Martin Luther University Halle-Wittenberg
Professor Dr. O. Holtemöller

Problem Set 3

Advanced Macroeconomics

Winter 2025/26

1 Unconstrained Optimization

Derive the maximum of the profit function

$$D = P_t Y_t - MC_t Y_t$$

with respect to Y_t.

2 Constrained Optimization

During this course, we will often use the so-called *Lagrange* approach. In this tutorial, we want to learn more about it.

2.1 General Simple Lagrange Approach

We want to start with the simplest case: image a function with one argument. You want to maximize this function but you know that there is a constraint, so the solution must be within certain limits. For example, if you get a strict positive utility by eating ice cream (you do not get sick after a certain number of scoops...) you still cannot simply eat an infinite number of scoops without infinite resources to pay for them. So, the optimal amount of scoops is limited given your available money and the price you must pay for one scoop.
So, how many scoops are optimal for us now? This is where we are very grateful to a very smart guy named Giuseppe Luigi Lagrangia, today well-known as *Lagrange*. Originally used in physics, his approach provides a simple method to solve constrained optimization problems. Let's illustrate this approach using the simple case, where our problem is given as follows:

$$\max_x f(x) \text{ s. t. } g(x) = \gamma$$

We want to maximize a function $f(x)$ with respect to (w. r. t.) an argument x, subject to (s. t.) the constraint $g(x) = \gamma$. Here (for simplicity and because we will always assume that in this course), the restriction is binding, so we will use our complete available resources. Otherwise, the problem would become a little more complicated (Karush-Kuhn-Tucker). The respective so-called Lagrange-function $\mathcal{L}$ can be written as:

$$\mathcal{L} = f(x) - \lambda\,(g(x) - \gamma)$$

The so-called Lagrange-multiplier λ tells us by how much the target function changes when the constraint changes marginally. It is also called *shadow-price* in this economic context. We get the first-order conditions by partial derivation of target function $\mathcal{L}$ w. r. t. argument x and setting it equal to zero:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f(x)}{\partial x} - \lambda \frac{\partial g(x)}{\partial x} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -g(x) + \gamma = 0$$

Hence, we get two equations for two unknowns x and λ. Note that we concentrate here on maximization, but it is also possible to minimize a function under constraint. For example, to minimize cost for consumption under the constraint that you want to achieve a certain level of utility. The duality of these two problems is usually discussed in microeconomics courses on both the household and the production sector.

2.2 Application: Utility Maximization

Apply this general approach now to these concrete examples:

1. * Given a utility function $u(c) = \ln(c)$ and a budget constraint $pc = y$ with disposable income $y = 3$ and price $p = 1$, derive the first-order conditions for a utility maximum and the optimal amount of consumption c.

2. Given a utility function $u(c_1, c_2) = \ln(c_1) + a\ln(c_2)$ and a budget constraint $p_1c_1 + p_2c_2 = y$ with disposable income $y = 5$, prices $p_1 = 1$ and $p_2 = 2$, derive the optimal amount of consumption c_1 and c_2 depending on the utility weight a and explain how the consumption allocation depends on this parameter.

3 Solving Systems of Equations

1. Write a code for solving the non-linear equation system

$$0 = x - 2$$
$$0 = y - x^2.$$

   Use the vector $init = [1, 1]$ as starting values and load the resulting values for x and y into the vector sol.

2. * Solve the following system of three non linear equations with three unknown variable values using the command fsolve. Assume the initial value vector $x_0 = [1, 1, 1]$.

$$
\begin{aligned}
2x_1 + \ln x_2 &= 3 \\
x_2 x_3 &= 1 \\
\ln x_3 &= 2
\end{aligned}
$$

3. * Solve the following system of equations for the endogenous variables r, w and y, assuming initial values $k = 1$, $n = 1$ and $A = 1.1$ for exogenously given variables and parameter values $\alpha = 0.6$ and $\beta = 0.4$.

$$
\begin{aligned}
\alpha A k^{\alpha-1} n^{\beta} &= r \\
\beta A k^{\alpha} n^{\beta-1} &= w \\
A k^{\alpha} n^{\beta} &= y
\end{aligned}
$$

*Note: tasks marked with * are examples for additional practice and intended for independent self-study.*