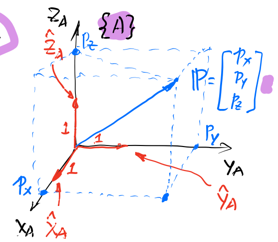
Chapter 2 Spatial Description and Homogeneous Transformation

Objective: To study representations of positions and orientations of a 3-D object.

A. Description of position + orientation and frame

a. Location of position : Position Vector

Given a 3D Cartesian coordinate $\{A\}$, any position in the 3-D space can be described by a position vector:



$${}^{A}\mathbb{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \text{ where }$$

P_x = The projection of $\mathbb{P}$ onto $\hat{X}_A$

P_y = The projection of $\mathbb{P}$ onto $\hat{Y}_A$

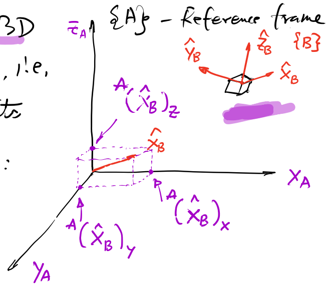
P_z = The projection of $\mathbb{P}$ onto $\hat{Z}_A$

b. Orientation of a 3D object (body) : Rotation Matrix $3\times3 = 9$

In order to describe the orientation of a 3D object, we will attach a frame (coordinate) to the 3D object in a known way and then give a description of the frame relative to the reference frame.

Given $\{A\}$, the orientation of the 3D object can be described by $\{B\}$, i.e., by the three unit vectors along its principal axes, in relation to $\{A\}$:

$${}^{A}\hat{X}_B, \quad {}^{A}\hat{Y}_B, \quad {}^{A}\hat{Z}_B, \quad \text{where}$$



$$\Rightarrow {}^{A}\hat{X}_B = \begin{bmatrix} {}^{A}(\hat{X}_B)_x \\ {}^{A}(\hat{X}_B)_y \\ {}^{A}(\hat{X}_B)_z \end{bmatrix} \quad \text{Similarly} \quad {}^{A}\hat{Y}_B = \begin{bmatrix} {}^{A}(\hat{Y}_B)_x \\ {}^{A}(\hat{Y}_B)_y \\ {}^{A}(\hat{Y}_B)_z \end{bmatrix}, \quad {}^{A}\hat{Z}_B = \begin{bmatrix} {}^{A}(\hat{Z}_B)_x \\ {}^{A}(\hat{Z}_B)_y \\ {}^{A}(\hat{Z}_B)_z \end{bmatrix}$$

which form the three columns of the Rotation Matrix:

$$\underset{3\times3}{{}^{A}_{B}R} \triangleq \underset{3\times3}{\begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix}} = \underset{3\times3}{\begin{bmatrix} {}^{A}(\hat{X}_B)_x & {}^{A}(\hat{Y}_B)_x & {}^{A}(\hat{Z}_B)_x \\ {}^{A}(\hat{X}_B)_y & {}^{A}(\hat{Y}_B)_y & {}^{A}(\hat{Z}_B)_y \\ {}^{A}(\hat{X}_B)_z & {}^{A}(\hat{Y}_B)_z & {}^{A}(\hat{Z}_B)_z \end{bmatrix}}$$

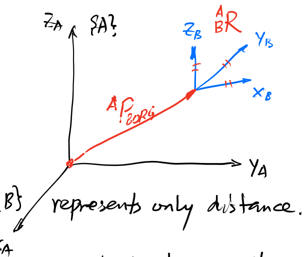
Rotation Matrix

The rotation matrix ${}^{A}_{B}R$ describes the orientation of frame $\{B\}$ relative to frame $\{B\}$, thus the orientation of the 3D object, on which frame $\{B\}$ is attached.

c. Description of a frame (position + orientation)

A frame $\{B\}$ is characterized by the position of the origin and the orientation of the frame relative to a reference frame, say $\{A\}$, i.e.

$$\{B\} = \{ {}^{A}_{B}R, \ {}^{A}P_{BORG} \}$$



Two special cases:

1° If ${}^{A}_{B}R = I$, $\{A\}$ & $\{B\}$ are parallel, $\{B\}$ represents only distance.

2° If ${}^{A}P_{BORG} = 0$, origins of $\{A\}$ & $\{B\}$ are coincident, on the same position. $\{B\}$ represents only orientation.

B Changing Description from frame to frame: Homogeneous Transformation

a. mapping involve translated frames:

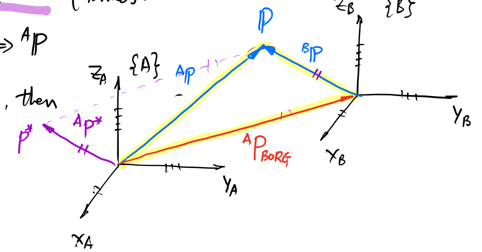
For a position P in {A} ⇒ ^{A}P

while in {B} ⇒ ^{B}P, then

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

$$\Downarrow$$

$$^{A}P^{*}$$



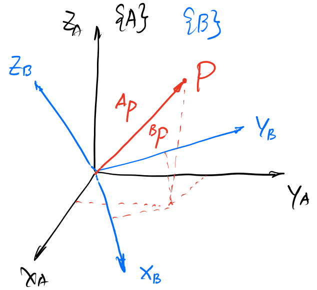
$$\Rightarrow \quad {}^{A}P = {}^{A}P^{*} + {}^{A}P_{BORG}$$

If {A} // {B}, $\Rightarrow$ $^{A}P = {}^{B}P + {}^{A}P_{BORG}$

b. mapping involve rotated frames:

Given {A} and {B}, where {B} is obtained by rotating {A} by certain angles about certain axis.

Fix a point P, then ^{A}P in {A}, ^{B}P in {B}. Let $^{A}P = \begin{bmatrix} ^{A}P_x \\ ^{A}P_y \\ ^{A}P_z \end{bmatrix}$ (*)



where $^{A}P_x$ = the projection of ^{A}P onto $\hat{X}_A$

$$= {}^{A}P \cdot {}^{A}\hat{X}_A = {}^{A}\hat{X}_A \cdot {}^{A}P = {}^{A}\hat{X}_A^{T}\,{}^{A}P = {}^{B}\hat{X}_A^{T}\,{}^{B}P$$

Similarly, $^{A}P_y = {}^{A}\hat{Y}_A \cdot {}^{A}P = {}^{B}\hat{Y}_A^{T}\,{}^{B}P$

$^{A}P_z = {}^{A}\hat{Z}_A \cdot {}^{A}P = {}^{B}\hat{Z}_A^{T} \cdot {}^{B}P$

For the rotation matrix ${}^{A}_{B}R$, it is always true that

$${}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T} \Rightarrow {}^{A}_{B}R \text{ is orthogonal.}$$

$$\Rightarrow {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}$$ (For orthogonal matrix, inverse = transpose)

$$\Rightarrow {}^{A}_{B}R = \begin{bmatrix} ^{A}\hat{X}_B & ^{A}\hat{Y}_B^{T} & ^{A}\hat{Z}_B \end{bmatrix} = {}^{B}_{A}R^{T} = \begin{bmatrix} ^{B}\hat{X}_A & ^{B}\hat{Y}_A & ^{B}\hat{Z}_A \end{bmatrix}^{T} = \begin{bmatrix} ^{B}\hat{X}_A^{T} \\ ^{B}\hat{Y}_A^{T} \\ ^{B}\hat{Z}_A^{T} \end{bmatrix}$$

Similarly,

$${}^{B}_{A}R = \begin{bmatrix} ^{B}\hat{X}_A & ^{B}\hat{Y}_A & ^{B}\hat{Z}_A \end{bmatrix} = {}^{A}_{B}R^{T}$$

$$\Rightarrow {}^{A}P = \begin{bmatrix} ^{A}P_x \\ ^{A}P_y \\ ^{A}P_z \end{bmatrix} = \begin{bmatrix} ^{B}\hat{X}_A^{T} \cdot {}^{B}P \\ ^{B}\hat{Y}_A^{T} \cdot {}^{B}P \\ ^{B}\hat{Z}_A^{T} \cdot {}^{B}P \end{bmatrix} = \begin{bmatrix} ^{B}\hat{X}_A^{T} \\ ^{B}\hat{Y}_A^{T} \\ ^{B}\hat{Z}_A^{T} \end{bmatrix} {}^{B}P = {}^{A}_{B}R \cdot {}^{B}P$$

$$\Rightarrow {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P \Longrightarrow {}^{B}P = {}^{B}_{A}R \cdot {}^{A}P$$

c. Mapping involving general frame:

Translation only: $^{A}P = {}^{B}P + {}^{A}P_{BORG}$, $^{B}P = {}^{A}P^{*}$

Rotation only: $^{A}P = {}^{A}_{B}R \cdot {}^{B}P$

Combine together:

rotation Translation

$$^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG}$$

matrix

$$\Rightarrow \begin{bmatrix} ^{A}P_{3\times1} \\ 1_{1\times1} \end{bmatrix}_{4\times1} = \begin{bmatrix} {}^{A}_{B}R_{3\times3} & {}^{A}P_{BORG\,3\times1} \\ 0\;0\;0_{1\times3} & 1_{1\times1} \end{bmatrix}_{4\times4} \cdot \begin{bmatrix} ^{B}P_{3\times1} \\ 1_{1\times1} \end{bmatrix}_{4\times1}$$

where this 4×4 matrix formed by ${}^{A}_{B}R$ & $^{A}P_{BORG}$ is defined as Homogeneous Transformation :

$$\boxed{{}^{A}_{B}T_{4\times4} \triangleq \left[\begin{array}{c|c} {}^{A}_{B}R_{3\times3} & {}^{A}P_{BORG\,3\times1} \\ \hline 0\ 0\ 0_{1\times3} & 1_{1\times1} \end{array}\right]_{4\times4}}$$