Advanced Macroeconomics

6. Optimal Consumption in a Centralized Economy

The Centralized Economy

Dynamic Adjustment

Summary and Literature

Basic Model I

- Ramsey (1928), Cass (1965), Koopmanns (1965)

- Closed economy, constant prices, constant population ($y = Y/N$, $k = K/N$)

- Output (y) can be consumed (c) or invested (i)

$$y_t = c_t + i_t$$

- Savings (s)

$$s_t = y_t - c_t \qquad \Rightarrow \qquad s_t = i_t$$

- Capital accumulation

$$k_{t+1} = (1 - \delta)k_t + i_t, \qquad \Delta k_{t+1} = k_{t+1} - k_t = i_t - \delta k_t$$

Basic Model II

- Production function

$$y_t = F(k_t)$$

- Positive marginal product

$$F' = \frac{\partial F}{\partial k} > 0$$

- Decreasing marginal product

$$F'' = \frac{\partial^2 F}{\partial k^2} \leq 0$$

- Inada conditions (Inada, 1964)

$$\lim_{k \to \infty} F'(k) = 0 \qquad \text{and} \qquad \lim_{k \to 0} F'(k) = \infty$$

The Basic Model in One Equation

- Model equations

  - Expenditure $y_t = c_t + i_t$

  - Capital accumulation $\Delta k_{t+1} = i_t - \delta k_t$

  - Production function $y_t = F(k_t)$

- Resource constraint

$$F(k_t) = c_t + \Delta k_{t+1} + \delta k_t = c_t + k_{t+1} - (1 - \delta)k_t \tag{1}$$

- The central planner maximizes utility from consumption subject to the resource constraint

Utility Maximization

- Lagrange function using resource constraint (1)

$$\mathcal{L}_t = \sum_{s=0}^{\infty} \left\{ \beta^s U(c_{t+s}) + \lambda_{t+s} \left[F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s} \right] \right\}$$

- First-order conditions

$$
\begin{aligned}
\frac{\mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0, \qquad s \geq 0 \\
\frac{\mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s} \left[F'(k_{t+s}) + 1 - \delta \right] - \lambda_{t+s-1} = 0, \qquad s > 0
\end{aligned}
$$

- In period t, the central planner chooses next period's capital stock

- Transversality condition

$$\lim_{s \to \infty} \beta^s U'(c_{t+s}) k_{t+s+1} = 0$$

Consumption Euler Equation

- From the first first-order condition we get

$$\lambda_{t+s} = \beta^s U'(c_{t+s})$$

- Substitute λ_{t+s} into the second condition

$$\beta^s U'(c_{t+s}) \left[F'(k_{t+s}) + 1 - \delta\right] = \beta^{s-1} U'(c_{t+s-1}), \qquad s > 0$$

- For $s = 1$

$$\beta \frac{U'(c_{t+1})}{U'(c_t)} \left[F'(k_{t+1}) + 1 - \delta\right] = 1$$

Static Equilibrium

- Euler equation ($c_t = c^*$, $\Delta c_t = 0$, $k_t = k^*$, $\Delta k_t = 0$)

$$\beta \frac{U'(c^*)}{U'(c^*)} \left[F'(k^*) + 1 - \delta \right] = 1$$

- Optimal capital stock

$$F'(k^*) = \frac{1}{\beta} + \delta - 1 = \delta + \rho$$

- Steady state consumption

$$F(k^*) = c^* + \Delta k^* + \delta k^* = c^* + \delta k^* \qquad \Leftrightarrow \qquad c^* = F(k^*) - \delta k^*$$

- The larger the time preference rate (ρ), the smaller k^* and the smaller c^*

Example: Power Utility and Cobb-Douglas Production Function

- Power utility

$$U(c) = \frac{c^{1-\theta} - 1}{1 - \theta}$$

  such that

$$U'(c) = c^{-\theta}, \qquad U''(c) = -\theta c^{-\theta-1}, \qquad \frac{-cU''}{U'} = c\frac{\theta c^{-\theta-1}}{c^{-\theta}} = \theta$$

- Cobb-Douglas production function

$$y_t = Zk_t^{\alpha}$$

- Euler equation

$$\beta \frac{U'(c_{t+1})}{U'(c_t)} \left[F'(k_{t+1}) + 1 - \delta\right] = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\theta} \left[\alpha Zk_{t+1}^{\alpha-1} + 1 - \delta\right] = 1$$

Example (continued)

- Steady state: $k_{t+1} = k_t = k^*$, $c_{t+1} = c_t = c^*$

- Euler equation

$$\beta \left(\frac{c^*}{c^*} \right)^{-\theta} \left[\alpha Z k^{*\alpha-1} + 1 - \delta \right] = 1$$

- Static equilibrium:

$$k^* = \left(\frac{\alpha Z}{\delta + \rho} \right)^{1/(1-\alpha)}$$

and from resource constraint:

$$c^* = Z k^{*\alpha} - \delta k^*$$

Numeric Solution

(Ramsey_Static.m, Ramsey_Steady.mod)

- Find c^*, k^*: roots of Euler equation and resource constraint

$$
\begin{aligned}
h_1(c, k) &= \beta \left[\alpha Z k^{\alpha-1} + 1 - \delta \right] - 1 \\
h_2(c, k) &= c - (Z k^{\alpha} - \delta k)
\end{aligned}
$$

- Guess initial values: k=10, c=2

- Output: k^*, c^*

6. Optimal Consumption in a Centralized Economy

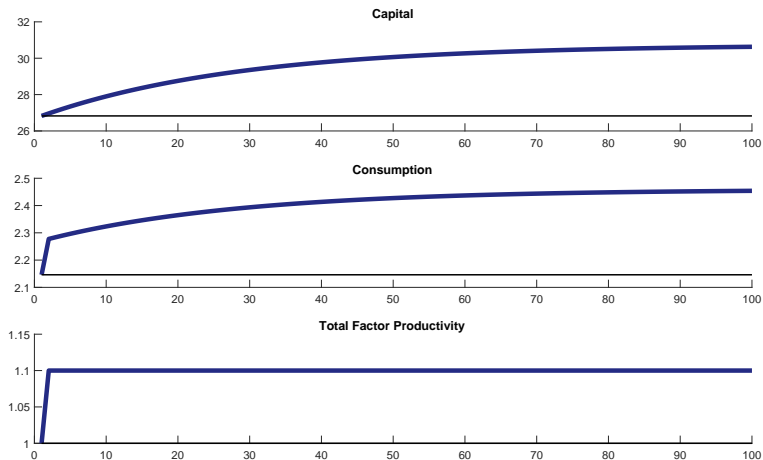The Centralized Economy
Dynamic Adjustment
Summary and Literature

Dynamic Adjustment

- Deterministic extended path (EP) method (Fair and Taylor 1983)

- Compute

  - Steady state k_0^* before shock

  - Steady state k_1^* after shock

- Solve dynamic model equations (Euler equation and capital accumulation) for equilibrium paths of c_t and k_t from k_0^* to k_1^*
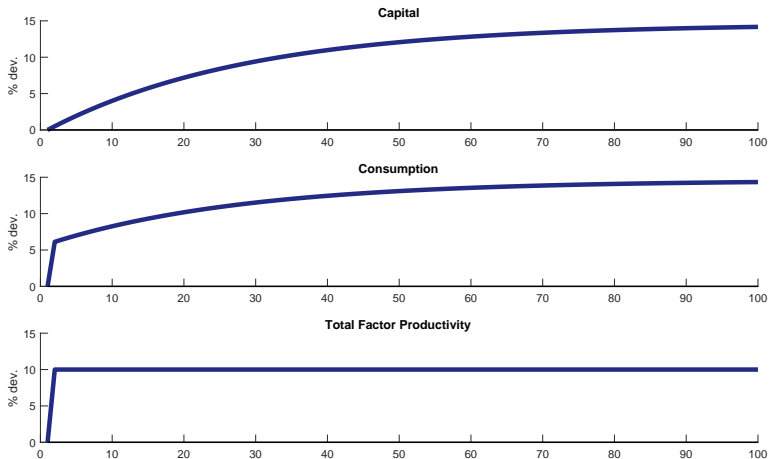
Permanent Productivity Shock

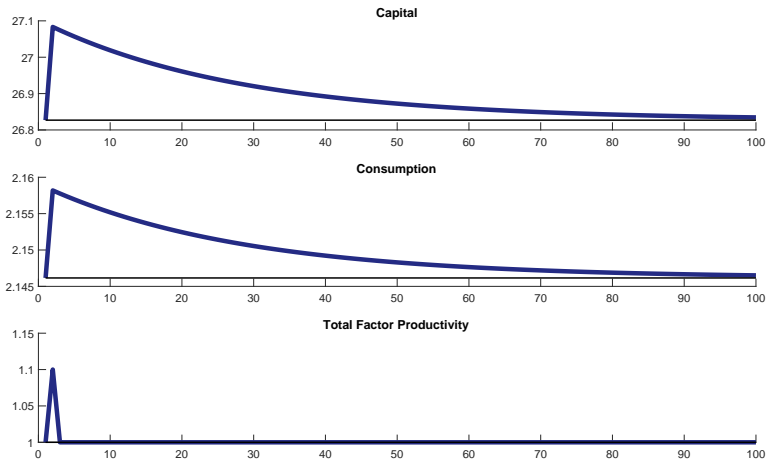(Ramsey_Dynamic.m, Ramsey_Perm.mod)

Permanent Productivity Shock
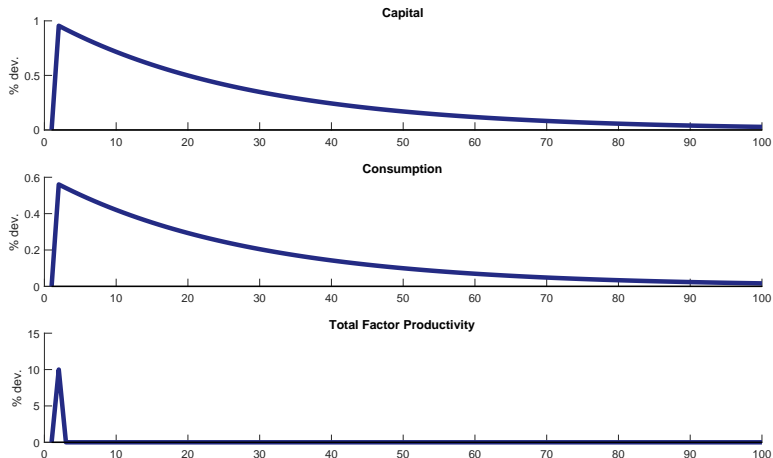
(Percentage deviation from initial steady state)

Temporary Productivity Shock
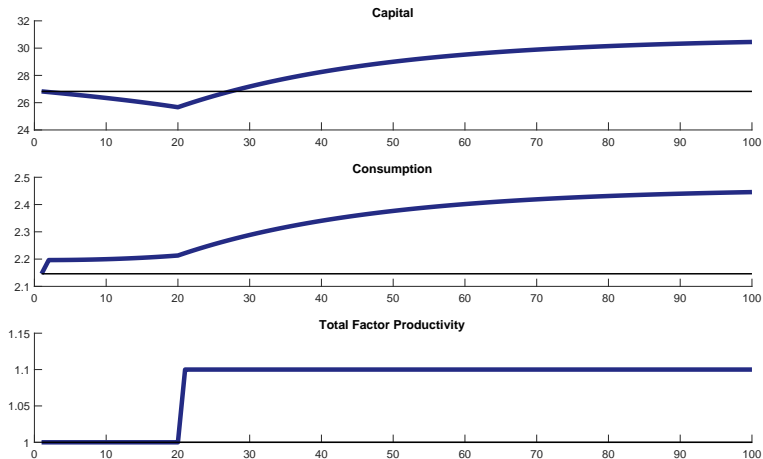
(`Ramsey_Dynamic.m, Ramsey_Temp.mod`)

Temporary Productivity Shock
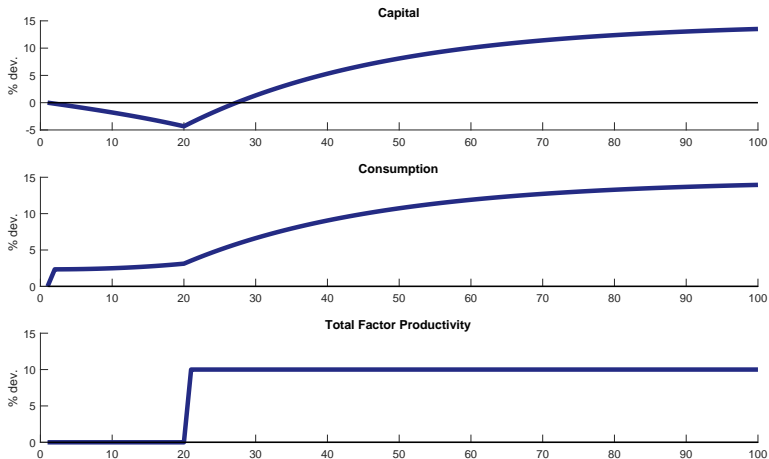
(Percentage deviation from initial steady state)

Expected Productivity Shock

(`Ramsey_Dynamic.m, Ramsey_Exp.mod`)

Expected Productivity Shock

(Percentage deviation from initial steady state)

6. Optimal Consumption in a Centralized Economy

Summary

- Ramsey model describes the optimal growth path depending on preferences, technology and endowment

- Limitations

  - Long-run growth rate of consumption per capita exogenous (technological progress) $\Rightarrow$ endogenous growth models (investment in human capital and ideas)

  - Heterogeneity not considered: young versus old households, heterogeneity in income and wealth $\Rightarrow$ models with heterogenous agents, overlapping generation models

Literature

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