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Facets of Descent, I

Dedicated to Nico Pumplün on the occasion of his sixtieth birthday

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Abstract. An elementary topological approach to Grothendieck's idea of descent is given. While being motivated by the idea of localization which is central in Sheaf Theory, we show how the theory of monads (= triples) provides a direct categorical approach to Descent Theory. Thanks to an important observation by Bénabou and Roubaud and by Beck, the monadic description covers descent also in the abstract context of a bifibred category satisfying the Beck–Chevalley condition. We present the fundamentals of fibrational descent theory without requiring any prior knowledge of fibred categories. The paper contains a number of new topological descent results as well as some new examples in the context of regular categories which demonstrate the subtlety of the descent problem in concrete situations.

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Introduction

Sheaf Theory is centred around the idea of defining functions *locally*. For instance, in order to obtain a continuous real-valued function f on an open subspace U of a topological space X, it is sufficient to have an open cover $(U_i)_{i\in I}$ of U and a continuous function f_i on each U_i such that the *matching condition*

$$f_i(x) = f_j(x) \text{ for all } x \in U_i \cap U_j \text{ and } i, j \in I$$

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holds; f is just the "collation" of the functions f_i. In fact, each f in the set $C(U)$ is obtained this way, by collating its restrictions $f|_{U_i}$, $i \in I$. In other words, the diagram

$$(*) \qquad C(U) \to \prod_{i\in I} C(U_i) \rightrightarrows \prod_{i,j} C(U_i \cap U_j)$$

is an equalizer diagram in **Set**, with all three arrows given by restriction: $f \mapsto (f|_{U_i})_{i\in I}$, $(f_i)_{i\in I} \mapsto (f_i|_{U_i\cap U_j})_{i,j}, (f_j|_{U_i\cap U_j})_{i,j}$. This describes the functor

$$C : \mathcal{O}(X)^{op} \to \mathbf{Set}$$

as a *sheaf* (of sets), with $\mathcal{O}(X)$ the category (in fact: partially ordered set) of open subsets of X.

By axiomatizing the notion of cover Grothendieck was led to bold generalizations of the notion of sheaf which, however, are still describable by an exactness condition as in $(*)$. The book [24] by Mac Lane and Moerdijk gives a thorough and well-motivated introduction to this subject and its applications in topos theory. We refer the interested reader also to Gray's historical summary [12] on the development of sheaf theory which he describes as "an octopus spreading itself throughout everyone else's history".

Descent Theory was developed by Grothendieck [13], [14] in the abstract context of fibred categories (see also Demazure [8] and Giraud [10]). However, in order to understand its connection with Sheaf Theory, very little knowledge of the latter subject is needed. In fact, we may go back to topological spaces and ask whether the sheaf condition $(*)$ may be helpful in order to investigate, for example, bundles over a "base" space B (i.e., continuous maps into B), rather than real-valued maps defined on B. Hence the *set* (or group, ring, $\mathbb{R}$-algebra) $C(B)$ gets traded for the "sliced" *category* **Top**/B. Are we able then to describe a space $(A, \alpha : A \to B)$ over B when we are given an open cover $(U_i)_{i\in I}$ of B and spaces $(E_i, \gamma_i : E_i \to U_i)$ over each U_i, analogously to $(*)$? Restriction should obviously be replaced by "pulling back": $(A, \alpha) \mapsto (\alpha^{-1}(U_i), \alpha^{-1}(U_i) \to U_i)$. Hence we have a diagram of functors

$$(**) \qquad \mathbf{Top}/B \to \prod_{i\in I} \mathbf{Top}/U_i \rightrightarrows \prod_{i,j} \mathbf{Top}/U_i \cap U_j$$

but we must be careful about its "commutativity", that is: about the matching condition. Since pullbacks are determined only up to isomorphism, "matching" should mean "matching up to isomorphism". Moreover, the isomorphisms should satisfy some obvious coherence conditions in order to make the "collation" work. Hence, when given the spaces (E_i, γ_i) over U_i, $i \in I$, from which we should build a space (A, α) over B whose pullbacks along $U_i \hookrightarrow B$ give (E_i, γ_i), we should also be supplied with homeomorphisms $\xi_{i,j} : \gamma_i^{-1}(U_i \cap U_j) \to \gamma_j^{-1}(U_i \cap U_j)$ over $U_i \cap U_j$ such that $\xi_{i,i} =$id and

$$(***) \qquad \xi^i_{j,k}\xi^k_{i,j} = \xi^j_{i,k}$$

holds for all $i, j, k \in I$, with $\xi_{i,j}^k : \gamma_i^{-1}(U_i \cap U_j \cap U_k) \to \gamma_j^{-1}(U_i \cap U_j \cap U_k)$ denoting the restriction of $\xi_{i,j}$. Such a system of homeomorphisms forms the so-called *descent data* which have to be carried with the family $(E_i, \gamma_i)_{i \in I}$. An easy application of the results of Sections 1 and 3 of this paper shows that the category **Top**/B is in fact equivalent to the category whose objects are the objects of $\prod_{i \in I} \mathbf{Top}/U_i$, together with given descent data. (Condition $(***)$ is just an instance of the commutative diagram (18) below, with the canonical isomorphisms being identity morphisms.)

Condition $(***)$ is known as the *cocycle condition* in *Cohomology Theory* and therefore appears to be very natural. But in fact, no background in Sheaf- or Cohomology Theory is needed in order to recognize its naturality. It is the goal of this paper to show that Descent Theory gives an immediate access to the fundamental idea of localization, mostly based on the fairly easy categorical *theory of monads* (= triples), a theory that was not yet in place when Grothendieck developed descent but that is well-described already in Mac Lane's book [23]. This may be explained in terms of the situation $(**)$, as follows. Consider the induced map $p : E \to B$ of the topological sum $E = \coprod_{i \in I} U_i$ which is the identity map on each summand. The category $\prod_{i \in I} \mathbf{Top}/U_i$ is equivalent to **Top**/E, and we have the pullback functor

$$p^* : \mathbf{Top}/B \to \mathbf{Top}/E.$$

Descent Theory now asks, whether the objects of **Top**/B can be presented algebraically in terms of objects of **Top**/E, more precisely: whether the functor p^* is monadic, so that **Top**/B is essentially the category of Eilenberg–Moore algebras over **Top**/E. In this setting, descent data are nothing but algebra structures, and the cocycle condition $(***)$ becomes just the *associative law* for an algebra over the monad induced by p^*. (It may also be interpreted as a *functoriality condition*, a point that is already apparent in Grothendieck's work and that will be made precise in [17] where we investigate descent in terms of internal categories and equivalence relations.)

We now see that the problem of descent appears in a much broader context: we may consider *any* morphism $p : E \to B$ in *any* category $\mathcal{C}$ with pullbacks and try to characterize those morphisms p for which $p^* : \mathcal{C}/B \to \mathcal{C}/E$ is monadic. In fact, one may replace the categories $\mathcal{C}/B$ by arbitrary categories $\mathcal{E}(B)$ of "structures over B", given by a *fibred category $\mathcal{E}$ over $\mathcal{C}$* (cf. [11], [4]) or by a *$\mathcal{C}$-indexed category* (cf. [28], [25]), and still define the *category* $\mathrm{Des}_{\mathcal{E}}(p)$ *of descent data relataive to* p and a comparison functor

$$\Phi^p : \mathcal{E}(B) \to \mathrm{Des}_{\mathcal{E}}(p)$$

which, under suitable conditions, is just the comparison functor into the Eilenberg–Moore category associated with p^*. This latter point was the main discovery of Bénabou and Roubaud [5] and of Beck (unpublished). More precisely, they showed for a *bifibred* category $\mathcal{E}$ over a category $\mathcal{C}$ with pullbacks which satisfies the so-called *Beck–Chevalley condition*, that the classically defined category $\mathrm{Des}_{\mathcal{E}}(p)$ is the category of algebras for the monad induced by p^*, so that p is *effective for descent* (i.e., Φ^p is an equivalence of categories) if and only if p^* is monadic. It is somewhat surprising that this important observation did not lead to substantially increased interest in Descent Theory among category theorists at that time.

However, after Joyal and Tierney [21] published their "Extension of the Galois theory of Grothendieck" in 1984, the idea of descent was taken up by many category- and topos theorists (see, in particular, Moerdijk [27] and Makkai [26]). Joyal and Tierney showed that the algebraic machinery leading to Grothendieck's descent theorem for modules (which, roughly, characterizes the morphisms $f : A \to B$ of commutative rings with the property that every B-module M has the form $M \cong B \otimes_A N$ with an A-module N) can be established quite analogously for "pointless spaces", i.e., for locales (see [20]). One of the main results in [21] says that open surjections of locales are effective for descent of sheaves, and they extend this even from locales to topoi.

In the presence of a powerful localic descent theorem it seemed natural to the authors to ask for a descent theorem for (ordinary) topological spaces, but our paper [16] appears to be the first to do so (although the term "descent" did not appear explicitly in that paper). We showed that all locally sectionable (and therefore all local homeomorphisms, in particular the maps $p : \coprod_{i\in I} U_i \to B$ considered above) are effective for descent. Moerdijk [27], as a prologue to his topos-theoretic investigations, provided sufficient conditions for effective descent in arbitrary categories which can be used to show that open surjections and proper maps of topological spaces are effective for descent (see also [34], [22]; effectiveness of proper maps has meanwhile been established constructively by J. Vermeulen [37] also for locales). A complete characterization of effective descent maps of topological spaces was given only recently by Reiterman and Tholen [33].

Instead of following the historical development, we begin this article by presenting an elementary account of topological descent theory, without using any advanced categorical methods. However, we generalize our previous accounts, by working with subcategories $\mathbb{E}(B)$ of bundles over B given by a suitable class $\mathbb{E}$ of continuous maps (i.e. with a subfibration of the basic fibration given by **Top**/B), so that when $\mathbb{E}$ is the class of local homeomorphisms, $\mathbb{E}(B)$ is actually the category of **Set**-valued sheaves on B. In analogy to Sheaf Theory we describe concretely the adjunction

$$\Psi^p \dashv \Phi^p : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p)$$

which determines when p is an (effective) descent map, i.e., when Φ^p is full and faithful (an equivalence of categories, respectively), see Theorem 1.10.

In Section 2 we trade the category **Top** for an arbitrary category $\mathcal{C}$ with pullbacks and provide various $\mathbb{E}$-modifications of criteria given previously for the basic fibration only (see, in particular, [16], [33], [34]). Effectiveness of certain equivalence relations associated with descent data plays a crucial role here. This leads to a number of user-friendly descent criteria, in particular in the realm of exact categories and of locally cartesian closed categories. We provide new examples of regular epimorphisms in regular categories (in the sense of Barr [2]) which fail to be effective, something that cannot happen in exact categories (see 2.7). We also consider the case when the left-adjoint $p_!$ is *not* given by composition with p (i.e., when the given subfibration is not a sub*bi*fibration of the basic fibration).

This last aspect gives additional justification for presenting the descent problem for *arbitrary* fibred categories, as Pavlović did in [30]. His very general descent theorem is recorded here (see 3.6; no prior knowledge of fibred categories is assumed as all needed notions and facts are being presented here). In case of a *bi*fibration satisfying the Beck–Chevalley condition, one may go back to the monadic machinery, thanks to the Bénabou-Roubaud and Beck result. We show that for the subfibration of the basic fibration as given by the class $\mathbb{E}$ belonging to a factorization system $(\mathbb{D}, \mathbb{E})$, the Beck–Chevalley condition is satisfied if and only if $\mathbb{D}$ is stable under pullback (see 3.8) and thereby obtain an easy (effective) descent theorem (see 3.9) under this condition. In case $\mathbb{D}$ fails to be stable under pullback, however, the category $\mathrm{Des}_{\mathbb{E}}(p)$ may be very far from the Eilenberg–Moore category given by $p_! \dashv p^*$, as we show in 3.10.

Since fibrations (over $\mathcal{C}$) are essentially equivalent to pseudo-functors $\mathcal{C}^{op} \to \mathbf{CAT}$, it is clear that the descent problem can be treated essentially equivalently in the context of indexed categories (see [28]). This aspect will be discussed in [17], but we may refer the interested reader also to the paper [6] by Bunge and Paré for further studies in this regard.

The last section of this paper is devoted to the investigation of (effective) $\mathbb{E}$-descent maps of **Top** when $\mathbb{E}$ is the class of all continuous maps ("*global-descent*"), of open-subspace embeddings ("*open-descent*"), or of local homeomorphisms ("*étale-descent*"). For the first choice of $\mathbb{E}$, we record the characterization given in [33], while for the second and third choice of $\mathbb{E}$ new criteria are being presented here. We give a complete characterization of étale-descent maps and show that effective global-descent maps are effective étale-descent maps (see 4.7), but do not have a complete characterization of the latter type of maps.

This paper's successor [17] will show the use of *internal category theory* for the descent problem and draw further connections with Sheaf Theory. For *2-dimensional and lax descent* we refer the reader to Makkai's recent Memoir [26] and the references given therein, and to [38].

1. An Elementary Account of Topological Descent Theory

1.1. The general aim of topological descent theory is to study the objects and morphisms of the category

$$\mathbb{E}(B)$$

of $\mathbb{E}$-*bundles over* B. Here B is a fixed topological ("base") space, $\mathbb{E}$ is a class of continuous functions which is assumed to be closed under composition with homeomorphisms (for instance the class of all "*étale*" maps, that is: all local homeomorphisms), and $\mathbb{E}(B)$ is the full subcategory of **Top**/B whose objects belong to $\mathbb{E}$. Hence objects of $\mathbb{E}(B)$ are pairs (A, α) with $\alpha : A \to B$ in $\mathbb{E}$, and a morphism $f : (A, \alpha) \to (A', \alpha')$ in $\mathbb{E}(B)$ is given by a commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & A' \arrow[dl, "\alpha'"] \\ & B & \end{tikzcd} \tag{1}$$

of continuous maps. The idea is now to replace B by some ("extension") space E over B and try to describe $\mathbb{E}$-bundles over B and their morphisms by $\mathbb{E}$-bundles over E which come equipped with an additional algebraic structure, so-called descent data, as defined below. (For instance, when one has an open cover $(U_i)_{i\in I}$ of B, it may be advantageous to "spread" the cover and consider the topological sum $E = \coprod_{i\in I} U_i$, as a bundle over B.) Whether this procedure can be successful obviously depends on the choice of E and the map $p : E \to B$ which makes E a bundle over B.

1.2. Let $p : E \to B$ be a continuous map, and let (C, γ) be an $\mathbb{E}$-bundle over E. The *fibred product*

$$E \times_B C = \{(x, c) | p(x) = p\gamma(c)\}$$

is considered a subspace of the topological product $E \times C$. We may think of $E \times_B C$ as the join of the fibres $\gamma^{-1}y$ $(y \in E)$, each of which gets embedded into $E \times_B C$ along every $x \in p^{-1}p(y)$; more precisely, for all points $x, y \in E$ with $p(x) = p(y)$ one has a canonical embedding

$$j_{x,y} : \gamma^{-1}y \to E \times_B C, \ \ c \mapsto (x, c),$$

and $E \times_B C$ is the union of the subspaces $j_{x,y}(\gamma^{-1}y)$.

Descent data for (C, γ) (*relative to* p) are given by a family of maps

$$\xi_{x,y} : \gamma^{-1}x \to \gamma^{-1}y \ \ (x, y \in E, \ p(x) = p(y))$$

such that the *functoriality* and the *glueing* conditions hold:

(F) $\xi_{x,x} = id$ and $\xi_{x,z} = \xi_{y,z}\xi_{x,y}$ for all x, y, z of the same p-fibre;

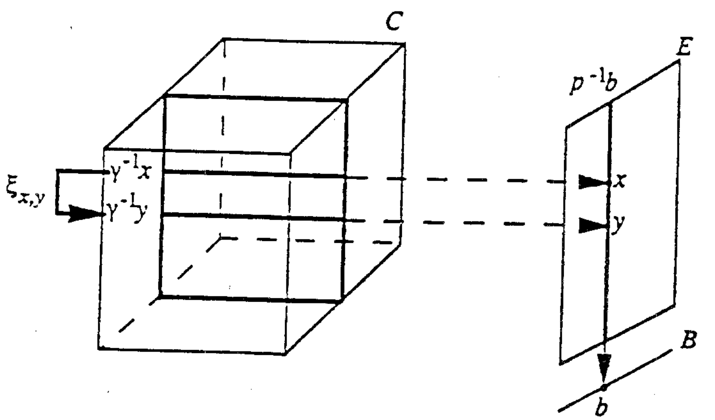
(G) the unique map $\overline{\xi}$ which makes all diagrams

$$\begin{tikzcd} \gamma^{-1}x \arrow[r, "\xi_{x,y}"] \arrow[d, "j_{y,x}"'] & \gamma^{-1}y \arrow[d, "j_{x,y}"] \\ E \times_B C \arrow[r, "\overline{\xi}"] & E \times_B C \end{tikzcd} \tag{2}$$

$(x, y) \in E, p(x) = p(y)$ commute is continuous; explicitly,

$$\overline{\xi}(y, c) = (x, \xi_{x,y}(c)) \text{ with } x = \gamma(c).$$

We note that each $\xi_{x,y}$ is a homeomorphism (since $\xi_{y,x}\xi_{x,y} = id$), hence also $\overline{\xi}$ is a homeomorphism (with $\overline{\xi}^{-1} = \overline{\xi}$).



(3)

If p is the induced map $E = \coprod_{i \in I} U_i \to B$ for an open cover $(U_i)_{i \in I}$ of B (see the Introduction), so that $E = \{(b, i) | b \in U_i\}$ with $p(b, i) = b$, then descent data for a space (C, γ) over E are given by maps

$$\xi_{(b,i),(b,j)} : \gamma^{-1}(b, i) \to \gamma^{-1}(b, j) \ (b \in U_i \cap U_j)$$

which, when "glued along b", give maps

$$\xi_{i,j} : \gamma_i^{-1}(U_i \cap U_j) \to \gamma_j^{-1}(U_i \cap U_j) \ (i, j \in I);$$

here $\gamma_i : \gamma^{-1}(U_i) \to U_i$ is the restriction of γ, with U_i considered a subspace of E. Condition (G) makes sure that each ξ_{ij} is continuous (while the second part of) (F) is equivalent to the *cocycle condition* mentioned in the Introduction.

1.3. If $\mathbb{E}$ is *stable under pullback along* $p : E \to B$, then every $\mathbb{E}$-bundle (A, α) over B induces the $\mathbb{E}$-bundle

$$p^*(A, \alpha) := (E \times_B A, \pi_1)$$

over E, by pulling back along p:

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"] & B \end{tikzcd} \tag{4}$$

(The stability requirement precisely means that with α in $\mathbb{E}$ also the projection π_1 belongs to $\mathbb{E}$.) The bundle $p^*(A, \alpha)$ comes equipped with *canonical descent data*:

$$\varphi_{x,y} : \pi_1^{-1}x \to \pi_1^{-1}y, \ (x, a) \mapsto (y, a);$$

hence $\overline{\varphi} : E \times_B (E \times_B A) \to E \times_B (E \times_B A)$ is the involution $(y, (x, a)) \mapsto (x, (y, a))$.

1.4. $\mathbb{E}$-bundles over E equipped with descent data form the objects $(C, \gamma; \overline{\xi})$ of the category

$$\mathrm{Des}_{\mathbb{E}}(p).$$

A morphism $h : (C, \gamma; \overline{\xi}) \to (C', \gamma'; \overline{\xi}')$ is a morphism $h : (C, \gamma) \to (C', \gamma')$ of $\mathbb{E}(E)$ which is *compatible with descent data*:

$$h(\xi_{x,y}(c)) = \xi'_{x,y}(h(c))$$

for all x, y in the same p-fibre and $c \in \gamma^{-1}x$; equivalently,

$$\overline{h}\,\overline{\xi} = \overline{\xi}'\,\overline{h}$$

with $\overline{h} = (1_E \times_B h) : E \times_B C \to E \times_B C', \ (y, c) \mapsto (y, h(c))$.

For $\mathbb{E}$ stable under pullback along p, one has the "pullback functor"

$$p^* : \mathbb{E}(B) \to \mathbb{E}(E)$$

which sends a morphism f to $1_E \times_B f$. It can be "lifted" to the *comparison functor*

$$\Phi^p : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p), \ (A, \alpha) \mapsto (E \times_B A, \pi_1; \overline{\varphi})$$

(since $1_E \times_B f$ is compatible with the canonical descent data as defined in 1.3). Hence Φ^p is a functor over $\mathbb{E}(E)$:

$$\begin{tikzcd}
\mathbb{E}(B) \arrow[rr, "\Phi^p"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathbb{E}}(p) \arrow[dl, "U^p"] \\
& \mathbb{E}(E) &
\end{tikzcd} \tag{5}$$

(with U^p the obvious forgetful functor).

1.5. DEFINITION. One says that p is an $\mathbb{E}$-*descent map* iff Φ^p is full and faithful, and it is an *effective* $\mathbb{E}$-*descent map* iff Φ^p is an equivalence of categories.

Hence for an $\mathbb{E}$-descent map p, morphisms $f : (A, \alpha) \to (A', \alpha')$ of $\mathbb{E}(B)$ are completely described by morphisms $h : (E \times_B A, \pi_1) \to (E \times_B A', \pi_1')$ which are compatible with the canonical descent data, so that h is necessarily of the form $h = 1 \times_B f$. For an effective $\mathbb{E}$-descent map p one knows in addition that, up to isomorphism, every $\mathrm{Des}_{\mathbb{E}}(p)$-object $(C, \gamma; \xi)$ is of the form $(E \times_B A, \pi_1; \overline{\varphi})$. In what follows we give some elementary sufficient (and partly necessary) conditions for p to be an $\mathbb{E}$-descent map or even an effective $\mathbb{E}$-descent map.

1.6. The map $p : E \to B$ is called an $\mathbb{E}$-*universal quotient map* if for every pullback diagram (4) with $\alpha \in \mathbb{E}$, the map π_2 is a quotient map. (In particular, when $1_B \in \mathbb{E}$, p must be a quotient map in this case.) We say that $\mathbb{E}$ is *transferable along* p if, for every pullback diagram (4) with $\pi_1 \in \mathbb{E}$ and π_2 a quotient map, one has $\alpha \in \mathbb{E}$. Certainly, for $p \in \mathbb{E}$, the latter condition holds if $\mathbb{E}$ is closed under composition and satisfies the right-cancellation property ($\alpha q \in \mathbb{E}$ with q quotient map $\Rightarrow \alpha \in \mathbb{E}$). Now we are ready to prove:

PROPOSITION. *Let* $\mathbb{E}$ *be stable under pullback along p. Then, for the statements*

(i) p *is an* $\mathbb{E}$-*universal quotient map*,

(ii) p *is an* $\mathbb{E}$-*descent map*,

(iii) *a morphism f in* $\mathbb{E}(B)$ *is an isomorphism if* $1_E \times_B f$ *is an isomorphism in* $\mathbb{E}(E)$,

one has the implications (i) $\Rightarrow$(ii)$\Rightarrow$(iii), *and all are equivalent if* $\mathbb{E}$ *is transferable along p.*

Proof. (i) $\Rightarrow$ (ii) For a morphism $h : \Phi^p(A, \alpha) \to \Phi^p(A', \alpha')$ we can define $f : (A, \alpha) \to (A', \alpha')$ by putting

$$f(a) := \pi_2' h(x, a)$$

for every $a \in A$ and any $x \in p^{-1}(\alpha(a))$. This definition is independent of the choice of x since h is compatible with the canonical descent data. Since, by hypothesis,

π_2 is a quotient map and $f\pi_2 = \pi_2'h$ is continuous, also f is continuous, in fact a map in $\mathbb{E}(B)$ with $1_E \times_B f = h$, as required.

(ii) $\Rightarrow$ (iii). In (5), U^p reflects isomorphisms. By hypothesis, Φ^p is full and faithful and therefore reflects isomorphisms. Hence p^* reflects isomorphisms.

(iii) $\Rightarrow$ (i). We consider the pullback diagram (4) with $\alpha \in \mathbb{E}$. Let A' be the space obtained by providing the set $\pi_2(E \times_B A)$ with the quotient topology with respect to the restricted projection $\pi_2' : E \times_B A \to A'$. The inclusion map $f : A' \to A$ and the restriction $\alpha' = \alpha f$ are continuous, and one easily checks that the fibred products $E \times_B A$ and $E \times_B A'$ coincide; more precisely,

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2'"] \arrow[d, "\pi_1"'] & A' \arrow[d, "\alpha'"] \\ E \arrow[r, "p"] & B \end{tikzcd} \tag{6}$$

is a pullback diagram with π_2' a quotient map. From the assumptions on $\mathbb{E}$ we conclude ($\alpha \in \mathbb{E} \Rightarrow \pi_1 \in \mathbb{E} \Rightarrow \alpha' \in \mathbb{E}$). Hence $f : (A, \alpha) \to (A', \alpha')$ is a morphism in $\mathbb{E}(B)$, and since $1_E \times_B f = 1_{E\times_B A}$ is an isomorphism, (iii) yields that f is an isomorphism, in fact: $f = 1_A$. Consequently, $\pi_2 = \pi_2'$ is a quotient map. □

1.7. In case $\mathbb{E}$ is the class all continuous maps, one usually omits the prefix $\mathbb{E}$. For emphasis, we sometimes speak of *(effective) global descent* then. In this case $\mathbb{E}$ is trivially transferable along every map p. Hence the Proposition gives that the descent maps are exactly the universal quotient maps (cf. [16], Theorem 1.1). Universal quotient maps were characterized by Day and Kelly [7] as the maps $p : E \to B$ with the property that for every family $(D_i)_{i\in I}$ of open sets E covering a fibre $p^{-1}b$ there are finitely many $i_1, \ldots, i_n \in I$ with $b \in \mathrm{int} p(D_{i_1} \cup \cdots \cup D_{i_n})$; equivalently, for every adherence point b of a filter $\mathcal{F}$ in B there is an adherence point x of the filterbase $p^{-1}\mathcal{F}$ belonging to $p^{-1}b$ (cf. Reiterman and Tholen [33], 1.6).

1.8. We note that for two classes $\mathbb{E}_0 \subseteq \mathbb{E}_1$ (both stable under pullback along p and under composition with homeomorphisms), we have the implication

$$p\ \mathbb{E}_1\text{-descent} \quad\Rightarrow\quad p\ \mathbb{E}_0\text{-descent},$$

since $\Phi_0^p : \mathbb{E}_0(B) \to \mathrm{Des}_{\mathbb{E}_0}(p)$ is just a restriction of Φ_1^p. Consequently, 1.6 and 1.7 give:

COROLLARY. *Every universal quotient map is an $\mathbb{E}$-descent map.* □

Specific classes $\mathbb{E}$ are being discussed in 3.10 and, more intensively, in Section 4.

1.9. In order to find criteria for effective $\mathbb{E}$-descent, we try to construct a left adjoint

$$\Psi^p : \mathrm{Des}_{\mathbb{E}}(p) \to \mathbb{E}(B)$$

of the comparison functor Φ^p. Guided by the mapping behaviour of the canonical descent data (cf. 1.4), for an arbitrary object $(C, \gamma; \bar{\xi}) \in \mathrm{Des}_{\mathbb{E}}(p)$ we consider the equivalence relation $\tilde{\xi}$ on C given by

$$c\tilde{\xi}d \Leftrightarrow p\gamma(c) = p\gamma(d) \text{ and } d = \xi_{\gamma(c),\gamma(d)}(c).$$

(The functoriality condition (F) gives that $\tilde{\xi}$ is in fact an equivalence relation.) By definition of $\tilde{\xi}$, the map $p\gamma$ factors through the quotient map $q : C \to C/\tilde{\xi}$. Hence we have a commutative diagram

$$\begin{tikzcd} C \arrow[r, "q"] \arrow[d, "\gamma"'] & C/\tilde{\xi} \arrow[d, "\delta"] \\ E \arrow[r, "p"] & B \end{tikzcd} \tag{7}$$

Let us call $\mathbb{E}$ *descent stable w.r.t. p* if $\mathbb{E}$ is stable under pullback along p and, for all $(C, \gamma; \bar{\xi})$ in $\mathrm{Des}_{\mathbb{E}}(p)$, the map δ belongs to $\mathbb{E}$ again.

The assignment

$$(C, \gamma; \bar{\xi}) \mapsto (C/\tilde{\xi}, \delta)$$

then yields the object-part of the functor Ψ^p. There is a continuous map

$$\eta = \langle \gamma, q \rangle : C \to E \times_B (C/\tilde{\xi})$$

which serves as the unit of the adjunction; η is a bijective map, with the **Set**-inverse given by $((y, q(c)) \mapsto \xi_{\gamma(c),y}(c))$. Hence, at the **Set**-level, one has the commutative diagram

$$\begin{tikzcd} E \times_B C \arrow[r, "1_E \times_B q"] \arrow[d, "\bar{\xi}"'] & E \times_B (C/\tilde{\xi}) \arrow[d, "\eta^{-1}"] \\ E \times_B C \arrow[r, "\pi_2"] & C \end{tikzcd} \tag{8}$$

Using the glueing condition (G) we conclude that η^{-1} is a continuous map if $1_E \times_B q$

is a quotient map. This last condition is actually necessary for the continuity of η^{-1} (that is: for η being a homeomorphism) since the map

$$\xi = \pi_2\overline{\xi} : E \times_B C \to C$$

is *always* a quotient map. In fact, ξ is a retraction since

$$\xi\langle\gamma, 1_C\rangle = 1_C.$$

It is interesting to note that, for every $(A, \alpha) \in \mathbb{E}(B)$, the co-unit

$$\varepsilon : (E \times_B A)/\tilde{\varphi} \to A, \quad q(x, a) \mapsto a,$$

is, like η, a continuous *bijective* map, provided p is surjective. In fact, the equivalence relation $\tilde{\varphi}$ (induced by the canonical descent data $\overline{\varphi}$) is given by

$$(x, a)\tilde{\varphi}(y, a') \Leftrightarrow p(x) = p(y) \text{ and } a = a',$$

hence, given $a \in A$, any $x, y \in E$ with $p(x) = \alpha(a) = p(y)$ yield the same $\tilde{\varphi}$-equivalence class $q(x, a) = q(y, a)$.

1.10. Up to filling in the details of the adjoint machinery we have proved:

THEOREM. *Let $\mathbb{E}$ be descent stable w.r.t. $p : E \to B$. Then there is a pair of adjoint functors*

$$\Psi^p \dashv \Phi^p : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p)$$

whose units η are continuous bijections; this is also true for the co-units ε, provided p is a surjective map. Furthermore, in this case

(1) *p is an $\mathbb{E}$-descent map if and only if for every $(A, \alpha) \in \mathbb{E}(B)$ the map ε (as defined above) is open; and*

(2) *if p is an $\mathbb{E}$-descent map, then the category $\mathbb{E}(B)$ is equivalent to the full subcategory of those $(C, \gamma; \xi) \in \mathrm{Des}_{\mathbb{E}}(p)$, for which the map*

$$1 \times_B q : E \times_B C \to E \times_B (C/\tilde{\xi}), \ (x, c) \mapsto (x, q(c)),$$

(with q as in diagram (7)) *is a quotient map; consequently,*

(3) *p is an effective $\mathbb{E}$-descent map if and only if p is an $\mathbb{E}$-descent map and the quotient condition of* (2) *holds for all objects of* $\mathrm{Des}_{\mathbb{E}}(p)$. □

We remark that $\mathbb{E}$ is descent stable w.r.t. p if $\mathbb{E}$ is stable under pullback along p, if $\mathbb{E}$ is transferable along p, and if $\mathbb{E}$ is *right cancellable with respect to epimorphisms* (i.e., $\gamma = \gamma'\eta \in \mathbb{E}$ with η epic implies $\gamma' \in \mathbb{E}$).

Clearly, the conditions of the Theorem may not be easy to check, even for easy classes $\mathbb{E}$. For instance, effective descent maps (with respect to the class of all continuous maps) are quite difficult to describe. A better description of the abstract structure at hand as given in the following sections turns out to be helpful and is being applied in concrete cases in [33] and in Section 4 of this paper.

2. A First Generalization: Monadic Descent Theory

2.1. We now work in an arbitrary category $\mathcal{C}$ with pullbacks and consider a class $\mathbb{E}$ of morphisms which is closed under composition with isomorphims. For an object B in $\mathcal{C}$, $\mathbb{E}(B)$ is the full subcategory of the sliced category $\mathcal{C}/B$ with objects in $\mathbb{E}$. Let $p : E \to B$ be a morphism such that $\mathbb{E}$ is stable under pullback along p and under composition with p from the left. There is then a pair of adjoint functors

$$p_! \dashv p^* : \mathbb{E}(B) \to \mathbb{E}(E)$$

(with p^* pulling back along p and with $p_!$ composing with p from the left), hence there is an induced monad T^p on $\mathbb{E}(E)$. An Eilenberg–Moore algebra structure on an object $(C, \gamma) \in \mathbb{E}(E)$ w.r.t. T^p is given explicitly by a $\mathcal{C}$-morphism

$$\xi : E \times_B C \to C$$

such that the diagrams

$$\begin{tikzcd} C \arrow[rr, "\langle \gamma, 1_C \rangle"] \arrow[dd, "1_C"'] & & E \times_B C \arrow[dd, "\pi_1"] \arrow[ddll, "\xi"'] \\ & \text{②} \quad \text{①} & \\ C \arrow[rr, "\gamma"'] & & E \end{tikzcd} \tag{9}$$

$$\begin{tikzcd} E \times_B (E \times_B C) \arrow[r, "1_E \times_B \xi"] \arrow[d, "1_E \times_B \pi_2"'] \arrow[dr, phantom, "\text{③}"] & E \times_B C \arrow[d, "\xi"] \\ E \times_B C \arrow[r, "\xi"'] & C \end{tikzcd}$$

commute. (① makes ξ a morphism of $\mathbb{E}(E)$, and ② and ③ represent the algebra conditions.) A T^p-homomorphism $h : (C, \gamma; \xi) \to (C', \gamma'; \xi')$ is a morphism

$h : (C, \gamma) \to (C', \gamma')$ of $\mathbb{E}(E)$ such that

$$\begin{tikzcd} E \times_B C \arrow[r, "1_E \times_B h"] \arrow[d, "\xi"'] & E \times_B C' \arrow[d, "\xi'"] \\ C \arrow[r, "h"] & C' \end{tikzcd} \tag{10}$$

commutes.

2.2. We already saw in 1.9 (in case $\mathcal{C} = \mathbf{Top}$) how to pass from descent data on (C, γ) (as in 1.2) to a τ^p-algebra structure; simply put

$$\xi := \pi_2 \overline{\xi}.$$

It is easy to check that this way one obtains a bijective correspondence between descent data and algebra structures. Compatibility with descent data translates into the homomorphism condition (10) under this correspondence.

We note that the bijective correspondence ($\xi \leftrightarrow \overline{\xi}$) persists in the abstract context of 2.1: given a τ^p-structure ξ on (C, γ) one can define

$$\overline{\xi} := \langle \gamma\pi_2, \xi \rangle : E \times_B C \to E \times_B C;$$

$\overline{\xi}$ is an involution ($\overline{\xi}\overline{\xi} = 1_{E \times_B C}$, see [16] 2.4 for a proof), and its characteristic equations (which, in turn, make $\xi := \pi_2\overline{\xi}$ a τ^p-structure) are easily obtained from (9).

Therefore, for any category $\mathcal{C}$ with pullbacks and for any class $\mathbb{E}$ of morphisms stable under pullback, we define the category

$$\mathrm{Des}_{\mathbb{E}}(p)$$

to have objects $(C, \gamma; \xi)$ with $(C, \gamma) \in \mathbb{E}(E)$ and $\xi : E \times_B C \to C$ in $\mathcal{C}$ making the diagrams (9) commutative, and to have morphisms as in diagram (10). There is the comparison functor

$$\Phi^p : \mathbb{E}(B) \to \mathrm{Des}_{\mathbb{E}}(p), \ (A, \alpha) \mapsto (E \times_B A, \ \pi_1; 1_E \times_B \pi_2)$$

and, as in 1.5, we call p an *(effective)* $\mathbb{E}$*-descent morphism* iff Φ^p is full and faithful (an equivalence of categories).

In 2.1 we showed:

PROPOSITION. *If* $\mathbb{E}$ *is closed under composition with* p *from the left, then* $\mathrm{Des}_{\mathbb{E}}(p)$ *is exactly the Eilenberg–Moore category of the monad induced by the adjunction* $p_! \dashv p^*$*, and* p *is an (effective)* $\mathbb{E}$*-descent morphism iff* p^* *is premonadic (monadic).* □

We show in 3.10 below that the hypothesis of the Proposition is essential.

2.3. One can now exploit the various versions of Beck's monadicity criterion. We denote by

$$\mathbb{E}^*(p)$$

the class of all morphisms which are pullbacks of p along a morphism in $\mathbb{E}$ (i.e., the morphisms π_2 of every pullback diagram (4) with $\alpha \in \mathbb{E}$, and their composites with isomorphisms). We say that p is an $\mathbb{E}$-*universal regular epimorphism* if $\mathbb{E}^*(p)$ is contained in the class of regular epimorphisms.

For the following theorem we assume, in addition to the standard assumption that $\mathbb{E}$ be stable under pullback along $p : E \to B$ in $\mathcal{C}$ and under composition with p from the left, that $\mathcal{C}$ has coequalizers of parallel pairs of morphisms in $\mathbb{E}^*(p)$.

THEOREM. *The morphism p is an $\mathbb{E}$-descent morphism of $\mathcal{C}$ if and only if p is an $\mathbb{E}$-universal regular epimorphism of $\mathcal{C}$. The $\mathbb{E}$-descent morphism p is effective, if $\mathbb{E}$ is right cancellable w.r.t. those regular epimorphisms of $\mathcal{C}$ which are coequalizers of $\mathbb{E}^*(p)$-morphisms over B and if these coequalizers are stable under pullback along p.*

Proof. It is well-known that the comparison functor Φ^p is full and faithful if and only if the counits of the adjunction $p_! \dashv p^*$ are regular epimorphisms in $\mathbb{E}(B)$. But the latter are given by the projections π_2 of pullback diagrams (4) with $\alpha \in \mathbb{E}$. Furthermore, due to the existence of certain coequalizers, there is no need to distinguish between being regularly epic in $\mathbb{E}(B)$ or in $\mathcal{C}$ (see the more detailed argument below). This proves the first assertion.

For the second assertion one first constructs the coequalizer

$$E \times_B C \underset{\xi}{\overset{\pi_2}{\rightrightarrows}} C \xrightarrow{q} Q \tag{11}$$

for every $(C, \gamma; \xi) \in \mathrm{Des}_{\mathbb{E}}(p)$. We note that both π_2 and $\xi = \pi_2\bar{\xi}$ belong to $\mathbb{E}^*(p)$. There is a morphism $\delta : Q \to B$ with $\delta q = p\gamma$ (see (7)) which, by right cancellability of $\mathbb{E}$, makes the coequalizer live in $\mathbb{E}(B)$. Now $\Psi^p(C, \gamma; \xi) := (Q, \delta)$ defines the left adjoint of Φ^p, and one knows that the unit of the adjunction $\Psi^p \dashv \Phi^p$ (given by $\eta = \langle \gamma, q \rangle : C \to E \times_B Q$) is an isomorphism if and only if p^* preserves the coequalizer (11). □

2.4. In case $\mathbb{E}$ is the class of all morphisms of $\mathcal{C}$ one obtains:

COROLLARY (cf. [16], [27], [33]). *A morphism p is a descent morphism if and only if p is a universal regular epimorphism. A descent morphism $p : E \longrightarrow B$ is an effective descent morphism if the coequalizer of every parallel pair of universal regular epimorphisms over B exists and is stable under pullback under p.*

The needed preservation of coequalizers is certainly guaranteed if p^* has a *right* adjoint. Hence, *if our category (with pullbacks and coequalizers) is locally cartesian closed, then the effective descent morphisms are exactly the (necessarily universal) regular epimorphisms.*

2.5. In order to refine these results one must, as in Theorem 1.10, actively use the equivalence relations $\tilde{\xi}$. For simplicity, we continue to consider the case $\mathbb{E}$ = all morphisms of a category $\mathcal{C}$ with pullbacks and coequalizers (with the existence of coequalizers being assumed for the sole purpose of making sure that regular epis in $\mathcal{C}$ and in its slices look the same).

It was observed in [34] that, for a τ^p-algebra (C, γ, ξ), the pair $\tilde{\xi} = (\pi_2, \xi)$ is an equivalence relation (in the sense of [19]). This equivalence relation is effective (i.e., is the kernel pair of its coequalizer) if and only if the units of the adjunction $\Psi^p \dashv \Phi^p$ are monic. This fact can be used to prove:

THEOREM (cf. [34]). *A regular epimorphism p is an effective descent morphism if and only if, for every $(C, \gamma; \xi) \in \mathrm{Des}(p)$, the equivalence relation $\tilde{\xi}$ is effective and its coequalizer is a descent morphism.* □

COROLLARY (cf. [34]). *If regular epimorphisms are universal and if equivalence relations are effective, then the effective descent morphisms are exactly the regular epimorphisms. In particular, descent implies effective descent.* □

Hence in every (Barr-)exact category, descent theory is easy. However, a regular category may have regular epimorphisms which are not effective for descent: see 2.7 below. In other words, effectiveness of (some) equivalence relations is an essential condition in the Theorem and the Corollary above.

2.6. Since the results of 2.5 were stated only for the largest class $\mathbb{E}$, the following proposition will be useful to get back to the general setting. We consider two classes $\mathbb{E}_0, \mathbb{E}_1$ of morphisms of a category with pullbacks, both stable under pullback along $p : E \longrightarrow B$ and under composition with isomorphisms, with

$$\mathbb{E}_0 \subseteq \mathbb{E}_1 .$$

PROPOSITION. *$\mathbb{E}_1$-descent for p implies $\mathbb{E}_0$-descent for p. The effective $\mathbb{E}_1$-descent morphism p is an effective $\mathbb{E}_0$-descent morphism if and only if the following (modified transferability) condition holds: for every pullback diagram* (4), *$\pi_1 \in \mathbb{E}_0$ and $\alpha \in \mathbb{E}_1$ implies $\alpha \in \mathbb{E}_0$.*

Proof. The first statement follows as in 1.8. Let now p be an effective $\mathbb{E}_1$-descent morphism. Then the stated condition suffices to show that $\Phi_0^p : \mathbb{E}_0(B) \to \mathrm{Des}_{\mathbb{E}_0}(p)$ is an equivalence of categories: we already know that Φ_0^p is full and faithful, and for every $(C,\gamma;\xi) \in \mathrm{Des}_{\mathbb{E}_0}(p)$ there is $(A,\alpha) \in \mathbb{E}_1(B)$ with $\Phi_1^p(A,\alpha) \cong (C,\gamma;\xi)$; hence $p^*(A,\alpha) \cong (C,\gamma) \in \mathbb{E}_0(E)$ implies $(A,\alpha) \in \mathbb{E}_0(B)$ and $\Phi_0^p(A,\alpha) \cong (C,\gamma;\xi)$. On the other hand, the condition is necessary for Φ_0^p to be an equivalence of categories: if, in the pullback diagram (4), we have $\pi_1 \in \mathbb{E}_0$, then $\Phi_1^p(A,\alpha) \in \mathrm{Des}_{\mathbb{E}_0}(p)$; hence there is $(A',\alpha') \in \mathbb{E}_0(B)$ with

$$\Phi_1^p(A,\alpha) \cong \Phi_0^p(A',\alpha') \cong \Phi_1^p(A',\alpha'),$$

so that $(A,\alpha) \cong (A',\alpha') \in \mathbb{E}_0(B)$ follows. □

2.7. There are two instances of Proposition 2.6 of particular interest:

COROLLARY. 1. *For $\mathcal{C}$ with pullbacks and $\mathbb{E}$ stable under pullback along the effective global descent morphism p of $\mathcal{C}$, p is an effective $\mathbb{E}$-descent morphism if and only if in every pullback square* (4), $\pi_1 \in \mathbb{E}$ *implies* $\alpha \in \mathbb{E}$.

2. *For $\mathcal{D}$ with pullbacks and $\mathcal{C}$ a full subcategory closed under pullbacks in $\mathcal{D}$, a morphism p of $\mathcal{C}$ which is an effective descent morphism in $\mathcal{D}$ is also an effective descent morphism in $\mathcal{C}$ if and only if in every pullback square* (4) *of* $\mathcal{D}$, $E \times_B A \in \mathcal{C}$ *implies* $A \in \mathcal{C}$. □

While the first part of the Corollary allows us to exploit the criteria given in 2.5 also in the general $\mathbb{E}$-case, its second part is useful when our category $\mathcal{C}$ is embedded in a larger category $\mathcal{D}$ where effective descent morphisms are easily characterized (for instance when $\mathcal{D}$ is locally cartesian closed or exact, see 2.4, 2.5). An application of this procedure is given in [33] and in the following examples (see the remark at the end of 2.5.).

EXAMPLES *of regular categories in which regular epimorphisms may fail to be effective for descent.* 1. We consider the full subcategory $\mathcal{C}$ of semigroups which have at most one idempotent element. It is closed under products and subsemigroups, hence it is (finitely) complete and has (regular-epi, mono)-factorizations, and the regular epimorphisms are, as in the category $\mathcal{D}$ of all semigroups, stable under pullback. The regular epimorphism $p : E \to 1$ with E a non-empty semigroup without any idempotents is an effective descent morphism of $\mathcal{D}$ but, according to Corollary 2, not of $\mathcal{C}$. Indeed, with A any semigroup with at least two idempotents, one has $E \times A \in \mathcal{C}$ but $A \notin \mathcal{C}$.

2. In 1. $\mathcal{C}$ is a regular-epireflective subcategory of a monadic category $\mathcal{D}$ over **Set**. A similar example is possible *with $\mathcal{D}$ not only regular, but even a Grothendieck topos*: let $\mathcal{D}$ be the category of G-sets (with a fixed non-trivial monoid G, and let $\mathcal{C}$ be the full subcategory of those G-sets which have at most one element x with trivial orbit (i.e., $Gx = \{x\}$). The argumentation is as above.

3. Finally, we may choose $\mathcal{C}$ to be *a regular-epireflective subcategory of an abelian category*, in fact, of the category of abelian groups: for a fixed natural number $n \geq 2$ consider those abelian groups satisfying the property

$$n^2x = 0 \Rightarrow nx = 0. \qquad (*)$$

In this category the regular epimorphism $p : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ fails to be effective for descent in $\mathcal{C}$. To wit, let $\alpha : A = \mathbb{Z}/n^2\mathbb{Z} \to B = \mathbb{Z}/n\mathbb{Z}$ be the projection. Then the pullback $P = \mathbb{Z} \times_B A$ satisfies $(*)$ although A clearly does not. In fact, for $(a, b) \in P$ with $n^2(a, b) = 0$, one has $a = 0$ and therefore $\alpha(b) = p(a) = 0$. Hence we can write $b = nk + n^2\mathbb{Z}$ with $k \in \mathbb{Z}$ and conclude $nb = 0$ in A.

2.8. It's time for a "positive" application of Corollary 2.7.2!

COROLLARY. *In the category $\mathcal{C}$ of torsion-free abelian groups, every surjective homomorphism is an effective descent morphism.*

Proof. In Corollary 2 above, let B, E, and $E \times_B A$ be torsion-free. For $a \in A$ with $na = 0$ and $n \neq 0$, one has $n\alpha(a) = 0$ and therefore $\alpha(a) = 0$ since B is torsion-free. Hence $(0, a) \in E \times_B A$, and $n(0, a) = 0$. Since $E \times_B A$ is torsion-free, one concludes $(0, a) = 0$ and then $a = 0$. □

2.9. So far we have been considering exclusively the case that the adjunction

$$p_! \dashv p^* : \mathbb{E}(B) \to \mathbb{E}(E)$$

is a restriction of the corresponding adjunction between $\mathcal{C}/B$ and $\mathcal{C}/E$. But there are important cases when p^* has a left adjoint $p_!$ which is *not* given by "composition with p".

Let $\mathbb{E}$ be the *second* factor of a factorization system $(\mathbb{D}, \mathbb{E})$ of the category $\mathcal{C}$ with pullbacks. Hence every morphism f factors as $f = \alpha q$ with $q \in \mathbb{D}$ and $\alpha \in \mathbb{E}$, and the $(\mathbb{D}, \mathbb{E})$-diagonalization property holds (see [9], [31], [1]). $\mathbb{E}$ is then stable under pullback (and composition), and the left adjoint $p_!$ of the pullback functor p^* is given by taking $\mathbb{E}$-images: the morphism α of $(A, \alpha) = p_!(C, \gamma)$ is the $\mathbb{E}$-part of a $(\mathbb{D}, \mathbb{E})$ factorization of $p\gamma$:

$$p\gamma = \alpha q \quad (q \in \mathbb{D}, \alpha \in \mathbb{E}).$$

As in 2.1 we can consider the Eilenberg–Moore category with respect to the induced monad T^p, *but it is important to note that it may not be equivalent to the category* $\mathrm{Des}_{\mathbb{E}}(p)$ *as defined in* 2.2 (see 3.10 for a counter-example). A sufficient condition for coincidence will be given in 3.9, while in 2.10 we give a criterion for premonadicity (monadicity) of p^*, i.e. for the comparison functor into the Eilenberg–Moore category to be full and faithful (an equivalence of categories).

2.10. The following two conditions on $\mathbb{E}$ will be of interest:

(I) for all morphisms f, g with $fg = 1$, $f \in \mathbb{E}$ and $g \in \mathbb{E}$ implies f iso;

(II) for all morphisms f, g with $fg = 1$, $f \in \mathbb{E}$ implies f iso.

Obviously, (II) implies (I), and (I) holds if $\mathbb{E}$ is a class of monos or a class of epis in $\mathcal{C}$. Condition (II) holds if and only if $\mathbb{E}$ is a class of monos in $\mathcal{C}$: see [9], Proposition 2.1.4.

PROPOSITION. *Let $\mathcal{C}$ have pullbacks, and let $(\mathbb{D}, \mathbb{E})$ be a factorization system for morphisms of $\mathcal{C}$. Under condition* (I), *the monad τ^p induced by p^* is idempotent. Idempotency of the monad τ^p implies that the following statements are equivalent:*

(i) $p^* : \mathbb{E}(B) \to \mathbb{E}(E)$ *reflects isomorphisms,*

(ii) p^* *is full and faithful,*

(iii) p^* *is premonadic,*

(iv) p^* *is monadic.*

Each of (i)–(iv) *is implied by*

(v) p *is an $\mathbb{E}$-universal $\mathbb{D}$-morphism,*

and (v) *is equivalent to each of* (i) - (iv) *under condition* (II).

Proof. We first note that $\mathbb{E}$, as a second factor of a factorization system, satisfies the weak cancellation rule ($fg \in \mathbb{E}$ and $f \in \mathbb{E}$ implies $g \in \mathbb{E}$). Hence, under condition (I) we obtain immediately that retractions (and sections) in the category $\mathbb{E}(E)$ are isomorphisms. The triangular equations of the adjunction $p_! \dashv p^*$ then show that τ_p is idempotent.

The general adjunction machinery also shows that one always has

$$\text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)},$$

and that for an idempotent monad all four statements are equivalent.

If condition (v) holds, that is: if the pullback of p along any morphism in $\mathbb{E}$ belongs to $\mathbb{D}$, then (ii) holds, and therefore each of (i)–(iv). In fact, under condition (v) the $(\mathbb{D}, \mathbb{E})$-diagonalization property produces immediately for every $h : p^*(A, \alpha) \to p^*(A', \alpha')$ in $\mathbb{E}(E)$ a unique $f : (A, \alpha) \to (A', \alpha')$ in $\mathbb{E}(B)$ with $1_E \times_B f = h$.

It remains to be shown that (i) $\Rightarrow$ (v) holds under condition (II). Hence we consider the pullback diagram (4) with $\alpha \in \mathbb{E}$ and a $(\mathbb{D}, \mathbb{E})$-factorization $\pi_2 = fq$ $(q \in \mathbb{D}, f \in \mathbb{E})$. Since $\mathbb{E}$ is closed under composition, $\alpha' := \alpha f : A' \to B$ belongs to $\mathbb{E}$, and one has $f : (A', \alpha') \to (A, \alpha)$ in $\mathbb{E}(B)$. Since $\mathbb{E}$ is stable under pullback, $1_E \times_B f$ belongs to $\mathbb{E}$. Finally, since

$$(1_E \times_B f)\langle 1_{E \times_B A}, q\rangle = 1_{E \times_B A},$$

condition (II) gives that $1_E \times_B f$ is an isomorphism, hence f is an isomorphism under condition (i). Consequently, π_2 belongs to $\mathbb{D}$. □

Of course, the setting of 2.9 includes the case $\mathbb{E}$ = all morphisms in $\mathcal{C}$ so that $\mathbb{D}$ is the class of isomorphisms and $p_!$ is given, as before, by composition, but Proposition 2.10 is not tailored for that case. In fact, condition (I) (and therefore (II)) does not hold in this case (unless every retraction of $\mathcal{C}$ is an isomorphism) and, as a consequence of this, condition (v) is strictly stronger than each of (i) – (iv).

3. Descent Theory with Respect to a Fibration

3.1. The categories $\mathbb{E}(C)$ $(C \in \mathcal{C})$ as considered in 2.1 are the fibres of the codomain functor

$$P_{\mathbb{E}} : \mathbb{E}^2 \to \mathcal{C};$$

here $\mathbb{E}^2$ is the category whose objects are all morphisms belonging to the class $\mathbb{E}$ and whose morphisms $(p', p) : \alpha' \to \alpha$ (with $\alpha, \alpha' \in \mathbb{E}$) are given by commutative squares in $\mathcal{C}$:

$$\begin{tikzcd} \cdot \arrow[r, "p'"] \arrow[d, "\alpha'"'] & \cdot \arrow[d, "\alpha"] \\ \cdot \arrow[r, "p"] & \cdot \end{tikzcd} \qquad (12)$$

Both, the property that (12) is *cartesian* (i.e., a pullback square) and the property that for morphisms p, α with the same codomain and $\alpha \in \mathbb{E}$ there exists a pullback square (12) with $\alpha' \in \mathbb{E}$, can be expressed abstractly in terms of the functor $P_{\mathbb{E}}$. It is therefore not surprising that also the problem of descent can be expressed with respect to quite general types of functors, see 3.5 below; however, it is surprising that the basic criteria for descent survive the passage to the more abstract context, see 3.6 below.

3.2. We consider an arbitrary functor $P : \mathcal{E} \to \mathcal{C}$ and a morphism $p : E \to B$ in $\mathcal{C}$. Recall that the *fibre*

$$\mathcal{E}(B) := P^{-1}B$$

of P *at* B is the (non-full) subcategory of $\mathcal{E}$ whose morphisms $f : A \to A'$ satisfy $Pf = 1_B$. Given $A \in \mathcal{E}(B)$, a pair (C, c) with $C \in \mathcal{E}(E)$ and $c : C \to A$ a morphism in $\mathcal{E}$ with $Pc = p$ is called a (P)*-lifting of* p at A. A morphism $c : C \to A$ in $\mathcal{E}$ is called (P)*-cartesian* if it is a terminal P-lifting of Pc at A, in the following sense:

given any $d : D \to A$ in $\mathcal{E}$ and any $q : PD \to PC$ with $(Pc)q = Pd$, then there is a unique $g : D \to C$ with $cg = d$ and $Pg = q$.

Let us now suppose that for every $A \in \mathcal{E}(B)$ we are given a cartesian lifting $(p^*A, \vartheta_p A)$ of p at A.

$$\begin{array}{ccc} p^*A & \xrightarrow{\vartheta_p A} & A \\ & \Downarrow P & \\ E & \xrightarrow{p} & B \end{array} \tag{13}$$

Then we obtain the *inverse-image functor*

$$p^* : \mathcal{E}(B) \to \mathcal{E}(E)$$

and a *cleavage*

$$\vartheta_p : J_E p^* \to J_B$$

(here $J_B : \mathcal{E}(B) \to \mathcal{E}$ is the inclusion functor) with $P\vartheta_p = \Delta p : \Delta E \to \Delta B$ the constant natural transformation.

The functor P is a *(cloven) fibration* if every $p : E \to B$ admits a (specified) cartesian lifting at every $A \in \mathcal{E}(B)$. We note that

$$(\ )^* : \mathcal{C}^{op} \to \mathbf{CAT}$$

becomes a pseudo-functor in this case. In fact, for all $p : E \to B$ and $q : X \to E$ in $\mathcal{C}$ there are uniquely determined natural equivalences

$$\begin{aligned} i_B &: Id_{\mathcal{E}(B)} \to (1_B)^* \text{ with } \vartheta_{1_B} \cdot J_B i_B = 1_{J_B}, PJ_B i_B = \Delta 1_B, \\ j_{p,q} &: q^* p^* \to (pq)^* \text{ with } \vartheta_{pq} \cdot J_X j_{p,q} = \vartheta_p \cdot \vartheta_q p^*,\ PJ_X j_{p,q} = \Delta 1_X. \end{aligned}$$

If these natural equivalences are identities, so that $(\ )^*$ is a functor, then P is called a *split fibration*.

We also note that the cleavages of a fibration define a right-adjoint right-inverse of every functor

$$P_A : \mathcal{E}/A \to \mathcal{C}/PA \quad (A \in \mathcal{E})$$

induced by P. Vice versa, having right-adjoint right-inverses of these functors, one establishes P as a cloven fibration.

3.3. EXAMPLES. 1. For a class $\mathbb{E}$ of morphisms of a category $\mathcal{C}$, every pullback square (12) with $\alpha, \alpha' \in \mathbb{E}$ represents a $P_{\mathbb{E}}$-cartesian morphism of $\mathbb{E}^2$. Vice versa, if

$\mathbb{E}$ contains all isomorphisms of $\mathcal{C}$, then every $P_{\mathbb{E}}$-cartesian morphism of $\mathcal{C}$ is given by a pullback diagram.

Let $p : E \to B$ be in $\mathcal{C}$, and suppose that for every $\alpha : A \to B$ in $\mathbb{E}$ the fibred product (4) exists in $\mathcal{C}$. This means that for every $(A, \alpha) \in \mathbb{E}(B)(= \mathbb{E}^2(B))$ a cartesian $P_{\mathbb{E}}$-lifting of p at (A, α) exists; the inverse image functor p^* and the cleavage ϑ_p are obtained from a choice of the fibred product (4). Hence, if $\mathcal{C}$ is a category with (chosen) pullbacks and if $\mathbb{E}$ is stable under pullback, then $P_{\mathbb{E}}$ is a (cloven) fibration.

2. Let $P : \mathcal{E} \to \mathcal{C}$ be an (amnestic) topological functor (in the sense of [1]). Then P is a (split) fibration, in fact: a split bifibration (see 3.7 below). The P-cartesian morphisms are the P-initial morphisms.

3. For a topological space X, let πX be the *fundamental groupoid* of X : its objects are the points of X, and its morphisms are homotopy equivalence classes of paths in X. Every continuous map $p : E \to B$ gives a functor $\pi p : \pi E \to \pi B$. (In fact, $\pi : \mathbf{Top} \to \mathbf{Cat}$ is a 2-functor, with $\mathbf{Top}$ considered a groupoid-enriched category.) If p is a *Hurewicz fibration* (cf. [35]) then πp is a cloven fibration.

3.4. Example 1 of 3.3 provides guidance for defining descent data w.r.t. an arbitrary fibration $P : \mathcal{E} \to \mathcal{C}$, for $\mathcal{C}$ with pullbacks. For $p : E \to B$ in $\mathcal{C}$, let (p_1, p_2) be the kernelpair of p, so that one has a pullback diagram

$$\begin{tikzcd} & E \times_B E \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ E \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{14}$$

Descent data for $C \in \mathcal{E}(E)$ (*relative to* p) will be given by *certain* morphisms $\hat{\xi} : p_1^* C \to p_2^* C$ in $\mathcal{E}(E \times_B E)$. In fact, if $P = P_{\mathbb{E}}$ is given as in Example 1 then we have a commutative diagram of canonical isomorphisms, as follows:

$$\begin{tikzcd} & E \times_B C & \\ p_1^* C = C \times_E (E \times_B E) \arrow[ur, "\pi_{3,1}"] \arrow[rr] & & (E \times_B E) \times_E C = p_2^* C \arrow[ul, "\pi_{1,3}"'] \end{tikzcd} \tag{15}$$

Hence every descent structure $\overline{\xi} : E \times_B C \to E \times_B C$ gives a morphism

$$\hat{\xi} = \pi_{1,3}^{-1} \overline{\xi} \pi_{3,1} : p_1^* C \to p_2^* C,$$

and vice versa. However, "translating" the relevant equations for $\overline{\xi}$ into equivalent conditions on $\hat{\xi}$ in the abstract context of a fibration needs careful considerations.

In order to mimick the commutativity of ② in (9), one considers the "diagonal" $\delta : E \to E \times_B E$ with $p_1 \delta = p_2 \delta = 1_E$. With a cartesian lifting $(p_i^* C, \vartheta_{p_i} C)$ of p_i for $i = 1, 2$, we obtain unique morphisms

$$\delta_i : C \to p_i^* C \text{ with } P\delta_i = \delta \text{ and } \vartheta_{p_i} C \cdot \delta_i = 1_C.$$

$\hat{\xi}$ is required to render the diagram

$$\begin{tikzcd} p_1^* C \arrow[rr, "\hat{\xi}"] & & p_1^* C \arrow[dl, "\vartheta_{p2} C"] \\ & C \arrow[ul, "\delta_1"] & \end{tikzcd} \tag{16}$$

commutative (at C). Mimicking the commutativity of ③ in (9) takes greater effort. In addition to the pullback (14), one forms the pullback diagram

$$\begin{tikzcd} & (E \times_B E) \times_E (E \times_B E) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E \times_B E \arrow[dr, "p_2"'] & & E \times_B E \arrow[dl, "p_1"] \\ & E & \end{tikzcd} \tag{17}$$

and considers the morphism

$$\pi := \pi_{1,4} = \langle p_1 \pi_1, p_2 \pi_2 \rangle : (E \times_B E) \times_E (E \times_B E) \to E \times_B E.$$

One can now substitute ③ by requiring $\hat{\xi}$ to render

$$
\begin{tikzcd}
 & \pi_1^*p_2^*C \arrow[r, "j"] & \pi_2^*p_1^*C \arrow[dr, "\pi_2^*\hat{\xi}"] & \\
\pi_1^*p_1^*C \arrow[ur, "\pi_1^*\hat{\xi}"] \arrow[dr, "j_1^{-1}"'] & & & \pi_2^*p_2^*C \\
 & \pi^*p_1^*C \arrow[r, "\pi^*\hat{\xi}"] & \pi^*p_2^*C \arrow[ur, "j_2"'] &
\end{tikzcd}
\tag{18}
$$

commutative. Here the canonical isomorphisms j and j_i arise from the identities $p_1\pi_2 = p_2\pi_1$ and $p_i\pi = p_i\pi_i$ $(i = 1,2)$ as in 3.2; explicitly, $j = (j^{-1}_{p_1,\pi_2}C)(j_{p_2,\pi_1}C)$ and $j_i = (j^{-1}_{p_i,\pi_i}C)(j_{p_i,\pi}C)$.

One can show that $\hat{\xi}$ must *necessarily* be an isomorphism (see [29], Remark 44 for an explicit proof), hence all morphisms occuring in (18) are isos.

3.5. Descent data $(C, \hat{\xi})$ relative to p (with $C \in \mathcal{E}(E)$ and $\hat{\xi} : p_1^*C \to p_2^*C$ in $\mathcal{E}(E \times_B E)$ such that (16) and (18) commute) form the objects of the category

$$\mathrm{Des}_{\mathcal{E}}(p)$$

whose morphisms $h : (C, \hat{\xi}) \to (C', \hat{\xi}')$ are morphisms $h : C \to C'$ in $\mathcal{E}(E)$ such that

$$
\begin{tikzcd}
p_1^*C \arrow[r, "p_1^*h"] \arrow[d, "\hat{\xi}"'] & p_1^*C' \arrow[d, "\hat{\xi}'"] \\
p_2^*C \arrow[r, "p_2^*h"] & p_2^*C'
\end{tikzcd}
\tag{19}
$$

commutes. For every $A \in \mathcal{E}(B)$, p^*A can be provided with *canonical descent data*

$$\hat{\varphi} = (j^{-1}_{p,p_2}A)(j_{p,p_1}A) : p_1^*p^*A \to p_2^*p^*A$$

so that p^* can be lifted to a functor Φ^p which makes

$$\begin{tikzcd} \mathcal{E}(B) \arrow[rr, "\Phi^p"] \arrow[dr, "p^*"'] & & \mathrm{Des}_{\mathcal{E}}(p) \arrow[dl, "U^p"] \\ & \mathcal{E}(E) & \end{tikzcd} \tag{20}$$

commute (with U^p the obvious forgetful functor). In accordance with Grothendieck [13] p is called an *(effective) $\mathcal{E}$-descent morphism* if Φ_p is full and faithful (an equivalence of categories).

3.6. The following notions are due to Pavlović [30] and are needed to characterize (effective) $\mathcal{E}$-descent morphisms. A morphism g in $\mathcal{C}$ is a *P-coequalizer* of a pair k_1, k_2 of parallel morphisms if $gk_1 = gk_2$ and if every h with $hk_1 = hk_2$ and $Ph = Pg$ gives a unique t with $tg = h$ and $Pt = 1$. A *(natural) interpolant* for $(C, \hat{\xi}) \in \mathrm{Des}_{\mathcal{E}}(p)$ is a triple (a, A, c) with $A \in \mathcal{E}(E \times_B E)$ and

$$(C, \hat{\xi}) \xrightarrow{a} (p^*A, \hat{\varphi}) \xrightarrow{c} (C, \hat{\xi})$$

in $\mathrm{Des}_{\mathcal{E}}(p)$ such that the diagram

$$\begin{tikzcd} p_1^*p^*A \arrow[r, "\hat{\varphi}"] & p_2^*p^*A \arrow[d, "p_2^*a"] \\ p_1^*C \arrow[u, "p_1^*a"] \arrow[r, "\hat{\xi}"] & p_2^*C \end{tikzcd} \tag{21}$$

commutes. The interpolant (a, A, c) is *simple* if every morphism $f : A \to A$ in $\mathcal{E}(E \times_B E)$ with $(p^*f)a = a$ and $c(p^*f) = c$ is an identity morphism. Now one can prove:

THEOREM (Pavlović [30]). *Let $P : \mathcal{E} \to \mathcal{C}$ be a fibration, and let $\mathcal{C}$ have pullbacks. Then a morphism $p : E \to B$ in $\mathcal{C}$ is an $\mathcal{E}$-descent morphism if and only if every cartesian lifting of p is a P-coequalizer of its kernelpair, and it is effective if and only if all $\mathcal{E}$-descent data relative to p have simple interpolants.* □

We observe that the characterization of $\mathcal{E}$-descent morphisms is in fact a straight generalization of the characterization of $\mathbb{E}$-descent morphisms given in Theorem 2.3: in the presence of (enough) coequalizers (as assumed in 2.3), there is no difference between $P_{\mathbb{E}}$-coequalizers and (ordinary) coequalizers (see the corresponding remark in the proof of 2.3). We also observe that, in the general context of fibrations, the kernelpair of a cartesian lifting of p automatically exists, as a cartesian lifting of a kernelpair of p in $\mathcal{C}$.

3.7. Definition 3.4 of descent data makes it clear that for $P = P_{\mathbb{E}}$, there is a bijective correspondence ($\hat{\xi} \leftrightarrow \overline{\xi} \leftrightarrow \xi$) which leads to an isomorphism between the category $\mathrm{Des}_{\mathbb{E}}(p)$ as defined in 2.2 and the category $\mathrm{Des}_{\mathcal{E}}(p)$ with $\mathcal{E} = \mathbb{E}^2$ as defined in 3.5. It was a fundamental observation of Bénabou and Roubaud [5] and of Beck (unpublished) that $\mathrm{Des}_{\mathcal{E}}(p)$ can be viewed as the Eilenberg–Moore category of p^*, in a much more general context than that of Proposition 2.2, as follows.

Of course, one first needs to have a left adjoint $p_!$ to p^*. In order to guarantee this, one assumes $P : \mathcal{E} \to \mathcal{C}$ to be a (cloven) *bifibration* so that both P and $P^{op} : \mathcal{E}^{op} \to \mathcal{C}^{op}$ are (cloven) fibrations. Hence, dually to the inverse image functor p^* and the cleavage γ_p one has a *direct image functor*

$$p_! : \mathcal{E}(E) \to \mathcal{E}(B)$$

and a *cocleavage*

$$\delta_p : J_E \to J_B p_!.$$

There are uniquely determined natural transformations

$$\rho_p : Id_{\mathcal{E}(E)} \to p^* p_! \quad \text{with} \quad (\vartheta_p p_!)(J_E \rho_p) = \delta_p,$$

$$\sigma_p : p_! p^* \to Id_{\mathcal{E}(B)} \quad \text{with} \quad (J_B \sigma_p)(\delta_p p^*) = \vartheta_p$$

which serve as unit and counit of the adjunction $p_! \dashv p^*$. Furthermore, with the kernelpair (p_1, p_2) of p, one has the *Beck transformation*

$$\beta_p : (p_2)_! p_1^* \to p^* p_! \quad \text{with} \quad (J_E \beta_p)(\delta_{p_2} p_1^*) = (J_E \rho_p)\vartheta_{p_1}.$$

One says that P *satisfies the Beck–Chevalley condition for* p if β_p is a natural equivalence. Equipped with this property it is easy to see how to establish the bijective correspondence ($\hat{\xi} \leftrightarrow \xi$) : given $\mathcal{E}$-descent data $\hat{\xi}$ for $C \in \mathcal{E}(E)$, one obtains an algebra structure ξ w.r.t. the monad given by $p_! \dashv p^*$, as the composite

$$p^* p_! C \xrightarrow{\beta_p^{-1} C} (p_2)_! p_1^* C \xrightarrow{(\hat{\xi})^{\#}} C,$$

with $(\hat{\xi})^{\#} = (\sigma_{p_2} C)((p_2)_! \hat{\xi})$ corresponding to $\hat{\xi}$ by adjunction. This (essentially) shows:

THEOREM (Bénabou–Roubaud [5], Beck). *For $\mathcal{E}$ bifibred over a category $\mathcal{C}$ with pullbacks and for $p : E \to B$ in $\mathcal{C}$ such that the Beck–Chevalley condition is satisfied for p,* $\mathrm{Des}_{\mathcal{E}}(p)$ *is isomorphic to the Eilenberg–Moore category of the* *monad induced by $p_! \dashv p^*$. Hence p is an (effective) $\mathcal{E}$-descent morphism if and only if p^* is premonadic (monadic).* □

Consequently, descent criteria may again be obtained from criteria for monadicity, provided the hypotheses of the Theorem hold (see [5], [6]).

3.8. We wish to clarify the meaning of the hypotheses of Theorem 3.7 in case $P = P_{\mathbb{E}}$ (see 3.1, 3.3). $P_{\mathbb{E}}$ is a subfibration of the basic fibration if $\mathbb{E}$ is stable under pullback in the category $\mathcal{C}$. This condition holds automatically if $\mathbb{E}$ is part of a $(\mathbb{D}, \mathbb{E})$-factorization system of $\mathcal{C}$, and then $P_{\mathbb{E}}$ is a bifibration (but *not* in general a subbifibration of the basic fibration). More precisely, it was shown in [36] (Cor. 3.3 dually) that $P_{\mathbb{E}}$ is a cofibration (i.e., $P_{\mathbb{E}}^{op}$ is a fibration) if and only if $\mathcal{C}$ has "locally coorthogonal $\mathbb{E}$-factorizations", with the latter condition being equivalent to the existence of an (ordinary) $(\mathbb{D}, \mathbb{E})$-factorization system exactly when $\mathbb{E}$ is closed under composition. But to keep things easy, we will continue to work only with ordinary factorization systems.

The Beck–Chevalley condition may be considered for any commutative diagram

$$\begin{tikzcd}
& S \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\
E \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\
& B &
\end{tikzcd} \tag{22}$$

in $\mathcal{C}$, rather than for the kernelpair (14), even in case of an arbitrary bifibration $P : \mathcal{E} \to \mathcal{C}$ since one always has a natural transformation

$$\beta : \varphi_! \psi^* \to q^* p_!.$$

P is said to have the *Beck–Chevalley property* if β is an isomorphism for every *pullback* square (22) in $\mathcal{C}$.

PROPOSITION. *Let $(\mathbb{D}, \mathbb{E})$ be a factorization system of a category $\mathcal{C}$ with pullbacks. Then the bifibration $P_{\mathbb{E}}$ has the Beck–Chevalley property if and only if $\mathbb{D}$ is stable under pullback in $\mathcal{C}$.*

Proof. First let $\mathbb{D}$ be stable under pullback. For every $\gamma \in \mathbb{E}$, we consider the pullback diagrams

$$\begin{tikzcd} \cdot \arrow[r, "\psi^*(\gamma)"] \arrow[d, "\psi'"'] & \cdot \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "\gamma"] & \cdot \arrow[r, "p"] & \cdot \end{tikzcd} \tag{23}$$

and the $(\mathbb{D}, \mathbb{E})$-factorizations

$$p_!(\gamma) \cdot p' = p\gamma, \quad \varphi_!(\psi^*(\gamma)) \cdot \varphi' = \varphi \cdot \psi^*(\gamma)$$

with $p', \varphi' \in \mathbb{D}$. Since $p_!(\gamma) \cdot (p'\psi') = (q \cdot \varphi_!(\psi^*(\gamma))) \cdot \varphi'$ with $p_!(\gamma) \in \mathbb{E}$ and $\varphi' \in \mathbb{D}$, there is a unique t with $t\varphi' = p'\psi'$ and $p_!(\gamma) \cdot t = q \cdot q^*(p_!(\gamma))$. The Beck morphism β is determined by the equations $q'\beta = t$ and $q^*(p_!(\gamma)) \cdot \beta = \varphi_!(\psi^*(\gamma))$, and since both $q^*(p_!(\gamma))$ and $\varphi_!(\psi^*(\gamma))$ belong to $\mathbb{E}$, we have $\beta \in \mathbb{E}$. Now the (combined) pullback diagram (23) is factored as

$$\begin{tikzcd} \cdot \arrow[r, "\beta\varphi'"] \arrow[d, "\psi'"'] & \cdot \arrow[r, "q^*(p_!(\gamma))"] \arrow[d, "q'"] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "p'"] & \cdot \arrow[r, "p_!(\gamma)"] & \cdot \end{tikzcd} \tag{24}$$

with the right square a pullback, hence the left square is also a pullback. Since $p' \in \mathbb{D}$ we have $\beta\varphi' \in \mathbb{D}$ by hypothesis, and since $\varphi' \in \mathbb{D}$ this implies $\beta \in \mathbb{D}$. Consequently, β is an isomorphism.

Vice versa, we consider the pullback diagram (22) with $p \in \mathbb{D}$ and exploit the Beck–Chevalley property for $\gamma = 1$. Then all $p_!(\gamma)$, $q^*(p_!(\gamma))$ and $\psi^*(\gamma)$ can be chosen to be identity morphisms as well, and the Beck–Chevalley condition amounts to saying that in the $(\mathbb{D}, \mathbb{E})$-factorization of φ the $\mathbb{E}$-part is an isomorphism. Hence $\varphi \in \mathbb{D}$. □

3.9. From Theorem 3.7 and Propositions 2.10 and 3.8 we obtain immediately:

THEOREM. *Let $(\mathbb{D}, \mathbb{E})$ be a factorization system of a category $\mathcal{C}$ with pullbacks, with $\mathbb{D}$ stable under pullback. Under condition* (I) *of* 2.10, *the following statements are equivalent for a morphism $p : E \to B$ in $\mathcal{C}$:*

(i) $p^* : \mathbb{E}(B) \to \mathbb{E}(C)$ *reflects isomorphisms,*

(ii) p^* *is full and faithful,*

(iii) p *is an* $\mathbb{E}$*-descent morphism,*

(iv) p *is an effective* $\mathbb{E}$*-descent morphism.*
Under condition II *of* 2.10, *that is, if* $\mathbb{E}$ *is a class of monomorphisms, these are also equivalent to*

(v) p *belongs to the class* $\mathbb{D}$. □

3.10. We remark that condition (I) of 2.10 is essential for the validity of Theorem 3.9: consider the trivial factorization system (Iso, All) in **Top**; see [33] and 4.1 below.

The following two easy examples illustrate the significance of pullback stability of $\mathbb{D}$.

Let $\mathbb{E}_1$ be the class of subspace embeddings in **Top**. With $\mathbb{D}_1$ = surjective maps, it is part of a factorization system, and Theorem 3.9 immediately gives the equivalences

$$(*) \quad p \text{ surjective} \Leftrightarrow p \text{ (effective) } \mathbb{E}_1\text{-descent} \Leftrightarrow p^* \text{ reflects isos}$$

for every p, since $\mathbb{D}_1$ is stable under pullback.

For $\mathbb{E}_0$ the class of closed-subspace embeddings in **Top**, we still have a factorization system $(\mathbb{D}_0, \mathbb{E}_0)$, with $\mathbb{D}_0$ = dense maps (= maps whose image is dense in their codomain), but $\mathbb{D}_0$ is not stable under pullback. According to Proposition 2.10, the closed-hereditary dense maps are exactly the maps p for which $p^*_{\mathbb{E}_0}$ is (pre-)monadic. Such maps need not be surjective (just consider any non-surjective map $p : E \to B$ with $E \neq \emptyset$ and B indiscrete), and dense maps need not be closed hereditary (consider any dense map $p : E \to B$ with a closed point in $B \backslash p(E)$). From $(*)$ and Proposition 2.6 one knows that surjective maps are $\mathbb{E}_0$-descent, and that quotient maps are effective $\mathbb{E}_0$-descent. But a surjective map $p : E \to B$ may fail to be effective $\mathbb{E}_0$-descent, although $p^*_{\mathbb{E}_0}$ is monadic. To wit, let X be a set with at least two points, and let E be the discrete structure on X and B be the indiscrete structure on X, and let p be the identity map. Up to categorical equivalence, $\mathbb{E}_0(E)$ is the (partially ordered) power set of X while $\mathbb{E}_0(B)$ is the 2-element chain, and an easy inspection of 2.2 yields that $\mathrm{Des}_{\mathbb{E}_0}(p)$ is equivalent to $\mathbb{E}_0(E)$, hence non-equivalent to $\mathbb{E}_0(B)$.

4. Global, Open and Étale Descent

We return to the category **Top** of topological spaces and continuous maps and investigate (effective) $\mathbb{E}$-descent maps $p : E \to B$ for specific classes $\mathbb{E}$ stable under pullback.

4.1. *(Effective) global-descent.* As mentioned previously (see 1.7), it is known that in case $\mathbb{E}$ = all (continuous) maps, the $\mathbb{E}$-descent (= global-descent) maps are exactly the universal quotient maps p which are also characterized by the property that for every ultrafilter $\mathcal{U}$ on B converging to $b \in B$ there is an ultrafilter $\mathcal{V}$ on E converging to some $x \in p^{-1}b$ with $p\mathcal{V} = \mathcal{U}$. Hence, for these maps it is possible to describe bundles over B equivalently by certain bundles over E which come equipped with descent data (see Theorem 1.10(2)).

It was shown in [33] that there are global-descent maps which fail to be effective, and the following filter-theoretic characterization of effective global-descent maps was given.

THEOREM (Reiterman and Tholen [33]). *A surjective map $p : E \to B$ is an effective global-descent map if and only if for every family of ultrafilters $\mathcal{F}_i$ on B converging to $b_i \in B$, $i \in I$, such that the b_i's converge to $b \in B$ with respect to an ultrafilter $\mathcal{U}$ on I, there is an ultrafilter $\mathcal{V}$ on E converging to a point $x \in p^{-1}b$ such that $\bigcup_{i \in U} A_i \in \mathcal{V}$ for all $U \in \mathcal{U}$; here A_i is the set of those adherence points of the filterbase $p^{-1}\mathcal{F}_i$ which belong to $p^{-1}b_i$.* □

Although the proof of this theorem uses sophisticated filter-theoretic techniques, its basic idea is simple: embed **Top** into the locally cartesian closed category of pseudo-topological (= Choquet) spaces (where effective global-descent maps are simply quotient maps) and then "interpret" this characterization in terms of the category **Top**, using Corollary 2.7, 2. Each of the following classes of maps is (properly contained in the class of effective global-descent maps:

- *open surjections* (Moerdijk [27]; Sobral, see [34]),
- *proper* (= stably closed) *maps* (Moerdijk, Vermeulen [37]; see also [33]),
- *locally sectionable maps* (Janelidze–Tholen [16], [17]), in particular surjective local homeomorphisms.

4.2. *Open-embedding descent.* From 2.6 and 3.10 we derive immediately that every surjective map $p : E \to B$ in **Top** is $\mathbb{E}$-descent for $\mathbb{E}$ the class of open-subspace embeddings; we briefly speak about *open-descent* in this case. A complete characterization of open-descent maps can be given as follows. First observe that, for every space B, the category $\mathbb{E}(B)$ is equivalent to the (complete) lattice $\mathcal{O}(B)$ of open sets of B. Therefore we may consider p^* a monotone map $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$. Now assume p to be open-descent. Then p^* reflects isos (i.e., $p^{-1}U_1 = p^{-1}U_2$ with $U_1 \subseteq U_2 \subseteq B$ open implies $U_1 = U_2$); we say that p is *open-conservative* in this case. Since $\mathcal{O}(B)$ is a lattice, this implies that the map p^* is injective, and then that the functor p^* is full and faithful (i.e., $p^{-1}U_1 \subseteq p^{-1}U_2$ with $U_1, U_2 \subseteq B$ open implies $U_1 \subseteq U_2$). Finally, if p^* is full and faithful, then also the comparison functor Φ^p is full and faithful. Hence one has a string of equivalent conditions for open-descent maps, of which we record only the easiest one.

PROPOSITION. *A map $p : E \to B$ in* **Top** *is open-descent if and only if the map $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is injective. An open-descent map p is effective if and only if every open set $V \subseteq E$ with $V = p^{-1}(p(V))$ is of the form $V = p^{-1}U$ with $U \subseteq B$ open. Hence the surjective effective open-descent maps are exactly the quotient maps.*

Proof. We still have to show the second part of the Proposition. For an open subset $V \subseteq E$ to carry a descent structure $\xi : E \times_B V \to V$ means exactly that $V = p^{-1}(p(V))$ holds, as an easy inspection of ① of diagram (9) reveals. Now the criterion given rephrases just the fact that an open-descent map p is effective if and only if Φ^p is (essentially) surjective on objects. □

Note that an effective open-descent map need not be surjective: for p the embedding of a singleton space E into any indiscrete space B, $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is bijective, hence it is an effective open-descent map.

4.3. From now on we consider the case

$$\{\text{ open embeddings}\} \subseteq \mathbb{E} \subseteq \{\text{ local homeomorphisms}\},$$

with $\mathbb{E}$ stable under pullback. In case $\mathbb{E} = \{\text{local homeomorphisms}\}$, we speak of *étale-descent* instead of $\mathbb{E}$-descent and write étale(B) for $\mathbb{E}(B)$. We recall that the category étale(B) is equivalent to the category Sh(B) of *sheaves over B*. If $p : E \to B$ has the property that $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is bijective, then this map induces an isomorphism

$$\mathbf{Set}^{\mathcal{O}(E)^{op}} \longrightarrow \mathbf{Set}^{\mathcal{O}(B)^{op}}$$

of the presheaf categories, which restricts to an isomorphism $\mathrm{Sh}(E) \to \mathrm{Sh}(B)$ (sending each sheaf on E to its direct image under p; cf. [24], p. 68). Hence p^* : étale$(B) \to$ étale(E) is an equivalence of categories if $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is bijective.

A necessary condition for p to be an $\mathbb{E}$-descent map is that p is open-descent, hence that $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is injective (see 4.2). We may factor p as

$$(*) \qquad E \xrightarrow{\overline{p}} p(E) \overset{j}{\hookrightarrow} B$$

with j the subspace embedding. Then $j^* : \mathcal{O}(B) \to \mathcal{O}(p(E))$ is surjective, and it is also injective if $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is injective. With the previous remark we obtain that j^* : étale$(B) \to$ étale$(p(E))$ is an equivalence of categories, and that therefore its restriction $j^* : \mathbb{E}(B) \to \mathbb{E}(p(E))$ is full and faithful. Since $E \times_B E = E \times_{p(E)} E$, descent data with respect to p and to $\overline{p}$ coincide, and the diagram

$$\begin{array}{ccc} \mathbb{E}(B) & \xrightarrow{\Phi^p} & \mathrm{Des}_{\mathbb{E}}(p) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle Id} \\ \mathbb{E}(p(E)) & \xrightarrow{\Phi^{\overline{p}}} & \mathrm{Des}_{\mathbb{E}}(\overline{p}) \end{array} \tag{25}$$

commutes. This shows that for $\mathbb{E}$ = étale we may restrict our investigations to the case that p is surjective:

PROPOSITION. *A map $p : E \to B$ is an $\mathbb{E}$-descent map if p is an open-descent map and $\overline{p}$ as in $(*)$ is an $\mathbb{E}$-descent map. These conditions are also necessary in case* $\mathbb{E}$ = étale. *p is an effective étale-descent map if and only if p is an open-descent and $\overline{p}$ an effective étale-descent map. If $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is bijective, then p is an effective étale-descent map.*

Proof. We still have to prove the last statement, which does not follow automatically from the fact that p^* : étale$(B) \to$ étale(E) is an equivalence of categories if $p^* : \mathcal{O}(B) \to \mathcal{O}(E)$ is bijective. But under this condition also $\overline{p}^* : \mathcal{O}(p(E)) \to \mathcal{O}(B)$ is bijective, hence $\overline{p}$ is an open surjection and therefore an effective global-descent map (4.1). But since the class $\mathbb{E}$ = étale is easily shown to be transferable along open surjections (more generally and less trivially: along universal quotient maps, see 4.7 below), 2.7 gives that $\overline{p}$ is an effective étale-descent map. Consequently, with $\Phi^{\overline{p}}$ also Φ^p is an equivalence of categories. □

4.4. A complete characterization of étale-descent maps is given in 4.5 below. Here we first remark that étale-descent is *not* characterized by reflection of isomorphisms like global descent (see Proposition 1.6). Direct inspection yields the following easy Proposition.

PROPOSITION. *Every surjective map $p : E \to B$ is étale-conservative, i.e., p^* : étale$(B) \to$ étale(E) reflects isomorphisms.* □

With the example given at the end of 4.2, and with Proposition 4.3 we conclude that there are non-surjective étale-conservative (even effective étale-descent) maps. More importantly, an étale-conservative map need not be étale-descent: a bijective map p with E a 2-point discrete and B a 2-point indiscrete space fails to be étale-descent, as can be seen by direct inspection or be derived from 4.6 below.

4.5. With $\mathbb{E}$ as in 4.3, and for a given topological space B, we call an (additional) topology τ on an open subset $U \subseteq B$ an *$\mathbb{E}$-topology* if there is a map $\delta : D \to U$ in $\mathbb{E}$ and a **Set**-map $\varphi : U \to D$ with $\delta\varphi = 1_U$ such that τ is the weak topology on

U with respect to φ, i.e., $\tau = \{\varphi^{-1}W : W \subseteq D \text{ open }\}$. When we consider the set U with the topology τ, we denote this space by U_τ.

PROPOSITION. *A surjective map $p : E \to B$ in* **Top** *is an $\mathbb{E}$-descent map if and only if the following condition holds: for every open subspace U of B and for every $\mathbb{E}$-topology τ on U such that the restriction $p_U : p^{-1}U \to U_\tau$ is continuous, all τ-open sets are open in B.*

Proof. For "if" we reconsider the proof of 1.6 (i) $\Rightarrow$ (ii) : given $h : \Phi^p(A, \alpha) \to \Phi^p(A', \alpha')$ one just needs to show that the set map $f : A \to A'$ with $f(a) = \pi_2' h(x, a)$ and any $x \in p^{-1}(\alpha(a))$ is continuous. For that it suffices to find for every $a \in A$ a neighbourhood V of a such that $f|_V$ is continuous. But for $a \in A$ we do have an open neighbourhood V such that α gives a homeomorphism $\alpha_V : V \to \alpha(V)$. Putting $U = \alpha(V)$, $D = (\alpha')^{-1}U$, δ the restriction of α', and $\varphi = (f|_V)\alpha_V^{-1}$, we obtain a **Set**-map $\varphi : U \to D$ such that $\delta\varphi = 1_U$ and the composite φp_U (with $p_U : p^{-1}U \to U$ the restriction of p) is continuous. In fact, since π_1 can be restricted to a homeomorphism $(\pi_1)_V : \pi_1^{-1}(p^{-1}(U)) = \pi_2^{-1}(V) \to p^{-1}(U)$, one has

$$\varphi p_U = (f|_V)\alpha_V^{-1} p_U = (f|_V)(\pi_2)_V(\pi_1)_V^{-1},$$

and $(f|_V)(\pi_2)_V$ is just a restriction of the continuous map $\pi_2' h$. Continuity of the composite φp_U equivalently means that $p_U : p^{-1}U \to U_\tau$ is continuous, with τ the weak topology w.r.t. φ. By hypothesis, then $\varphi : U \to D$ is continuous, hence also $f|_V = \varphi\alpha_V$ is.

Conversely, consider an $\mathbb{E}$-topology τ on an open subspace U of B such that $p_U : p^{-1}U \to U_\tau$ is continuous. Hence with φ and (D, δ) as in the definition of $\mathbb{E}$-topology, $\varphi p_U : p^{-1}U \to D$ is continuous. The map

$$h = \langle \varphi p_U, p^{-1}U \hookrightarrow E \rangle : \Phi^p(U, j : U \hookrightarrow B) \to \Phi^p(D, \delta j)$$

corresponds, by hypothesis, to a continuous map $U \to D$ which, at the **Set**-level, must agree with φ. Hence $\varphi : U \to D$ is continuous, that is: $\mathbb{E}$-open sets are open in B. $\square$

For $\mathbb{E} = \{$ open embeddings$\}$, an $\mathbb{E}$-topology coincides with the original topology. Hence, in this case the Proposition just shows that surjections in **Top** are open-descent maps (see 4.2). A more useful easy consequence of the Proposition for $\mathbb{E}$-étale is the following

COROLLARY. *Every quotient map is an étale-descent map.* $\square$

4.6. In what follows we would like to trade the notion of $\mathbb{E}$-topology (which still refers to all objects of $\mathbb{E}(B)$) for an equivalent notion which is more internal to the space B, in case $\mathbb{E}$ = étale.

An *étale system* for an open subset U of B is a family $X = (X_{u,v})_{(u,v)\in U\times U}$ of open subsets of U such that, for all $u, v, w \in U$,

1. $u \in X_{u,u}$,
2. $X_{u,v} = X_{v,u}$,
3. $X_{u,v} \cap X_{v,w} \subseteq X_{u,w}$.

A subset $V \subseteq U$ is *X-admissible* if $V = \{v \in Y : v \in X_{u,v}\}$ for some open subset $Y \subseteq X_{u,u}$ with $u \in U$.

THEOREM. *A surjective map $p : E \to B$ in* **Top** *is an étale-descent map if and only if for every open subspace U of B and every étale-system X for U, all X-admissible subsets are open in B whenever the inverse images under p of all these sets are open in E.*

Proof. According to Proposition 4.5 it suffices to show that each étale-topology on an open set U has a subbase given by the X-admissible subsets for some étale system X for U and, vice versa, each such system defines an étale-topology in the sense of 4.5.

From an étale-topology and from the data as in 4.5 we may construct X as follows. For every $u \in U$, let D_u be an open neighbourhood of $\varphi(u)$ in D such that the restriction $\delta_u : D_u \to \delta(D_u)$ of δ is a homeomorphism. Clearly, with $X_{u,v} := \delta(D_u \cap D_v)$ we obtain an étale system for U. For every open set $W \subseteq D$ and every $u \in \varphi^{-1}W$ one can find an X-admissible set $V \subseteq U$ with $u \in V \subseteq \varphi^{-1}W$; simply take $V := \varphi^{-1}(W \cap D_u)$. In fact, with the open set $Y = \delta_u(W \cap D_u) \subseteq X_{u,u}$ one has $V = \{v \in Y : v \in X_{u,v}\}$.

Conversely, given an étale system X for U, we consider the topological sum $S = \coprod_{u\in U} X_u$, with $X_u = X_{u,u}$. Conditions 1–3 allow us to define an equivalence relation $\sim$ on S with $((u,x) \sim (v,y) \Leftrightarrow x = y \in X_{u,v})$. The continuous(!) projection $\sigma : S \to U,\ (u,x) \mapsto x$, then yields a continuous map $\delta : D = S/\sim \to U$ with $\delta q = \sigma$, q the canonical projection. Also, the **Set**-map $\varphi : U \to D,\ u \mapsto q(u,u)$, satisfies $\delta\varphi = 1_U$. For every $u \in U$ the set $q(X_u)$ is open in D since

$$q^{-1}(q(X_u)) = \bigcup_{v\in U} X_{u,v}$$

is open in U. Furthermore, the restriction $\delta_u : q(X_u) \to \delta(q(X_u)) = X_u$ of δ is obviously bijective and also open. In fact, for every open subset $W \in D$,

$$\delta_u(W \cap q(X_u)) = q^{-1}(W) \cap X_u$$

is open as a subset of U. Finally, as before one shows that the X-admissible sets generate the weak topology on U w.r.t. φ. $\square$

If one provides a two-element set with the discrete, Sierpinski, and indiscrete topology to obtain spaces E, S, and B, respectively, then an easy application of the Theorem shows that the identity map $S \to B$ is étale-descent while the identity maps $E \to X$ and $E \to B$ are not.

4.7. The following Theorem gives a sufficient condition for effective étale-descent:

THEOREM. *Every effective global-descent map is an effective étale-descent map.*

According to Proposition 2.6, the Theorem follows from the following Lemma which is of interest in its own right:

LEMMA. *If the pullback of a map α along a universal quotient map p is a local homeomorphism, then α itself is a local homeomorphism.*

Proof. We first remark that the Lemma certainly holds true if we trade "local homeomorphism" for "surjective map" or for "open map". In the latter case, just observe that $\pi_1(\pi_2^{-1}(W)) = p^{-1}(\alpha(W))$ holds for every (open) $W \subseteq A$ with the notation as in diagram (4). It therefore suffices to show that, whenever π_1 is locally injective, also α is locally injective, i.e., for every $a \in A$ there is a neighbourhood W of a such that $\alpha|_W$ is an injective map.

Indeed, for every $x \in p^{-1}b$, with $b = \alpha(a)$, by hypothesis on π_1 we can find open neighbourhoods V_x and W_x of x and a, respectively, such that π_1 maps $V_x \times_B W_x$ injectively. Since the sets V_x cover $p^{-1}b$ and since p is a universal quotient map, we can actually find finitely many sets $V_1, \ldots, V_n$ among the V_x's with $b \in U := \operatorname{int} p(V_1 \cup \cdots \cup V_n)$. With the corresponding sets $W_1, \ldots, W_n$, one obtains a neighbourhood $W = \alpha^{-1}(U) \cap W_1 \cap \cdots \cap W_n$ of a in A. We claim that α maps W one-to-one. Indeed, for $c, d \in W$ with $\alpha(c) = \alpha(d)$ one has $\alpha(c) = \alpha(d) \in U$ and therefore $\alpha(c) = \alpha(d) = p(e)$ for some $e \in V_i$. Hence $(e, c), (e, d) \in V_i \times_B W_i$ which, by hypothesis on π_1, implies $c = d$. $\square$

Since there are non-surjective effective étale descent maps (see 4.2, 4.3), effective étale-descent does not imply effective global-descent. We do not have an appropriate surjective counterexample.

4.8. The following chart summarizes some of the statements given in this section.

$$\begin{array}{ccccc}
\text{eff. global-descent} & & \Longrightarrow & & \text{eff. étale-descent} \\
\Downarrow & & & & \\
\text{global-descent} & \Leftrightarrow & \text{universal quotient} & & \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{surj. eff. open-descent} & \Leftrightarrow & \text{quotient} & \Rightarrow & \text{étale-descent} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{surjective} & \Rightarrow & \text{étale-conservative} \\
\Downarrow & & & & \Downarrow \\
\text{eff. open-descent} & & \Longrightarrow & & \text{open-descent}
\end{array}$$

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