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A NOTE ON NUCLEI OF QUANTALE ALGEBRAS*

Abstract

The paper considers the role of quantale algebra nuclei in representation of quotients of quantale algebras, and in factorization of quantale algebra homomorphisms. The set of all nuclei on a given quantale algebra is endowed with the structure of quantale semi-algebra.

Keywords: Extremal quotient, quantale, quantale algebra, quantale algebra nucleus, quantale module, sup-algebra, sup-algebra nucleus, $\bigvee$-semilattice.

1. Introduction

This paper is a continuation of our study on *quantale algebras*, introduced in [26] as a mixture of the notions of *quantale* [20] and *quantale module* [11]. Both concepts have a rich history and applications in a variety of contexts.

The study of quantale-like structures goes back up to the 1930's, when M. Ward and R. P. Dilworth [30] started their research on residuated structures, motivated by ring-theoretic considerations. The term quantale was coined much later in 1986 by C. J. Mulvey [14], in his attempt to provide a possible setting for constructive foundations of quantum mechanics, and to study the spectra of non-commutative C^*-algebras, which are locales in the commutative case. The combination of "quantum logic" and "locale" gave rise to "quantale". The latter notion generalizes the well-known concept of *complete residuated lattice*, dropping off commutativity of its binary operation (that results in two residuals), and removing the condition on the

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top element of the lattice to be the unit. One of the most important applications of quantale theory is the area of linear logic of J.-Y. Girard [6]. In particular, [1] backs the slogan "quantales are to linear logic as frames are to intuitionistic logic".

The concept of *quantale module* was motivated by the notion of module over a ring [2]. It replaces rings by quantales and abelian groups by complete lattices. The first lattice analogy of ring module appeared in the paper of A. Joyal and M. Tierney [10], in connection with the analysis of descent theory. Despite the fact that the authors work with commutative structures, most of their results are valid for non-commutative case, as well. The idea of quantale module appeared for the first time as the key notion in the unified treatment of process semantics done by S. Abramsky and S. Vickers [1]. Modules over a commutative unital quantale provide a family of models of full linear logic as was shown by K. I. Rosenthal [21]. Moreover, quite recently, modules over a unital quantale found an application in the theory of lattice-valued topological spaces [25].

By analogy with the concept of algebra over a ring [2], we decided to unite the above-mentioned two notions into the concept of *quantale algebra* [26], which is a quantale and also a quantale module. The motivation for the new notion came from the realm of lattice-valued topology, namely, from the need of a common framework for two different topological settings: that of J. A. Goguen [7] (extending the approach of C. L. Chang [4]) and that of R. Lowen [13]. Recall that J. A. Goguen defines a *Q-topological space* over a unital quantale Q as a pair (X, τ), where X is a set and τ is a unital subquantale of the powerset quantale Q^X, whereas R. Lowen puts an additional restriction of all constant maps (maps of the form $X \xrightarrow{\underline{q}} Q$ for some $q \in Q$, with $\underline{q}(x) = q$ for every $x \in X$) being elements of τ (*stratification*). It appears that if Q is a commutative quantale, then the latter condition results precisely in the notion of subalgebra of the quantale algebra Q^X.

Since the most important features of many-valued topological spaces depend on the underlying algebraic structure used, we decided to investigate the properties of quantale algebras more closely. Bearing in mind that every quantale is based on a $\bigvee$*-semilattice* (partially ordered set having arbitrary joins), the study of which relies heavily on the concept of *closure operator* [20], we decided to follow the same path in the new setting. In particular, motivated by the famous representation theorem for quantales of K. I. Rosenthal [20], we proved a similar result for quantale algebras [26], which was based on the notion of *quantale algebra nucleus* (induced by *quantale* or *quantic nuclei* [20] and *module nuclei* [16], which in their turn were motivated by *frame nuclei* [9]) generalizing the above-mentioned closure operator. Later on, we used the concept to construct coproducts of quantale algebras [24], thereby obtaining some properties of products of pointless lattice-valued topological spaces in the sense of P. T. Johnstone [9]. Moreover, there exists a relation between nuclei and (*existential*) *quantifiers* on quantale algebras (every quantifier is a nucleus but not vice versa) [23], the latter being motivated by monadic logic of P. Halmos [8]. We even tried to generalize the famous topological representation for monadic Boolean algebras of the latter author, employing the tools of lattice-valued topology, the task, however, encountering some difficulties had not been brought to the end. Further generalizations of the topic of quantale algebras and its applications to many-valued topology can be found in [22, 27, 28].

The above-mentioned results motivated the need for a thorough build up of the theory of quantale algebra nuclei. Some of its elements (in a scattered way) has already appeared in the aforesaid papers. It is the main purpose of this manuscript to develop the theory in one place in its full extent. Based on the results of [18], which provide the tools for representation of quotients of frames (or sublocales) in terms of frame nuclei, we consider representations of quotients of quantale algebras as well as factorizations of quantale algebra homomorphisms. The latter goal is achieved by employing the technique of producing subalgebras of quantale algebras by means of extending a relation to a congruence.

In [9] P. T. Johnstone showed that the set of all nuclei on a given frame (or locale) can be equipped with the structure of complete Heyting algebra. Following the line, S. B. Niefield and K. I. Rosenthal [15] suggested a structure on the set of all nuclei on a given quantale, which turns this set into a quantale. The proposal, unluckily, appeared to be a mistake as was shown by S.-H. Sun [29]. One of the last sections of this paper provides a partial solution to the problem. It endows the set of all nuclei on a given quantale algebra with the structure of *quantale semi-algebra*. The restriction of this structure to a particular subset gives a quantale algebra.

2. Quantale algebras

In this section we recall from [26] the notion of *quantale algebra* over a given unital commutative quantale Q. Start by recalling the definition of *quantale* [20].

Definition 1. A *quantale* is a triple $(Q, \leqslant, \otimes)$ such that

1. $(Q, \leqslant)$ is a $\bigvee$-semilattice, i.e., a partially ordered set having arbitrary joins;
2. $(Q, \otimes)$ is a semigroup;
3. $q \otimes (\bigvee S) = \bigvee_{s \in S}(q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S}(s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

A quantale Q is said to be *unital* provided that there exists an element $\imath \in Q$ such that $(Q, \otimes, \imath)$ is a monoid. Q is said to be *commutative* provided that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in Q$. ■

Every quantale, being a complete lattice, has the largest element $\top$ and the smallest element $\perp$. The following are basic examples of quantales.

Example 2. If $\mathbf{2} = \{\perp, \top\}$, then $(\mathbf{2}, \leqslant, \wedge, \top)$ is a unital commutative quantale. ■

Example 3. Let $(A, \cdot)$ be a semigroup. The powerset $\mathcal{P}(A)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{s \cdot t \mid s \in S,\ t \in T\}$. If $(A, \cdot, \imath)$ is a monoid, then $\mathcal{P}(A)$ is unital, with the unit $\{\imath\}$. If $(A, \cdot)$ is commutative, then so is $\mathcal{P}(A)$. ■

Example 4. Let X be a set and let $\mathcal{R}(X)$ stand for the set of all binary relations on X. $\mathcal{R}(X)$ is a quantale, where $\bigvee$ are unions and $\otimes$ is given by $S \otimes T = \{(x, y) \in X \times X \mid (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in X\}$. $\mathcal{R}(X)$ is unital, with the diagonal relation $\triangle = \{(x, x) \mid x \in X\}$ being the unit. ■

Definition 5. Let Q_1 and Q_2 be quantales. A map $Q_1 \xrightarrow{f} Q_2$ is a *quantale homomorphism* provided that f preserves $\otimes$ and $\bigvee$. ■

On the next step, we recall from [11] the notion of (*left*) (*unital*) *Q-module* over a given quantale Q. The definition is motivated by the concept of (left) (unital) R-module over a given ring R [2].

Definition 6. Let Q be a quantale. A (*left*) *Q-module* over Q is a pair $(A, *)$, where A is a $\bigvee$-semilattice and $Q \times A \xrightarrow{*} A$ is a map (*action* of Q on A) such that

1. $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$ and every $S \subseteq A$;
2. $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ for every $a \in A$ and every $S \subseteq Q$;
3. $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every q_1, $q_2 \in Q$ and every $a \in A$.

If Q is unital, then $(A, *)$ is called a *unital* (left) Q-module provided that

(4) $\imath * a = a$ for every $a \in A$.

A (*left*) *Q-module homomorphism* $(A, *) \xrightarrow{f} (B, *)$ is a $\bigvee$-preserving map $A \xrightarrow{f} B$ such that $f(q * a) = q * f(a)$ for every $a \in A$ and every $q \in Q$. ■

For the sake of shortness, from now on, "Q-module" means "left Q-module". Some properties of Q-modules are considered in [25]. In particular, one can easily see that every $\bigvee$-semilattice is a **2**-module (recall that an abelian group is a unital module over the integers $\mathbb{Z}$ [2]). Also notice that every quantale is a module over itself, with the action given by multiplication.

Now we are ready to define the notion of *Q-algebra* over a given unital commutative quantale Q. The definition is motivated by the concept of K-algebra over a unital commutative ring K [2].

Definition 7. Let Q be a unital commutative quantale. A *Q-algebra* over Q is a triple $(A, *, \otimes)$ such that

1. $(A, *)$ is a unital Q-module;
2. $(A, \otimes)$ is a quantale;
3. $q * (a \otimes b) = (q * a) \otimes b = a \otimes (q * b)$ for every $a, b \in A$ and every $q \in Q$.

A *Q-algebra homomorphism* $(A, *, \otimes) \xrightarrow{f} (B, *, \otimes)$ is a quantale homomorphism $(A, \otimes) \xrightarrow{f} (B, \otimes)$ which is also a Q-module homomorphism. ■

It is clear that every quantale is a **2**-algebra. Also notice that every unital commutative quantale is an algebra over itself, with the action given by multiplication.

From now on, we consider algebras over a fixed unital commutative quantale Q. The following definition and subsequent lemma will be extremely useful for us throughout the rest of the paper.

Definition 8. A Q-algebra homomorphism $A \xrightarrow{e} B$ is called *final* provided that given a Q-algebra homomorphism $A \xrightarrow{f} C$ and a map $B \xrightarrow{h} C$ such that $h \circ e = f$, it follows that h is a Q-algebra homomorphism. ■

Lemma 9. *Every surjective Q-algebra homomorphism is final.*

Proof: Suppose we are given Q-algebra homomorphisms $A \xrightarrow{e} B$, $A \xrightarrow{f} C$ with e being surjective, and a map $B \xrightarrow{h} C$ such that $h \circ e = f$. Straightforward computations show that h is a Q-algebra homomorphism. For example, given $q \in Q$ and $b \in B$, there exists $a \in A$ such that $e(a) = b$ and, therefore, $h(q * b) = h(q * e(a)) = h \circ e(q * a) = f(q * a) = q * f(a) = q * (h \circ e)(a) = q * h(b)$. □

Lemma 9 can be readily applied to the case of $\bigvee$-semilattice homomorphisms or quantale homomorphisms, that will be done in Theorem 28 and its generalizations.

3. Extremal quotients of quantale algebras

In this section, we consider extremal quotients of quantale algebras, which arise from the notion of *extremal Q-algebra epimorphism*. For convenience of the reader, we begin with the definition of the latter concept (cf. [18, Section 2.2]).

Definition 10. A Q-algebra homomorphism e (resp. m) is called an *epimorphism* (resp. *monomorphism*) provided that for every Q-algebra homomorphisms g, h such that $g \circ e = h \circ e$ (resp. $m \circ g = m \circ h$), it follows that $g = h$. A Q-algebra epimorphism e is called *extremal* provided that for every Q-algebra homomorphisms f, m with $e = m \circ f$, if m is a monomorphism, then m must be an isomorphism. ■

Notice that the above-mentioned concepts of epimorphism and monomorphism deviate from their usual definitions in universal algebra. Our point here is to develop a rather category-theoretic viewpoint on the topic, to achieve its full generality. Definition 10 suggests a clarification of the nature of Q-algebra monomorphisms.

Proposition 11. *Q-algebra monomorphisms are precisely the homomorphisms with injective underlying maps.*

Proof: It will be enough to show the necessity. Start by constructing a particular Q-algebra. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the additive semigroup of all positive integers. The set $Q^{\mathbb{N}}$ of all maps $\mathbb{N} \xrightarrow{\alpha} Q$ is a unital Q-module with the operations given point-wise. Define a binary operation $\otimes$ on $Q^{\mathbb{N}}$ by $(\alpha \otimes \beta)(z) = \bigvee_{x+y=z} \alpha(x) \otimes \beta(y)$ (cf. *monoid rings* of [12]). Straightforward computations show that $Q^{\mathbb{N}}$ is a Q-algebra. As an example, check Item (3) of Definition 7. Given $q \in Q$, $\alpha, \beta \in Q^{\mathbb{N}}$ and $z \in \mathbb{N}$, $(q * (\alpha \otimes \beta))(z) = q \otimes ((\alpha \otimes \beta)(z)) = q \otimes (\bigvee_{x+y=z} \alpha(x) \otimes \beta(y)) = \bigvee_{x+y=z} (q \otimes \alpha(x) \otimes \beta(y)) = \bigvee_{x+y=z} ((q * \alpha)(x) \otimes \beta(y)) = ((q * \alpha) \otimes \beta)(z)$. It follows that $q * (\alpha \otimes \beta) = (q * \alpha) \otimes \beta$.

Let $A \xrightarrow{f} B$ be a Q-algebra monomorphism with $f(a) = f(b)$ for some $a, b \in A$. Define the following two maps: $Q^{\mathbb{N}} \xrightarrow{h_a} A$, $h_a(\alpha) = \bigvee_{x \in \mathbb{N}} (\alpha(x) * a^x)$ and $Q^{\mathbb{N}} \xrightarrow{h_b} A$, $h_b(\alpha) = \bigvee_{x \in \mathbb{N}} (\alpha(x) * b^x)$. Straightforward computations show that both maps are homomorphisms of Q-algebras. As an example, check that h_a is a quantale homomorphism. Given $\alpha, \beta \in Q^{\mathbb{N}}$, $h_a(\alpha \otimes \beta) = \bigvee_{x \in \mathbb{N}} ((\alpha \otimes \beta)(x) * a^x) = \bigvee_{x \in \mathbb{N}} ((\bigvee_{y+z=x} \alpha(y) \otimes \beta(z)) * a^x) = \bigvee_{x \in \mathbb{N}} (\bigvee_{y+z=x} \alpha(y) * (\beta(z) * (a^y \otimes a^z))) = \bigvee_{x \in \mathbb{N}} (\bigvee_{y+z=x} \alpha(y) * (a^y \otimes (\beta(z) * a^z))) = \bigvee_{x \in \mathbb{N}} (\bigvee_{y+z=x} (\alpha(y) * a^y) \otimes (\beta(z) * a^z)) = (\bigvee_{x \in \mathbb{N}} \alpha(x) * a^x) \otimes (\bigvee_{x \in \mathbb{N}} \beta(x) * a^x) = h_a(\alpha) \otimes h_a(\beta)$.

Since $f \circ h_a = f \circ h_b$, then $h_a = h_b$. Define $\alpha \in Q^{\mathbb{N}}$ by $\alpha(x) = \imath$ for $x = 1$; otherwise, $\alpha(x) = \bot$. It follows that $a = \bigvee_{x \in \mathbb{N}} (\alpha(x) * a^x) = h_a(\alpha) = h_b(\alpha) = b$. $\square$

Proposition 11 implies the following characterization of extremal Q-algebra epimorphisms (notice that every map $X \xrightarrow{f} Y$ extends to the standard image and preimage operators $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$).

Proposition 12. *Extremal Q-algebra epimorphisms are precisely the homomorphisms with surjective underlying maps.*

Proof: Given an extremal Q-algebra epimorphism $A \xrightarrow{e} B$, factor it through its image $e^{\rightarrow}(A)$. Since the embedding $e^{\rightarrow}(A) \hookrightarrow B$ must be an isomorphism, e is surjective. On the other hand, every surjective Q-algebra homomorphism $A \xrightarrow{e} B$ is obviously an epimorphism, and if we have a factorization $e = m \circ f$ with a monomorphism m, then m is bijective and hence a Q-algebra isomorphism. □

Given Q-algebra A, the class of all *extremal quotients* of A consists of pairs (e, B), where $A \xrightarrow{e} B$ is an extremal Q-algebra epimorphism. The family has a natural preorder $(e_1, B_1) \leqslant (e_2, B_2)$ iff there exists a Q-algebra homomorphism $B_2 \xrightarrow{f} B_1$ such that $f \circ e_2 = e_1$. Extremal quotients (e_1, B_1), (e_2, B_2) are called *equivalent* provided that $(e_1, B_1) \leqslant (e_2, B_2)$ and $(e_2, B_2) \leqslant (e_1, B_1)$ or, in other words, if there exists a Q-algebra isomorphism $B_2 \xrightarrow{f} B_1$ such that $f \circ e_2 = e_1$. The ensuing partially ordered class is denoted by $\mathcal{EQ}(A)$ and its elements by $[(e, B)]$.

In the following, we give different representations of the class $\mathcal{EQ}(A)$. In particular, we show that the class $\mathcal{EQ}(A)$ is a set.

4. Quantale algebra nuclei and their properties

In this section, we recall from [26] the notion of *quantale algebra nucleus*. Notice that it generalizes the notions of *quantic nucleus* [20, Definition 3.1.1] and *module nucleus* [16, Definition 2.1], which in their turn were induced by the concept of *frame* (or *locale*) *nucleus* [9, Section II.2.2].

Definition 13. Let A be a Q-algebra. A map $A \xrightarrow{j} A$ is called a *quantale algebra nucleus* on A provided that for every $a, b \in A$ and every $q \in Q$, the following hold:

1. $a \leqslant b$ implies $j(a) \leqslant j(b)$;
2. $a \leqslant j(a)$;
3. $j \circ j(a) \leqslant j(a)$;
4. $j(a) \otimes j(b) \leqslant j(a \otimes b)$;
5. $q * j(a) \leqslant j(q * a)$. ■

For the sake of brevity, from now on, "nucleus" will mean "quantale algebra nucleus". The following proposition shows some consequences of Definition 13.

PROPOSITION 14. *Let A be a Q-algebra and let $A \xrightarrow{j} A$ be a nucleus on A. For every $a, b \in A$, every $q \in Q$ and every subset $S \subseteq A$, the following hold:*

1. $j \circ j(a) = j(a)$;
2. $j(\bigvee S) = j(\bigvee j^{\rightarrow}(S))$;
3. $j(a \otimes b) = j(j(a) \otimes j(b))$;
4. $j(q * a) = j(q * j(a))$.

PROOF: The proof consists of straightforward computations. As an example, we show Items (1), (3). For (1): Given $a \in A$, $j(a) \leqslant j \circ j(a)$ and, therefore, $j \circ j(a) = j(a)$. For (3): Given $a, b \in A$, it follows that $j(a) \otimes j(b) \leqslant j(a \otimes b)$ and, therefore, $j(j(a) \otimes j(b)) \leqslant j \circ j(a \otimes b) = j(a \otimes b)$. On the other hand, $a \leqslant j(a)$ and $b \leqslant j(b)$ imply $a \otimes b \leqslant j(a) \otimes j(b)$ and, therefore, $j(a \otimes b) \leqslant j(j(a) \otimes j(b))$. $\square$

Proposition 14 induces a useful construction mentioned in the following proposition and its corollary.

PROPOSITION 15. *Let A be a Q-algebra and let $A \xrightarrow{j} A$ be a nucleus on A. Define $A_j = \{a \in A \,|\, j(a) = a\}$. Then $A_j = j^{\rightarrow}(A)$ and, moreover, A_j is a Q-algebra with the following structure:*

1. $\bigvee_j S = j(\bigvee S)$ *for every* $S \subseteq A_j$;
2. $a \otimes_j b = j(a \otimes b)$ *for every* $a, b \in A_j$;
3. $q *_j a = j(q * a)$ *for every* $q \in Q$ *and every* $a \in A_j$.

PROOF: Straightforward computations and Proposition 14 show that A_j with the aforesaid structure is a Q-algebra. As an example, we show Item (3) of Definition 7. If $a, b \in A$ and $q \in Q$, then $q *_j (a \otimes_j b) = j(q * j(a \otimes b)) = j(q * (a \otimes b)) = j((q * a) \otimes b) = j(j(q * a) \otimes j(b)) = j((j \circ j)(q * a) \otimes j(b)) = j(j(q * a) \otimes b) = (q *_j a) \otimes_j b$, and, similarly, $(q *_j a) \otimes_j b = a \otimes_j (q *_j b)$. $\square$

COROLLARY 16. *Let A be a Q-algebra and let $A \xrightarrow{j} A$ be a nucleus on A. Define a map $A \xrightarrow{\nu} A_j$ by $\nu(a) = j(a)$. Then $A \xrightarrow{\nu} A_j$ is an extremal Q-algebra epimorphism.*

Proof: Use straightforward computations together with Propositions 14, 15. As an example, we show that ν is $\bigvee$-preserving. If $S \subseteq A$, then $\nu(\bigvee S) = j(\bigvee S) = j(\bigvee j^{\rightarrow}(S)) = \bigvee_j j^{\rightarrow}(S) = \bigvee_j \nu^{\rightarrow}(S)$. $\square$

Given a Q-algebra A, $\mathcal{N}(A)$ stands for the set of all nuclei on A, partially ordered by $j_1 \leqslant j_2$ iff $j_1(a) \leqslant j_2(a)$ for every $a \in A$. By analogy with [9], one can easily show the following result.

Proposition 17. *The partially ordered set $\mathcal{N}(A)$ is a $\bigwedge$-semilattice.*

Proof: Given a subset $J \subseteq \mathcal{N}(A)$, define $A \xrightarrow{\bigwedge J} A$ by $(\bigwedge J)(a) = \bigwedge_{j \in J} j(a)$. Straightforward computations show that $\bigwedge J$ is a nucleus on A. As an example, we show Item (5) of Definition 13. Given $q \in Q$ and $a \in A$, $q * ((\bigwedge J)(a)) = q * (\bigwedge_{j \in J} j(a)) \leqslant \bigwedge_{j \in J} q * j(a) \leqslant \bigwedge_{j \in J} j(q * a) = (\bigwedge J)(q * a)$. $\square$

Joins of the lattice $\mathcal{N}(A)$ are harder to describe explicitly. Some additional results from Section 7 will help us to answer the question in Section 8.

We are going to end the section by establishing a relation between the partially ordered sets $\mathcal{EQ}(A)$ and $\mathcal{N}(A)$. Start with the observation that given a Q-algebra homomorphism $A \xrightarrow{e} B$, since e is $\bigvee$-preserving, it has the *upper adjoint* (in the sense of partially ordered sets [5]) $B \xrightarrow{e_*} A$. Recall that $1_A \leqslant \overset{e}{\to} \circ e$ and $e \circ \overset{e}{\to} \leqslant 1_B$ (in particular, $e \circ \overset{e}{\to} \circ e = e$). The explicit formula for $\overset{e}{\to}$ is given by $\overset{e}{\to}(b) = \bigvee e^{\leftarrow}(\downarrow b)$.

Proposition 18. *Given a Q-algebra homomorphism $A \xrightarrow{e} B$, the definition $j = \overset{e}{\to} \circ e$ provides a nucleus on A.*

Proof: Items (1)-(3) of Definition 13 are trivial. Items (4), (5) are similar and, therefore, we show the last one. If $a \in A$, then $e(q * j(a)) = q * (e \circ j)(a) = q * e(a) = e(q * a)$ and, therefore, $q * j(a) \leqslant \overset{e}{\to} \circ e(q * a) = j(q * a)$. $\square$

With the help of Proposition 18, we are able to show the following useful result.

Proposition 19. *For every Q-algebra A, the partially ordered sets $\mathcal{EQ}(A)$ and $\mathcal{N}(A)$ are dually isomorphic.*

PROOF: Define the following two maps: $\mathcal{EQ}(A) \xrightarrow{F} \mathcal{N}(A)$, $F([(e,B)]) = A \xrightarrow{\xrightarrow{e}\circ e} A$ (use the adjunction properties of [5] to show that the map is correct) and $\mathcal{N}(A) \xrightarrow{G} \mathcal{EQ}(A)$, $G(j) = [(\nu, A_j)]$. We prove that the above-mentioned maps establish an order-reversing one-to-one correspondence between $\mathcal{EQ}(A)$ and $\mathcal{N}(A)$.

To show $F \circ G = 1_{\mathcal{N}(A)}$, notice that given $j \in \mathcal{N}(A)$ and $a \in A$, $(F \circ G(j))(a) = \xrightarrow{\nu} \circ \nu(a) = \bigvee \nu^{\leftarrow}(\downarrow \nu(a)) = \bigvee_{j(x) \leqslant j(a)} x = j(a)$ since $j \circ j(a) = j(a)$.

To check $G \circ F = 1_{\mathcal{EQ}(A)}$, use the fact that given $[(e,B)] \in \mathcal{EQ}(A)$, $G \circ F([(e,B)]) = [(\nu, A_{\xrightarrow{e}\circ e})]$. It is enough to show that (e,B) and $(\nu, A_{\xrightarrow{e}\circ e})$ are equivalent. Define a map $B \xrightarrow{f} A_{\xrightarrow{e}\circ e}$ by $f(b) = \xrightarrow{e} (b)$. Then, $f \circ e = \nu$. Since e is surjective, $\xrightarrow{e}$ must be injective [5] and thus, f is bijective. By Lemma 9, f is a Q-algebra homomorphism.

To show that F is order-reversing, notice that given $[(e_1, B_1)], [(e_2, B_2)] \in \mathcal{EQ}(A)$ such that $[(e_1, B_1)] \leqslant [(e_2, B_2)]$, there exists a Q-algebra homomorphism $B_2 \xrightarrow{f} B_1$ such that $f \circ e_2 = e_1$. From the adjunctions $f \dashv \xrightarrow{f}$ and $e_2 \dashv \xrightarrow{e_2}$ one gets $f \circ e_2 \dashv \xrightarrow{e_2} \circ \xrightarrow{f}$ and then, $e_1 \dashv \xrightarrow{e_1}$ implies $\xrightarrow{e_2} \circ \xrightarrow{f} = \xrightarrow{e_1}$. Moreover, $1_{B_2} \leqslant \xrightarrow{f} \circ f$ implies $\xrightarrow{e_2} \circ e_2 \leqslant \xrightarrow{e_2} \circ \xrightarrow{f} \circ f \circ e_2 = \xrightarrow{e_1} \circ e_1$ and thus, $F([(e_2, B_2)]) \leqslant F([(e_1, B_1)])$.

To show that G is order-reversing, use the fact that given $j_1, j_2 \in \mathcal{N}(A)$ such that $j_1 \leqslant j_2$, one can define a map $A_{j_1} \xrightarrow{f} A_{j_2}$, $f(j_1(a)) = j_2(a)$. To show that the map is correct, notice that given $a, b \in A$ such that $j_1(a) = j_1(b)$, $b \leqslant j_1(b) = j_1(a) \leqslant j_2(a)$ implies $j_2(b) \leqslant j_2(a)$, and similarly, $j_2(a) \leqslant j_2(b)$. Clearly, $f \circ \nu_1 = \nu_2$. By Lemma 9, f is a Q-algebra homomorphism and, therefore, $G(j_2) \leqslant G(j_1)$. □

COROLLARY 20. *For every Q-algebra A, the class $\mathcal{EQ}(A)$ is a set.*

5. Factorization of quantale algebra homomorphisms

Based on a generalization of the tools of [18], this section considers a technique of producing subalgebras of quantale algebras by means of extending a relation to a congruence. Start with some preliminary remarks. Given a Q-algebra A, every $a \in A$ and every $q \in Q$ induce the following adjunctions:

1. $A \longrightarrow^{a \to_l \cdot} A$ and $A \longrightarrow^{\cdot \otimes a} A$ with $a \to_l b = \bigvee\{x \in A \,|\, x \otimes a \leqslant b\}$;
2. $A \longrightarrow^{a \to_r \cdot} A$ and $A \longrightarrow^{a \otimes \cdot} A$ with $a \to_r b = \bigvee\{x \in A \,|\, a \otimes x \leqslant b\}$;
3. $A \longrightarrow^{q \twoheadrightarrow \cdot} A$ and $A \longrightarrow^{q * \cdot} A$ with $q \twoheadrightarrow b = \bigvee\{x \in A \,|\, q * x \leqslant b\}$.

The next proposition shows a crucial property of nuclei, which will be useful for us later.

Proposition 21. *Given a Q-algebra A, for every subset $S \subseteq A$, equivalent are:*

1. *there exists a nucleus $A \xrightarrow{j} A$ on A such that $S = A_j$;*
2. *given $T \subseteq S$, $s \in S$, $a \in A$ and $q \in Q$, it follows that $\bigwedge T$, $a \to_l s$, $a \to_r s$, and $q \twoheadrightarrow s$ are elements of S.*

Proof: (1) $\Rightarrow$ (2): For $T \subseteq S$, $\bigwedge T \leqslant j(\bigwedge T) \leqslant \bigwedge j^{\to}(T) \leqslant \bigwedge T$ and thus, $\bigwedge T = j(\bigwedge T)$.

Since $a \otimes (a \to rs) \leqslant s$, $a \otimes j(a \to rs) \leqslant j(a) \otimes j(a \to rs) \leqslant j(a \otimes (a \to rs)) \leqslant j(s) = s$ and, therefore, $j(a \to rs) \leqslant a \to rs$. From $a \to rs \leqslant j(a \to rs)$, one gets $j(a \to rs) = a \to rs$. Similarly, one establishes that $j(a \to ls) = a \to ls$.

Since $q * (q \twoheadrightarrow s) \leqslant s$, $q * j(q \twoheadrightarrow s) \leqslant j(q * (q \twoheadrightarrow s)) \leqslant j(s) = s$ and, therefore, $j(q \twoheadrightarrow s) \leqslant q \twoheadrightarrow s$. From $q \twoheadrightarrow s \leqslant j(q \twoheadrightarrow s)$, one gets $j(q \twoheadrightarrow s) = q \twoheadrightarrow s$.

(2) $\Rightarrow$ (1): Define $A \xrightarrow{j} A$ by $j(a) = \bigwedge(\uparrow a \bigcap S)$. It follows that $S = j^{\to}(A)$. Show that j is a nucleus on A. Items (1), (2) of Definition 13 are clear. For (3): Since $j \circ j(a) = \bigwedge(\uparrow j(a) \bigcap S)$ and $j(a) \in \uparrow j(a) \bigcap S$, $j \circ j(a) \leqslant j(a)$. For (4): $a \otimes b \leqslant j(a \otimes b)$ implies $a \leqslant b \to lj(a \otimes b)$ implies $j(a) \leqslant b \to lj(a \otimes b)$ implies $j(a) \otimes b \leqslant j(a \otimes b)$ implies $b \leqslant j(a) \to rj(a \otimes b)$ implies $j(b) \leqslant j(a) \to rj(a \otimes b)$ implies $j(a) \otimes j(b) \leqslant j(a \otimes b)$. For (5): $q * a \leqslant j(q * a)$ implies $a \leqslant q \twoheadrightarrow j(q * a)$ implies $j(a) \leqslant q \twoheadrightarrow j(q * a)$ implies $q * j(a) \leqslant j(q * a)$. $\square$

By analogy with [18, Section 2.15], we introduce the following definition.

Definition 22. Let A be a Q-algebra and let $R \subseteq A \times A$ be a binary relation on A. An element $s \in A$ is called (R-)*saturated* provided that for every $a, b, c, d \in A$ and every $q \in Q$, aRb implies the following:

1. $c \otimes a \leqslant s$ iff $c \otimes b \leqslant s$;
2. $a \otimes c \leqslant s$ iff $b \otimes c \leqslant s$;
3. $c \otimes a \otimes d \leqslant s$ iff $c \otimes b \otimes d \leqslant s$;
4. $q * a \leqslant s$ iff $q * b \leqslant s$.

A/R stands for the set of all saturated elements of A. ■

The next proposition shows some properties of the set A/R of Definition 22.

Proposition 23. *Let A be a Q-algebra and let R be a binary relation on A. Given $S \subseteq A/R$, $t \in A/R$, $x \in A$ and $q \in Q$, it follows that $\bigwedge S$, $x \to lt$, $x \to rt$ and $q \twoheadrightarrow t$ are elements of A/R.*

Proof: Take $a, b, c \in A$ with aRb and $p \in Q$.

Since $c \otimes a \leqslant \bigwedge S$ iff $c \otimes a \leqslant s$ for every $s \in S$ iff $c \otimes b \leqslant s$ for every $s \in S$ iff $c \otimes b \leqslant \bigwedge S$, and similarly the other properties, it follows that $\bigwedge S \in A/R$.

Since $c \otimes a \leqslant x \to lt$ iff $(c \otimes a) \otimes x \leqslant t$ iff $(c \otimes b) \otimes x \leqslant t$ iff $c \otimes b \leqslant x \to lt$ (Items (2), (3) of Definition 22 follow similarly) and $p * a \leqslant x \to lt$ iff $(p * a) \otimes x \leqslant t$ iff $a \otimes (p * x) \leqslant t$ iff $b \otimes (p * x) \leqslant t$ iff $(p * b) \otimes x \leqslant t$ iff $p * b \leqslant x \to lt$, it follows that $x \to lt \in A/R$. Similarly, one shows that $x \to rt \in A/R$.

Since $c \otimes a \leqslant q \twoheadrightarrow t$ iff $q * (c \otimes a) \leqslant t$ iff $(q * c) \otimes a \leqslant t$ iff $(q * c) \otimes b \leqslant t$ iff $q * (c \otimes b) \leqslant t$ iff $c \otimes b \leqslant q \twoheadrightarrow t$ (Items (2), (3) of Definition 22 follow similarly) and $p * a \leqslant q \twoheadrightarrow t$ iff $q * (p * a) \leqslant t$ iff $(q \otimes p) * a \leqslant t$ iff $(q \otimes p) * b \leqslant t$ iff $q * (p * b) \leqslant t$ iff $p * b \leqslant q \twoheadrightarrow t$, it follows that $q \twoheadrightarrow t \in A/R$. □

Let A be a Q-algebra and let R be a binary relation on A. By Propositions 21, 23, there exists a nucleus j on A such that $j^{\rightarrow}(A) = A/R$ and defined by $j(a) = \bigwedge(\uparrow a \bigcap A/R)$. By Corollary 16, $A \xrightarrow{\nu_R} A/R$ given by $\nu_R(a) = j(a)$ is an extremal Q-algebra epimorphism. Moreover, the following lemma shows that ν_R respects R.

Lemma 24. *Given $a, b \in A$ with aRb, it follows that $\nu_R(a) = \nu_R(b)$.*

Proof: Since $a \leqslant \nu_R(a)$ and aRb, $b \leqslant \nu_R(a)$ and then $\nu_R(b) \leqslant \nu_R \circ \nu_R(a) = \nu_R(a)$. Similarly, one gets that $\nu_R(a) \leqslant \nu_R(b)$. □

We show that the homomorphism ν_R is *universal* among all Q-algebra homomorphisms with the property of Lemma 24.

THEOREM 25. *Let A be a Q-algebra and let $R \subseteq A \times A$ be a binary relation on A. For every Q-algebra homomorphism $A \xrightarrow{h} B$ such that $h(a) = h(b)$ whenever aRb, there exists a unique Q-algebra homomorphism $A/R \rightarrow^{\overline{h}} B$ such that $\overline{h} \circ \nu_R = h$.*

PROOF: Take any Q-algebra homomorphism $A \xrightarrow{h} B$ such that $h(a) = h(b)$ whenever aRb, and let $a \in A$. Define $\tau(a) = \bigvee\{x \in A \,|\, h(x) \leqslant h(a)\}$. Clearly, $a \leqslant \tau(a)$ and $h \circ \tau(a) = h(a)$. Easy calculations show that $\tau(a) \in A/R$. As an example, check Item (1) of Definition 22. Suppose $a_1, a_2 \in A$ are such that $a_1 R a_2$. Given $c \in A$, $c \otimes a_1 \leqslant \tau(a)$ implies $h(c \otimes a_2) = h(c) \otimes h(a_2) = h(c) \otimes h(a_1) = h(c \otimes a_1) \leqslant h \circ \tau(a) = h(a)$ and, therefore, $c \otimes a_2 \leqslant \tau(a)$. The converse implication follows similarly.

Since $\tau(a) \in A/R$, $a \leqslant \nu_R(a) \leqslant \tau(a)$. Thus, $h(a) \leqslant h \circ \nu_R(a) \leqslant h \circ \tau(a) = h(a)$ and, therefore, $h \circ \nu_R(a) = h(a)$. Define $A/R \xrightarrow{\overline{h}} B$ by $\overline{h}(a) = h(a)$. Then $\overline{h} \circ \nu_R = h$. By Lemma 9, $\overline{h}$ is a Q-algebra homomorphism. Uniqueness follows from the fact that ν_R is a Q-algebra epimorphism. $\square$

6. Factorization of $\bigvee$-semilattice and quantale module homomorphisms

In this section we show some modifications of Theorem 25. They arise by imposing certain restrictions on the binary relation R. Start with the following definition, which was induced by [18, Section 2.15].

DEFINITION 26. Let A be a Q-algebra and let $R \subseteq A \times A$ be a binary relation on A. R is called *($*$-$\otimes$)-stable* provided that there exists a subset $M \subseteq A$ such that:

1. given $a \in A$, $a = \bigvee\{x \in M \,|\, x \leqslant a\}$ (the latter set is denoted by M_a);
2. given $a, b \in A$, $q \in Q$ and $x, y \in M$, aRb implies
   (a) $(x \otimes a)R(x \otimes b)$;
   (b) $(a \otimes x)R(b \otimes x)$;
   (c) $(x \otimes a \otimes y)R(x \otimes b \otimes y)$;
   (d) $(q * a)R(q * b)$. ■

By analogy with Definition 22, one has the notion of (R-)saturated element. Now, however, it has a simpler characterization shown in the next proposition.

Proposition 27. *Let A be a Q-algebra and let R be a ($*$-$\otimes$)-stable binary relation on A. An element $s \in A$ is saturated iff for every $a, b \in A$, whenever aRb, it follows that $a \leqslant s$ iff $b \leqslant s$.*

Proof: The necessity is clear. Easy calculations show the sufficiency. As an example, check Item (1) of Definition 22. Take $a, b \in A$ with aRb. Given $c \in A$ with $c \otimes a \leqslant s$, $c \otimes b = (\bigvee M_c) \otimes b = \bigvee_{x \in M_c} (x \otimes b)$. Given $x \in M_c$, $x \otimes a \leqslant s$ and $(x \otimes a) R (x \otimes b)$ together imply $x \otimes b \leqslant s$. It follows that $c \otimes b \leqslant s$. The converse implication is similar. $\square$

Proposition 27 gives rise to the following modification of Theorem 25.

Theorem 28. *Let A be a Q-algebra and let $R \subseteq A \times A$ be a ($*$-$\otimes$)-stable binary relation on A. For every $\bigvee$-semilattice homomorphism $A \xrightarrow{h} B$ such that $h(a) = h(b)$ whenever aRb, there exists a unique $\bigvee$-semilattice homomorphism $A/R \xrightarrow{\overline{h}} B$ such that $\overline{h} \circ \nu_R = h$.*

Proof: See the proof of Theorem 25. To show that $\tau(a) \in A/R$, proceed as follows. Take $x, y \in A$ with xRy. If $x \leqslant \tau(a)$, then $h(y) = h(x) \leqslant h \circ \tau(a) = h(a)$ implies $y \leqslant \tau(a)$. The converse implication goes similarly. $\square$

To replace $\bigvee$-semilattices with Q-modules in Theorem 28, proceed as follows.

Definition 29. Let A be a Q-algebra, let $R \subseteq A \times A$ be a binary relation on A. R is called $\otimes$-*stable* provided that it fulfils all items of Definition 26 except (2).(d). ■

By analogy with Proposition 27, one shows the following one.

Proposition 30. *Let A be a Q-algebra and let R be a $\otimes$-stable binary relation on A. An element $s \in A$ is saturated iff for every $a, b \in A$ and every $q \in Q$, whenever aRb, it follows that $q * a \leqslant s$ iff $q * b \leqslant s$.*

The modification of Theorem 28 is now obvious as well as the case of replacing $\bigvee$-semilattices by quantales in it, where the notion of $*$-*stable* relation arises.

7. Representations of extremal quotients of quantale algebras

In this section we show two more representations of the set $\mathcal{EQ}(A)$, apart from that already given in Section 4. We begin with the following definition, motivated by [18, Section 2.15].

Definition 31. Let A be a Q-algebra. A subset $S \subseteq A$ is a called a *sub(Q-)algebra set* provided that the following hold:

1. $T \subseteq S$ implies $\bigwedge T \in S$;
2. $a \in A$, $s \in S$ implies $a \to ls$, $a \to rs \in S$;
3. $q \in Q$, $s \in S$ implies $q \twoheadrightarrow s \in S$. ■

Let A be a Q-algebra A and define $\mathcal{S}(A) = \{S \subseteq A \,|\, S$ is a subalgebra set$\}$. Then $(\mathcal{S}(A), \subseteq)$ is a $\bigwedge$-semilattice, where $\bigwedge$ are given by the usual set-theoretic intersections. The next proposition shows a relation between the partially ordered sets $\mathcal{S}(A)$ and $\mathcal{N}(A)$.

Proposition 32. *For every Q-algebra A, the partially ordered sets $\mathcal{S}(A)$ and $\mathcal{N}(A)$ are dually isomorphic.*

Proof: The proof of Proposition 21 induces two maps: $\mathcal{S}(A) \xrightarrow{F} \mathcal{N}(A)$, $F(S) = A \xrightarrow{j_S} A$, $j_S(a) = \bigwedge(\uparrow a \bigcap S)$ and $\mathcal{N}(A) \xrightarrow{G} \mathcal{S}(A)$, $G(j) = A_j$. Show that the maps establish an order-reversing one-to-one correspondence between $\mathcal{S}(A)$ and $\mathcal{N}(A)$.

Clearly, $G \circ F = 1_{\mathcal{S}(A)}$. To show that $F \circ G = 1_{\mathcal{N}(A)}$, take $j \in \mathcal{N}(A)$ and $a \in A$. Then $(F \circ G(j))(a) = j_{A_j}(a) = \bigwedge(\uparrow a \bigcap A_j)$. Since $a \leqslant j(a)$ and $j \circ j(a) = j(a)$, $j(a) \in \uparrow a \bigcap A_j$. Given $b \in \uparrow a \bigcap A_j$, $a \leqslant b$ and thus, $j(a) \leqslant j(b) = b$. It follows that $j_{A_j}(a) = j(a)$.

Clearly, F is order-reversing. To show that G is order-reversing, take $j_1, j_2 \in \mathcal{N}(A)$ such that $j_1 \leqslant j_2$. Given $s \in A_{j_2}$, $j_1(s) \leqslant j_2(s) = s$ and, therefore, $j_1(s) = s$. Thus, $s \in A_{j_1}$. □

One should not forget that *congruences* on a Q-algebra A (equivalence relations respecting the structure of quantale algebra) give another representation of $\mathcal{EQ}(A)$. Define $\mathcal{C}(A) = \{R \subseteq A \times A \,|\, R \text{ is a congruence on } A\}$. Then $(\mathcal{C}(A), \subseteq)$ is a $\bigwedge$-semilattice, the set-theoretic intersection of congruences being again a congruence.

Proposition 33. *For every Q-algebra A, the partially ordered sets $\mathcal{C}(A)$ and $\mathcal{EQ}(A)$ are dually isomorphic.*

Proof: It is possible to define two maps: $\mathcal{C}(A) \xrightarrow{F} \mathcal{EQ}(A)$, $F(R) = [A \xrightarrow{q} A/R]$ (notice that A/R denotes the quotient algebra and not the set of saturated elements) and $\mathcal{EQ}(A) \xrightarrow{G} \mathcal{C}(A)$, $G([(e, B)]) = \{(a, b) \in A \times A \,|\, e(a) = e(b)\}$ (notice that the map is correct). It is easy to show that the maps establish an order-reversing one-to-one correspondence between the partially ordered sets $\mathcal{C}(A)$ and $\mathcal{EQ}(A)$. □

Given a Q-algebra A, Propositions 19, 32, 33 give the following representations of extremal quotients of A (notice that given a poset X, X^d stands for its dual): $(\mathcal{C}(A))^d \cong \mathcal{EQ}(A) \cong (\mathcal{N}(A))^d \cong \mathcal{S}(A)$.

8. Algebraic structure on the set of quantale algebra nuclei

For a given Q-algebra A, Section 7 shows different representations of the set $\mathcal{EQ}(A)$. The lattice structure of the above-mentioned representations is particularly transparent in $\mathcal{S}(A)$. We already know that $\bigwedge$ are intersections. The following proposition clarifies the nature of $\bigvee$ (cf. [18, Section 2.15]).

Proposition 34. *Let A be a Q-algebra and let $\{S_i \,|\, i \in I\} \subseteq \mathcal{S}(A)$. The definition $S = \{\bigwedge T \,|\, T \subseteq \bigcup_{i \in I} S_i\}$ provides the join of $(S_i)_{i \in I}$ in $\mathcal{S}(A)$.*

Proof: It will be enough to show that S is an element of $\mathcal{S}(A)$. Clearly, S is $\bigwedge$-closed. Moreover, given $a \in A$ and $s \in S$, $a \to ls = a \to l(\bigwedge T) = \bigwedge\{a \to lt \,|\, t \in T\}$, and every $a \to lt$ is in $\bigcup_{i \in I} S_i$. All other properties follow similarly. □

Let us notice that there exists the concept of Q-*coalgebra*, in which $\bigvee$-semilattices are replaced by $\bigwedge$-semilattices. Since $\mathcal{S}(A)$ is dually isomorphic to $\mathcal{N}(A)$, and one would like the latter set to be a Q-algebra, there is a temptation to define a Q-coalgebra structure on $\mathcal{S}(A)$. In the following, we show such an attempt, in which, however, we have to restrict ourselves to a particular subset of S(A). Recall from [3] that a *po-semigroup* is a semigroup $(G, \otimes)$ equipped with a partial order $\leqslant$, which is preserved by $\otimes$ in each argument. A *cozero* of a po-semigroup $(G, \otimes, \leqslant)$ is the largest element of $(G, \leqslant)$, which is also is the unit of $(G, \otimes)$.

Proposition 35. *Let A be a Q-algebra. There exists a binary operation $\circledcirc$ on $\mathcal{S}(A)$ such that $(\mathcal{S}(A), \circledcirc)$ is a po-semigroup with $\top$ being a cozero.*

Proof: Given $S, T \in \mathcal{S}(A)$ let $S \circledcirc T = \uparrow (S \otimes T) = \{a \in A \,|\, s \otimes t \leqslant a \text{ for some } (s,t) \in S \times T\}$. Show that $S \circledcirc T \in \mathcal{S}(A)$. Given $P \subseteq S \circledcirc T$, $\bigwedge P \geqslant \bigwedge_{p\in P}(s_p \otimes t_p) \geqslant (\bigwedge_{p\in P} s_p) \otimes (\bigwedge_{p\in P} t_p) \in S \otimes T$ and then $\bigwedge P \in S \circledcirc T$. Given $a \in A$ and $x \in S \circledcirc T$, $a \to rx \geqslant a \to r(s_x \otimes t_x) \geqslant (a \to rs_x) \otimes t_x \in S \otimes T$ and then $a \to rx \in S \circledcirc T$. The other conditions of Definition 31 follow similarly.

Clearly, $\circledcirc$ is order-preserving and associative. To show that $\top \circledcirc S = \top = S \circledcirc \top$ for every $S \in \mathcal{S}(A)$, notice that $\uparrow (A \otimes S) \supseteq \uparrow (\bot_A \otimes \top_A) = \uparrow \bot_A = A = \top$. □

Proposition 36. *Every Q-algebra A has a map $Q \times \mathcal{S}(A) \xrightarrow{\circledast} \mathcal{S}(A)$ such that for every S, S_1, $S_2 \in \mathcal{S}(A)$, every q, q_1, $q_2 \in Q$ and every $P \subseteq Q$ the following hold:*

1. $q_1 \circledast (q_2 \circledast S) = (q_1 \otimes q_2) \circledast S$;
2. $(\bigvee P) \circledast S = \bigwedge_{p\in P}(p \circledast S)$;
3. $q \circledast \top = \top$;
4. $S \leqslant 1 \circledast S$;
5. $q \circledast (S_1 \circledcirc S_2) = (q \circledast S_1) \circledcirc S_2 = S_1 \circledcirc (q \circledast S_2)$.

Proof: For $q \in Q$, $S \in \mathcal{S}(A)$ let $q \circledast S = \uparrow (q * S) = \{a \in A \,|\, q * s \leqslant a \text{ for some } s \in S\}$. Show that $q \circledast S \in \mathcal{S}(A)$. Given $P \subseteq q \circledast S$, $\bigwedge P \geqslant \bigwedge_{p\in P}(q * s_p) \geqslant q * (\bigwedge_{p\in P} s_p) \in q * S$ implies $\bigwedge P \in q \circledast S$. Given $a \in A$ and $x \in q \circledast S$, $a \to rx \geqslant a \to r(q * s_x) \geqslant q * (a \to rs_x) \in q * S$ implies $a \to rx \in q \circledast S$. The other conditions of Definition 31 follow similarly.

Easy computations show the properties. As an example, we check Item (2). If $P = \varnothing$, then $(\bigvee P) \circledast S = \bot_Q \circledast S = \uparrow (\bot_Q * S) = \uparrow \bot_A = A = \bigwedge_{p \in P}(p \circledast S)$. Suppose $P \neq \varnothing$. To see that $\bigwedge_{p \in P}(p \circledast S) \subseteq (\bigvee P) \circledast S$ let $x \in \bigwedge_{p \in P}(p \circledast S)$. Then $x \geqslant p * s_p$ for every $p \in P$. Define $s = \bigwedge_{p \in P} s_p$. Then $x \geqslant \bigvee_{p \in P}(p * s) = (\bigvee P) * s \in (\bigvee P) * S$ and thus, $x \in (\bigvee P) \circledast S$. The converse inclusion is clear. $\square$

Propositions 35, 36 show that $\mathcal{S}(A)$ can be endowed with the structure of Q-*semi-coalgebra* (delete $\bigwedge$-distributivity and unitality). In the following, we show that S(A) contains a Q-coalgebra closed under arbitrary meets in S(A). Its definition is easily suggested by Item (4) of Proposition 36.

Definition 37. Let A be a Q-algebra and let $S \subseteq A$ be a subalgebra set. S is called an *upper subalgebra set* provided that $S = \uparrow S$. $\mathcal{S}_u(A)$ stands for the set of all upper subalgebra sets of A. ■

One can easily see that $\mathcal{S}_u(A)$ is closed under the formation of meets in S(A) and, therefore, is a $\bigwedge$-semilattice.

Proposition 38. *For every Q-algebra A, $\mathcal{S}_u(A)$ is a Q-coalgebra.*

Proof: In view of Propositions 35, 36, it is enough to show that both $\circledcirc$ and $\circledast$ distribute over non-empty meets. Consider the former case, the latter one being similar. Take $\{S_i \mid i \in I\} \subseteq \mathcal{S}_u(A)$ and $T \in \mathcal{S}_u(A)$. To see that $\bigwedge_{i \in I}(S_i \circledcirc T) \subseteq (\bigwedge_{i \in I} S_i) \circledcirc T$ let $x \in \bigwedge_{i \in I}(S_i \circledcirc T)$. Then $x \geqslant s_i \otimes t_i$ for every $i \in I$. Set $t = \bigwedge_{i \in I} t_i$. Then $x \geqslant s_i \otimes t$ and, therefore, $t \to lx \geqslant s_i$ for every $i \in I$. Since $x \geqslant (t \to lx) \otimes t \in (\bigwedge_{i \in I} S_i) \otimes T$, $x \in (\bigwedge_{i \in I} S_i) \circledcirc T$. The converse inclusion is clear. $\square$

Going back to the set $\mathcal{N}(A)$, consider the following definition.

Definition 39. Let A be a Q-algebra and let j be a nucleus on A. j is called an *u-nucleus* provided that $A_j = \uparrow A_j$. $\mathcal{N}_u(A)$ denotes the set of all u-nuclei on A. ■

One can easily check that the restriction of maps of Proposition 32 to $\mathcal{S}_u(A)$ and $\mathcal{N}_u(A)$, respectively, establishes a one-to-one correspondence between them. Proposition 38 immediately implies the following result.

Proposition 40. *For every Q-algebra A, $\mathcal{N}_u(A)$ is a Q-algebra.*

Notice that $\mathcal{N}_u(A)$ is closed under the formation of $\bigvee$ in $\mathcal{N}(A)$.

9. Conclusion

In the paper we have presented a technique of characterizing quotients of quantale algebras in terms of particular maps called *quantale algebra nuclei*. The idea stems from the well-known representation of quotients of ring modules in terms of particular subsets called *ideals*. The crucial shift from subsets to maps is possible by the existence of order relation on the underlying lattices of quantale algebras. The latter property opens a door for further generalizations of the theory. In particular, P. Resende [19] (see also the results of J. Paseka [17]) introduced the notion of *sup-algebra*, which is an algebra, whose carrier is a $\bigvee$-semilattice, and whose operations are $\bigvee$-semilattice homomorphisms in each variable separately. Given a sup-algebra A, a *sup-algebra nucleus* on A is a *closure operator* $A \xrightarrow{j} A$ (order-preserving map j such that $1_A \leqslant j$ and $j \circ j = j$), which satisfies $\omega(j(a_1), \ldots, j(a_n)) \leqslant j(\omega(a_1, \ldots, a_n))$ for every n-ary operation ω of A and every $a_1, \ldots, a_n \in A$. In his PhD Thesis [19], P. Resende developed the general representation theory of quotients of sup-algebras in terms of sup-algebra nuclei. Later on, he illustrated the results with the examples of $\bigvee$-semilattices (closure operators), frames (nuclei), quantales (quantic nuclei) and quantale modules (module nuclei). Our paper backs the theory with another (and previously unstudied) example of quantale algebras (quantale algebra nuclei).

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