Advanced Macroeconomics

7. Decentralized Economy and Optimal Growth

Household Behavior

- Labor supply is exogenous ($N = 1$)
- The representative household maximizes intertemporal utility

$$\max_{(c_{t+s}, a_{t+s+1})_{s \geq 0}} \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

subject to the budget constraint

$$\Delta a_{t+s+1} + c_{t+s} = w_{t+s} + r_{t+s} a_{t+s}, \quad \forall s \geq 0$$

where a_t is financial wealth, w_t is income and r_t is interest rate

Consumption Euler Equation

- LAGRANGE function

$$\mathcal{L} = \sum_{s=0}^{\infty} \{\beta^s U(c_{t+s}) + \lambda_{t+s} [w_{t+s} + (1 + r_{t+s}) a_{t+s} - c_{t+s} - a_{t+s+1}]\}$$

- First-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0, \qquad \forall s \geq 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+s}} &= \lambda_{t+s}(1 + r_{t+s}) - \lambda_{t+s-1} = 0, \qquad \forall s > 0 \end{aligned}$$

- For $s = 1$ we get after elimination of λ_{t+s} and λ_{t+s-1}

$$\beta \frac{U'(c_{t+1})}{U'(c_t)} (1 + r_{t+1}) = 1$$

Equal to Euler of central planned economy since $r_{t+1} = F'(k_{t+1}) - \delta$

Profit Maximization

- Present value of profits

$$\mathcal{P}_t = \sum_{s=0}^{\infty} (1+r)^{-s} \left\{ F(k_{t+s}) - w_{t+s} n_{t+s} - k_{t+s+1} + (1-\delta) k_{t+s} + b_{t+s+1} - (1+r) b_{t+s} \right\}$$

- First-order conditions

$$\frac{\partial \mathcal{P}_t}{\partial k_{t+s}} = (1+r)^{-s} [F_{k,t+s} + 1 - \delta] - (1+r)^{-(s-1)} = 0, \qquad s > 0$$

$$\frac{\partial \mathcal{P}_t}{\partial b_{t+s}} = -(1+r)^{-s}(1+r) + (1+r)^{-(s-1)} = 0, \qquad s > 0$$

- Optimal capital stock

$$F_{k,t+1} = r + \delta \qquad \Rightarrow \qquad i_t = F_{k,t+1}^{-1}(r+\delta) - (1-\delta) k_t$$

Static Equilibrium

- Households: consumption

$$\beta \frac{U'(c^*)}{U'(c^*)}(1+r) = 1 \qquad \Rightarrow \qquad \frac{1}{1+\rho} \equiv \beta = \frac{1}{1+r} \qquad \Rightarrow \qquad \rho = r$$

- Firms: capital

$$F'(k^*) = r + \delta = \rho + \delta$$

- Optimal capital stock equal to central planner's solution

$$k^* = \left(\frac{\alpha Z}{\delta + \rho}\right)^{1/(1-\alpha)}$$

- Households' financial assets $(a) =$ firms' debt (b)

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Intertemporal Utility

- Population and efficiency of labor grow at constant rates

$$N_t = N_0(1+n)^t \qquad \text{and} \qquad A_t = A_0(1+a)^t$$

- Intertemporal utility function

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(C_t/N_t)N_t = N_0 \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t u(C_t/N_t)$$

- Power utility function for consumption per effective worker ($c_t = C_t/(A_t N_t)$)

$$U = N_0 \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t \frac{(C_t/N_t)^{1-\theta}}{1-\theta} = \underbrace{A_0^{1-\theta} N_0}_{B} \sum_{t=0}^{\infty} \left(\underbrace{\frac{1+n}{1+\rho}(1+a)^{1-\theta}}_{\beta}\right)^t \frac{c_t^{1-\theta}}{1-\theta}$$

Capital Accumulation

- Capital accumulation

$$K_{t+1} = (1-\delta)K_t + Z_t F(K_t, A_t N_t) - C_t$$

- Divided by efficiency units of labor ($A_t N_t$)

$$\frac{K_{t+1}}{A_t N_t} = \frac{(1+n)(1+a)K_{t+1}}{A_{t+1}N_{t+1}} = (1+n)(1+a)k_{t+1} = (1-\delta)k_t + Z_t f(k_t) - c_t$$

- Constant returns to scale and competitive markets ($Z_t f(k_t) = (r_t + \delta)k_t + w_t$)

$$k_{t+1} = \frac{1}{(1+n)(1+a)}\left((1+r_t)k_t + w_t - c_t\right)$$

Optimization

- Lagrange function

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(B \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t \left((1+r_t)k_t + w_t - c_t - (1+n)(1+a)k_{t+1} \right) \right)$$

- First-order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t \left(B c_t^{-\theta} - \lambda_t \right) = 0 \quad \Rightarrow \quad \lambda_t = B c_t^{-\theta}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \beta^{t+1} \lambda_{t+1} (1 + r_{t+1}) - \beta^t \lambda_t (1+n)(1+a) = 0$$

- Together with $\beta = \frac{1+n}{1+\rho}(1+a)^{1-\theta}$

$$\lambda_t = \beta \frac{1 + r_{t+1}}{(1+n)(1+a)} \lambda_{t+1} = \frac{1 + r_{t+1}}{(1+\rho)(1+a)^{\theta}} \lambda_{t+1}$$

Euler Equation

- Combining the two first-order conditions

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r_{t+1}}{(1+\rho)(1+a)^{\theta}}\right)^{1/\theta} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} \frac{1}{1+a}$$

- Plugging in $r_{t+1} = Zf'(k_{t+1}) - \delta$

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+Zf'(k_{t+1})-\delta}{1+\rho}\right)^{1/\theta} \frac{1}{1+a}$$

Steady State

- Derive steady state capital per efficiency unit of labor from Euler equation ($c_{t+1}/c_t = c^*/c^* = 1$)

$$Zf'(k^*) = (1+\rho)(1+a)^\theta - (1-\delta)$$

- For $f(k_t) = Zk_t^\alpha$

$$k^* = \left(\frac{\alpha Z}{(1+\rho)(1+a)^\theta - (1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

- Derive steady state consumption per efficiency unit of labor from capital accumulation equation

$$c^* = Zf(k^*) - (1+n)(1+a)k^* + (1-\delta)k^* = Z(k^*)^\alpha - \left((1+n)(1+a) - (1-\delta)\right) k^*$$

- Competitive markets

$$r^* = (1+\rho)(1+a)^\theta - 1 \approx \rho + \theta a \qquad \text{and} \qquad w^* = (1-\alpha)Z(k^*)^\alpha$$

Dynamic Adjustment

- Capital

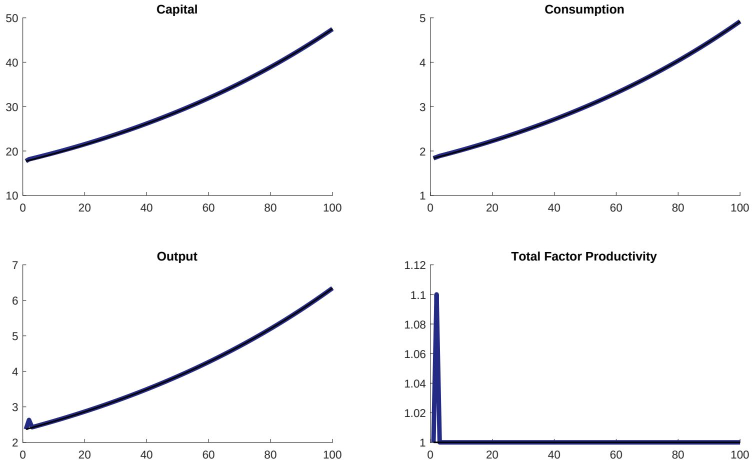
$$k_{t+1} = \frac{1}{(1+n)(1+a)} \left(Zk_t^{\alpha} + (1-\delta)k_t - c_t\right)$$

- Consumption

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+\alpha Z k_{t+1}^{\alpha-1} - \delta}{1+\rho}\right)^{\frac{1}{\theta}} \frac{1}{1+a}$$

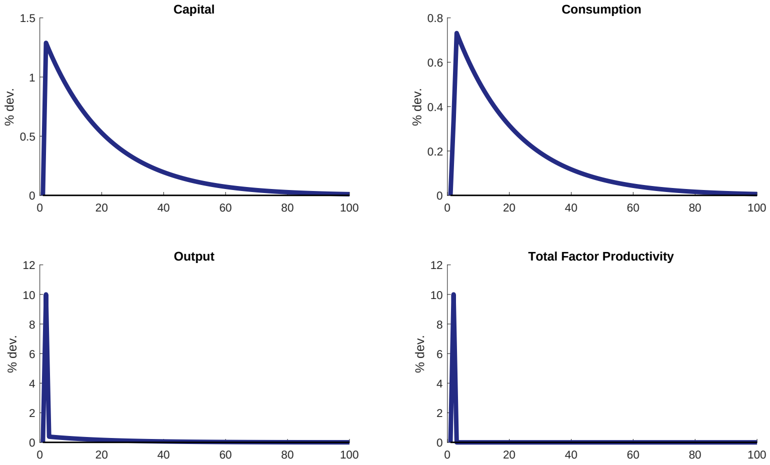
Temporary Productivity Shock

(`Ramsey_Growth.m`)



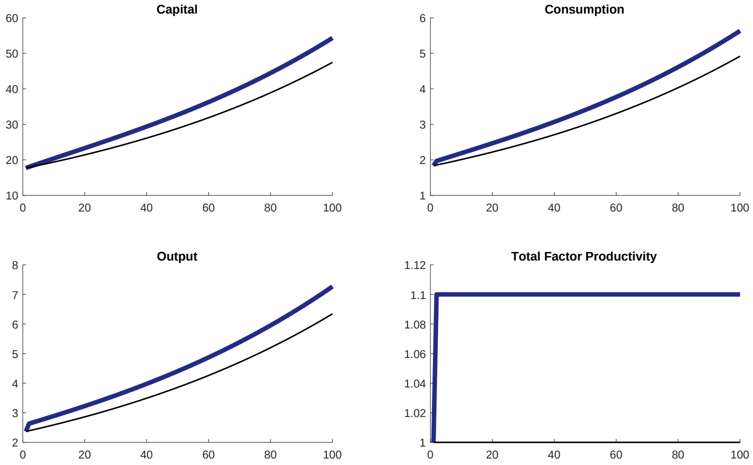
Temporary Productivity Shock

(Percentage deviation from initial steady state)



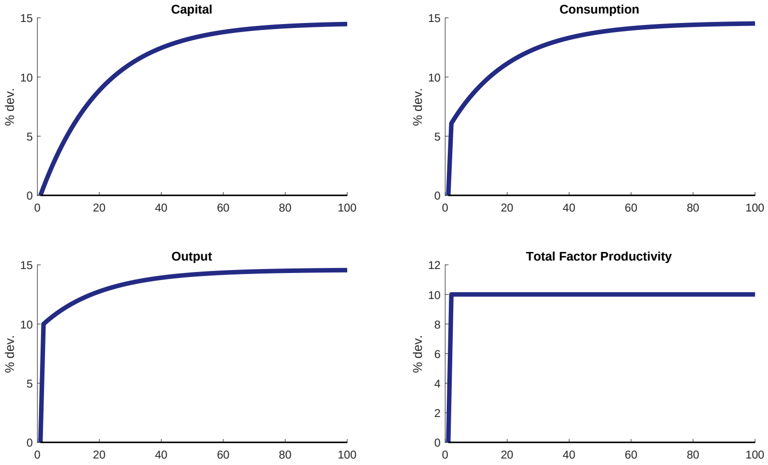
Permanent Productivity Shock

(Ramsey_Growth.m)



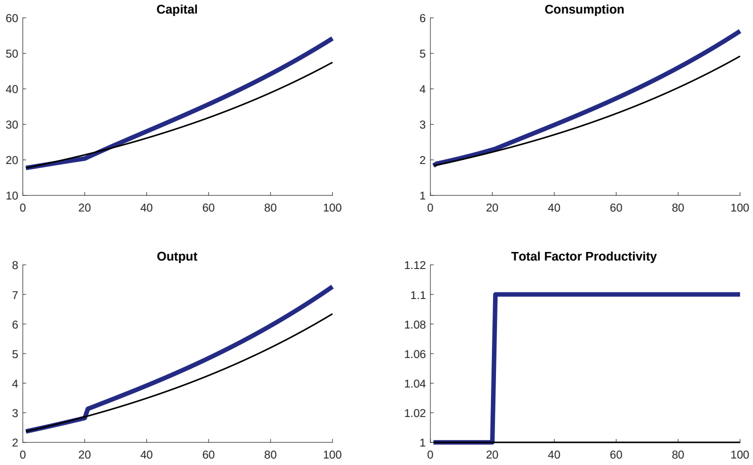
Permanent Productivity Shock

(Percentage deviation from baseline)



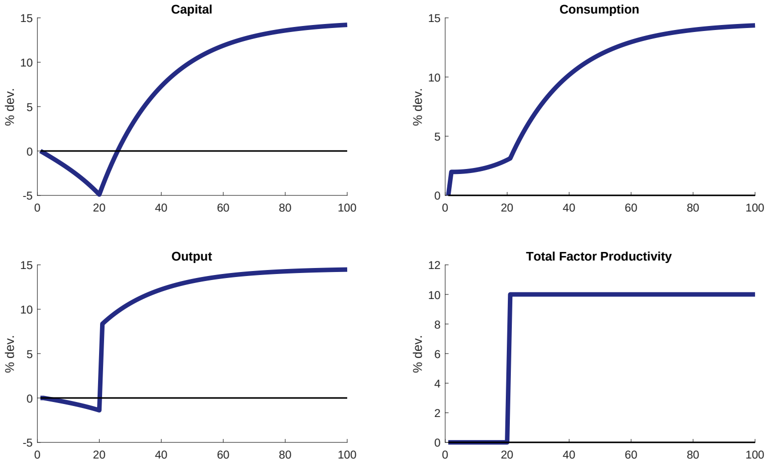
Expected Productivity Shock

(`Ramsey_Growth.m`)



Expected Productivity Shock

(Percentage deviation from baseline)



7. Decentralized Economy and Optimal Growth

Summary

- Ramsey model describes the optimal growth path depending on preferences, technology and endowment
- Reference model: without frictions central planner solution and decentralized economy yield identical growth paths
- Limitations
  - Long-run growth rate of consumption per capita exogenous (technological progress) $\Rightarrow$ endogenous growth models (investment in human capital and ideas)
  - Heterogeneity not considered: young versus old households, heterogeneity in income and wealth $\Rightarrow$ models with heterogenous agents, overlapping generation models

Literature

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