Advanced Macroeconomics

5. Saving and Investment in a Recursive Model

The Treatment of Time

- Extension of the two-period model to a larger horizon T

$$U = \sum_{t=0}^{T} \left(\frac{1}{1+\rho}\right)^{t} u(c_t)$$

- Discrete time

$$\Delta_{t+1} = x_{t+1} - x_t$$

- Continuous time

$$\Delta x_t = x_{t+\Delta t} - x_t, \qquad \dot{x} = \lim_{\Delta t \to 0} \frac{x_{t+\Delta t} - x_t}{\Delta t}$$

- Intertemporal utility function with continuous time

$$U = \int_{t=0}^{T} \exp(-\rho t) u\left(c(t)\right) \, dt, \qquad t \in [0, T]$$

- The two types of intertemporal analysis are equivalent, we will only use discrete time, but some of the literature uses continuous time

5. Saving and Investment in a Recursive Model

The Neoclassical Growth Model

Robert M. Solow (1924-2023)

1987 The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel

https://www.nobelprize.org/prizes/economic-sciences/1987/solow/biographical/

A CONTRIBUTION TO THE THEORY OF ECONOMIC GROWTH

By Robert M. Solow

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I. Introduction

All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.[1] A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.

I wish to argue that something like this is true of the Harrod-Domar model of economic growth. The characteristic and powerful conclusion of the Harrod-Domar line of thought is that even for the long run the economic system is at best balanced on a knife-edge of equilibrium growth. Were the magnitudes of the key parameters — the savings ratio, the capital-output ratio, the rate of increase of the labor force — to slip ever so slightly from dead center, the consequence would be either growing unemployment or prolonged inflation. In Harrod's terms the critical question of balance boils down to a comparison between the natural rate of growth which depends, in the absence of technological change, on the increase of the labor force, and the warranted rate of growth which depends on the saving and investing habits of households and firms.

But this fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of *fixed proportions.* There is no possibility of substituting labor for capital in production. If this assumption is abandoned, the knife-edge notion of unstable balance seems to go with it. Indeed it is hardly surprising that such a gross

1. Thus transport costs were merely a negligible complication to Ricardian trade theory, but a vital characteristic of reality to von Thünen.

65

(Quarterly Journal of Economics 70(1), 1956)

Solow-Model in Discrete Time

- Robert Solow (1956) and T.W. Swan (1956)
- Exogenous variables
  - Labor supply (N_t)
  - Initial capital endowment (K_0)
  - Total factor productivity (Z_t) and efficiency of labor (A_t)
  - Constant savings ratio (s)
- Endogenous variables
  - Investment ($I_t = sY_t$) and capital stock (K_t, $t > 0$)
  - Consumption ($C_t = (1 - s)Y_t$)
  - Output ($Y_t = C_t + I_t$)

Production

- Cobb-Douglas production function

$$Y_t = Z_t K_t^{\alpha} (A_t N_t)^{1-\alpha}, \qquad 0 < \alpha < 1$$

- Intensive form

$$y_t = Z_t f(k_t) = Z_t k_t^{\alpha}, \qquad y_t = \frac{Y_t}{A_t N_t}, \quad k_t = \frac{K_t}{A_t N_t}$$

- Labor supply

$$N_t = (1+n) N_{t-1}$$

- Efficiency of labor

$$A_t = (1+a) A_{t-1}$$

Capital Accumulation

- Capital accumulation

$$K_{t+1} = (1-\delta)K_t + I_t \qquad \Leftrightarrow \qquad \Delta K_{t+1} = sY_t - \delta K_t$$

- Growth rate of capital stock

$$\frac{\Delta K_{t+1}}{K_t} = \frac{sY_t}{K_t} - \delta = \frac{sY_t/(A_t N_t)}{K_t/(A_t N_t)} - \delta = s\frac{y_t}{k_t} - \delta$$

- Growth rate of capital stock per efficient unit of labor

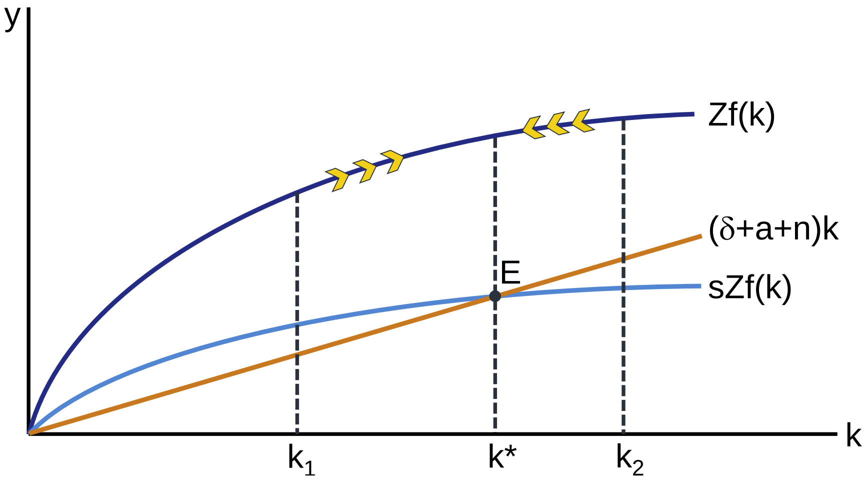
$$\frac{\Delta k_{t+1}}{k_t} \approx \frac{\Delta K_{t+1}}{K_t} - \frac{\Delta N_{t+1}}{N_t} - \frac{\Delta A_{t+1}}{A_t} = s\frac{y_t}{k_t} - (\delta + n + a)$$

- Capital accumulation per efficient unit of labor

$$\Delta k_{t+1} = sy_t - (\delta + n + a)k_t$$

Production and Savings in the SOLOW-Modell

(Alogoskoufis 2019, Figure 3.2)



Steady State

- Constant capital stock per efficient unit of labor

$$\Delta k_{t+1} = 0 \qquad \Rightarrow \qquad sy_t = (\delta + n + a)k_t$$

- Solve for steady state of k

$$sy = (\delta + n + a)k \qquad \Rightarrow \qquad sZf(k) = (\delta + n + a)k$$

- Cobb-Douglas case: $f(k) = k^\alpha$

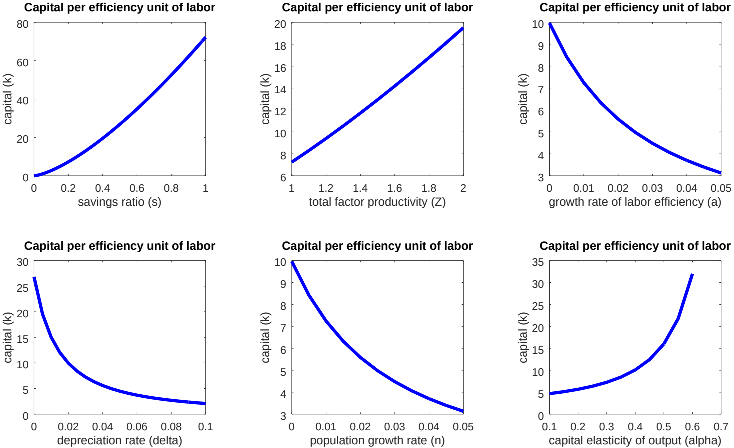
$$sZk^\alpha = (\delta + n + a)k$$

- Steady state capital stock per efficient unit of labor

$$k^* = \left(\frac{sZ}{\delta + n + a}\right)^{\frac{1}{1-\alpha}}$$

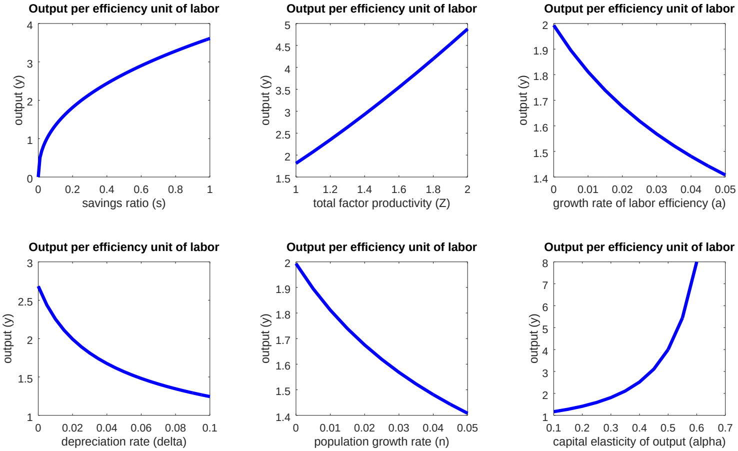
Solow Model Steady State Sensitivity: Capital

(`Solow_Sensitivity.m`)



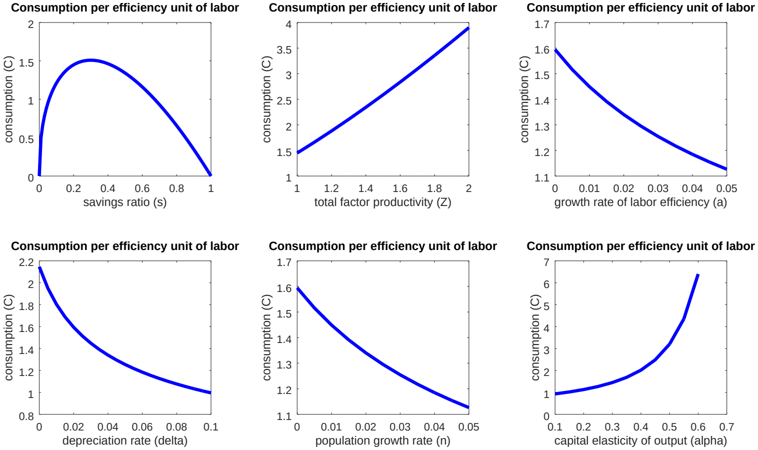
Solow Model Steady State Sensitivity: Output

(`Solow_Sensitivity.m`)



Solow Model Steady State Sensitivity: Consumption

(Solow_Sensitivity.m)



The Rate of Growth of Capital and Output

- Growth rate of capital stock

$$\gamma(k_t) = s\frac{y_t}{k_t} - (\delta + n + a)$$

- The larger the capital stock, the smaller its growth rate

$$\frac{\partial \gamma(k_t)}{\partial k_t} = \frac{s}{k_t} \cdot \frac{\partial y_t}{\partial k_t} - s \cdot \frac{y_t}{k_t^2} = -\frac{s y_t}{k_t^2}\left(1 - \frac{\partial y_t}{\partial k_t} \cdot \frac{k_t}{y_t}\right) < 0$$

with $(\partial y_t/\partial k_t)/(y_t/k_t) = \alpha < 1$

Implications

- Advanced economies (large capital stock) grow slower than emerging economies (small capital stock): $\partial\gamma/\partial k < 0$
- The higher the savings rate, the more grows the capital stock: $\partial\gamma/\partial s = y/k > 0$
- The larger the depreciation rate, the population growth rate and the growth rate of labor efficiency, the lower the growth rate of k: $\partial\gamma/\partial n = \partial\gamma/\partial\delta = \partial\gamma/\partial a < 0$
- Technological progress increases capital productivity (y_t/k_t) and growth rate of k: $\partial\gamma/\partial(y/k) = s > 0$

Steady State Growth Rates

- Capital stock

$$\gamma(k) = \gamma(K/(AN)) = 0 \qquad \Rightarrow \qquad \gamma(K) = \gamma(AN) = \gamma(A) + \gamma(N) = a + n$$

- Output

$$s\frac{y}{k} = \delta \qquad \Rightarrow \gamma(y) = \gamma(k) \qquad \Rightarrow \qquad \gamma(Y) = a + n$$

- Consumption per worker

$$\gamma\left(\frac{C}{N}\right) = \gamma\left(A\frac{C}{AN}\right) = \gamma\left(A(1-s)\frac{Y}{AN}\right) = a$$

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Competitive Markets

- Profit maximizing conditions
  - Marginal productivity of capital equals user cost of capital (depreciation rate plus interest rate)

$$r_t = Zf'(k_t) - \delta$$

  - Marginal productivity of labor equals real wage

$$w_t = Zf(k_t) - k_t Zf'(k_t)$$

- Total gross output per efficiency unit of labor

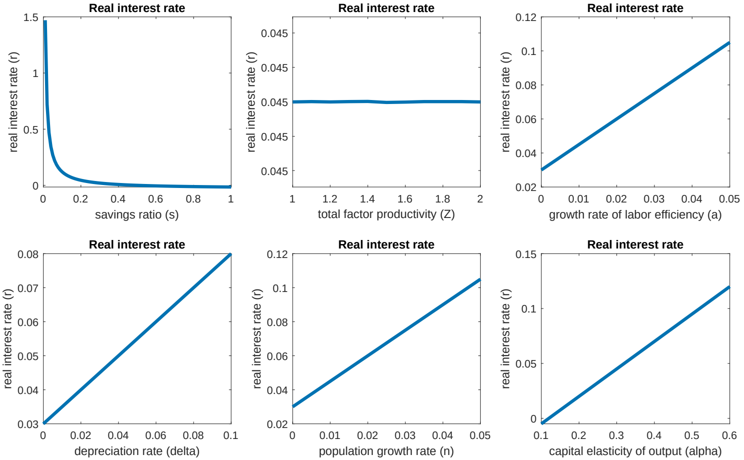
$$y_t = Zf(k_t) = r_t k_t + w_t + \delta k_t$$

- Steady state growth rate of real wage per worker

$$\gamma\left(\frac{W}{N}\right) = \gamma(wa) = \gamma(w) + \gamma(a) = a$$

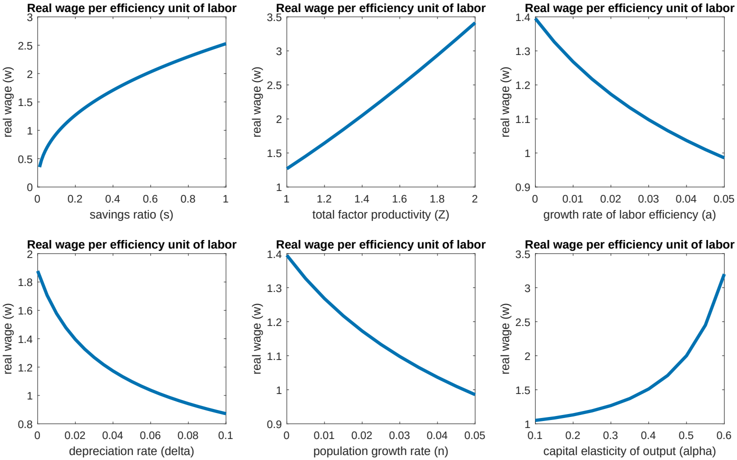
Solow Model Steady State Sensitivity: Real Interest Rate

(`Solow_Competitive_Sensitivity.m`)



Solow Model Steady State Sensitivity: Real Wage

(`Solow_Competitive_Sensitivity.m`)



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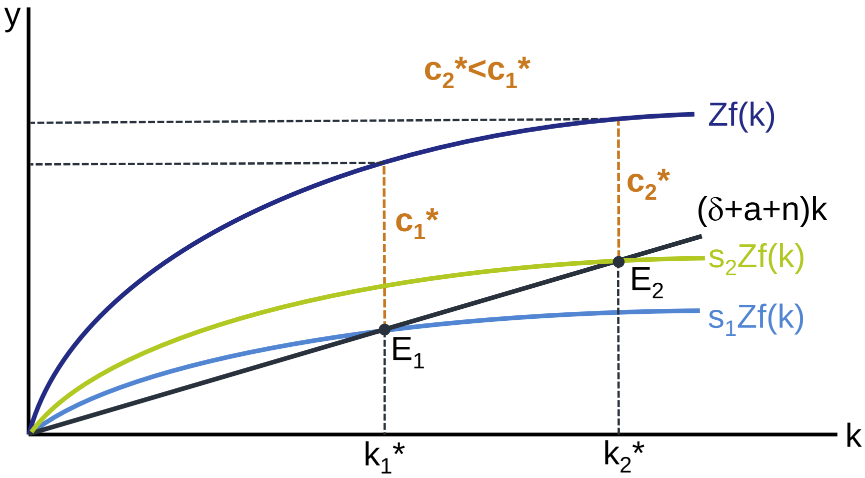
Implications of a Rise in the Savings Rate

(Alogoskoufis 2019, Figure 3.4)



Optimal Consumption

(Optimal_Consumption.m)

$$\max_s c^* = Zf(k^*) - (n+a+\delta)k$$

- First-order condition

$$\frac{\partial c^*}{\partial s} = \left(Zf'(k^*) - (n+a+\delta)\right)\frac{\partial k^*}{\partial s} = 0$$

- Since $\frac{\partial k^*}{\partial s} > 0$

$$Zf'(k^*) = (n+a+\delta) \quad \Rightarrow \quad Zf'(k^*) - \delta = n+a \quad \Rightarrow \quad r^* = n+a$$

- Cobb-Douglas case

$$Z\alpha k^{\alpha-1} - \delta = n+a \quad \Rightarrow \quad k = \left(\frac{n+a+\delta}{\alpha Z}\right)^{\frac{1}{\alpha-1}}$$

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Recursive Modeling: Timing

- For production, the capital stock at the beginning of a period is relevant
- Capital stock at the beginning of period t: K_{t-1}
- Production function

$$Y_t = Z_t F(K_{t-1}, N_t)$$

- Capital accumulation

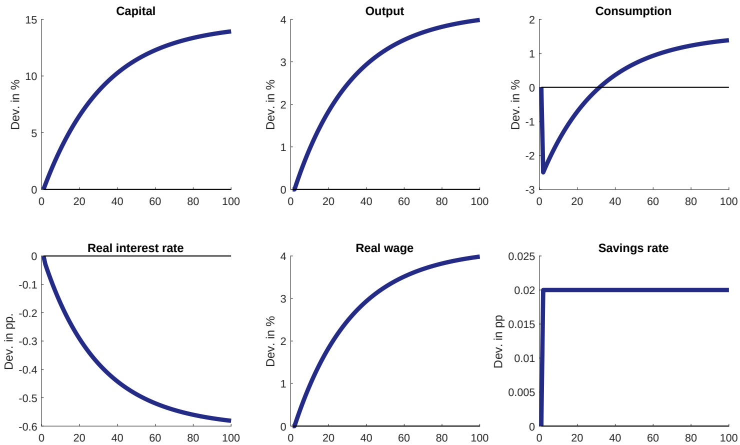
$$K_t = (1-\delta)K_{t-1} + sY_t \quad \Rightarrow \quad \frac{K_t}{Y_t} = (1-\delta)\frac{K_{t-1}}{Y_t} + s = (1-\delta)\frac{K_{t-1}}{Y_{t-1}(1+a)(1+n)} + s$$

- Steady state capital-output ratio

$$\frac{K}{Y} \equiv ky = (1-\delta)\frac{ky}{(1+a)(1+n)} + s \qquad \Rightarrow \qquad ky = \frac{(1+a)(1+n)s}{(1+a)(1+n) - 1 + \delta}$$

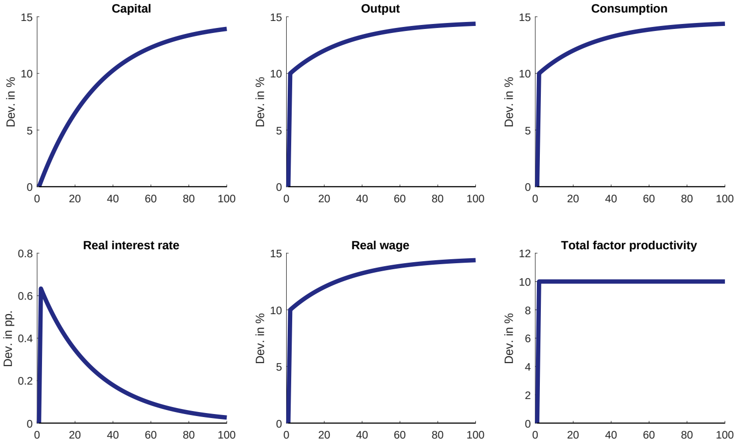
Experiment: Increase in s

(`Solow_Dynamic.m`)



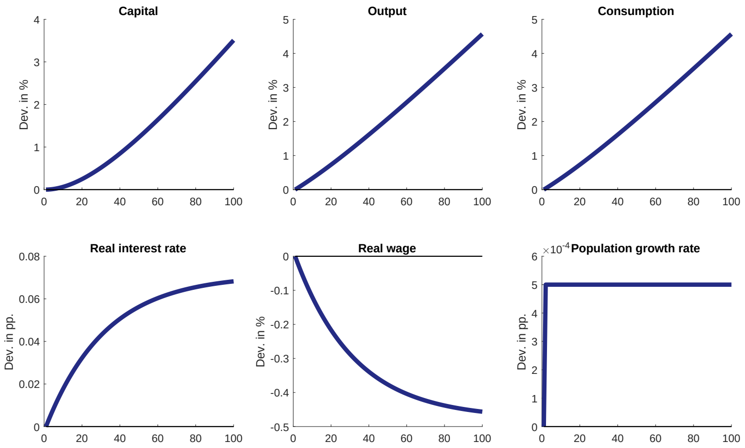
Experiment: Increase in Z

(`Solow_Dynamic.m`)



Experiment: Increase in n

(`Solow_Dynamic.m`)



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Summary

- Capital accumulation is an important driver of economic growth.
- In the Solow growth model, capital per efficiency unit of labor converges to a steady state value which increases in the savings ratio.
- On a balanced growth path, output per worker grows with an exogenous rate (growth rate of labor efficiency).
- Golden rule: consumption-maximizing savings rate can be derived. The savings rate can be higher than optimal (dynamic inefficiency).
- Convergence to steady state takes many years.

Literature

- Alogoskoufis, George (2019): Dynamic Macroeconomics, MIT Press, Chapter 3
- Romer, David (2018): Advanced Macroeconomics, 5th Edition, Chapter 1
- Solow, Robert M. (1956): A Contribution to the Theory of Economic Growth, Quarterly Journal of Economics 70(1), 65-94