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GENERAL TOPOLOGY — THE MONADIC CASE, EXAMPLES, APPLICATIONS

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Dedicated to Professor Ákos Császár on the occasion of his 75th birthday

Abstract. The paper deals with monadic as well as monadic-free topological notions. For defining these monadic-free notions the notion of basic triple Φ is introduced. A lot of monadic-free topological notions are presented, for instance that of Φ-convergence structure, Φ-hull operator and Φ-uniform structure. By means of a generalized metric, e.g. a probabilistic metric, and the general notion of Φ-zero approach introduced in this paper, a Φ-uniform structure is generated. In case of a fuzzy metric the related Φ-uniform structure defines in a canonic way a fuzzy topology which is used for developing a fuzzy analysis and fuzzy calculus.

1. The basic triples $\Phi = (\varphi, \leqq, \eta)$

In our theory we are interested in partially ordered sets $(X, \leqq)$ in which all non-empty suprema exist. They are called *almost complete semilattices.* Let acSLAT denote the category of almost complete semilattices where the morphisms are the mappings between almost complete semilattices which preserve non-empty suprema.

For each acSLAT-morphism $f : (X, \leqq) \to (Y, \leqq)$ a type of inverse, called the *sup-inverse* of f, can be introduced. We mean the acSLAT-morphism $g : (D, \leqq) \to (X, \leqq)$ with $D = \{ y \in Y \mid \exists x \in X \; f(x) \leqq y \}$ such that for each $y \in D$, $g(y)$ is the greatest element x of X with $f(x) \leqq y$. Here D is equipped with the induced partial ordering of $(Y, \leqq)$. In case of $D = Y$, g preserves also all infima, as far as these infima exist. If f is surjective, then $D = Y$.

An important notion in general topology is that of a *basic triple,* that is of a triple $\Phi = (\varphi, \leqq, \eta)$ with the following properties:

(A) Φ consists of a covariant functor $(\varphi, \leqq) : \mathsf{SET} \to \mathsf{acSLAT}$, $X \mapsto (\varphi X, \leqq)$ with $\varphi : \mathsf{SET} \to \mathsf{SET}$ the underlying set functor, and of a natural transformation $\eta = (\eta_X)_{X \in \mathrm{Ob}\mathsf{SET}}$ of mappings $\eta_X : X \to \varphi X$.

(B) If X is the empty set, then φX is empty.

(C) For each set X and all $x, y \in X$ the infimum $\eta_X(x) \wedge \eta_X(y)$ exists only in case of $x = y$.

For each set X the elements of φX are called *φ-objects* on X. For each non-empty subset M of a set X, we write $\eta_X[M]$ instead of $\bigvee_{x \in M} \eta_X(x)$. A

φ-object $\mathcal{M}$ of a set X is said to be *stratified* [9] provided that $\mathcal{M} \leqq \eta_X[X]$ holds.

For each mapping $f : X \to Y$, the sup-inverse of φf will be written as $\varphi^- f$. $\mathcal{D} = \{\mathcal{N} \in \varphi Y \mid \exists \mathcal{M} \in \varphi X \ \varphi f(\mathcal{M}) \leqq \mathcal{N}\}$ is the domain of $\varphi^- f$. If f is surjective, then $\mathcal{D} = \varphi Y$.

$\Phi = (\mathsf{P}_0 \circ \mathsf{id}^n, \leqq, \eta)$ is an example of a basic triple. Here φ is the composition of the n-*th power* id^n of the identity set functor with the proper powerset functor P_0, where n is any positive cardinal. Hence, for each set X, $\varphi X = \{M \subseteqq X^n \mid M \neq \emptyset\}$. As partial ordering on each set φX we take the inclusion. For each set X and each $x \in X$ let $\eta_X(x) = \{c_x\}$ with $c_x : n \to X$ the constant mapping with value x.

In classical topology, $(\mathsf{F}, \leqq, \eta)$ is taken as basic triple where F is the *filter functor,* which assigns to each set X the set $\mathsf{F}X$ of all (proper) filters on X. As partial ordering $\leqq$ of each set $\mathsf{F}X$ we choose the finer relation of filters, that is, the inversion of the inclusion. For each element x of a set X, $\eta_X(x)$ means the filter $\{M \subseteqq X \mid x \in M\}$.

In the *fuzzy filter case* we mean as Φ the basic triple $(\mathcal{F}_L, \leqq, \eta)$, defined as follows: L is a fixed non-degenerate *frame,* that is, an infinitely distributive complete lattice with different smallest element 0 and largest element 1. If no misinterpretation seems possible, for each set X and any $\alpha \in L$ we denote by $\bar{\alpha}$ the constant mapping of X into L with value α.

For each set X, $\mathcal{F}_L X$ consists of all *L-fuzzy filters* on X, that is of all mappings $\mathcal{M} : L^X \to L$ such that

(F1) $\mathcal{M}(\bar{0}) = 0$ and $\mathcal{M}(\bar{1}) = 1$,

(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

We define $\mathcal{M} \leqq \mathcal{N} \iff \mathcal{M}(f) \geqq \mathcal{N}(f)$ for all $f \in L^X$. For each mapping $g : X \to Y$, each $\mathcal{M} \in \mathcal{F}_L X$ and each $h \in L^Y$ let $\mathcal{F}_L g(\mathcal{M})(h) = \mathcal{M}(h \circ g)$. Moreover, for each $x \in X$ and $f \in L^X$ let $\eta_X(x)(f) = f(x)$.

The notion of fuzzy filter, defined as above, was proposed first by U. Höhle [12]. There are also interesting more restricted notions of fuzzy filters. Two of them were proposed first in [2, 6]. In these cases the fuzzy filters $\mathcal{M}$ are *homogeneous* and *bounded,* that is, they have the property that for each $\alpha \in L$, $\mathcal{M}(\bar{\alpha}) = \alpha$ and $\mathcal{M}(\bar{\alpha}) \leqq \alpha$, respectively. Homogeneous fuzzy filters are called *tight* in [13]. *Stratified* fuzzy filters $\mathcal{M}$ are those for which $\mathcal{M}(\bar{\alpha}) \geqq \alpha$ holds for all $\alpha \in L$. All these special types of fuzzy filters define basic triples. For more details on these fuzzy filters we refer to [9].

2. Φ-convergence structures

Let $\Phi = (\varphi, \leqq, \eta)$ be a basic triple. A Φ-*convergence structure* on a set X is a subset T of $\varphi X \times X$ such that, writing $\mathcal{M} \to x$ instead of $(\mathcal{M}, x)$

$\in T$, we have: (1) $\eta_X(x) \to x$ for all $x \in X$, (2) $\mathcal{M} \to x$ and $\mathcal{N} \leqq \mathcal{M}$ imply $\mathcal{N} \to x$, and (3) $\mathcal{M} \to x$ implies $\mathcal{M} \vee \eta_X(x) \to x$.

For $\mathcal{M} \to x$ we say that $\mathcal{M}$ *converges to* x. In the filter case a Φ-convergence structure is a *convergence structure* in sense of D. C. Kent, introduced first in [14].

A *Φ-pretopology* is a mapping $p : X \to \varphi X$ such that $\eta_X \leqq p$ holds. Each Φ-pretopology p will be identified with the Φ-convergence structure $T = \{ (\mathcal{M}, x) \mid \mathcal{M} \leqq p(x) \}$. In the fuzzy filter case the values of p are bounded fuzzy filters.

For each Φ-convergence structure T the mapping $p : X \to \varphi X$ with $p(x) = \bigvee_{\mathcal{M} \to x} \mathcal{M}$ for all $x \in X$ is the finest Φ-pretopology which is coarser than T, called the *associated Φ-pretopology.*

Clearly, a set X equipped with a Φ-convergence structure T on X is called a *Φ-convergence space.* Analogously *Φ-pretopological spaces* are defined.

A mapping $f : (X, T) \to (Y, T')$ between Φ-convergence spaces is said to be *continuous* provided $(\mathcal{M}, x) \in T$ implies $(\varphi f(\mathcal{M}), f(x)) \in T'$. A mapping $f : (X, p) \to (Y, p')$ between Φ-pretopological spaces is continuous if and only if $\varphi f \circ p \leqq p' \circ f$ holds.

Let $\Phi\mathsf{CON}$ denote the concrete category of Φ-convergence spaces with the continuous mappings between these spaces as morphisms. $\Phi\mathsf{CON}$ and the concrete subcategory of all Φ-pretopologies are topological categories.

3. Φ-hull operators

Let $\Phi = (\varphi, \leqq, \eta)$ be a basic triple. For each set X an acSLAT-morphism $\mathsf{op} : (\varphi X, \leqq) \to (\varphi X, \leqq)$, for which $\mathsf{op} \circ \mathsf{op} = \mathsf{op}$ and $1_{\varphi X} \leqq \mathsf{op}$ holds, will be called a *Φ-hull operator* on X. In some sense Φ-hull operators are generalized monadic-free topologies.

According to the following proposition each Φ-hull operator on X can be identified with a subset $\mathcal{O}$ of φX which fulfils the following condition:

(O) $\mathcal{O}$ is closed with respect to all non-empty suprema and all infima, as far as these infima exist.

PROPOSITION 1. *There is a one-to-one correspondence between the Φ-hull operators* op *on X and the subsets $\mathcal{O}$ of φX which fulfill condition* (O)*. This correspondence can be realized by taking $\mathcal{O} = \{\mathcal{M} \in \varphi X \mid \mathsf{op}\mathcal{M} = \mathcal{M}\}$ on one hand and* $\mathsf{op}\mathcal{M} = \bigwedge_{\mathcal{M} \leqq \mathcal{N} \in \mathcal{O}} \mathcal{N}$ *for all $\mathcal{M} \in \varphi X$ on the other hand.*

A set X, equipped with a Φ-hull operator on X, will be called a *Φ-hull space.* A mapping $f : (X, \mathsf{op}) \to (Y, \mathsf{op}')$ between Φ-hull spaces is said to be a *Φ-hull morphism* provided that $\varphi f \circ \mathsf{op} \leqq \mathsf{op}' \circ \varphi f$ holds.

PROPOSITION 2. *Let $f : (X, \mathsf{op}) \to (Y, \mathsf{op}')$ be a mapping between Φ-hull spaces. Moreover, let $\mathcal{O}$ be the subset of φX defined by* op *as in Proposition* 1 *and let $\mathcal{O}'$ be the subset of φY defined analogously by* op'. *Then f is a Φ-hull morphism if and only if for each $\mathcal{N} \in \mathcal{O}'$, for which the preimage $\varphi^{-} f(\mathcal{N})$ exists, we have $\varphi^{-} f(\mathcal{N}) \in \mathcal{O}$.*

The concrete category $\Phi\mathsf{OP}$ of Φ-hull spaces with the Φ-hull morphisms between these spaces as morphisms, is a topological category.

For each Φ-hull operator op on a set X the composition $p = \mathsf{op} \circ \eta_X$ is a Φ-pretopology, called the *associated Φ-pretopology* of op. Convergence in a Φ-hull space (X, op) means the convergence with respect to the associated Φ-pretopology.

PROPOSITION 3. *Each Φ-hull morphism $f : (X, \mathsf{op}) \to (Y, \mathsf{op}')$ is continuous as a mapping of (X, p) into (Y, p'), where p and p' are the associated Φ-pretopologies of* op *and* op', *respectively.*

4. The notion of partially ordered monad

By means of partially ordered monads generalized topologies in a proper sense can be defined. By a *partially ordered monad* (cf. [4, 9]) we mean a quadrupel $(\varphi, \leqq, \eta, \mu)$ such that $(\varphi, \leqq, \eta)$ is a basic triple and the following conditions are fulfilled:

(D) $\mu = (\mu_X)_{X \in \mathrm{Ob}\mathsf{SET}}$ is a natural transformation consisting of mappings $\mu_X : \varphi\varphi X \to \varphi X$ such that (φ, η, μ) is a monad over SET.

(E) For all mappings $f, g : Y \to \varphi X$, $f \leqq g$ implies $\mu_X \circ \varphi f \leqq \mu_X \circ \varphi g$, where $\leqq$ is defined argumentwise with respect to the partial ordering of φX.

(F) For each set X, $\mu_X : (\varphi\varphi X, \leqq) \to (\varphi X, \leqq)$ preserves non-empty suprema.

A non-extendable basic triple. For any cardinal $n > 1$ the triple $\Phi = (\mathsf{P}_0 \circ \mathsf{id}^n, \leqq, \eta)$ cannot be extended to a partially ordered monad (cf. [2] for $n = 2$). Clearly, in all these cases of n there is no problem to work with Φ-convergence structures, Φ-pretopologies and Φ-hull operators.

5. Examples of partially ordered monads

The partially ordered proper powerset monad. For $n = 1$ the triple $(\mathsf{P}_0 \circ \mathsf{id}^n, \leqq, \eta)$, that is $(\mathsf{P}_0, \leqq, \eta)$, is extendable. The extension is unique, as is shown in [2]. As extension we obtain the *partially ordered proper power-*

set monad $(\mathsf{P}_0, \leqq, \eta, \mu)$ where for each set X and each $\mathcal{L} \in \mathsf{P}_0\mathsf{P}_0X$, $\mu_X(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} A$.

In this case, topological interpretations in graph theory are of interest, noting that each $(\mathsf{P}_0, \leqq, \eta)$-pretopology p can be identified with the binary relation $R = \{(x,y) \mid y \in p(x)\}$.

The partially ordered filter monad. Many classical topological structures can be described by means of the *partially ordered filter monad* $(\mathsf{F}, \leqq, \eta, \mu)$. $(\mathsf{F}, \leqq, \eta)$ is the basic triple in the filter case and μ is the natural transformation consisting of all mappings $\mu_X : \mathsf{FF}X \to \mathsf{F}X$, where for each filter $\mathcal{L}$ on $\mathsf{F}X$, $\mu_X(\mathcal{L}) = \bigcup_{A \in \mathcal{L}} \bigcap_{\mathcal{M} \in A} \mathcal{M}$.

The partially ordered fuzzy filter monad. The *partially ordered fuzzy filter monad* $(\mathcal{F}_L, \leqq, \eta, \mu)$ with L a fixed non-degenerate frame is important in general fuzzy topology. $(\mathcal{F}_L, \leqq, \eta)$ is the basic triple in the fuzzy filter case and μ is the natural transformation of $\mathcal{F}_L \circ \mathcal{F}_L$ to $\mathcal{F}_L$ defined for each $\mathcal{L} \in \mathcal{F}_L\mathcal{F}_LX$ and $f \in L^X$ by $\mu_X(\mathcal{L})(f) = \mathcal{L}(e^f)$, where $e^f : \mathcal{F}_LX \to L$ is the mapping $\mathcal{M} \mapsto \mathcal{M}(f)$. For the homogeneous, the bounded and the stratified fuzzy filters analogously defined partially ordered submonads exist.

6. Monadic-free topological notions

A lot of topological notions in general topology depend only on a fixed basic triple $\Phi = (\varphi, \leqq, \eta)$, that is, they do not depend on an extension of this triple to a partially ordered monad, which, as we already noticed, even may not exist.

Some examples of monadic-free topological notions related to a Φ-convergence space are the following:

(1) *Compactness of a φ-object $\mathcal{M}$.* It means that for each $\mathcal{L} \leqq \mathcal{M}$ there are $\mathcal{N} \leqq L$ and x with $\mathcal{N} \to x$ and $\eta_X(x) \leqq \mathcal{M}$. *Compactness of a subset M* means compactness of $\eta_X[M]$, and *compactness of the space* means compactness of the underlying set of this space.

(2) *Local compactness* of a Φ-convergence space. It means that for any converging φ-object $\mathcal{M}$ there exists a compact subset M of X with $\mathcal{M} \leqq \eta_X[M]$.

(3) *Adherence point* x of a φ-object $\mathcal{M}$: x is an element of X such that $\mathcal{M} \wedge \mathcal{N}$ exists for some $\mathcal{N} \to x$.

(4) *Projective closedness of a φ-object $\mathcal{M}$.* It means that for each adherence point x of $\mathcal{M}$ we have $\eta_X(x) \leqq \mathcal{M}$.

(5) The *separation axioms* T_0, T_1 and T_2. They mean that (a) $\eta_X(x) \to y$ and $\eta_X(y) \to x$ imply $x = y$, that (b) $\eta_X(x) \to y$ implies $x = y$, and that (c) $\mathcal{M} \to x$ and $\mathcal{M} \to y$ imply $x = y$, respectively.

On these notions a lot of results are given, for instance, a generalized Tychonoff Theorem (cf. e.g. [5, 7]). A further monadic-free topological notion is that of a Φ-*Cauchy structure* (cf. [1]).

7. The monadic notions of neighborhood and closure operators

Let $(\varphi, \leqq, \eta, \mu)$ be a partially ordered monad and T a $(\varphi, \leqq, \eta)$-convergence structure on a set X. As *neighborhood operator* of T ([4, 9]) we mean the composition $\mathsf{nb} = \mu_X \circ \varphi p$, where p is the associated $(\varphi, \leqq, \eta)$-pretopology of T. $\mathsf{nb} : (\varphi X, \leqq) \to (\varphi X, \leqq)$ is an acSLAT-morphism and a hull operator. Since $\mathsf{nb} \circ \eta_X = p$ holds, there is a one-to-one correspondence between the neighborhood operators and the related $(\varphi, \leqq, \eta)$-pretopologies.

Classical case: For each filter $\mathcal{M}$ on a usual topological space, $\mathsf{nb}\mathcal{M}$ has the set of all open sets $M \in \mathcal{M}$ as a base.

A φ-object $\mathcal{M}$ on X is called *open* provided that $\mathcal{M} = \mathsf{nb}\mathcal{M}$. Since nb is an acSLAT-morphism which is a hull-operator, the set $\mathcal{O}$ of all open φ-objects fulfils the condition (O), introduced in Section 4. For the related Φ-hull operator op we have $\mathsf{nb} \leqq \mathsf{op}$.

In the following let t_1 and t_2 denote the first and the second projections $(\mathcal{M}, x) \mapsto \mathcal{M}$ and $(\mathcal{M}, x) \mapsto x$ of T into φX and X, respectively.

PROPOSITION 4 [9]. *The sup-inverse $\varphi^- t_2$ of φt_2 has φX as domain and* $\mathsf{nb} = \mu_X \circ \varphi t_1 \circ \varphi^- t_2$.

By a similar procedure, a monadic notion of closure can be defined. We have that the sup-inverse of $\mu_X \circ \varphi t_1$, denoted in the following by $\varphi_\mu^- t_1$, has φX as domain. The composition $\mathsf{cl} = \varphi t_2 \circ \varphi_\mu^- t_1$ is called the *closure operator* of T [4].

PROPOSITION 5 (cf. [4, 9]). $\mathsf{cl} : (\varphi X, \leqq) \to (\varphi X, \leqq)$ *is an* acSLAT-*morphism and* $1_{\varphi X} \leqq \mathsf{cl}$ *holds.*

Classical case: For each filter $\mathcal{M}$ on a usual topological space, $\mathsf{cl}\mathcal{M}$ is the filter which has the set of all closures of the sets $M \in \mathcal{M}$ as a base.

A φ-object $\mathcal{M}$ is said to be *closed* provided that $\mathcal{M} = \mathsf{cl}\mathcal{M}$.

By means of the operators nb and cl the following *monadic separation axioms* can be introduced: (1) The *separation axiom* T_1^+. It means that $\eta_X(x)$ is closed for all $x \in X$. (2) The *regularity*. It means that $\mathcal{M} \to x$ implies $\mathsf{cl}\mathcal{M} \to x$. (3) The *normality*. It means that $\mathsf{cl}(\mathsf{nb}\mathcal{M}) \leqq \mathsf{nb}(\mathsf{cl}\mathcal{M})$ holds for all $\mathcal{M} \in \varphi X$.

PROPOSITION 6 [9]. *For topological spaces we have the following: Both axioms T_1^+ and T_1 coincide with the usual first separation axiom and separatedness, regularity and normality coincide with the corresponding classical notions.*

In general, T_1^+ implies T_1 and for a $(\varphi, \leqq, \eta)$-pretopological space regularity and T_1 imply T_2 and normality and T_1^+ imply regularity.

8. Monadic topologies

Let $\Phi = (\varphi, \leqq, \eta, \mu)$ be a partially ordered monad. By a *Φ-topology*, also called a *monadic topology*, we mean a $(\varphi, \leqq, \eta)$-pretopology p such that $\mu_X \circ \varphi p \circ p = p$. The property of p being a monadic topology can be written as $\mathsf{nb} \circ p = p$ and is equivalent to $\mathsf{nb} \circ \mathsf{nb} = \mathsf{nb}$ [4].

Monadic topologies can be characterized as follows.

PROPOSITION 7. *A $(\varphi, \leqq, \eta)$-hull operator* op *is, up to an identification, a monadic topology p if and only if it coincides with the neighborhood operator* nb *of p, that is, if and only if* $\mathsf{nb}\mathcal{M} = \bigwedge_{\mathcal{N} \in \mathcal{O}, \mathcal{M} \leqq \mathcal{N}} \mathcal{N}$ *for all $\mathcal{M} \in \varphi X$, where $\mathcal{O}$ is the subset of φX identified with* op.

Thus, a monadic topology is on one hand a special $(\varphi, \leqq, \eta)$-pretopology and on the other hand a special $(\varphi, \leqq, \eta)$-hull operator. The following result says that the morphisms between monadic topological spaces have the analogous property.

PROPOSITION 8. *A mapping $f : (X, p) \to (Y, p')$ between Φ-topological spaces is continuous if and only if $f : (X, \mathsf{nb}) \to (Y, \mathsf{nb}')$ is a Φ-hull morphism.*

9. Generalized uniform structures

Let $\Phi = (\varphi, \leqq, \eta)$ be a basic triple and X a set. By π_{12}, π_{23} and π_{13} we mean the mappings of X^3 into X^2 which assign to each triple $(x, y, z) \in X^3$ the pair (x, y), (y, z) and (x, z), respectively.

PROPOSITION 9. *Let $\mathcal{U}$ and $\mathcal{V}$ be φ-objects on X^2 and let x, y and z be elements of X such that $\eta_{X^2}(x, y) \leqq \mathcal{U}$ and $\eta_{X^2}(y, z) \leqq \mathcal{V}$ hold. Then the infimum $\varphi^{-}\pi_{12}(\mathcal{U}) \wedge \varphi^{-}\pi_{23}(\mathcal{V})$ exists. For $\mathcal{V} \circ \mathcal{U} = \varphi\pi_{13}\big(\varphi^{-}\pi_{12}(\mathcal{U}) \wedge \varphi^{-}\pi_{23}(\mathcal{V})\big)$, called the relational product of $\mathcal{U}$ and $\mathcal{V}$, we have $\eta_{X^2}(x, z) \leqq \mathcal{V} \circ \mathcal{U}$.*

For each φ-object $\mathcal{U}$ on X^2, $\mathcal{U}^{-1} = \varphi s(\mathcal{U})$ with $s : X^2 \to X^2$ the mapping $(x, y) \mapsto (y, x)$, is called the *inverse* of $\mathcal{U}$.

PROPOSITION 10. *If $\mathcal{U}$ and $\mathcal{V}$ are φ-objects on X^2 and there are $x, y, z \in X$ such that $\eta_{X^2}(x, y) \leqq \mathcal{U}$ and $\eta_{X^2}(y, z) \leqq \mathcal{V}$ hold, then $(\mathcal{V} \circ \mathcal{U})^{-1} = \mathcal{U}^{-1} \circ \mathcal{V}^{-1}$.*

Let Δ denote the diagonal $\{(x, x) \mid x \in X\}$ of X^2.

PROPOSITION 11. *For each stratified φ-object $\mathcal{M}$ on X we have $\varphi d(\mathcal{M}) \leqq \eta_{X^2}[\Delta] \wedge \varphi^{-}\pi_1(\mathcal{M})$, where π_1 is the first projection of X^2 and $d : X \to X^2$ is the mapping $x \mapsto (x,x)$.*

By means of this proposition we obtain the following.

PROPOSITION 12. *Let $\mathcal{M}$ be a stratified φ-object on X and $\mathcal{U}$ a φ-object on X^2 for which $\eta_{X^2}[\Delta] \leqq \mathcal{U}$ holds. Then the infimum $\mathcal{U} \wedge \varphi^{-}\pi_1(\mathcal{M})$ exists, its image $\mathcal{U}[\mathcal{M}] = \varphi\pi_2\big(\mathcal{U} \wedge \varphi^{-}\pi_1(\mathcal{M})\big)$ is also stratified and we have $\mathcal{M} \leqq \mathcal{U}[\mathcal{M}]$.*

PROPOSITION 13. *Let $\mathcal{U}$ and $\mathcal{V}$ be φ-objects on X^2 such that $\eta_{X^2}[\Delta] \leqq \mathcal{U}$ and $\eta_{X^2}[\Delta] \leqq \mathcal{V}$ hold. Moreover, let $\mathcal{M}$ be a stratified φ-object on X. Then we have $\big[\mathcal{V}\big[\mathcal{U}[\mathcal{M}]\big] \leqq (\mathcal{V} \circ \mathcal{U})[\mathcal{M}]$.*

In the fuzzy filter case, *relational φ-objects*, that is, φ-objects on squares X^2 of sets, were investigated in detail in the joint paper [10] with F. Bayoumi et al.

By a *Φ-uniform structure* on X we mean a φ-object $\mathcal{U}$ on X^2 such that (1) $\eta_{X^2}[\Delta] \leqq \mathcal{U}$, (2) $\mathcal{U}^{-1} = \mathcal{U}$, and (3) $\mathcal{U} \circ \mathcal{U} \leqq \mathcal{U}$ hold.

In the following let a Φ-uniform structure $\mathcal{U}$ on X be fixed. A stratified Φ-object $\mathcal{M}$ on X is called *uniformly open* provided that $\mathcal{U}[\mathcal{M}] = \mathcal{M}$. For each stratified Φ-object $\mathcal{M}$, $\mathcal{U}[\mathcal{M}]$ is uniformly open.

We define a Φ-pretopology p on X by taking $p(x) = \mathcal{U}\big[\eta_X(x)\big]$ for all $x \in X$. The φ-objects $p(x)$ are uniformly open. p is called the *associated Φ-pretopology* of $\mathcal{U}$.

Clearly, a set X equipped with a Φ-uniform structure on X is called a *Φ-uniform space.*

A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between Φ-uniform spaces is called *uniformly continuous* provided that $\varphi(f \times f)(\mathcal{U}) \leqq \mathcal{V}$ holds.

PROPOSITION 14. *Each uniformly continuous mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between Φ-uniform spaces is continuous as a mapping between the associated Φ-pretopological spaces.*

10. Monadic uniform structures and the associated monadic topologies

In the following let a partially ordered monad $(\varphi, \leqq, \eta, \mu)$ be fixed and let $\Phi = (\varphi, \leqq, \eta)$. Let $\mathcal{U}$ be a Φ-uniform structure on a set X, p be the associated Φ-pretopology of $\mathcal{U}$ and $\mathsf{nb} = \mu_X \circ \varphi p$. $\mathcal{U}$ will be called a *monadic uniform structure* if for each stratified φ-object $\mathcal{M}$ on X we have $\mathsf{nb}\mathcal{M} \leqq \mathcal{U}[\mathcal{M}]$.

PROPOSITION 15. *For each monadic uniform structure $\mathcal{U}$ the associated Φ-pretopology p is a monadic topology (that is, $\mathsf{nb} \circ p = p$).*

Fuzzy filter case: In this case monadic uniform structures were treated first in the joint paper [10] with F. Bayoumi et al.

11. Φ-uniform structures generated by D-metrics and Φ-zero approaches

Let $\mathbf{D} = (D, \leqq, 0, +)$ be a partially ordered commutative monoid, that is, $\mathbf{D}$ is a commutative monoid $(D, 0, +)$ equipped with a partial ordering $\leqq$ such that 0 is the smallest element of D and for all $a, b, c, d \in D$ from $a \leqq b$ and $c \leqq d$ it follows $a + c \leqq b + d$.

By a **D**-*metric* on a set X we mean a mapping $\rho : X^2 \to D$ such that for all $x, y, z \in X$ we have: (1) $\rho(x, y) = 0$ if and only if $x = y$, (2) $\rho(x, y) = \rho(y, x)$, and (3) $\rho(x, z) \leqq \rho(x, y) + \rho(y, z)$. This notion appeared already in Trillas [15].

Let $\Phi = (\varphi, \leqq, \eta)$ be a basic triple. It is important to notice that for generating a Φ-uniform structure by using a **D**-metric, an additional structure on **D** is needed, called a *Φ-zero approach* on **D**. By this we mean a φ-object $\mathcal{E}$ on D such that the following conditions are fulfilled:

(1) $\eta_D(0) \leqq \mathcal{E}$.

(2) For all mappings f, g, h of a set M into D, for which $h \leqq f + g$ holds and there is an element e of M with $f(e) = g(e) = h(e) = 0$, we have $\varphi^- f(\mathcal{E}) \wedge \varphi^- g(\mathcal{E}) \leqq \varphi^-(h)(\mathcal{E})$.

Clearly, the partial ordering and the addition appearing in the inequality $h \leqq f + g$ are defined argumentwise by means of the partial ordering and the addition on D. Notice that the preimages and the infimum in (2) always exist.

In the following let a **D**-metric ρ on a set X and a Φ-zero approach $\mathcal{E}$ on **D** be fixed. Because of property (1) of $\mathcal{E}$ the preimage $\mathcal{U} = \varphi^- \rho(\mathcal{E})$ exists. Applying property (2) of $\mathcal{E}$ on the triangle inequality of ρ, which can be written as $\rho \circ \pi_{13} \leqq \rho \circ \pi_{12} + \rho \circ \pi_{23}$, shows that $\varphi^- \pi_{12}(\mathcal{U}) \wedge \varphi^- \pi_{23}(\mathcal{U}) \leqq \varphi^- \pi_{13}(\mathcal{U})$ holds. We even have the following.

PROPOSITION 16. *$\mathcal{U}$ is a Φ-uniform structure on X.*

12. A standard example of a Φ-zero approach

Let L be a non-degenerate frame with smallest element 0 and largest element 1, and let $(K, \leqq, 0_K, \oplus)$ be a partially ordered commutative monoid such that K (with respect to $\leqq$) is a non-degenerate frame. The largest element of K will be denoted by 1_K.

We introduce a partially ordered commutative monoid $\mathbf{D} = (D, \leqq, \mathrm{O}, +)$ as follows:

D is the set of all isotone mappings $x : K \to L$ such that for each $\alpha \in L$ there exists the greatest element ξ of K with $x(\xi) \leqq \alpha$ and we have $x(1_K) = 1$. For each $x \in D$, $x(0_K) = 0$ holds. Each $x \in D$ and its *sup-inverse* $x^- : L \to K$, given by $x^-(\alpha) = \bigvee_{x(\xi) \leqq \alpha} \xi$ for all $\alpha \in L$, define a *Galois-connection*, that is, for all $\xi \in K$ and $\alpha \in L$ we have $x(\xi) \leqq \alpha \Leftrightarrow \xi \leqq x^-(\alpha)$. Clearly, $x(\xi) = \bigwedge_{\xi \leqq x^-(\alpha)} \alpha$ for all $\xi \leqq K$.

The partial ordering on D is given by $x \leqq y \Leftrightarrow x(\xi) \geqq y(\xi)$ for all $\xi \in K$. Hence, $x \leqq y$ holds if and only if we have $x^-(\alpha) \leqq y^-(\alpha)$ for all $\alpha \in L$.

The addition on D is given by $(x + y)^-(\alpha) = x^-(\alpha) \oplus y^-(\alpha)$ for all $\alpha \in L$ and the zero-element O on D by $\mathrm{O}(\xi) = 1$ for all $\xi > 0_K$ and $\mathrm{O}(0_K) = 0$ or equivalently by $\mathrm{O}^-(\alpha) = 0_K$ for all $\alpha < 1$ and $\mathrm{O}^-(1) = 1_K$.

Let $\mathcal{E} : L^D \to L$ be the mapping defined by $\mathcal{E}(f) = \bigvee_{s^\varepsilon \wedge \bar{\alpha} \leqq f,\, \varepsilon > 0_K} \alpha$ for all $f \in L^D$, where for each $\varepsilon \in K$ with $\varepsilon > 0_K$, $s^\varepsilon : D \to L$ is the mapping $x \mapsto x(\varepsilon)$ and for each $\alpha \in L$, $\bar{\alpha} : D \to L$ is the constant mapping $x \mapsto \alpha$.

PROPOSITION 17. *$\mathcal{E}$ is an $(\mathcal{F}_L, \leqq, \eta)$-zero approach. Hence, for each $\mathbf{D}$-metric ρ on a set X, $\mathcal{U} = \mathcal{F}_L^- \rho(\mathcal{E})$ is an $(\mathcal{F}_L, \leqq, \eta)$-uniform structure on X.*

Proposition 17 remains true if we additionally assume the elements x of D to have the *finite property:* $\alpha < 1$ implies $x^-(\alpha) < 1_K$, or even to have the *boundedness property:* $\bigvee_{\alpha < 1} x^-(\alpha) < 1_K$ holds.

13. The real fuzzy case

Let L be a non-degenerate frame equipped with an order-reversing involution $'$ and let K be the closed interval $[0, \infty]$ equipped with the usual ordering and the usual addition of the non-negative real numbers extended by ∞. We restrict D, defined as in the preceding section, to the mappings which additionally have the boundedness property. Instead of the mappings $x \in D$ we can take the mappings $x' : K \to L$, $\xi \mapsto x(\xi)'$, which can be interpreted as the non-negative *fuzzy numbers* with value 1 at 0 ([3]). The addition on D is then given by the *addition of the level sets.* In our case, each $\mathbf{D}$-metric ρ is, up to an identification, a *fuzzy metric* in sense of [3], the related $(\mathcal{F}_L, \leqq, \eta)$-zero approach appear already in [3]. The associated $(\mathcal{F}_L, \leqq, \eta)$-uniform structure is monadic, hence the associated $(\mathcal{F}_L, \leqq, \eta)$-pretopology is a monadic topology, called the *canonic fuzzy topology* of ρ ([3]). Using this fuzzy topology, in [8, 11] basic results on fuzzy analysis and on fuzzy calculus are given.

14. The probabilistic case

In the following let $L = [0, \infty]$ and let $(K, \leqq, 0, \oplus)$ be a partially ordered commutative monoid with $K = [0, 1]$ and $\leqq$ the usual ordering of $[0, 1]$. $\oplus$ is called a *triangle conorm* [16]. By means of L and $(K, \leqq, 0, \oplus)$ let $\mathbf{D} = (D, \leqq, \mathrm{O}, +)$ be defined as in Section 12. Instead of each $x \in D$ we can take its sup-inverse x^-. The addition of these sup-inverses is the *argumentwise defined addition* with respect to $\oplus$. The $\mathbf{D}$-metrics are up to identifications the *probabilistic metrics* in sense of Schweitzer and Sklar [16]. Such identifications can be obtained by taking instead of each sup-inverse $x^- : L \to K$ the mapping $x^* : L \to K$, $\alpha \mapsto 1 - x^-(\alpha^{-1})$ and by changing from $\oplus$ to the triangular norm $\otimes$ given by $1 - (\xi \otimes \nu) = (1 - \xi) \oplus (1 - \nu)$.

The $(\mathcal{F}_L, \leqq, \eta)$-uniform structure defined as in Section 12 by a $\mathbf{D}$-metric ρ from above can be considered as a *probabilistic uniform structure* with respect to ρ.

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