Assignment #1

Start Assignment

- 기한 1월27일 4오후까지
- 배점 50
- 제출 방식 파일 업로드
- 파일 유형 pdf
- 사용 가능 2024 1월15일 12오전 이후

The assignment is about (in the following order): Vector calculus, non-convex optimization, high-dimensional space.

This assignment is supposed to be submitted on paper (to the instructor) or PDF (on Canvas) by Jan. 27th 4PM (start of the class).

1. (Optimization) Compute the gradient $\nabla f(\mathbf{x})$ and Hessian $\nabla^2 f(\mathbf{x})$ of the function (5 points)

$$f(\mathbf{x}) = (x_1 + x_2)(x_1 x_2 + x_1 x_2^2)$$

Find at least 3 stationary points of this function (3 points). Show that $[3/8, -6/8]^\top$ is a local maximum of this function (2 point).

2. (Optimization) Show that the function $f(\mathbf{x}) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$ has only one stationary point (4 points), and that it is neither a minimum nor a maximum, but is a saddle point (4 points).

3. (Linear Algebra) If **A** and **B** are positive definite matrices, prove that the matrix $\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{B} \end{bmatrix}$ is also positive definite (7 points).

4. (Chain Rule Calculus) Consider this function: $f(\mathbf{x}) = \mathbf{w}_2^\top sigmoid(\mathbf{W}_1 \mathbf{x})$, where $sigmoid(x) = \frac{1}{1+e^{-x}}$ applies to each entry of the vector, please compute the derivatives of $\frac{\partial f}{\partial \mathbf{w}_2}, \frac{\partial f}{\partial \mathbf{W}_1}, \frac{\partial f}{\partial \mathbf{x}}$ (15 points), $\mathbf{W}_1$ is $c \times d$, $\mathbf{x}$ is $d \times 1$, $\mathbf{w}_2$ is $c \times 1$.

5. (High Dimensional Statistics ("Curse of Dimensionality")) Consider N data points independent and uniformly distributed in a p-dimensional unit ball B (for every $x \in B$, $\|x\|^2 \leq 1$), centered at the origin. The median distance from the origin to the closest data point is given by the expression:

$$d(p, N) = \left(1 - \frac{1}{2}^{\frac{1}{N}}\right)^{\frac{1}{p}}$$

Prove this expression (8 points). Compute the median distance $d(p, N)$ for $N = 10,000$, $p = 1,000$ (2 points).

Hint: The volume of a ball in p dimensions is $V_p(R) = \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}+1\right)} R^p$, where R is the radius of the ball, and Γ is the Gamma function (the exact form of it does not matter for this assignment). A point being the **closest** point to the origin means that there is **no** point that has a smaller distance to the origin than itself. What is the **probability** for that to happen with a uniform distribution in a unit ball?