Homework 1. Preliminaries

Hyuntaek Oh

ohhyun@oregonstate.edu

Due: April 16, 2025

ECE599/ AI539 Nonlinear Optimization (Spring 2025)
Homework 1. Preliminaries (Due: 11:59pm, Wednesday, April 16)

Instruction: Students should provide enough detail of the logical procedure of deriving answers. Answers without sufficient justification will receive partial or no credit.

Reading: Before working on the homework questions, read Appendices A and B of the textbook (Luenberger and Ye).

1. Consider a subspace $\mathcal{V}$ of $\mathbb{R}^n$ with dimension K. Let $\{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_K\}$ denote a basis of V, i.e., $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_K$ are linearly independent, and they span $\mathcal{V}$. Let $\Phi \triangleq [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_K] \in \mathbb{R}^{n\times K}$. Then, any element a of V can be written as $a = \Phi\mathbf{c}$ for some $\mathbf{c} \in \mathbb{R}^K$. Now, let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^n$. The *orthogonal projection* of $\mathbf{x}$ onto $\mathcal{V}$, denoted by $\mathbf{x}_{\mathcal{V}}$, is an element of $\mathcal{V}$ that is closest to $\mathbf{x}$ (why? think of the geometry of orthogonal projection). In other words, the projection is a solution of the following problem:

$$\min_{\mathbf{y}\in\mathcal{V}} ||\mathbf{x} - \mathbf{y}||^2 \tag{1}$$

   (a) Verify that $\mathbf{x}_{\mathcal{V}} = \Phi(\Phi^T\Phi)^{-1}\Phi^T\mathbf{x}$.
   (Hint: Consider a minimization problem $\min_{c\in\mathbb{R}^K} ||\mathbf{x} - \Phi\mathbf{c}||^2$. An optimal solution of such an unconstrained problem satisfies the first-order condition, which is that the gradient of the objective function is zero at the optimal solution. For this case, the first-order condition can be written as:

$$2\Phi^T\Phi\mathbf{c} - 2\Phi^T\mathbf{x} = \mathbf{0}.$$

   For this question, you are allowed to use the above first-order condition without deriving it.)

   To verify the given equation, we need to show:

$$\mathbf{x}_{\mathcal{V}} = \min_{\boldsymbol{y}\in\mathcal{V}} ||\mathbf{x} - \boldsymbol{y}||^2$$

   Any element $\mathbf{a}$ *of* $\mathbf{V}$ *can be written as* $\mathbf{a} = \Phi\mathbf{c}$*, meaning that any element* $\boldsymbol{y}$ *in* $\mathbf{V}$ *is:*

$$\boldsymbol{y} = \Phi\mathbf{c}, \ \textit{where } \mathbf{c} \in \mathbb{R}^K$$

   Then, we need to consider a minimization problem $\min_{c\in\mathbb{R}^K} ||\mathbf{x} - \Phi\mathbf{c}||^2$*. By using this, the objective is defined as:*

$$f(c) = ||\mathbf{x} - \Phi\mathbf{c}||^2 = (\mathbf{x} - \Phi\mathbf{c})^T(\mathbf{x} - \Phi\mathbf{c})$$

From the hint, the gradient of the objective function at the optimal solution is when the objective function is equal to zero. This means:

$$2\Phi^T\Phi\mathbf{c} - 2\Phi^T\mathbf{x} = \mathbf{0}.$$

The equation of the gradient can be simplified as:

$$\Phi^T\Phi\mathbf{c} = \Phi^T\mathbf{x}$$

The "$\Phi^T\Phi$" is invertible since Φ is linearly independent and the size of its vector is $(K \times n)(n \times K) = K \times K$, meaning that the determinant of it is not equal to zero. So, the inverse matrix $(\Phi^T\Phi)^{-1}$ exists. Based on this, the equation would be:

$$\mathbf{c} = (\Phi^T\Phi)^{-1}\Phi^T\mathbf{x}$$

A solution of minimization problem $\mathbf{c}$ can be applied to:

$$\mathbf{x}_{\mathcal{V}} = \Phi\mathbf{c} = \Phi(\Phi^T\Phi)^{-1}\Phi^T\mathbf{x}$$

Thus, $\mathbf{x}_{\mathcal{V}} = \Phi(\Phi^T\Phi)^{-1}\Phi^T\mathbf{x}$ holds.

(b) Prove that $\mathbf{x}-\mathbf{x}_{\mathcal{V}}$ is orthogonal to any vector in $\mathcal{V}$; in other words, show that $\mathbf{v}^T(\mathbf{x}-\mathbf{x}_{\mathcal{V}}) = 0$ for all $\mathbf{v} \in \mathcal{V}$.

Any vector $\mathbf{v}$ in $\mathcal{V}$ can be written as:

$$\mathbf{v} = \Phi\mathbf{c}$$

$\mathbf{v}^T$ is:

$$\mathbf{v}^T = (\Phi\mathbf{c})^T = \mathbf{c}^T\Phi^T$$

$\mathbf{v}^T$ is applied to $\mathbf{v}^T(\mathbf{x} - \mathbf{x}_{\mathcal{V}}) = 0$:

$$\mathbf{v}^T(\mathbf{x} - \mathbf{x}_{\mathcal{V}}) = \mathbf{c}^T\Phi^T(\mathbf{x} - \Phi(\Phi^T\Phi)^{-1}\Phi^T\mathbf{x}) = 0$$

Extending and simplifying the equation above:

$$\begin{aligned} &= \mathbf{c}^T\Phi^T\mathbf{x} - \mathbf{c}^T(\Phi^T\Phi)(\Phi^T\Phi)^{-1}\Phi^T\mathbf{x} \\ &= \mathbf{c}^T\Phi^T\mathbf{x} - \mathbf{c}^T\Phi^T\mathbf{x} \\ &= 0 \end{aligned}$$

Thus, $\mathbf{v}^T(\mathbf{x} - \mathbf{x}_{\mathcal{V}}) = 0$ holds, meaning that $\mathbf{x} - \mathbf{x}_{\mathcal{V}}$ is orthogonal to $\mathcal{V}$.

2. For each of the following sets, determine whether the set is open, whether the set is closed, and whether the set is compact (provide rationale behind your answers). In addition, identify the closure, interior, and boundary of the set.

   (a) $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$

   i. *"Not Open": S consists of only boundary, meaning that the ball around any points x includes the points that are $||x|| < 1$ and $||x|| > 1$, which are necessarily out of S.*

   ii. *"Closed": If $\{x_k\} \subset S$ and $x_k \to x$, $||x_k|| = 1 \Rightarrow ||x|| = 1 \to x \in S$, including the limit.*

   iii. *"Compact": S is bounded since the distance at which all elements are exactly 1 from the origin, and S is closed as mentioned above.*

   iv. *"Closure of S": Since S is closed, closure of S is itself.*

   v. *"Interior of S": $\{\ \}$. The ball around some $\mathbf{x} \in S$ does not include in S.*

   vi. *"Boundary of S": S. All the points in S is boundary since there exists the points that has a boundary between $||\mathbf{x}|| < 1$ and $||\mathbf{x}|| > 1$, which is $||\mathbf{x}|| = 1$.*

   (b) $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \geq b\}$ where $\mathbf{a} \in \mathbb{R}^n$ ($\mathbf{a} \neq \mathbf{0}$) and $b \in \mathbb{R}$.

   i. *"Not Open": the boundary hyperplane $\mathbf{a}^T\mathbf{x} = b$ is included, and the ball around $\mathbf{x}$ at the boundary includes points out of S.*

   ii. *"Closed": the set [b, ∞) is closed in $\mathbb{R}$. If $\{x_k\} \subset S$ with $x_k \to x$, $\mathbf{a}^T x_k \to \mathbf{a}^T x \Rightarrow \mathbf{a}^T x \geq b$. So, the limit point $x \in S \Rightarrow S$ is closed.*

   iii. *"Not Compact": For example, if $\mathbf{a} = (1, 0, ..., 0)$ and $b = 0$, then $S = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, which extends infinitely in all directions except the negative x_1-axis. This means S is unbounded. Thus, it is not compact (closed, but not bounded)*

   iv. *"Closure of S": S is already a closed set, so its closure is itself.*

   v. *"Interior of S": $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} > b\}$, which lies strictly within the half-space and can be surrounded by a small open ball that stays entirely within S. However, any points where $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$ do not have neighborhoods entirely contained in S.*

   vi. *"Boundary of S": $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$. All points on this hyperplane are boundary points since the points on the boundary can contain $\mathbf{a}^T\mathbf{x} > b$ (in S) and $\mathbf{a}^T\mathbf{x} < b$ (in S^c).*

   (c) $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$ where $\mathbf{a} \in \mathbb{R}^n$ ($\mathbf{a} \neq \mathbf{0}$) and $b \in \mathbb{R}$.

   i. *"Not Open": S is a boundary, and the ball around some points in S includes $\mathbf{a}^T\mathbf{x} \neq b$.*

   ii. *"Closed": If $x_k \in S$ and $x_k \to x$, $\mathbf{a}^T x_k = b \Rightarrow \mathbf{a}^T x = b$ (by continuity). So, S is closed.*

   iii. *"Not Compact": S is closed, but not bounded. The points where $\mathbf{a}^T x = b$ extends infinitely in two directions if $n = 2$ without any boundaries.*

iv. *"Closure of S": As mentioned above, S is already closed, so closure of S is itself.*

v. *"Interior of S": $S = \{\ \}$. Any points where $\mathbf{a}^T\mathbf{x} = b$ do not have neighborhoods entirely contained in S, meaning that if exists, the ball would have any points which are $\mathbf{a}^T\mathbf{x} \neq b$.*

vi. *"Boundary of S": All points in S are boundaries thus the boundary of S is itself.*

(d) $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : |x_i| < 1, i = 1, ..., n\}$ where x_i is the i-th row of $\mathbf{x}$

i. *"Open": each coordinate in S lies in the open interval (-1, 1). For any point $x \in S$, there exists the open ball which is entirely contained in S.*

ii. *"Not Closed": S does not include its boundary points, meaning that it does not contain all of its limit points.*

iii. *"Not Compact": S is not closed, but bounded since all coordinates are between -1 and 1. So, it is not compact.*

iv. *"Closure of S": $\{x \in \mathbb{R}^n : -1 \leq x_i \leq 1$ for all i $\}$. The closure of an open box $(-1, 1)^n$ includes all its boundary points.*

v. *"Interior of S": $S = (-1, 1)^n$. every point in S is an interior point, meaning that its interior is the set itself.*

vi. *"Boundary of S": $[-1, 1]^n(-1, 1)^n$ It varies, depending on the value of n. If $n = 1$, the boundary of S would be $\{-1, 1\}$. If $n = 2$, the boundary of S would be $[-1, 1]^2$. If $n = 3$, the boundary of S would be $[-1, 1]^3$.*

3. **Limit Point** Consider a sequence $\{\mathbf{x}_k\}$ with $\mathbf{x}_k \in \mathbb{R}^n$ for all k. Suppose that $\mathbf{y}$ is a limit point of $\{\mathbf{x}_k\}$, i.e., there exists a subsequence of $\{\mathbf{x}_k\}$ that converges to $\mathbf{y}$. Prove that given any $\epsilon > 0$ and any positive integer *M*, there exist infinitely many j's satisfying (i) $j \geq M$, and (ii) $||\mathbf{x}_j - \mathbf{y}|| < \epsilon$.

(*Looking ahead*: most algorithms for solving nonlinear optimization are iterative algorithms that provide a sequence of solutions $\{\mathbf{x}_k\}$ wherein $\mathbf{x}_k$ denotes the solution at the end of the k-th iteration. A typical convergence property of these algorithms is that *any* limit point of $\{\mathbf{x}_k\}$ is a local optimum. Therefore, understanding the concept of limit point is important for this course.)

As described, $\mathbf{y}$ is a limit point of a sequence $\{\mathbf{x}_k\}$, meaning that there exists a subsequence $\{\mathbf{x}_{k_i}\}$ such that:

$$\lim_{i\to\infty} \mathbf{x}_{k_i} = \mathbf{y}$$

This means:

$$\exists\epsilon > 0,\ \exists I \in \mathbb{N} \text{ such that } i \geq I \Rightarrow ||\mathbf{x}_{k_i} - \mathbf{y}|| < \epsilon, \text{ where } I \text{ is large enough integer}$$

In addition, since $\{k_i\}$ is an increasing (unbounded) sequence of integers:

$$k_1 < k_2 < k_3 < ..., \text{ and } \lim_{i\to\infty} k_i = \infty$$

That also means that there exists some i_0 such that $k_i \geq M$ whenever $i \geq i_0$.

Therefore, if $i \geq max(I, i_0)$, $k_i \geq M$ and $||\mathbf{x}_{k_i} - \mathbf{y}|| < \epsilon$. Since i can be taken arbitrarily large, there are infinitely many distinct values of k_i that satisfy $k_i \geq M$ and $||\mathbf{x}_{k_i} - \mathbf{y}|| < \epsilon$.

Then, setting $k_i = j$ fulfill the goal that there are infinitely many such j.

4. Recall the matrix norm induced by Euclidean norm: for $A \in \mathbb{R}^{m\times n}$,

$$||A|| := \max_{\mathbf{x}\in\mathbb{R}^n : ||\mathbf{x}||\leq 1} ||A\mathbf{x}||. \tag{2}$$

(a) Prove that $||A\mathbf{x}|| \leq ||A|| \cdot ||\mathbf{x}||$

Assume that $x \in \mathbb{R}^n$ and $x \neq 0$. For example, the normalized vector u is:

$$u := \frac{x}{||x||} \Rightarrow ||u|| = 1$$

That is:

$$x = ||x|| \cdot u \Rightarrow Ax = A(||x|| \cdot u) = ||x|| \cdot Au$$

When taking the norm of both sides:

$$||Ax|| = ||\ ||x|| \cdot Au\ || = ||x|| \cdot ||Au||$$

Then, the definition of the matrix norm is used:

$$||Au|| \leq \max_{||v||\leq 1} ||Av|| = ||A|| \Rightarrow ||Au|| \leq ||A||$$

By applying it to the problem:

$$||Ax|| = ||x|| \cdot ||Au|| \leq ||x|| \cdot ||A|| \Rightarrow ||Ax|| \leq ||A|| \cdot ||x||$$

If $x = 0$, it would be:

$$||Ax|| = ||A \cdot 0|| = ||0|| = 0 \leq ||A|| \cdot ||0|| = 0$$

Therefore, for all $x \in \mathbb{R}^n$, the following inequality holds:

$$||Ax|| \leq ||A|| \cdot ||x||$$

(b) Suppose that A is an $n \times n$ real symmetric matrix. Let $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ denote the n eigenvalues of A. Prove that

$$||A|| = \max_{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y}|| \leq 1} ||\text{diag}(\lambda_1, ..., \lambda_n) \cdot \mathbf{y}|| \tag{3}$$

where $\text{diag}(\lambda_1, ..., \lambda_n)$ denotes the n by n diagonal matrix with its diagonal entries being $\lambda_1, \lambda_2, ..., \lambda_n$.
(Hint: $||A\mathbf{x}|| = \sqrt{\mathbf{x}^T A^T A\mathbf{x}}$. Use an eigen-decomposition of A.)
Given the hint $||Ax|| = \sqrt{x^T A^T Ax}$, matrix A is symmetric. So, we get:

$$||Ax|| = \sqrt{x^T A^T Ax} = \sqrt{x^T A^2 x}$$

This is applied to induced norm above:

$$||A|| = \max_{||x|| \leq 1} ||Ax|| = \max_{||x|| \leq 1} \sqrt{x^T A^2 x}$$

Since A is symmetric, we can write:

$$A = Qdiag(.)Q^T \text{ (orthogonal eigen-decomposition)}, \text{ where Q is orthogonal}$$

Then:

$$A^2 = AA = Qdiag(.)Q^T Qdiag(.)Q^T = Q(diag(.))^2 Q^T$$

So, we get:

$$x^T A^2 x = x^T Q(diag(.))^2 Q^T x$$

By using $y = Q^T x$:

$$y := Q^T x \Rightarrow x = Qy \quad (\text{ since } Q^T = Q^{-1})$$

Since Q is orthogonal, we get:

$$||y|| = ||Q^T x|| = ||x|| \leq 1$$

This means:

$$x^T A^2 x = y^T (diag(.))^2 y \Rightarrow ||Ax|| = \sqrt{y^T (diag(.))^2 y} = ||diag(.) \cdot y||$$

Thus, the following mathematical expressions hold.

$$||Ax|| = ||diag(\lambda_1, ..., \lambda_n) \cdot y||, \text{ where } ||y|| \leq 1$$

$$||A|| = \max_{||x|| \leq 1} ||Ax|| = \max_{||y|| \leq 1} ||diag(\lambda_1, ..., \lambda_n) \cdot y||$$

(c) Again, suppose that A is an n by n real symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Prove that $||A|| = max\{|\lambda_1|, |\lambda_n|\}$.
(Hint: you can use the result you proved in part (b))
From (b), we know $||A|| = \max_{||x||\leq 1} ||Ax|| = \max_{||y||\leq 1} ||diag(\lambda_1, ..., \lambda_n) \cdot y||$.
Then, for any vector $y = (y_1, y_2, ..., y_n)^T$*, the mathematical expression above would be:*

$$diag(\lambda_1, ..., \lambda_n) \cdot y = (\lambda_1 y_1, \lambda_2 y_2, ..., \lambda_n y_n)^T$$

For Euclidean norm, the expression above is squared:

$$||diag(.) \cdot y||^2 = \sum_{i=1}^{n} \lambda_i^2 y_i^2$$

Maximizing the Euclidean norm of the vector $diag(.) \cdot y$ *is to find the largest possible value of* $\sqrt{\sum \lambda_i^2 y_i^2}$ *under the constraint that* $||y|| \leq 1$ ($y_1^2 + y_2^2 + ... + y_n^2 \leq 1$)*. So, the goal is:*

$$\max_{||y||\leq 1} \sqrt{\sum_{i=1}^{n} \lambda_i^2 y_i^2}$$

In the expression above, if each λ_i^2 *is different, then to maximize the total value, the best way to achieve the goal is concentrating the entire norm in the direction corresponding to the largest* λ_i^2.
So, we assume:

$$|\lambda_k| = \max_i |\lambda_i|$$

By using the unit vector y *along the* k*-th coordinate direction,* $diag(.) \cdot y$ *would be:*

$$y = e_k \Rightarrow ||y|| = 1, \quad diag(.) \cdot y = \lambda_k e_k$$

Since e_k *is the unit vector,* $||e_k|| = 1$*:*

$$||diag(\lambda_1, ..., \lambda_n) \cdot y|| = ||\lambda_k e_k|| = |\lambda_k| \cdot ||e_k|| = |\lambda_k| \cdot 1 = |\lambda_k|$$

Making $||diag(\lambda_1, ..., \lambda_n) \cdot y||$ *larger by distributing mass across multiple components. This means:*

$$||diag(\lambda_1, ..., \lambda_n) \cdot y||^2 = \sum_{i=1}^{n} \lambda_i^2 y_i^2 \leq \max_i \lambda_i^2 \cdot \sum_{i=1}^{n} y_i^2 \leq \max_i \lambda_i^2$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i^2 y_i^2 \leq \sum_{i=1}^{n} \max_i \lambda_i^2 y_i^2$$

Since $\sum y_i^2 = ||y||^2 \leq 1$, *that would be:*

$$||diag(\lambda_1, ..., \lambda_n) \cdot y||^2 \leq \sum_{i=1}^{n} \max_i \lambda_i^2$$

Next, taking square roots on both sides:

$$||diag(\lambda_1, ..., \lambda_n) \cdot y|| \leq \max_i |\lambda_i|$$

Based on the assumption that eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, *we expect that* $|\lambda_1|$ *or* $|\lambda_n|$ *have potentially maximum value. As a result, we get:*

$$||A|| = \max_{||y|| \leq 1} ||diag(\lambda_1, ..., \lambda_n) \cdot y|| = \max_i |\lambda_i| = \max\{|\lambda_1|, |\lambda_n|\}$$

Thus:

$$||A|| = max\{|\lambda_1|, |\lambda_n|\}$$

5. Consider the following minimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} (\mathbf{x} - \mathbf{a})^T A (\mathbf{x} - \mathbf{a}) + b \text{ subject to } ||\mathbf{x} - \mathbf{a}|| \leq \epsilon$$

where **a** is a constant vector in $\mathbb{R}^n$, b is a real scalar, and ϵ is a small positive real number.

(a) Suppose that A is positive semidefinite. What would be a global minimum of this minimization?

For simplifying minimization problem, y *is used to make it become a quadratic form:*

$$y := \mathbf{x} - \mathbf{a} \Rightarrow \min_{y \in \mathbb{R}^n,\ ||y|| \leq \epsilon} y^T A y + b$$

b *is negligible since it is a real scalar, which may slightly affect minimum value (but it will be added in the end). So, the objective of minimization problem is:*

$$\min_{||y|| \leq \epsilon} y^T A y$$

Based on the assumption that A *is positive semidefinite,* A *is a symmetric* $(n \times n)$ *and* $x^T A x \geq 0$ *for all* $x \in \mathbb{R}^n$, *meaning that eigen-decomposition can be applied to a symmetric matrix* A. *In this case, eigen-values are non-negative* $(\lambda_i \geq 0)$.

According to lect4_April_9 slide, y is applied to the following inequalities:

$$\lambda_1||y||^2 \leq y^T Ay \leq \lambda_n||y||^2, \text{ for all } ||y|| \leq \epsilon$$

, where λ_1 is the smallest eigenvalue (lower boundary) of A and λ_n is the largest eigenvalue (upper bound) of A.
So, minimum value would be when $y = 0$ if $\lambda_1 = 0$. This means:

$$\min_{||y|| \leq \epsilon} y^T Ay = 0 \text{ (at } y = 0)$$

Since $y := \mathbf{x} = \mathbf{a}$, it would be:

$$\Rightarrow \mathbf{x} = \mathbf{a}$$

Then, a real scalar b is applied to:

$$f(a) = y^T Ay + b = 0 + b = b$$

Thus, a global minimum of this minimization would be b.

(b) Suppose that A is *not* positive semidefinite. Is **a** a global minimum of this minimization? Provide a justification for your answer. (Hint: use the eigen-decomposition of A)
Similar to (a), simplifying minimization problem:

$$y := \mathbf{x} - \mathbf{a} \Rightarrow \min_{y \in \mathbb{R}^n,\ ||y|| \leq \epsilon} y^T Ay + b$$

Since A is symmetric, it is orthogonally diagonalizable even if it is not positive semidefinite. So, A is:

$$A = Q \cdot diag(\lambda_1, ..., \lambda_n) \cdot Q^T$$

, where $Q \in \mathbb{R}^{n \times n}$ is orthogonal matrix, and $diag(\lambda_1, ..., \lambda_n)$ is a diagonal matrix of real eigenvalues, and some $\lambda_i < 0$ because of A is not positive semidefinite.
By using the eigenbasis, the mathematical expression would be:

$$g(y) = y^T Ay \Rightarrow y^T Q \cdot diag(\lambda_1, ..., \lambda_n) \cdot Q^T y$$

$$z := Q^T y \Rightarrow y = Qz \text{ and } ||z|| = ||y|| \leq \epsilon$$

Then, the objective becomes:

$$f(x) = f(a + y) = y^T Ay + b = z^T diag(.)z + b = \sum_{i=1}^{n} \lambda_i z_i^2 + b$$

$$f(x) = \sum_{i=1}^{n} \lambda_i z_i^2 + b \quad \textit{subject to } ||z||^2 = \sum z_i^2 \leq \epsilon^2$$

If $z = 0$, which means that $y = 0$ $(x = a)$, the result of the objective is:

$$f(a) = \sum \lambda_i \cdot 0^2 + b = b$$

However, if a eigenvalue is less than zero, there may exist the value that is lower than b.
Suppose $\lambda_1 = min_i \lambda_i < 0$. With the first standard basis vector, we set:

$$z = \epsilon \cdot e_1 = (\epsilon, 0, 0, ..., 0)^T$$

$$||z||^2 = z_1^2 + z_2^2 + ... + z_n^2 = \epsilon^2 + 0 + 0 + ... + 0 = \epsilon^2 \Rightarrow ||z|| = \epsilon$$

That satisfies the constraint $||z|| \leq \epsilon$, which is on the boundary of the feasible set.
Then, it would be applied to the objective:

$$f(a + Qz) = \sum_{i=1}^{n} \lambda_i z_i^2 + b = \lambda_1 \epsilon^2 + b < b$$

b, which is the result of when $x = a$, is not a global minimum. On the other hand, if z exists such that $||z|| = \epsilon$, and $f(a + Qz) < b$, there may exist a global minimum since it is lower than b.
Thus, $x = a$ is not a global minimizer if A has any negative eigenvalue.