Advanced Macroeconomics

3. Static General Equilibrium Models

Models

- Models are imaginary, simplified versions of the aspect of the world being studied
- Models take the form of systems of equations
    - Behavioral relations (labor supply, consumption function, investment function, ...)
    - Accounting identities (expenditure-side definition of GDP, ...)
    - Equilibrium conditions (labor supply = labor demand, ...)
- Variables refer to economic magnitudes that can take different values
- Parameters constant and exogenously given numbers

3. Static General Equilibrium Models

General Equilibrium in a One-Period Competitive Model

- Agents:
  - Private households: representative household chooses consumption (c) and labor supply (n) according to preferences (utility maximization)
  - Private firms: representative firm maximizes its profit
- In a one-period world, the initial capital stock (k) is given as endowment (no investment)
- Technology: Production function

Household Behavior

- Utility function $u(c)$

$$u' = \frac{\partial u}{\partial c} > 0, \qquad u'' = \frac{\partial^2 u}{\partial c^2} < 0$$

- Constraints:
  - Time constraint: $n \leq 1$
  - Capital constraint (endowment of one): $k \leq 1$
  - Budget constraint: $c \leq (1+r)k + wn$ (capital stock may be consumed)

Utility Maximization

- Lagrange function

$$\mathcal{L} = u(c) - \lambda_1(c - (1 + r)k - wn) - \lambda_2(k - 1) - \lambda_3(n - 1)$$

- First-order conditions: nonsatiation

$$\begin{aligned} \lambda_1 &= u' \\ 0 &= c - k - rk - wn \\ k &= 1 \\ n &= 1 \end{aligned}$$

- Optimal consumption: $c = 1 + r + w$

Production Function

- Production function: $y = AF(k, n)$
- Marginal productivity

$$F_k = \frac{\partial F}{\partial k} > 0, \qquad F_n = \frac{\partial F}{\partial n} > 0$$

- Marginal productivity of capital diminishing and increasing in n

$$F_{kk} = \frac{\partial^2 F}{\partial k^2} < 0, \qquad F_{kn} = \frac{\partial^2 F}{\partial k \partial n} > 0$$

- Marginal productivity of labor diminishing and increasing in k

$$F_{nn} = \frac{\partial^2 F}{\partial n^2} < 0, \qquad F_{nk} = \frac{\partial^2 F}{\partial n \partial k} > 0$$

- Constant returns to scale

$$\mu y = AF(\mu k, \mu n)$$

Per-worker Production Function

- Output per worker: $\widehat{y} = y/n$
- Capital intensity: $\widehat{k} = k/n$
- Per-worker production function

$$\widehat{y} = AF(\widehat{k}, 1) = Af(\widehat{k})$$

satisfying

$$f'(\widehat{k}) = \frac{\partial f}{\partial \widehat{k}} > 0, \qquad f''(\widehat{k}) = \frac{\partial^2 f}{\partial \widehat{k}^2} < 0$$

Profit Maximization by Firms

- Profit function

$$\mathcal{P} = AF(k,n) - rk - wn = n\left(Af(\widehat{k}) - r\widehat{k} - w\right)$$

- First-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial k} &= AF_k - r = 0 \quad \text{and} \quad r = Af'(\widehat{k}) \\ \frac{\partial \mathcal{P}}{\partial n} &= AF_n - w = 0 \quad \text{and} \quad w = Af(\widehat{k}) - \widehat{k}Af'(\widehat{k}) \end{aligned}$$

- Labor and capital income add up to total output

$$rk + wn = Af'(\widehat{k})k + Af(\widehat{k})n - kAf'(\widehat{k}) = Af(\widehat{k})n = AF(k,n)$$

Cobb-Douglas Production Function

- Cobb-Douglas production function:

$$y = Ak^{\alpha} n^{1-\alpha}, \qquad 0 < \alpha < 1$$

- Profit-maximizing first-order conditions:

$$\begin{aligned} r &= \alpha A k^{\alpha-1} n^{1-\alpha} = \alpha A \widehat{k}^{\alpha-1} \\ w &= (1-\alpha) A k^{\alpha} n^{-\alpha} = (1-\alpha) A \widehat{k}^{\alpha} \end{aligned}$$

- Labor share:

$$\frac{n \cdot w}{y} = \frac{n(1-\alpha)Ak^{\alpha}n^{-\alpha}}{y} = \frac{(1-\alpha)Ak^{\alpha}n^{1-\alpha}}{y} = 1-\alpha$$

General Equilibrium in the One-Period Model

- Capital stock and labor supply exogenous

$$k = 1, \qquad n = 1, \qquad \widehat{k} = 1$$

- Equilibrium production: $\overline{y} = A$
- Equilibrium rental price of capital (Cobb-Douglas production function)

$$\overline{r} = \alpha A = \alpha \overline{y}$$

- Equilibrium wage

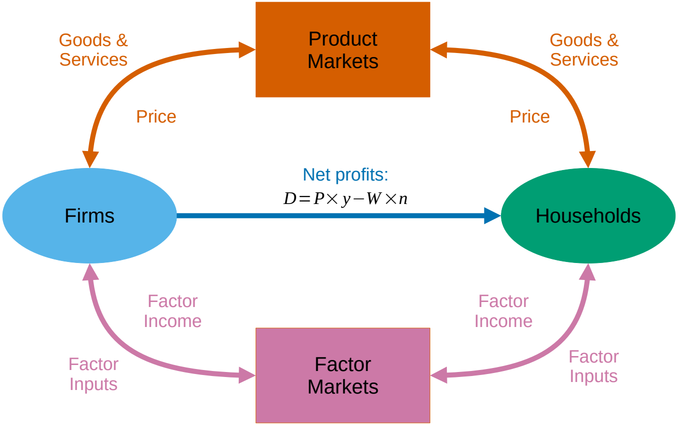
$$\overline{w} = (1-\alpha)A = (1-\alpha)\overline{y}$$

- Equilibrium consumption

$$\overline{c} = 1 + \overline{r} + \overline{w} = 1 + \alpha A + (1-\alpha)A = 1 + A$$

3. Static General Equilibrium Models

Walrasian Equilibrium



Constrained Static Optimization I

(Sydsæter et al. 2005, Chapter 3)

- LAGRANGE-Optimization: minimize or maximize a function of one or more variables subject to constraints
- Function: $f(x_1, x_2, \ldots, x_n)$ with n arguments and m constraints:

$$\begin{aligned} g_1(x_1, x_2, \ldots, x_n) &= \gamma_1 \\ &\cdots \\ g_m(x_1, x_2, \ldots, x_n) &= \gamma_m \end{aligned}$$

- Example:
  1. Function f: utility function $u(c)$
  2. Constraints g: time constraint, capital endowment, budget constraint
  3. Variable: consumption c

Constrained Static Optimization II

(Sydsæter et al. 2005, Chapter 3)

- First step: LAGRANGE function

$$\mathcal{L} = f(x_1, x_2, \ldots, x_n) - \sum_{j=1}^{m} \lambda_j \left(g_j(x_1, x_2, \ldots, x_n) - \gamma_j \right)$$

where λ_j are called LAGRANGE-multipliers

- First-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x_1, \ldots, x_n)}{\partial x_i} &= 0, \qquad i = 1, 2, \ldots, n \\ \frac{\partial \mathcal{L}}{\partial \lambda_j} = g_j(x_1, x_2, \ldots, x_n) - \gamma_j &= 0, \qquad j = 1, 2, \ldots, m. \end{aligned}$$

- $n + m$ equations to be solved for $n + m$ unknowns $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m$

Walrasian Equilibrium

- Household maximizes utility

$$\max_{c,n} u(c,n) \quad \text{s.t.} \quad Pc = Wn + D$$

- Firm maximizes profit

$$\max_{y,n} D = Py - Wn \quad \text{s.t.} \quad y = f(n)$$

- Four endogenous variables: w, n, y, c
- Four equations
  - Combined household first-order condition (labor supply)
  - Combined firm first-order condition (labor demand)
  - Goods market equilibrium ($y = c$)
  - Production function $y = f(n)$
- Nominal wage (W) and price level (P) are not determined

Walrasian Equilibrium: Example – Household

- Household maximizes utility

$$\max_{c,n} u(c,n) \quad \text{s.t.} \quad Pc = wPn + D \quad \text{where} \quad u(c,n) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{n^{1+\varphi}}{1+\varphi}$$

  - Lagrangian

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} - \frac{n^{1+\varphi}}{1+\varphi} + \lambda\left(wPn + D - Pc\right)$$

  - Consumption first-order condition

$$\frac{\partial \mathcal{L}}{\partial c} = c^{-\sigma} - \lambda P = 0$$

  - Labor first-order condition

$$\frac{\partial \mathcal{L}}{\partial n} = -n^{\varphi} + \lambda w P = 0$$

  - Both FOCs combined

$$w = c^{\sigma} n^{\varphi}$$

Walrasian Equilibrium: Example – Firm

- Firm maximizes profit

$$\max_{y,n} D = Py - wPn \quad \text{s.t.} \quad y = f(n)$$

  - Lagrangian for $y = an^{\alpha}$

$$\mathcal{L} = Py - wPn - \kappa\left(y - an^{1-\alpha}\right)$$

  - Output first-order condition

$$\frac{\partial \mathcal{L}}{\partial y} = P - \kappa = 0$$

  - Labor first-order condition

$$\frac{\partial \mathcal{L}}{\partial n} = -wP + \kappa a(1-\alpha)n^{-\alpha} = 0$$

  - Both FOCs combined

$$w = a(1-\alpha)n^{-\alpha}$$

Walrasian Equilibrium: Example – Equilibrium Conditions

- Four endogenous variables: w, n, y, c
- Four equations
  - Combined household FOC (labor supply)
  - Combined firm FOC (labor demand)
  - Goods market equilibrium ($y = c$)
  - Production function ($y = an^{1-\alpha}$)
- Nominal wage (W) and price level (P) are not determined

Walrasian Equilibrium: Log-Linearized System

- Labor supply

$$\ln w = \sigma \ln c + \varphi \ln n$$

- Labor demand

$$\ln w = \ln a + \ln(1 - \alpha) - \alpha \ln n$$

- Goods market equilibrium

$$\ln y = \ln c$$

- Production function

$$\ln y = \ln a + (1 - \alpha) \ln n$$

Solving Systems of Linear Equations

`linsyssolv.m`

- 2-dimensional system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 18 \\ 3x_1 - 4x_2 &= -7 \end{aligned}$$

- Matrix notation

$$Ax = b \qquad A = \begin{bmatrix} 2 & 3 \\ 3 & -4 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad b = \begin{bmatrix} 18 \\ -7 \end{bmatrix}$$

- Solution

$$x = A^{-1}b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Numerical Solution Linearized System

- Linear system in matrix notation

$$\begin{bmatrix} 0 & -\sigma & 1 & -\varphi \\ 0 & 0 & 1 & \alpha \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(1-\alpha) \end{bmatrix} \begin{bmatrix} \ln y \\ \ln c \\ \ln w \\ \ln n \end{bmatrix} = \begin{bmatrix} 0 \\ \ln a + \ln(1-\alpha) \\ 0 \\ \ln a \end{bmatrix}$$

- The linear system can be solved analytically and numerically
- Numerical example: `linwalras.m`

Solving for the Walrasian Equilibrium Using Dynare

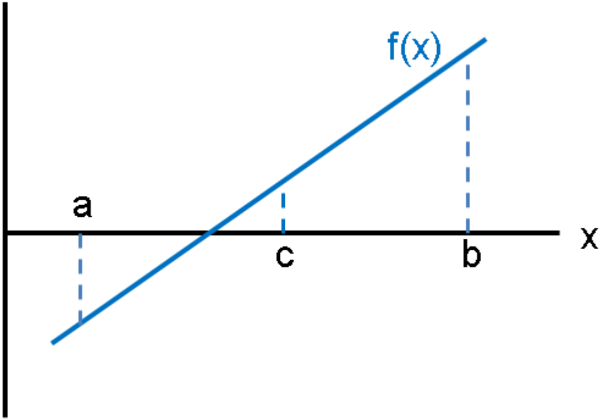
- Software for solving (dynamic) models with (forward-looking variables): `www.dynare.org`
- Dynare `.mod`-files can be executed in Octave, Matlab or Julia
- Example: `walras.mod`

Solving the Non-linear System

- Example in Octave/Matlab using: `nlwalras.m`
- Dynare can also solve the non-linear system `walrasnl.mod`

Solving Non-linear Equations: Bisection

(Judd 1998, Figure 5.1)



Bisection: Algorithm

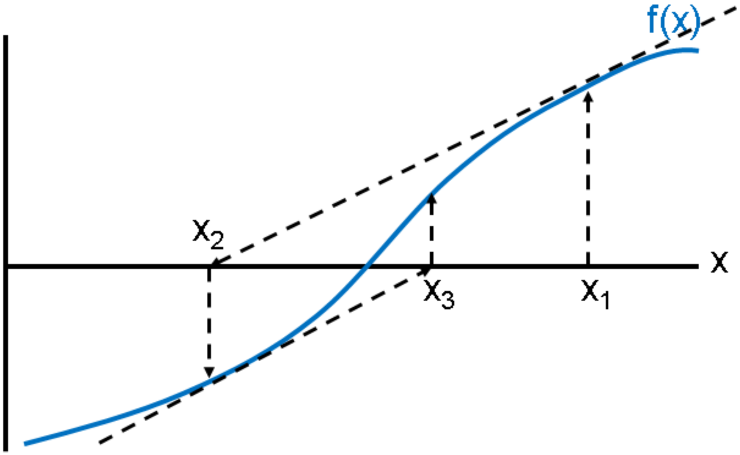
- Find the root (x_0) of $f(x)$: $f(x_0) = 0$
- Initialization
  - Find an interval that includes a root: Find $x^L < x^R$ such that $f(x^L) \cdot f(x^R) < 0$
  - Specify stop criteria ($\epsilon_x = 10^{-5}$, $\epsilon_f = 10^{-5}$, e.g.)
- Step 1: Compute mean: $x^M = (x^L + x^R)/2$
- Step 2: Adjust interval

$$\begin{aligned} x^R &= x^M && \text{if } f(x^L) \cdot f(x^M) < 0 \\ x^L &= x^M && \text{otherwise} \end{aligned}$$

- Step 3: Check stop criteria
  - Stop: $x_0 = x^M$ if $x^R - x^L \leq \epsilon_x(1 + |x^L| + |x^R|)$ or $|f(x^M)| \leq \epsilon_f$
  - Otherwise go back to step 1

Newton-Method

(Judd 1998, Figure 5.2)



Newton-Method: Algorithm

- Bisection is slow
- Linear approximation $g(x)$ of $f(x)$ at x_k has a root close to the root of $f(x)$

$$g(x) \equiv f'(x_k)(x - x_k) + f(x_k)$$

- Root of $g(x)$:

$$0 = f'(x_k)(x - x_k) + f(x_k)$$

- Iteration:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

- Stop if

$$|x_k - x_{k+1}| \leq \epsilon_x(1 + |x_k|)$$

- Success only if $|f(x_{k+1})| \leq \epsilon_f$

Walras' Law

- Plug profit equation into private household's budget constraint

$$Pc^d = Wn + D = Wn^s + Pc^s - Wn^d$$

- If the goods market or the labor market is cleared, then the other market is cleared as well

$$P(c^d - c^s) + W(n^d - n^s) = 0$$

- Generalization to m markets: if $m - 1$ markets are cleared, then the remaining market is also cleared

3. Static General Equilibrium Models

Summary

- The competitive general equilibrium is determined by
  - household preferences (utility function)
  - resource constraint (production function)
  - initial endowment (capital stock in $t = 0$, labor force)
- Dynare solves linear and non-linear models

Literature

- Alogoskoufis, George (2019): Dynamic Macroeconomics, MIT Press, Chapter 2
- Judd, Kenneth L. (1998): Numerical Methods in Economics, MIT Press
- Sydsæter, Knut; Hammond, Peter; Seierstad, Atle; Strøm, Arne (2005): Further Mathematics for Economic Analysis, Prentice Hall