal{E}^C}$, hence $\Delta(C) = \mathrm{Sep}_{\mathcal{X}}(\mathcal{E}^C)$ by the commutativity of diagram (7.3). In case $C = \mathrm{epi}^{\mathcal{A}} = (\mathrm{reg}^{\mathcal{A}})\check{}$ (cf. Theorem 6.2), this shows $\mathrm{E}_{\mathcal{X}}(\mathcal{A}) = \Delta(\mathrm{epi}^{\mathcal{A}})$.

(i) $\Leftrightarrow$ (ii) holds since we have a Galois correspondence, and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are trivial since $\mathrm{epi}^{\mathcal{A}}$ is both weakly hereditary and idempotent. Finally, for (v) $\Rightarrow$ (i) and C weakly hereditary, one has $C \leq \mathrm{epi}^{\Delta(C)} \leq \mathrm{reg}^{\Delta(C)}$, hence $\Delta(C) = \Delta(\mathrm{epi}^{\Delta(C)}) = \mathrm{E}_{\mathcal{X}}(\Delta(C))$. □

EXAMPLES

(1) Since **Haus** $= \Delta(K)$ with K (weakly) hereditary, **Haus** is epi-closed. Similarly, $\Delta(\sigma) =$ **US** is the category of spaces in which convergent sequences have uniquely determined limits, hence (weak) hereditariness of the sequential closure yields epi-closedness of **US**.

(2) Every disconnectedness of **Top** in the sense of Exercise 6.T has a weakly hereditary regular closure operator and is therefore epi-closed.

(3) For every full and replete epireflective subcategory $\mathcal{A}$ of $\mathbf{Mod}_R$ and the $\mathcal{A}$-regular radical $\mathbf{r}$, one has $\mathcal{A} = \{X \in \mathbf{Mod}_R : \mathbf{r}(X) = 0\}$ (cf. Proposition 6.7) and $\mathrm{reg}^{\mathcal{A}} = C^{\mathbf{r}}$. Since $\mathrm{epi}^{\mathcal{A}} = (\mathrm{reg}^{\mathcal{A}})_\infty$, with the Corollary and Exercise 4.G(d) one computes

$$\mathrm{E}(\mathcal{A}) = \Delta(C^{(\mathbf{r}^\infty)}) = \{X \in \mathbf{Mod}_R : \mathbf{r}^\infty(X) = 0\}.$$

Since with $\mathbf{r}$ also $\mathbf{r}^\infty$ is a radical, $\mathbf{r}^\infty$ is the $\mathrm{E}(\mathcal{A})$-regular radical. Hence with $\mathrm{reg}^{\mathrm{E}(\mathcal{A})} = C^{(\mathbf{r}^\infty)}$ and $\mathrm{reg}^{\mathcal{A}} = C^{\mathbf{r}}$ one concludes that $\mathcal{A}$ *is epi-closed if and only if* $\mathbf{r}$ *is idempotent*, and this is the case exactly when $\mathcal{A}$ is closed under extensions (see Example 6.8(1)).

(4) The criterion of (3) can be used to detect failure of epi-closedness, as in the following two examples where $R = \mathbb{Z}$, i.e., $\mathbf{Mod}_R = \mathbf{AbGrp}$. For $\mathcal{A} = \{A : \mathbf{f}(A) = 0\}$ the subcategory of groups with trivial Frattini subgroup, since $\mathbf{f}_\infty = \mathbf{d}$ (see Example 3.4(3)), $\mathrm{E}(\mathcal{A}) = \{A : \mathbf{d}(A) = 0\}$ is the category of reduced groups. For $\mathcal{A} = \{A : p(A) = 0\}$ with a fixed prime p, since $\mathbf{p}_\infty = \mathbf{d}_p$ (see Example 4.6(2)), $\mathrm{E}(\mathcal{A}) = \{A : \mathbf{d}_p(A) = 0\}$ is the category of groups without p-divisible subgroups.

PROBLEM *For a regular closure operator* C*, the identity* $\mathrm{E}_{\mathcal{X}}(\Delta(C)) = \Delta(\check{C})$ *holds true. Does it hold true for every closure operator? For the* θ*-closure in* **Top***?*

7.7 The maximal epi-preserving extension

The general hypotheses in this section are as in 7.6. For any subcategory $\mathcal{A}$ of $\mathcal{X}$, the inclusion functor of $\mathcal{A}$ into its epi-closure $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$ surely preserves epimorphisms, i.e., an epimorphism of the category $\mathcal{A}$ is also an epimorphism in $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$, just by the definition of $\mathrm{E}_{\mathcal{X}}(\mathcal{A}) = \mathrm{Sep}(\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}))$. Furthermore, if $\mathcal{A}$ is reflective in $\mathcal{X}$, then it is actually epireflective in $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$ since the $\mathcal{A}$-reflexions are $\mathcal{A}$-epimorphisms of $\mathcal{X}$. Actually, they are $\mathcal{A}$-epimorphisms with the additional property that their codomains belong to $\mathcal{A}$. Hence, the argumentation for epi-preservation and epireflectivity is still valid if we replace $\mathrm{E}_{\mathcal{X}}(\mathcal{A})$ by the larger subcategory

$$\mathrm{D}(\mathcal{A}) = \mathrm{D}_{\mathcal{X}}(\mathcal{A}) := \mathrm{Sep}_{\mathcal{X}}(\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathrm{Cod}_{\mathcal{X}}(\mathcal{A})),$$

with $\mathrm{Cod}_{\mathcal{X}}(\mathcal{A})$ denoting the class of those morphisms in $\mathcal{X}$ with codomain in $\mathcal{A}$. This proves the first part of the following Proposition. Its second part describes a maximality property of $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$.

PROPOSITION *Let* $\mathcal{A}$ *be a full subcategory of* $\mathcal{X}$*. Then:*

(1) *The inclusion functor* $\mathcal{A} \to \mathrm{D}_{\mathcal{X}}(\mathcal{A})$ *preserves epimorphisms, and* $\mathcal{A}$ *is epireflective in* $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ *if* $\mathcal{A}$ *is reflective in* $\mathcal{X}$*.*

(2) *For every full and replete reflective subcategory* $\mathcal{B}$ *of* $\mathcal{X}$ *containing* $\mathcal{A}$ *such that* $\mathcal{A}$ *is epireflective in* $\mathcal{B}$ *and the inclusion functor* $\mathcal{A} \to \mathcal{B}$ *preserves epimorphisms, one has* $\mathcal{B} \subseteq \mathrm{D}_{\mathcal{X}}(\mathcal{A})$*.*

Proof We still have to prove (2). Let $B \in \mathcal{B}$ and consider any $\mathcal{A}$-epimorphism $d : X \to A$ in $\mathcal{X}$ with $A \in \mathcal{A}$ and morphisms $f, g : A \to B$ with $f \cdot d = g \cdot d$. Then d factors as

$$X \xrightarrow{\sigma_X} SX \xrightarrow{\rho_{SX}} RSX \xrightarrow{d'} A,$$

with a $\mathcal{B}$-reflexion σ_X and an $\mathcal{A}$-reflexion ρ_{SX}. Since σ_X is $\mathcal{B}$-epic, one has $f \cdot d' \cdot \rho_{SX} = g \cdot d' \cdot \rho_{SX}$, and this implies $f \cdot d' = g \cdot d'$ since ρ_{SX} is an epimorphism of $\mathcal{B}$. But d' is an epimorphism of $\mathcal{A}$ since d is $\mathcal{A}$-epic, so that d' is also an epimorphism of $\mathcal{B}$, by hypothesis. Therefore $f = g$, as desired. $\square$

REMARKS

(1) Note that $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ is closed under mono-sources (see Remark 7.6(1)) and therefore (strongly epi-) reflective in $\mathcal{X}$, under mild assumptions on $\mathcal{X}$ (see Exercise 7.A). If $\mathcal{A}$ and $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ are reflective in $\mathcal{X}$, then the Proposition characterizes $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ as the largest (strongly epi-) reflective subcategory $\mathcal{B}$ of $\mathcal{X}$ containing $\mathcal{A}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ preserves epimorphisms and $\mathcal{A}$ is epireflective in $\mathcal{B}$.

(2) One always has the inclusions

$$\mathcal{A} \subseteq S_{\mathcal{X}}(\mathcal{A}) \subseteq \mathrm{E}_{\mathcal{X}}(\mathcal{A}) \subseteq \mathrm{D}_{\mathcal{X}}(\mathcal{A}).$$

But as an operator on $SUB(\mathcal{X})$, D behaves very differently from S and E, which are monotone while D is not. For instance, for $\mathcal{X} = \mathbf{Top}$, Proposition (2) shows $\mathrm{D}(\mathbf{Top}_1) = \mathbf{Top}$ (since epimorphisms are surjective in $\mathbf{Top}_1$, so that $\mathbf{Top}_1 \hookrightarrow \mathbf{Top}$ preserves them; see Example 6.5(1)), while $\mathrm{D}(\mathbf{Top}_0) = \mathbf{Top}_0$ (since epimorphisms are not surjective in $\mathbf{Top}_0$, so that $\mathrm{D}(\mathbf{Top}_0)$ cannot be $\mathbf{Top}$, and any proper strongly epireflective subcategory of $\mathbf{Top}$ is already contained in $\mathbf{Top}_0$; see Remark 6.9).

(3) $\mathrm{D}_{\mathcal{X}}$ is in fact order reversing for those full and replete epireflective subcategories $\mathcal{A} \subseteq \mathcal{B}$ of $\mathcal{X}$ for which $\mathrm{D}_{\mathcal{X}}(\mathcal{B})$ is reflective in $\mathcal{X}$ and $\mathcal{A} \hookrightarrow \mathcal{B}$ preserves epimorphisms. In fact, since then also $\mathcal{A} \hookrightarrow \mathrm{D}_{\mathcal{X}}(\mathcal{B})$ preserves epimorphisms, the Proposition gives $\mathrm{D}_{\mathcal{X}}(\mathcal{B}) \subseteq \mathrm{D}_{\mathcal{X}}(\mathcal{A})$. When applying this property in case $\mathcal{B} = \mathrm{D}_{\mathcal{X}}(\mathcal{A})$, we see that

$$\mathrm{D}_{\mathcal{X}}(\mathcal{A}) = \mathrm{D}_{\mathcal{X}}(\mathrm{D}_{\mathcal{X}}(\mathcal{A}))$$

holds whenever $\mathcal{A}$ and each full subcategory of $\mathcal{X}$ closed under monosources is epireflective in $\mathcal{X}$.

Despite its rather unpredictable size, membership in the subcategory $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ can be tested reasonably easily.

COROLLARY *The following three conditions are equivalent for a full subcategory $\mathcal{A}$ of $\mathcal{X}$ and every object B of $\mathcal{X}$, if $\mathcal{X}$ has finite products:*

(i) $B \in \mathrm{D}_{\mathcal{X}}(\mathcal{A})$;

(ii) $\delta_B : B \to B \times B$ *belongs to* $\mathcal{D} := (\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathrm{Cod}_{\mathcal{X}}(\mathcal{A}))_{\perp}$;

(iii) *for all* $f, g : X \to B$ *in* $\mathcal{X}$, *the equalizer* $\mathrm{equ}(f, g)$ *of* f, g *belongs to* $\mathcal{D}$.

If $\mathcal{A}$ is reflective in $\mathcal{X}$ or if $\mathcal{X}$ is closed under $\mathcal{M}$-subobjects in $\mathcal{X}$, these conditions are also equivalent to

(iv) *for all* $f, g : A \to B$ *in* $\mathcal{X}$ *with* $A \in \mathcal{A}$, $\mathrm{equ}(f, g)$ *is an* $\mathcal{A}$-*extremal monomorphism (cf. Theorem* 6.2*).*

Proof The equivalence (i) $\Leftrightarrow$ (iii) follows from Proposition 7.6. For (iv) $\Rightarrow$

(ii), assume $\delta_B \cdot k = h \cdot d$ with $d : U \to A \in \mathcal{A}$ an $\mathcal{A}$-epimorphism. Then d factors through $t := h^{-1}(\delta_B)$, so that also t is $\mathcal{A}$-epic. On the other hand, t is the equalizer of $p_1 h_1, p_2 h : A \to B$ with $p_1, p_2 : B \times B \to B$ the projections, hence an $\mathcal{A}$-extremal monomorphism and therefore iso. Composition of its inverse with a pullback projection yields the desired "diagonal".

Only for (iii) $\Leftrightarrow$ (iv) we need the additional hypotheses on $\mathcal{A}$. For $f, g : A \to B$ and $t = \mathrm{equ}(f, g)$, we already have that t belongs to $(\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathrm{Cod}_{\mathcal{X}}(\mathcal{A}))_\perp$ and must show that t is actually $\mathcal{A}$-extremal. Consider a factorization $t = k \cdot e$ with $k : X \to A$ in $\mathcal{M}$ and e $\mathcal{A}$-epic. If $\mathcal{A}$ is closed under $\mathcal{M}$-subobjects, then $X \in \mathcal{A}$, hence $e \in (\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathrm{Cod}_{\mathcal{X}}(\mathcal{A}))_\perp$, and orthogonality gives that e must be an isomorphism. If $\mathcal{A}$ is reflective, the k factors through the $\mathcal{A}$-reflexion $\rho_X : X \to RX$ as $k = k' \cdot \rho_X$, and $\rho_X \cdot e$ is an $\mathcal{A}$-epimorphism with codomain in $\mathcal{A}$, which therefore must be orthogonal to t. Hence there is a morphism s with $s \cdot \rho_X \cdot e = 1$ and $t \cdot s = k'$. This implies $k \cdot e \cdot s \cdot \rho_X = t \cdot s \cdot \rho_X = k' \cdot \rho_X = k$, hence $e \cdot s \cdot \rho_X = 1$ since k is monic. Consequently, e is an isomorphism. □

We wish to describe objects of $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$ in terms of a closure operator. Of course, one approach would be to take the idempotent closure operator that has the class $(\mathrm{Epi}_{\mathcal{X}}(\mathcal{A}) \cap \mathrm{Cod}_{\mathcal{X}}(\mathcal{A}))_\perp$ as its closed $\mathcal{M}$-subobjects. But this operator would hardly be computationally accessible. Since the $\mathcal{A}$-extremal monomorphisms in $\mathcal{M}$ are exactly the $\mathrm{epi}^{\mathcal{A}}$-closed subobjects, condition (iv) of the Theorem suggests to consider

$$\mathfrak{d}_X^{\mathcal{A}}(m) :\cong m \vee \bigvee \{h(\mathrm{epi}_A^{\mathcal{A}}(h^{-1}(m))) : A \in \mathcal{A}, h : A \to X\}$$

for all $m \in \mathcal{M}/X$. Of course, we may take a more general approach and consider any closure operator C of $\mathcal{X}$ in lieu of $\mathrm{epi}^{\mathcal{A}}$. Hence we define the *$\mathcal{A}$-comodification of C* by

$$^{\mathcal{A}}c_X(m) :\cong m \vee \bigvee \{h(c_A(h^{-1}(m))) : A \in \mathcal{A}, h : A \to X\}$$

for all $m \in \mathcal{M}/X$. This gives indeed a closure operator, and the terminology fits with the one introduced in 5.12:

LEMMA *For any closure operator C of $\mathcal{X}$ and every subcategory $\mathcal{A}$ of $\mathcal{X}$, $^{\mathcal{A}}C$ is a closure operator of $\mathcal{X}$ whose closed subobjects are precisely those $m \in \mathcal{M}/X$ for which $h^{-1}(m)$ is C-closed in A for all $h : A \to X, A \in \mathcal{A}$. If $\mathcal{A}$ is coreflective in $\mathcal{X}$, with coreflexion $\varepsilon : S \to Id_{\mathcal{X}}$, then $^{\mathcal{A}}C$ is the (S, ε)-comodification of C in the sense of* 5.12, *i.e.,*

$$^{\mathcal{A}}c_X(m) \cong m \vee \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m))) \cong {}^{S}c_X(m).$$

Proof The continuity condition for $^{\mathcal{A}}C$ follows from preservation of joins by $f(-)$ for every $f : X \to Y$, as follows:

$$\begin{aligned} f(^{\mathcal{A}}c_X(m)) &\cong f(m) \vee \bigvee \{(f \cdot h)(c_A(h^{-1}(m))) : A \in \mathcal{A}, h : A \to X\}, \\ &\leq f(m) \vee \bigvee \{(f \cdot h)(c_A((f \cdot h)^{-1}(f(m))) : A \in \mathcal{A}, h : A \to X\}, \end{aligned}$$

$$\leq \quad f(m) \vee \bigvee \{k(c_A(k^{-1}(f(m))) : A \in \mathcal{A}, k : A \to Y\}\,,$$
$$\leq \quad {}^{\mathcal{A}}c_Y(f(m)),$$

since $h^{-1}(m) \leq h^{-1}(f^{-1}(f(m))) \cong (f \cdot h)^{-1}(f(m))$ for all $h : A \to X$.

If m is ${}^{\mathcal{A}}C$-closed, then $h(c_A(h^{-1}(m))) \leq m$ and therefore $c_A(h^{-1}(m)) \leq h^{-1}(m)$ for all $h : A \to X$, i.e., $h^{-1}(m)$ is C-closed. Conversely, the C-closedness of each $h^{-1}(m)$ gives

$${}^{\mathcal{A}}c_X(m) \cong f(m) \vee \bigvee \{h(h^{-1}(m)) : A \in \mathcal{A}, h : A \to X\} \leq m$$

so that m must be ${}^{\mathcal{A}}C$-closed.

In case X has an $\mathcal{A}$-coreflexion $\varepsilon_X : SX \to X$, then ε_X can take the place of h and we have

$$m \vee \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m))) \leq {}^{\mathcal{A}}c_X(m).$$

On the other hand, each h factors as $h = \varepsilon_X \cdot h_0$, hence C-continuity of $h_0 : A \to SX$ gives

$$h(c_A(h^{-1}(m))) \leq \varepsilon_X(c_{SX}(h_0(h^{-1}(m)))) \leq \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m)))$$

and therefore

$${}^{\mathcal{A}}c_X(m) \leq m \vee \varepsilon_X(c_{SX}(\varepsilon_X^{-1}(m))).$$

□

With

$$\mathfrak{d}^{\mathcal{A}} = {}^{\mathcal{A}}(\mathrm{epi}^{\mathcal{A}}),$$

the Corollary and the Lemma provide an effective description of the category $\mathrm{D}_{\mathcal{X}}(\mathcal{A})$:

THEOREM *If the full and replete subcategory $\mathcal{A}$ of the finitely complete $\mathcal{X}$ is reflective or closed under $\mathcal{M}$-subobjects, then*

$$\mathrm{D}_{\mathcal{X}}(\mathcal{A}) = \Delta(\mathfrak{d}^{\mathcal{A}}).$$

Proof Since pairs $f, g : A \to B$ correspond bijectively to morphisms $h : A \to B \times B$, with the equalizer of f, g corresponding to $h^{-1}(\delta_B)$, the equivalence (i) $\Leftrightarrow$ (ii) of the Corollary tells us that $B \in \mathrm{D}_{\mathcal{X}}(\mathcal{A})$ if and only if δ_B is $\mathrm{epi}^{\mathcal{A}}$-closed. According to the Lemma, this means that $B \in \mathrm{D}_{\mathcal{X}}(\mathcal{A})$ if and only if δ_B is $\mathfrak{d}^{\mathcal{A}}$-closed. Hence the assertion of the Theorem follows from Proposition 7.1 □

In many cases, the $\mathcal{A}$-comodification of a closure operator can be described effectively, as the following examples show.

EXAMPLES

(1) If $\mathcal{X}$ is a topological category over **Set** such that constant maps are $\mathcal{X}$-morphisms and if $\mathcal{A}$ contains a non-empty space, the formula for ${}^{\mathcal{A}}C$ can be simplified as

$${}^{\mathcal{A}}c_X(M) = \bigcup \{h(c_A(h^{-1}(M))) : A \in \mathcal{A}, h : A \to X\}.$$

For $\mathcal{X} = \mathbf{Top}$ and $\mathcal{A} = \{\mathbb{N}_\infty\}$ the one-point Alexandroff compactification of the discrete space $\mathbb{N}$ (i.e., the converging sequence $n \to \infty$ with each n isolated), and for $C = K$ the Kuratowski closure operator,

$$^{\{\mathbb{N}_\infty\}}K = \sigma$$

is the sequential closure operator. For $\mathcal{A} = \mathbf{CompTop}$ the subcategory of compact topological spaces (not necessarily Hausdorff!), which is closed under images, we obtain the $\mathfrak{k}$-closure:

$$^{\mathbf{CompTop}}k_X(M) = \bigcup\{k_B(M \cup B) : B \subseteq X \text{ compact }\} = \mathfrak{k}_X(M).$$

(2) Let $\mathcal{A}$ be epireflective in $\mathcal{X} = \mathbf{Mod}_R$, with $\mathcal{A}$-regular radical $\mathbf{r}$. We wish to describe the $\mathcal{A}$-comodification of a maximal closure operator $C = C^{\mathbf{s}}$, for some preradical $\mathbf{s}$.

First we consider the submodule (in fact: two-sided ideal) $I := \mathbf{r}(R)$ and observe that every object $A \in \mathcal{A}$ is a quotient of a copower of $R/I \in \mathcal{A}$, i.e., R/I is a generator of $\mathcal{A}$. With the help of I one defines the preradical

$$\mathbf{r}^{\#}(X) := \{x \in X : Ix = 0\} = \{x \in X : (\exists h : R/I \to X)\ h(1) = x\} = \sum_{h:A\to X, A\in\mathcal{A}} h(A)$$

which is easily seen to be hereditary. For every surjective R-linear map $f : A \to X$ and $M \leq X$, since $A/f^{-1}(M) \cong X/M$, the maximal closure operator C satisfies $f(c_A(f^{-1}(M))) = c_X(M)$. Therefore,

$$\begin{aligned} ^{\mathcal{A}}c_X(M) &= M + \sum h(c_A(h^{-1}(M)) \,, \\ &= M + \sum h(c_{h(A)}(h(A) \cap (M)) \,, \\ &= M + c_{\mathbf{r}^{\#}(X)}(\mathbf{r}^{\#}(X) \cap M) \end{aligned}$$

(since always $h(A) \subseteq \mathbf{r}^{\#}(X)$, and there are enough such maps h – just consider for A the $|\mathbf{r}^{\#}(X)|$-th copower of R/I). This shows immediately that the preradical induced by $^{\mathcal{A}}C$ is

$$\pi(^{\mathcal{A}}C) = \mathbf{s}\,\mathbf{r}^{\#}.$$

(If $I = 0$, hence $\mathbf{r}^{\#} = \mathbf{1}$, then this formula gives $\pi(^{\mathcal{A}}C) = \mathbf{s}$.) For $C = \mathrm{epi}^{\mathcal{A}} = (C^{\mathbf{r}})_\infty = C^{(\mathbf{r}^\infty)}$, so that $\mathbf{s} = \mathbf{r}^\infty$, we obtain

$$\pi(\mathfrak{d}^{\mathcal{A}}) = \mathbf{r}^\infty \mathbf{r}^{\#}.$$

Consequently, with Example 7.1(1) one concludes

$$X \in \mathrm{D}(\mathcal{A}) \Leftrightarrow X \in \Delta(\mathfrak{d}^{\mathcal{A}}) \Leftrightarrow 0 \text{ is } \mathfrak{d}^{\mathcal{A}}\text{-closed in } X$$

$$\Leftrightarrow \pi(\mathfrak{d}^{\mathcal{A}})(X) = 0 \Leftrightarrow \mathbf{r}^\infty(\{x \in X : Ix = 0\}) = 0.$$

(If $I = 0$, the last condition just means $\mathbf{r}^\infty(X) = 0$.)

(3) We apply the characterization of $\mathrm{D}(\mathcal{A})$-objects of (2) in the case $R = \mathbb{Z}$ and prove that *for* $\mathcal{A}$ *epireflective in* **AbGrp**, *either* $\mathrm{D}(\mathcal{A}) = \mathrm{E}(\mathcal{A})$ *or* $\mathrm{D}(\mathcal{A}) = \mathbf{AbGrp}$. In fact, for the subgroup $I \leq \mathbb{Z}$ as in (2), either $I = 0$ or $I = n\mathbb{Z}$ for positive n. The first case gives with (2) and Example 7.6 (3)

$$X \in \mathrm{D}(\mathcal{A}) \Leftrightarrow \mathbf{r}^{\infty}(X) = 0 \Leftrightarrow X \in \mathrm{E}(\mathcal{A}).$$

In case $I = n\mathbb{Z}$ one has $nX \subseteq \mathbf{r}(X)$ for all X (since, in general, $IX \subseteq \mathbf{r}(X)$) and $\mathbf{r}(\mathbb{Z}/n\mathbb{Z}) = 0$. This implies $\mathbf{r}(X/nX) = 0$ for every abelian group X since, by *Prüfer's Theorem* on bounded torsion abelian groups, there is an embedding

$$X/nX \to (\mathbb{Z}/n\mathbb{Z})^{\alpha}$$

into some power of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Consequently, $\mathbf{r}(X) = nX$ for every X, and

$$\mathcal{A} = \{X : \mathbf{r}(X) = 0\} = \{X : (\forall x \in X)\, nx = 0\}$$

is the category of abelian groups of exponent n. In this category, being closed under quotients in **AbGrp**, epimorphisms are surjective, which, by the Proposition, means $\mathrm{D}(\mathcal{A}) = \mathbf{AbGrp}$.

7.8 Nabla categories

In the setting of 7.1, we assume in addition that $\mathcal{X}$ has finite products. As usual, $\mathcal{E}$ denotes the class for which $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorizations. We have seen in the preceding section that the Delta-subcategories

$$\Delta(C) = \{X \in \mathcal{X} : \delta_X : X \to X^2 \text{ is } C\text{-closed}\}$$

give notions of separation or disconnectedness, depending on the given closure operator C of $\mathcal{X}$. It seems natural to introduce the category

$$\nabla(C) = \{X \in \mathcal{X} : \delta_X : X \to X^2 \text{ is } C\text{-dense}\},$$

which we call *the Nabla-subcategory of* C, and to associate with it a notion of connectedness. The following examples confirm this:

EXAMPLES

(1) In the category **PoSet**, $\nabla(\downarrow)$ contains exactly those posets in which any two points have upper an bound. $\nabla(\mathrm{conv}) = \nabla(\downarrow) \cap \nabla(\uparrow)$ contains those posets in which any two points have an upper bound and a lower bound.

(2) In the category **Top**, $\nabla(K) = \mathbf{IrrTop}$ is the category of *irreducible spaces* (i.e., those spaces X in which $X = F \cup G$ with (K-) closed subsets is possible only for $X = F$ or $X = G$). In fact, to say that $\Delta_X := \delta(X)$ is dense in $X \times X$ is the same as to say that any non-empty set is dense in X .

(3) For the largest proper closure operator Q of **Top**, $\nabla(Q) = C\mathbf{Top}$ is the category of connected spaces. In fact, if we assume $q_{X\times X}(\Delta_X) \neq X \times X$ for a space X, then we could find a non-empty proper clopen subset of $X\times X$, that means $X\times X$ is not connected, hence X is not connected. On the other hand, for a non-connected space $X = A\cup B$ with disjoint non-empty clopen sets A, B, the set $(A\times A)\cup(B\times B)$ would be a proper clopen subset of $X \times X$ containing Δ_X, hence $X \notin \nabla(Q)$. (See also Exercise 7.L.)

(4) For a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$, a submodule $M \leq X$ is $C_{\mathbf{r}}$-dense if and only if $\mathbf{r}(X)+M = X$. Since $\mathbf{r}(X\times X) = \mathbf{r}(X)\times\mathbf{r}(X)$, the R-module X belongs to $\nabla(C_{\mathbf{r}})$ if and only if $\mathbf{r}(X) = X$, so that $\nabla(C_{\mathbf{r}})$ is the *radical class* (torsion class, I think) associated with $\mathbf{r}$.

The following properties of Nabla-subcategories are also typical for categories of "connected" objects.

PROPOSITION *Let C be a closure operator of $\mathcal{X}$. Then :*

(1) *$\nabla(C)$ is closed under $\mathcal{E}$-images, so that for $e : X \to Y$ in $\mathcal{E}$ with $X \in \nabla(C)$ also $Y \in \nabla(C)$, provided that $\mathcal{E}$ is closed under finite direct products.*

(2) *$\nabla(C)$ is closed under C-dense extensions, so that for $m : M \to X$ in $\mathcal{E}^C \cap \mathcal{M}$ with $M \in \nabla(C)$ also $X \in \nabla(C)$, provided that C is finitely productive and idempotent.*

(3) *$\nabla(C)$ is closed under (finite) direct products in $\mathcal{X}$, provided that C is (finitely) productive .*

Proof For $e : X \to Y$ in $\mathcal{X}$, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[d, "e"'] & X \times X \arrow[d, "e\times e"] \\ Y \arrow[r, "\delta_Y"] & Y \times Y \end{tikzcd} \tag{7.5}$$

Under the hypotheses of (1), one has $\delta_Y \in \mathcal{E}^C$ and $e \times e \in \mathcal{E}$, hence $(e\times e)\cdot\delta_X = \delta_Y \cdot e \in \mathcal{E}^C$ and therefore $\delta_Y \in \mathcal{E}^C$ (see Exercise 2.F(b) and Corollary* of 2.3). In the situation of (2), one has $e \times e \in \mathcal{E}^C \cap \mathcal{M}$ by the finite productivity of C (see Theorem 2.7) and

(3) follows with Theorem 2.7. □

Recall that if finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ are covered by their sections, then an idempotent closure operator is already finitely productive (see Proposition 4.11; the sufficient condition is certainly satisfied if $\mathcal{X}$ is a topological category over

Set such that any constant map between (the underlying sets of) two $\mathcal{X}$-objects lift to a morphism of $\mathcal{X}$. Recall further that in this situation the idempotent closure operator C is even productive, if there exists a closure operator $D \leq C$ with the finite structure property of products (like K in **Top**; see Theorem 4.11). We therefore obtain from the Proposition:

THEOREM *Let finite products of $\mathcal{M}$-subobjects in $\mathcal{X}$ be covered by their sections, and let C be an idempotent closure operator of $\mathcal{X}$. Then $\nabla(C)$ is closed under $\mathcal{E}$-images, C-dense extensions, and under finite products. It is even closed under non-trivial direct products if there is a closure operator $D \leq C$ with the finite structure property of products.* □

COROLLARY *For every idempotent closure operator C of* **Top**, *$\nabla(C)$ is closed under $\mathcal{E}$-images, C-dense extensions, and under finite products. In case $C \geq K$, it is even closed under arbitrary direct products.* □

The Corollary gives in particular all closedness properties for $\nabla(K)$ = **IrrTop** and $\nabla(Q)$ = **CTop** mentioned at the beginning (see (Examples (2) and (3)).

REMARKS

(1) $\nabla(C)$ does not change when passing to the weakly hereditary core of C, which is idempotent whenever C is idempotent (see Theorem* of 5.4). Hence, when dealing with $\nabla(C)$, one may always assume C to be weakly hereditary,also when C is supposed to be idempotent.

(2) For $D = \bigwedge_i C_i$ one has $\mathcal{E}^D = \bigcap_i \mathcal{E}^{C_i}$ (see Proposition 4.4), hence $\nabla(D) = \bigcap_i \nabla(C_i)$. In other words, the functor

$$\nabla : CL(\mathcal{X},\mathcal{M}) \to SUB(\mathcal{X})$$

preserves arbitrary meets. It therefore has a left adjoint which assigns to a full subcategory $\mathcal{A}$ its (let's call it) *coregular* closure operator $\mathrm{coreg}^{\mathcal{A}}$. We do not have a good description of this operator, other than the one given by Theorem 1.3, i.e., by the characteristic property

$$\mathcal{A} \subseteq \nabla(C) \Leftrightarrow \mathrm{coreg}^{\mathcal{A}} \leq C.$$

(3) "Dually" to Theorem 7.6 one can construct a right adjoint $\sigma^{(-)}$ to the functor $C \mapsto \mathcal{M}^C : CL(\mathcal{X},\mathcal{M}) \to MOR(\mathcal{X})^{op}$ and then consider the composite adjunctions

$$SUB(\mathcal{X}) \quad \underset{\mathrm{coreg}}{\overset{\nabla}{\rightleftarrows}} CL(\mathcal{X},\mathcal{M}) \underset{\mathcal{M}^{(-)}}{\overset{\sigma^{(-)}}{\rightleftarrows}} \quad MOR(\mathcal{X})^{op} \tag{7.6}$$

which represents a kind of dual of the PR-correspondence. But again, we do not have a good explicit description of it.

7.9 Companions of $\Delta(C)$ in topological categories

For a topological category $\mathcal{X}$ over **Set** such that constant **Set**-maps between $\mathcal{X}$-objects are morphisms in $\mathcal{X}$, we provide "bounds"

$$T_2(C) \subseteq \Delta(C) \subseteq T_1(C)$$

for additive closure operators C of $\mathcal{X}$ which may help to characterize the objects of $\Delta(C)$ in concrete cases. Notationally we do not distinguish between objects in $\mathcal{X}$ and their underlying sets. Recall from 5.10 that every closure operator C of $\mathcal{X}$ is equivalently described by a concrete functor

$$\mathbf{C} : \mathcal{X} \to \mathbf{CS}, \quad X \mapsto (X, c_X),$$

which takes values in **PrTop** if and only if C is grounded and additive. We remind the reader of the intricacies of the functor $\mathbf{C}$: it does not preserve subobjects unless C is hereditary (see Proposition 5.10), and even for productive closure operators, $\mathbf{C}$ may not preserve products ! Nevertheless, to some extent $\mathbf{C}$ is useful in "transporting" properties back and forth between $\mathcal{X}$ and **PrTop**.

As in **PrTop**, for any closure operator C of $\mathcal{X}$ and every $x \in X \in \mathcal{X}$, one calls $M \subseteq X$ a *C-neighbourhood* of x if $x \notin c_X(X \setminus M)$. Now the full subcategory $T_2(C)$ of $\mathcal{X}$ contains, by definition, all objects X in which distinct points can be separated by disjoint C-neighbourhoods. $T_1(C)$ contains those objects X in which singleton subsets are C-closed.

PROPOSITION *Let C be a closure operator of $\mathcal{X}$. Then:*

(1) *All $T_2(C)$, $\Delta(C)$ and $T_1(C)$ are strongly epireflective in $\mathcal{X}$;*

(2) $\Delta(C) \subseteq \Delta(C^{\oplus}) = T_1(C)$;

(3) *$T_2(C) \subseteq \Delta(C)$ holds for additive C, and both categories coincide if $\mathbf{C} : \mathcal{X} \to$ **PrTop** preserves finite products.*

Proof (1) Only closedness under mono-sources needs to be checked (cf. Exercise 7.A), and this only for $T_2(C)$ and $T_1(C)$ (cf. Proposition 7.1). Mono-sources are point-separating families $p_i : X \to X_i$ $(i \in I)$ of morphisms in $\mathcal{X}$. Hence, if all X_i lie in $T_2(C)$, for $x \neq y$ in X we have an $i \in I$ with $p_i(x) \neq p_i(y)$ and therefore disjoint C-neighbourhoods in X_i which separate these points. Their inverse images along p_i give disjoint C-neighbourhoods of x and y, due to the C-continuity of p_i. Hence $X \in T_2(C)$.

If all $X_i \in T_1(C)$, and if we assume $y \in c_X(\{x\}) \subseteq X$, then $p_i(y) \in p_i(c_X(\{x\})) \subseteq c_{X_i}(\{p_i(x)\}) = \{p_i(x)\}$ for all $i \in I$, hence $x = y$, and therefore $X \in T_1(C)$.

(2) We first show $\Delta(C) \subseteq T_1(C)$ and consider $x \in X \in \Delta(C)$. Our category $\mathcal{X}$ satisfies the hypotheses of Lemma 4.11, which gives us

$$c_X(\{x\}) \times \{x\} \subseteq c_{X \times X}(\{x\} \times \{x\}) \subseteq c_{X \times X}(\Delta_X) = \Delta_X,$$

hence $c_X(\{x\}) = \{x\}$. Consequently $X \in T_1(C)$.

Next we show $\Delta(C) \subseteq T_1(C)$ for fully additive C. In fact, in that case we have

$$c_{X\times X}(\Delta_X) = \bigcup_{x\in X} c_{X\times X}(\{x\}\times\{x\}),$$

and since for $X \in T_1(C)$ also $X \times X \in T_1(C)$ (by (1)), C-closedness of the diagonal follows.

Since trivially $T_1(C) = T_1(C^{\oplus})$, and since Δ is order-reversing, this completes the proof of (2).

(3) Without loss of generality we may assume C to be grounded. (As in **Top**, the trivial operator T is the only non-grounded closure operator of $\mathcal{X}$ – see Example 2.H, and for $C = T$, $T_2(C) = \Delta(C)$ is the full subcategory of objects of cardinality at most 1 - see Exercise 7.B.) We then have a concrete functor $\mathbf{C} : \mathcal{X} \to \mathbf{PrTop}$. As in the case of **Top**, also in **PrTop** one has $\Delta(K) = T_2(C)$, with K the Čech closure operator (cf. Example (1) below). Furthermore, by definition of $T_2(C)$,

$$T_2(C) = \mathbf{C}^{-1}(T_2(K)),$$

hence it suffices to show $\mathbf{C}^{-1}(\Delta(K)) \subseteq \Delta(C)$ in $\mathcal{X}$. But by the functoriality of $\mathbf{C}$,

$$id_{X\times X} : \mathbf{C}(X\times X) \to \mathbf{C}(X)\times\mathbf{C}(X)$$

is a map in **PrTop**, hence K-closedness of Δ_X in $\mathbf{C}(X)\times\mathbf{C}(X)$ implies C-closedness of Δ_X in $X \times X$, as desired. The converse statement holds true if $id_{X\times X}$ is iso in **PrTop**, i. e., if $\mathbf{C}$ preserves the product $X \times X$. □

REMARKS

(1) In case $\mathcal{X} = \mathbf{Top}$, Corollary 6.10 provides an explicit description of the $T_1(C)$-reflexion for every idempotent and symmetric closure operator C of **Top** (since $T_1(C) = T_1(C^{\oplus})$, and since the passage $C \mapsto C^{\oplus}$ preserves idempotency and symmetry). This description remains valid for arbitrary topological categories over **Set** in which constant maps are morphisms.

(2) Lemma 6.9 gives immediately for every (full and replete) strongly epireflective subcategory of **Top** (or any other topological category over **Set** with constant morphisms) the identity $\mathcal{A} = T_1(\mathrm{reg}^{\mathcal{A}})$. Hence every such $\mathcal{A}$ can be written as $\mathcal{A} = T_1(C)$. From the Generating Diagonal Theorem 7.2, we already know that $\mathcal{A}$ can be presented in the form $\Delta(C)$. But we do not know the answer to:

PROBLEM *Can every strongly epireflective subcategory of* **Top** *be presented in the form $T_2(C)$? As $T_2(\mathrm{reg}^{\mathcal{A}})$? What about the case $\mathcal{A} = \mathbf{US}$?*

EXAMPLES

(1) The category **Haus** can be presented via operators T_1 and T_2 in **Top**:

$$T_2(K) = \Delta(K) = \mathbf{Haus} = T_1(\theta).$$

However,

$$T_2(\theta) = \Delta(\theta) = \mathbf{Ury} \text{ (cf. Example 7.1(5))}$$

is properly smaller.

(2) The σ-closure in **Top** provides an example, that in general, the inclusion $T_2(C) \subseteq \Delta(C)$ may be *proper*, which is shown by the following example due to I. Gotchev. For an uncountable set Z and $a \neq b$ outside $Z \times Z$, topologize $X = \{a\} \cup \{b\} \cup Z \times Z$ by declaring points of $Z \times Z$ to be isolated, and by taking as basic neighbourhoods of a and b the sets

$$U_a = \{a\} \cup (\bigcup \{\{z\} \times A_z : z \in Z, A_z \subseteq Z \text{ cofinite}\}),$$

$$V_b = \{b\} \cup (\bigcup \{B_z \times \{z\} : z \in Z, B_z \subseteq Z \text{ cofinite}\}).$$

Then σ-neighbourhoods of a and b contain open neighbourhoods, but X is not Hausdorff, hence $X \notin T_2(\sigma)$. On the other hand, X clearly belongs to $\Delta(\sigma) = \mathbf{US}$; see Example 7.6(1).

The Proposition provides the possibility of computing the regular closure operator of $T_2(C)$, as follows:

LEMMA *Every closure operator C of $\mathcal{X}$ satisfies*

$$(C^+)^\infty \leq \mathrm{reg}^{T_2(C)} \text{ and } \mathrm{reg}^{T_2(C)}|_{T_2(C)} \leq (\tilde{C})^\infty|_{T_2(C)}.$$

In case C is additive, the second inequality becomes an equality.

Proof Since T_2 is order-reversing (see Exercise 7.M), Proposition (3) gives $T_2(C) \subseteq T_2(C^+) \subseteq \Delta(C^+)$, hence $C^+ \leq \mathrm{reg}^{T_2(C)}$ and then even $(C^+)^\infty \leq \mathrm{reg}^{T_2(C)}$. For the second inequality, we must show that every $\tilde{C}$-closed $M \subseteq X \in T_2(C)$ is $T_2(C)$-closed, i.e., we must prove $K = X +_M X \in T_2(C)$. In the notation of Frolík's Lemma (Theorem 6.5), for every $a \in X \setminus M$, the points $i(a)$, $j(a)$ have the disjoint C-neighbourhoods $K \setminus j(X)$, $K \setminus i(X)$ respectively. For $x, y \in K$ which are mapped to distinct points in X by the common retraction $\varepsilon : K \to X$, these points have disjoint C-neighbourhoods in X, the preimages of which along ε provide disjoint C-neighbourhoods of x, y in K. Hence $K \in T_2(C)$.

For additive C one has $T_2(C) \subseteq \Delta(C)$ by the Proposition, hence $C^{\mathrm{reg}} \leq \mathrm{reg}^{T_2(C)}$ which, by Proposition 7.4, implies $(\tilde{C})^\infty \leq \mathrm{reg}^{T_2(C)}$. □

The impact of the Lemma on epimorphisms is as follows.

THEOREM *Let C be a closure operator of $\mathcal{X}$. Then:*

(1) *the epimorphisms of each $T_2(C)$ and $\Delta(C^+)$ are $(\tilde{C})^\infty$-dense maps;*

(2) *the $(\tilde{C})^\infty$-dense maps in $\Delta(C)$ are epimorphisms of $\Delta(C)$;*

(3) *if C is additive, then the epimorphisms of $T_2(C)$ and $\Delta(C)$ are precisely the $(\tilde{C})^\infty$-dense maps.*

Proof (1) The second inequality of the Lemma gives that epimorphisms of $T_2(C)$ are $(\tilde{C})^\infty$-dense maps. For $\Delta(C^+)$, the corresponding statement follows from the first inequality given in Remark 7.4 (1).

(2) By Proposition 7.4, $(\tilde{C})^\infty \leq C^{\mathrm{reg}}$, so that $(\tilde{C})^\infty$-dense maps are $\Delta(C)$-dense.

(3) follows from Theorem 7.4(1) and from the Lemma. □

COROLLARY *For C additive, the inclusion functor $T_2(C) \hookrightarrow \Delta(C)$ preserves epimorphisms. If, in addition, C is essentially strong, then $\mathrm{reg}^{T_2(C)}$ and C^∞ are essentially equivalent, and the epimorphisms in each $T_2(C)$ and $\Delta(C)$ are precisely the C^∞-dense maps.* □

Applications of the Theorem and its Corollary will be given in Chapter 8.

Exercises

7.A (*Strongly epireflective hulls*) Let $\mathcal{A}$ be a full subcategory of a category $\mathcal{X}$. Prove:

(a) If every source $(f_i : X \to Y_i)_{i \in I}$ in $\mathcal{X}$ has a (strong epi, mono-source)-factorization (so that $f_i = m_i \cdot e$ for all $i \in I$ with a strong epimorphism e and a mono-source $(m_i)_{i \in I}$; cf. Exercise 1.E), then the closure $S(\mathcal{A})$ of $\mathcal{A}$ under mono-sources is strongly epireflective in $\mathcal{X}$; moreover, $S(\mathcal{A})$ is contained in any strongly epireflective full and replete subcategory of $\mathcal{X}$ that contains $\mathcal{A}$. (*Hint:* For every object X, consider the source of all morphisms with domain X and codomain in $\mathcal{A}$.)

(b) The hypothesis of (a) is satisfied if and only if the category $\mathcal{X}$ has coequalizers and is $\mathcal{E}$-cocomplete with $\mathcal{E}$ the class of strong epimorphisms.

(c) If $\mathcal{A}$ is reflective in $\mathcal{X}$, then $\mathcal{A}$ is bireflective in $S(\mathcal{A})$ (so that the reflexions are both monic and epic in $S(\mathcal{A})$), and the existence of (strong epi, mono)-factorization (for morphisms) in $\mathcal{X}$ suffices to conclude that $S(\mathcal{A})$ is the least strongly epireflective subcategory of $\mathcal{X}$ containing $\mathcal{A}$ (cf. Exercise 3.L).

(d) Let $\mathcal{X}$ have direct products and be $\mathcal{E}$-cowellpowered for $\mathcal{E}$ as in (b). Then the existence of (strong epi, mono)-factorizations for morphisms implies the existence of the same type of factorizations for all sources of $\mathcal{X}$.

7.B (*Preterminal object*) Recall that an object A of a category $\mathcal{X}$ is *preterminal* if each hom-set $\mathcal{X}(X, A)$ contains at most one morphism (cf. Exercise 6.F). Prove:

(a) Every preterminal object of a finitely $\mathcal{M}$-complete category $\mathcal{X}$ belongs to $\Delta(T)$, with T the trivial closure operator (cf. Exercise 2.A). Conversely, if $\mathcal{X}$ has equalizers which belong to $\mathcal{M}$, then every object in $\Delta(T)$ is preterminal.

(b) For an object $A \in \mathcal{X}$, let the square $A \times A$ exist in $\mathcal{X}$, with projections p_1, p_2. Then the following are equivalent:

(1) A is preterminal,

(2) $p_1 = p_2$,

(3) $\delta_A = \langle 1_A, 1_A \rangle : A \to A \times A$ is an isomorphism,

(4) δ_A is epic.

(c) Let $\mathcal{X}$ have all squares, and consider any functor $U : \mathcal{X} \to \mathcal{Y}$. If U preserves squares, then U preserves preterminal objects, and if U is faithful, then U reflects preterminal objects (so that $A \in \mathcal{X}$ must be preterminal whenever $UA \in \mathcal{Y}$ is preterminal). Conclude that if $\mathcal{X}$ admits a faithful square-preserving functor into **Set**, then $A \in \mathcal{X}$ is preterminal if and only if UA has at most one element.

(d) Find one example of an $\mathcal{M}$-complete category in which least $\mathcal{M}$-subobjects are not necessarily preterminal.

7.C (*Closed Graph Theorem*) Prove for a closure operator C of a category $\mathcal{X}$ (as in 2.1) with finite products, that for every morphism $f : X \to Y$ in $\mathcal{X}$ with $Y \in \Delta(C)$ the graph $\langle 1_X, f \rangle : X \to X \times Y$ is C-closed.

7.D (*C and $\hat{C}$ have the same Delta-subcategory*) For every closure operator C and its idempotent hull $\hat{C}$, show $\Delta(C) = \Delta(\hat{C})$, *without* assuming the existence of finite products (as in Proposition 7.1) or $\mathcal{M}$-wellpoweredness of $\mathcal{X}$ (so that $\hat{C}$ could be constructed as C^∞, see 4.6). *Hint:* Use the adjointness property ($C \leq \mathrm{reg}^{\mathcal{A}} \Leftrightarrow \mathcal{A} \subseteq \Delta(C)$).

7.E (*Normal-valued regular closure in* **Grp**) Prove that a full subcategory $\mathcal{A}$ of **Grp** is contained in **AbGrp** if and only if the $\mathcal{A}$-regular closure of a subgroup is always normal.

7.F (*Regular hull of maximal closure operators*) For a closure operator C of **Grp**, let $C^{\max} = C^{\pi(C)}$ be its *maximal hull* (cf. 5.5). Let $\mathcal{X}$ be a strongly epireflective subcategory of **Grp** with $\mathcal{A}$-regular radical $\mathbf{r}$. Prove:

(a) $\Delta(C^{\mathbf{r}}) = \mathcal{A} \cap \mathbf{AbGrp}$ and $(C^{\mathbf{r}})^{\mathrm{reg}} = \mathrm{reg}^{\mathcal{A} \cap \mathbf{AbGrp}} = C^{\mathbf{r} \vee \mathbf{k}}$;

(b) $\nu \vee \mathrm{reg}^{\mathcal{A}} \leq \nu \cdot \mathrm{reg}^{\mathcal{A}} \leq \mathrm{reg}^{\mathcal{A}} \cdot \nu = \nu \cdot \mathrm{reg}^{\mathcal{A}} \cdot \nu \leq \mathrm{reg}^{\mathcal{A} \cap \mathbf{AbGrp}} = ((\mathrm{reg}^{\mathcal{A}})^{\max})^{\mathrm{reg}}$;

(c) the closure operators of (b) induce the same Delta-subcategory.

7.G (*Essential equivalence*) In the context of 7.4, confirm that essential equivalence is an equivalence relation on $CL(\mathcal{X}, \mathcal{M})$. Furthermore, if $C_0 \leq C_1 \leq C_2$

with C_0, C_2 essentially equivalent, then all three operators are essentially equivalent. Now prove that if C^{reg} and $(\tilde{C})^{\infty}$ are essentially equivalent, then $\tilde{C}$ is essentially strong.

7.H (*σ is not essentially strong*) Confirm the claim of Example 7.6(1) that $\Delta(\sigma)$ is the category **US** and then show that σ is not essentially strong, using the following example due to J. Pelant: let $X = \beta\mathbb{N} \cup \{\infty\}$ be such that each point of $\mathbb{N}$ is isolated; for an ultrafilter $\Phi \in \beta\mathbb{N} \setminus \mathbb{N}$ a basic neighbourhood is $\{\Phi\} \cup U$ $(U \in \Phi)$, and for ∞ a basic neighbourhood has the form $\{\infty\} \cup A \cup W$, with A a cofinite subset of $\mathbb{N}$ and W a cocountable subset of $\beta\mathbb{N} \setminus \mathbb{N}$; now prove $X \in$ **US** and that $M := \beta\mathbb{N} \setminus \mathbb{N}$ is σ-closed in X, but that for the cokernelpair $i, j : X \to K = X +_M X$ of $M \hookrightarrow X$, $i(X)$ is not σ-closed in K. *Hint :* M is σ-closed in X since $n \to \infty$ and its subsequences are the only non-stationary convergent sequences in X. But $i(X)$ is not σ-closed in K since $i(n) \to j(\infty)$. For the latter property first confirm that, for every $A \subseteq \mathbb{N}$,

$$k_X(A) = k_{\beta\mathbb{N}}(A) \cup \{\infty\};$$

here $\beta\mathbb{N}$ has the usual compact topology. Conclude that for every open subset V of X with $V \cap M$ cocountable in M, $V \cap \mathbb{N}$ is cofinite in $\mathbb{N}$, using the well-known fact that for $A \subseteq \mathbb{N}$ infinite, $k_{\beta\mathbb{N}}(A)$ is uncountable (in fact, $|k_{\beta\mathbb{N}}(A)| = 2^{2^{\aleph_0}}$). A typical open neighbourhood of $j(\infty)$ in K has the form $W = i(V) \cup j(U)$ with U, V open in X, $\infty \in U$ and $V \cap M = U \cap M$. The definition of the topology yields that $V \cap M$ is cocountable in M.

7.I (*Regular hull in additive categories*) For any closure operator C of $\mathbf{Mod}_R$, prove that C^{reg} and $(\tilde{C})^{\infty}$ are essentially equivalent and that these operators actually coincide if C is essentially strong. *Hint:* Recall that $\tilde{C}$ is simply the maximal closure operator of the preradical induced by C; see Example 6.6(2).

7.J (*Preservation of initiality by $(C \mapsto C^+)$*) For a closure operator C of a topological category $\mathcal{X}$ over **Set**, show that C^+ is initial if C is initial. Provide sufficient conditions which yield the same result for closure operators of an abstract category (see 4.8).

7.K (*Minimal and maximal epireflective extension subcategories*) A mono-source $(p_i : B \to A_i)_{i \in I}$ is *strong* if for every epimorphism $e : U \to V$ and for all u, v_i with $p_i \cdot u = v_i \cdot e$ for all $i \in I$ there is a $w : V \to B$ with $p_i \cdot w = v_i$ for all $i \in I$. For a full and replete subcategory $\mathcal{A}$ of $\mathcal{X}$, denote by

$$\overline{S}(\mathcal{A}) = \overline{S}_{\mathcal{X}}(\mathcal{A}) := \{B \in \mathcal{X} : B \text{ is the domain of a strong mono-source with codomains in } \mathcal{A}\}.$$

(a) Exhibit $\overline{S}(\mathcal{A})$ as the *epireflective hull of* $\mathcal{A}$ in $\mathcal{X}$ whenever sources factor as (epi, strong mono-sources); cf. Exercise 7.A. Formulate and prove statements analogous to those of Exercise 7.A.

(b) Prove that $\mathcal{A}$ is bireflective in $\overline{S}(\mathcal{A})$ (so that the reflexions are both monic and epic) whenever it is reflective in $\mathcal{X}$.

(c) For $\mathcal{A}$ reflective in $\mathcal{X}$, let $EPI(\mathcal{A},\mathcal{X})$ denote the conglomerate of all full and replete epireflective subcategories $\mathcal{B}$ of $\mathcal{X}$ with $\mathcal{A}$ epireflective in $\mathcal{B}$. If $\mathcal{X}$ has source-factorizations as in (a), then $EPI(\mathcal{A},\mathcal{X})$ (partially ordered by inclusion) has the structure of a large complete lattice, with bottom element $\overline{S}(\mathcal{A})$.

(d) (Cf. Baron [1969]) Under the hypotheses of (c), denote the $\mathcal{A}$-reflexion by ρ and prove that the top element in $EPI(\mathcal{A},\mathcal{X})$ is

$$B_{\mathcal{X}}(\mathcal{A}) := \mathrm{Sep}_{\mathcal{X}}(\{\rho_Y : Y \in \mathcal{B}_0\}),$$

for any $\mathcal{B}_0 \in EPI(\mathcal{A},\mathcal{X})$.

7.L (*Nabla-presentations of* **CTop**) With the preradicals $\mathbf{c}$ (=connected component) and $\mathbf{q}$ (=quasi-component) of $\mathbf{Top}_*$ (see Example 6.9) and their induced closure operators $\overline{C}_{\mathbf{c}}$ and $\overline{C}_{\mathbf{q}}$ of **Top** (see Theorem 6.10(1)), show

$$Q_\infty = \overline{C}_{\mathbf{c}} < \overline{C}_{\mathbf{q}} = Q^{\oplus} < Q$$

and conclude that all these closure operators induce the same Nabla-subcategory, namely the category of connected spaces.

7.M (*Lattice rules for $T_j(C)$*) Show that $C \mapsto T_j(C)$ is order reversing for $j = 1,2$. "Compute" $T_j(\bigvee_{i\in I})$.

7.N (*The "dual" of $T_1(C)$*) In the category **Top**, for every closure operator C, let $T_1^*(C)$ be the full subcategory of spaces X in which each point is C-dense. Show:

(a) $T_1^*(C) = T_1^*(C^{\oplus}) \subseteq \nabla(C^{\oplus}) \subseteq \nabla(C)$;

(b) $T_1^*(K)$ is the category of indiscrete spaces;

(c) $T_1^*(Q) = \nabla(Q) = \mathbf{CTop}$;

(d) $T_1^*(\theta) = \nabla(K) =$**IrrTop**.

7.O (*$\tilde{C}$ need not be additive for additive C*) In the categories $\mathbf{Mod}_R$ and **Top**, find additive closure operators whose strong modification is not additive. *Hint:* In $\mathbf{Mod}_R$, take a non-cohereditary radical $\mathbf{r}$ and consider its minimal closure. Then use Exercise 3.M(b) and Example 6.6 (2). In **Top**, consider the fully additive core of a non-additive regular closure operator (see Example 7.5(2)) and apply the formula of Exercise 6.U.

7.P (*Topological coreflection à la Herrlich [1969]*)

(a) For every (full and replete) coreflective subcategory $\mathcal{A}$ of **Top** one obtains the closure operator $C = {}^{\mathcal{A}}K$ of **Top**, i.e., the comodification of the Kuratowski operator along the $\mathcal{A}$-coreflexion. Show: C is an additive, grounded and idempotent closure operator with $C \leq K$.

(b) For every additive, grounded and idempotent closure operator $C \leq K$ there is a coreflective subcategory $\mathcal{A}$ of **Top** with $C = {}^{\mathcal{A}}K$. *Hint:* let $\mathcal{A}$ contain the spaces X with $c_X = k_X$. The closure operator C gives a **CS**-valued functor (cf. 5.10) which actually takes values in **Top**, and for every $X \in$ **Top**, $id_X : \mathbf{C}X \to X$ is continuous. Now consider all ordinal powers of **C** to eventually arrive at the $\mathcal{A}$-coreflexion of X.

7.Q (*Nabla versus Delta*) In the setting of 7.8, show that $\nabla(C) \cap \Delta(C)$ is the subcategory of preterminal objects in $\mathcal{X}$, for every closure operator C of $\mathcal{X}$. Conclude that if the class $\mathcal{E}$ belonging to $\mathcal{M}$ is closed under finite products, then every morphism $f : X \to Y$ in $\mathcal{X}$ with $X \in \nabla(C)$ and $Y \in \Delta(C)$ factors through a preterminal object of $\mathcal{X}$ (i.e., f is "constant" morphism).

7.R (*Bing's [1953] space*) Let $R := \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ be the upper half-plane, and let X be its subset of rational points. For $\varepsilon > 0$ and $(x, y) \in R$, set $B_\varepsilon(x, y) = \{(z, y) : x - \varepsilon < z < x + \varepsilon\} \cap X$. Now define a topology on X by taking as a base of neighbourhoods of $(x, 0) \in X$ the sets $B_\varepsilon(x, 0)$ $(\varepsilon > 0)$, and as a base of neighbourhoods of $(x, y) \in X$ with $y > 0$ the sets $U_\varepsilon(x, y) = \{(x, y)\} \cup B_\varepsilon(u, 0) \cup B_\varepsilon(v, 0)$ $(\varepsilon > 0)$, with u and v such that the triangle with vertices $\{(x, y), (u, 0), (v, 0)$ is regular. Show:

(a) X is Hausdorff, hence $X \in T_1(\theta)$ (cf. Example 7.9(1));

(b) no pair of distinct points in X can be separated by disjoint closed neighbourhoods, hence $X \in \nabla(\theta)$.

Conclude:

(c) while the identities $\mathrm{reg}^{\mathbf{Ury}} = \theta^\infty$ and $\mathrm{epi}^{\mathbf{Ury}} = (\theta^\infty)_\infty$ hold true when restricted to **Ury**, they *fail* when considered for the whole category **Top**;

(d) while, for any closure operator C of **Top**, $\nabla(C) \cap \Delta(C)$ contains only trivial spaces, $\nabla(C) \cap T_1(C)$ may contain non-trivial spaces.

7.S (*Alternative description of the left adjoint $\tau^{(-)} \dashv \mathcal{E}^{(-)}$ of Theorem 7.6*) Let $\mathcal{X}$ be $\mathcal{M}$-complete, with $\mathcal{M}$ as in 2.1, and let $\mathcal{D}$ be a class of morphisms in $\mathcal{X}$. Using the orthogonality relation for morphisms (as defined in 1.8), define a closure operator $\gamma^{\mathcal{D}}$, as follows : given $m : M \to X$ in $\mathcal{M}$, for every $d : U \to V$ in $\mathcal{D}$, let $\mathcal{F}(d)$ be the set of all morphisms $v : V \to X$ for which there is $u : U \to M$ with $m \cdot u = v \cdot d$; now put

$$\gamma_X^{\mathcal{D}}(m) = \bigvee \{v(1_V) : v \in \mathcal{F}(d) \text{ for some } d : U \to V \text{ in } \mathcal{D}\}$$

and show:

(a) $\gamma^{\mathcal{D}}$ is a closure operator of $\mathcal{X}$ with $\mathcal{D} \subseteq \mathcal{E}^{\gamma^{\mathcal{D}}}$;

(b) for every closure operator C of $\mathcal{X}$ with $\mathcal{D} \subseteq \mathcal{E}^C$, $\gamma^{\mathcal{D}} \leq C$. Conclude $\gamma^{\mathcal{D}} \cong \tau^{\mathcal{D}}$, in particular: $\gamma^{\mathcal{D}}$ is weakly hereditary.

7.T (*Weak hereditariness of* $\mathrm{reg}^{\mathrm{E}(\mathcal{A})}$) Let $\mathcal{A}$ be a subcategory of a category $\mathcal{X}$ as in 7.7. Show:

(a) $(\mathrm{reg}^{\mathcal{A}})_\infty \leq \mathrm{reg}^{\mathrm{E}(\mathcal{A})} \leq \mathrm{reg}^{\mathcal{A}}$. The first inequality becomes an equality iff $\mathrm{reg}^{\mathrm{E}(\mathcal{A})}$ is weakly hereditary.

(b) $\mathrm{reg}^{\mathrm{E}(\mathcal{A})}$ is weakly hereditary in case $\mathcal{X} = Mod_R$. *Hint:* See Ex.7.6 (3).

(c) In case $\mathcal{X} =$ **Grp** or **Top**, $\mathrm{reg}^{\mathrm{E}(\mathcal{A})}$ is weakly hereditary iff the epimorphisms in $\mathcal{A}$ are surjective.

Notes

The Salbany Correspondence of Theorem 7.1 appears in Tholen [1988], while the Generating Diagonal Theorem was proved by Giuli and Hušek [1986] for the category **Top** and by Giuli, Mantovani and Tholen [1988] in categorical generality. The notion of essential equivalence for closure operators was introduced (under a different name) by Dikranjan [1992] who also proved the crucial Theorems 7.4 and 7.5 in the context of topological categories over Set. The PR-Correspondence appears in the paper by Pumplün and Röhrl [1985], with its factorization through the conglomerate of idempotent closure operators being discussed by Castellini, Koslowski and Strecker [1992a]. Hoffmann[1982] introduced the maximal epi-preserving extension of Section 7.7, with more general categorical studies appearing in Giuli, Mantovani and Tholen [1988]; its description via a closure operator has its origins in the paper [1987b] by Dikranjan and Giuli. Nabla subcategories were defined but hardly studied in the Dikranjan-Giuli paper [1987a], while the "companions" 7.8 of $\Delta(C)$ appear for the first time in Dikranjan [1992].

8 Epimorphisms and Cowellpoweredness

Characterizing the epimorphisms of a concrete category and settling the question whether the category is cowellpowered can be a challenging problem and has been the theme of many research papers (see the Notes at the end of this chapter). In many cases, closure operators offer themselves as a natural tool to tackle the problem. We concentrate here on results for those categories of topology and algebra where this approach proves to be successful. These include criteria for epimorphisms in subcategories of modules and fields, recent or new results on cowellpowered and non-cowellpowered subcategories of topological spaces, and a rather direct proof of Uspenskij's recent discovery of a non-dense epimorphism in the category of Hausdorff topological groups.

8.1 Categorical preliminaries

We consider an arbitrary category $\mathcal{X}$ and any subclass $\mathcal{E}$ of morphisms in $\mathcal{X}$ which is closed under composition with isomorphisms. Two morphisms $e, d \in \mathcal{E}$ with common domain are *isomorphic* ($e \cong d$) if there is an isomorphism j in $\mathcal{X}$ with $j \cdot e = d$. For every object $X \in \mathcal{X}$, this defines an equivalence relation on the class $X \backslash \mathcal{E}$ of morphisms in $\mathcal{E}$ with domain X. The category $\mathcal{X}$ is called *$\mathcal{E}$-cowellpowered* or *cowellpowered w.r.t.* $\mathcal{E}$ if for every $X \in \mathcal{X}$, the conglomerate of $\cong$-equivalence classes can be labeled by a small set, i.e., if there is a small set I_X and a map

$$\varphi_X : I_X \to X \setminus \mathcal{E}$$

such that for every $e \in \mathcal{E}$ with codomain X there is $i \in I$ with $\varphi_X(i) \cong e$. Note that $\mathcal{E}$ is not assumed to be necessarily a class of epimorphisms in $\mathcal{X}$. If it is, then $\mathcal{E}$-cowellpoweredness is dual to $\mathcal{M}$-wellpoweredness as defined in 1.1, i.e., $\mathcal{X}$ is $\mathcal{E}$-cowellpowered if and only if $\mathcal{X}^{op}$ is $\mathcal{E}$-wellpowered. The prefix $\mathcal{E}$ is omitted if $\mathcal{E} = \mathrm{Epi}(\mathcal{X})$ is precisely the class of epimorphisms in $\mathcal{X}$.

As usual in Category Theory, *we assume the category $\mathcal{X}$ to have small hom-sets*. The advantage of this assumption for the notion of $\mathcal{E}$-cowellpoweredness is that it suffices to consider the codomains of morphisms in $X \setminus \mathcal{E}$. Every object Y which appears as such a codomain is called an *$\mathcal{E}$-image* of X. (If $\mathcal{E}$ belongs to an $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathcal{X}$, this terminology is in accordance with the notion introduced in 1.3.) We then have:

PROPOSITION *$\mathcal{X}$ is $\mathcal{E}$-cowellpowered if and only if every object has only a small set of non-isomorphic $\mathcal{E}$-images.*

Proof With a representative system $(Y_j)_{j \in J}$ of non-isomorphic $\mathcal{E}$-images of X one obtains for every $e : X \to Y$ in $\mathcal{E}$ a unique $j \in J$ with $Y_j \cong Y$. Hence the inclusion map

$$I_X := \bigcup_{j \in J} \mathcal{E}(X, Y_j) \hookrightarrow X \setminus \mathcal{E}$$

with $\mathcal{E}(X,Y_j) := \mathcal{E} \cap \mathcal{X}(X,Y_j)$ shows the "if" part. The converse statement is trivial. □

We now consider any functor $F : \mathcal{A} \to \mathcal{X}$. For an object $X \in \mathcal{X}$, let $F^{-1}X := \{A \in \mathcal{A} : FA = X\}$ be the (class of objects of the) *fibre of F at X*, and let $\tilde{F}^{-1}X := \{B \in \mathcal{A} : FB \cong X\}$ be its *replete closure* in $\mathcal{X}$. F is called (*strongly*) *fibre-small* if (the replete closure of) each fibre of F contains only a small set of non-isomorphic objects. Finally, F is called *transportable* if for every $B \in \tilde{F}^{-1}X$ there is $A \in F^{-1}X$ with $A \cong B$.

LEMMA *Every strongly fibre-small functor F is fibre-small, and the converse holds true if F is transportable.*

Proof If F is transportable, then a representative system of non-isomorphic objects in $F^{-1}X$ is also representative for $\tilde{F}^{-1}X$. □

We obtain the following simple Theorem which, however, proves to be very useful for applications:

THEOREM *Let $F : \mathcal{A} \to \mathcal{X}$ be a functor, and let $\mathcal{D}, \mathcal{E}$ be classes of morphisms of $\mathcal{A}$, $\mathcal{X}$, respectively, both closed under composition with isomorphisms, such that $F(\mathcal{D}) \subseteq \mathcal{E}$. Then $\mathcal{E}$-cowellpoweredness of $\mathcal{X}$ implies $\mathcal{D}$-cowellpoweredness of $\mathcal{A}$, provided F is strongly fibre-small; even fibre-small suffices if F is transportable.*

Proof For $A \in \mathcal{A}$, let $(Y_j)_{j \in J}$ represent the non-isomorphic $\mathcal{E}$-images of FA, and for every $j \in J$, let $(B_{jk})_{k \in K_j}$ represent the non-isomorphic objects in $\tilde{F}^{-1}Y_j$. Since $F(\mathcal{D}) \subseteq \mathcal{E}$, for every $\mathcal{D}$-image B of A, FB is isomorphic to some Y_j, hence $B \in \tilde{F}^{-1}Y_j$ is isomorphic to some B_{jk}. Consequently, the size of a representative system of $\mathcal{D}$-images of X cannot exceed the cardinality of

$$\bigcup_{j \in J} \{j\} \times K_j .$$

□

COROLLARY *If $\mathcal{A}$ admits a fibre-small, transportable functor F into* **Set**, *then $\mathcal{A}$ is $\mathcal{D}$-cowellpowered if and only if there is a cardinal function ρ for the objects of $\mathcal{A}$ such that for every $\mathcal{D}$-image B of A,* $\mathrm{card}(FB) \le \rho(A)$.

Proof Taking for $\mathcal{E}$ the closure of $F(\mathcal{E})$ under composition with isomorphisms, we see with the Theorem that the given condition is sufficient for $\mathcal{D}$-cowellpoweredness of $\mathcal{A}$. Trivially, it is also a necessary condition. □

REMARKS

(1) Transportability is an essential condition for both, the Theorem and the Corollary. For instance, if $\mathcal{A}$ is the ordered class *Ord* of all ordinals, considered as a

category (see Example 1.11(2)), we may define a faithful and fibre-small functor $F : Ord \to \mathbf{Set}$ sending each ordinal α to the singleton set $\{\alpha\}$. Every morphism in Ord is epic, hence Ord is *not* cowellpowered. Still, the constant cardinal function 1 satisfies the criterion of the Corollary. Note also that F maps every (epi)morphism of Ord to an isomorphism in $\mathbf{Set}$.

(2) While faithfulness is irrelevant for the validity of the Theorem and the Corollary, it is the only essential condition in the following useful statement: *if any category $\mathcal{A}$ admits a faithful functor into* $\mathbf{Set}$*, then $\mathcal{A}$ is both wellpowered w.r.t. the regular monomorphisms of $\mathcal{A}$ and cowellpowered w.r.t. the regular epimorphisms of $\mathcal{A}$* (with regular epimorphism defined dually to regular monomorphism): see Exercise 8.A.

(3) Statement (2) can be strengthened: one only needs a *collectively faithful small set* of functors $(F_i : \mathcal{A} \to \mathbf{Set})_{i\in I}$, so that two morphisms $f, g : A \to B$ in $\mathcal{A}$ with $F_i f = F_i g$ for all $i \in I$ must coincide. Any category $\mathcal{A}$ with a small generating set $(G_i)_{i\in I}$ of objects provides this environment.

(4) Having achieved cowellpoweredness w.r.t. regular epimorphisms fairly easily, one may ask about cowellpoweredness w.r.t. the larger class of strong epimorphisms or even of extremal epimorphisms (for definitions in the dual case, see Exercises 1.D, 1.E). Since

$$\{\text{regular epis}\} \subseteq \{\text{strong epis}\} \subseteq \{\text{extremal epis}\},$$

with strong epis the only class always being closed under composition, it is natural to consider *large chains of regular epimorphisms*, i.e., functors $E : Ord \to \mathcal{A}$ such that $E(\alpha) \to E(\alpha+1)$ is a non-isomorphic regular epimorphism for every $\alpha \in Ord$, and $E(\lambda) \cong \mathrm{colim}_{\alpha<\lambda} E(\alpha)$ for every limit ordinal λ. Clearly, *for the conditions*

(i) *$\mathcal{A}$ is cowellpowered w.r.t. extremal epimorphisms,*

(ii) *$\mathcal{A}$ is cowellpowered w.r.t. strong epimorphisms,*

(iii) *$\mathcal{A}$ is cowellpowered w.r.t. regular epimorphisms and $\mathcal{A}$ has no large chains of regular epimorphisms,*

one obviously has (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). However, whether these conditions are equivalent for categories $\mathcal{A}$ with "sufficiently many" colimits, depends on our set-theoretic hypotheses, as we shall explain in more detail next.

One needs a "large version" of *König's Lemma* which asserts that an infinite tree with finite levels has an infinite branch. Recall that a *tree* is a poset with a bottom element such that for every element x, the set of all predecessors of x is well-ordered; the ordinal type of this set is called the *level* of x. Any well-ordered set of a tree is called a *branch*. Observe that a large tree has small levels if and only if every element has only a small set of immediate successors. Now call the *universe* (the class which has as its elements all (small) sets) *weakly compact* if every large tree with small levels has a large branch. In this terminology one can prove (cf. Adámek and Tholen [1990]):

(I) *If the universe is weakly compact, then for every cocomplete category $\mathcal{A}$ with a* *small generating set, conditions (i)-(iii) are equivalent.* (Actually, $\mathcal{A}$ needs to have only coequalizers and small-indexed cointersections of strong epimorphisms.)

(II) *If the universe is not weakly compact, then there is a cocomplete category $\mathcal{A}$ with a small generating set which satisfies (iii) but not (ii).*

(5) Characterizing the extremal epimorphisms in a category is normally an easy task, provided one "knows" sufficiently many monomorphisms. But in a concrete category $\mathcal{A}$ with a faithful functor $F : \mathcal{A} \to \mathbf{Set}$, morphisms with injective underlying **Set**-maps are certainly monic. Hence, if the (epi,mono)-factorization of $Ff : FA \to FB$ in **Set** can be "lifted" to $\mathcal{A}$, so that $f = m \cdot e$ with Fm injective and Fe surjective, then every extremal epimorphism of $\mathcal{A}$ has surjective underlying **Set**-map. (The converse statement holds true if F reflects isomorphisms.) Certainly, a mono-fibration F has the needed lifting property and is transportable. Hence, a trivial application of the Corollary gives the statement that *a category $\mathcal{A}$ which admits a faithful, fibre-small mono-fibration $\mathcal{A} \to$ **Set**, is cowellpowered w.r.t. extremal epimorphisms.*

Unfortunately, the gap between extremal epimorphisms and all epimorphisms is generally big, hence settling the question of cowellpoweredness becomes considerably more difficult. Consequently, for general results, fairly strong assumptions on the category are needed. The next section contains such results for categories of type $S(\mathcal{A})$ and $\overline{S}(\mathcal{A})$, with $\mathcal{A}$ small or $\mathcal{A}$ closed under limits.

We close this section with an immediate application of the Corollary:

EXAMPLE **Haus** *is cowellpowered.* In fact, epimorphisms in **Haus** are dense (Example 6.5 (2)), and for every dense subspace X of a Hausdorff space Y, $\mathrm{card}Y \leq 2^{2^{\mathrm{card}X}}$. For the latter property, consider the injective map that assigns to every $y \in Y$ the set $\{U \cap X : U \text{ neighbourhood of } y \text{ in } Y\}$.

8.2 Reflectivity and cowellpoweredness

A class of objects $\mathcal{A}$ in a category $\mathcal{X}$ is *cogenerating* if $\mathcal{X} = S_{\mathcal{X}}(\mathcal{A})$, i.e., for every object X there is a mono-source with domain X and codomain in $\mathcal{A}$; equivalently, the source $\mathcal{X}(X, \mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathcal{X}(X, A)$ is monic (cf. 7.2 in the dual situation). $\mathcal{A}$ is called *strongly cogenerating* if $\mathcal{X} = \overline{S}_{\mathcal{X}}(\mathcal{A})$ (cf. Exercise 7.K).

The existence of a *small* cogenerating set in a category usually enforces cowellpoweredness:

PROPOSITION *Let $\mathcal{X}$ admit a transportable and fibre-small functor $F : \mathcal{X} \to$ **Set** which preserves mono-sources. If $\mathcal{X}$ has a small cogenerating set $\mathcal{A}$ of objects, then $\mathcal{X}$ is cowellpowered.*

Proof For an epimorphism $e : X \to Y$ in $\mathcal{X}$, the **Set**-map

$$\mathcal{X}(e, \mathcal{A}) : \mathcal{X}(Y, \mathcal{A}) \to \mathcal{X}(X, \mathcal{A}), \ \mapsto h \cdot e$$

is injective. As a source in $\mathcal{X}$, $\mathcal{X}(Y,\mathcal{A})$ is monic, and so is its F-image. Hence the canonical map

$$FY \to \prod_{h\in\mathcal{X}(Y,\mathcal{A})} F(\text{codomain}(h)),\ z \mapsto ((Fh)(z))_h,$$

is injective. Consequently, if $\mathcal{X}(Y,\mathcal{A}) = \emptyset$, the cardinality of FY is at most 1. Otherwise the injective map $\mathcal{X}(e,\mathcal{A})$ has a retraction which induces an injective map

$$\prod_{h\in\mathcal{X}(Y,\mathcal{A})} F(\text{codomain}(h)) \to \prod_{k\in\mathcal{X}(X,\mathcal{A})} F(\text{codomain}(k));$$

hence the cardinality of FY is bounded by the cardinality of the codomain of the last map. Consequently, the assertion follows from Corollary 8.1 □

Of course, smallness of the cogenerating set $\mathcal{A}$ is essential for the validity of the Proposition (since every category $\mathcal{X}$ has $\mathcal{X}$ as its own cogenerating class). Still, for not necessarily small $\mathcal{A}$, something can be said about the interplay between $\mathcal{A}$, $\overline{S}_{\mathcal{X}}(\mathcal{A})$, and $S_{\mathcal{X}}(\mathcal{A})$, as far as reflectivity and cowellpoweredness are concerned, provided $\mathcal{A}$ is closed under limits:

THEOREM *Let $\mathcal{A}$ be a full and replete subcategory of a complete wellpowered and cowellpowered category $\mathcal{X}$. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ *is reflective in* $\mathcal{X}$ *and cowellpowered;*

(ii) $\mathcal{A}$ *is limit-closed in* $\mathcal{X}$ *and* $S_{\mathcal{X}}(\mathcal{A})$ *is cowellpowered;*

(iii) $\mathcal{A}$ *is limit-closed in* $\mathcal{X}$ *and* $\overline{S}_{\mathcal{X}}(\mathcal{A})$ *is cowellpowered.*

Proof (i) $\Rightarrow$ (ii) Reflectivity always implies closedness under limits, hence cowellpoweredness of $S(\mathcal{A})$ is the only issue. For an epimorphism $e : B \to C$ in $S(\mathcal{A})$, consider its reflection:

$$\begin{tikzcd} B \arrow[r, "e"] \arrow[d, "\rho_B"'] & C \arrow[d, "\rho_C"] \\ RB \arrow[r, "Re"] & RC \end{tikzcd} \tag{8.1}$$

Since $C \in S(\mathcal{A})$, ρ_C is monic in $\mathcal{X}$ and in $S(\mathcal{A})$, hence an epimorphism in $S(\mathcal{A})$ (cf. Exercise 3.L, dual). But since $\rho_C \cdot e = Re \cdot \rho_B$ is epic, also Re is epic in $S(\mathcal{A})$, in fact epic in $\mathcal{A}$. Hence every epic image of B in $S(\mathcal{A})$ is a monic subobject in $\mathcal{X}$ of an epic image of RB in $\mathcal{A}$. Consequently, wellpoweredness of $\mathcal{X}$ and cowellpoweredness of $\mathcal{A}$ give cowellpoweredness of cowellpoweredness of $S_{\mathcal{X}}(\mathcal{A})$

(i) $\Rightarrow$ (iii) In (i) $\Rightarrow$ (ii) replace $S(\mathcal{A})$ by $\overline{S}(\mathcal{A})$. ("Monic" may be replaced by "strongly monic"; hence wellpoweredness of $\mathcal{X}$ with respect to strong monomorphisms suffices for this implication.)

(ii) $\Rightarrow$ (i) Since $\mathcal{A} \hookrightarrow S_{\mathcal{X}}(\mathcal{A})$ preserves epimorphisms, cowellpoweredness of $\mathcal{A}$ follows trivially with Theorem 8.1. Since $S_{\mathcal{X}}(\mathcal{A})$ is reflective in $\mathcal{X}$, due to co-wellpoweredness of $\mathcal{X}$ (cf. Exercise 7.A), only reflectivity of $\mathcal{A}$ in $S(\mathcal{A})$ is left to be shown. Since $\mathcal{A}$ is complete, with limits formed as in $S(\mathcal{A})$ and $\mathcal{X}$, according to the *General Adjoint Functor Theorem*, it suffices to provide a solution set for each $B \in S(\mathcal{A})$. We claim that a representative system of $S(\mathcal{A})$-epimorphisms with codomain in $\mathcal{A}$ gives such a set. In fact, every morphism $f : B \to A \in \mathcal{A}$ factors as $f = m \cdot e$, with $m : A' \to A$ monic in $\mathcal{A}$ and $e : B \to A'$ epic in $S(\mathcal{A})$: take m to be the intersection of all monos in $\mathcal{A}$ through which f factors (cf. Theorem 1.10); the resulting morphism is epic since $\mathcal{A}$ is closed under equalizers.

(iii) $\Rightarrow$ (i) In (ii) $\Rightarrow$ (i), replace $S(\mathcal{A})$ by $\overline{S}(\mathcal{A})$. □

REMARKS

(1) Cowellpoweredness of $\mathcal{X}$ is used in the Theorem only to derive reflectivity of $S_{\mathcal{X}}(\mathcal{A})$ and $\overline{S}_{\mathcal{X}}(\mathcal{A})$. But in the case of $S_{\mathcal{X}}(\mathcal{A})$, cowellpoweredness of $\mathcal{X}$ w.r.t. strong epimorphisms suffices for that.

(2) For (ii) $\Rightarrow$ (i), only cowellpoweredness of $S_{\mathcal{X}}(\mathcal{A})$ w.r.t. those $\mathcal{A}$-epimorphisms with codomain in $\mathcal{A}$ is needed which factor only trivially through a monomorphism of $\mathcal{A}$.

EXAMPLES

(1) The category **CBoo** of complete Boolean algebras is an orthogonal full subcategory of the category **Frm**, hence closed under limits (cf. Exercise 5.B). But it is not reflective in **Frm** since, otherwise, free complete Boolean algebras (on countably many free generators) would have to exist – but they don't (see Johnstone [1982], p. 33). Still, **CBoo** and **Frm** are intimately connected: every frame is isomorphic to a subframe of a Boolean algebra (see Johnstone [1982], p. 53); hence $S(\mathbf{CBoo}) = \mathbf{Frm}$. *From these two facts alone one concludes with the Theorem that* **Frm** *is not cowellpowered*, as follows: **Frm** is complete and trivially wellpowered and cowellpowered w.r.t. strong epimorphisms (see Remark (1)!; strong epis are surjective in **Frm**, according to Remark 8.1(5)), hence the Theorem is applicable, and failure of condition (i) yields failure of (ii). (An explicit construction of a large chain of epimorphisms can be found in Johnstone [1982], p. 53.) Conversely, having non-cowellpoweredness of **Frm** and using the fact that epimorphisms in the category **CBoo** are surjective, the Theorem also permits to derive non-reflectivity of **CBoo** and therefore non-existence of (certain) free complete Boolean algebras!

(2) Let $\mathcal{A}$ be the full subcategory of **Top** described in Example 7.5 (i.e., a proper rigid class). We claim that $S(\mathcal{A})$ *is not cowellpowered*. In fact, by Lemma 7.1, $\mathcal{A}$ and $S(\mathcal{A})$ give the same regular closure. Since $\mathcal{A}$ does not consist of singleton spaces only, $S(\mathcal{A})$ contains the discrete doubleton D. As shown in Example 7.5, every injective map $D \to X \in \mathcal{A}$ is $\mathcal{A}$-dense, hence $S(\mathcal{A})$-dense. But the cardinality of spaces in a proper class cannot be bounded. Hence, by Corollary 8.1, $S(\mathcal{A})$ is

not cowellpowered. With the Theorem one obtains that $\overline{S}(\mathcal{A})$ is not cowellpowered either.

8.3 Epimorphisms in subcategories of Top – a first summary

In this section we give a brief synopsis of those results on epimorphisms and cowellpoweredness for subcategories of **Top** which can be obtained fairly easily from the general techniques presented so far. We begin with a list of:

EXAMPLES

(1) *For $\mathcal{A}$ epireflective with weakly hereditary regular closure operator, epimorphisms of $\mathcal{A}$ are surjective, hence $\mathcal{A}$ is cowellpowered* (Proposition 6.9, Corollary 8.1). This applies to $\mathbf{Top}_1$ (Examples 6.5(2), 6.9(2)), $\mathcal{H}_\alpha$ (Example 6.9(4)), and every disconnectedness (in the sense of Exercise 6.T).

(2) *For $\mathcal{A}$ epireflective with* $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = K|_{\mathcal{A}}$*, one has $\mathcal{A} \subseteq$* **Haus***, and epimorphisms are (K-)dense, hence $\mathcal{A}$ is cowellpowered* (Corollary 6.9, Example 8.1). This applies to **Reg**, **Tych**, 0-**Top** and **DHaus** (=totally disconnected Hausdorff spaces), see Examples 6.9(4), 6.9(1).

(3) The (idempotent, additive and grounded) b-closure can be considered as a concrete functor $\mathbf{b} : \mathbf{Top} \to \mathbf{Top}$ (cf. 5.10). Now $\mathbf{b}$ maps $\mathbf{Top}_0$ and its b-dense maps to **Haus** and its K-dense maps. Hence the restriction $\mathbf{Top}_0 \to \mathbf{Haus}$ of $\mathbf{b}$ preserves epimorphisms (cf. Example 6.5(2),(3)), so that $\mathbf{Top}_0$ must be cowellpowered (Corollary 8.1, Example 8.1).

(4) Cowellpoweredness of $\mathbf{Top}_0 = S_{\mathbf{Top}}(\{\text{ Sierpinski space}\})$, 0-$\mathbf{Top} = \overline{S}_{\mathbf{Top}}(\{$2-point discrete space$\})$, $\mathbf{Tych} = \overline{S}_{\mathbf{Top}}(\{$unit interval$\})$ can also be derived with Proposition 8.2.

Let us now assume $\mathcal{A}$ to be strongly epireflective in **Top** with $\mathcal{A} \subseteq$ **Haus**. Then $(K$-)dense maps are certainly $\mathcal{A}$-epic (Corollary 6.9). We are interested in a criterion for the converse proposition and, for that purpose, we try to define the "position" of its maximal epi-preserving extension $\mathrm{D}_{\mathbf{Top}}(\mathcal{A})$ (see 7.7). For that it is useful to have the following criterion for containment in **Haus**.

For every cardinal κ, consider a set X of cardinality κ and an ultrafilter Φ on X. Then one generates a topology on $Y_{\kappa,\Phi} = X \cup \{a, b\}$ with $a \neq b$ outside X by declaring the sets $\{x\}$ $(x \in X)$, $U \cup \{a\}$ $(U \in \Phi)$, $U \cup \{b\}$ $(U \in \Phi)$ to be open.

LEMMA *For $\mathcal{A}$ strongly epireflective subcategory in* **Top***, $\mathcal{A} \subseteq$* **Haus** *holds if and only if* no *space $Y_{\kappa,\Phi}$ belongs to $\mathcal{A}$.*

Proof The necessity is obvious since the space $Y_{\kappa,\Phi}$ is non-Hausdorff. Assume $\mathcal{A} \not\subseteq$ **Haus**. If $\mathcal{A} \not\subseteq \mathbf{Top}_1$ then $\mathcal{A}$ contains the Sierpiński dyad, consequently $\mathcal{A}$ contains $\mathbf{Top}_0$. Since each space $Y_{\kappa,\Phi}$ is T_1, we conclude $Y_{\kappa,\Phi} \in \mathcal{A}$ in this case. Now suppose that $Z \in \mathcal{A}$ is a non-Hausdorff T_1 space. Then there exist distinct points $a, b \in Z$ which cannot be separated by disjoint neighbourhoods. With $X = Z \setminus \{a, b\}$ the intersections $U \cap V \cap X$, where U is a neighbourhood of a and V is a neighbourhood of b, form a filter-base $\mathcal{F}$ on X. Let Φ be an ultrafilter on X containing $\mathcal{F}$. Then the identity $1_Z : Y_{\kappa,\Phi} \to Z$ is obviously continuous, so that $Y_{\kappa,\Phi} \in \mathcal{A}$ follows from $Z \in \mathcal{A}$ since $\mathcal{A}$ is strongly epireflective. □

REMARK There are similar (in fact: easier) containment criteria for other subcategories in lieu of **Haus**: for any strongly epireflective $\mathcal{A}$ in **Top**, one has:

(a) $\mathcal{A} \subseteq \mathbf{Top}_0$ if and only if the 2-point indiscrete space does not belong to $\mathcal{A}$;

(b) $\mathcal{A} \subseteq \mathbf{Top}_1$ if and only if the Sierpiński dyad does not belong to $\mathcal{A}$;

(c) $\mathcal{A} \subseteq \mathcal{H}_\alpha$ if and only if the space X_α of Example 6.9(5) does not belong to $\mathcal{A}$;

(d) $\mathcal{A} \subseteq \mathbf{US}$ if and only if the following space $Y = \mathbb{N} \cup \{a, b\}$ does not belong to $\mathcal{A}$: basic open sets are $\{n\}$, $A \cup \{a\}$, $B \cup \{b\}$ with $n \in \mathbb{N}$ and $A, B \subseteq \mathbb{N}$ cofinite.

We leave the verification of these statements as Exercise 8.D.

PROPOSITION *Let $\mathcal{A}$ be non-trivial and strongly epireflective in* **Top** *with $\mathcal{A} \subseteq$* **Haus**. *Then also $\mathrm{D}_{\mathbf{Top}}(\mathcal{A}) \subseteq$* **Haus**, *and both categories coincide if and only if the epimorphisms of $\mathcal{A}$ are precisely the K-dense maps in $\mathcal{A}$; in this case $\mathcal{A}$ is cowellpowered.* □

Proof Since $\mathrm{D}(\mathcal{A})$ is strongly epireflective in **Top** (Remark (1) of 7.7), if we assume $\mathrm{D}(\mathcal{A}) \not\subseteq$ **Haus**, then $\mathrm{D}(\mathcal{A})$ must contain a **Haus**-test space $Y_{\kappa,\Phi} = X \cup \{a, b\}$, by the Lemma. The subspace $Z \cup \{a\}$ is Hausdorff and actually zero-dimensional. But since $\mathcal{A} \neq \mathcal{T}$ is non-trivial, the strongly epireflective $\mathcal{A}$ contains 0-**Top**, hence Z and its subspace X belong to $\mathcal{A}$. The inclusion $X \hookrightarrow Z$ is dense, i.e., epic in **Haus**, hence epic in $\mathcal{A}$ and therefore in $\mathrm{D}(\mathcal{A})$. On the other hand, the distinct maps $f, g : Z \to Y$ with $f|_X = g|_X$, $f(a) = a$, $g(a) = b$ are continuous – a contradiction. This proves $\mathrm{D}(\mathcal{A}) \not\subseteq$ **Haus**.

Since $\mathcal{A} \subseteq$ **Haus**, K-dense maps in $\mathcal{A}$ are always $\mathcal{A}$-dense (Corollary 6.9). Hence, if $\mathrm{D}(\mathcal{A}) =$ **Haus**, epimorphisms in $\mathcal{A}$ are epic in $\mathrm{D}(\mathcal{A})$ and therefore K-dense. On the other hand, having the latter condition, the maximality of $\mathrm{D}(\mathcal{A})$ yields $\mathrm{D}(\mathcal{A}) =$ **Haus**. □

The Proposition gives $\mathrm{D}(\mathcal{A}) =$ **Haus** if and only if $\mathrm{epi}^{\mathcal{A}}|_{\mathcal{A}} = K|_{\mathcal{A}}$.

PROBLEM *For strongly epireflective $\mathcal{A}$ as in the Proposition, is the statement $\mathrm{D}(\mathcal{A}) =$* **Haus** *also equivalent to $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = K|_{\mathcal{A}}$, or to $\mathrm{E}(\mathcal{A}) =$* **Haus***?*

Let us now move to categories of type $\Delta(C)$ and $T_2(C)$ for an additive closure operator C of **Top**. (Due to the Additive Diagonal Theorem, every strongly epireflective subcategory is of this type; see Proposition 7.5.) From Theorem 7.9(3) one knows that the epimorphisms in each $\Delta(C)$ and $T_2(C)$ are precisely the $(\tilde{C})^\infty$-dense maps, so that the inclusion functor $\Delta(C) \to T_2(C)$ preserves epimorphisms. This proves the first statement of:

THEOREM *Let C be an additive closure operator of* **Top**. *Then $T_2(C)$ is cowellpowered under each of the following two conditions:*

(a) *$\Delta(C)$ is cowellpowered,*

(b) *C is essentially strong and bounded on $T_2(C)$.*

Proof We are left with having to consider condition (b) so that $(\tilde{C})^\infty = C^\infty = C^\alpha$ for some $\alpha \in Ord$; cf. Corollary 6.6. First note that for every C-dense subspace X of $Y \in T_2(C)$, $\mathrm{card}Y \leq 2^{2^{\mathrm{card}X}}$; this is shown exactly as in the case $C = K$ (cf. Example 8.1), by trading (K-) neighbourhoods for C-neighbourhoods. From the definition of C^α one obtains by ordinal induction

$$\mathrm{card}Y \leq \rho^\alpha(X)$$

whenever X is C^α-dense in Y, with ρ^α the α-th iteration of the function $\rho(\kappa) = 2^{2^\kappa}$. With Corollary 8.1, this proves the claim. □

Without prior knowledge that C is essentially strong, or if C fails to be essentially strong, like in the case of the sequential closure σ (see Exercise 7.H), we need effective computational methods for its strong modification $\tilde{C}$. These are provided in particular cases in the following section.

8.4 Projective closure operators and the categories Haus($\mathcal{P}$)

For every class $\mathcal{P}$ of topological spaces and every closure operator C of **Top** one has the $\mathcal{P}$-modification of C, as defined in 7.7. For $C = K$ the Kuratowski closure operator, and if $\mathcal{P}$ contains a non-empty space, then this is the operator given by

$${}^{\mathcal{P}}k_X(M) = \bigcup\{h(\overline{h^{-1}(M)}) : P \in \mathcal{P}, h : P \to X\}.$$

Obviously, ${}^{\mathcal{P}}K \leq K$, with $K = {}^{\mathbf{Top}}K$. We have seen in Example 7.7 that, for instance, σ and $\mathfrak{k}$ are such comodifications of K. In this section we study specific closure operators C with ${}^{\mathcal{P}}K \leq C \leq K$ and their induced Delta-subcategories.

DEFINITION For a class $\mathcal{P} \subseteq$ **Top** containing a non-empty space, define the *$\mathcal{P}$-projective closure operator* $\mathrm{pro}^{\mathcal{P}}$ by

$$\mathrm{pro}^{\mathcal{P}}_X(M) := \bigcup\{\overline{M \cap h(P)} : P \in \mathcal{P}, h : P \to X\}$$

and its *image restriction* $\mathrm{ipro}^{\mathcal{P}}$ by

$$\mathrm{ipro}^{\mathcal{P}}_X(M) := \bigcup\{\overline{M \cap h(P)} \cap h(P) : P \in \mathcal{P}, h : P \to X\}.$$

PROPOSITION

(1) $\mathrm{pro}^{\mathcal{P}}$ *and* $\mathrm{ipro}^{\mathcal{P}}$ *are, like* ${}^{\mathcal{P}}K$*, additive closure operators of* **Top** *with*

$${}^{\mathcal{P}}K \leq \mathrm{ipro}^{\mathcal{P}} \leq \mathrm{pro}^{\mathcal{P}} \leq K.$$

(2) *The Delta-subcategory of* $\mathrm{pro}^{\mathcal{P}}$ *is the subcategory* $\mathbf{Haus}(\mathcal{P})$ *of spaces* X *such that, for every* $h : P \to X$, $P \in \mathcal{P}$*, the subspace* $\overline{h(P)}$ *is Hausdorff.*

(3) *If* $\mathcal{P}$ *is closed-hereditary (=closed under closed subspaces), then* $\mathrm{pro}^{\mathcal{P}}$ *and* $\mathrm{ipro}^{\mathcal{P}}$ *are weakly hereditary, and* $\mathbf{Haus}(\mathcal{P})$ *is epi-closed.*

Proof The verification of (1) and (3) is straightforward. For (2), we first show

$$\mathrm{pro}^{\mathcal{P}} \leq \mathrm{reg}^{\mathbf{Haus}(\mathcal{P})}$$

which implies $\mathbf{Haus}(\mathcal{P}) \subseteq \Delta(\mathrm{pro}^{\mathcal{P}})$. For this we must prove that for all $h : P \to X$, $P \in \mathcal{P}$ and $M \subseteq X$,

$$\overline{M \cap h(P)} \subseteq \mathrm{reg}_X^{\mathbf{Haus}(\mathcal{P})}(M).$$

In fact, we shall prove a stronger condition, namely

$$\overline{M \cap \overline{h(P)}} \subseteq \mathrm{reg}_X^{\mathbf{Haus}(\mathcal{P})}(M) \qquad (+)$$

Let $Z := \overline{h(P)}$ and $N := M \cap Z$. Since Z is closed in X, and since the Kuratowski closure is (weakly) hereditary, we obtain (+) once we have shown the even stronger statement

$$\overline{N} = k_Z(N) \subseteq \mathrm{reg}_Z^{\mathbf{Haus}(\mathcal{P})}(M). \qquad (++)$$

Finally, in order to show (++), we consider maps $f, g; Z \to Y \in \mathbf{Haus}(\mathcal{P})$ with $f|_N = g|_N$ and prove $f|_{\overline{N}} = g|_{\overline{N}}$. Since $Y \in \mathbf{Haus}(\mathcal{P})$, the set $B := \overline{f(h(P))} \cap \overline{g(h(P))}$ is Hausdorff. Indeed, for every $x \in \overline{N}$,

$$f(x) \in f(\overline{N}) \subseteq \overline{f(N)} = \overline{g(N)} \subseteq B$$

since $N \subseteq h(P)$. Analogously, $g(x) \in B$. With the map $s := \langle f, g \rangle : Z \to Y \times Y$, one therefore has $s(x) \in B \times B$. With $s(N) \subseteq \Delta_B$, continuity of s gives

$$s(x) \in k_{Y \times Y}(\Delta_B) \cap B \times B = k_{B \times B}(\Delta_B) = \Delta_B$$

since B is Hausdorff. This shows $f(x) = g(x)$ and completes the proof of the inclusion "$\subseteq$" of $\mathbf{Haus}(\mathcal{P}) = \Delta(\mathrm{pro}^{\mathcal{P}})$.

For "$\supseteq$", let $X \notin \mathbf{Haus}(\mathcal{P})$, so that there is a map $h : P \to X$ with $\overline{h(P)}$ non-Hausdorff. There are then two points $x, y \in \overline{h(P)}$ which cannot be separated. We claim that $(x, y) \in \mathrm{pro}^{\mathcal{P}}(\Delta_X)$ and show $(x, y) \in \overline{h'(P)} = \overline{\Delta_X \cap h'(P)}$ with $h' = \langle h, h \rangle : P \to X \times X$. In fact, any neighbourhood of $(x, y) \in X \times X$ contains a product $W = U \times V$ of open neighbourhoods of x and y. The choice of x and y

enforces $U \cap V \cap \overline{f(P)} \neq \emptyset$, even $U \cap V \cap f(P) \neq \emptyset$ since $U \cap V$ is open. This implies $W \cap h'(P) \neq \emptyset$. Consequently Δ_X is not $\mathrm{pro}^{\mathcal{P}}$-closed, i.e., $X \notin \Delta(\mathrm{pro}^{\mathcal{P}})$. □

REMARKS

(1) Clearly, if $\mathcal{P}$ is closed under images, then $\mathrm{pro}^{\mathcal{P}} = {}^{\mathcal{P}}K$.

(2) The closure operators $\mathrm{pro}^{\mathcal{P}}$ and $\mathrm{ipro}^{\mathcal{P}}$ often coincide (but not always: see Example (1) below). More precisely, if

$$(\forall h : P \to X, P \in \mathcal{P})(\forall x \in \overline{h(P)})(\exists j : Q \to X, Q \in \mathcal{P})\ h(P) \cup \{x\} \subseteq j(Q), \tag{$*$}$$

then $\mathrm{pro}^{\mathcal{P}} = \mathrm{ipro}^{\mathcal{P}}$. Condition (*) is certainly satisfied if the following condition holds:

$$(\forall P \in \mathcal{P})(\exists Q \in \mathcal{P} \setminus \{\emptyset\})\ P + Q \in \mathcal{P}. \tag{$**$}$$

(3) As a consequence of Proposition (2), under condition (*) and, a fortiori, under (**), the Delta-subcategory of $\mathrm{ipro}^{\mathcal{P}}$ is $\mathbf{Haus}(\mathcal{P})$. In general we put

$$\mathbf{Haus}_i(\mathcal{P}) := \Delta(\mathrm{ipro}^{\mathcal{P}}).$$

(4) The assignment $(\mathcal{P} \mapsto \mathrm{pro}^{\mathcal{P}})$ is monotone (actually, $\mathrm{pro}^{\mathcal{P}} = \bigvee_{P \in \mathcal{P}} \mathrm{pro}^{\{P\}}$). More generally, for two classes $\mathcal{P}_0$ and $\mathcal{P}_1$ (each containing a nonempty space), if every $P_0 \in \mathcal{P}_0$ can be covered by a surjective map $q : P_1 \to P_0$ with $P_1 \in \mathcal{P}_1$, then $\mathrm{pro}^{\mathcal{P}_0} \leq \mathrm{pro}^{\mathcal{P}_1}$ and $\mathrm{ipro}^{\mathcal{P}_0} \leq \mathrm{ipro}^{\mathcal{P}_1}$.

Our next goal is to compute the strong modification of $\mathrm{pro}^{\mathcal{P}}$ and $\mathrm{ipro}^{\mathcal{P}}$. In consideration of inclusion (+) in the proof of the Proposition , it seems reasonable to look at

$$\mathrm{esp}^{\mathcal{P}}_X(M) = \bigcup \{\overline{M \cap \overline{h(P)}} : P \in \mathcal{P}, h : P \to X\},$$

$$\mathrm{iesp}^{\mathcal{P}}_X(M) = \bigcup \{\overline{M \cap \overline{h(P)}} \cap h(P) : P \in \mathcal{P}, h : P \to X\}.$$

It is checked easily that these are in fact additive closure operators which, in the lattice of all closure operators, give this picture:

$$\begin{tikzcd}
 & \mathrm{esp}^{\mathcal{P}} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{pro}^{\mathcal{P}} \arrow[dr, dash] & & \mathrm{iesp}^{\mathcal{P}} \arrow[dl, dash] \\
 & \mathrm{ipro}^{\mathcal{P}} &
\end{tikzcd}$$

If $\mathcal{P}$ satisfies (*) of Remark (2), then $\mathrm{ipro}^{\mathcal{P}} = \mathrm{pro}^{\mathcal{P}}$ and $\mathrm{iesp}^{\mathcal{P}} = \mathrm{esp}^{\mathcal{P}}$. The proof that the two new operators play (in most cases) the anticipated role, requires some topological propositions, as given by the following Lemma. For $M \subseteq X \in \mathbf{Top}$, as usual we denote the cokernelpair injections by $i, j : X \to X +_M X =: Y$ and let $\varepsilon : Y \to X$ be their common retraction.

LEMMA *For every $B \subseteq Y$,*

$$j^{-1}(\overline{i(X) \cap \overline{B}}) \cup i^{-1}(\overline{j(X) \cap \overline{B}}) = \overline{M \cap \overline{\varepsilon(B)}} = j^{-1}(\overline{i(X) \cap B}) \cup i^{-1}(\overline{j(X) \cap B}). \quad (\dagger)$$

Proof Let us see first that for $A \subseteq X$ and $x \in X$ the condition $x \notin \overline{M \cap \overline{A}}$ is equivalent to the existence of an open neighbourhood U of x and an open subset $V \subseteq U$ with $V \cap M = U \cap M$ and $V \cap A = \emptyset$. In fact, if the latter condition holds, then also $V \cap \overline{A} = \emptyset$, so that $U \cap (M \cap \overline{A}) = \emptyset$, thus $x \notin \overline{M \cap \overline{A}}$. Conversely, if $x \notin \overline{M \cap \overline{A}}$, then there exists an open neighbourhood U of x such that $U \cap (M \cap \overline{A}) = \emptyset$. Then for every $z \in U \cap M$ there exists an open neighbourhood $z \in V_z \subseteq U$ which avoids $\overline{A}$. Then $V = \bigcup\{V_z : z \in U \cap M\}$ is an open subset of U satisfying $V \cap M = U \cap M$ and $V \cap A = \emptyset$.

Now assume that $x \notin \overline{M \cap \overline{\varepsilon(B)}}$. Then by the above argument there exist open sets $V \subseteq U$ of X such that $x \in U$, $V \cap M = U \cap M$ and $V \cap \varepsilon(B) = \emptyset$. Then $W = i(V) \cup j(U)$ is an open neighbourhood of $j(x)$, which has the property that $W \cap i(X) \cap \overline{B} = \emptyset$. In fact, $W \cap i(X) = i(V)$, so it suffices to prove $i(V) \cap \overline{B} = \emptyset$. Let $z \in X$ and $i(z) \in \overline{B}$. Then $z = \varepsilon i(z) \in \varepsilon(\overline{B}) \subseteq \overline{\varepsilon(B)}$. Now the choice of V gives $z \notin V$. Consequently $W \cap i(X) \cap \overline{B} = \emptyset$ is established, which implies that $j(x) \notin \overline{i(X) \cap \overline{B}}$. Arguing by symmetry we get also $i(x) \notin \overline{j(X) \cap \overline{B}}$. This proves the inclusion $\subseteq$ of the first identity claimed.

To prove the inclusion $\subseteq$ of the second identity claimed, note that $\varepsilon(B) = i^{-1}(B) \cup j^{-1}(B)$ and that therefore

$$\overline{M \cap \overline{\varepsilon(B)}} = \overline{M \cap \overline{i^{-1}(B)}} \cup \overline{M \cap \overline{j^{-1}(B)}}$$

holds. Next we show $\overline{M \cap \overline{i^{-1}(B)}} \subseteq j^{-1}(\overline{i(X) \cap B})$. In fact, if $x \in X \backslash j^{-1}(\overline{i(X) \cap B})$, then there exists a neighbourhood W of $j(x)$ with $W \cap i(X) \cap B = \emptyset$. One can take W of the form $W = i(V) \cup j(U)$, where U is an open neighbourhood of x in X, V is an open subset of U and $V \cap M = U \cap M$. Now $W \cap i(X) \cap B = \emptyset$ yields $V \cap i^{-1}(B) = \emptyset$, which implies $x \notin \overline{M \cap \overline{i^{-1}(B)}}$ according to our initial remark. A similar argument proves $\overline{M \cap \overline{j^{-1}(B)}} \subseteq i^{-1}(\overline{j(X) \cap B})$, and the claimed inclusion is shown. Since the right hand side in (†) is obviously contained in the left hand side, this proves (†). $\square$

We are now ready to state the main result of this section.

THEOREM

(1) $\widetilde{\mathrm{pro}}^{\mathcal{P}} = \widetilde{\mathrm{esp}}^{\mathcal{P}} = \mathrm{esp}^{\mathcal{P}}$ *and* $\widetilde{\mathrm{ipro}}^{\mathcal{P}} \le \widetilde{\mathrm{iesp}}^{\mathcal{P}} = \mathrm{iesp}^{\mathcal{P}}$; *in particular,* $\mathrm{esp}^{\mathcal{P}}$ *and* $\mathrm{iesp}^{\mathcal{P}}$ *are essentially strong.*

(2) $(\mathrm{esp}^{\mathcal{P}})^{\infty}$ *coincides with the regular closure operator of* $\mathbf{Haus}(\mathcal{P})$ *when restricted to* $\mathbf{Haus}(\mathcal{P})$, *and* $(\mathrm{iesp}^{\mathcal{P}})^{\infty}$ *coincides with the regular closure operator of* $\mathbf{Haus}_i(\mathcal{P})$ *when restricted to* $\mathbf{Haus}_i(\mathcal{P})$.

(3) *The epimorphisms of* $\mathbf{Haus}(\mathcal{P})$ *and of* $\mathbf{Haus}_i(\mathcal{P})$ *are precisely the* $(\mathrm{esp}^{\mathcal{P}})^{\infty}$*- and the* $(\mathrm{iesp}^{\mathcal{P}})^{\infty}$*-dense maps, respectively.*

Proof (1) We trivially have $\mathrm{pro}^{\mathcal{P}} \leq \mathrm{esp}^{\mathcal{P}} \leq \widetilde{\mathrm{esp}}^{\mathcal{P}}$. Hence we must show $\widetilde{\mathrm{esp}}^{\mathcal{P}} \leq \mathrm{esp}^{\mathcal{P}} \leq \widetilde{\mathrm{pro}}^{\mathcal{P}}$ in order to conclude $\widetilde{\mathrm{pro}}^{\mathcal{P}} = \mathrm{esp}^{\mathcal{P}} = \widetilde{\mathrm{esp}}^{\mathcal{P}}$, which by Theorem 7.4, yields essential strength of $\mathrm{esp}^{\mathcal{P}}$. But for $M \subseteq X \in \mathbf{Top}$ and any map $h : P \to X +_M X$ with $P \in \mathcal{P}$, we can consider the set $B := h(P)$ and, with $g := \varepsilon \cdot h : P \to X$, obtain from the Lemma:

$$i^{-1}(\overline{j(X) \cap \overline{h(P)}}) \subseteq \overline{M \cap \overline{g(P)}} \subseteq \mathrm{esp}^{\mathcal{P}}_X(M).$$

But the left-hand side represents a typical member of the union defining $\widetilde{\mathrm{esp}}^{\mathcal{P}}$, which is therefore contained in $\mathrm{esp}^{\mathcal{P}}_X(M)$. In order to prove $\mathrm{esp}^{\mathcal{P}}_X(M) \subseteq \widetilde{\mathrm{pro}}^{\mathcal{P}}_X(M)$, consider a map $f : P \to X$ with $P \in \mathcal{P}$, and this time let $B := i(f(P))$ in the Lemma. Then $\varepsilon(B) = f(P)$, and

$$\overline{M \cap \overline{f(P)}} \subseteq i^{-1}(\overline{j(X) \cap \overline{i(f(P))}}),$$

with the right-hand side representing a typical member of the union contributing to $\widetilde{\mathrm{pro}}^{\mathcal{P}}_X(M)$.

The proof of $\widetilde{\mathrm{iesp}}^{\mathcal{P}} \leq \mathrm{iesp}^{\mathcal{P}}$ proceeds similarly to the one given for $\widetilde{\mathrm{esp}}^{\mathcal{P}} \leq \mathrm{esp}^{\mathcal{P}}$ and is therefore omitted.

(2) and (3) follow from (1) and from Theorem 7.4. □

EXAMPLES

(1) Let $\mathcal{P} = \mathcal{T} = \{X : \mathrm{card}(X) \leq 1\}$ be the subcategory of trivial spaces (which does not satisfy condition (*) of the Remark!). Then $\widetilde{\mathrm{ipro}}^{\mathcal{T}} = {}^{\mathcal{T}}K = S$ is discrete and $\mathrm{pro}^{\mathcal{T}} = K^{\oplus}$, so that the Theorem gives $\mathrm{esp}^{\mathcal{T}} = \widetilde{K^{\oplus}}$. The operator $\mathrm{iesp}^{\mathcal{T}}$ is easily seen to be the b-closure. Since trivially $\tilde{S} = S$, this shows that in general $\widetilde{\mathrm{ipro}}^{\mathcal{P}} < \mathrm{iesp}^{\mathcal{P}}$. Consequently, $\mathbf{Haus}(\mathcal{T}) = \Delta(K^{\oplus}) = \mathbf{Top}_1$, while $\Delta(\mathrm{ipro}^{\mathcal{T}}) = \mathbf{Top}$, and $\mathbf{Haus}_i(\mathcal{T}) = \mathbf{Top}_0$.

According to the Theorem, the epimorphisms in $\mathbf{Top}_1 = \mathbf{Haus}(\mathcal{T})$ are the $\widetilde{K^{\oplus}}$-dense maps in $\mathbf{Top}_1$, which are surjective (cf. Example 6.5) since $\widetilde{K^{\oplus}}|_{\mathbf{Top}_1} = S|_{\mathbf{Top}_1}$. Again by the Theorem, the epimorphisms in $\mathbf{Top}_0 = \Delta(\mathrm{iesp}^{\mathcal{T}}) = \mathbf{Haus}_i(\mathcal{T})$ are the b-dense maps (cf. Example 6.5).

(2) Let $\mathcal{P}$ be a class of indiscrete spaces, containing at least one non-trivial space. Then

$$\mathrm{pro}^{\mathcal{P}} = K^{\oplus},\ \mathrm{esp}^{\mathcal{P}} = \widetilde{K^{\oplus}},\ \mathrm{iesp}^{\mathcal{P}} = b,\ \mathrm{ipro}^{\mathcal{P}} = {}^{\mathcal{P}}K = \mu,$$

with μ defined by

$$\mu_X(M) := \bigcup\{I \subseteq X : I \text{ indiscrete } \& \ I \cap M \neq \emptyset\}$$

(see Exercise 8. F). Actually, $\mu = b^{\oplus}$, hence with Exercise 6.U one obtains

$$\widetilde{\mathrm{ipro}}^{\mathcal{P}} = \tilde{\mu} = ((\mathrm{reg}^{\mathbf{Top}_0})^{\oplus})\tilde{} = \mathrm{reg}^{\mathbf{Top}_0} = b = \mathrm{iesp}^{\mathcal{P}}.$$

As in (1), one has $\mathbf{Haus}(\mathcal{P}) = \mathbf{Top}_1$, and

$$\Delta(\mathrm{ipro}^{\mathcal{P}}) = \Delta({}^{\mathcal{P}}K) = \mathbf{Top}_0 = \mathbf{Haus}_i(\mathcal{P}) = \Delta(\mathrm{iesp}^{\mathcal{P}}).$$

(3) Let $\mathcal{P}$ consist of finite spaces, with at least one non-discrete space. In fact, we can then assume that the Sierpiński dyad S belongs to $\mathcal{P}$ (see Remark (4)). Then

$$\mathrm{pro}^{\mathcal{P}} = \mathrm{ipro}^{\mathcal{P}} = K^{\oplus} \text{ and } \mathrm{esp}^{\mathcal{P}} = \mathrm{iesp}^{\mathcal{P}} = \widetilde{K^{\oplus}},$$

and $\mathbf{Haus}(\mathcal{P}) = \mathbf{Haus}_i(\mathcal{P}) = \mathbf{Top}_1$.

(4) Since we are mostly interested in computing the projective closure operators in $\mathbf{Haus}(\mathcal{P})$-spaces (which are T_1-spaces), we shall consider from now on only classes $\mathcal{P} \subseteq \mathbf{Top}_1$ (since every $h : P \to X \in \mathbf{Haus}(\mathcal{P})$ factorizes through the $\mathbf{Top}_1$-reflexion of P). Every infinite T_1-space P admits a continuous bijection $P \to X_\alpha$, where $\alpha = |P|$ and X_α is the "test-space" given in Example 6.9 (5) (see also Remark 8.3(c)), hence $\mathrm{pro}^{\{P\}} \geq \mathrm{pro}^{\{X_\alpha\}}$. Note that $\{X_\alpha\}$ satisfies (*) of Remark (2), so that $\mathrm{pro}^{\{X_\alpha\}} = \mathrm{ipro}^{\{X_\alpha\}}$ and $\mathrm{esp}^{\{X_\alpha\}} = \mathrm{iesp}^{\{X_\alpha\}}$. One has $x \in \mathrm{pro}_X^{\{X_\alpha\}}(M)$ if and only if there exists an embedding $X_\alpha \hookrightarrow X$, such that $x \in X_\alpha$, and $X_\alpha \cap M$ is infinite. Note, that $\mathbf{Haus}(\{X_\alpha\}) = \mathcal{H}_\alpha$ since every continuous T_1-image of X_α is homeomorphic to X_α. Hence $\mathcal{H}_\alpha = \Delta(\mathrm{pro}^{\{X_\alpha\}})$.

(5) $\mathcal{P} = \{\mathbb{N}_\infty\}$. Now ${}^{\{\mathbb{N}_\infty\}}K = \sigma$, as already mentioned. Since (*) is fulfilled, $\mathrm{pro}^{\{\mathbb{N}_\infty\}} = \mathrm{ipro}^{\{\mathbb{N}_\infty\}}$ and $\mathrm{esp}^{\{\mathbb{N}_\infty\}} = \mathrm{iesp}^{\{\mathbb{N}_\infty\}}$. The category $\mathbf{Haus}(\{\mathbb{N}_\infty\})$ coincides with the category $\mathbf{SUS}$ of topological spaces in which each convergent sequence has a unique accumulation point, and $\mathrm{pro}^{\{\mathbb{N}_\infty\}}|_{\mathbf{SUS}} = \sigma_{\mathbf{SUS}}$ (see Exercise 8.I). Consequently, $\mathrm{reg}^{\mathbf{SUS}}|_{\mathbf{SUS}} = \sigma^\infty|_{\mathbf{SUS}}$.

(6) $\mathcal{P} = \{\kappa\}$, where κ is a cardinal provided with the discrete topology. Now (*) holds and $\mathrm{pro}^\kappa = \mathrm{ipro}^\kappa$ is idempotent. (This closure operator is known also as *κ-closure*; spaces X with $\mathrm{pro}^\kappa_X = k_X$ are usually called *spaces of tightness* $\leq \kappa$.) Furthermore, esp^κ is not idempotent (so that pro^κ is not essentially strong, see Exercise 8.K). $\mathbf{Haus}(\kappa)$ is the category of spaces in which every subspace of cardinality $\leq \kappa$ is Hausdorff.

(7) $K = \mathrm{pro}^{\mathbf{Top}} = \mathrm{pro}^{\mathbf{Discr}}$, where $\mathbf{Discr}$ is the category of discrete spaces, and in both cases (*) is fulfilled, hence also $K = \mathrm{ipro}^{\mathbf{Top}}$. Obviously, also $K = \mathrm{esp}^{\mathbf{Top}}$, which proves, in view of the Theorem, that K is strong.

(8) $\mathcal{P} = \mathbf{CTop}$. By Remark (1) and (*), which holds now, $\mathrm{pro}^{\mathcal{P}} = \mathrm{ipro}^{\mathcal{P}} = {}^{\mathcal{P}}K$ and $\mathrm{esp}^{\mathcal{P}} = \mathrm{iesp}^{\mathcal{P}}$. Since the closure of a connected set is connected, we have $\mathrm{esp}^{\mathcal{P}} = \mathrm{pro}^{\mathcal{P}}$ as well, so that all four closure operators coincide and are obviously idempotent. $\mathbf{Haus}(\mathbf{CTop})$ is the category of spaces where each connected subspace is Hausdorff (in particular, $\mathbf{Haus}(\mathbf{CTop})$ contains all totally disconnected spaces).

Obviously, **Haus(CTop)** $\subseteq \mathcal{H}_\omega$ and **Haus(CTop)** is not comparable with **US**. A map $f : X \to Y$ in **Haus(CTop)** is an epimorphism if and only if for each $y \in Y$ there exists a closed subset $C \subseteq Y$, such that $y \in \overline{C \cap f(X)}$.

8.5 Cowellpowered subcategories of Top

Here we add to the Examples 8.3 some new examples of cowellpowered subcategories of **Top**. We begin with the following fact which one can easily isolate from the proof of Theorem 8.3.

Let C be an additive closure operator of **Top**. *Then $T_2(C)$ is $\mathcal{E}^C$-cowellpowered.*

We note that this statement cannot be extended to Delta-subcategories. In fact, as we show in the next section, there are non-cowellpowered strongly epireflective subcategories $\mathcal{A}$ of **Top** with additive $\mathcal{A}$-regular closure (so that $\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}})$, and now $\mathcal{E}^{\mathrm{reg}^{\mathcal{A}}}$-cowellpoweredness means simply cowellpoweredness). Nevertheless, we have the following easy lemma which can be proved by just mimicking the proof of Theorem 8.3. We say that a closure operator C is *bounded on* $\mathcal{A}$ if there exists an ordinal α such that c^α_X is idempotent for every $X \in \mathcal{A}$.

LEMMA *Let $\mathcal{A}$ be an epireflective subcategory of* **Top** *with* $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = C^\infty|_{\mathcal{A}}$ *for some closure operator C which is bounded on $\mathcal{A}$. Then $\mathcal{A}$ is cowellpowered if and only if $\mathcal{A}$ is $\mathcal{E}^C$-cowellpowered.* □

PROPOSITION *Let C be an additive and essentially strong closure operator of* **Top**.

(a) *If C is bounded on $T_2(C)$, then $T_2(C)$ is cowellpowered.*

(b) *If C is bounded on $\Delta(C)$, then $\Delta(C)$ is cowellpowered if and only if $\Delta(C)$ is $\mathcal{E}^C$-cowellpowered.*

Proof In both cases essential strength of C yields essential equivalence of the respective regular closure and C^∞, so that the Lemma applies. □

Item (b) of the Proposition rephrases Theorem 8.3 (b). Note that it trivially holds when C is regular. Hence regularity is one of the conditions that solves the first of the following

PROBLEMS

(1) *Find conditions under which $\Delta(C)$ is $\mathcal{E}^C$-cowellpowered, for an additive closure operator C.*

(2) *Does there exist an additive closure operator C of* **Top** *such that $T_2(C)$ is cowellpowered, but C is not bounded on $T_2(C)$?*

(3) *Is $\tilde{\sigma}$ bounded on* **US** *?*

(4) *Is* **US** *$\tilde{\sigma}$-cowellpowered?*

(5) *Is* **US** *cowellpowered?*

(6) *When does the inclusion* **Haus**$(\mathcal{P}) \hookrightarrow \Delta({}^{\mathcal{P}}K)$ *preserve epimorphisms?*

(7) *If $\mathcal{A}$ is a cowellpowered subcategory of* **Top**, *is then the subcategory $S(\mathcal{A} \cup \{X\})$ cowellpowered for every space X?*

(8) *If $\mathcal{A}$ and $\mathcal{B}$ are cowellpowered subcategory of* **Top**, *is then the subcategory $S(\mathcal{A} \cup \mathcal{B})$ cowellpowered?*

Note that for $\mathcal{P} = \{\mathbb{N}_\infty\}$ the inclusion from (6) preserves epimorphisms (see Exercise 8.Q). In connection with (7) we recall that a subcategory of **Top** cogenerated by a single space X is always cowellpowered (cf. Proposition 8.2).

Returning to Problem (1), other than regularity, also $\Delta(C) = T_2(C)$ would be a sufficient condition for $\Delta(C)$ to be $\mathcal{E}^C$-cowellpowered. However, it is not a necessary condition as the case $C = \sigma$ shows; see Examples 7.9 (2)). Another important instance when Problem (1) has a positive solution is given by:

THEOREM **Haus**$(\mathcal{P})$ *is $\mathcal{E}^{\mathrm{esp}^{\mathcal{P}}}$-cowellpowered.*

Proof It suffices to prove, that for $M \subseteq X \in$ **Haus**$(\mathcal{P})$,

$$|\mathrm{esp}_X^{\mathcal{P}}(M)| \leq 2^{2^{|M|}}. \tag{*}$$

In fact, $X \in$ **Haus**$(\mathcal{P})$ implies that for each map $h : P \to X$, with $P \in \mathcal{P}$, $\overline{h(P)}$ is a Hausdorff subspace of X. Now $M_1 = M \cap \overline{h(P)}$ is a subset of this Hausdorff space, hence $|\overline{M_1}| \leq 2^{2^{|M_1|}} \leq 2^{2^{|M|}}$. Note that $\overline{M_1}$ is a typical member of the union defining $\mathrm{esp}_X^{\mathcal{P}}(M)$. Since there are at most $2^{|M|}$ such members, (*) is now obvious. □

REMARK The reader should observe that for a weakly hereditary unbounded closure operator C there exist a proper class of embeddings $m_\alpha : X_\alpha \hookrightarrow Y_\alpha$, such that for every ordinal α, m_α is $C^{\alpha+1}$-dense but not C^α-dense. In particular, every m_α is C^∞-dense but the domains X_α are distinct (unlike the case of testing $\mathcal{E}^{C^\infty}$-cowellpoweredness). In other words, for a weakly hereditary closure operator C, $\mathcal{E}^{C^\infty}$-cowellpoweredness should be considered a weak form of boundedness of C.

COROLLARY **Haus**$(\mathcal{P})$ *is cowellpowered whenever* $\mathrm{esp}^{\mathcal{P}}$ *is bounded on* **Haus**$(\mathcal{P})$.

Proof Apply the Theorem and the Proposition. □

Now we provide examples when $\mathrm{esp}^{\mathcal{P}}$ is bounded.

EXAMPLES

(1) If there exists a cardinal γ such that all spaces of $\mathcal{P}$ have a dense subset of cardinality at most γ, then the order of $\mathrm{esp}^{\mathcal{P}}$ is bounded (by $(2^{2^\gamma})^+$) on **Haus**($\mathcal{P}$). In fact, if $M \subseteq X \in \mathbf{Haus}(\mathcal{P})$ and α is an ordinal of cardinality at least $(2^{2^\gamma})^+$, then for $M_1 = \overline{(\mathrm{esp}^{\mathcal{P}})^{\alpha}_X(M)}$ and $x \in \mathrm{esp}^{\mathcal{P}}_X(M_1)$ there exists $h : P \to X$, with $P \in \mathcal{P}$ and $x \in \overline{h(P)} \cap M_1$. Since $\overline{h(P)}$ is Hausdorff (by $X \in \mathbf{Haus}(\mathcal{P})$) and has a dense subset of cardinality $\leq \gamma$, we have $|\overline{h(P)}| \leq 2^{2^\gamma}$. Hence $\overline{h(P)} \subseteq M_1$, since M_1 is a union of an α-chain with $\alpha > |\overline{h(P)}|$. This gives $x \in M_1$. Thus M_1 is $\mathrm{esp}^{\mathcal{P}}$-closed.

(2) Clearly the condition from (1) is satisfied when $\mathcal{P}$ is a set. We are particularly interested in the following cases when $\mathcal{P}$ consists of a single space: a) $\mathcal{P} = \{\mathbb{N}_\infty\}$, b) $\mathcal{P} = \{\kappa\}$ (see Examples 8.4). This immediately yields that the categories **SUS** and **Haus**(κ) are cowellpowered.

8.6 Non-cowellpowered subcategories of Top

The non-cowellpowered subcategories of **Top** considered in this section are divided in two groups, depending on their position with respect to the subcategory **Haus**, which plays a pivotal role in **Top**. We begin first with "large" subcategories of **Top**, i.e., subcategories containing **Haus**. They will be of the form **Haus**($\mathcal{P}$) for some proper class $\mathcal{P}$ of topological spaces, since otherwise **Haus**($\mathcal{P}$) is cowellpowered according to the sufficient conditions given in Corollary 8.5 and Example 8.5. Now we give sufficient conditions for non-cowellpoweredness of **Haus**($\mathcal{P}$).

Non-cowellpoweredness of **Haus**($\mathcal{P}$)

The following generalization of compactness will be used in this part of the section.

DEFINITION

(1) A subset B of a topological space X is said to be *bounded* (*w.r.t.* X) if every open cover of X admits a finite subfamily covering B.

(2) A topological space X is *e-compact* if X admits a dense bounded subset.

Clearly, compact subsets are bounded and compact spaces are e-compact. For further properties of bounded sets and e-compact spaces, see Exercise 8.O.

For an ordinal α we denote by α also the space of all ordinals less than α equipped with the order topology. Then the space $\alpha + 1$ is compact for each α.

Now we prove the following general result:

THEOREM *The category* **Haus**($\mathcal{P}$) *is non-cowellpowered if the class* $\mathcal{P}$ *satisfies the following conditions:*

(1) *for a proper class of ordinals* α, *the class* $\mathcal{P}$ *contains the space* $\alpha + 1$;

(2) *every space in $\mathcal{P}$ is e-compact.*

Proof For each ordinal κ we shall provide the set $(\kappa+1)\times\omega$ with a topology such that the resulting space X_κ satisfies:

A) X_α is an open subset of X_κ whenever $\alpha < \kappa$;

B) $X_\kappa \in \mathbf{Haus}(\mathcal{P})$;

C) the embedding $X_0 \hookrightarrow X_\kappa$ is an epimorphism in $\mathbf{Haus}(\mathcal{P})$.

Obviously, B) and C) will imply that $\mathbf{Haus}(\mathcal{P})$ is non-cowellpowered.

Let $\{N_n\}$ be a partition of ω with N_n infinite for every $n \in \omega$. The topology on X_κ will be defined by transfinite induction on κ such that:

- Each point $(0,n)$ is isolated.
- Assume that a neighbourhood base of all points (γ,n) with $\gamma < \beta$ is already defined such that A) holds.

  i) If $\beta = \gamma+1$, then a neighborhood base of (β,n) is composed of the sets $\{(\beta,n)\} \cup \bigcup\{V_x : x \in \{\gamma\}\times N_n \setminus F\}$ for finite sets $F \subseteq N_n$, and for neighbourhoods V_x of x in X_γ.

  ii) A neighborhood base of (β,n), for β limit, is composed of the sets $\{(\beta,n)\}\cup\bigcup\{V_\delta : \gamma < \delta < \beta\}$ for $\delta < \beta$, and for neighbourhoods V_δ of (δ,n).

Clearly A) holds. To verify B) and C) we need to prove the following:

Claim: *For each $\beta \le \kappa$, let G_β be a subset of ω such that $G_\beta \cap N_n$ is finite for each n. Then $G = \bigcup_{\beta\le\kappa}\{\beta\}\times G_\beta$ is closed in X_κ whenever one of the following conditions hold:*

(a) *all G_β coincide;*

(b) *the sets G_β are pairwise disjoint.*

Moreover, G has the product topology in case (a), *while in case* (b) *G is discrete.*

Proof of the Claim Take $z = (\beta,n) \in X_\kappa \setminus G$, $\beta \le \kappa$. To find a neighbourhood U of z which misses G we proceed by transfinite induction on β.

If $z \in X_0$, then z is isolated and we are through. Assume all $z = (\gamma,n)$, with $\gamma < \beta$, have a neighbourhood in X_κ disjoint from S. Consider first the case when β is a limit ordinal. Now $(\gamma,n) \in G$ may occur for at most one $\gamma < \beta$. Choose a $\gamma_0 < \beta$ such that $(\gamma,n) \notin G$ for all γ with $\gamma_0 < \gamma < \beta$. Now by the induction hypothesis each $x_\gamma = (\gamma,n)$ has a neighbourhood V_γ in X_κ disjoint from G. Then $\{z\}\cup\bigcup\{V_\gamma : \gamma_0 < \gamma < \beta\}$ is a neighbourhood of z which misses G. In case $\beta = \gamma+1$ is a successor ordinal, the subset $F = G_\gamma \cap N_n$ of N_n is finite by hypothesis. Now each point $x_k \in \{\gamma\}\times(N_n\setminus F)$ has a neighbourhood V_k which misses G by induction hypothesis. Now $V = \{z\}\cup\bigcup_k V_k$ works as before.

The discreteness of G in case (b) follows easily from the definition of the topology of X_κ. In the case (a) let all G_β coincide with some $G_0 \subseteq \omega$. To see that $G = \kappa\times G_0$ has the product topology fix $n \in G_0$ and note that for $G_1 = G_0 \setminus \{n\}$ the set $G = \kappa \times G_1$ is closed in X_κ by the first part of our claim. Therefore, the set

$C_{\kappa,n} = \kappa \times \{n\}$ is clopen in G. It remains to observe that $C_{\kappa,n}$ is homeomorphic to the compact ordinal space $\kappa + 1$. This proves the Claim.

Call *diagonal set* in X_κ every subset G of X_κ satisfying (b) of the above Claim. By the Claim, every diagonal set is closed and discrete. Hence, by Exercise 8.O, every bounded set of X_κ can contain only finite diagonal sets. Consequently a bounded subset B of X_κ is either contained in a product $(\kappa + 1) \times F$, where F is a finite subset of ω, or contained in a product $K \times \omega$, where K is a finite subset of $\kappa + 1$. In the first case $(\kappa + 1) \times F$ has the product topology according to the Claim, so that $(\kappa+1) \times F$, and consequently B, is Hausdorff. In the second case, we note that finiteness of the diagonal subsets of B implies that for each $\beta \in K$, B meets only finitely many sets $\{\beta\} \times N_n$. Therefore B is contained in a subspace of X_κ which is homeomorphic to a finite coproduct of converging sequences, hence Hausdorff. In this way we have proved that every bounded subset of X_κ is Hausdorff.

Now consider a map $h : P \to X$ with $P \in \mathcal{P}$ and $a, b \in \overline{h(P)}$. By our hypothesis (2), P is e-compact. Hence there exists a bounded dense subset B of P. Now $j(B)$ will be a dense bounded subset of $j(P)$ by Exercise 8.O. Since $B' = j(B) \cup \{a, b\}$ is still bounded (see Exercise 8.O), our previous argument yields that B' is Hausdorff. Thus a and b can be separated by disjoint open neighbourhoods in B'. It remains to note that $B' \cap \overline{h(P)}$ is dense in $\overline{h(P)}$, so that a and b can be separated in $\overline{h(P)}$ as well. This finishes the proof of $X_\kappa \in \mathbf{Haus}(\mathcal{P})$.

To check C) denote by $\mathcal{Q}$ the class of all spaces κ, with $\kappa \in Ord$. Since for $\kappa' \leq \kappa$ one can find easily a continuous surjection $\kappa \to \kappa'$, Remarks 8.4 (4) and our hypothesis (1) imply that $\mathrm{pro}^{\mathcal{Q}} \leq \mathrm{pro}^{\mathcal{P}}$. Hence, according to Theorem 8.4 (3) and since $\mathrm{pro}^{\mathcal{Q}}_{X_\kappa} = \mathrm{esp}^{\mathcal{Q}}_{X_\kappa}$, it suffices to prove that for $\alpha \leq \kappa$,

$$X_\alpha \subseteq (\mathrm{pro}^{\mathcal{Q}})^{\alpha+1}_{X_\kappa}(X_0). \qquad (*)$$

But the proof of (*) is a standard application of transfinite induction in view of the homeomorphism $C_{\kappa,n} \cong (\kappa + 1)$. □

COROLLARY *For the following classes $\mathcal{P}$ of topological spaces the category* $\mathbf{Haus}(\mathcal{P})$ *is not cowellpowered:*

(1) *compact spaces;*

(2) *compact Hausdorff spaces;*

(3) *e-compact spaces.*

□

There are other classes $\mathcal{P}$ of topological spaces which satisfy the hypotheses of the Theorem, such as compact totally disconnected Hausdorff spaces. However, since each compact Hausdorff space is a continuous image of such a space, this new class gives no impact on $\mathbf{Haus}(\mathcal{P})$, according to Remark 8.4(4) (while (1)-(3) give different categories $\mathbf{Haus}(\mathcal{P})$, see Exercise 8.Q).

Non-cowellpoweredness of **Ury** *and of other subcategories of* **Haus**

Now we define in an appropriate way a generalization of the Urysohn separation axiom (distinct points can be separated by disjoint closed neighbourhoods). To this end we define the notion of $S(n)$-neighbourhood in **Top**.

DEFINITION*

(1) *Let n be a natural number and let $k = n/2$ in case n is even and $k = (n+1)/2$ otherwise. If X is a topological space, $U \subset X$ and $x \in X$, then we say that U is a* $S(n)$-neighborhood *of x if there is a family of open sets $\{U_s : 1 \leq s \leq k\}$ such that*

- $x \in U_1$
- $1 \leq s < k \Rightarrow \overline{U}_s \subset U_{s+1}$
- $U_k \subseteq U$ *if n is odd, otherwise* $\overline{U_k} \subseteq U$.

(2) *A topological space is an* $S(n)$-space *if any pair of distinct points can be separated by disjoint closed $S(n)$-neighbourhoods.*

In the sequel we denote by $S(n)$ the full subcategory of **Top** having as objects all $S(n)$-spaces. Clearly, $S(1) =$ **Haus** and $S(2) =$ **Ury**.

REMARKS

(1) Every zero-dimensional space is an $S(n)$-space for each $n \in \mathbb{N}$, since any pair of distinct points can be separated by disjoint clopen neighbourhoods.

(2) For every map $f : X \to Y$ in **Top** and $x \in X$, the inverse image of an $S(n)$-neighbourhood U of $f(x)$ in Y is an $S(n)$-neighbourhood of x in X (see Exercise 8.L (b)). This permits to define $S(n)$-closure, which turns out to be a closure operator due to this property (a point $x \in X$ is in the $S(n)$-closure of a subspace $M \subseteq X$ if and only if each $S(n)$-neighbourhood of x meets M). Denote by θ_k the $S(n)$-closure in case $n = 2k$. Then clearly $\theta_1 = \theta$ and $T_2(\theta_k) = S(2k)$. Although θ_k is essentially strong (see Exercise 8.L(e)), Corollary 8.5 cannot be applied to get cowellpoweredness of $S(2k)$ since θ_k is unbounded (this is witnessed by the spaces X_κ constructed in the proof of Theorem* below).

(3) For infinite ordinals α one can define the notion of $S(\alpha)$-neighborhood by taking *decreasing* ordinal chains of size α of neighbourhoods U_γ ($\gamma < \alpha$) as before; or the notion of $S(\alpha^*)$-neighbourhood by taking *increasing* chains. In both cases no distinction is needed as in the third item of Definition* (1) when α is a limit ordinal. Now define $S(\alpha)$-spaces and $S(\alpha^*)$-spaces in a similar way.

THEOREM* *Every subcategory $\mathcal{A}$ of* **Top** *satisfying $\bigcap_n S(n) \subseteq \mathcal{A} \subseteq$* **Ury** *is non-cowellpowered.*

Proof For each ordinal κ we shall provide the set $(\kappa + 1) \times \mathbb{Q}$ with a topology such that some appropriate subspaces X_κ of the resulting spaces Z_α will satisfy:

A) $X_\kappa \in S(n)$ for each $n \in \mathbb{N}$ (hence $X_\kappa \in \mathcal{A}$);

B) the embedding $X_0 \hookrightarrow X_\kappa$ is θ^∞-dense, hence an epimorphism in **Ury** (and, a fortiori, an epimorphism in $\mathcal{A}$).

Clearly, A) and B) will imply that $\mathcal{A}$ is non-cowellpowered.

To define the topology of Z_α we need to modify the topology of the ordinal space $\kappa + 1$, as follows. A non-zero ordinal β is said to be *odd*, if β can be presented as $\beta = \beta_0 + n$, where β_0 is a limit ordinal or zero, and n is an odd natural number. Non-odd ordinals will be called *even*. The new topology τ of the ordinal space $\kappa+1$ leaves unchanged the basic neighbourhoods of the even ordinals in κ, while every odd ordinal $\beta \leq \kappa$ has as least τ-neighbourhood the set $\{\beta - 1, \beta, \beta + 1\} \cap \kappa$. This topology is not even T_1, but *distant ordinals* α and β (i.e. such that either $\alpha+\omega \leq \beta$ or $\beta + \omega \leq \alpha$) have disjoint $S(n)$-neighbourhoods. We leave the easy check of this fact to the reader.

Now let $Z_\kappa = (\kappa + 1) \times \mathbb{Q}$ be the product space, where $\mathbb{Q}$ is equipped with the usual topology, and $\kappa + 1$ is equipped with the topology τ. Let $p : Z_\kappa \to \mathbb{Q}$ and $q : Z_\kappa \to \kappa + 1$ be the canonical projections. In order to define the appropriate subspace X_α of Z_α fix a partition $\mathbb{Q} = \bigcup_{n=0}^\infty \mathbb{Q}_n$ into disjoint dense subsets. For an arbitrary ordinal $\beta = \beta_0 + n$, with β_0 limit or zero, set $\mathbb{Q}_\beta := \mathbb{Q}_n$. Now let

$$X_\kappa := \bigcup_{\beta\leq\kappa} \{\beta\} \times \mathbb{Q}_\beta \subseteq Z_\kappa$$

be equipped with the topology induced by Z_α. Making use of both projections p and q, we shall show that $X_\kappa \in S(n)$ for each $n \in \mathbb{N}$. Fix $z, z' \in X_\kappa$. In case $p(z) \neq p(z')$ in $\mathbb{Q}$, we can $S(n)$-separate $p(z)$ and $p(z')$ in $\mathbb{Q}$ (which is zero-dimensional, see Remark (1)) and then take inverse images along p to get disjoint $S(n)$-neighbourhoods of z and z' in X_κ. In case $p(z) = p(z')$, it follows by the definition of X_κ and $\mathbb{Q}_\beta$, that $\beta = q(z)$ and $\beta' = q(z')$ are distant. Hence we can $S(n)$-separate β and β' in the ordinal space $(\kappa + 1, \tau)$ and then, as before, take inverse images under q of the disjoint $S(n)$-neighbourhoods in $(\kappa + 1, \tau)$ to get disjoint $S(n)$-neighbourhoods of z and z' in X_κ. This finishes the verification of A).

To prove B) and finish the proof of the Theorem, we need the following:

Claim: *For each $\beta + 1 \leq \kappa$, $X_{\beta+1} \subseteq \theta_{X_\kappa}(X_\beta)$. Consequently, $X_\kappa = \theta^\infty_{X_\kappa}(X_0)$.*

Let $z = (\beta + 1, x) \in X_{\beta+1}$. If β is either a limit ordinal or odd, then each neighbourhood of β in the space (κ, τ) hits the subspace β. Hence, taking into account also the density of each $\mathbb{Q}_\beta$ in $\mathbb{Q}$, we conclude that actually $z \in k_{X_\kappa}(X_\beta)$ since every neighbourhood of z in X_κ hits X_β. Assume now that β is an even non-limit ordinal. Then a basic open neighbourhood of z in X_κ has the form $W = \{\beta + 1\} \times (V \cap \mathbb{Q}_{\beta+1})$, where V is an open neighbourhood of x in $\mathbb{Q}$. By the density of $\mathbb{Q}_{\beta+1}$ in $\mathbb{Q}$, there exists $r \in V \cap \mathbb{Q}_\beta$. Let us see that $(\beta, r) \in X_\beta \cap \overline{W}$. In fact, a basic open neighbourhood of (β, r) in X_κ always contains $\{\beta + 1\} \times (O \cap \mathbb{Q}_{\beta+1})$, where O is an open neighbourhood of r in $\mathbb{Q}$. (By the density of $\mathbb{Q}_{\beta+1}$ in $\mathbb{Q}$, always $O \cap \mathbb{Q}_{\beta+1} \neq \emptyset$.) Thus $\overline{W} \cap X_\beta \neq \emptyset$ is proved. This finishes the first part of the proof. The second part of the proof follows from the first one by transfinite induction. □

A careful analysis of the proof of the Claim shows that X_0 is actually C^∞-dense in X_κ, where C is defined as follows. For a space X and $M \subseteq X$ a point $x \in X$ is in the closure $c_X(M)$ if for every open neighbourhood U of x in X, $\sigma_X(U)$ meets M. Denote by **sUry** the full subcategory of **Top** having as objects those spaces X in which distinct points x and y have neighbourhoods U and V respectively, such that $\sigma_X(U) \cap \sigma_X(V) = \emptyset$. Clearly, $\mathbf{Ury} \subseteq \mathbf{sUry} = T_2(C)$. Our observation shows that the hypothesis of the Theorem can be weakened to

$$\bigcap_n S(n) \subseteq \mathcal{A} \subseteq \mathbf{sUry}.$$

In the sequel we consider another generalization of the Urysohn separation axiom. It depends on a sequence $\{h_n\}$ of closure operators defined for each $n \geq 0$, as follows: h_0 is the discrete closure operator, and for a topological space X and $M \subseteq X$ define

$$(h_{n+1})_X(M) = \{x \in X : \text{ for each neighbourhood } U \text{ of } x, h_n(U) \cap h_n(M) \neq \emptyset\}.$$

Now define h'_n by

$$(h'_n)_X(M) = \{x \in X : \text{ for each neighbourhood } U \text{ of } x, h_n(U) \cap M \neq \emptyset\}.$$

The reader can easily check that h_n and h'_n are closure operators for each $n \geq 0$, with $h_1 = K$ and $h'_1 = \theta$. A space X is an S_n-space if for each pair x and y of distinct points of X, there exist open neighbourhoods U and V, of x and y respectively, such that $h_n(U) \cap h_n(V) = \emptyset$. Denote by S_n the full subcategory of **Top** of S_n-spaces. Then one obtains:

COROLLARY* *The categories S_n $(n \in \mathbb{N}, n \geq 1)$ are non-cowellpowered.*

Proof It is easy to see that $S_n \subseteq S_1 = \mathbf{Ury}$ for each $n \geq 1$. Moreover, $S(2^n) \subseteq S_n$ (see Exercise 8.N). Now we can apply Theorem*. □

The categories $S(\alpha)$ $(\alpha > 1)$ as defined in Remark (3) represent a *proper class* of non-cowellpowered subcategories of **Top**. Actually, one can define $S(\eta)$-separation depending on an arbitrary order type η and characterize those order types η for which the category $S(\eta)$ is cowellpowered (see Dikranjan and Watson [1994]; some particular cases are given in Exercise 8.M).

8.7 Quasi-uniform spaces

A *quasi-uniform space* is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a filter of reflexive relations on X such that for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ with $V \circ V \subseteq U$. A *uniformly continuous map* $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a set-map such that for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $(f \times f)(U) \subseteq V$. This defines the category **QUnif** of quasi-uniform spaces which contains the category **Unif** of uniform spaces (see 5.11) as a full subcategory. The forgetful functor $G : \mathbf{QUnif} \to \mathbf{Set}$ is topological; in particular, the quasi-uniformities on a set form a complete lattice (see Exercise 8.R).

For a quasi-uniform space $(X,\mathcal{U})$ one can define a topology $T(\mathcal{U})$ on X by taking as a base of neighbourhoods at $x \in X$ the filter base $\{U(x)\}_{U \in \mathcal{U}}$, where $U(x) = \{y \in X : (x,y) \in U\}$. Every uniformly continuous map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ gives a continuous map with respect to the induced topologies. Hence one has a concrete functor T : **QUnif** $\to$ **Top** which composed with the forgetful functor V : **Top** $\to$ **Set** (see diagram (5.31)) gives $G = V \circ T$.

The functor T admits many sections, i.e., functorially chosen quasi-uniformities on the underlying set X of a topological space (X,τ) inducing the given topology τ, of which the following two are the most relevant ones. The *finest quasi-uniformity* $F(X,\tau)$ is defined as the finest quasi-uniformity $\mathcal{U}$ on X with $T(\mathcal{U}) = \tau$ (see Exercise 8.R for its existence). The *Pervin quasi-uniformity* $P(X,\tau)$ of a topological space (X,τ) is generated by the filter base of binary relations $\{S_G\}$, with $S_G := (G \times G) \cup ((X \setminus G) \times X)$ and G running through the family of all open subsets of X. In this way one obtains two functors F,P : **Top** $\to$ **QUnif** of which F is left adjoint to T. Then, in the notation of 5.13, the counit $\varepsilon_X : FTX \to X$ is the identity map, where FTX is the set X equipped with the finest quasi-uniformity of the topological space TX.

Consider the natural involutive endofunctor ι : **QUnif** $\to$ **QUnif** defined by $(X,\mathcal{U}) \overset{\iota}{\mapsto} (X,\mathcal{U}^{-1})$, where $\mathcal{U}^{-1}$ denotes the filter $\{U^{-1} : U \in \mathcal{U}\}$. The composite $T^\iota := T \circ \iota$: **QUnif** $\to$ **Top** has $F^\iota := \iota \circ F$ as a left adjoint with counit $\varepsilon^\iota_X := \iota(\varepsilon_{\iota X}) : F^\iota T^\iota X \to X$. The category **Unif** of uniform spaces is simply the full subcategory of **QUnif** of all quasi-uniform spaces left fixed by ι. Actually, **Unif** is coreflective in **QUnif**, with the coreflector defined by $(X,\mathcal{U}) \mapsto (X,\mathcal{U}^*)$, with $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$ and the join taken in the lattice of all quasi-uniformities on X. The restriction of the functor T : **QUnif** $\to$ **Top** to **Unif** coincides with the forgetful functor W : **Unif** $\to$ **Top** defined in 5.11. The composition of the coreflector $*$: **QUnif** $\to$ **Unif** with W is usually denoted by T^* : **QUnif** $\to$ **Top**. Hence one has a commutative diagram:

$$\begin{tikzcd} \mathbf{QUnif} \arrow[rr, "*"] \arrow[dr, "T^*"'] & & \mathbf{Unif} \arrow[dl, "W"] \\ & \mathbf{Top} & \end{tikzcd} \tag{8.2}$$

With respect to the class $\mathcal{M}$ of all embeddings, **QUnif** is $\mathcal{M}$-complete. As described in 5.13, with the help of the adjunctions $F \dashv T$, $F^\iota \dashv T^\iota$, Incl $\dashv^*$ one can "push" closure operators of **Top** and **Unif** along the respective counits to obtain closure operators of **QUnif** w.r.t. $\mathcal{M}$. In particular, for the Kuratowski operator K of **Top** and its initial lifting to **Unif** (see 5.11), we obtain the closure operators

$$\gamma := {}^{\varepsilon^\iota}K \quad \text{and} \quad \beta := {}^{\varepsilon^*}K,$$

where ε^* is the coreflexion of the coreflector $*$: **QUnif** $\to$ **Unif**. Actually, $\beta = (\mathcal{K} \wedge \gamma)^+$, so that β becomes the quasi-uniform counterpart of the $\mathfrak{b}$-closure. But it should be mentioned that the restrictions of the closure operators γ, β and $\mathcal{K}$ to

Unif coincide ; in particular, $\beta|_{\mathbf{Unif}}$ coincides with the Kuratowski closure lifted from **Top** to **Unif**. Furthermore, $\beta|_{\mathbf{Unif}}$ is regular, while neither ${}^{\varepsilon}K$ nor γ are regular closure operators of **QUnif** (see Examples below).

Now we define a series of closure operators of **QUnif** which fully exploit the specific features of **QUnif**. Let $n \in \mathbb{N}$ and $\mathcal{U}$ be a quasi-uniformity in X. For $V \in \mathcal{U}$ consider the relation $w_n(V) = V_n \circ \ldots \circ \ldots V_2 \circ V_1$, on X, where $V_1 = V, V_2 = V^{-1}, \ldots, V_{k+1} = V^{(-1)^k} \ldots$. Then $\{w_n(V)(x) : V \in \mathcal{U}, n \in \mathbb{N}\}_{x \in X}$ is a neighbourhood system of a pretopological space $\overline{\boldsymbol{\theta}}_n(X)$ satisfying the condition from Exercise 8.L(b)(i). Hence $\overline{\boldsymbol{\theta}}_n : \mathbf{QUnif} \to \mathbf{PrTop}$ is a concrete functor, which defines an additive closure operator $\overline{\theta}_n$ for every $n \in \mathbb{N}$. Obviously, $\overline{\theta}_1 = {}^{\varepsilon}K$. More generally, a $\overline{\theta}_n$-neighbourhood in $X \in \mathbf{QUnif}$ is also an $S(n)$-neighbourhood in TX (cf. Definition* (1) of 8.6). If the Pervin quasi-uniformity of TX is contained in the quasi-uniformity of X, then both neighbourhood systems coincide on TX. Hence, with $\overline{S}(n) = T_2(\overline{\theta}_n)$ and T_2 defined as in 7.11, one has $T(\overline{S}(n)) = S(n)$ (see also Exercise 8.R(e)).

EXAMPLES

(1) Making use of the definitions and Exercise 8.R(b) one can show that $\bigcap\{(V \cap V^{-1}) : V \in \mathcal{U}\} = \Delta_X$ holds for $X \in \mathbf{QUnif}$ iff $X \in T^{-1}(\mathbf{Top}_0) = (T^{\iota})^{-1}(\mathbf{Top}_0) = \Delta(\beta)$. We denote by $\mathbf{QUnif}_0$ this subcategory of **QUnif**. The reflector $\mathbf{QUnif} \to \mathbf{QUnif}_0$ is defined as follows: for $(X, \mathcal{U}) \in \mathbf{QUnif}$ set $L_X := \bigcap\{(V \cap V^{-1}) : V \in \mathcal{U}\}$. Clearly L_X is an equivalence relation on X, such that the quotient map $\rho : X \to RX = (X/L_X, \rho(\mathcal{U}))$ is the $\mathbf{QUnif}_0$-reflexion of X. Note that ρ sends β-closed sets of X to β-closed sets of RX. Since $\mathbf{QUnif}_0$ is the biggest proper strongly epireflective subcategory of **QUnif**, the Generating Diagonal Theorem implies that its regular closure operator $\mathrm{reg}^{\mathbf{QUnif}_0}$ is the finest non-discrete regular closure operator of **QUnif**. We claim $\beta = \mathrm{reg}^{\mathbf{QUnif}_0}$. To prove the inequality $\beta \leq \mathrm{reg}^{\mathbf{QUnif}_0}$, note that both closure operators are idempotent, hence it suffices to show that, each equalizer $\mathrm{eq}(f, g)$ with $f, g : X \to Y \in \mathbf{QUnif}_0$ is β-closed. In fact, $\mathrm{eq}(f, g) = h^{-1}(\Delta_Y)$ for $h = < f, g >: X \to Y \times Y$, and the diagonal Δ_Y is β-closed in $Y \times Y$. To prove the opposite inequality, consider a β-closed subset M of X. By Exercise 8.R, $i(x)$ and $j(x)$ are separated in $X +_M X$. In case $X \in \mathbf{QUnif}_0$ the other pairs of distinct points of $X +_M X$ can be separated via the projection $X +_M X \to X$ as in Example 6.5, so that by Frolík's Lemma, M is $\mathbf{QUnif}_0$-closed. In the general case one has to observe that the $\mathbf{QUnif}_0$-reflexion $\rho : X \to RX$ sends β-closed sets to β-closed sets.

(2) Analogously, $T^{-1}(\mathbf{Top}_1) = (T^{\iota})^{-1}(\mathbf{Top}_1) = \Delta({}^{\varepsilon}(K^*))$. Let $\mathbf{QUnif}_1$ denote this category. One can show that $\bigcap\{V : V \in \mathcal{U}\} = \Delta_X$ (or, equivalently, $\bigcap\{V^{-1} : V \in \mathcal{U}\} = \Delta_X$) holds for $X \in \mathbf{QUnif}$ iff $X \in \mathbf{QUnif}_1$. Similarly to (1) one can show that the regular closure operator of $\mathbf{QUnif}_1$ in **QUnif** coincides with β on $\mathbf{QUnif}_1$. But note that, unlike the case of $\mathbf{QUnif}_0$, $\mathrm{reg}^{\mathbf{QUnif}_1}$ and β do not coincide on **QUnif**, due to the Generating Diagonal Theorem.

Since a uniform T_0-space is also Hausdorff, we have $\mathbf{Unif}_0 := \mathbf{Unif} \cap \mathbf{QUnif}_0 = \mathbf{Unif} \cap \mathbf{QUnif}_1$. Note that for $X \in \mathbf{Unif}_0$, the topological space TX is a Tychonoff space (see Exercise 8.R (d)). One can easily find examples to distinguish $\mathbf{QUnif}_0$

and $\mathbf{QUnif}_1$. To see how big the difference between $\beta = ({}^{\mathfrak{c}}K \wedge \gamma)^+$ and ${}^{\mathfrak{c}}K \wedge \gamma$ is, compare $\mathbf{QUnif}_0 = \Delta(\beta)$ with $\Delta({}^{\mathfrak{c}}K \wedge \gamma) \subset \mathbf{QUnif}_0$. (In fact, if X is the Sorgenfrey Line and $M = \mathbb{Q}$ the set of rationals, then M is β-closed, so that the amalgamation $Y = X +_M X$ belongs to $\mathbf{QUnif}_1$ by Exercise 8.R, while for each irrational point $x \in X$ the pair $(i(x), j(x))$ is in the $({}^{\mathfrak{c}}K \wedge \gamma)$-closure of the diagonal of $Y \times Y$, but not in the diagonal Δ_Y, so $Y \in \mathbf{QUnif}_1 \setminus \Delta({}^{\mathfrak{c}}K \wedge \gamma)$.)

The following theorem helps to find non-cowellpowered subcategories of **QUnif**.

THEOREM

(1) *Let $\mathcal{B}$ be a strongly epireflective subcategory of* **Top**. *Then the subcategory $\mathcal{A} = T^{-1}(\mathcal{B})$ of* **QUnif** *is strongly epireflective and* $\mathrm{reg}^{\mathcal{A}} \geq {}^{\mathfrak{c}}\mathrm{reg}^{\mathcal{B}}$. *The functor T is* $(\mathrm{reg}^{\mathcal{A}}, \mathrm{reg}^{\mathcal{B}})$*-continuous (cf. 5.7) iff* $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = {}^{\mathfrak{c}}\mathrm{reg}^{\mathcal{B}}|_{\mathcal{A}}$. *This implies that the restriction $T|_{\mathcal{A}} : \mathcal{A} \to T(\mathcal{A})$ preserves epimorphisms. If $\mathcal{B} \subseteq$* **Haus**, *then T is* $(\mathrm{reg}^{\mathcal{A}}, \mathrm{reg}^{\mathcal{B}})$*-continuous.*

(2) *Let $\mathcal{A}$ be a strongly epireflective subcategory of* **QUnif**. *Then the restriction $T|_{\mathcal{A}} : \mathcal{A} \to T(\mathcal{A})$ reflects epimorphisms. Consequently, $\mathcal{A}$ is non-cowellpowered if $T(\mathcal{A})$ is non-cowellpowered. In particular, if $\mathcal{A} = T^{-1}(T(\mathcal{A}))$ and $T(\mathcal{A}) \subseteq$* **Haus**, *then $T|_{\mathcal{A}}$ also preserves epimorphisms, so that $T(\mathcal{A})$ is cowellpowered iff $\mathcal{A}$ is cowellpowered.*

Proof (1) In order to prove $\mathrm{reg}^{\mathcal{A}} \geq {}^{\mathfrak{c}}\mathrm{reg}^{\mathcal{B}}$, it suffices to show that an $\mathcal{A}$-closed subspace $M \subseteq X \in \mathbf{QUnif}$ is $\mathcal{B}$-closed in TX. By Frolík's Lemma it suffices to see that $TX +_M TX \in \mathcal{B}$. Note that by Frolík's Lemma again $X +_M X \in \mathcal{A}$, so that $T(X +_M X) \in \mathcal{B}$. Since the canonical (bijective) map $TX +_M TX \to T(X +_M X)$ is continuous, strong epireflectivity of $\mathcal{B}$ gives $TX +_M TX \in \mathcal{B}$. The second assertion follows from this inequality and the definition of continuity of a functor (see (*) in Definition 5.7). To prove the last assertion of (1), assume $\mathcal{B} \subseteq$ **Haus** and proceed with the proof of the inequality $\mathrm{reg}^{\mathcal{A}} \leq {}^{\mathfrak{c}}\mathrm{reg}^{\mathcal{B}}$, which will imply continuity of T. In view of the idempotency of the regular closure operators it suffices to see that if $M \subseteq X \in \mathbf{QUnif}$ is $\mathcal{B}$-closed in TX, then it is also $\mathcal{A}$-closed. By Frolík's Lemma, $\mathcal{B}$-closedness of M gives $TX +_M TX \in \mathcal{B}$. Since $\mathcal{B} \subseteq$ **Haus**, $\mathcal{B}$-closedness of M yields also closedness of M in TX. Now Exercise 8.R(b) guarantees that both topologies on the amalgamation $X +_M X$ coincide, so that also $T(X +_M X) \in \mathcal{B}$. Thus $X +_M X \in \mathcal{A}$, consequently M is $\mathcal{A}$-closed by Frolík's Lemma.

(2) Follows from (1) and Theorem 8.1. □

COROLLARY *For every $n > 1$, the categories $\overline{S}(n) = T_2(\overline{\theta}_n) \subset T^{-1}(S(n))$ are non-cowellpowered, while the categories* $\mathbf{QUnif}_0, \mathbf{QUnif}_1$ *and* $\mathbf{Unif}_0$ *are cowellpowered.* □

Proof For $\mathcal{A} = \overline{S}(n)$ the category $T(\mathcal{A}) = S(n)$ is not cowellpowered according to Theorem* of 8.6. Thus by (1) of the Theorem, both $\mathcal{A}$ and $T^{-1}(S(n))$ are not cowellpowered. For the properness of the inclusion $\overline{S}(n) \subset T^{-1}(S(n))$ see Exercise

8.R(e). To prove the last assertion, note that the functor T^* : **QUnif** $\to$ **Top** sends $\mathbf{QUnif}_0$ and its subcategories $\mathbf{QUnif}_1$ and $\mathbf{Unif}_0$ into the cowellpowered category **Haus** and preserves epimorphisms. Now Theorem 8.1 applies. □

REMARKS

(1) Exercise 8.R shows that the functor T need not preserve epimorphisms if $\mathcal{B}$ is not contained in **Haus**. In fact, while the epimorphisms in $\mathbf{QUnif}_0$ are the β-dense maps, the epimorphisms in $\mathbf{Top}_0 = T(\mathbf{QUnif}_0)$ are the b-dense maps (it is easy to find a β-dense map f such that Tf is not b-dense). Another example is provided by the smaller subcategory $\mathbf{QUnif}_1$. Here epimorphisms are (again!) the β-dense maps, while the epimorphisms in $\mathbf{Top}_1 = T(\mathbf{QUnif}_1)$ are *surjective* (see Exercise 8.R(c)).

(2) It follows from the Corollary, Exercise 8.R(f) and from the obvious counterpart of Theorem 8.3 for **QUnif**, that $\overline{\theta}_n$ is unbounded for $n > 1$ (compare with Problems 8.5 (2) in the case of **Top**).

(3) As the examples above show the deficiencies of the amalgamation in **QUnif** (see Exercise 8.R) prevent an easy "lifting" of the known results in **Top** along the functor T : **QUnif** $\to$ **Top**. This fact makes it necessary to study problems of cowellpoweredness in **QUnif** separately.

(4) The following fact shows that in the Theorem one cannot replace T by the functor $B = T \times T^\iota$: **QUnif** $\to$ **2Top**. Let $\mathcal{A}$ be the full subcategory of **2Top** of *pairwise-*T_2 spaces, i.e., spaces (X, τ_1, τ_2) such that for each pair x, y of distinct points of X there exist disjoint τ_i-open sets O_i $(i = 1, 2)$ such that either $x \in O_1$ and $y \in O_2$, or $x \in O_2$ and $y \in O_1$. Hence $\mathbf{QUnif}_0 = B^{-1}(\mathcal{A})$, and this category is cowellpowered by the Corollary. Recently Schröder [1995] proved that $\mathcal{A}$ is non-cowellpowered.

PROBLEM *Do there exist non-cowellpowered full subcategories of* **Unif** *?*

8.8 Topological groups

It is easy to see that for the categories of Hausdorff abelian groups or Hausdorff topological modules over a ring R, the epimorphisms coincide with maps with dense range (see Exercise 8.R). As mentioned in Example 6.1, this remains true also for the category of compact groups. Now consider the category **HausGrp** of all Hausdorff topological groups and continuous homomorphisms. Every morphism with dense range is an epimorphism in **HausGrp**, since the forgetful functor **HausGrp** $\to$ **Haus** reflects the epimorphisms. It was an open problem formulated already in the late sixties whether the converse proposition is true. This problem is equivalent to the following problem of K. H. Hofmann (see Comfort [1990], Problem 512).

If G is a Hausdorff topological group and H a proper closed subgroup of G, can H be $\mathrm{reg}^{\mathbf{HausGrp}}$*-dense ?*

In other words, does $\mathrm{epi}^{\mathbf{HausGrp}}$ coincide with the Kuratowski closure operator K ? It was proved by Nummela [1978] that $\mathrm{epi}_G^{\mathbf{HausGrp}}$ coincides with k_G for locally compact and other classes of topological groups G. It was shown recently by Uspenskij [1994] that, in general, the answer to this question is negative, by means of the following example. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group equipped with the Euclidean metric ρ, and let G be the group of all self-homeomorphisms of $\mathbb{T}$, equipped with the topology of uniform convergence. The filter $\mathcal{N}(G)$ of neighbourhoods of the neutral element e in G has as basic neighbourhoods the sets

$$\{g \in G : (\forall x \in \mathbb{T})\ \rho(g(x), x) < n^{-1}\}$$

for $n \in \mathbb{N}$. Denote by $H = \{g \in G : g(1) = 1\}$ the stabilizer of the point $1 \in \mathbb{T}$. Then H is a proper closed subgroup of G since G is Hausdorff. We can now prove:

THEOREM *The inclusion $H \to G$ is an epimorphism in* **HausGrp**.

Sketch of Proof. Consider an arbitrary pair of morphisms $f, g : G \to K$ to a Hausdorff group K with $f|H = g|H$. In order to prove $f = g$, define $j : G \times G \to K$ by $j(x, y) = f(x)g(y)^{-1}$. Let $\mathcal{V}$ be the *lower uniformity* on K. This uniformity has a base $\{(x, y) \in K^2 : x \in VyV\}$, where V runs over $\mathcal{N}(K)$, and is compatible with the topology of K. The main steps of the proof are:

1. $G \times G$ admits a uniformity $\mathcal{U}$ for which j is $(\mathcal{U}, \mathcal{V})$-uniformly continuous.

2. Every uniformly continuous function $h : (K, \mathcal{V}) \to [0, 1]$ is constant on the image D of the diagonal Δ_G of $G \times G$ in K. By Exercise 8.R(d) this means that D is a singleton, i.e., j is constant on the diagonal of $G \times G$. Thus $f = g$.

Proof **Step 1.** We define the uniformity $\mathcal{U}$ on $G \times G$. For $U \in \mathcal{N}(G)$, define a binary reflexive relation W_U on $G \times G$ as follows: $((x, y), (x', y')) \in W_U$ for $x, x', y, y' \in G$ iff there exists $u \in U$ such that (x', y') equals one of the following four pairs: (ux, y), (x, uy), $(xy^{-1}uy, y)$ or $(x, yx^{-1}ux)$. If $U = U^{-1}$, which we shall assume, the relation W_U is symmetric. Define the uniformity $\mathcal{U}$ on $G \times G$ as the finest uniformity with the following property: for every entourage $W \in \mathcal{U}$ there exists $U \in \mathcal{N}(G)$ such that $W_U \subseteq W$. For $V \in \mathcal{N}(K)$, let us say that two points $z_1, z_2 \in V$ are *V-close* if either $z_2 \in z_1 V$ or $z_2 \in V z_1$. To prove that *the map $j : G \times G \to K$ defined by $j(x, y) = f(x)g(y)^{-1}$ is $(\mathcal{U}, \mathcal{V})$-uniformly continuous*, it suffices to prove the following assertion:

For any $V \in \mathcal{N}(K)$ there exists $U \in \mathcal{N}(G)$ with the following property: If $(x, y), (x', y') \in W_U$, then $z_1 = j(x, y)$ and $z_2 = j(x', y')$ are V-close.

This implies that the coarsest uniformity $\mathcal{U}$ on $G \times G$ for which j is $(\mathcal{W}, \mathcal{V})$-uniformly continuous, is coarser than $\mathcal{U}$ or, equivalently, that j is $(\mathcal{U}, \mathcal{V})$-uniformly continuous.

If (x', y') is (ux, y) or (x, uy), then $z_2 z_1^{-1} = f(u)$ or $z_1^{-1} z_2 = g(u)^{-1}$, respectively, and the assertion follows from the continuity of f and g. If $(x', y') = (xy^{-1}uy, y)$, then

$$z_1^{-1} z_2 = g(y)f(x)^{-1}f(xy^{-1}uy)g(y)^{-1} = k^{-1}f(u)k,$$

where $k = j(y, y)$. Let $F = \{j(y, y) : y \in G\}$. This is a compact subset in K: indeed, since the map $y \mapsto j(y, y)$ is constant on left H-cosets, F is a continuous image of the quotient space G/H, which can be identified with the compact space $\mathbb{T}$. It follows that there exists $W \in \mathcal{N}(K)$ such that $k^{-1}Wk \subset V$ for all $k \in F$. Pick $U \in \mathcal{U}(G)$ so that $f(U) \subset W$. Then $z_1^{-1}z_2 = k^{-1}f(u)k \in k^{-1}Wk \subset V$, which means that z_1 and z_2 are V-close. If $(x', y') = (x, yx^{-1}ux)$, the argument is similar: in this case, $z_2z_1^{-1} = kg(u)^{-1}k^{-1}$ with $k = j(x, x)$.

Step 2. The diagonal embedding $i : H \to G \times G$, $i(h) = (h, h)$ for $h \in H$, induces a right action of H on $G \times G$ via $(x, y) \cdot h = (xh, yh)$. Note that $j(xh, yh) = j(x, y)$ for all $h \in H$. We show that any uniformly continuous function $(K, \mathcal{V}) \to [0, 1]$ is constant on D. Since j is $(\mathcal{U}, \mathcal{V})$-uniformly continuous it suffices to prove that any $\mathcal{U}$-uniformly continuous function $d : G \times G \to [0, 1]$ which is constant on H-orbits, is constant on the diagonal. Fix such a function $d : G \times G \to [0, 1]$. To check that d is constant on Δ_G, pick an arbitrary $a \in G$. We must verify $d(a, a) = d(e, e)$. To this end we show that for every $\varepsilon > 0$, one has

$$|d(a, a) - d(e, e)| < 4\varepsilon. \qquad (*)$$

Let $t = aH \in G/H$. By the uniform continuity of d, for every $k = 0, 1, \dots$ there exists $U_k \in \mathcal{N}(G)$ such that, if $p_1, p_2 \in W_{U_k}$, then $|d(p_1) - d(p_2)| < 2^{-k}\varepsilon$. We claim that for the sequence $U_0, U_1, \dots$ and the coset t we can find a sequence $g_1 = e, \dots, g_n$ in G and $v \in U_0$ such that $g_n \in t$ and

$$(\forall k \in \mathbb{N}, 1 \le k < n)(\exists u_k \in U_k)(g_{k+1} = u_k g_k \text{ or } g_{k+1} = v^{-1}u_k v g_k), \qquad (**)$$

i. e., $g_n = ah$ for some $h \in H$ and $g_{k+1} = u_k g_k$, or $g_{k+1} = v^{-1}u_k v g_k$ for some $u_k \in U_k$. If $a \in H$ this is trivially verified, so that from now on we assume $a \in G \setminus H$.

It follows from the definition of G that for every $U \in \mathcal{N}(G)$ we can choose a positive δ so that for every $n \in \mathbb{N}$ and every pair of n-tuples $a_1, \dots, a_n$ and $b_1, \dots, b_n$ of distinct points in $\mathbb{T}$ with $\rho(a_i, b_i) < \delta$, $i = 1, \dots, n$, there exists $u \in U$ with $u(a_i) = b_i$ for each i. For each $k = 0, 1, \dots$, let δ_k denote the positive δ determined as above for U_k. Consider the point $b = a(1)$ of $\mathbb{T}$, we can assume without loss of generality that $0 < \operatorname{Arg} b \le \pi$. Let for $n \in \mathbb{N}$ the point $z_n \in \mathbb{T}$ be determined by $\operatorname{Arg} z_n = \operatorname{Arg} b/2n$. Choose n such that $\rho(z_n^2, 1) < \delta_0$ and set $\delta = \min(\delta_0, \dots, \delta_{2n-1})$. Pick a point w on the arc $\overset{\frown}{1z_n}$ of $\mathbb{T}$ with $\rho(w, 1) < \delta$. For $0 \le k \le n$ set $a_{2k} = z_n^{2k}$ (so that in particular $a_{2n} = b$) and for $0 \le k \le n-1$ set $a_{2k+1} = a_{2k}w$, $b_{2k+1} = z_n^{2k+1}$ and $b_{2k+2} = b_{2k+1}w$. Choose $v \in U_0$ with $v(a_i) = b_i$, $i = 1, \dots, 2n$ and $u_k \in U_k$ $(k = 1, \dots, 2n-1)$ with $u_k(a_k) = a_{k+1}$ if k is even and $u_k(b_k) = b_{k+1}$ if k is odd. This is possible since $\rho(a_k, a_{k+1}) < \delta \le \delta_k$ if k is even, and $\rho(b_k, b_{k+1}) < \delta \le \delta_k$ if k is odd. Define a sequence $g_1, \dots, g_{2n} \in G$ as follows: $g_1 = e$, $g_{k+1} = u_k g_k$ if k is even, $g_{k+1} = v^{-1}u_k v g_k$ if k is odd. Then $g_k(a) = a_k$. In particular, $g_{2n}(a) = b$, so $g_{2n} \in t$ and the sequence $g_1, \dots, g_{2n}$ is as in (**). Our claim is verified.

Let $p_k = (g_k, vg_k) \in G \times G$ for $0 < k < n$. Let us see that (**) implies $(p_k, p_{k+1}) \in W_{U_k} \circ W_{U_k}$. If $g_{k+1} = u_k g_k$, then $p_{k+1} = (u_k g_k, vu_k g_k) = (u_k x, (vg_k)(g_k^{-1}u_k g_k))$, so that $(p_k, p_{k+1}) \in W_{U_k} \circ W_{U_k}$ follows by the definition of W_{U_k}. In the case

$g_{k+1} = v^{-1}u_k v g_k$ it suffices to exploit the relation $p_{k+1} = (v^{-1}u_k v g_k, u_k v g_k) = (g_k(vg_k)^{-1}u_k(vg_k), u_k(vg_k))$.

By the choice of the U_k's and $(p_k, p_{k+1}) \in W_{U_k} \circ W_{U_k}$, we have $|d(p_k) - d(p_{k+1})| < 2^{1-k}\varepsilon$. Consequently,

$$|d(p_1) - d(p_n)| < 2\sum_{k=1}^{n-1} 2^{-k}\varepsilon < 2\varepsilon.$$

Since for $p_1 = (e, v)$ and $p_n = (g_n, vg_n)$ $(p_1, (e,e)) \in W_{U_0}$ and $(p_n, (g_n, g_n)) \in W_{U_0}$, we have $|d(e,e) - d(g_n, g_n)| < 4\varepsilon$. Since $(g_n, g_n) = (ah, ah)$ and d is constant on H-orbits, (g_n, g_n) can be replaced by (a, a) in the last inequality. This proves (*).
□

The following questions remain still open:

PROBLEMS

(1) *Does the regular closure of* **HausGrp** *coincide with the epi-closure ?*

(2) *Is* **HausGrp** *cowellpowered?*

As we know, (1) is equivalent to asking if the regular closure of **HausGrp** is weakly hereditary (see 6.2). It was shown by Uspenskij [1995] that the regular closure of **HausGrp** is not hereditary.

8.9 Epimorphisms and cowellpoweredness in algebra

In this section we give a synopsis of epimorphism-related results for R-modules and Abelian groups obtained with the help of closure operators presented in previous sections, and we briefly summarize known results for algebraic categories, referring to the literature for proofs. It turns out that with respect to cowellpoweredness the situation in algebra is fundamentally different from that in topology: under the assumption of a large-cardinal axiom, any conceivable "ranked" algebraic category becomes cowellpowered, and the cowellpoweredness statement for such categories is actually equivalent to that axiom; for reflective subcategories of groups or R-modules, we sketch proofs for cowellpoweredness within ZFC, without any additional set-theoretic axioms.

Epimorphisms in categories of R-modules and Abelian groups

Let $\mathcal{A}$ be a full and replete epireflective subcategory of $\mathbf{Mod}_R$ with induced $\mathcal{A}$-regular radical $\mathbf{r}$; hence $\mathcal{A}$ contains exactly the $\mathbf{r}$-torsion-free R-modules, i.e., for all $X \in \mathbf{Mod}_R$,

$$X \in \mathcal{A} \Leftrightarrow \mathbf{r}(X) = 0.$$

(cf. Proposition 6.7). Its epi-closure $\mathrm{E}(\mathcal{A})$ contains exactly those R-modules X such that X^2 does not contain any proper submodule M containing the diagonal such that X^2/M is $\mathbf{r}$-torsion, i.e.,

$$X \in \mathrm{E}(\mathcal{A}) \Leftrightarrow (\forall M \leq X^2)(\Delta_X \leq M \;\&\; \mathbf{r}(X^2/M) = X^2/M \Rightarrow M = X^2).$$

In fact, the last condition simply means that $\Delta_X \hookrightarrow X^2$ is an $\mathcal{A}$-extremal monomorphism, hence epi$^{\mathcal{A}}$-closed (cf. Theorem 6.2), and this characterizes the $\mathrm{E}(\mathcal{A})$-objects, by Corollary 7.6. But we have already shown in Example 7.6 (3) that, fortunately, this condition simplifies to:

$$X \in \mathrm{E}(\mathcal{A}) \Leftrightarrow \mathbf{r}^{\infty}(X) = 0,$$

so that $\mathrm{E}(\mathcal{A})$ is the $\mathbf{r}^{\infty}$-torsionfree class. With Example 6.7 (1), we conclude from this:

PROPOSITION *The following conditions are equivalent:*

(i) $\mathcal{A}$ *is epi-closed,*

(ii) $\mathbf{r}$ *is idempotent,*

(iii) $\mathcal{A}$ *is closed under extensions.*

□

COROLLARY *If $\mathcal{A}$ is closed under images, then epimorphisms of $\mathcal{A}$ are surjective. Conversely, surjectivity of epimorphisms of $\mathcal{A}$ forces $\mathcal{A}$ to be closed under images, provided $\mathcal{A}$ is closed under extensions.*

Proof Trivially, if $\mathcal{A}$ is closed under images, for every $f : A \rightarrow B$ in $\mathcal{A}$, $f(A) \hookrightarrow B$ is the kernel of $B \rightarrow B/f(A)$ in $\mathcal{A}$, which must be iso if f is epic. Conversely, for every surjective $f : A \rightarrow B$ with $A \in \mathcal{A}$, since $\mathbf{r}$ is idempotent by the Proposition, so that $C^{\mathbf{r}}$ is weakly hereditary, $K = \ker f$ is $\mathcal{A}$-dense in its maximal closure $c^{\mathbf{r}}_{A}(K) \cong f^{-1}(\mathbf{r}(B))$, hence $K = f^{-1}(\mathbf{r}(B))$ since epis in $\mathcal{A}$ are surjective. Consequently, $\mathbf{r}(B) = 0$, i.e., $B \in \mathcal{A}$. □

A more intricate criterion follows from Theorem 7.7. Let $I := \mathbf{r}(R)$, and for every R-module X, let

$$\mathbf{a}(X) := \mathrm{Ann}_X I := \{x \in X : Ix = 0\}$$

be the preradical given by the *annihilator of I in X*. We then have:

THEOREM

(1) *Epimorphisms in $\mathcal{A}$ are surjective if and only if $\mathbf{a}(X) \in \mathrm{E}(\mathcal{A})$ for all $X \in$ $\mathbf{Mod}_R$ or, equivalently, if $\mathbf{r}^{\infty}\mathbf{a} = \mathbf{0}$.*

(2) *If $I = 0$, and if $\mathcal{A} \neq \mathbf{Mod}_R$ is closed under extensions, then there are non-surjective epimorphisms in $\mathcal{A}$.*

(3) *In case $R = \mathbb{Z}$, if $\mathcal{A}$ is closed under extensions and contains non-surjective epimorphisms, then there is no epireflective subcategory $\mathcal{B}$ properly containing $\mathcal{A}$ such that $\mathcal{A} \hookrightarrow \mathcal{B}$ preserves epimorphisms.*

Proof (1) From Proposition 7.7 one knows that epimorphisms in $\mathcal{A}$ are surjective if and only if $\mathrm{D}(\mathcal{A}) = \mathbf{Mod}_R$, and modules X in $\mathrm{D}(\mathcal{A})$ were characterized in Example 7.7 (2) as those with $\mathbf{r}^\infty(\mathbf{a}(X)) = 0$.

(2) If $I = 0$, then $\mathbf{a}(X) = X$ for all X, hence $\mathrm{D}(\mathcal{A}) = \mathrm{E}(\mathcal{A})$. Consequently, if $\mathrm{E}(\mathcal{A}) = \mathcal{A} \neq \mathbf{Mod}_R$, also $\mathrm{D}(\mathcal{A}) \neq \mathbf{Mod}_R$.

(3) As shown in Example 7.7 (3), in case $R = \mathbb{Z}$, one has either $\mathrm{D}(\mathcal{A}) = \mathrm{E}(\mathcal{A})$ or $\mathrm{D}(\mathcal{A}) = \mathbf{AbGrp}$, hence $\mathcal{A} = \mathrm{D}(\mathcal{A})$ under the given conditions on $\mathcal{A}$. The assertion of (3) therefore rephrases the maximality property of $\mathrm{D}(\mathcal{A})$ given in Proposition 7.7. □

Epimorphisms in categories of universal algebras

By a category of universal algebras we understand a full subcategory $\mathcal{A}$ of the category $\mathcal{X}$ of all algebras of a fixed finitary type, i.e., of sets which come equipped with operations of given finitary arities. We suppose that $\mathcal{A}$ is closed under the formation of subalgebras and homomorphic images, and that finite coproducts exist in $\mathcal{A}$. Certainly, every *Birkhoff variety* satisfies these assumptions, i.e., every class $\mathcal{A}$ which is closed under subalgebras, homomorphic images, and direct products; equivalently, every class that is definable by axioms having the form of identically valid equations. For instance, groups are presented as sets with a binary operation $x \cdot y$, a unary operation x^{-1}, and a nullary operation e, subject to the equations $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $x \cdot x^{-1} = e$, $x \cdot e = x$; similarly for rings, R-modules (consider the unary operations αx for each $\alpha \in R$), R-algebras, lattices, etc. (The epireflective subcategories of $\mathbf{Mod}_R$ considered above are only so-called *quasi-varieties*, being characterized by closedness under the formation of subalgebras and direct products – but not necessarily under homomorphic images; their axiomatic description must permit implications between equations, with a typical example given by torsionfree Abelian groups: $(\forall n \geq 1)(\forall x)(nx = 0 \;\Rightarrow\; x = 0)$.)

The $\mathcal{A}$-regular closure $\mathrm{reg}_B^{\mathcal{A}}(A)$ of a subalgebra A of an algebra B in $\mathcal{A}$ is called the *dominion* of A in B, and its elements are characterized by Isbell's [1966]

LEMMA *(Domination Lemma) With the coproduct injections $k, l : B \longrightarrow B * B$, the following conditions are equivalent for an element $d \in B$:*

(i) $d \in \mathrm{reg}_B^{\mathcal{A}}(A)$;

(ii) *there exists a finite sequence $w_0 = k(d), \ldots, w_n = l(d)$ in $B * B$ such that for each $0 \leq i < n$, the element (w_i, w_{i+1}) belongs to the subalgebra of $(B * B) \times (B * B)$ generated by all elements of three forms, (x, x), $(k(a), l(a))$, $(l(a), k(a))$, with $a \in A$;*

(iii) *$(k(d), l(d))$ belongs to the congruence of $B * B$ generated by the pairs $(k(a), l(a))$, $a \in A$.*

Proof The dominion $\mathrm{reg}_B^{\mathcal{A}}(A)$ is the equalizer of the cokernelpair of $A \hookrightarrow B$ in $\mathcal{A}$. But since the coproduct $B * B$ exists in $\mathcal{A}$, and since $\mathcal{A}$ is closed under homomorphic images, the cokernelpair can be constructed as

$$B \overset{k}{\underset{l}{\rightrightarrows}} B * B \to B * B/\sim,$$

with $\sim$ the least congruence relation on $B*B$ which contains $R := \{(k(a), l(a)) : a \in A\}$. Therefore, (i) and (iii) are equivalent. Condition (ii) is just an elaboration of how to reach $\sim$ from R: first enforce reflexivity and symmetry, then the congruence property, and finally transitivity. □

The real work starts when one considers

EXAMPLES Of the many published results concerning the characterization of epimorphisms in varieties of universal algebras, we mention only two non-trivial characterization theorems. For proofs of these, we refer to the literature:

(1) Isbell's [1966] *Zig-Zag Theorem* for the category $\mathcal{A} = \mathbf{SGrp}$ of semigroups characterizes the elements $d \in \mathrm{reg}_B^{\mathcal{A}}(A) \setminus A$ as those which have two factorizations $d = a_0 y_1 = x_m a_{2m}$ connected by $2m$ relations

$$a_0 = x_1 a_1, a_1 y_1 = a_2 y_2, x_1 a_2 = x_2 a_3, \ldots, a_{2m-1} y_m = a_{2m},$$

for some $m \geq 1$, and with all a_i in A.

(2) In the category $\mathcal{A} = \mathbf{Rng}$ of rings (with unit element) the elements $d \in \mathrm{reg}_B^{\mathcal{A}}(A) \setminus A$ are characterized as those $d \in B$ with $d \otimes 1 = 1 \otimes d$ in $B \otimes_A B$ or, equivalently, those which can be written as

$$d = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$$

with $x_i, y_j \in B, a_{i,j} \in A$ and $\sum_{i=1}^{m} x_i a_{ij} \in A$ $(1 \leq j \leq n)$, $\sum_{j=1}^{n} a_{ij} y_j \in A$ $(1 \leq i \leq m)$ (cf. Storrer [1973]).

The survey article by Kiss, Márki, Pröhle and Tholen [1983] contains an extensive list to the literature on epimorphisms (predominantly, but not exclusively) in algebra, and on related problems (particularly: amalgamation).

Cowellpoweredness of locally presentable categories and accessible categories

Already in Isbell's [1966] paper it is noted that, as a consequence of the Domination Lemma, *the categories of universal algebras satisfying the required hypotheses for the Lemma, in particular the Birkhoff varieties, are cowellpowered.* As far as varieties are concerned, this statement allows for considerable generalization, to all *locally presentable categories* in the sense of Gabriel and Ulmer [1971], and further. (For a regular infinite cardinal κ, a category $\mathcal{A}$ is *locally κ-presentable* if it is cocomplete and contains a small set $\mathcal{G}$ of κ-presentable objects such that every object is a κ-filtered colimit of objects in $\mathcal{G}$; recall that an object $A \in \mathcal{A}$ is κ-presentable if the hom-functor $\mathcal{A}(A,\text{-}) : \mathcal{A} \to \mathbf{Set}$ preserves κ-filtered colimits.

$\mathcal{A}$ is called *locally presentable* if it is locally κ-presentable for some κ.) The extent to which locally presentable categories generalize varieties becomes clearer by their characterization as model categories of *essentially algebraic theories* (a concept due to Freyd [1972], with an equivalence proof given by Adámek and Rosický [1994]). The passage from the theories having Birkhoff varieties as their categories of models to essentially algebraic theories has three features: allow for multi-sorted signatures (so that algebras may have more than just one "underlying set"), allow operations to be infinitary, but insist that there is only a small set of them, and allow operations to be partial, but in an equationally-controlled way: the domains of definition of partial operations are given by equations involving only total operations. A typical example is the category **Cat** of all small categories and functors: an object C in this category has two underlying sets, $\operatorname{ob} C$ and $\operatorname{mor} C$; three totally defined operations, domain, codomain : $\operatorname{mor} C \to \operatorname{ob} C$, and $1 : \operatorname{ob} C \to \operatorname{mor} C$; and one partial operation $(\operatorname{mor} C)^2 \to \operatorname{mor} C$ whose domain of definition is the set $\{(x, y) : \mathrm{domain}(x) = \mathrm{codomain}(y)\}$.

It is not difficult to show that locally presentable categories are complete, cocomplete and wellpowered. It is much harder to prove:

Every locally presentable category is cowellpowered.

This fact goes back to Gabriel and Ulmer [1971], with a very readable version of the proof being presented in the book of Adámek and Rosický [1994]. Consequently, every variety of groups or of R-modules, say, is cowellpowered. What about quasi-varieties then, in a locally finitely presentable category ($\kappa = \aleph_0$), say? Such subcategories are still reflective (in fact: strongly epireflective), so the question is simply whether the given subcategory is closed under directed colimits; if it is, then it also locally finitely presentable and therefore cowellpowered. Makkai and Pitts [1987] proved a remarkable generalization of this fact:

Every full subcategory of a locally finitely presentable category $\mathcal{X}$ closed under limits and directed colimits is reflective in $\mathcal{X}$ and therefore locally finitely presentable and cowellpowered.

In this statement, closedness under limits is trivially a necessary condition while the role of directed colimits is much more delicate. It was shown by Rosický, Trnková and Adámek [1990] that under a certain large-cardinal principle, called the *Vopěnka Principle*, every reflective subcategory of a locally presentable category is closed under κ-filtered colimits for some κ; hence it is locally presentable in its own right and therefore cowellpowered. Of course, in concrete examples one would try to avoid recourse to the Vopěnka Principle whenever possible; see for instance Exercise 8.U.

As far as quasi-varieties of groups or of R-modules are concerned, one can easily prove (without any recourse to the Vopěnka Principle or any other set-theoretic assumption) that they are locally presentable, hence cowellpowered (see Ex.8.V-Y; for more details and generality, see Dikranjan and Tholen [1995]). The key difference here is that one does not need to prove that such a quasi-variety is closed under κ-filtered colimits. (In fact, it needn't be, even for $\kappa = \aleph_0$ and $R = \mathbb{Z}$, while such a quasi-variety is surely finitely presentable; see Ex.8.X(d).)

The cowellpoweredness result can be generalized once again, from locally presentable categories to *accessible categories*. (*κ-accessible categories* are defined like locally κ-presentable categories, except that the cocompleteness condition is relaxed to the existence requirement for κ-filtered colimits. A category is accessible if it is κ-accessible for some κ.) These categories can be described by certain so-called *sketches*, as first formulated by Lair [1981] and (largely independently) by Makkai and Paré [1989]; sketches themselves go back to Ehresman [1968] and are treated in considerable detail in the book of Barr and Wells [1985]. Although accessible categories still have (certain kinds of) directed colimits, they are in general no longer complete or cocomplete: an accessible category is complete (or cocomplete) if and only if it is locally presentable. In the realm of algebra, a typical example of an accessible but not locally presentable category is the category **Fld** of fields: (even finite) products are strikingly absent here; consequently it is impossible to give an essentially algebraic description (in particular of the domain of definition for the multiplication of a field). Usually, the categories of general topology, like **Top** itself, are not accessible. Makkai and Paré [1989] gave an elegant proof of the remarkable fact that *accessible categories with pushouts are cowellpowered* (see also Adámek and Rosický [1994]). *Under the set-theoretical assumption that there are arbitrarily large compact cardinals, one can abandon even the existence condition for pushouts and show that every accessible category is cowellpowered.*

Is there hope then to find, within the average-mathematicians's set-theoretic horizon, a constructively-defined accessible category which fails to be cowellpowered? The short answer to this question is *No*: Adámek and Rosický [1994] have shown that the statement "Every full subcategory of a locally presentable category is cowellpowered" is equivalent to the Vopěnka Principle mentioned above; it implies the existence of measurable cardinals and, on the other hand, the existence of huge cardinals implies its consistency. Accessible categories are (easily) fully embeddable into locally presentable categories, and the statement "Every accessible category is cowellpowered" implies the existence of large measurable cardinals.

Finally then, how does the non-cowellpoweredness result for the category **Frm** fare in light of the previous statements (cf. Example 8.2 (1)) ? Well, although equationally defined, the arities of the operations here $(\bigvee, \wedge)$ are not bounded by a fixed cardinal, since one permits joins to range over arbitrarily-large indexing sets. The non-cowellpoweredness result therefore implies that **Frm** is not locally presentable, hence there exists no "bounded" equational description for this category.

8.10 The Frobenius closure operator of fields

Every field K of positive characteristic p admits a fundamental endomorphism

$$\Phi_K : K \to K,\ \alpha \mapsto \alpha^p,$$

the so-called *Frobenius morphism* of K. If we put $\Phi_K := id_K$ in case $\operatorname{char} K = 0$, then this defines a natural transformation $\Phi : Id_{\mathbf{Fld}} \to Id_{\mathbf{Fld}}$ for the identity functor of the category **Fld** and its homomorphisms. Now define the *Frobenius closure* of K in an extension field L of K by the formula

$$\mathrm{fro}_L(K) = \Phi_L^{-1}(K) = \begin{cases} \{\alpha \in L : \alpha^p \in K\} & \text{for } p = \operatorname{char} L > 0, \\ K & \text{otherwise.} \end{cases}$$

Recall that every morphism in **Fld** is monic, and our subobject structure is given by $\mathcal{M} = \mathrm{Mor}\,\mathbf{Fld}$.

LEMMA fro *is a hereditary but non-idempotent closure operator of countable order* ω, *with* $\mathrm{fro}^{\omega} \leq \mathrm{int}$ (*the algebraic closure, cf.* 3.5(3)). *A field extension* $K \leq L$ *is purely inseparable if and only if it is* fro^{ω}*-dense, and it is separable if and only if it is* int*-dense and* fro*-closed.*

Proof fro is obviously a hereditary closure operator of **Fld** – it is in fact the "modified modification" of the discrete operator w.r.t. the pointed endofunctor $(Id_{\mathbf{Fld}}, \Phi)$, in the sense of Exercise 5.V. For positive characteristic p, $\mathrm{fro}_L(K)$ contains exactly the roots of the polynomials $x^p - \beta \in K[x]$ in L, hence $\mathrm{fro}_L(K) \leq \mathrm{int}_L(K)$. Since int is idempotent, $\mathrm{fro}^{\omega} \leq \mathrm{int}$ follows. Since

$$\mathrm{fro}_L^{\omega}(K) = \{\alpha \in L : \alpha^{p^n} \in K \text{ for some } n \geq 1\},$$

we obviously have $\mathrm{fro}_L(\mathrm{fro}_L^{\omega}(K)) = \mathrm{fro}_L^{\omega}(K))$. For the fields

$K_0 := \mathbb{Z}_p(x)$ (=rational functions over the prime field $\mathbb{Z}_p$),

$K_n := K_{n-1}(\sqrt[p^n]{x})$, $n \geq 1$

$M := \bigcup_{n\geq 1} K_n$,

one easily shows $\mathrm{fro}_M^{n+1}(K_0) = K_0^{n+1} \neq K_0^n = \mathrm{fro}_M^n(K_0)$. This proves non-idempotency of fro, with $o(\mathrm{fro}) = \omega$.

Recall that a field extension $K \leq L$ is *purely inseparable* if for every $\alpha \in L$ one has $\alpha^{p^n} \in K$ for some $n \geq 1$, with $p = \mathrm{char}\,L$, or if $K = L$. This is the case exactly if $\mathrm{fro}_L^{\omega}(K) = L$. The field extension is *separable* if it is algebraic and has no purely inseparable intermediate extensions (other than K); in other words, if $\mathrm{int}_L(K) = L$ and if $K \leq M \leq L$ with $K \leq M$ fro^{ω}-dense is possible only in case $M = K$. The latter condition certainly holds if $K \leq L$, and then also $K \leq M$ is fro^{ω}-closed; on the other hand, taking for $M = \mathrm{fro}_L^{\omega}(K)$, one sees that it also implies fro-closedness. □

THEOREM fro^{ω} *is the regular closure operator* $\mathrm{reg}^{\mathbf{Fld}}$. *Consequently, the epimorphisms of* **Fld** *are exactly the purely inseparable field extensions.*

Proof To show $\mathrm{fro}_L^{\omega}(K) \subseteq \mathrm{reg}_L^{\mathbf{Fld}}(K)$, we need to consider only the case $\mathrm{char}\,L = p > 0$. But for any morphisms $\sigma, \tau : L \to M$ with $\sigma|_K = \tau|_K$ and for every $\alpha \in \mathrm{fro}_L^{\omega}(K)$ we find $n \geq 1$ with $\alpha^{p^n} \in K$, hence $\sigma(\alpha^{p^n}) = \tau(\alpha^{p^n})$. This implies $(\sigma(\alpha) - \tau(\alpha))^{p^n} = 0$, hence $\sigma(\alpha) = \tau(\alpha)$.

For the converse inclusion, it suffices to show that every fro-closed extension $K \leq L$ is a regular monomorphism of **Fld**. In fact, this follows from our claim that for every $\alpha \in L \setminus K$ we can find a morphism $\sigma : L \to \overline{L}$ of L into its algebraic closure $\overline{L}$ with $\sigma(\alpha) \neq \alpha$, so that α does not belong to the equalizer of σ and $\mathrm{incl} : L \to \overline{L}$.

To prove the claim, first assume α to be transcendental over K. In this case we can take any automorphism τ of the field $K(\alpha)$ of rational functions with $\tau(\alpha) \neq \alpha$ and extend it to an automorphism $\overline{\tau}$ of $\overline{L}$; now put $\sigma = \overline{\tau}|_L$. If α is algebraic over K, we can consider the splitting field K_1 of its minimal polynomial over K. Then

$K \leq K_1$ is a Galois extension with $K_1 \leq \overline{L}$. Now we let τ to be any automorphism of K_1 with $\tau(\alpha) \neq \alpha$ and extend it as in the first case. □

COROLLARY *Let $\mathcal{A}$ be a class of fields such that for every field K there is a morphism $K \to A \in \mathcal{A}$. Then* fro^ω *is the regular closure operator of $\mathcal{A}$.*

Proof For a field L, let $\overline{L}$ be its algebraic closure. Then the proof of the Theorem shows $\mathrm{reg}_L^{\{\overline{L}\}}(K) \leq \mathrm{fro}_L^\omega(K)$ for all $K \leq L$ (in fact: the two closures coincide). With any (mono)morphism $\overline{L} \to A \in \mathcal{A}$, one then has

$$\mathrm{reg}_L^{\mathcal{A}}(K) \leq \mathrm{reg}_L^{\{A\}}(K) \leq \mathrm{reg}_L^{\{\overline{L}\}} \leq \mathrm{fro}_L^\omega(K) \leq \mathrm{reg}_L^{\mathbf{Fld}}(K) \leq \mathrm{reg}_L^{\mathcal{A}}(K).$$

□

With the natural transformation $\Phi : Id_{\mathbf{Fld}} \to Id_{\mathbf{Fld}}$, the identity functor becomes both, pointed and copointed. Its fixed class $\mathrm{Fix}(Id, \phi) = \{K : \Phi_K \text{ iso}\}$ contains exactly the *perfect* fields, i.e., those fields K with $\Phi_K(K) = K$.

PROPOSITION

(1) *A field K is perfect if and only if K has no proper epimorphic extension fields.*

(2) *The subcategory* **PerFld** *of perfect fields is both coreflective and reflective in* **Fld**. *It is in fact the least full and replete reflective subcategory of* **Fld**.

Proof (1) By the Theorem, for K not to have any proper epic extension is the same as to say that K is fro-closed in every extension field L. But this implies that $K \cong \Phi_K(K) \leq K$ is fro-closed. Since always $\mathrm{fro}_K(\Phi_K(K)) = \Phi_K^{-1}(\Phi_K(K)) = K$, under the given condition, $K = \Phi_K(K)$ is perfect. Conversely, for any extension $K \leq L$ of a perfect filed K, one has

$$\mathrm{fro}_L(K) = \Phi_L^{-1}(\Phi_K(K)) = \Phi_L^{-1}(\Phi_L(K)) = K.$$

(2) For coreflectivity, observe that $\mathbf{r}(K) := \Phi_K(K) \leq K$ defines a preradical. Hence the (mono-) coreflexion of K can be obtained by forming the idempotent core of $\mathbf{r}$. As in the case of the idempotent hull of fro, one has $\mathbf{r}^\infty(K) = \mathbf{r}^\omega(K)$; explicitly, for $\mathrm{char}\, K = p > 0$,

$$\mathbf{r}^\omega(K) = \bigcap_{n \geq 1} \mathbf{r}^n(K) = \bigcap_{n \geq 1} K^{p^n}.$$

For reflectivity, one proceeds analogously, by countable iteration of a suitable epiprereflection (S, σ) of **Fld** with $\mathrm{Fix}(S, \sigma) = \mathbf{Fld}$: for $\mathrm{char}\, K = 0$, let $SK = K$, and for $\mathrm{char}\, K = p > 0$, let SK be the splitting field of the polynomials $\{x^p - \alpha : \alpha \in K\}$. Note that the inclusion $\sigma_K : K \to SK$ is purely inseparable, hence epic in **Fld**. Now the **PerFld**-reflexion of K is the inclusion map of K into the direct colimit

$$S^\omega(K) = \bigcup_{n\geq 1} S^n(K).$$

Note that this is an epimorphism of **Fld**.

Finally, consider any full and replete reflective subcategory $\mathcal{A}$ of **Fld**. Since $\mathcal{A}$ is monoreflective, it must be bireflective, i.e., the $\mathcal{A}$-reflexion of any field K is epic. But if K is perfect, by (1), there are no proper epic extensions, so that the $\mathcal{A}$-reflexion is iso and K belongs to $\mathcal{A}$. □

Exercises

8.A (*Cowellpoweredness w.r.t. regular epimorphisms*) Show that every concrete category $\mathcal{A}$ is wellpowered w.r.t. regular monomorphisms and cowellpowered w.r.t. regular epimorphisms. *Hint:* With $F : \mathcal{A} \to$ **Set** faithful, to every regular monomorphism m of $\mathcal{A}$, assign the image of Fm, and to every regular epimorphism e of $\mathcal{A}$, assign the equivalence relation induced by Fe.

8.B (*Cowellpoweredness w.r.t. strong epimorphisms for categories with cogenerator*) Prove: every category $\mathcal{X}$ with pushouts and a small cogenerating set $\mathcal{A}$ is cowellpowered w.r.t. strong epimorphisms. *Hint:* For non-isomorphic strong epimorphisms $e : X \to Y$, $e' : X \to Y'$, show that the maps $\mathcal{X}(e, \mathcal{A})$, $\mathcal{X}(e', \mathcal{A})$ (as in the proof of Proposition 8.2) have distinct images, as follows. At least one of the pushout projections $p : Y \to P$, $p' : Y' \to P$ must be non-monic. If $pf = pg$ with $f \neq g$, pick h with codomain in $\mathcal{A}$ with $hf \neq hg$ and show that there is no h' with $h \cdot e = h' \cdot e'$.

8.C (*Reflectivity of small limit-closures*) Let $\mathcal{A}$ be a full subcategory of a complete and wellpowered category $\mathcal{X}$. Let $L_{\mathcal{X}}(\mathcal{A})$ denote the closure of $\mathcal{A}$ under limits in $\mathcal{X}$ (i.e., the intersection of all limit-closed full subcategories of $\mathcal{X}$ containing $\mathcal{A}$). Show:

(a) If $\mathcal{A}$ is small, then $L_{\mathcal{X}}(\mathcal{A})$ is the least full and replete reflective subcategory of $\mathcal{X}$ containing $\mathcal{A}$, and $\mathcal{A}$ is strongly generating in $L_{\mathcal{X}}(\mathcal{A})$. *Hint:* $L_{\mathcal{X}}(\mathcal{A})$ is contained in $\overline{S}_{\mathcal{X}}(\Pi(\mathcal{A}))$, with $\Pi(\mathcal{A})$ denoting the full subcategory consisting of direct products of objects in $\mathcal{A}$. $\mathcal{A}$ is strongly generating in $\overline{S}_{\mathcal{X}}(\Pi(\mathcal{A}))$ and therefore in $L(\mathcal{A})$. Now apply the *Special Adjoint Functor Theorem*.

(b) Give an example in which $L_{\mathcal{X}}(\mathcal{A})$ fails to be reflective in $\mathcal{X}$.

8.D (*The non-T_0 syndrom, etc*) Prove the claims of Remark 8.3. Find test spaces for **SUS**. *Hint:* Let Φ be a non-fixed ultrafilter on $\mathbb{N}$ and let $X = X_{\infty,\Phi}$ be the space $\mathbb{N} \cup \{\infty\} \cup \{\Phi\}$ with all points of $\mathbb{N}$ isolated, open neighbourhoods of ∞ are $\{\infty\} \cup A$, $A \subseteq \mathbb{N}$ co-finite, and open neighbourhoods of $\{\Phi\}$ are $\{\Phi\} \cup U$, $U \subseteq \mathbb{N}$ and $U \in \Phi$. Show that $\mathcal{A} \subseteq$ **SUS** for a strongly epireflective $\mathcal{A}$ of **Top** iff $\mathcal{A}$ does not contain a space homeomorphic to $X_{\infty,\Phi}$ for some ultrafilter Φ.

8.E (*Staying properly within* **Haus**) Prove: every full and replete epireflective subcategory $\mathcal{A}$ of **Top** which is properly contained in **Haus**, has a strongly epireflective hull in **Top** which is properly contained in **Haus**. *Hint:* Consider $X \in \mathbf{Haus} \setminus \mathcal{A}$ and assume $S(\mathcal{A}) = \mathbf{Haus}$. Then the $\mathcal{A}$-reflexion $\rho_X : X \to RX$ can be taken as the identity map, i.e., RX and X have the same underlying set, but the topology of RX is properly coarser than that of X. Now compare $\mathrm{reg}^{\mathcal{A}} = \mathrm{reg}^{S(\mathcal{A})}$ with K for the spaces X and RX, using also Theorem 6.3.

PROBLEM *Are there strongly epireflective subcategories of* **Top** *other than* **Haus** *with this property ?*

8.F (*Housekeeping on projective closure operators*) Verify the claims made in the Examples 8.4 (2), (3). Show in particular: μ is the least proper closure operator of **Top**.

8.G (*Surjectivity of epis in* **Haus**$(\mathcal{P})$) For a class $\mathcal{P}$ of topological spaces denote by $\mathrm{Dis}(\mathcal{P})$ the class of all topological spaces X such that for every map $h : P \to X$ with $P \in \mathcal{P}$ the subset $f(P)$ of X is closed and discrete.

(a) Show that $\mathrm{Dis}(\mathcal{P})$ is closed under taking subspaces and finite products.

(b) $X \in \mathbf{Haus}(\mathcal{P})$ belongs to $\mathrm{Dis}(\mathcal{P})$ if and only if $\mathrm{epi}_X^{\mathbf{Haus}(\mathcal{P})}$ is discrete (if and only if $\mathrm{reg}_X^{\mathbf{Haus}(\mathcal{P})}$ is discrete). Conclude that for $Y \in \mathbf{Haus}(\mathcal{P})$ every epimorphism $f : X \to Y$ in $\mathbf{Haus}(\mathcal{P})$ is surjective if and only if $Y \in \mathrm{Dis}(\mathcal{P})$.

(c) Show that for $\mathcal{A} = \mathbf{Haus}$(compact spaces) and $Y \in \mathcal{A}$ every epimorphism $f : X \to Y$ in $\mathcal{A}$ is surjective if and only if the only compact subsets of Y are finite.

(d) Show that for $\mathcal{B} = \mathbf{Haus}$(connected spaces) and $Y \in \mathcal{B}$ every epimorphism $f : X \to Y$ in $\mathcal{B}$ is surjective if and only if T is totally disconnected (i.e., the only connected subsets of Y are singletons).

8.H (*Generalizing projective closure operators*) Taking the definition of the $\mathcal{P}$-modification ${}^{\mathcal{P}}C$ of a closure operator C as a model (see 7.7) define projective closure operators in $\mathcal{M}$-complete categories, depending on $\mathcal{P}$ and a closure operator C, and investigate their properties.

8.I (*Projective closure operators and* σ)

(a) Show $\mathrm{pro}^{\{\mathbb{N}_\infty\}} > \sigma$. *Hint:* Let $X = X_{\infty,\Phi}$ be the test space of Exercise 8.D. Now $\lim n = \infty$, and for $M = \mathbb{N} \subseteq X$ one has $\sigma_X(M) = M \cup \{\infty\}$. On the other hand, for the one-point compactification $\mathbb{N}_\infty$ of the naturals $\mathbb{N}$, one can easily find a surjective map $f : \mathbb{N}_\infty \to X$, so that $\Phi \in \overline{M} = \overline{f(\mathbb{N}_\infty) \cap M \cap f(\mathbb{N}_\infty)} \subseteq \mathrm{pro}^{\{\mathbb{N}_\infty\}}(M)$.

(b) Show that for a topological space X the following are equivalent:

(i) every converging sequence in X has a unique accumulation point;

(ii) $\sigma_X = \mathrm{pro}_X^{\{\mathbb{N}_\infty\}}$;

(iii) $X \in \Delta(\mathrm{pro}^{\{\mathbb{N}_\infty\}})$.

8.J (*An essentially strong but non-strong closure operator of* **Top**)

(a) Show that $K^{\oplus}$ is essentially strong but not strong. *Hint:* Consider the space $X = \widetilde{\mathbb{N} \cup \{\infty\}}$ with $\emptyset$ and $[n, \infty]$ for $n \in \mathbb{N}$ the only open sets. Then $M = \mathbb{N}$ is $K^{\oplus}$-dense and $K^{\oplus}$-closed. Hence $K^{\oplus}$ is not strong. On the other hand, $K^{\oplus}$ is essentially strong since every subspace of a space in $\mathbf{Top}_1 = \Delta(K^{\oplus})$ is $\Delta(K^{\oplus})$-closed, hence strongly $K^{\oplus}$-closed.

(b) Show that the inequality $b \leq K^{\oplus}$ does not hold in **Top**. *Hint:* Use the same example as above; M is b-dense, hence $M = k_X^{\oplus}(M) \subset b_X(M) = X$.

8.K (*A proper class of non-strong closure operators in* **Top**) Show that for each infinite cardinal κ the closure operator $\mathrm{pro}^{\{\kappa\}}$ is not strong. *Hint:* For each infinite cardinal κ define the space X_κ with underlying set $\beta\kappa \cup \{\infty\}$ following the idea of Exercise 7.H. Then the subset $M := \beta\kappa \setminus \kappa \subseteq X$ is $\mathrm{pro}^{\{\kappa\}}$-closed, but $i(X) \hookrightarrow X +_M X$ is not $\mathrm{pro}^{\{\kappa\}}$-closed.

8.L (*A proper class of essentially strong closure operators in* **Top**)

(a) Let C be an additive and essentially strong closure operator of **Top** with $C^\infty \geq K$. Then the closure operators $C_1 = CK$ and $C_2 = KC$ are essentially strong, and the following hold for $i = 1, 2$:

(i) $T_2(C_i) \subseteq T_2(C)$ and $C^\infty = C_i^\infty$;

(ii) $\Delta(C_i) = \Delta(C)$;

(iii) the $T_2(C_i)$-closure coincides on $T_2(C_i)$ with the $T_2(C)$-closure.

(b) For a topological category $\mathcal{X}$ over **Set** as in 7.9 a **C**-*neighbourhood structure* is a concrete functor $\mathbf{C} : \mathcal{X} \to \mathbf{PrTop}$, i.e., an additive grounded closure operator C of $\mathcal{X}$ (see 5.10).

(i) An assignment $(X, x) \mapsto \mathcal{V}_x^X$ that gives for every $X \in \mathcal{X}$ and $x \in X$ a filter $\mathcal{V}_x^X$ on X, induces a neighbourhood structure $\mathbf{C} : \mathcal{X} \to \mathbf{PrTop} \hookrightarrow \mathbf{FC}$ via ($\mathcal{F} \to x \Leftrightarrow \mathcal{V}_x \subseteq \mathcal{F}$; cf. 3.2) if and only if for every morphism $f : X \to Y$ in $\mathcal{X}$ and every $x \in X$ one has $f^{-1}(U) \in \mathcal{V}_x^X$ for every $U \in \mathcal{V}_{f(x)}^Y$.

(ii) Show that $\mathbf{S(n)} : \mathbf{Top} \to \mathbf{PrTop}$ defined by means of $S(n)$-neighbourhoods as in Definition* of 8.6 is a (concrete) functor and therefore defines an additive grounded closure operator of **Top** (see Remark 8.6 (2)).

(c) Let η be an arbitrary order-type (i.e., an isomorphism class of totally ordered sets). For $X \in \mathbf{Top}$, $M \subseteq X$ and $x \in X$ define U to be a $S[\eta]$-*neighborhood* of x if there is a family of open sets $\{U_\gamma : \gamma \in \eta\}$ contained in U and containing x such that $\overline{U}_\lambda \subseteq U_{\lambda'}$, whenever $\lambda > \lambda'$.

(i) Show that this defines a $S[\eta]$-neighbourhood structure as in (b), hence an additive grounded closure operator Θ_η of **Top**.

(ii) Define $S[\eta] = T_2(\Theta_\eta)$ and show that for $\eta = n > 0$ **finite**, $S[n] = S(2n-1)$.

(d) For $X \in$ **Top** and $M \subseteq X$ define $\theta_\eta(M)$ by $x \notin \theta_\eta(M)$ iff there exists a family $\{U_\lambda : \lambda \in \eta\}$ of open subsets of X containing x such that $\overline{U}_\lambda \subseteq U_{\lambda'}$, whenever $\lambda > \lambda'$ and $\overline{U}_\lambda \cap M = \emptyset$ for each $\lambda \in L$.

(i) Show that θ_η is an additive grounded closure operator of **Top** coinciding with θ_n as defined in Remark 8.6, when $\eta = n > 0$ is finite.

(ii) Show $\Theta_\eta K = \Theta_\eta \leq \theta_\eta \leq \Theta_{1+\eta}$. If $\eta \cong 1 + \eta$, then these inequalities become equalities. When η has no bottom element then $\Theta_\eta = \theta_\eta$. If η has a bottom element, i. e., if $\eta = 1 + \eta'$, then $\Theta_\eta = \theta_{\eta'} K$ and $\Theta_\eta^\infty = \theta_{\eta'}^\infty$.

(iii) If $\eta = \alpha$ is an infinite ordinal, $\Theta_\alpha = \theta_\alpha$ and $T_2(\theta_\alpha) = S[\alpha] = S(\alpha)$, as given by Porter and Votaw [1973] (see Remark 8.6);

(iv) (PorterVotaw [1973], Example 2.10) Show that different ordinals α give different categories $S(\alpha)$, hence different closure operators θ_α.

(e) Show that the closure operators θ_η and Θ_η are essentially strong for every order type η. Conclude that the epimorphisms in $S[\eta]$ and $S(\eta) := T_2(\theta_\eta)$ are the θ_η^∞-dense maps, and that the inclusion $S[1+\eta] \hookrightarrow S(\eta)$ preserves epimorphisms.

(f) Generalize the construction of θ_η and Θ_η from totally ordered sets to arbitrary partially ordered sets and closure operators, as follows. Let $(L, \leq)$ be a partially ordered set and let $\mathcal{C} = \{C_\lambda : \lambda \in L\}$ and $\mathcal{D} = \{D_\lambda : \lambda \in L\}$ be collections of additive grounded closure operators of **Top**. Define $C = \theta(L, \mathcal{C}, \mathcal{D})$ by declaring that, for $X \in$ **Top**, $M \subseteq X$, and $x \in X$ one has $x \notin c_X(M)$ iff there exists a family $\{U_\lambda : \lambda \in L\}$ of subsets of X containing x such that:

(i) U_λ is a C_λ-neighbourhood of x (in the sense of 7.9) for each $\lambda \in L$;

(ii) $U_{\lambda'}$ is a $C_{\lambda'}$-neighbourhood of $(d_\lambda)_X(U_\lambda)$ whenever $\lambda > \lambda'$;

(iii) $(d_\lambda)_X(U_\lambda) \cap M = \emptyset$ for each $\lambda \in L$.

Show that $C = \theta(L, \mathcal{C}, \mathcal{D})$ is an additive grounded closure operator of **Top**. C is essentially strong whenever each C_λ is essentially strong and if there exists $\lambda_o \in L$ such that $D_\lambda \leq C_{\lambda_o}$ for each $\lambda \in L$. *Hint:* Apply (a).

8.M (*The categories $S(\alpha)$ are not cowellpowered*) Let α be an infinite ordinal and let α^* denote the reversed order of α.

(a) Show that the epimorphisms in $S(\alpha)$ and $\Delta(\theta_\alpha)$ are the θ_α^∞-dense maps (see Remark 8.6(3) and Exercise 8.L(d)).

(b) Show that the epimorphisms in $S(\alpha^*)$ and $\Delta(\theta_{\alpha^*})$ are the $\theta_{\alpha^*}^\infty$-dense maps (see Exercise 8.L (e)).

(c) (Dikranjan and Watson [1994]) Show that the category $S(\alpha)$ is not cowellpowered. In particular, the category $S(\omega)$ is not cowellpowered.

(d) (Dikranjan and Watson [1994]) Show that the category $S(\alpha^*)$ is cowellpowered if and only if α is indecomposable (i.e., whenever α is expressed as the ordinal sum of two ordinals, the latter ordinal equals α). In particular, the category $S(\omega^*)$ is cowellpowered.

(e) Let η be the order type of the reals or of the rationals. Prove that $S(\eta) = S[\eta]$ is cowellpowered. *Hint:* Show that θ_η is idempotent (Dikranjan and Watson [1994]).

8.N (*Housekeeping on S_n*) Show that $\theta_n\theta_s \leq \theta_{n+s}$ and $h_{n+1} \leq \theta_{2^n}$ for each $n, s \geq 0$ and conclude $S(2^n) \subseteq S_n$ for each $n \geq 0$.

8.O (*Boundedness and e-compactness*) Let B be a bounded subset of a topological space X. Prove:

(a) every subset of B is bounded;

(b) for every continuous map $f : X \to Y$, the subset $f(B)$ of Y is bounded;

(c) if B is closed, then B is compact; in particular, if B is closed and discrete, then B is finite. *Hint:* For an open cover $B = \bigcup_{\beta\in I} U_\beta$ of B consider the open cover $X = (X \setminus B) \cup \bigcup_{b\in B} W_\beta$, where W_β is an open subset of X such that $W_\beta \cap B = U_\beta$ for each $\beta \in I$;

(d) e-compactness is preserved under continuous surjective maps in **Top**.

8.P (*"Cowellpowered core" of a subcategory of* **Top**) For a reflective subcategory $\mathcal{A}$ of **Top** denote the subcategory $T_2(\mathrm{reg}^{\mathcal{A}})$ for brevity by $\mathcal{A}^c$. Prove the following properties of $\mathcal{A}^c$.

(a) $\mathcal{A}^c$ is a strongly epireflective subcategory of **Top**.

(b) $\mathcal{A}^c = (S(\mathcal{A}))^c \subseteq S(\mathcal{A}) = \Delta(\mathrm{reg}^{\mathcal{A}})$. *Hint:* $\mathcal{A}^c$ depends only on the $\mathcal{A}$-regular closure which coincides with the $S(\mathcal{A})$-regular closure.

(c) $\mathcal{A}^c = \mathbf{Top}$ iff $S(\mathcal{A}) = \mathbf{Top}$. *Hint:* If S(**A**) = **Top**, then the $\mathcal{A}$-closure coincides with the discrete closure operator, so $\mathcal{A}^c = \mathbf{Top}$. The other implication follows from (b).

(d) $\mathrm{reg}^{\mathcal{A}^c}$ coincides with $\mathrm{reg}^{\mathcal{A}}$ on $\mathcal{A}^c$. *Hint:* see Corollary 7.9.

(e) $\mathcal{A}^{cc} = \mathcal{A}^c$.

(f) The inclusion $\mathcal{A}^c \hookrightarrow \mathcal{A}$ preserves the epimorphisms.

(g) If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^c \subseteq \mathcal{B}^c$.

(h) If the $\mathcal{A}$-closure is additive and grounded, then the restriction $\mathbf{reg}^{\mathcal{A}}|_{\mathcal{A}^c} : \mathcal{A}^c \to$ **Haus** of the functor $\mathbf{reg}^{\mathcal{A}} : \mathbf{Top} \to \mathbf{PrTop}$ preserves epimorphisms.

(i) If the $\mathcal{A}$-regular closure is additive and grounded then $\mathcal{A}^c$ is cowellpowered.

(j) If the $\mathcal{A}$-regular closure is additive and the restriction $\mathbf{reg}^{\mathcal{A}} : \mathcal{A} \to \mathbf{Haus}$ of the functor $\mathbf{reg}^{\mathcal{A}} : \mathbf{Top} \to \mathbf{PrTop}$ to $\mathcal{A}$ preserves finite products, then $\mathcal{A}^c = S(\mathcal{A})$, so that both $\mathcal{A}$ and S($\mathcal{A}$) are cowellpowered. *Hint:* Follows from (h).

(k) If $\mathrm{reg}^{\mathcal{A}}|_{\mathcal{A}} = K|_{\mathcal{A}}$, then both $\mathcal{A}$ and S($\mathcal{A}$) are cowellpowered.

(l) $\mathbf{Haus}^c = \mathbf{Haus}$, $\mathbf{FHaus}^c = \mathbf{FHaus}$, $\mathbf{Top}_0^c = \mathbf{Top}_0$, $\mathbf{0\text{-}Top}^c = S(\mathbf{0\text{-}Top})$, $\mathbf{Reg}^c = S(\mathbf{Reg})$.

(m) The following subcategories of **Top** are cowellpowered: Hausdorff spaces, regular spaces, functionally Hausdorff spaces, Tychonoff spaces, zero-dimensional spaces, T_0-spaces.

(n) (Watson [1990]) Show that $\mathbf{reg}^{\mathbf{FHaus}} : \mathbf{FHaus} \to \mathbf{Haus}$ does not preserve finite products.

(o) $\mathbf{Ury}^c \neq \mathbf{Ury}$, and more generally, $S(\alpha)^c \neq S(\alpha)$ for each ordinal $\alpha > 1$. Show that $\mathbf{Ury}^c$ contains all spaces which admit a regular topology coarser than the given topology.

(p) (*Open problem*) Assume that the $\mathcal{A}$-regular closure is additive. Is $\mathcal{A}^c$ the biggest cowellpowered subcategory of $\mathcal{A}$? In particular, does $\mathcal{A}^c$ coincide with $\mathcal{A}$ in case the latter is cowellpowered ?

8.Q (*Preservation of epimorphisms*) Show:

(a) the inclusion $\mathbf{SUS} \hookrightarrow \mathbf{US}$ preserves epimorphisms;

(b) the inclusion **Haus**(compact spaces) $\hookrightarrow$ **Haus**(compact Hausdorff spaces) preserves epimorphisms;

(c) **Haus**(compact spaces) $\hookrightarrow$ **Haus**(e-compact spaces) does not preserve epimorphisms. *Hint:* Let $Z = \beta\mathbb{N} = X \setminus \{\infty\}$ be the subspace of the space X defined in Example 7.H. Then Z is Hausdorff and $D = \mathbb{N}$ is dense and bounded in Z. Therefore, for $\mathcal{P} = \{e\text{-compact spaces}\}$, D is $\mathrm{pro}^{\mathcal{P}}$-dense in Z, so the inclusion $i : D \hookrightarrow Z$ is an epimorphism in $\mathbf{Haus}(\mathcal{P})$ by Theorem 8.4. Since the only compact subsets of Z are finite, $X \in \mathrm{Dis}(\mathcal{Q})$ (see Exercise 8.G) with $\mathcal{Q} = \{\text{compact spaces}\}$, so that i is not an epimorphism in $\mathbf{Haus}(\mathcal{Q})$.

In particular, the inclusion in (c) is proper. Show that also the inclusions (a) and (b) are proper. *Hint:* For (a) it suffices to note that the test-spaces $X_{\infty,\Phi}$ from Exercise 8.D are always in **US**. For (b) take the Alexandroff one-point compactification of the rationals $\mathbb{Q}$ equipped with the usual topology.

8.R (*Housekeeping on quasi-uniform spaces*)

(a) The functor $G : \mathbf{QUnif} \to \mathbf{Set}$ is topological, while the functor $T : \mathbf{QUnif} \to \mathbf{Top}$ is not topological. If $\{\mathcal{U}_i\}_{i\in I}$ is a family of quasi-uniformities on a set X inducing the same topology, then also $\bigvee_i \mathcal{U}_i$ induces this topology on X. *Hint:* A topological space need not admit a coarsest quasi-uniformity inducing the given topology.

(b) For $(X,\mathcal{U}) \in \mathbf{QUnif}$ and a subspace $M \subseteq X$, one defines the adjunction space $X +_M X$ in **QUnif** as follows. The underlying set of $X +_M X$, the canonical embeddings $i, j : X \to X +_M X$ and the natural projection map $p : X + X \to X +_M X$ are defined as in Example 6.5. The quasi-uniform structure $\mathcal{U} +_M \mathcal{U}$ of $X +_M X$ has as a base the sets (for $V \in \mathcal{U}$) $W_V := (i \times i)(V) \cup (j \times j)(V) \cup (i \times j)(R_{M,V} \cup R_{M,V}^{-1}) \cup (j \times i)(R_{M,V} \cup R_{M,V}^{-1})$, where $R_{M,V} = \{(x,y) \in X \times X : V^{-1}(y) \cap V(x) \cap M \neq \emptyset\}$ (i.e., $y \in V(V(x) \cap M)$).

Then $\mathcal{U} +_M \mathcal{U}$ is a uniformity whenever $\mathcal{U}$ is. If $x \notin {}^{\varepsilon}k_X(M)$ (with ε as in 8.7), then $i(X \setminus M)$ is an open neighbourhood of $i(x)$ in $T(X +_M X)$, so that the topologies of $T(X +_M X)$ and of $TX +_M TX$ coincide at $i(x)$ (hence also at $j(x)$). In particular, both topologies coincide on $X +_M X$ when M is closed in TX. Finally, if $X \in$ **Unif** and $x \in {}^{\varepsilon}k_X(M)$, then $i(x)$ and $j(x)$ have the same neighbourhoods in the space $T(X+_M X)$. *Hint:* Use the relations $(p\times p)(W_V) \subseteq V^2$ and $(i \times i)(V) = (i(X) \times i(X)) \cap W_V$ and $(j \times j)(V) = (j(X) \times j(X)) \cap W_V$ for $V \in \mathcal{U}$. Note that for a point $x \in \overline{M} \setminus M$ in TX the following are equivalent: i) $x \in V(V(x) \cap M)$ for each $V \in \mathcal{U}$; ii) $(V \cap V^{-1})(x)$ meets M for each $V \in \mathcal{U}$; iii) $x \in \bigcap\{[V \cap V^{-1}](M) : V \in \mathcal{U}\}$ (i. e., $x \in {}^{\varepsilon}\beta_X(M)$). Choose V symmetric to obtain the final assertion.

(c) Show that the subset $\mathbb{Q}$ of $\mathbb{R} \in \mathbf{QUnif}_1$ is $\mathbf{QUnif}_1$-dense, while $\mathbb{Q}$ is $\mathbf{Top}_1$-closed in $T\mathbb{R}$. Conclude that ${}^{\varepsilon}\mathrm{reg}^{\mathbf{Top}_1}$ does not coincide with $\mathrm{reg}^{\mathbf{QUnif}_1}$ on $\mathbb{R}$. *Hint:* $\mathbb{Q}$ is $\mathbf{Top}_1$-closed in $T\mathbb{R}$ since $\mathrm{reg}^{\mathbf{Top}_1}$ is discrete on $\mathbf{Top}_1$ (cf. Example 6.5(1)). By (b), for every $x \in \mathbb{R}$ the neighbourhoods of the points $i(x)$ and $j(x)$ of $T(\mathbb{R} +_M \mathbb{R})$ coincide, hence the canonical map $\mathbb{R} +_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ is in fact the $\mathbf{QUnif}_1$-reflexion of $\mathbb{R} +_{\mathbb{Q}} \mathbb{R}$.

(d) For $(X,\mathcal{U}) \in \mathbf{Unif}_0$ and $a \neq b$ in X, there exists a uniformly continuous function $f : X \to [0,1]$ with $f(a) \neq f(b)$. For the closure operator Z^u of **Unif** defined in analogy with Z (see Example 6.9(4)), but w.r.t. *uniformly* continuous functions $f : X \to [0,1]$, show that $Z^u = {}^{\varepsilon}K = {}^{\varepsilon}Z$ on $\mathbf{Unif}_0$. *Hint:* Choose $U \in \mathcal{U}$ such that $(a,b) \notin U$ and a sequence of symmetric $U_n \in \mathcal{U}$, $n \in \mathbb{N}$, such that $U_0 = U$ and $U_n \circ U_n \circ U_n \subseteq U_{n-1}$ for $n > 1$. Define a pseudometric d (i.e., a function $d : X \to [0,1]$ such that $d(x,y) = d(y,x)$ and $d(x,y)+d(y,z) \leq d(x,z)$ for $x,y,z \in X$) with $U_n \subseteq \{(x,y) \in X \times X : d(x,y) \leq 2^{-n}\} \subseteq U_{n-1}$ for each $n > 1$. Then $f(x) = d(x,a)$ is uniformly continuous and $f(a) = 0, f(b) > 0$.

(e) For $n > 1$, show that $T^{-1}(S(n))$ properly contains $\overline{S}(n)$. *Hint:* Set $X = N \cup \{\infty\}$, and for each n set $L_n(\infty) = \{\infty\}$, $L_n(k) = \{k\}$ if $k \leq n$, and $L_n(k) = X$ otherwise. Then the sets $L_n = \bigcup_{x\in X}\{x\} \times L_n(x)$ generate a quasi-uniformity $\mathcal{U}$ on X whose induced topology is discrete, so that $TX \in S(n)$ with $n \leq \omega$. On the other hand, for each n obviously $n+1 \in L_n^{-1}(1) \cap L_n^{-1}(\infty)$, so that $X \notin \overline{S}(n)$ for any $n > 1$.

(f) Show that $\overline{\theta}_n$ is essentially strong for every n and conclude that $\mathrm{reg}^{\overline{S}(n)}$ coincides on $\overline{S}(n)$ with the idempotent hull of $\overline{\theta}_n$.

8.S (*Exotic properties of pretopological spaces*)

(a) Let $\mathbf{PrTop}_0$ be the category of T_0-pretopological spaces. Show that the epimorphisms in $\mathbf{PrTop}_0$ are surjective. *Hint:* Apply Frolík's Lemma.

(b) (Cf. Kneis [1986] and Dikranjan, Giuli and Tholen [1988].) Let $\mathbf{PrTop}_2 = T_2(K)$ be the category of T_2-pretopological spaces. Show that $\mathbf{PrTop}_2$ is not cowellpowered. *Hint:* Consider the restriction F of the functor $\boldsymbol{\theta} : \mathbf{Top} \to \mathbf{PrTop}$ to **Ury**. Observe that F sends **Ury** into $\mathbf{PrTop}_2$ and preserves epimorphisms, since the epimorphisms in $\mathbf{PrTop}_2$ are precisely the K^{∞}-dense maps according to Theorem 7.9. According to Theorem 8.1, $\mathbf{PrTop}_2$ cannot be cowellpowered since **Ury** is not cowellpowered, by Theorem* of 8.6.

8.T (*Epimorphisms of topological modules*) Let R be a unital ring, let $\mathbf{TopMod}_R$ be the category of topological R-modules and continuous module homomorphisms, let $V : \mathbf{TopMod}_R \to \mathbf{Top}$ be the forgetful functor and let K_V be the initial lifting of K along V (in analogy with Theorem 5.9; see also Theorem 5.8 and Remark 5.8 (4)). Show that K_V coincides with the regular closure operator of the full subcategory $\mathbf{HausMod}_R$ of Hausdorff topological modules of $\mathbf{TopMod}_R$. Conclude that the epimorphisms in $\mathbf{HausMod}_R$ are the homomorphisms with dense image.

8.U (*Cowellpoweredness of module categories and hereditariness of preradicals*)

(a) Let $\mathbf{r}$ be a hereditary preradical of $\mathbf{Mod}_R$. Then the subcategory $\mathcal{A}_\mathbf{r}$ of $\mathbf{r}$-torsion-free modules of $\mathbf{Mod}_R$ is cowellpowered. *Hint:* Show that if $m : M \to X$ is an epimorphism in $\mathcal{A}_r$, then necessarily M is an essential submodule of X. (If N is a non-zero submodule of X, then the quotient $N/(M \cap N)$, being isomorphic to a submodule of the $\mathbf{r}$-torsion quotient X/M, is $\mathbf{r}$-torsion as well, so that $M \cap N$ cannot be 0.) Conclude that X is isomorphic to a submodule of the injective hull $E(M)$ of M, so that $\mathrm{card} X$ is bounded by a cardinal function *depending only on M and R.*

(b) Let (m_n) be a sequence of natural numbers $m_n > 1$. Define a preradical $\mathbf{r}$ in **AbGrp** by declaring $\mathbf{r}(G)$ to be the intersection of the family of subgroups $\{m_n G\}$ for every abelian group G. Then the subcategory $\mathcal{A}_\mathbf{r}$ of $\mathbf{r}$-torsion-free modules of $\mathbf{Mod}_R$ is cowellpowered, but $\mathbf{r}$ fails to be hereditary. *Hint.* Consider the functor $T : \mathbf{AbGrp} \to \mathbf{TopGrp}$ which sends a group G to the *topological group* (G, τ), where τ is the group topology on G obtained by taking as a prebase of neighbourhoods of 0 the family $\{m_n G\}$. Clearly, T sends a group G to **HausGrp** iff G is in $\mathcal{A}_\mathbf{r}$. Since the epimorphisms in $\mathcal{A}_\mathbf{r}$ are the $C^\mathbf{r}$-dense homomorphisms, the restriction of the functor T sends $\mathcal{A}_\mathbf{r}$ to the category **HausAbGrp** of Hausdorff abelian groups and preserves epimorphisms. From the cowellpoweredness of **HausAbGrp** (see Exercise 8.T) and Theorem 8.1 we conclude that also $\mathcal{A}_\mathbf{r}$ is cowellpowered. To see that $\mathbf{r}$ is not hereditary, consider the inclusion $\mathbb{Z} \to \mathbb{Q}$ and observe that $\mathbf{r}(\mathbb{Q}) = \mathbb{Q}$, while $\mathbf{r}(\mathbb{Z}) \neq \mathbb{Z}$.

8.V (*Local presentability for categories of groups or modules;* cf. Adámek and Rosický [1994]) Let $\mathcal{X}$ be the category of groups or of R-modules (or any variety of finitary universal algebras). Show:

(a) an object $A \in \mathcal{X}$ has less than κ generators iff it is κ-generated in the following sense: every morphism $f : A \to Y$, where $Y = \lim Y_\alpha$ is a κ-filtered colimit such that all canonical morphisms $i_\alpha : Y_\alpha \to Y$ are mono, factors through some $i_\alpha(Y_\alpha) \to Y$;

(b) A is κ-presentable iff it has a presentation with less than κ generators and less than κ relations in these generators, in the sense that there exists a short exact sequence $0 \to N \to F \to A \to 0$ where F is a free object of $\mathcal{X}$ of less than κ generators and N is κ-generated.

Extend (a) and (b) to any epireflective subcategory $\mathcal{A}$ of $\mathcal{X}$, replacing $\mathcal{X}$ by $\mathcal{A}$ everywhere (cf. Dikranjan and Tholen [1995], the free object F is now "free in $\mathcal{A}$", hence obtained by applying the reflector to a free object of $\mathcal{X}$).

8.W (*Epireflective subcategories of* **Grp**) Show that every epireflective subcategory $\mathcal{A}$ of **Grp** is locally $\aleph_1$-presentable. *Hint:* Prove first that every countable group $G \in \mathcal{A}$ is $\aleph_1$-presentable: since G has countably many (i.e., $< \aleph_1$) generators $\{x_n\}_{n \in \mathbb{N}}$, it suffices to note that there exists a short exact sequence $1 \to N \to F \to A \to 1$ where F is a free group of $\mathcal{A}$ of $\aleph_0$ generators, hence both F and N are *countable.* Now note that every group of $\mathcal{A}$ is a $\aleph_1$-directed colimit of $\aleph_1$-generated (hence $\aleph_1$-presentable in $\mathcal{A}$) subgroups, of which there are at most $2^{\aleph_0}$. Finally observe that reflective subcategories of **Grp** are cocomplete.

8.X (*Epireflective subcategories of* $\mathbf{Mod}_R$)

(a) Let R be a right noetherian ring. Show that every finitely generated right R-module is $\aleph_0$-presentable. Show furthermore that the hypothesis on R is essential for the previous statement. Extend the result to any epireflective subcategory of $\mathbf{Mod}_R$. *Hint:* It suffices to argue as in 8.W, replacing *countable* by *finitely generated*, and using the fact that submodules of finitely generated modules are finitely generated. A counterexample for general R can be found in Bourbaki [1961].

(b) Let κ be a regular cardinal with $\mathrm{card}R < \kappa$. Show that every κ-generated module is also κ-presentable. *Hint:* Let M be a κ-generated module. Then there exists a set of generators $\{x_i\}_{i \in I}$ of M with $\mathrm{card}\, I < \kappa$. Let F be a free R-module of I generators. Then $\mathrm{card}F < \kappa$ and there exists a surjective homomorphism $f : F \to A$. Since obviously $\mathrm{card}(\ker f) < \kappa$ and since the relations between the generators $\{x_i\}$ of A are given by the elements of $\ker f$, it follows that M is κ-presentable.

(c) Show that every epireflective subcategory of $\mathbf{Mod}_R$ is locally κ-presentable if $\mathrm{card}R < \kappa$. *Hint:* Argue as in 8.W replacing *countable* by *κ-generated* and *$\aleph_1$-directed colimit* by *κ-directed colimit.*

(d) Show that the strongly epireflective subcategory $\mathcal{R}$ of **AbGrp** of reduced groups is locally $\aleph_0$-presentable, but not closed under directed (hence $\aleph_0$-filtered) colimits in **AbGrp**. *Hint:* The first part follows from (a), for the counterexample observe that $\mathbb{Q} \notin \mathcal{R}$ is a (directed) union of cyclic subgroups.

8.Y (*Cowellpoweredness of quasi-varieties of groups or modules*) Let $\mathcal{X}$ be as in 8.V. Show that every reflective subcategory of $\mathcal{X}$ is cowellpowered. *Hint:* For such a subcategory $\mathcal{A}$ the strongly epireflective hull $S_{\mathcal{X}}(\mathcal{A})$ is locally presentable by

8.W and 8.X (c). Then $S_{\mathcal{X}}(\mathcal{A})$ is cowellpowered by Gabriel and Ulmer's theorem. Finally, Theorem 8.2 permits to conclude that $\mathcal{A}$ is cowellpowered.

8.Z (*Non-additivity of Frobenius closure and algebraic closure*)

(a) Consider the field of rational functions $\mathbb{Z}_2(x,y)$ in two variables over the prime field of characteristic 2 and its algebraic extension $L = \mathbb{Z}_2(x,y)(z)$ with $z = \sqrt{x+y}$. Show that $K_1 = \mathbb{Z}_2(x)$ and $K_2 = \mathbb{Z}_2(y)$ are fro-closed in L. Since $z \in \mathrm{fro}_L(\mathbb{Z}_2(x,y))$, conclude that fro is not additive.

(b) Use the same example as in (a) to show that also the algebraic closure is not additive.

Notes

The surjectivity of epimorphisms in $\mathbf{Top}_1$ was noted by Burgess [1965] who also described the epimorphisms in **Haus**, while Baron [1968] characterized the epimorphisms in $\mathbf{Top}_0$.

The closure operators $\mathrm{ipro}^{\mathcal{P}}$ and $\mathrm{iesp}^{\mathcal{P}}$ were defined in Dikranjan and Giuli [1987], while $\mathrm{esp}^{\mathcal{P}}$ appears in Dikranjan and Giuli [1986] and in Giuli and Hušek [1986], in two particular cases.

The axioms $S(n)$ were introduced by Viglino [1969], and the axioms $S(\alpha)$, for infinite ordinals α, by Porter and Votaw [1973], while S_n appears in Arens [1978]. The axioms $S(\eta)$ and $S[\eta]$, depending on an arbitrary order type η (with $S(\eta) = S[\eta] = S(\alpha)$ when $\eta = \alpha$ is an ordinal) were introduced by Dikranjan and Watson [1994].

The first example of a non-cowellpowered subcategory of **Top** (see Example 8.2(2)) was given by Herrlich [1975]. Schröder [1983] proved that **Ury** and S_n, $n > 1$, are not cowellpowered. Giuli and Hušek [1986] established non-cowellpoweredness of **Haus**(compact spaces). This was extended to the smaller subcategory **Haus**(e-compact spaces) by Giuli and Simon [1990]; the proof of Theorem 8.6 follows essentially the proof given there. Tozzi [1986] proved that the category **SUS** is cowellpowered. Non-cowellpoweredness of **Haus**(compact Hausdorff spaces), as well as cowellpoweredness of certain subcategories of **Top** (see Example 8.5), was established in Dikranjan and Giuli [1986]. Cowellpoweredness of $S(n)$ and of **sUry** was shown by Dikranjan, Giuli and Tholen [1989]. The proof of Theorem 8.6*, isolated essentially from Dikranjan and Watson [1994], is substantially simpler than all its predecessors given for **Ury** and $S(n)$. Cowellpoweredness of $S[\eta]$ can be characterized in terms of properties of the order type η (see Dikranjan and Watson [1994] and Exercise 8.M). The epimorphisms in $\mathbf{QUnif}_0$ were described by Holgate [1992], while Theorem and Corollary 8.7 come from Dikranjan and Künzi [1995]. The proof of Theorem 8.8 is taken from Uspenskij [1994], where a compact connected manifold without boundary (either finite-dimensional or a Hilbert cube manifold) is considered instead of $\mathbb{T}$.

The description of the epimorphisms of **Fld** belongs to general categorical knowledge but seems hard to track down in the literature. Theorem 8.9 appears to be new, and so do the assertions of Exercises 8.W, X, Y.

9 Dense Maps and Pullback Stability

In this chapter we briefly discuss a particular type of closure operator, called Lawvere-Tierney topology, which generalizes the notion of Grothendieck topology and is a fundamental tool in Sheaf- and Topos Theory: Lawvere-Tierney topologies are simply idempotent and weakly hereditary closure operators (with respect to the class of monomorphisms) such that dense subobjects are stable under pullback. Localizations (=reflective subcategories with finite-limit preserving reflector) give rise to such closure operators. A Lawvere-Tierney topology allows for an effective construction of the reflector into its Delta-subcategory, which we describe in detail.

9.1 Hereditariness revisited

In an $\mathcal{M}$-complete category $\mathcal{X}$ with $\mathcal{M}$ a class of monomorphisms closed under composition and for a closure operator C, every morphism $f : X \to Y$ in $\mathcal{X}$ gives a "lax-commutative" diagram

$$\begin{tikzcd} \mathcal{M}/Y \arrow[r, "f^{-1}(\text{-})"] \arrow[d, "c_Y"'] \arrow[dr, phantom, "\geq"] & \mathcal{M}/X \arrow[d, "c_X"] \\ \mathcal{M}/Y \arrow[r, "f^{-1}(\text{-})"] & \mathcal{M}/X \end{tikzcd} \tag{9.1}$$

due to the C-continuity of f. In this chapter we consider situations when (9.1) or its restriction to certain subsets of $\mathcal{M}/Y$ commutes up to isomorphism, i.e., when

$$f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$$

holds true for *certain* morphisms f and *certain* subobjects $n \in \mathcal{M}/Y$; we say that f^{-1} *preserves the C-closure of n* in this situation. In this terminology we can rephrase the results of 2.5, as follows:

PROPOSITION

(1) *C is weakly hereditary if and only if for every $m \in \mathcal{M}$ and with $f = c(m)$, f^{-1} preserves the C-closure of every $n \leq m$;*

(2) *An idempotent closure operator C is weakly hereditary if and only if for every $f \in \mathcal{M}^C$, f^{-1} preserves the C-closure of every $n \leq f$;*

(3) *For C weakly hereditary, $\mathcal{E}^C \cap \mathcal{M}$ is left cancellable w.r.t. $\mathcal{M}^C$, so that $f \cdot m$ C-dense with $f \in \mathcal{M}^C$ and $m \in \mathcal{M}$ implies m C-dense.*

(4) *The following are equivalent:*

(i) *C is hereditary;*

(ii) *for every $f \in \mathcal{M}$, f^{-1} preserves the C-closure of every $n \leq f$;*

(iii) *C is weakly hereditary and, for every $f \in \mathcal{E}^C \cap \mathcal{M}$, f^{-1} preserves the C-closure of every $n \leq f$;*

(iv) *C is weakly hereditary and $\mathcal{E}^C \cap \mathcal{M}$ is left cancellable w.r.t. $\mathcal{M}$.*

(5) *C is hereditary and idempotent if and only if C is weakly hereditary and $\mathcal{E}^C \cap \mathcal{M}$ is closed under composition and left cancellable w.r.t. $\mathcal{M}$.*

Proof These are reformulations or minor ramifications of the statements proved in detail in 2.5. The reader must keep in mind that

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "m"'] & \cdot \arrow[d, "f \cdot m"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \tag{9.2}$$

is a pullback diagram whenever f is monic. □

Next we want to show that to some extent, idempotency and hereditariness are competing properties, at least in categories of topology. Briefly, we shall show that for a non-idempotent closure operator C (of **Top**, say), hereditariness can *never* be achieved by passing to its idempotent hull, regardless of whether C was hereditary or not.

For the remainder of this section, we let $\mathcal{P}$ be the class of $\vee$-prime elements in $\mathcal{M}$ (cf. 6.5) and assume that

(A) $p \leq q$ implies $p \cong q$ for all $p, q \in \mathcal{P}$,

(B) $m \cong \bigvee\{p \in \mathcal{P}/X : p \leq m\}$ for every $m \in \mathcal{M}/X$.

LEMMA *For $C, D \in CL(\mathcal{X}, \mathcal{M})$, the composite CD is hereditary only if $CD \cong C \vee D$.*

Proof One always has $C \vee D \leq CD$. Now suppose that for some $m : M \to X$, the morphism $k : c_X(M) \vee d_X(M) \to c_X(d_X(M))$ is not iso, so that by (B) there must exist $p \in \mathcal{P}/X$ with $p \leq c_X(d_X(m))$ but $p \not\leq c_X(m) \vee d_X(m)$. For $y := m \vee p : Y \to X$, in the notation of 2.5, one has $y \cdot c_Y(m_Y) \leq c_X(m) \leq c_X(m) \vee d_X(m)$, hence $p \not\leq y \cdot c_Y(m_Y)$. But this implies $m_Y \cong c_Y(m_Y)$. In fact, for every $q \in \mathcal{P}/X$ with $q \leq y \cdot c_Y(m_Y)$ one has $q \leq y$, so that $q \leq m$ or $q \leq p$ follows, with the second case impossible by property (A) and by the choice of p; consequently, from (B) one has $y \cdot c_Y(m_Y) \leq m = y \cdot m_Y$, hence $c_Y(m_Y) \cong m_Y$. Similarly, $d_Y(m_Y) \cong m_Y$, hence $c_Y(d_Y(m_Y)) \cong m_Y$. On the other hand, trivially $y \leq c_X(d_X(m))$, hence $y^{-1}(c_X(d_X(m))) \cong 1_Y$, while $m_Y \not\cong 1_Y$. Therefore, CD cannot be hereditary. □

THEOREM *For a non-idempotent closure operator C, no proper power C^α ($2 \leq \alpha \leq \infty$) is hereditary.*

Proof In case $C = D$, since $C^2 > C \vee C \cong C$, the proof of the Lemma provides subobjects $m < y : Y \to X$ with m_Y C^2-closed in Y but $y \leq c_X^2(m)$. Since $\mathcal{M}^C = \mathcal{M}^{C^2} = \mathcal{M}^{C^\infty}$ (see the proof of Theorem 4.6), m_Y is in fact C^α-closed in Y but $y \leq c_X^\alpha(m)$, which implies non-hereditariness of C^α. □

COROLLARY *Let C be weakly hereditary but not idempotent. Then for every $\alpha \in Ord \cup \{\infty\}$, $\alpha \geq 2$, the class of C^α-dense morphisms in $\mathcal{M}$ is left-cancellable w.r.t. $\mathcal{M}^C$, but not w.r.t. $\mathcal{M}$.*

Proof C^α is weakly hereditary (by the proof of Theorem 4.6), but not hereditary (by the Theorem). Hence the assertion of the Corollary follows from the Proposition. □

REMARKS

(1) The general assumptions for the Theorem are certainly satisfied when there is a monofibration $U : \mathcal{X} \to \mathbf{Set}$ such that $\mathcal{M}$ is the class of U-embeddings. Hence in such a category, the idempotent hull of a non-idempotent closure operator is never hereditary. Phrased differently, this means that a hereditary operator is not presentable as the idempotent hull (or any proper power) of a non-idempotent closure operator. For example, the Kuratowski closure operator K in **Top** cannot be a power of a non-idempotent operator; consequently, the idempotent hull of the sequential closure σ must be properly smaller than K

(2) The general assumptions for the Theorem are essential for its validity. In fact, in the category $\mathbf{Mod}_R$, the situation is completely different: the idempotent hull of a maximal hereditary closure operator is always hereditary; see Exercise 9.A.

9.2 Initial and open morphisms

In this section we discus two topologically important notions for morphisms and study their behaviour under composition, cancellation and pullback. We work in an $\mathcal{M}$-complete category $\mathcal{X}$ with $\mathcal{M}$ closed under composition and consider a closure operator C.

DEFINITION A morphism $f : X \to Y$ is *C-initial* if

$$c_X(m) \cong f^{-1}(c_Y(f(m)))$$

for all $m \in \mathcal{M}/X$. f is *C-open* if f^{-1} preserves the C-closure of every $n \in \mathcal{M}/Y$, i.e., if

$$f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$$

for all $n \in \mathcal{M}/Y$.

Note that the closure operator C is initial in the sense of 7.5 if and only if each $\Delta(C)$-reflexion ρ_X-reflexion is a C-initial morphism.

There is the following immediate connection between the notions of C-initiality and C-openness:

LEMMA

(1) *Every C-open morphism in $\mathcal{M}$ is C-initial.*

(2) *Every C-initial morphism in $\mathcal{E}$ is C-open, provided $\mathcal{E}$ is stable under pullback.*

Proof (1) Let $m : M \to X$, $f : X \to Y$ both be in $\mathcal{M}$, with f C-open. Then

$$f^{-1}(c_Y(f(m))) \cong c_X(f^{-1}(f(m))) \cong c_X(m)$$

(see Exercise 1.K (c)).

(2) If $f \in \mathcal{E}$ is C-initial and if $\mathcal{E}$ is stable under pullback, then $f(f^{-1}(n)) \cong n$ for all $n \in \mathcal{M}/Y$ (see Exercise 1.K(d)), hence

$$f^{-1}(c_Y(n)) \cong f^{-1}(c_Y(f(f^{-1}(n)))) \cong c_X(f^{-1}(n)). \qquad \square$$

EXAMPLES

(1) In **Top**, with $C = K$ the Kuratowski closure operator, the C-open morphisms are exactly the open maps, i.e., those continuous maps $f : X \to Y$ with $f(U) \subseteq Y$ open whenever $U \subseteq X$ is open. The map f is C-initial if and only if X carries the coarsest topology making f continuous; hence f is C-initial if and only if it is U-initial, with $U : \mathbf{Top} \to \mathbf{Set}$ the forgetful functor (cf. Exercise 5.P).

(2) If $\mathcal{X}$ has equalizers contained in $\mathcal{M}$, then for every full and replete reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, each $\mathcal{A}$-reflexion is $\mathrm{reg}^{\mathcal{A}}$-initial (see Theorem 6.3 (1)), hence it is also $\mathrm{reg}^{\mathcal{A}}$-open is case $\mathcal{A}$ is $\mathcal{E}$-reflective.

(3) If C is the trivial closure operator of $\mathcal{X}$, then every morphism is C-initial and C-open. If C is the discrete closure operator, every morphism is C-open and every morphism in $\mathcal{M}$ is C-initial.

PROPOSITION

(1) *The classes of C-initial morphisms and of C-open morphisms are closed under composition.*

(2) *C-initial morphisms are left-cancellable, while C-open morphisms are left-cancellable w.r.t. $\mathcal{M}$.*

(3) *Both C-initial morphisms and C-open morphisms are right-cancellable w.r.t. $\mathcal{E}$, provided $\mathcal{E}$ is stable under pullback.*

Proof (1) Consider morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{X}$. If f and g are C-initial for all $m \in \mathcal{M}/X$, then one has

$$\begin{aligned} c_X(m) &\cong f^{-1}(c_Y(f(m))) \\ &\cong f^{-1}(g^{-1}(c_Z(g(f(m))))) \\ &\cong (g \cdot f)^{-1}(c_Z((g \cdot f)(m))). \end{aligned}$$

The statement for C-open morphisms is trivial.

(2) Since $c_X(m) \leq f^{-1}(c_Y(f(m)))$ holds just by C-continuity, the following proves the claim for C-initial morphisms:

$$c_X(m) \cong f^{-1}(g^{-1}(c_Z(g(f(m))))) \geq f^{-1}(c_Y(f(m))).$$

Now assume $g \cdot f$ to be C-open with $g \in \mathcal{M}$. Then

$$f^{-1}(c_Y(n)) \leq f^{-1}\left(g^{-1}\left(c_Z(g(n))\right)\right) \cong c_X(f^{-1}\left(g^{-1}\left(g(n)\right)\right)) \cong c_X(f^{-1}\left(n\right))$$

holds for all $n \in \mathcal{M}/Y$; this proves C-openness of $g \cdot f$ since "$\geq$" is always true.

(3) If the composite $(X \xrightarrow{f} Y \xrightarrow{g} Z)$ is C-initial with $f \in \mathcal{M}$, then

$$c_Y(n) \geq f(c_X(f^{-1}(n))) \cong f(f^{-1}(g^{-1}(c_Z(g(f(f^{-1}(n))))))) \cong g^{-1}(c_Z(g(n)))$$

for all $n \in \mathcal{M}/Y$. The relevant computation for C-initiality is:

$$g^{-1}\left(c_Z(k)\right) \cong f(f^{-1}\left(g^{-1}\left(c_Z(k)\right)\right)) \cong f(c_X(f^{-1}\left(g^{-1}\left(k\right)\right))) \leq c_X(g^{-1}\left(k\right))$$

for all $k \in \mathcal{M}/Z$. □

We now turn to pullback stability for C-open morphisms. First observe that *any* commutative diagram

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[d, "\varphi"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"] & Y \end{tikzcd} \tag{9.3}$$

defines two functors

$$\psi(\varphi^{-1}(\text{-})),\ f^{-1}(h(\text{-})) : \mathcal{M}/Z \to \mathcal{M}/X,$$

and they are connected by a natural transformation since

$$\psi(\varphi^{-1}(k)) \leq f^{-1}(h(k)) \text{ for all } k \in \mathcal{M}/Z.$$

We say that the commutative diagram (9.3) satisfies the *Beck-Chevalley Property* (BCP) if this transformation is an isomorphism.

For pullback diagrams, the status of (BCP) is clarified by the first assertion of the following Theorem.

THEOREM

(1) *The Beck-Chevalley Property holds for every pullback diagram in $\mathcal{X}$ if and only if the class $\mathcal{E}$ is stable under pullback.*

(2) *Let the commutative diagram* (9.3) *satisfy (BCP), and let ψ be C-initial. Then, if f is C-open, also φ is C-open.*

Proof (1) Let $\mathcal{E}$ be stable under pullback and consider the commutative diagram

$$\begin{tikzcd}
& W \arrow[rr, "\psi"] \arrow[dd, "\varphi"] & & X \arrow[dd, "f"] \\
\varphi^{-1}(K) \arrow[ur] \arrow[rr, "h''"] \arrow[dd] & & f^{-1}(h(K)) \arrow[ur] \arrow[dd] & \\
& Z \arrow[rr, "h"] & & Y \\
K \arrow[ur, "k"] \arrow[rr, "h'"'] & & h(K) \arrow[ur] &
\end{tikzcd} \tag{9.4}$$

The front face is a pullback diagram since all the back, left, and right face are pullback diagrams. Consequently, since $h' \in \mathcal{E}$, also $h'' \in \mathcal{E}$, which implies $\psi(\varphi^{-1}(k)) \cong f^{-1}(h(k))$.

Conversely, consider the pullback diagram (9.3) with $h \in \mathcal{E}$ and let $k \in 1_Z$. Then $h(k) \cong 1_Y$ and $\varphi^{-1}(k) \cong 1_W$, hence

$$\psi(1_W) \cong f^{-1}(h(k)) \cong f^{-1}(1_Y) \cong 1_X.$$

Consequently, $\psi \in \mathcal{E}$ (see Exercise 2.K(b)).

(2) Under the given assumptions, C-continuity of h, C-openness of f, and C-initiality of ψ give for every $n \in \mathcal{M}_Z$

$$\varphi^{-1}(c_Z(n)) \quad \leq \quad \varphi^{-1}(h^{-1}(c_Y(h(n))))$$

$$
\begin{aligned}
&\cong \psi^{-1}(f^{-1}(c_Y(h(n)))) \\
&\cong \psi^{-1}(c_X(f^{-1}(h(n)))) \\
&\cong \psi^{-1}(c_X(\psi(\varphi^{-1}(n)))) \\
&\cong c_W(\varphi^{-1}(n)) .
\end{aligned}
$$

□

REMARK In **Top**, with $C = K$, where C-open maps are characterized by preservation of open sets, C-open maps are stable under pullback, i.e., the assumption of C-initiality of ψ in Theorem (2) is not needed in this case. In fact, due to BCP, for a basic open set $\varphi^{-1}(U) \cap \psi^{-1}(V)$ in the pullback W, with U, V open in Z, X, respectively, one has

$$\varphi(\varphi^{-1}(U) \cap \psi^{-1}(V)) = U \cap \varphi(\psi^{-1}(V)) = U \cap h^{-1}(f(V)),$$

with $f(V)$ open by hypothesis on f.

9.3 Modal closure operators

Let $\mathcal{X}$ be finitely $\mathcal{M}$-complete with $\mathcal{M} \subseteq \mathrm{Mono}(\mathcal{X})$ closed under composition, and let C be a closure operator of $\mathcal{X}$ w.r.t. $\mathcal{M}$.

DEFINITION C is called *modal* if every morphism in $\mathcal{X}$ is C-open, that is, if $f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$ holds for all $f : X \to Y$ in $\mathcal{X}$ and all $n \in \mathcal{M}/Y$. An idempotent modal closure operator is called *Lawvere-Tierney topology* (*LT*-topology, for short) of $\mathcal{X}$ w.r.t. $\mathcal{M}$.

PROPOSITION

(1) *Every modal closure operator is hereditary.*

(2) *C is modal if and only if C is weakly hereditary and $\mathcal{E}^C \cap \mathcal{M}$ is stable under pullback.*

(3) *With C also C^∞ is modal (if it exists), provided that $f^{-1}(\text{-})$ preserves joins (of ascending chains) for every f in $\mathcal{X}$.*

Proof (1) is trivial (cf. Proposition 9.1(4)). (2) If C is modal, then C is weakly hereditary (by (1)) and $f^{-1}(\text{-})$ preserves the C-closure of every $n \in \mathcal{M}/Y$, for each $f : X \to Y$ in $\mathcal{X}$; consequently, if n is C-dense, also $f^{-1}(n)$ is C-dense. Conversely, assume C to be weakly hereditary and $\mathcal{E}^C \cap \mathcal{M}$ to be stable under pullback. In diagram (9.5), the upper square is a pullback diagram since the lower square and the outer diagram are pullbacks.

$$\begin{tikzcd}
f^{-1}(N) \arrow[r] \arrow[d, "d"'] & N \arrow[d, "j_n"] \\
f^{-1}(c_Y(N)) \arrow[r, "f'"] \arrow[d, "k"'] & c_Y(N) \arrow[d, "c_Y(n)"] \\
X \arrow[r, "f"] & Y
\end{tikzcd} \tag{9.5}$$

By hypothesis, with j_n also $d \cong (f')^{-1}(j_n)$ is C-dense, and the following diagram commutes

$$\begin{tikzcd}
f^{-1}(n) \arrow[r, "d"] \arrow[d] & f^{-1}(c_Y(N)) \arrow[d, "f^{-1}(c_Y(n))"] \\
c_X(f^{-1}(N)) \arrow[r, "c_X(f^{-1}(n))"] & X
\end{tikzcd} \tag{9.6}$$

With the Diagonalization Property, $f^{-1}(c_Y(n)) \leq c_X(f^{-1}(n))$ follows.

(3) Inductively, one shows that the (existing) powers C^α ($\alpha \in Ord \cup \{\infty\}$) are modal if C is modal. Indeed, the composite of two modal closure operators is obviously modal; and if every inverse-image functor preserves joins (of ascending chains), then the join of (an ascending chain of) modal closure operators is modal. □

Modal closure operators are rare in topology but quite common in algebra, as the following two observations indicate.

EXAMPLES

(1) The only modal closure operators of **Top** are the discrete and the trivial operator. Since the trivial operator is the only non-grounded closure operator of **Top** (cf. Exercise 2.H), this follows from the following more general fact: *if $\mathcal{X}$ admits a faithful monofibration $U : \mathcal{X} \to$ **Set** with $\mathcal{M} = \mathcal{M}_U$, then the only grounded and modal closure operator of $\mathcal{X}$ is the discrete operator.*

In fact, if $c_X(M) \neq M$ for some $X \in \mathcal{X}$ and subobject M, then (without distinguishing between $\mathcal{X}$-objects and their underlying sets) for the inclusion morphism $f : \{x\} \to X$ of some $x \in c_X(M) \setminus M$, groundedness and modality of C lead to the contradictory statement

$$\emptyset = c_{\{x\}}(f^{-1}(M)) = f^{-1}(c_X(M)) = \{x\}.$$

(2) In $\mathbf{Mod}_R$, maximal closure operators are often modal. More precisely, *for a preradical* $\mathbf{r}$, *the following statements are equivalent:*

i) $C^{\mathbf{r}}$ *is modal,*

ii) $C^{\mathbf{r}}$ *is hereditary,*

iii) $\mathbf{r}$ *is hereditary.*

Consequently, $C^{\mathbf{r}}$ *is an LT-topology if and only if* $\mathbf{r}$ *is a hereditary radical.*

Since (ii)$\Leftrightarrow$(iii) was shown in Theorem 3.4(4), in light of the Proposition it suffices to show that for $\mathbf{r}$ hereditary, $C^{\mathbf{r}}$-density is stable under pullback. Hence let $N \leq Y$ be $C^{\mathbf{r}}$-dense, i.e., let $\mathbf{r}(Y/N) = Y/N$ be $\mathbf{r}$-torsion. Since for every $f : X \to Y$ one has

$$X/f^{-1}(N) \cong f(X)/N \hookrightarrow Y/N,$$

hereditariness of $\mathbf{r}$ yields that also $X/f^{-1}(N)$ is $\mathbf{r}$-torsion. This means that $f^{-1}(N)$ is $C^{\mathbf{r}}$-dense in X.

Minimal closure operators can be modal only if they are maximal: see Exercise 9.E and Remark 5.12.

The most important examples of modal closure operators arise as follows. Let (T, η) be a pointed endofunctor of $\mathcal{X}$ such that T *preserves subobjects* (so that $Tm \in \mathcal{M}$ for all $m \in \mathcal{M}$) and T *preserves inverse images* ($T(f^{-1}(n)) \cong (Tf)^{-1}(Tn)$ for all $f : X \to Y$ and $n \in \mathcal{M}/Y$; cf. Lemma 5.7). One can then define the (T, η)-*pullback closure* of $m \in \mathcal{M}/X$ by

$$\mathrm{pb}_X(m) = \eta_X^{-1}(Tm).$$

It is elementary to show that pb is a closure operator of $\mathcal{X}$; in fact, pb $=_T \tilde{S}$ is the "modified modification" of the discrete closure operator S of $\mathcal{X}$, as introduced in Exercise 5.V. Obviously, an $\mathcal{M}$-subobject m is pb-closed if and only if

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\eta_M"'] & X \arrow[d, "\eta_X"] \\ TM \arrow[r, "Tm"] & TX \end{tikzcd} \tag{9.7}$$

is a pullback diagram, and it is pb-dense if and only if the commutative square (9.7) admits a diagonal, i.e., there is a morphism $t : X \to TM$ with $Tm \cdot t = \eta_X$ and (necessarily) $t \cdot m = \eta_M$. Furthermore:

THEOREM *The pullback closure of a pointed endofunctor* (T, η) *with* T *preserving subobjects and inverse images is modal. It is idempotent if* (T, η) *is idempotent, i.e., if* (T, η) *is given by the reflector of a full reflective subcategory of* $\mathcal{X}$. *In this case* *the C-dense subobjects $\mathcal{M}$ are precisely those with Tm iso, i.e.,*

$$\mathcal{E}^C \cap \mathcal{M} = T^{-1}(\mathrm{Iso}(\mathcal{X})) \cap \mathcal{M}.$$

Proof Modality follows from

$$f^{-1}(\mathrm{pb}_Y(n)) \cong f^{-1}\left(\eta_Y^{-1}(Tn)\right) \cong \eta_X^{-1}((Tf)^{-1}(Tn)) \cong \eta_X^{-1}(T(f^{-1}(n))).$$

If (T,η) is idempotent, so that $\eta T (= T\eta)$ is iso, then Tm is obviously pb-closed, hence also its pullback $\eta_X^{-1}(Tm)$. Therefore, pb is idempotent. Furthermore, if m is C-dense, so that there is a morphism t with $Tm \cdot t = \eta_X$, hence $TTm \cdot Tt = \eta_{TX}$, one has

$$Tm \cdot \eta_{TM}^{-1} \cdot Tt = \eta_{TX}^{-1} \cdot TTm \cdot Tt = 1.$$

As a monomorphism Tm is therefore iso. Conversely, if Tm is iso, m is trivially pb-dense. □

In the particular case $\mathcal{M} = \mathrm{Mono}(\mathcal{X})$ we can conclude:

COROLLARY *The pullback closure induced by a localization of $\mathcal{X}$ (that is: by a full reflective subcategory of $\mathcal{X}$ whose reflector preserves finite limits) is a Lawvere-Tierney topology on $\mathcal{X}$ w.r.t. the class of all monomorphisms. When restricted to regular monomorphisms (which are equalizers), the pullback closure is simply the regular closure of the localization.* □

Proof The reflector of a localization preserves in particular monomorphisms and equalizers as well as pullbacks of such subobjects. Regularity of the pullback closure follows with the formula given in Theorem 6.3(2). □

9.4 Barr's reflector

For an LT-topology C on $\mathcal{X}$ we give an explicit description of the reflector $S : \mathcal{X} \to \Delta(C)$ (cf. 7.1) and prove (in partial conversion of Theorem 9.3) that it preserves subobjects. *We assume $\mathcal{X}$ to be finitely $\mathcal{M}$-complete with* $\mathrm{Reg}(\mathcal{X}) \subseteq \mathcal{M} \subseteq \mathrm{Mono}(\mathcal{X})$ *and $\mathcal{M}$ closed under composition, and we let $\mathcal{X}$ have finite products and coequalizers of equivalence relations (as defined below). Furthermore, the companion $\mathcal{E}$ of $\mathcal{M}$ is assumed to be stable under pullback throughout this section.*

An $\mathcal{M}$-subobject $r : R \to X \times X$ is called an $\mathcal{M}$-relation of $X \in \mathcal{X}$. It is an *equivalence relation* if it is

- *reflexive*: $\delta_X \leq r$, with $\delta_X : X \to X \times X$ the diagonal of X;
- *symmetric*: $r^* \leq r$, with $r^* = \langle p_2 \cdot r, p_1 \cdot r \rangle$ the converse of r, where $p_1, p_2 : X \times X \to X$ are product projections;
- *transitive*: $r \circ r \leq r$, with the composite $r \circ r$ given by the $\mathcal{M}$-part of an $(\mathcal{E}, \mathcal{M})$-factorization of $\langle p_1 \cdot r \cdot \pi_1, p_2 \cdot r \cdot \pi_2 \rangle$, where π_1, π_2 are defined by the pullback diagram

$$\begin{tikzcd} \bullet \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & R \arrow[d, "p_1 \cdot r"] \\ R \arrow[r, "p_2 \cdot r"] & X \end{tikzcd} \tag{9.8}$$

The equivalence relation r is called *effective* if $(p_1 \cdot r, p_2 \cdot r)$ is the kernel pair of some morphism $f : X \to Y$. (Note that the kernelpair (r_1, r_2) of a morphism $f : X \to Y$ always defines an equivalence relation $<r_1, r_2>$ on X.) In the category **Set**, equivalence relations are always effective, and the same is true in every variety of universal algebras (since, as subalgebras, equivalence relations are actually congruence relations).

In order to construct the reflector into the Delta-subcategory of an LT-topology C on $\mathcal{X}$, one considers for every $X \in \mathcal{X}$ the closure

$$r_X = c_{X \times X}(\delta_X) : RX \to X \times X$$

of the diagonal δ_X in $X \times X$.

PROPOSITION *For every $X \in \mathcal{X}$, r_X is an equivalence relation on X.*

Proof It is a categorical routine exercise to show that an $\mathcal{M}$-relation $r : R \to X \times X$ is an equivalence relation in $\mathcal{X}$ if and only if , for every $Z \in \mathcal{X}$, the relation

$$\mathcal{X}(X, r) : \mathcal{X}(Z, R) \to \mathcal{X}(Z, X \times X) \cong \mathcal{X}(Z, X) \times \mathcal{X}(Z, X)$$

is an equivalence relation in **Set** (cf. Exercise 9.F). Hence we must show that, for all $X, Z \in \mathcal{X}$, the relation

$$u \sim v \;:\Leftrightarrow\; (<u, v>: Z \to X \times X \text{ factors through } r_X)$$

is an equivalence relation on the hom-set $\mathcal{X}(Z, X)$. We first show:

$$u \sim v \;\Leftrightarrow\; e := \text{equalizer}(u, v) \text{ is } C\text{-dense in } X. \tag{$*$}$$

In fact, e is a pullback of the diagonal δ_X:

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d] & Z \arrow[d, "<u, v>"] \\ X \arrow[r, "\delta_X"] & X \times X \end{tikzcd} \tag{9.9}$$

Modality of C gives the pullback diagrams

$$\begin{array}{ccccc}
E & \longrightarrow & c_Z(E) & \xrightarrow{c(e)} & Z \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle <u,v>} \\
X & \longrightarrow & RX & \xrightarrow{r_X} & X \times X
\end{array} \tag{9.10}$$

Hence, if $< u, v >$ factors through r_X, the pullback property of the right half of (9.10) gives an inverse of $c(e)$, so that e is C-dense. Conversely, C-density of e leads trivially to a factorization of $<u, v>$ through r_X.

Right-cancellability of C-dense morphisms gives with (*):

$$u \sim v \;\Leftrightarrow\; (\exists m \in \mathcal{M}/Z)\; m \text{ is } C\text{-dense and } u \cdot m = v \cdot m. \tag{**}$$

From (**) we now see that the relation $\sim$ is trivially reflexive and symmetric, but also transitive since for C-dense subobjects m, n of Z also $m \wedge n$ is C-dense in Z (due to pullback stability and closedness of composition of C-dense subobjects). □

We can now proceed with the construction of the reflector into $\Delta(C)$, by forming the coequalizer q_X of the projections p_1, p_2 restricted by r_X:

$$RX \xrightarrow{r_X} X \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{q_X} SX$$

Our main difficulty is to show that SX belongs to $\Delta(C)$. Let us first observe that the Diagonalization Lemma 2.4 makes R functorial, hence S is a functor as well, pointed by q. We now show:

Lemma *If the equivalence relation r_X is effective, then there is a pullback diagram*

$$\begin{array}{ccc}
RX & \xrightarrow{e} & SX \\
{\scriptstyle r_X}\downarrow & & \downarrow{\scriptstyle \delta_{SX}} \\
X \times X & \xrightarrow{q_X \times q_X} & SX \times SX
\end{array} \tag{9.11}$$

Proof Let us observe that the diagonal morphism Rq_X of

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[d, "j_X"'] & SX \arrow[d, "j_{SX}"] \\
RX \arrow[r, "Rq_X"] \arrow[d, "r_X"'] & RSX \arrow[d, "r_{SX}"] \\
X \times X \arrow[r, "q_X \times q_X"] & SX \times SX
\end{tikzcd} \tag{9.12}$$

factors through $e := q_X \cdot p_1 \cdot r_X = q_X \cdot p_2 \cdot r_X$. In fact, with the projections $s_i : SX \times SX \to SX$ $(i = 1, 2)$ one has

$$s_i \cdot \delta_{SX} \cdot e = e = s_i \cdot (q_X \times q_X) \cdot r_X,$$

hence

$$r_{SX} \cdot j_{SX} \cdot e = \delta_{SX} \cdot e = (q_X \times q_X) \cdot r_X = r_{SX} \cdot Rq_X,$$

with r_{SX} monic. Consequently, (9.11) commutes and we have

$$r_X \leq (q_X \times q_X)^{-1}(\delta_X) =: d : D \to X \times X.$$

In order to show $d \leq r_X$ one uses the fact that $(p_1 \cdot r_X, p_2 \cdot r_X)$ is a kernelpair, actually: the kernelpair of its coequalizer q_X. In fact, with the pullback projection $h : D \to SX$ one has

$$q_X \cdot p_i \cdot d = s_i \cdot (q_X \times q_X) \cdot d = s_i \cdot \delta_{SX} \cdot h = h \ \ (i = 1, 2),$$

so that the universal property of $(p_1 \cdot r_X, p_2 \cdot r_X)$ gives the desired morphism $D \to RX$.
□

REMARKS

(1) In a (reasonably interpreted) commutative diagram

$$\begin{tikzcd}
U \arrow[r, shift left, "u_1"] \arrow[r, shift right, "u_2"'] \arrow[d, "g"] & X \arrow[r, "p"] \arrow[d, "f"] & P = X/U \arrow[d, "h"] \\
V \arrow[r, shift left, "v_1"] \arrow[r, shift right, "v_2"'] & Y \arrow[r, "q"] & Q = Y/V
\end{tikzcd} \tag{9.13}$$

in **Set** where the rows are coequalizers, if f and g are monic, h may *fail* to be monic (cf. Exercise 1.M). However, if $u =< u_1, u_2 >$ and $v =< v_1, v_2 >$ are equivalence relations with $u \cong (f \times f)^{-1}(v)$, then f (and therefore necessarily g) being monic implies that also h is a monomorphism.

(2) In our abstract category $\mathcal{X}$, we say that $\mathcal{M}$ is *stable under coequalizers of equivalence relations* if for every commutative diagram (9.13) with equivalence relations $u =< u_1, u_2 >$, $v =< v_1, v_2 >$ such that $u \cong (f \times f)^{-1}(v)$, and with $p = \text{coequalizer}(u_1, u_2)$, $q = \text{coequalizer}(v_1, v_2)$, from $f \in \mathcal{M}$ follows $h \in \mathcal{M}$.

(3) *For $\mathcal{M} = \text{Mono}(\mathcal{X})$, if equivalence relations are effective in $\mathcal{X}$, then $\mathcal{M}$ is always stable under coequalizers of equivalence relations.* In fact, in the notation (9.13), if we let $(w_1, w_2 : W \to P)$, $(z_1, z_2 : Z \to X)$ be the kernelpairs of h and $h \cdot p$ respectively, then there is a unique morphism $e : Z \to W$ with $w_i \cdot e = p \cdot z_i$ $(i = 1, 2)$. We can think of e as the diagonal morphism in

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
A_1 \arrow[r] \arrow[d] & W \arrow[r, "w_1"] \arrow[d, "w_2"] & P \arrow[d, "h"] \\
X \arrow[r, "p"] & P \arrow[r, "h"] & Q
\end{tikzcd} \tag{9.14}$$

where each square is a pullback diagram; as the composite of two pullbacks of p, it therefore belongs to $\mathcal{E}$ and is epic.

Since (v_1, v_2) is the kernelpair of q, there is a morphism $k : Z \to V$ with $v_i \cdot k = f \cdot z_i$; obviously, $v \cdot k = (f \times f) \cdot z$. Since $u \cong (f \times f)^{-1}(v)$, there is also a morphism $l : Z \to U$ with $u \cdot l = z$ and $g \cdot l = k$. The first identity leads to $p \cdot z_1 = p \cdot z_2$, hence $w_1 \cdot e = w_2 \cdot e$. Since e is epic, $w_1 = w_2$ follows and h must be a monomorphism.

THEOREM *Let C be an LT-topology of the category $\mathcal{X}$ in which equivalence relations are effective. Then S as constructed above is the reflector of the strongly epireflective subcategory $\Delta(C)$ of $\mathcal{X}$, and it preserves subobjects whenever $\mathcal{M}$ is stable under coequalizers of equivalence relations; the latter condition always holds for $\mathcal{M} = \text{Mono}(\mathcal{X})$.*

Proof Since (9.11) is a pullback diagram, modality of C shows that also the lower part of (9.12) is a pullback diagram. Consequently, Rq_X is a pullback of the morphism

$$q_X \times q_X = (q_X \times 1_{SX})(1_X \times q_X),$$

with each factor being a pullback of the regular epimorphism q_X. Since $\mathcal{E}$ contains q_X and is stable under pullback and closed under composition, also Rq_X belongs to $\mathcal{E}$. Consequently, as a second factor of an $\mathcal{E}$-morphism, also j_{SX} belongs to $\mathcal{E}$, hence it must be iso, and $SX \in \Delta(C)$ follows.

For an arbitatry morphism $f : X \to A \in \Delta(C)$, the Diagonalization Lemma 2.4 gives a morphism $w : RX \to A$ with $\delta \cdot w = (f \times f) \cdot r_X$. Therefore, with the projections $t_i : A \times A \to A$, one obtains

$$f \cdot p_i \cdot r_X = t_i \cdot (f \times f) \cdot r_X = t_i \cdot \delta_A \cdot w = w \quad (i = 1, 2).$$

Consequently, by the preservation property, f factors uniquely through q_X.

Concerning the preservation by S of an $\mathcal{M}$-subobject m, first note that

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "\delta_M"'] & X \arrow[d, "\delta_X"] \\ M \times M \arrow[r, "m \times m"] & X \times X \end{tikzcd} \tag{9.15}$$

is a pullback diagram since m is monic. Modality of C gives then the pullback diagram

$$\begin{tikzcd} RM \arrow[r, "Rm"] \arrow[d, "r_M"'] & RX \arrow[d, "r_X"] \\ M \times M \arrow[r, "m \times m"] & X \times X \end{tikzcd} \tag{9.16}$$

Since $\mathcal{M}$ is stable under coequalizers of equivalence relations, the construction of S gives immediately that Sm must belong to $\mathcal{M}$. □

In the case $\mathcal{M} = \mathrm{Reg}(\mathcal{X})$, such that every regular monomorphism is an equalizer (which holds true in particular when $\mathcal{X}$ has cokernelpairs), one obtains:

Corollary *Under the hypothesis of the Theorem, a morphism in $\mathcal{X}$ is $\Delta(C)$-epic if and only if it is C-dense. Hence C is the $\Delta(C)$-epi-closure of $\mathcal{X}$. Consequently, for $\mathcal{X}$ $\mathcal{M}$-complete, C is the $\Delta(C)$-regular closure of $\mathcal{X}$ if and only if the $\Delta(C)$-regular closure is weakly hereditary.* □

Proof By definition of $\Delta(C)$, every C-dense subobject is $\Delta(C)$-epic. Conversely, let $m = \text{eqializer}(u, v)$ with $u, v : X \to Y$ be $\Delta(C)$-epic. Then

$$Su \cdot q_X \cdot m = Su \cdot Sm \cdot q_M = Sv \cdot Sm \cdot q_M = Sv \cdot q_X \cdot m$$

implies $Su = Sv$ since q_X is a reflexion. Therefore, the diagram

$$\begin{tikzcd}
X \arrow[r, "q_X"] \arrow[d, "<u,v>"'] & SX \arrow[r, "Su"] & SY \arrow[d, "\delta_{SY}"] \\
Y \times Y \arrow[rr, "q_X \times q_X"] & & SY \times SY
\end{tikzcd} \tag{9.17}$$

commutes. Consequently, $<u,v>$ factors through the pullback of δ_{SX} along $q_X \times q_X$, which is r_X (according to the Lemma). But this means that the equalizer m of u, v is C-dense (see the proof of the Proposition).

The additional statements follow with Theorem 6.2. □

9.5 Total density

Rather than asking whether the inverse image for a given morphism preserves the closure of certain subobjects, one may also investigate the problem whether the closure of a given subobject is preserved by the inverse image of certain morphisms. In this section we discuss one particular instance of this problem which has its origins in topological group theory (see Example below). We assume our category $\mathcal{X}$ to be $\mathcal{M}$-complete, with $\mathcal{M}$ closed under composition, and consider a closure operator C. The notion of total C-density as in Example 5.4 for $\mathcal{X} = \mathbf{Top}$ can be given for an arbitrary category $\mathcal{X}$, as follows.

DEFINITION An $\mathcal{M}$-subobject $m : M \to X$ is called *totally C-dense* if $k^{-1}(m) : K \wedge M \to K$ is C-dense for every C-closed $\mathcal{M}$-subobject $k : K \to X$.

As usual, $\mathcal{E}$ denotes the factorization companion of $\mathcal{M}$. Under suitable hypotheses on $(\mathcal{E}, \mathcal{M})$, total C-density can be described as density w.r.t. a closure operator:

PROPOSITION *If $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms, and if every subobject lattice $\mathcal{M}/X$, $X \in \mathcal{X}$, has the structure of a frame, then there is a weakly hereditary closure operator C^{tot} of $\mathcal{X}$, uniquely determined (up to isomorphism) by the property that the C^{tot}-dense subobjects are precisely the totally C-dense subobjects. Moreover, C^{tot} is idempotent whenever C is idempotent.*

Proof It suffices to show that the class $\mathcal{D}$ of totally C-dense subobjects satisfies the conditions (a)-(c) of Theorem* of 5.4, and that $\mathcal{D}$ is closed under composition whenever C is idempotent. But the latter property follows immediately from the Definition and the fact that $\mathcal{E}^C$ is closed under composition for C idempotent (Proposition 2.4). Similarly, right cancellability of $\mathcal{D}$ w.r.t. $\mathcal{M}$ follows from the corrsponding property of $\mathcal{E}^C$ (Corollary* of 2.3).

In order to check the $\wedge$-$\vee$-preservation property (b) for $\mathcal{D}$, we consider the pullback diagram 5.13 with $n \in \mathcal{M}/X$ and $f(1_X) \vee n = 1_Y$ and assume $f^{-1}(n) \in \mathcal{D}$. For every $k \in \mathcal{M}^C/X$, one then has the cube

$$
\begin{tikzcd}
 & f^{-1}(N) \arrow[rr] \arrow[dd, "f^{-1}(n)"'] & & N \arrow[dd, "n"] \\
f^{-1}(L) \arrow[ur] \arrow[rr] \arrow[dd, "j"'] & & L \arrow[ur] \arrow[dd, "k^{-1}(n)"' near start] & \\
 & X \arrow[rr, "f"' near start] & & Y \\
f^{-1}(K) \arrow[ur, "f^{-1}(k)"'] \arrow[rr] & & K \arrow[ur, "k"'] &
\end{tikzcd}
\tag{9.18}
$$

with $L = K \wedge N$, and all faces given by pullback. Since $f^{-1}(k)$ is C-closed, j must be C-dense, by hypothesis on n. Since $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms, the $\mathcal{E}$-images of all horizontal arrows of (9.18) are given by the horizontal arrows of (9.19):

$$
\begin{tikzcd}
 & f^{-1}(N) \arrow[rr] \arrow[dd] & & f(X) \wedge N \arrow[dd] \\
f^{-1}(L) \arrow[ur] \arrow[rr, "d"] \arrow[dd, "j"'] & & L \arrow[ur] \arrow[dd, "i" near start] & \\
 & X \arrow[rr] & & f(X) \\
f^{-1}(K) \arrow[ur, "f^{-1}(k)"'] \arrow[rr, "e"'] & & f(X) \wedge K \arrow[ur] &
\end{tikzcd}
\tag{9.19}
$$

Since $e \in \mathcal{E}$, one obtains $i \cdot d = e \cdot j \in \mathcal{E}^C$ (see Exercise 2.F(b)), hence $i \in \mathcal{E}^C$. With the distributivity of $\mathcal{M}/X$ the morphism $f^{-1}(k)$ is easily recognized as the join of the C-dense morphisms i and 1_L as in Exercise 2.F(d) and is therefore C-dense, as desired.

For (c), it suffices to consider $\mathcal{M}$-subobjects $m_i : M_i \to X$ $(i \in I)$ with $\bigvee_{i \in I} m_i \cong 1_X$, and $l : L \to X$ with $l \leq m_i$ such that the resulting morphism $l_i : L \to M_i$ belongs to $\mathcal{D}$, for all $i \in I$. But in the pullback diagrams

$$\begin{tikzcd} K \wedge L \arrow[r] \arrow[d, "l'_i"'] & L \arrow[d, "l_i"] \\ K \wedge M_i \arrow[r, "k_i"] \arrow[d, "m'_i"'] & M_i \arrow[d, "m_i"] \\ K \arrow[r, "k"] & X \end{tikzcd} \tag{9.20}$$

every k_i is C-closed, hence l'_i is C-dense. Frame distributivity gives

$$\bigvee_{i \in I} k \wedge m_i \cong k \wedge \bigvee_{i \in I} m_i \cong k,$$

hence $\bigvee_{i \in I} m'_i \cong 1_K$, and this implies that also $k^{-1}(l) : K \wedge L \to K$ is C-dense. □

Distributivity of the subobject lattices is an essential hypothesis of the Proposition; without it, total C-density may in fact not be describable as density w.r.t. a closure operator, as the following example shows. It gives the context in which the notion of total density was investigated first.

EXAMPLE (Tonolo [1995a]) In the category **TopAbGrp** of abelian topological groups, a subgroup $M \leq X$ is called totally dense if it is totally K-dense, with K the Kuratowski closure operator lifted from **Top** (see 5.9). Totally dense subgroups fail to satisfy the $\wedge$-$\vee$-preservation property, so that by Theorem* 5.4 there is no closure operator C whose C-dense subgroups are precisely the totally dense subgroups. In fact, with $\mathcal{I}_p$ the group of p-adic integers endowed with the p-adic topology and $\mathbb{Z}$ the subgroup of rational integers of $\mathcal{I}_p$, $M = \mathbb{Z} \times \mathbb{Z}$ fails to be totally dense in $X = \mathcal{I}_p \times \mathbb{Z}$, while its pullback $\mathbb{Z} \times 0 \to \mathcal{I}_p \times 0$ along the embedding $K = \mathcal{I}_p \times 0 \to X$ is totally dense (note that $K + M = X$ holds). In order to verify that M is not totally dense in X, pick $\xi \in \mathcal{I}_p$ such that $k\xi \notin \mathbb{Z}$ for each $0 \neq k \in \mathbb{Z}$. Then the cyclic subgroup L of Y generated by $(\xi, 1)$ is closed and $L \neq 0$, while $L \cap N = 0$, so that $L \cap N$ cannot be dense in L.

In case of its existence it would be desirable to have a handier description of the *total closure* C^{tot} than that given by the proof of Theorem* of 5.4. If "subobjects are given by points", as in Section 4.9, such a description is available, as we want to show next. Hence we consider a subclass $\mathcal{P} \subseteq \mathcal{M}$ of $\bigvee$-prime elements which is left-cancellable w.r.t. $\mathcal{M}$ and satisfies the following conditions:

(A) $f(p) \in \mathcal{P}/Y$ for every $f : X \to Y$ and $p \in \mathcal{P}/X$,

(B) $m \cong \bigvee\{p \in \mathcal{P}/X : p \leq m\}$ for every $m \in \mathcal{M}/X$,

(C) $p \cong q$ whenever $p \leq q$ in $\mathcal{P}/X$.

According to Remark 4.9(1), conditions (A), (B) give each subobject lattice the structure of a frame. Furthermore, using the left cancellability of $\mathcal{P}$ w.r.t. $\mathcal{M}$ and Exercise 1.K(b), these conditions also guarantee stability of $\mathcal{E}$ under pullback along $\mathcal{M}$-morphisms. Hence we are assured of the existence of C^{tot} for every closure operator C in this setting, by the Proposition.

For every $p : P \to X$ in $\mathcal{P}$, let $\hat{p} : \hat{P} \to X$ be the "point closure" $\hat{c}_X(p)$, with $\hat{C}$ the idempotent hull of C, and put

$$l(p,m,X) := \hat{p} \cdot c_{\hat{P}}(\hat{p}^{-1}(m)) : L(p,m,X) \to X.$$

Quite similarly to the $\mathcal{A}$-comodification of C as defined in 7.7, we now define

$$c^*_X(m) := \bigvee\{p \in \mathcal{P} : p \leq l(p,m,X)\}$$

for every $m \in \mathcal{M}/X, X \in \mathcal{X}$, and prove:

THEOREM *C^* is a closure operator with $C^* \leq C$ such that the C^*-dense subobjects are precisely the totally C-dense subobjects. Hence C^{tot} is the weakly hereditary core of C^*. If C is hereditary and if $c_X(p)$ is C-closed for all $p \in \mathcal{P}/X$, $X \in \mathcal{X}$, then C^* is weakly hereditary, and one has $C^* = C^{\mathrm{tot}}$.*

Proof For $m \in \mathcal{M}/X$, $p \in \mathcal{P}/X$ with $p \leq m$ one has

$$p \cong p \cdot p^{-1}(m) \leq \hat{p} \cdot \hat{p}^{-1}(m) \leq \hat{p} \cdot c_{\hat{P}}(\hat{p}^{-1}(m)) = l(p,m,X),$$

hence C^* is extensive. If $m \leq m'$, one sees immediately $l(p,m,X) \leq l(p,m',X)$, so that C^* is monotone. For a morphism $f : X \to Y$, from $p \leq l(p,m,X)$ one obtains

$$f(p) \leq f(l(p,m,X)) \leq l(f(p),f(m),Y),$$

due to C- and $\hat{C}$-continuity of f. Since $f(-)$ preserves arbitrary joins, this implies C^*-continuity of f. Hence C^* is a closure operator which, due to

$$l(p,m,X) \leq c_X(\hat{p} \cdot \hat{p}^{-1}(m)) \leq c_X(m),$$

satisfies $C^* \leq C$.

Let now $m \in \mathcal{M}/X$ be totally C-dense. Then, for every $p \in \mathcal{P}/X$, $\hat{p}^{-1}(m)$ is C-dense in $\hat{P}$, hence

$$l(p,m,X) \cong \hat{p} \cdot 1_{\hat{P}} = \hat{p} \geq p.$$

Consequently, $c^*_X(m) \cong 1_X$. Conversely, for m C^*-dense and every $k \in \mathcal{M}^C/X$, we must show that $k^{-1}(m)$ is C-dense. But for every $p \in \mathcal{P}/X$ with $p \leq k$ one has $\hat{p} \leq k$ and therefore

$$l(p,m,X) = \hat{p} \cdot c_{\hat{P}}(\hat{p}^{-1}(m)) \leq k \cdot c_K(k^{-1}(m));$$

furthermore, since $c_X^*(m) \cong 1_X$, from the $\bigvee$-primeness of p and conditions (B), (C) one has $p \le l(p, m, X)$, hence $p \le k \cdot c_K(k^{-1}(m))$. This gives

$$k \cong \bigvee\{p : p \le k\} \le k \cdot c_K(k^{-1}(m))$$

and then $1_K \cong c_K(k^{-1}(m))$, as desired.

Finally, let C be hereditary and let $c_X(p)$ be C-closed for all $p \in \mathcal{P}/X$, so that $\hat{p} \cong c_X(p)$. For $m \in \mathcal{M}/X$, let

$$y = c_X^*(m) : Y = c_X^*(M) \to X;$$

we must show that the morphism $m_Y : M \to Y$ with $y \cdot m_Y = m$ is C^*-dense, i.e., $c_X^*(m_Y) \cong 1_Y$. For that it will be enough to show the implication

$$p \le y \Rightarrow p_Y \le l(p_Y, m_Y, Y),$$

with $p_Y : P \to Y$ given by $y \cdot p_Y = p$, for all $p \in \mathcal{P}/X$, since then one has

$$1_Y \cong \bigvee\{p_Y : p \le y\} \le \bigvee\{p_Y : p_Y \le l(p_Y, m_Y, Y)\} \le c_Y^*(m_Y).$$

Hence assume $p \le y$ which, under conditions (B), (C), means $p \le l(p, m, X)$, and consider the following diagram:

$$\begin{tikzcd}
& \hat{P}_Y \arrow[rr, "y'"] \arrow[dd, "\hat{p}_Y"] & & \hat{P} \arrow[dd, "\hat{p}"] \\
\bullet \arrow[ur, "\hat{p}_Y^{-1}(m_Y)"] \arrow[rr, "t"] \arrow[dd] & & \bullet \arrow[ur, "\hat{p}^{-1}(m)"] \arrow[dd] & \\
& Y \arrow[rr, "y"] & & X \\
M \arrow[ur, "m_Y"] \arrow[rr, "1"'] & & M \arrow[ur, "m"'] &
\end{tikzcd} \tag{9.21}$$

Here the left and right faces are pullback by definition, and the bottom face is trivially a pullback. The back face is one by hereditariness of C:

$$\hat{p}_Y = c_Y(p_Y) \cong y^{-1}(c_X(p)) = y^{-1}(\hat{p}).$$

Hence also the front face is a pullback, with the induced morphism t being iso. Consequently, the top face of (9.21) is a pullback diagram. Applying hereditariness of C again we obtain

$$c_{\hat{P}_Y}(\hat{p}_Y^{-1}(m_Y)) \cong (y')^{-1}(c_{\hat{P}}(\hat{p}^{-1}(m))).$$

This means that also the top part of the following diagram is a pullback:

$$\begin{tikzcd} L(p_Y, m_Y, Y) \arrow[r] \arrow[d] & L(p, m, X) \arrow[d] \\ \hat{P}_Y \arrow[r] \arrow[d, "\hat{p}_Y"'] & \hat{P} \arrow[d, "\hat{p}"] \\ Y \arrow[r, "y"] & X \end{tikzcd} \tag{9.22}$$

Since the vertical composites are $l(p_Y, m_Y, Y)$ and $l(p, m, X)$, the implication

$$p \leq l(p, m, X) \;\Rightarrow\; p_Y \leq l(p_Y, m_Y, Y)$$

is now immediate, and this finishes the proof. □

REMARKS

(1) For applications of the Theorem to the category **Top**, see the Examples of 5.4, i.e., $C = K, b$, or σ, all of which satisfy the assumption of the Theorem: note that, although σ is not idempotent, σ-closures of points are σ-closed.

(2) Idempotency of C does not guarantee weak hereditariness of C^*, even in **Top**, as the idempotent hull of the θ-closure shows. For that topologize the set $X = \mathbb{R} \cup \{\infty\}$ by taking $\mathbb{R}$ to be open and basic neighbourhoods of ∞ to be of the form $\{\infty\} \cup U$, where U is an open dense subset of $\mathbb{R}$. Then for $M = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ one can easily show $\theta_X(\{\infty\}) = X$; consequently $\theta^*_X(M) = M \cup \{\infty\}$. But the last space is a topological sum of M and $\{\infty\}$, so that M is θ-closed in $\theta^*_X(M)$, hence not θ^∞-dense in $\theta^*_X(M)$.

(3) The notion of total density given in the Example can be extended to the category **TopGrp** of *all* topological groups, as total K-density, with K the Kuratowski closure operator lifted from **Top** (see 5.9). However, another extension turns out to be equally (if not more) relevant here: a subgroup $M \leq G \in$ **TopGrp** is called *weakly totally dense* if for every closed normal subgroup N of G the intersection $M \cap N$ is (K-) dense in N (see Exercise 9.M for the connection with the open mapping theorem for topolgical groups). It is natural to ask whether weak total density can be presented as total C-density w.r.t. some closure operator C. Since the C-closed subgroups must be the (K-) closed normal subgroups, the most natural candidates seem to be the closure operators $K \vee \nu \leq \nu K \leq K\nu$ which give as closed subgroups precisely the (K-) closed normal subgroups. It can be shown that none of these three closure operators does the job (see Exercise 9.M).

PROBLEM *Does there exist a closure operator C of the category* **TopGrp** *such that weak total density coincides with total C-density?*

Exercises

9.A (*Hereditariness of idempotent hulls*) For a hereditary preradical $\mathbf{r}$ of $\mathbf{Mod}_R$, show that all its copowers $\mathbf{r}_\alpha$ ($\alpha \in Ord$) are hereditary (cf. Exercise 4.G). Conclude that the idempotent hull of the maximal closure operator $C^{\mathbf{r}}$ is hereditary whenever $C^{\mathbf{r}}$ is hereditary itself. Does a corresponding statement hold for the minimal operator $C_{\mathbf{r}}$?

9.B (*Fully hereditary closure operators*) Call a closure operator C of an $\mathcal{M}$-complete category $\mathcal{X}$ (with $\mathcal{M}$ closed under composition) *fully hereditary* if every morphism in $\mathcal{X}$ is C-initial. Show that fully hereditary closure operators are hereditary, but not viceversa. Then show that every fully hereditary closure operator C has an *fully hereditary hull* C^{fh} which, if $\mathcal{X}$ has pushouts, can be constructed as

$$c_X^{\mathrm{fh}}(m) = \bigvee\{h^{-1}(c_Z(h(m))) : h : X \to Z\}$$

(cf. the construction of C^{he} in 4.10).

9.C (*C-open maps in* **Top**) Let C be a closure operator of **Top**. Prove for every $f : X \to Y$:

(a) f is C-open if and only if f maps a C-neighbourhood of $x \in X$ to a C-neighbourhood of $f(x)$ (cf. 7.9).

(b) If f is C-open, then f is also C^α-open for every $\alpha \in Ord \cup \{\infty\}$; in particular, f is C^∞-open.

(c) If f is bijective and σ^∞-open, then f is also σ-open. *Hint:* f is σ^∞-open if and only if $f : sX \to sY$ is open (with sX the sequential modification of X, i.e., with s the coreflector into the category of sequential spaces). Hence X and Y have the "same" converging sequences.

9.D (*Openness w.r.t. minimal and maximal closure operators*) For a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$, every surjective module homomorphism $f : X \to Y$ is $C^{\mathbf{r}}$-open; it is $C_{\mathbf{r}}$-open if and only if $f(\mathbf{r}(X)) = \mathbf{r}(Y)$.

9.E (*Modality of minimal closure operators*) Show that for a preradical $\mathbf{r}$ of $\mathbf{Mod}_R$, the following conditions are equivalent:

(i) $C_{\mathbf{r}}$ is modal,

(ii) for all $f : X \to Y$, $f^{-1}(\mathbf{r}(Y)) = \mathbf{r}(X)$,

(iii) $\mathbf{r}$ is hereditary and cohereditary (cf. Remark 5.12).

9.F (*Equivalence relations*) Check that an $\mathcal{M}$-relation $r : R \to X \times X$ is an equivalence relation in the finitely $\mathcal{M}$-complete category $\mathcal{X}$ with finite products if and only if for every $Z \in \mathcal{X}$

$$\mathcal{X}(X,r) : \mathcal{X}(Z,R) \to \mathcal{X}(Z,X \times X) \cong \mathcal{X}(Z,X) \times \mathcal{X}(Z,X)$$

is an equivalence relation in **Set**.

9.G ($\Delta(C)$*-reflection via prereflection*) Let C be any closure operator of the finitely $\mathcal{M}$-complete category $\mathcal{X}$ with finite products and coequalizers, and construct the pointed endofunctor (S,q) as in 9.4.

(a) Show $\mathrm{Fix}(S,q) = \Delta(C)$ (cf. 5.1).

(b) Conclude form (a) that $\Delta(C)$ is strongly epireflective in $\mathcal{X}$ whenever $\mathcal{X}$ is cowellpowered w.r.t. regular epimorphisms and has colimits of chains of regular epimorphisms. *Hint:* Iterate S.

(c) Describe the **Haus**-reflector of **Top**.

9.H (C*-initial sources and the finite structure property for products*) Call a source $\sigma = (f_i : X \to Y_i)_{i \in I}$ in $\mathcal{X}$ C*-initial* if $c_X(m) \cong \bigwedge_{i \in I} f_i^{-1}(c_{Y_i}(f_i(m)))$ for all $m \in \mathcal{M}/X$. If $\tau_i = (g_{ij} : Y_i \to Z_{ij})_{j \in J_i}$ are further sources ($i \in I$), the composite $(\tau_i \cdot \sigma)_{i \in I}$ is the source $(g_{ij} \cdot f_i : X \to Y_i)_{j \in J_i, i \in I}$. Show:

(a) If all τ_i are C-initial, then $(\tau_i \cdot \sigma)_{i \in I}$ is C-initial if and only if σ is C-initial.

(b) $C \cong T$ (the trivial closure operator of $\mathcal{X}$) if and only if every object of $\mathcal{X}$ (considered as an empty source) is C-initial.

(c) For any non-trivial closure operator C of **Top**, the only C-initial object is $\emptyset$.

(d) In **Top**, a source $(p_i : \prod_{j \in I} X_j \to X_i)_{i \in I}$ of product projections generally fails to be K-initial. What about **PrTop** and **CS** (cf. 5.10)?

(e) For $\mathcal{X}$ with direct products, the inverse-limit source

$$(p_F : \prod_{i \in I} X_i \to \prod_{i \in F} X_i)_{F \subseteq I \text{ finite}}$$

is C-initial if and only if C satisfies the finite structure property for products (cf. 4.11).

9.I (*Openness of product projections*) Call a closure operator C of a finitely $\mathcal{M}$-complete category $\mathcal{X}$ with finite products *semi-productive* if

$$c_{X \times Y}(m \times 1_Y) \cong c_X(m) \times 1_Y \qquad (*)$$

holds for all $X, Y \in \mathcal{X}$, $m \in \mathcal{M}/X$. Show:

(a) C is semi-productive if and only if every projection of a (finite) direct product is C-open.

(b) If C is idempotent, semi-productivity implies finite productivity of C.

(c) Every closure operator of a topological category over **Set** is semi-productive. *Hint*: For spaces X, Y and subspaces $M \subseteq X$ and $N \subseteq Y$, in order to prove the non-trivial inclusion in the formula $(*)$ note that for every $x \in c_X(M)$ and every $y \in Y$, one has $(x, y) \in c_{X\times\{y\}}(M \times \{y\}) \subseteq c_{X\times Y}(M \times Y)$.

(d) Find an example of a category $\mathcal{X}$ with a non-semi-productive closure operator.

9.J (*K-open morphisms in* **TopGrp**) Prove that K-open morphisms in **TopGrp**, unlike in **Top**, need not be open as maps between topological spaces. Conclude that the forgetful functor V : **TopGrp** $\to$ **Top** does not preserve K-openness of morphisms. (Here, for brevity, we denote each time by K the usual Kuratowski closure operator K of **Top** and its lifting from **Top** along V as in 5.9.) *Hint:* Note that if $\sigma \geq \tau$ are two distinct topologies on a group G, then the identity $1_G : (G, \sigma) \to (G, \tau)$ (as a morphism in **TopGrp**) is K-open if and only if both topologies have the same (K-) closed subgroups, while 1_G is K-open in **Top** if and only if both topologies coincide. We propose now two examples of a group G and two distinct topologies $\sigma > \tau$ on G with $1_G : (G, \sigma) \to (G, \tau)$ K-open. For the first one take $G = \mathbb{Z}$, σ the discrete topology of $\mathbb{Z}$ and τ the topology on G defined as in Exercise 8.U(b) with $m_n = n$ for each $n \in \mathbb{N}$. For the second one fix a prime p and take G to be the Prüfer group $\mathbb{Z}(p^\infty) := \mathbf{t}_p(\mathbb{T})$, with σ the discrete topology of $\mathbb{Z}(p^\infty)$ and τ the topology induced by $\mathbb{T}$. In both cases all subgroups of G are (K-) closed for both topologies.

9.K (*Total density vs essentiallity*) Call an $\mathcal{M}$-subobject $m : M \to X$ of a finitely $\mathcal{M}$-complete category $\mathcal{X}$ *essential* if for every morphism $f : X \to Y$ with $f \cdot m$ monic is a monomorphism. Call m *$\mathcal{E}$-essential* if the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $e \cdot m$ is essential for every $e \in \mathcal{E}$. Show that:

(a) A subgroup inclusion $M \hookrightarrow G$ in **HausGrp** is essential if and only if every proper closed normal subgroup of G non-trivially meets M.

(b) A (K-) dense subgroup M of an abelian group $G \in$ **HausGrp** is totally dense if and only if M is $\mathcal{E}$-essential.

9.L (*Weak total density and the open mapping theorem*) (cf. Dikranjan and Prodanov [1974]) A group $G \in$ **HausGrp** is said to satisfy the *open mapping theorem* if every morphism $G \to H$ in **HausGrp** is open. Prove that for a dense subgroup M of a group $G \in$ **HausGrp** the following two conditions are equivalent:

(a) M satisfies the open mapping theorem,

(b) G satisfies the open mapping theorem and M is weakly totally dense in G (cf. Remark 9.5(3)).

9.M (*Weak total density as total C-density*) Let C be one of the following three closure operators of **TopGrp**: $\nu \vee K$, νK and $K\nu$. Show:

(a) Weak total density implies total C-density in **TopGrp**.

(b) For the topological group G defined in Example 5.9 (4) total C-density does not imply weak total density. *Hint:* Note that G has no proper closed normal subgroups, so that a subgroup M of G is totally C-dense if and only if G is C-dense. Conclude that the stabilizer subgroup $M = \text{stab}(1)$ of G is totally C-dense. Since M is a proper (K-) closed subgroup of G, it is not weakly totally dense.

Notes

The notion of initial morphism (with respect to a closure operator) appears in Dikranjan [1992] while the notion of openness is intrinsic to the notion of Lawvere-Tierney topology (see Johnstone [1977]) which assumes every morphism to be open. Modal closure operators were investigated by Castellini, Koslowski and Strecker [1992b]; Theorem 9.3 is very much related to the work of Cassidy, Hébert and Kelly [1985]. The construction of Barr's reflector appears in Barr [1988] and has been used by various authors; see, for example, Carboni and Mantovani [1994]. The notion of total density appears for the first time in Soundararajan [1968] for abelian groups. The term weak total density was used in Dikranjan and Shakhmatov [1992], although the notion appeared much earlier (under the name total density) in Dikranjan and Prodanov [1974] (see Dikranjan, Prodanov and Stoyanov [1989] for further information). Total closure operators operators were constructed by Tonolo [1995b] for applications to topological groups.

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Index of Definitions

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Logical connections (see 2.4 - 2.7 and Exercises 3.M, 4.H):

$$\begin{array}{rcl}
\text{(ID)\&(CC)} \Longrightarrow \text{(WH)} \Longrightarrow \text{(CC)} & \text{and} & \text{(WH)\&(CD)} \Longrightarrow \text{(ID)} \Longrightarrow \text{(CD)} \\
\text{(CC)\&(CD)} \not\Longrightarrow \text{(ID)} & \text{and} & \text{(CC)\&(CD)} \not\Longrightarrow \text{(WH)} \\
\text{(HE)} \Longleftrightarrow \text{(WH)\&(LD)} & \text{and} & \text{(MI)} \Longleftrightarrow \text{(ID)\&(RC)} \\
\text{(MI)} \Longrightarrow \text{(FA)} & \Longleftrightarrow & \text{(AD) \& (DA)} \\
\text{(MI)} \not\Longrightarrow \text{(GR)} & \text{and} & \text{(DA)} \not\Longrightarrow \text{(AD)}\,.
\end{array}$$

Tables of Results

Table 1

Preservation of properties of closure operators under composition & cocomposition			
Property	Operation *comp.*	Operation *cocomp.*	Comments
Idempotency	–	+	Ex. 4.2(3); Prop. 4.3
Hereditariness	–	+	Ex. 4.B; Prop. 4.3
Productivity	+	+	Prop. 4.2; Prop. 4.3
Finite productivity	+	+	Prop. 4.2; Prop. 4.3
Regularity	?	–	+ for $\mathbf{Mod}_R$; consider $\mathrm{reg}^{\mathbf{Haus}}$
Weak hereditariness	+	–	Prop. 4.2; Ex. 4.3
Additivity	+	?[1]	Prop. 4.2;
Directed additivity	+	?	Prop. 4.2;
Full additivity	+	?[1]	Prop. 4.2;
Minimality	+	?[1]	Prop. 4.2;
Groundedness	+	+	Prop. 4.2; Prop. 4.3

1. We conjecture that for $\mathbf{Mod}_R$ "–" holds for minimality, in which case one example would work for all three cases (see Ex. 3.M(b)).

Table 2

Preservation of properties of closure operators under arbitrary meet & join			
Property	Operation *meet*	Operation *join*	Comments
Idempotency	+	$-^{f}$	Prop. 4.5; Ex. 4.2(3)
Hereditariness	+	$+^{f}$	Prop. 4.5; Ex. 4.D(b) (with $\mathcal{M}/X$ distributive)
Productivity	+	$-^{f}$	Prop. 4.5; Ex. 4.U, Ex. 4.6(1)
Finite productivity	+	$-^{f}$	Prop. 4.5; Ex. 4.U
Regularity	+	?	easy; + for $\mathbf{Mod}_R$
Weak hereditariness	$-^{f}$	+	Ex. 4.K; Prop. 4.5
Additivity	$-^{f}$	+	Ex. 4.3(2); Prop. 4.5
Directed additivity	+	+	(meets distribute over dir. joins in $\mathcal{M}/X$); Prop. 4.5
Full additivity	–	+	Ex. 5.5; Prop. 4.5
Minimality	–	+	Ex. 5.5; easy
Groundedness	+	+	Prop. 4.7 (non-empty $\bigwedge$); Prop. 4.5

"$+^{f}$" means that the answer is positive for finite meet or join.
"$-^{f}$" means that the answer is negative even for binary meet or join.

Table 3

Preservation of properties of closure operators by hulls and cores

	C^{∞}	C_{∞}	C^{he}	$C^{\min}$	C^{+}	$C^{\oplus}$	C^{G}	$\tilde{C}$	C^{reg}
Idempotency	+	+ Th.4.6	+ Th.4.10(a)	+ Th.*2.5	+ Le.4.8	+ Th.4.9	+ Cor.4.7	– [5]	+
Finite productivity	+ [1]	+ Th.4.6	+	? [4]	? [7]	? [4]	? [4]	? [7]	+ Ex.6.E
Productivity	– Ex.4.6	+ Th.4.6	+	? [4]	? [7]	? [8]	? [4]	? [7]	+ Ex.6.E
Hereditariness	– Ex.4.B	+ Th.4.6	+	+ [2] Ex.4.N	+ [3] Th.4.8	+ Th.4.9	+ Cor.4.7	– σ	– K^{reg}
Weak hereditariness	+ Th.4.6	+	+	+	? [7]	+ [9]	– $\nu = T^{G}$	– σ	– K^{reg}
Additivity	+ Th.4.6	– Ex.5.5	+ Th.4.10(b)	+	+	+	? [4]	– [5]	– [5]
Directed additivity	+ Th.4.6	– Ex.5.5	+ Th.4.10(b)	+	+	+	? [4]	– [5]	– [5]
Full additivity	+ Th.4.6	– Ex.5.5	+ Th.4.10(b)	+	+	+	? [4]	– [5]	– [5]
Minimality	+ Th.*2.5	– Ex.5.5	+ [2]	+	+	+ Rem.2.6	– T	– [5]	– [5]
Groundedness	+ Th.4.6	+ Th.4.6	+	+	+ Le.4.8	+ Th.4.9	+	+	– [6]

1 In the presence of “points” (cf. 4.9) which are prime w.r.t. directed joins; this condition is satisfied for both, topological categories over **Set** and for $\mathbf{Mod}_R$.

2 For $\mathcal{M}/X$ modular.

3 For $\mathcal{M}/X$ distributive.

4 + for topological categories over **Set** and for $\mathbf{Mod}_R$.

5 Consider the minimal closure operator C_{d} of **AbGrp** and Ex. 5.5.

6 $\tilde{G} = G^{\mathrm{reg}} = T$ is not grounded in **Top**.

7 + for $\mathbf{Mod}_R$ (Theorem 3.4).

8 + for topological categories over **Set**.

9 Under the conditions (A), (B) of 4.9.