Available online at www.sciencedirect.com
SCIENCE DIRECT®
Annals of Pure and Applied Logic 137 (2006) 413–438
ANNALS OF PURE AND APPLIED LOGIC
www.elsevier.com/locate/apal

Compactness in locales and in formal topology

Steven Vickers

School of Computer Science, The University of Birmingham, Birmingham, B15 2TT, UK

Available online 15 June 2005

Abstract

If a locale is presented by a "flat site", it is shown how its frame can be presented by generators and relations as a dcpo. A necessary and sufficient condition is derived for compactness of the locale (and also for its openness). Although its derivation uses impredicative constructions, it is also shown predicatively using the inductive generation of formal topologies. A predicative proof of the binary Tychonoff theorem is given, including a characterization of the finite covers of the product by basic opens. The discussion is then related to the double powerlocale.
© 2005 Elsevier B.V. All rights reserved.

MSC: 03F65; 06D22; 54A05; 54D30; 54B20

Keywords: Locale; Formal topology; Predicative systems; Tychonoff; Powerlocale

1. Introduction

The objective of this paper is to strengthen the connections between two constructive approaches to topological compactness: on the one hand the topos-valid approach of locale theory, choice-free but impredicative, and on the other the predicative approach of formal topology, embodying certain choice principles. We do this through a study of compactness, proving a criterion that is valid in both.

Both approaches find themselves handling topological spaces in similar ways, in that they both use point-free methods: methods that describe the behaviour of open sets independently of the points that they are meant to be sets of. The reason is that constructively there may not always be enough points available.

E-mail address: s.j.vickers@cs.bham.ac.uk.

0168-0072/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2005.05.028

In *locale theory* the entire topology, the entire lattice of opens, is taken as the concrete embodiment of the space. This is axiomatized lattice theoretically as a *frame*, a complete lattice in which binary meet distributes over arbitrary joins.

In *formal topology* [14] on the other hand, taking a predicative point of view, frames are objectionable. This is most obvious in the special case of discrete spaces, for the discrete topology on a set X is its powerset $\mathcal{P}X$ and that is not a legitimate set in predicative type theory. Less obvious but still true is that other frames are just as bad, essentially because they have joins of arbitrary subsets.

The predicative approach is compelled to use not the full topology but just a *base*: a generating set of opens so that all other opens are joins of basics. But of course, this device is also well known in locale theory, in the use of *sites* in the sense of Johnstone. In the simplest form of site (as in [6]), the base is taken to be closed under finite meets. In both locale theory and formal topology, we then see a meet semilattice equipped with a cover relation to describe when one basic open is covered by a set of others.

Just as the basic opens generate all others (as joins), there is a similar issue with the cover relation. In the definition of formal topology the cover relation $\lhd$ is expected to be the *full* cover relation expressing all instances of $a \leq \bigvee U$. This is enforced by the *transitivity* axiom

$$\frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

(where $U \lhd V$ means that $u \lhd V$ for every $u \in U$). In practice, however, it is common to want to describe only a generating part of the full cover relation, and we shall typically write this as $\lhd_0$. We shall think of this as a “cover base”, and refer to the instances $a \lhd_0 U$ as *basic covers*. In locale theory, the full cover relation is then generated impredicatively via Johnstone’s concrete construction of the entire frame as the set of “C-ideals”: $a \lhd U$ iff a is contained in the least C-ideal that includes U. In formal topology on the other hand, $a \lhd U$ is generated by an inductive construction of its proofs [4].

The difference shows up rather strongly in compactness proofs. In formal topology, compactness of X is normally proved quite directly: whenever $X \lhd U$ then $X \lhd U_0$ for some finite subset $U_0 \subseteq U$. This will typically rely on a structural induction on the proof of $X \lhd U$. In locale theory, as we shall see, there are often quite different proofs using *presentations by generators and relations* that rely only on knowledge of $\lhd_0$ but which have an impredicative justification.

We shall show here that these compactness results derived using impredicative methods can also be justified predicatively, using an inductive generation [4] of $\lhd$ from $\lhd_0$. Our main result is proved in both locale theory and formal topology (as Theorems 10 and 15). It characterizes compactness in such a way that to verify the criteria one does not need to attend explicitly to the inductive analysis of proofs of $X \lhd U$.

2. Locale theory: Presentations

At its most uncompromising, locale theory is just the study of frames, but under a mask of categorical duality that allows them to pretend to be topological spaces.

Such a study of impredicative objects would appear to make it quite incompatible with type theory, but in practice many of the techniques of locale theory are predicative constructions.

In particular, the algebraic techniques of *presentation by generators and relations* creates connections between the two philosophies. This is because while the frame of *all* opens may be impredicative, there may yet be a predicative set of generators. In its most general form, a presentation gives a set G of generators, from which all opens can be constructed as joins of finite meets. (In topological terms it is a *subbase*.) The presentation also provides a set R of relations, each of the form

$$\bigvee_{i\in I} \bigwedge S_i = \bigvee_{j\in J} \bigwedge T_j \tag{$*$}$$

(inequality $\leq$ is also possible here), where each S_i and T_j is a finite subset of G. These relations are required to hold in the presented frame $\Omega X = \mathbf{Fr}\langle G \mid R\rangle$.

What it means to "present" is defined by a universal property. For any frame A, there is a bijection between frame homomorphisms $\mathbf{Fr}\langle G \mid R\rangle \to A$ and functions $G \to A$ that respect the relations (make them hold when translated into A). Thus though the presentation gives little explicit information about the *elements* of $\mathbf{Fr}\langle G \mid R\rangle$, it does tell you very precisely about the *frame homomorphisms* with $\mathbf{Fr}\langle G \mid R\rangle$ as domain. In fact this implies that it tells you precisely about the *points* of the corresponding locale, for they are just the homomorphisms $\mathbf{Fr}\langle G \mid R\rangle \to \Omega$.

We are writing Ω for the powerset $\mathcal{P}1$, the *subobject classifier* in topos terminology. We shall often treat it as the lattice of truthvalues. Since a function $G \to \Omega$ is just a subset of G, we thus get another description of what a point is. It is a subset $U \subseteq G$ such that every relation ($*$) is respected in the following sense: if $i \in I$ and $S_i \subseteq U$, then there is some $j \in J$ for which $T_j \subseteq U$; and conversely.

It is worth remarking that in topos-valid mathematics, this ability to derive the points from the presentation is very powerful. The universal property describes frame homomorphisms to *any* frame ΩY, not just Ω, and it turns out that the description of points just given still works when interpreted in the internal logic of the topos of sheaves over Y. In other words, the presentation describes the "generalized points at any stage of definition". (These are the same as continuous maps $Y \to X$.)

This goes a long way to overcoming the embarrassing fact that not all locales have enough points; that is to say *global* points, homomorphisms to Ω, for they do have enough *generalized* points. In particular, the generic point (in the topos of sheaves over the locale X, corresponding to the identity map $X \to X$) is enough for many purposes. This brings the practice of locale theory much closer to ordinary topology, so long as one reasons (constructively) by "geometric" principles that transfer well between toposes. We shall not pursue the idea here. It is implicit in the work of many topos theorists and has been expounded and exploited in some detail in [20] and [21]. However, it is worth pointing out in this context that the techniques would not be expected to work readily in type theory. This is because the typical topos is choice-free and therefore its internal logic, even a fragment that avoids impredicativity, will not be a model for the choice principles intrinsic in type theory.

Presentations are not always given in the general form $\mathbf{Fr}\langle G \mid R\rangle$ just described. Frequently the generators and relations are implicit in some other structure. In Johnstone's sites, for example, implicit relations say that the finite meets of generators are preserved in the frame. Other relations are more explicitly given by the *coverage*: if U is stipulated as covering a ($a \lhd_0 U$), then there is a corresponding relation $a \leq \bigvee U$. Note that these do not need to mention finite meets, since they can be absorbed by the semilattice structure on the generators. The general open is just a join of generators, and so the generators form a *base*. In summary, a site $(P, 1_P, \wedge_P, \lhd_0)$ can be taken as shorthand for a presentation

$$\begin{aligned}\mathbf{Fr}\langle P\ |1 &= 1_P\\ a \wedge b &= (a \wedge_P b) \quad (a, b \in P)\\ a &\leq \bigvee U \quad (a \lhd_0 U)\rangle\end{aligned}$$

or, more briefly,

$$\mathbf{Fr}\langle P \text{ (qua } \wedge\text{-semilatice)} \mid a \leq \bigvee U \quad (a \lhd_0 U)\rangle.$$

As usual, this can also be taken as a description of the points: they are the filters F of P such that if $a \lhd_0 U$ and $a \in F$, then there is an element in $F \cap U$. (But this can be taken the other way round too. If you say explicitly that those are the points, then it is clear what the implicit relations have to be.)

A site is also required to have another property, *meet stability*: if $a \lhd_0 U$ then $a \wedge b \lhd_0 \{u \wedge b \mid u \in U\}$ for every b. We shall see the significance of this in Section 3.

2.1. Flat sites

Rather than pursue the technicalities of sites, we shall take as our canonical presentations a generalization of site, widely used in both topos theory and in formal topology. Though our notation is different, these *flat sites* are just the *covering systems* discussed in [9] Section III.4; the notion is also found more recently in the *localized axiom-sets* of [4]. It relies on the fact that the notion of "meet preserving function" from P to a frame A can be defined sensibly even if P does not have meets; it just needs a preorder. Using a notion that is well known in topos theory [8], we say that a function $f : P \to A$ is *flat* iff

$$\begin{aligned}1_A &= \bigvee\nolimits_{x\in P} f(x)\\ f(x) \wedge f(y) &= \bigvee\{f(z) \mid z \leq x, z \leq y\}.\end{aligned}$$

It is obvious that if P is a meet semilattice, then f is flat precisely if it preserves finite meets.

Definition 1. A *flat site* is a structure $(P, \leq, \lhd_0)$ where $(P, \leq)$ is a preorder (i.e., transitive and reflexive), and $\lhd_0 \subseteq P \times \mathcal{P}P$ has the following *flat stability* property: if $a \lhd_0 U$ and $b \leq a$, then there is some $V \subseteq b \downarrow U$ such that $b \lhd_0 V$.

(For subsets or elements U and V, we write $U \downarrow V$ for $\downarrow U \cap \downarrow V$.)

Note that we do not assume that if $a \lhd_0 U$ then $U \subseteq \downarrow a$. This allows us some notational flexibility, but no more expressive power. For if $a \lhd_0 U$ then by considering $a \leq a$ we have some $U' \subseteq a \downarrow U$ such that $a \lhd_0 U'$.

If P is a meet semilattice, then flat stability is a mild weakening of the meet stability already mentioned.

The implicit frame presentation is

$$\begin{aligned}\mathbf{Fr}\langle P \text{ (qua preorder) } | 1 \leq \bigvee P \\ a \wedge b \leq \bigvee (a \downarrow b) \quad (a, b \in P) \\ a \leq \bigvee U \quad (a \lhd_0 U)\rangle.\end{aligned}$$

Just as with ordinary sites, we can see from this presentation that the points of the corresponding locale are the filters F of P such that if $a \lhd_0 U$ and $a \in F$, then F meets U.

This can be immediately related to the *localized axiom-sets* of [4].

Definition 2 (*[4]*). Let P be a preordered set. Then an *axiom-set* on P is a set indexed family $I(a)$ set $[a : P]$ together with a family of subsets $C(a, i) \subseteq P$ $[a : P, i : I(a)]$. The axiom-set is *localized* if, for any $a \leq c$ and $i \in I(c)$, there exists $j \in I(a)$ such that $C(a, j) \subseteq a \downarrow C(c, i)$.

This is equivalent to the notation for a flat site: $I(a)$ then is a set indexing the collection of pairs (a, U) with $a \lhd_0 U$, and if i is an index for (a, U) then $C(a, i)$ is just U. All this is on the understanding, of course, that these sets can be formed in a predicatively acceptable way, but that is why we work with a base and a cover base. The property of being localized is identical to our flat stability.

Proposition 3. *The structure of a flat site is, if described in a predicatively acceptable way, equivalent to that of a localized axiom-set on a preorder.*

Impredicatively, the full coverage $a \lhd U$ is defined as $a \leq \bigvee U$ in the frame. Predicatively, it must be defined instead by inductive generation from $\lhd_0$, and this is done in [4].

Theorem 4 (*[4]*). *Let $(P, \leq, \lhd_0)$ be a flat site. Let $\lhd$ be generated by rules*

- $\dfrac{a \in U}{a \lhd U}$ *(*reflexivity*)*
- $\dfrac{a \leq b \quad b \lhd U}{a \lhd U}$ *(*$\leq$-left*)*
- $\dfrac{a \lhd_0 V \quad V \lhd U}{a \lhd U}$ *(*infinity*).*

Then $\lhd$ is a cover (i.e., a formal topology but without positivity), and is the least such containing $\lhd_0$.

A cover relation by definition satisfies the rules of reflexivity and $\leq$-left, and in addition

- $\dfrac{a \lhd V \quad V \lhd U}{a \lhd U}$ (*transitivity*)
- $\dfrac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V}$ ($\leq$-*right*).

Notice the crucial difference between the transitivity rule for a cover relation, and the infinity rule used in generating it. In the infinity rule, the first premise $a \lhd_0 V$ must be a *basic* cover. This was recognized in [4]. The restriction was already recognized in [3] (in the definition of "hereditary" set), though with less emphasis on the inductive generation. Another paper [13] used inductive generation but with the transitivity rule, and this turned out not to work in type theory.

In the inductive generation U is fixed. The rules generate proofs of $a \lhd U$ for more and more opens a. The way we shall exploit the Theorem is that if we wish to show $a \lhd U$ implies some property $\Phi(a)$, then we shall verify three rules:

- $\dfrac{a \in U}{\Phi(a)}$
- $\dfrac{a \leq b \quad \Phi(b)}{\Phi(a)}$
- $\dfrac{a \lhd_0 V \quad \forall v \in V.\ \Phi(v)}{\Phi(a)}$.

These will then show that any proof of $a \lhd U$ can be transformed into a proof of $\Phi(a)$.

3. Suplattice presentations

As we discussed above, presenting a frame as $\mathbf{Fr}\langle P \mid R\rangle$ gives one a good grip on frame homomorphisms out of it. But one is commonly also interested in other kinds of functions out of it, an example being suplattice homomorphisms. (A *suplattice* [9] is a complete lattice, and a suplattice homomorphism preserves all joins.)

Suppose we wish to define a suplattice homomorphism f out of a frame A presented by a flat site, i.e.,

$$\begin{aligned}\mathbf{Fr}\langle P \text{ (qua preorder) } | 1 \leq \textstyle\bigvee P & \\ & a \wedge b \leq \textstyle\bigvee(a \downarrow b) \quad (a, b \in P) \\ & a \leq \textstyle\bigvee U \quad (a \lhd_0 U)\rangle.\end{aligned}$$

Since P is a base, every element of A is a join of generators. Hence if f is known on the generators, then for $W \subseteq P$ it has to be defined by

$$f(\textstyle\bigvee W) = \textstyle\bigvee_{w \in W} f(w).$$

However, there is no a priori guarantee that this is well defined. Clearly f must be monotone on generators and the relations $a \leq \bigvee U$ (for $a \lhd_0 U$) must be respected; remarkably, it is enough just to check those. This relies very much on our requirement of flat stability.

We can express the result by giving a suplattice presentation of A. The idea is implicit in [9] (in Chapter VI.1) and is stated and proved explicitly for ordinary sites in [1]. The suplattice universal property is not hard to prove for Johnstone's concrete construction [6] of the frame as a set of C-*ideals*, and so we call this kind of result a "coverage theorem".

Theorem 5 (*Coverage Theorem for Flat Sites*). *If* $(P, \leq, \lhd_0)$ *is a flat site, then its frame is order isomorphic to*

$$\mathbf{SupLat}\langle P \text{ (qua preorder)} \mid a \leq \bigvee U \quad (a \lhd_0 U)\rangle.$$

Proof. We sketch a proof much as given for ordinary sites in [1]. First, the suplattice presentation does indeed present a suplattice. (In fact all suplattice presentations do, or it could be constructed concretely using a method of "C-ideals" as in [6].) Let us write A for this suplattice. We must show first that A is a frame. Since, as a suplattice, it is in fact a complete lattice, the main task is to show frame distributivity.

Given $a \in P$, we can define a suplattice homomorphism $\alpha_a : A \to A$ by

$$\alpha_a(b) = \bigvee(a \downarrow b).$$

α_a is obviously monotone in b; we must also check that it respects the relations. In other words, if $b \lhd_0 U$ then we must have

$$\bigvee(a \downarrow b) \leq \bigvee_{u \in U} \bigvee(a \downarrow u).$$

Suppose, then, that $c \in a \downarrow b$. By flat stability we can find $V \subseteq c \downarrow U$ such that $c \lhd_0 V$, so $c \leq \bigvee V$. If $v \in V$ then $v \leq c \leq a$ and $v \leq u$ for some $u \in U$, so $v \in a \downarrow u$. Hence $\bigvee V \leq \bigvee_{u \in U} \bigvee(a \downarrow u)$ in A.

Now $\alpha_a(b) = a \wedge b$ in A, because it is the greatest lower bound of a and b. But we know that α_a is a suplattice homomorphism, and it follows that binary meet distributes over all joins and A is a frame.

It remains to prove the frame universal property. Suppose that B is a frame and $f : P \to B$ is a monotone function that respects the relations in the frame presentation. From A's suplattice presentation we know that f extends uniquely to a suplattice homomorphism $\overline{f} : A \to B$. Now

$$\begin{aligned}
\overline{f}(1_A) &= \overline{f}(\bigvee P) = \bigvee\{f(g) \mid g \in P\} = 1_B \\
\overline{f}(\bigvee U \wedge \bigvee V) &= \overline{f}(\bigvee_{u \in U} \bigvee_{v \in V} \bigvee(u \downarrow v)) \\
&= \bigvee_{u \in U} \bigvee_{v \in V} \bigvee\{f(c) \mid c \in (u \downarrow v)\} \\
&= \bigvee_{u \in U} \bigvee_{v \in V} f(u) \wedge_B f(v) \\
&= \overline{f}(\bigvee U) \wedge \overline{f}(\bigvee V)
\end{aligned}$$

so $\overline{f}$ is a frame homomorphism. $\square$

As an immediate application, we can give an analysis of openness of locales.

Definition 6. A locale X is *open* [9] iff the unique frame homomorphism $!^* : \Omega \to \Omega X$ has a left adjoint $\exists_! : \Omega X \to \Omega$.

The terminology arises because X is open iff the unique map $X \to 1$ is open in the sense that openness of sublocales is preserved by direct image: cf. open maps in topology.

Classically, every locale is open, but constructively this is not so. Following Paul Taylor, open locales are also called *overt*.

Ref. [7] defines the *positivity predicate* on ΩX for which a is positive iff whenever $a \leq \bigvee U$ in ΩX then U is inhabited. This is defined for arbitrary X, but the paper also shows that X is open iff every a is the join of the positive opens below it. It is then the case that a is positive iff $\exists_! a$ holds.

The positivity predicate is also found useful in formal topology, and it is known [12] that the positivity predicate as axiomatized in formal topologies is equivalent to openness of the corresponding locale. We can use the coverage theorem to show this.

Proposition 7. *Let X be a locale presented via a flat site $(P, \leq, \lhd_0)$. Then the following are equivalent.*

(1) *X is open.*
(2) *There is an upper closed subset Pos of P such that*
   (a) *If $a \lhd_0 U$ and $a \in Pos$ then Pos meets U.*
   (b) *For each a in P we have $a \lhd \{a' \mid a' = a$ and $a \in Pos\}$.*
(3) *P has a* positivity predicate*, i.e., a predicate $\mathsf{Pos}(a)$ satisfying the rules*
   (a) $\dfrac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b \in U)\ \mathsf{Pos}(b)}$ *(monotonicity)*
   (b) $\dfrac{a \lhd U \quad [\mathsf{Pos}(a)]}{a \lhd U}$ *(positivity).*

Proof. We prove (1) $\Leftrightarrow$ (2) impredicatively, since openness of X is defined explicitly in terms of the frame. On the other hand, we prove (2)$\Leftrightarrow$ (3) predicatively. In fact, this was essentially already done in [4]. Condition (3) is taken from their definition of a positivity predicate, and in (2) the upper closedness is their "monotonicity on $\leq$", while 2(a) is their "monotonicity on axioms".

(1) $\Leftrightarrow$ (2) (impredicatively): X is open iff there is a suplattice homomorphism $\theta : \Omega X \to \Omega$ that is left adjoint to $!^*$. By the coverage theorem, a suplattice homomorphism θ is equivalent to a monotone function $P \to \Omega$ that respects the relations, and this is equivalent to an upper closed subset *Pos* satisfying 2(a). It therefore remains only to show that θ being left adjoint to $!^*$ is equivalent to 2(b).

The left adjointness amounts to two inequations:

$$\theta(!^*(p)) \leq p \quad (p \in \Omega)$$
$$a \leq !^*(\theta(a)) \quad (a \in \Omega X)$$

Now $!^*(p) = \bigvee\{1 \mid p\}$, so the first inequation says $\bigvee\{\theta(1) \mid p\} \leq p$, i.e., if p then $\theta(1) \leq p$. This always holds. For the second inequation, it suffices to check it for $a \in P$ and so it says $a \leq \bigvee\{1 \mid \theta(a)\}$. This is equivalent to

$$a \leq a \wedge \bigvee\{1 \mid \theta(a)\} = \bigvee\{a' \mid a' = a \text{ and } a \in Pos\},$$

in other words 2(b).

For (2) $\Leftrightarrow$ (3), $\mathsf{Pos}(a)$ is just the predicate $a \in Pos$.

(3) $\Rightarrow$ (2): 3(a) implies 2(a) a fortiori. For 2(b), we can prove it by reflexivity on the assumption that $a \in Pos$, for then a is an element of $\{a' \mid a' = a$ and $a \in Pos\}$. Then 3(b)

tells us that it holds even without the assumption. To show *Pos* is upper closed, suppose $a \le a'$ and $\mathsf{Pos}(a)$. We have $a \lhd \{a'\}$ and $\mathsf{Pos}(a')$ follows by 3(a).

(2) ⇒ (3): For 3(b), suppose we can prove $a \lhd U$ on the assumption of $\mathsf{Pos}(a)$. This tells us that $\{a' \mid a' = a \text{ and } a \in Pos\} \lhd U$. Now 2(b) and transitivity for $\lhd$ give us that $a \lhd U$.

For 3(a) we must use induction on the proof of $a \lhd U$. Given U, define the property $\Phi_U(a)$ to hold iff $\mathsf{Pos}(a) \to (\exists b \in U)\, \mathsf{Pos}(b)$. We show that if $a \lhd U$ then $\Phi_U(a)$.

$\dfrac{a \in U}{\Phi_U(a)}$: This is obvious (take $b = a$).

$\dfrac{a \le a' \quad \Phi_U(a')}{\Phi_U(a)}$: If $\mathsf{Pos}(a)$ then by upper closure of *Pos* we have $\mathsf{Pos}(a')$, and we can use $\Phi_U(a')$.

$\dfrac{a \lhd_0 V \quad \forall v \in V.\ \Phi_U(v)}{\Phi_U(a)}$: If $\mathsf{Pos}(a)$ then by 2(a) there is some $v \in V$ with $\mathsf{Pos}(v)$. Now we can use $\Phi_U(v)$. □

4. Compactness: In locale theory

As is well-known, the compactness property for topological spaces can be expressed as a property of the topology (the lattice of opens) and adapts well to locales: the locale X is compact iff, whenever $1 = \bigvee U$ in the frame ΩX, then $1 = \bigvee U_0$ for some finite $U_0 \subseteq U$. It is also well known that this can be expressed in terms of directed joins: X is compact iff, whenever $1 = \bigvee^{\uparrow} U$ for U a directed subset of ΩX (we shall use the notation $\bigvee^{\uparrow}$ to indicate that the join is directed), then $1 = u$ for some $u \in U$.

Now consider the function $\forall_! : \Omega X \to \Omega$ defined by letting $\forall_!(a)$ be the proposition $(1 = a)$: it is right adjoint to $!^*$ and always exists (at least in topos-valid mathematics). The characterization of compactness using directed joins can now be rephrased: X is compact iff its $\forall_!$ preserves directed joins.

The question arises of how we can get sufficient information to prove compactness starting from a *presentation* of a frame. Let us say (for definiteness) we are given a flat site $(P, \le, \lhd_0)$. *If* $\forall_!$ is to preserve directed joins, then for every $U \subseteq P$ it must satisfy

$$\forall_!(\bigvee U) = \bigvee^{\uparrow}\{\forall_!(\bigvee U_0) \mid U_0 \text{ a finite subset of } U\}.$$

One might hope, therefore, for an approach similar to that used for openness of locales. Define a function $\forall_!$ that preserves directed joins by defining its action on *finite* joins of basics, and use the definition to show that it is indeed the desired right adjoint $\forall_!$. Again, we are trying to define a non-frame homomorphism out of the frame, but this time the coverage theorem is no help: unlike $\exists_!$, $\forall_!$ does not in general preserve finite joins. We now show how a presentation can allow us to define dcpo morphisms out of a frame. (A *dcpo* — a directed complete poset — is a poset with all directed joins, and a dcpo morphism is a function that preserves directed joins.)

In order to prove it we shall need the following proposition from [24].

Proposition 8. *Let L be a join semilattice and let $\lhd_0$ be a relation from L to $\mathcal{P}L$ such that if $a \lhd_0 U$ then U is directed, and (*join stability*) for each b in L we also have*

$a \vee b \lhd_0 \{u \vee b \mid u \in U\}$. *Then*

$$\mathbf{SupLat}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid a \leq {\bigvee}^{\uparrow} U \quad (a \lhd_0 U)\rangle$$
$$\cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid a \leq {\bigvee}^{\uparrow} U \quad (a \lhd_0 U)\rangle.$$

Here and later we shall use the symbol $\sqsubseteq_L$ for the *lower preorder* on the finite powerset $\mathcal{F}P$ of a preorder P, defined by

$$S \sqsubseteq_L T \text{ iff } \forall s \in S.\, \exists t \in T.\, s \leq t.$$

We also write $\mathcal{F}P/\sqsubseteq_L$ for the set of equivalence classes for $\sqsubseteq_L \cap (\sqsubseteq_L)^{op}$. A simple but useful result is that this is the free join semilattice over P qua preorder, with joins represented by union. (See e.g. [22, Proposition 19].)

Theorem 9. *If* $(P, \leq, \lhd_0)$ *is a flat site, then its frame is order isomorphic to*

$$\mathbf{dcpo}\langle \mathcal{F}P \text{ (qua } \sqsubseteq_L \text{ preorder)} \mid$$
$$\{a\} \cup T \leq {\bigvee}^{\uparrow}\{U_0 \cup T \mid U_0 \in \mathcal{F}U\} \quad (a \lhd_0 U)\rangle.$$

Here "qua $\sqsubseteq_L$ *preorder" indicates implicit relations to say that the inclusion of generators is monotone with respect to* $\sqsubseteq_L$ *(in* $\mathcal{F}P$*) and* $\leq$ *(in the dcpo).*

Proof. By Theorem 5 the frame is isomorphic to

$$\begin{aligned}
&\mathbf{SupLat}\langle P \text{ (qua preorder)} \mid a \leq \bigvee U \quad (a \lhd_0 U)\rangle \\
\cong\ &\mathbf{SupLat}\langle \mathcal{F}P/\sqsubseteq_L \text{ (qua } \vee = \cup\text{-semilattice)} \mid \\
&\qquad a \leq {\bigvee}^{\uparrow}\{U_0 \mid U_0 \in \mathcal{F}U\} \quad (a \lhd_0 U)\rangle \\
\cong\ &\mathbf{SupLat}\langle \mathcal{F}P/\sqsubseteq_L \text{ (qua } \vee = \cup\text{-semilattice)} \mid \\
&\qquad \{a\} \cup T \leq {\bigvee}^{\uparrow}\{U_0 \cup T \mid U_0 \in \mathcal{F}U\} \quad (a \lhd_0 U, T \in \mathcal{F}P)\rangle.
\end{aligned}$$

Now apply Proposition 8. □

Theorem 10. *Let* $(P, \leq, \lhd_0)$ *be a flat site presenting a locale* X*. Then* X *is compact iff there is a subset* F *of* $\mathcal{F}P$ *such that*

(1) F *is upper closed with respect to* $\sqsubseteq_L$.
(2) *If* $a \lhd_0 U$ *and* $\{a\} \cup T \in F$*, then* $U_0 \cup T \in F$ *for some* $U_0 \in \mathcal{F}U$.
(3) F *is inhabited.*
(4) *If* $S \in F$ *then* $P \lhd S$ *(i.e.,* $\forall g \in P.\, g \lhd S$*).*

In that case, F *necessarily comprises all finite covers of* X *by basics, i.e., all finite subsets of the base* P *that cover* P.

Proof. By Theorem 9, conditions (1) and (2) are equivalent to a dcpo morphism θ from ΩX to Ω, defined by $\theta(\bigvee U)$ iff $U_0 \in F$ for some $U_0 \in \mathcal{F}U$. We show that, in that situation, conditions (3) and (4) are equivalent to θ being right adjoint to $!^*$, in other words that $p \leq \theta(!^*(p))$ for all p in Ω, and $!^*(\theta(S)) \leq S$ for all S in $\mathcal{F}P$. The first of these amounts to saying that $\theta(1)$ holds, and since $1 = {\bigvee}^{\uparrow} \mathcal{F}P$ this is equivalent to condition (3).

The second amounts to saying that if $\theta(S)$ holds, i.e., if $S \in F$, then $1 \leq S$ in the frame, i.e., $P \lhd S$. Hence this is equivalent to condition (4).

Hence the conditions are equivalent to there being a dcpo morphism right adjoint to $!^*$, i.e., to compactness of X.

If $U \in \mathcal{F}P$ and $P \lhd U$, then $1 \leq \bigvee U$ in ΩX and so $\theta(1) \leq \theta(\bigvee U)$. It follows that $U \in F$. $\square$

For some examples, consider flat sites $(P, \leq, \lhd_0)$ in which all the cover axioms $a \lhd_0 U$ have U is finite. If P is a meet semilattice, then in the frame presentation derived from an ordinary site (Section 2) we see that all the joins are finite. Hence the corresponding locale is spectral (i.e., the frame is the ideal completion of a distributive lattice) and hence is compact. We can weaken this condition on P.

Proposition 11. *Let $(P, \leq, \lhd_0)$ be a flat site in which P has a top element 1 and if $a \lhd_0 U$ then U is finite. Then the corresponding locale X is compact.*

Proof. Consider finite trees with the following properties.

1. Every node is labelled with an element of P.
2. The root is labelled with 1.
3. If a branch node is labelled with a and its children are labelled with the elements of U, then $a \lhd_0 U$.
4. Each leaf node is marked (in addition to its label) as either "null" or "non-null". If a null leaf node is labelled with a then $a \lhd_0 \emptyset$.

Let us call such a tree a *cover tree*. We write $L(\tau)$ for the finite subset of P comprising the non-null leaf labels of τ; clearly this covers 1. Let F be the subset of $\mathcal{F}P$ comprising those finite subsets T with $L(\tau) \sqsubseteq_L T$ for some cover tree τ. In Theorem 10, all the conditions are obvious except for (2). For this, suppose $a \lhd_0 U$ and $L(\tau) \sqsubseteq_L \{a\} \cup T$ for some cover tree τ. We can construct (non-deterministically) a new cover tree τ' by modifying the non-null leaf nodes as follows.

- If a non-null leaf label is less than an element of T then we may leave the node unchanged.
- If a non-null leaf node is labelled by $b \leq a$, then we have $b \lhd_0 U_0$ for some $U_0 \in \mathcal{F}(b \downarrow U)$. If U_0 is inhabited then we may convert the leaf node into a branch node, with children non-null leaves labelled by the elements of U_0. If U_0 is empty, then we may mark the leaf node as null instead of non-null.

Then $L(\tau') \sqsubseteq_L U \cup T$, so $U \cup T \in F$. $\square$

Note that some condition does have to imposed on P. This is clear if one realizes that for an arbitrary preorder P, if there are no cover axioms at all then the site presents the localic equivalent of the algebraic dcpo $\mathrm{Idl}(P^{op})$ — its points are the filters of P — and these are not compact in general. For a particular example, take P to be the set $\mathbb{N}$ of natural numbers, with the discrete order. The site presents the discrete locale $\mathbb{N}$ (its frame is the powerset of $\mathbb{N}$) and this is not compact.

The proof of Theorem 10 was highly impredicative, but the statement was not. We now work towards showing, as Theorem 15, that the same result holds predicatively.

5. Remarks on finiteness

Before moving on to formal topologies, we pause to examine some issues of finiteness. We have assumed throughout that *finite* means, in topos theoretic terms, *Kuratowski finite*: in other words, a set X is finite iff, in the powerset $\mathcal{P}X$, X itself is in the $\cup$-subsemilattice generated by the singletons. In fact for any set X, that subsemilattice is the *finite powerset* $\mathcal{F}X$. (See [8]; $\mathcal{F}X$ is there called $K(X)$. It is also the notion of finiteness used in [3].) That appears very impredicative, but in fact $\mathcal{F}X$ can alternatively be characterized as the free semilattice over X and that gives access to inductive constructions.

To represent $\mathcal{F}X$ in predicative type theory one uses the fact that every Kuratowski finite set can be described by a finite enumeration of its elements (possibly with repetitions; this is unavoidable). Thus $\mathcal{F}X$ can be handled using the list monoid X^* with a defined equality by which two lists are considered equal iff each contains all the elements of the other. This is described in [13], where $\mathcal{F}X$ is denoted by $\mathcal{P}_\omega(X)$.

Some constructive issues in reasoning with these finite sets are discussed in [20]. In many of these there are quite explicit calculations, treated there by an "$\mathcal{F}$-recursion principle" but translatable into computations on finite lists that can quite easily be implemented in functional programming languages. For example, if X is finite then so is $\mathcal{F}X$. This is proved by defining, for arbitrary X, a function $f : \mathcal{F}X \to \mathcal{F}\mathcal{F}X$ whose specification is that $T \in f(S) \Leftrightarrow T \subseteq S$. The recursive implementation of f is

$$\begin{aligned} f(\emptyset) &= \{\emptyset\} \\ f(\{x\} \cup S) &= f(S) \cup \{\{x\} \cup T \mid T \in f(S)\}. \end{aligned}$$

A little inductive reasoning is then required to show that the implementation satisfies the specification. For instance, a finite subset of $\{x\} \cup S$ is either a finite subset of S and hence (by induction) in $f(S)$, or is of the form $\{x\} \cup T$ where T is a finite subset of S.

One can see how this could be implemented with lists (with no attempt whatsoever to avoid repetitions):

$$\begin{aligned} f(\langle\rangle) &= \langle\langle\rangle\rangle \\ f(\langle x\rangle^\frown S) &= f(S)^\frown g(x, f(S)) \\ g(x, \langle\rangle) &= \langle\rangle \\ g(x, \langle T\rangle^\frown Ts) &= \langle\langle x\rangle^\frown T\rangle^\frown g(x, Ts). \end{aligned}$$

Here $\langle$ and $\rangle$ are list brackets, so $\langle\rangle$ denotes the empty list, and $^\frown$ is list concatenation. The variable x has type X, S and T have type X^*, and Ts has type X^{**}. $g(x, Ts)$ is an auxiliary function to calculate the list of terms $\langle x\rangle^\frown T$ for T in Ts.

We summarize here some of the properties of finite sets that we shall need.

(1) If X is finite then so is $\mathcal{F}X$.
(2) If X is finite then emptiness of X is a decidable property.
(3) There is a simple induction principle for finite sets. Suppose Φ is a property of finite subsets of X such that (i) $\Phi(\emptyset)$, and (ii) $\Phi(S) \Rightarrow \Phi(\{x\} \cup S)$. Then Φ holds for all finite subsets of X.

(4) There is a simple mode of recursive definition of functions $f : \mathcal{F}X \to Y$,

$$f(\emptyset) = y_0$$
$$f(\{x\} \cup S) = e(x,\ f(S))$$

where $y_0 \in Y$ and $e : X \times Y \to Y$, provided e satisfies two conditions

$$e(x, e(x, y)) = e(x, y)$$
$$e(x_1, e(x_2, y)) = e(x_2, e(x_1, y)).$$

These are to respect the fact that $\{x\} \cup (\{x\} \cup S) = \{x\} \cup S$ and $\{x_1\} \cup (\{x_2\} \cup S) = \{x_2\} \cup (\{x_1\} \cup S)$. (cf. the elimination rule in [13].)

(5) (See [3].) Suppose X is finite and ϕ and ψ are two predicates on X such that for every x in X either $\phi(x)$ or $\psi(x)$ holds. Then there can be found finite sets X' and X'' such that $X = X' \cup X''$, every $x \in X'$ has $\phi(x)$, and every $x \in X''$ has $\psi(x)$.

(6) As a corollary, suppose X is a set, A and B are subsets and V a finite subset of $A \cup B$. Then there can be found finite subsets $V' \subseteq A$ and $V'' \subseteq B$ such that $V = V' \cup V''$.

(7) Suppose X, ϕ and ψ are as in (5). Then either every $x \in X$ has $\phi(x)$ or there is some $x \in X$ with $\psi(x)$. (Decompose X as above, and consider whether X'' is empty or not.)

We shall later prove binary Tychonoff, and for that we shall need to work with decompositions $X = X' \cup X''$ of a finite set X. We shall only consider *finite* decompositions, i.e., ones in which X' and X'' are also finite (constructively, subsets of a Kuratowski finite set are not necessarily finite).

Lemma 12. *If X is finite then so is its set of finite decompositions.*

Proof. For arbitrary X, we define a function $\mathsf{decomp} : \mathcal{F}X \to \mathcal{F}(\mathcal{F}X \times \mathcal{F}X)$ such that $(T', T'') \in \mathsf{decomp}(T)$ iff $T = T' \cup T''$.

$$\begin{aligned}
\mathsf{decomp}(\emptyset) &= \{(\emptyset, \emptyset)\} \\
\mathsf{decomp}(\{x\} \cup T) &= \{(\{x\} \cup T', T'') \mid (T', T'') \in \mathsf{decomp}(T)\} \\
&\quad \cup \{(T', \{x\} \cup T'') \mid (T', T'') \in \mathsf{decomp}(T)\} \\
&\quad \cup \{(\{x\} \cup T', \{x\} \cup T'') \mid (T', T'') \in \mathsf{decomp}(T)\}.
\end{aligned}$$

(There is a proof obligation to be checked here, to show that the calculation gives the same result for $\mathsf{decomp}(\{x\} \cup (\{y\} \cup T))$ as for $\mathsf{decomp}(\{y\} \cup (\{x\} \cup T))$, and the same for $\mathsf{decomp}(\{x\} \cup (\{x\} \cup T))$ as for $\mathsf{decomp}(\{x\} \cup T)$.)

To show that it satisfies its specification, we can assume an induction hypothesis that $\mathsf{decomp}(T)$ is correct. It is then clear that if $(T', T'') \in \mathsf{decomp}(\{x\} \cup T)$, then $\{x\} \cup T = T' \cup T''$.

Conversely, suppose $\{x\} \cup T = U' \cup U''$. Since $U' \subseteq \{x\} \cup T$, we can find a finite decomposition $U' = U'_x \cup U'_0$ with $U'_x \subseteq \{x\}$ and $U'_0 \subseteq T$. Similarly, we can find $U'' = U''_x \cup U''_0$ with $U''_x \subseteq \{x\}$ and $U''_0 \subseteq T$. Moreover, since $x \in U' \cup U''$, we can assume that at least one of U'_x and U''_x contains x: for instance, if $x \in U'$ we can replace U'_x by $U'_x \cup \{x\}$. On the other hand, since $T \subseteq U' \cup U''$ we can find a decomposition $T = T' \cup T''$ with $T' \subseteq U'$ and $T'' \subseteq U''$. It follows that $T = (T' \cup U'_0) \cup (T'' \cup U''_0)$, so without loss of generality $U'_0 \subseteq T'$ and $U''_0 \subseteq T''$. We have $(T', T'') \in \mathsf{decomp}(T)$.

Now $U' = U'_x \cup T'$. Since U'_x is finite, so that its emptiness is decidable, we must have U'_x equal to either $\emptyset$ or $\{x\}$. Hence U' is either T' or $\{x\} \cup T'$. Similarly, U'' is either T'' or $\{x\} \cup T''$. Since at least one of U'_x and U''_x is $\{x\}$ we deduce that $(U', U'') \in \mathsf{decomp}(\{x\} \cup T)$. $\square$

Our main use of such decompositions is in a distributivity result for distributive lattices.

Lemma 13. *Let L be a distributive lattice, let S be a finite set, and let a_i, b_i be elements of L indexed by elements i of S. Then*

$$\bigvee_{i \in S}(a_i \wedge b_i) = \bigwedge_{(T,U) \in \mathsf{decomp}(S)} \left(\bigvee_{i \in T} a_i \vee \bigvee_{i \in U} b_i\right).$$

Proof. Use induction on S. $\square$

We shall also need the following *Product Decomposition Lemma*.

Lemma 14. *Let X_i be a set and ϕ_i a predicate on it ($i = 1, 2$). Let $S \in \mathcal{F}(X_1 \times X_2)$ be such that for every finite decomposition $S = S' \cup S''$ there is either some $(x, y) \in S'$ with $\phi_1(x)$, or some $(x, y) \in S''$ with $\phi_2(y)$. Then there is some $(x, y) \in S$ with both $\phi_1(x)$ and $\phi_2(y)$.*

Proof. Classically this is easy. Let $S' = \{(x, y) \in S \mid \phi_1(x)\}$ and let $S'' = S - S'$. Then by considering the decomposition $S'' \cup S'$, we find either some $(x, y) \in S''$ with $\phi_1(x)$ or some $(x, y) \in S'$ with $\phi_2(y)$. The former is impossible, and the latter gives the result.

Constructively we use induction on S. If S is empty then the decomposition $\emptyset \cup \emptyset$ gives a contradiction.

Now suppose the result holds for S and we must prove it for $\{(x_0, y_0)\} \cup S$. Every decomposition $S' \cup S''$ of S gives two decompositions, $(\{(x_0, y_0)\} \cup S') \cup S''$ and $S' \cup (\{(x_0, y_0)\} \cup S'')$, of $\{(x_0, y_0)\} \cup S$. We deduce

$$\phi_1(x_0) \text{ or } \exists (x, y) \in S'.\ \phi_1(x) \text{ or } \exists (x, y) \in S''.\ \phi_2(y)$$

and

$$\exists (x, y) \in S'.\ \phi_1(x) \text{ or } \phi_2(y_0) \text{ or } \exists (x, y) \in S''.\ \phi_2(y).$$

It follows that for every decomposition of S we have *either* $\phi_1(x_0)$ and $\phi_2(y_0)$, *or* $\exists (x, y) \in S'.\ \phi_1(x)$ or $\exists (x, y) \in S''.\ \phi_2(y)$. Because the set of decompositions is finite, it therefore follows that *either* there is some decomposition with $\phi_1(x_0)$ and $\phi_2(y_0)$, *or* for every decomposition $S = S' \cup S''$ we have either $\exists (x, y) \in S'.\ \phi_1(x)$ or $\exists (x, y) \in S''.\ \phi_2(y)$. In the first case we are done, and in the second we can use induction. $\square$

6. Compactness: In formal topology

The conditions of Theorem 10 still make sense in the context of formal topology, and one can therefore ask whether the Theorem is still valid for formal topologies. However the calculations leading to it and its proof all relied on the impredicative construction of the frame and an analysis of its dcpo structure. In this section we shall see that the Theorem, though arrived at by impredicative considerations, is predicatively true.

In formal topology a compactness proof will proceed as follows (we stay with the notation of the flat sites): if $P \lhd U$ then we must prove that $\exists U_0 \in \mathcal{F}U$ such that $P \lhd U_0$. But this relies heavily on knowing the full $\lhd$ and often uses a result such as Theorem 4 to provide an inductive analysis of all possible proofs of $P \lhd U$. We now show how that inductive analysis can be used to justify the general criterion of Theorem 10.

Theorem 15. *Let $(P, \leq, \lhd_0)$ be a flat site generating a formal topology X with cover relation $\lhd$. Then X is compact iff there is a subset F of $\mathcal{F}P$ such that*

(1) *F is upper closed with respect to $\sqsubseteq_L$.*
(2) *If $a \lhd_0 U$ and $\{a\} \cup T \in F$, then $U_0 \cup T \in F$ for some $U_0 \in \mathcal{F}U$.*
(3) *F is inhabited.*
(4) *If $S \in F$ then $P \lhd S$ (i.e., $\forall g \in P.\ g \lhd S$).*

In that case, F is necessarily the subset of $\mathcal{F}P$ comprising all finite covers of X by basics.

Proof. $\Rightarrow$: Suppose the formal topology is compact. We define F to contain all finite covers of X by basics: if $S \in \mathcal{F}P$ then

$$S \in F \text{ iff } P \lhd S.$$

We now prove the four properties. (4) is immediate.

(1) If $S \sqsubseteq_L S'$ then $S \lhd S'$. If follows that if $S \in F$ then $S' \in F$.

(2) Since $a \lhd U$, it follows that $\{a\} \cup T \lhd U \cup T$. Then since $\{a\} \cup T \in F$, we have

$$P \lhd U \cup T.$$

Now by compactness there is a finite subset U' of $U \cup T$ such that $P \lhd U'$. We can then find finite subsets U_0 and U_1 of U and T respectively such that $U' = U_0 \cup U_1$, and it follows that $P \lhd U_0 \cup T$, i.e., $U_0 \cup T \in F$.

(3) We have $P \lhd P$, and by compactness it follows that there is some finite $S \subseteq P$ such that $S \in F$.

$\Leftarrow$: Given a subset $U \subseteq P$, let us say that a finite subset S has the property Φ_U iff, for every $T \in \mathcal{F}P$ with $S \cup T \in F$, there is some $U_0 \in \mathcal{F}U$ with $U_0 \cup T \in F$. (This is related to the predicate $P(x, y, Z)$ in [13, Definition 3.1]. Very roughly, $P(x, y, Z)$ corresponds to $\Phi_Z(\{x\})$.) We prove a couple of facts about Φ_U.

First, if $\Phi_U(\{a\})$ holds for every $a \in S$, then $\Phi_U(S)$ holds. This follows by induction on S. If $S = \emptyset$ and $S \cup T \in F$, then we can choose $U_0 = \emptyset$. Now suppose the claim holds for S and we want to prove it for $\{a\} \cup S$. If $\{a\} \cup S \cup T \in F$, then by using $\Phi_U(\{a\})$ we find U_0' with $S \cup U_0' \cup T \in F$; and then assuming $\Phi_U(S)$ by induction we find U_0'' with $U_0'' \cup U_0' \cup T \in F$. Take $U_0 = U_0'' \cup U_0'$.

Second, we show that if $a \lhd U$ then $\Phi_U(\{a\})$. For induction on the proof of $a \lhd U$, we verify the three rules. In each one, taking $\{a\} \cup T \in F$, we seek a suitable U_0.

- $\dfrac{a \in U}{\Phi_U(\{a\})}$: Take $U_0 = \{a\}$.

- $\dfrac{a \le b \quad \Phi_U(\{b\})}{\Phi_U(\{a\})}$: Since $\{a\} \cup T \sqsubseteq_L \{b\} \cup T$ we have $\{b\} \cup T \in F$, so we can use $\Phi_U(\{b\})$ to find U_0.
- $\dfrac{a \lhd_0 V \quad \forall v \in V.\ \Phi_U(\{v\})}{\Phi_U(\{a\})}$: By condition (2) of the Theorem, we have some $V_0 \in \mathcal{F}V$ such that $V_0 \cup T \in F$. But $\Phi_U(V_0)$ holds and that gives us our U_0.

By condition (3) we can find some S in F. Now suppose $P \lhd U$. Then $S \lhd U$ because $S \subseteq P$, so $\Phi_U(S)$ holds. Since $S \cup \emptyset \in F$, we can find $U_0 \in \mathcal{F}U$ with $U_0 \in F$, and by condition (4) U_0 is thus a finite subcover of U.

We have now shown that if F satisfies the conditions, then X is compact. Moreover, suppose $U \in \mathcal{F}P$ is a finite cover of X by basics. We have already shown that U has a finite subset U_0 in F; but then $U_0 \sqsubseteq_L U$ and so $U \in F$. Hence F comprises all finite covers of X by basics. □

Note a certain payoff from this Theorem. To use it to show compactness, we have to define F and it has to comprise all finite covers by basics. But we do not need to prove that fact. In practice we make an informed guess, often based on spatial intuitions, and then try to verify the conditions. If we can do that, then the Theorem confirms that our guess was right. The inductive analysis of proofs of $a \lhd U$ is done for us by the proof of the Theorem.

Note also the way that the full cover relation $\lhd$ appears, in condition (4). It is true that we need to know something of the inductive generation of $\lhd$ in order to prove $P \lhd S$. However, the Theorem saves us from having to analyse all possible ways that $a \lhd U$ might arise.

6.1. DL-sites

Our results are not specific to flat sites. In fact, we expect them to work quite generally for different modes of presentation. The results of [21] show that any frame presentation can be transformed geometrically (avoiding impredicative constructions) into a dcpo presentation.

As an illustration, we consider the *DL-sites* of [24]. In this the generators form a distributive lattice (DL) L, whose lattice structure is to be preserved in the frame, and the relations are all of the form $\bigvee^{\uparrow} I = \bigvee^{\uparrow} J$ where I and J are ideals of L (lower closed directed subsets). In other words, the relations are concerned only with directed joins, not with finite meets or finite joins. Also required are meet stability and join stability: given a relation $\bigvee^{\uparrow} I = \bigvee^{\uparrow} J$ and any $a \in L$, then the relations

$$\bigvee\nolimits^{\uparrow}\{x \wedge a \mid x \in I\} = \bigvee\nolimits^{\uparrow}\{x \wedge a \mid x \in J\}$$
$$\bigvee\nolimits^{\uparrow}\{x \vee a \mid x \in I\} = \bigvee\nolimits^{\uparrow}\{x \vee a \mid x \in J\}$$

must also be amongst the presented relations. For convenience here, we shall assume that the relations have been worked into the form $a \le \bigvee^{\uparrow} U$, so we have a relation $\lhd_0$ for which if $a \lhd_0 U$ then U is directed. $\lhd_0$ also has meet and join stability.

Ref. [24] show a coverage result for DL-sites,

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid a \leq \bigvee^{\uparrow} U \quad (a \lhd_0 U)\rangle$$
$$\cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid a \leq \bigvee^{\uparrow} U \quad (a \lhd_0 U)\rangle.$$

This is the frame of a corresponding locale, whose points are the prime filters F of L such that if $a \lhd_0 U$ and $a \in F$, then F meets U.

From the coverage theorem one can deduce impredicatively:

Proposition 16. *Let $(L, \lhd_0)$ be a DL-site presenting a locale X.*

(1) *X is compact iff there is an upper closed $F \subseteq L$ such that*
   (a) *if $a \lhd_0 U$ and $a \in F$, then F meets U;*
   (b) *$1 \in F$;*
   (c) *if $a \in F$ then $1 \lhd \{a\}$.*
(2) *X is open iff there is an upper closed $F \subseteq L$ such that*
   (a) *if $a \lhd_0 U$ and $a \in F$, then F meets U;*
   (b) *$0 \notin F$,*
   (c) *if $a \in L$ then $a \lhd \{1 \mid a \in F\}$.*

Proof. The proofs are analogous to that of Theorem 10. In each case upper closedness of F together with condition (a) are exactly what is needed to define a dcpo morphism $\Omega X \to \Omega$. Conditions (b) and (c) make it the appropriate adjoint of $!^*$.

We remark for part (2) that if F defines $\exists : \Omega X \to \Omega$ then one of the adjointness conditions is that $\exists(!^*(p)) \leq p$ for every $p \in \Omega$. Now

$$\exists(!^*(p)) = \exists(\bigvee^{\uparrow}(\{0\} \cup \{1 \mid p\}))$$
$$= \bigvee^{\uparrow}(\{\exists(0)\} \cup \{\exists(1) \mid p\}) = \exists(0) \vee (p \wedge \exists(1)),$$

so this condition is equivalent to $\exists(0) \leq \mathbf{false}$, i.e., $0 \notin F$. $\square$

Again, the Proposition is stated in predicative form, and can be proved predicatively. One way is to note that a DL-site can be expressed as a flat site $(L, \leq, \lhd_1)$, with $a \lhd_1 U$ whenever $a \lhd_0 U$, and also $\bigvee T \lhd_1 T$ for every $T \in \mathcal{F}L$. Then the conditions given in the Proposition can be related to those given in Theorem 15 and Proposition 7. For instance, for part (1) F here corresponds to $F' = \{T \in \mathcal{F}L \mid \bigvee T \in F\} \subseteq \mathcal{F}L$ as required for Theorem 15.

7. Products and Tychonoff

As a case study, let us consider products of locales.

Proposition 17. *Let $(P_1, \leq, \lhd_0)$ and $(P_2, \leq, \lhd_0)$ be two flat sites. Then the product of the corresponding locales is presented by a flat site $(P_1 \times P_2, \leq, \lhd_0)$ where the preorder is the product preorder, and the covers presented are*

$$(a, b) \lhd_0 U \times \{b\} \quad (a \lhd_0 U \text{ in } P_1)$$
$$(a, b) \lhd_0 \{a\} \times V \quad (b \lhd_0 V \text{ in } P_2).$$

Proof. First note that this is indeed a flat site.

The frame for the product is presented by putting together the presentations for the original frames. For clarity, let us write α_1 and α_2 for the two injections of generators. Then the frame is presented as

$$\begin{aligned}
\mathbf{Fr}\langle \alpha_1(a), \alpha_2(b)\ (a \in P_1, b \in P_2) \mid \\
\alpha_1(a) &\leq \alpha_1(a') \quad (a \leq a') \\
\alpha_2(b) &\leq \alpha_2(b') \quad (b \leq b') \\
1 &\leq \textstyle\bigvee_{a\in P_1} \alpha_1(a) \\
1 &\leq \textstyle\bigvee_{b\in P_2} \alpha_2(b) \\
\alpha_1(a) \wedge \alpha_1(a') &\leq \textstyle\bigvee \{\alpha_1(c) \mid c \leq a, c \leq a'\} \quad (a, a' \in P_1) \\
\alpha_2(b) \wedge \alpha_2(b') &\leq \textstyle\bigvee \{\alpha_2(c) \mid c \leq b, c \leq b'\} \quad (b, b' \in P_2) \\
\alpha_1(a) &\leq \textstyle\bigvee_{u\in U} \alpha_1(u) \quad (a \lhd_0 U) \\
\alpha_2(b) &\leq \textstyle\bigvee_{v\in V} \alpha_2(v) \quad (b \lhd_0 V)\rangle.
\end{aligned}$$

This is isomorphic to

$$\begin{aligned}
\mathbf{Fr}\langle P_1 \times P_2 \text{ (qua preorder)} \mid 1 &\leq \textstyle\bigvee (P_1 \times P_2) \\
(a, b) \wedge (a', b') &\leq \textstyle\bigvee ((a, b) \downarrow (a', b')) \\
(a, b) &\leq \textstyle\bigvee (U \times \{b\}) \quad (a \lhd_0 U \text{ in } P_1) \\
(a, b) &\leq \textstyle\bigvee (\{a\} \times V) \quad (b \lhd_0 V \text{ in } P_2)\rangle.
\end{aligned}$$

In one direction, the isomorphism takes $\alpha_1(a) \mapsto \bigvee_b (a, b)$ and $\alpha_2(b) \mapsto \bigvee_a (a, b)$, while in the other it takes $(a, b) \longmapsto \alpha_1(a) \wedge \alpha_2(b)$.

This second presentation corresponds to the product site described in the statement. □

We shall now give (yet) another proof of the binary Tychonoff theorem. Of course, this has been done before. Ref. [10] give a localic proof, relying on the impredicative construction of the frame, and [3,13] give proofs in type theory that avoid the use of choice principles. Here we shall examine how the technique of Theorem 10 applies: if we are given sets F_i describing compactness for the P_is, then we show how to construct a corresponding set F for the product. The main point of interest is that F itself can be defined without reference to the full coverage $\lhd$. The full coverage and its inductive generation only need to be considered when showing that every set in F covers the product space; but this is hardly surprising, because the corresponding facts for the F_is were described in terms of $\lhd$. In [23] we show how the same techniques can be used to prove infinitary Tychonoff in a general form.

First we prove a result about product coverings.

Proposition 18. *Let $(P_1, \leq, \lhd_0)$ and $(P_2, \leq, \lhd_0)$ be two flat sites. If $a_i \lhd U_i$ in each P_i, then $(a_1, a_2) \lhd U_1 \times U_2$.*

Proof. First note for arbitrary a_1 that we have

$$\frac{a_2 \leq a_2' \quad (a_1, a_2') \lhd U_1 \times U_2}{(a_1, a_2) \lhd U_1 \times U_2}$$

(obviously) and

$$\frac{a_2 \lhd_0 V \quad \forall v \in V.\, (a_1, v) \lhd U_1 \times U_2}{(a_1, a_2) \lhd U_1 \times U_2}.$$

This second follows because from the hypotheses we can deduce $(a_1, a_2) \lhd_0 \{a_1\} \times V \lhd U_1 \times U_2$.

If it happens that $a_1 \in U_1$, then we have

$$\frac{a_2 \in U_2}{(a_1, a_2) \lhd U_1 \times U_2}$$

and so in this case we can deduce by induction on the proof of $a_2 \lhd U_2$ that it implies $(a_1, a_2) \lhd U_1 \times U_2$.

Now by similar means we can use induction on a proof of $a_1 \lhd U_1$ to deduce the result. $\square$

We now prove Tychonoff's theorem. To apply Theorem 15, we must find a way to characterize the finite covers by basics. To motivate the argument, let us adopt a spatial notation. A pair (a, b) in Proposition 17 represents, as an open set, the cartesian product $a \times b$. Similarly, we write X and Y for "the entire spaces", the top elements of the frames. Then by distributivity, Lemma 13,

$$\begin{aligned} \bigvee_{i=1}^{n}(a_i \times b_i) &= \bigvee_{i=1}^{n}(a_i \times Y \wedge X \times b_i) \\ &= \bigwedge_{n=I\cup J}\left(\bigvee_{i\in I} a_i \times Y \vee \bigvee_{i\in J} X \times b_i\right). \end{aligned}$$

Hence for it to cover $X \times Y$ we must have for every finite decomposition $I \cup J$ of n (i.e., of $\{1, \ldots, n\}$) that $\bigvee_{i\in I} a_i \times Y \vee X \times \bigvee_{i\in J} b_i$ is the whole of $X \times Y$. In classical spatial reasoning we can see this happens iff either $X = \bigvee_{i\in I} a_i$ or $Y = \bigvee_{i\in J} b_i$, for if we have $x \notin \bigvee_{i\in I} a_i$ and $y \notin \bigvee_{i\in J} b_i$ then $(x, y) \notin \bigvee_{i\in I} a_i \times Y \vee X \times \bigvee_{i\in J} b_i$. This is no proof constructively, but if we use it as our definition of F we can set Theorem 10 to work on it.

Theorem 19 (*Binary Tychonoff*). *Let $(P_1, \leq, \lhd_0)$ and $(P_2, \leq, \lhd_0)$ be two flat sites for compact spaces, equipped with subsets $F_i \subseteq \mathcal{F}P_i$ satisfying the conditions of Theorem 10.*

Let $F \subseteq \mathcal{F}(P_1 \times P_2)$ be defined such that $T \in F$ iff for every finite decomposition $T = T' \cup T''$ we have either $\mathcal{F}\pi_1(T') \in F_1$ or $\mathcal{F}\pi_2(T'') \in F_2$.

($\pi_i : P_1 \times P_2 \to P_i$ is the projection. $\mathcal{F}f(T)$ for any function f is the direct image of the finite set T under f.)

Then F satisfies the conditions of Theorem 10 for $P_1 \times P_2$, and hence shows that $P_1 \times P_2$ is compact.

Proof. We verify the various conditions.

(1) F is upper closed with respect to $\sqsubseteq_L$.

Suppose $S \in F$ and $S \sqsubseteq_L T$. Let $T = T' \cup T''$ be a finite decomposition of T. Because every element of S is less than some element of T, we can find a (not necessarily unique) decomposition $S = S' \cup S''$ such that $S' \sqsubseteq_L T'$ and $S'' \sqsubseteq_L T''$. The result follows from upper closure of F_1 and F_2, since $\mathcal{F}\pi_1(S') \sqsubseteq_L \mathcal{F}\pi_1(T')$ and $\mathcal{F}\pi_2(S'') \sqsubseteq_L \mathcal{F}\pi_2(T'')$.

(2) F is inhabited.

Suppose $S_i \in F_i$. We show that $S_1 \times S_2 \in F$. Suppose we have a finite decomposition $S_1 \times S_2 = T' \cup T''$. If $a \in S_1$ then for every $b \in S_2$ we have either $(a, b) \in T'$ or $(a, b) \in T''$. Hence either $(a, b) \in T''$ for every $b \in S_2$, or $(a, b) \in T'$ for some $b \in S_2$. In the first case we have $\{a\} \times S_2 \subseteq T''$, so $S_2 \subseteq \mathcal{F}\pi_2(T'')$ and $\mathcal{F}\pi_2(T'') \in F_2$. In the second case, $a \in \mathcal{F}\pi_1(T')$.

We have thus shown for every $a \in S_1$ that either $a \in \mathcal{F}\pi_1(T')$ or $\mathcal{F}\pi_2(T'') \in F_2$. It follows that either every a is in $\mathcal{F}\pi_1(T')$ or there is some a for which $\mathcal{F}\pi_2(T'') \in F_2$. In the first case we have $S_1 \subseteq \mathcal{F}\pi_1(T')$ and $\mathcal{F}\pi_1(T') \in F_1$. In the second, we have $\mathcal{F}\pi_2(T'') \in F_2$.

(3) If $a \lhd_0 U$ and $\{(a, b)\} \cup T \in F$, then $(U_0 \times \{b\}) \cup T \in F$ for some $U_0 \in \mathcal{F}U$.

(The condition for covers deriving from P_2 is similar.) Every decomposition $T = T' \cup T''$ gives two decompositions of $\{(a, b)\} \cup T$, namely $(\{(a, b)\} \cup T') \cup T''$ and $T' \cup (\{(a, b)\} \cup T'')$. We therefore deduce *both*

$$\{a\} \cup \mathcal{F}\pi_1(T') \in F_1 \text{ or } \mathcal{F}\pi_2(T'') \in F_2$$

and

$$\mathcal{F}\pi_1(T') \in F_1 \text{ or } \{b\} \cup \mathcal{F}\pi_2(T'') \in F_2.$$

It follows that $\mathsf{decomp}(T)$ can itself be decomposed as $D' \cup D''$ where

$$\forall (T', T'') \in D'. \, (\{a\} \cup \mathcal{F}\pi_1(T') \in F_1 \text{ and } \{b\} \cup \mathcal{F}\pi_2(T'') \in F_2)$$
$$\forall (T', T'') \in D''. \, (\mathcal{F}\pi_1(T') \in F_1 \text{ or } \mathcal{F}\pi_2(T'') \in F_2)$$

(and D' and D'' are both finite).

Now for each $(T', T'') \in D'$ we can find $U_0 \in \mathcal{F}U$ such that $U_0 \cup \mathcal{F}\pi_1(T') \in F_1$. By taking the union of these, we can assume that a single U_0 caters for every $(T', T'') \in D'$. We show that $(U_0 \times \{b\}) \cup T \in F$.

Any decomposition of $(U_0 \times \{b\}) \cup T$ is given by decompositions $T' \cup T''$ of T and $U'_0 \cup U''_0$ of U_0. We must show that either $U'_0 \cup \mathcal{F}\pi_1(T') \in F_1$ or $\mathcal{F}\pi_2(U''_0 \times \{b\}) \cup \mathcal{F}\pi_2(T'') \in F_2$. If $(T', T'') \in D''$ this is clear. Now suppose $(T', T'') \in D'$. If U''_0 is inhabited (recall that this is decidable because U''_0 is finite), then $\mathcal{F}\pi_2(U''_0 \times \{b\}) \cup \mathcal{F}\pi_2(T'') = \{b\} \cup \mathcal{F}\pi_2(T'') \in F_2$. If $U''_0 = \emptyset$ then $U'_0 = U_0$ and so $U'_0 \cup \mathcal{F}\pi_1(T') = U_0 \cup \mathcal{F}\pi_1(T') \in F_1$.

(4) If $S \in F$ then $P_1 \times P_2 \lhd S$.

For every finite decomposition $S = T' \cup T''$ we have either $\mathcal{F}\pi_1(T') \in F_1$ or $\mathcal{F}\pi_2(T'') \in F_2$. It follows that $\mathsf{decomp}(S)$ can itself be finitely decomposed into a set D' of pairs (T', T'') for which $\mathcal{F}\pi_1(T') \in F_1$ and a set D'' of pairs for which $\mathcal{F}\pi_2(T'') \in F_2$. Let U_1 and U_2 be subsets of P_1 and P_2, not necessarily finite, defined by

$$U_1 = \bigcap_{(T', T'') \in D'} \downarrow \mathcal{F}\pi_1(T')$$
$$U_2 = \bigcap_{(T', T'') \in D''} \downarrow \mathcal{F}\pi_2(T'').$$

By the $\leq$-right rule we have $P_i \lhd U_i$, and it follows by Proposition 18 that $P_1 \times P_2 \lhd U_1 \times U_2$.

Now suppose $(u_1, u_2) \in U_1 \times U_2$. By definition of the U_is, we have that for every decomposition $S = T' \cup T''$ we have either some $(x, y) \in T'$ with $u_1 \leq x$, or some $(x, y) \in T''$ with $u_2 \leq y$. By Lemma 14 there is some $(x, y) \in S$ with $(u_1, u_2) \leq (x, y)$. We deduce that $P_1 \times P_2 \lhd S$. $\square$

Our proof is slightly shorter than that of [13]. The relative shortness is a little misleading, since theirs includes aspects of our Theorem 15 and Proposition 18. On the other hand, our proof contains an explicit finitary characterization of the finite covers of the product, which I believe is absent from theirs.

8. The double powerlocale

Proposition 7 and Theorem 15 characterized openness and compactness in terms that were very presentation dependent: they were tied to the particular form of the formal topology. The natural definition of compactness in formal topology (if $P \lhd U$ then $P \lhd U_0$ for some $U_0 \in \mathcal{F}U$) is also tied to the presentation: after all, the formal topology in effect *is* the presentation. By contrast the locale definition is presentation *in*dependent, but relies on being able to use the frame as a concrete embodiment of the locale. In this section we outline a localic technique that goes some way to reconciling these.

In locale theory, openness or compactness of a locale X can be characterized by the existence of certain points of *powerlocales* of X, i.e., locales whose points are certain "parts" (technically, sublocales) of X. Details can be found in [18] and (partly collecting older results) [19]. There are two parallel results, which [18] shows are dual. In each case, the idea is to characterize a powerlocale point that would represent X. In the lower powerlocale $P_L X$ each point (as a sublocale) is open as a locale in its own right (i.e., overt), and so if X (as sublocale of itself) appears as a point $P_L X$ then it is an open locale. The converse also holds. The upper powerlocale $P_U X$ is similar, but here the points are compact.

We shall not dwell on the details here, but let us remark the following. For the lower powerlocale, a consequence of Theorem 5 is that the frame for the lower powerlocale $P_L X$ can be presented as

$$\mathbf{Fr}\langle P \text{ (qua preorder)} \mid a \leq \bigvee U \quad (a \lhd_0 U)\rangle.$$

Thus a point of $P_L X$ is an upper closed subset F of P such that if $a \lhd_0 U$ and $a \in F$, then U meets F. These conditions relate to those of $\leq$-monotonicity and Pos-infinity in [17], where they arise in studying a binary generalization of the positivity predicate. A subset *Pos* as in Proposition 7 would have to be the biggest such subset F. (This is not the whole story. Ref. [11] shows that for *every* X there is a biggest such F, but condition 2 (b) in Proposition 7 corresponds to a stronger condition on it.) But by powerlocale theory (see [19]) these correspond to the "weakly closed sublocales of X with open domain". Classically these are exactly the closed sublocales.

The account for compactness has been studied in terms of *preframe* homomorphisms, which preserve directed joins and finite meets [2,10,16]. This is because the function $\forall_! : \Omega X \to \Omega$, mapping a to the truth value for $a = 1$, is a preframe homomorphism precisely when the locale is compact. Theorem 9 has analogues showing how to present the frame as a preframe, and these have been used to define functions such as $\forall_!$ in various cases.

All this tells us that by using the powerlocales openness and compactness can be abstracted away from explicit mention of the frames.

Now the same technique can also be used in formal topology, provided the powerlocales can be represented in it. This calls for predicative constructions on the formal topologies to give presentations of the powerlocales, and in essence this is the same idea as explored in some detail in [21], on the "geometricity" of the powerlocales. This is not a completely predicative story, since it works by relating those constructions to impredicative locale theory. In particular, to give a purely predicative account of presentation independence (homeomorphic formal topologies give homeomorphic power objects) one would have to work with the category of formal topologies (the morphisms, corresponding to continuous maps, are described in [14]) and prove functoriality. However, for the present we shall be content to show this relationship with the localic constructions.

Rather than use the lower and upper powerlocales separately for considerations of openness and compactness, we shall make use of a single construction, the *double powerlocale* $\mathbb{P}X$. This subsumes both lower and upper, and can be constructed as either $P_L P_U X$ or as $P_U P_L X$; they are homeomorphic. It was anticipated in [10] and examined more closely in [21,24].

Definition 20. If X is a locale, then its *double powerlocale* $\mathbb{P}X$ is defined by

$$\Omega\mathbb{P}X = \mathbf{Fr}\langle \Omega X \text{ (qua dcpo)}\rangle.$$

This definition in itself is impredicative. However, from Theorem 9 we see that if X is presented by a flat site $(P, \leq, \lhd_0)$, then $\mathbb{P}X$ can be presented by

$$\mathbf{Fr}\langle \mathcal{F}P \text{ (qua } \sqsubseteq_L \text{ preorder)} \mid \{a\} \cup T \leq \bigvee\nolimits^{\uparrow} \{U_0 \cup T \mid U_0 \in \mathcal{F}U\} \quad (a \lhd_0 U, T \in \mathcal{F}P)\rangle.$$

Though this is not in the form corresponding to a flat site, it can be manipulated into such a form by freely adjoining finite meets to the generators (this produces the free distributive lattice over P qua preorder) and augmenting the relations to make them meet stable. Thus $\mathbb{P}$ becomes a construction that can be performed on formal topologies. There is still an issue, of course, of whether the construction is presentation independent, i.e., functorial with respect to continuous maps between formal topologies. (In topos theory this is obvious, because the universal characterization depends only on the frame.)

If X is compact, then $\forall_! : \Omega X \to \Omega$ is a dcpo morphism and hence gives a frame homomorphism $\Omega\mathbb{P}X \to \Omega$, i.e., a point of $\mathbb{P}X$ (a map $1 \to \mathbb{P}X$). Similarly, if X is open then $\exists_!$ corresponds to a point of $\mathbb{P}X$. To characterize these points more precisely we shall need to know some more about the structure of $\mathbb{P}X$. It is helpful to picture its points as being a distributive lattice generated, in a suitably topological sense, by those of X (with their specialization order).

- $\mathbb{P}X$ has a top point $\top$ corresponding to the constant true dcpo morphism from ΩX to Ω. As a subset of $\mathcal{F}P$, it corresponds to the whole of $\mathcal{F}P$. (The reader can check that this respects the relations.) It is an open point corresponding to the basic open $\emptyset$: the subset of $\mathcal{F}P$ contains $\emptyset$ iff it is the whole of $\mathcal{F}P$. We shall write $\{\top\}$ for the corresponding open sublocale of $\mathbb{P}X$, and $\mathbb{P}X - \{\top\}$ for its closed complement.

- $\mathbb{P}X$ has a bottom point $\perp$ corresponding to the constant false dcpo morphism from ΩX to Ω. As a subset of $\mathcal{F}P$, it is empty. It is a closed point corresponding to the closed complement of $\bigvee \mathcal{F}P$. We shall write $\{\perp\}$ for the corresponding closed sublocale of $\mathbb{P}X$, and $\mathbb{P}X - \{\perp\}$ for its open complement.
- There is an embedding $\updownarrow : X \to \mathbb{P}X$, arising from the identity function (a dcpo morphism) $\Omega X \to \Omega X$.

What we show is that X is compact iff $\mathbb{P}X$ has a point that is almost but not quite $\perp$: it is less than $\updownarrow$ but still in $\mathbb{P}X - \{\perp\}$. Similarly, X is open iff it has a point that is almost but not quite $\top$.

Theorem 21. *Let X be a locale.*

(1) *X is compact iff $\mathbb{P}X$ has a point $\forall : 1 \to \mathbb{P}X$ for which $!; \forall \sqsubseteq \updownarrow : X \to \mathbb{P}X$, and $\forall$ is in $\mathbb{P}X - \{\perp\}$. If such a point exists, it is unique.*

(2) *X is open iff $\mathbb{P}X$ has a point $\exists : 1 \to \mathbb{P}X$ for which $!; \exists \sqsupseteq \updownarrow : X \to \mathbb{P}X$, and $\exists$ is in $\mathbb{P}X - \{\top\}$. If such a point exists, it is unique.*

Proof. Suppose X is presented by a flat site $(P, \leq, \lhd_0)$.

(1) A point of $\mathbb{P}X$ corresponds to an upper closed (under $\sqsubseteq_L$) subset $H \subseteq \mathcal{F}P$ such that if $a \lhd_0 U$ and $\{a\} \cup T \in H$, then $U_0 \cup T \in H$ for some $U_0 \in \mathcal{F}U$. The point is in $\mathbb{P}X - \{\perp\}$ iff H is inhabited. For such a point $\forall$, to analyse the condition $!; \forall \sqsubseteq \updownarrow$ we consider the inverse images of these two maps. The condition then says that for every $T \in \mathcal{F}P$, if $T \in H$ then $1 \leq T$ in ΩX; that is to say, $P \lhd T$. Hence the two conditions given here are equivalent to the conditions (3) and (4) given for compactness in Theorem 10.

Uniqueness follows from the uniqueness clause there.

(2) A point $\exists$ corresponding to $H \subseteq \mathcal{F}P$ is in the closed sublocale $\mathbb{P}X - \{\top\}$ iff $\emptyset \in H$ is contradictory. It satisfies $!; \exists \sqsupseteq \updownarrow$ iff, for every $T \in \mathcal{F}P$, we have $T \lhd \{S \in \mathcal{F}P \mid T \in H\}$.

Now suppose X is open. By Proposition 7, there is a positivity predicate $Pos \subseteq P$. Define $T \in H$ iff T meets Pos (so $a \in Pos$ iff $\{a\} \in H$). H satisfies the conditions needed to define a point of $\mathbb{P}X$, and it is in $\mathbb{P}X - \{\top\}$. If $a \in T$ and $a \in Pos$, then $T \in H$ and so $a \lhd \mathcal{F}P = \{S \in \mathcal{F}P \mid T \in H\}$. Hence by the properties of Pos we have $a \lhd \{S \in \mathcal{F}P \mid T \in H\}$ regardless of whether $a \in Pos$. Hence $T \lhd \{S \in \mathcal{F}P \mid T \in H\}$ for every T.

Conversely, suppose H has the conditions hypothesized for $\exists$ and $Pos = \{a \mid \{a\} \in H\}$.

We first show, for uniqueness, that $T \in H$ iff T meets Pos. If T meets Pos in a then $\{a\} \sqsubseteq_L T$ and so $T \in H$. On the other hand, suppose $T \in H$. For every $a \in T$ we have $\{a\} \lhd \{S \in \mathcal{F}P \mid \{a\} \in H\}$, and it follows that

$$T \lhd \{\emptyset\} \cup \{S \in \mathcal{F}P \mid \exists a \in T. \{a\} \in H\}.$$

We have put the $\{\emptyset\}$ in to make this a directed cover of T. Then because H corresponds to a dcpo morphism $\Omega X \to \Omega$ it follows that H contains some element of $\{\emptyset\} \cup \{S \in \mathcal{F}P \mid \exists a \in T. \{a\} \in H\}$. If it is $\emptyset$ then we get a contradiction; if it is in $\{S \in \mathcal{F}P \mid \exists a \in T. \{a\} \in H\}$ then T meets Pos as required.

We now show that *Pos* is a positivity predicate. Referring to Proposition 7, upper closure is immediate and condition 2(b) uses $\{a\} \lhd \{S \in \mathcal{F}P \mid \{a\} \in H\}$ and $\leq$-right. Now suppose $a \lhd_0 U$ and $a \in Pos$. Then there is some $U_0 \in \mathcal{F}U$ with $U_0 \in H$, and so U_0 meets *Pos*. Hence U meets *Pos*. $\square$

9. Conclusions

Predicativity implies that we cannot use frames, so we have to use presentations (of various kinds) instead. Some presentational proof techniques for compactness, and openness too, are justified impredicatively but nonetheless lead to predicatively valid arguments via inductive generation.

We finish with a speculative thought on the double powerlocale. The statement of Theorem 21 manages to be completely independent of representation: it characterizes compactness and openness without mentioning either frames or any specific form of presentation such as sites (or any specific definition of formal topology). This can most conveniently be expressed by using a category of formal topologies, with morphisms corresponding to continuous maps [14]. (In fact, one might say that this presentation independence is the same idea as topological invariance, and that to make this precise was the original purpose of categories.) In this form, the discussion is conducted in terms of objects (as "spaces"), morphisms ("maps"), the natural poset enrichment (specialization order on maps) and the double powerlocale functor (indeed, monad).

In locales there is already a body of work using this kind of categorical framework for discussing topology. Examples include [21] and [24] using the double powerlocale; and [5] and [15] using ideas of the lambda calculus. A particular link between the two [21] is that $\mathbb{P}X$ is isomorphic to the double exponential $\mathbb{S}^{\mathbb{S}^X}$ ($\mathbb{S}$ being the Sierpiński locale), and that this can be given sense [24] even when the locale X is not exponentiable and the exponential $\mathbb{S}^X$ does not exist as a locale.

The machinery of categorical logic can also be used in this categorical setting. This is similar machinery to that which allows one to reason about toposes as though they were just non-standard universes of sets, discussing the objects and morphisms as though they were sets and functions in a non-classical mathematics. In the category of locales it has the pleasant consequence that locales can be reasoned with as though they were spaces, with sufficient points. The basis of this is as follows. The standard ("global") points of a locale X are the morphisms $1 \to X$, which in general are insufficient. However, the categorical logic also deals with *generalized* points of X, morphisms to X from an arbitrary Y (the "stage of definition"). These are in effect points of X in the non-standard set theory of the topos of sheaves over Y. If one's reasoning about points is sufficiently constructive, then it also applies to the generalized points, and of these there are sufficient. "Sufficiently constructive" means complying with the constraints of *geometric* logic, so that the reasoning is not only valid in any (Grothendieck) topos, but can be transported from one to another in a well behaved way. The approach is set out in [21].

The logic has an intrinsic continuity. "Functions" defined using it are automatically continuous maps. An attractive idea therefore is that there might be some formal "logic of continuity", validly interpretable in topos theory, that expresses the mathematics of locales and continuous maps and includes the double powerlocale. But the evidence so far suggests that the geometric principles used in [21] are also predicative. Hence one might hope that such a logic could also be interpreted predicatively in a category of formal topologies, thus unifying them in a formal way with topos-valid locales. It would not capture the whole of conventional locale theory, since that includes features described using arbitrary functions between frames. However, by incorporating the double powerlocale it would capture that substantial part that can be expressed using Scott continuous functions between frames. This is because the locale maps from X to $\mathbb{P}Y$ are equivalent to the Scott continuous functions from ΩY to ΩX.

Acknowledgements

The author thanks Prof. Giovanni Sambin and the organizers of the Second Workshop in Formal Topology for the invitation to present this material there.

References

[1] S. Abramsky, S.J. Vickers, Quantales, observational logic and process semantics, Mathematical Structures in Computer Science 3 (1993) 161–227.

[2] B. Banaschewski, Another look at the localic Tychonoff theorem, Commentationes Mathematicae Universitatis Carolinae 24 (4) (1988) 647–656.

[3] T. Coquand, An intuitionistic proof of Tychonoff's theorem, The Journal of Symbolic Logic 57 (1992) 28–32.

[4] T. Coquand, G. Sambin, J. Smith, S. Valentini, Inductively generated formal topologies, Annals of Pure and Applied Logic 124 (2003) 71–106.

[5] M. Escardó, Synthetic topology of data types and classical spaces, in: P. Panangaden, J. Desharnais (Eds.), Bellairs Workshop: Domain Theoretic Methods in Probabilistic Processes, in: Electronic Notes in Theoretical Computer Science, vol. 87, Elsevier, 2004.

[6] P.T. Johnstone, Stone Spaces, in: Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, 1982.

[7] P.T. Johnstone, Open locales and exponentiation, Contemporary Mathematics 30 (1984) 84–116.

[8] P.T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, vol. 2, in: Oxford Logic Guides, vol. 44, Oxford University Press, 2002.

[9] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, Memoirs of the American Mathematical Society 309 (1984).

[10] P.T. Johnstone, S.J. Vickers, Preframe presentations present, in: A. Carboni, M.C. Pedicchio, G. Rosolini (Eds.), Category Theory — Proceedings, Como 1990, in: Lecture Notes in Mathematics, vol. 1488, Springer-Verlag, 1991, pp. 193–212.

[11] M.E. Maietti, S. Valentini, A structural investigation on formal topology: Coreflection of formal covers and exponentiability, The Journal of Symbolic Logic 69 (2004) 967–1005.

[12] S. Negri, Continuous domains as formal spaces, Mathematical Structures in Computer Science 12 (2002) 19–52.

[13] S. Negri, S. Valentini, Tychonoff's theorem in the framework of formal topologies, The Journal of Symbolic Logic 62 (4) (1997) 1315–1332.

[14] G. Sambin, Intuitionistic formal spaces — a first communication, in: D.G. Skordev (Ed.), Mathematical Logic and its Applications, Plenum, 1987, pp. 187–204.

[15] P. Taylor, Sober spaces and continuations, Theory and Applications of Categories 10 (2002) 248–299.

[16] C.F. Townsend, Preframe techniques in constructive locale theory, Ph.D. Thesis, Department of Computing, Imperial College, London, 1996.

[17] S. Valentini, The problem of the formalization of constructive topology, Archive for Mathematical Logic 44 (2005) 115–129.

[18] S.J. Vickers, Locales are not pointless, in: C.L. Hankin, I.C. Mackie, R. Nagarajan (Eds.), Theory and Formal Methods of Computing 1994, Imperial College Press, London, 1995, pp. 199–216.

[19] S.J. Vickers, Constructive points of powerlocales, Mathematical Proceedings of the Cambridge Philosophical Society 122 (1997) 207–222.

[20] S.J. Vickers, Topical categories of domains, Mathematical Structures in Computer Science 9 (1999) 569–616.

[21] S.J. Vickers, The double powerlocale and exponentiation: A case study in geometric reasoning, Theory and Applications of Categories 12 (2004) 372–422.

[22] S. Vickers, Entailment systems for stably locally compact locales, Theoretical Computer Science 316 (2004) 259–296.

[23] S. Vickers, Some constructive roads to Tychonoff, in: L. Crosilla, P. Schuster (Eds.), From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics, Oxford University Press (forthcoming).

[24] S.J. Vickers, C.F. Townsend, A universal characterization of the double powerlocale, Theoretical Computer Science 316 (2004) 297–321.