$$\Rightarrow \begin{bmatrix}\dot x(t)\\ \dot y(t)\end{bmatrix} = \begin{bmatrix}-l_1S_1-l_2S_{12} & -l_2S_{12}\\ l_1C_1+l_2C_{12} & l_2C_{12}\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}$$

linear

$$\Rightarrow {}^0v = {}^0J_v(\Theta)\cdot\dot\Theta$$

The above approach can be used for any robot manipulators. This is the most convenient way to calculate the LINEAR velocity part of the Jacobian. For angular velocity part of the Jacobian, we note the following alternative way to find ${}^0J_\omega(\Theta)$. In general, we have

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\cdot{}^{i}\omega_i + \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1}$$

Assume that all the joints are rotational, then

$${}^0\omega_n = {}^0_nR\cdot{}^n\omega_n = {}^0_nR\left({}^n_{n-1}R\cdot{}^{n-1}\omega_{n-1} + \dot\theta_n\,{}^n\hat Z_n\right) \qquad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= {}^0_nR\cdot{}^n_{n-1}R\cdot{}^{n-1}\omega_{n-1} + \underset{3\times3}{{}^0_nR}\cdot\dot\theta_n\cdot\underbrace{\underset{3\times1}{\begin{bmatrix}0\\0\\1\end{bmatrix}}}_{3\times1\ \text{vector}}$$

scalar number

$$= {}^0_{n-1}R\cdot{}^{n-1}\omega_{n-1} + {}^0_n\underbrace{\underset{3\times3}{\begin{bmatrix}r_{11}&r_{12}&r_{13}\\ r_{21}&r_{22}&r_{23}\\ r_{31}&r_{32}&r_{33}\end{bmatrix}}\underset{3\times1}{\begin{bmatrix}0\\0\\1\end{bmatrix}}}_{3\times1}\cdot\dot\theta_n$$

$$= {}^0_{n-1}R\cdot{}^{n-1}\omega_{n-1} + \begin{bmatrix}{}^0_nr_{13}\\ {}^0_nr_{23}\\ {}^0_nr_{33}\end{bmatrix}\cdot\dot\theta_n$$

${}^0_nr_3$

$$= {}^0_{n-1}R\cdot{}^{n-1}\omega_{n-1} + {}^0_nr_3\cdot\dot\theta_n$$

where ${}^0_nr_3$ is the third column of ${}^0_nR$.

$$\Rightarrow {}^0\omega_n = {}^0_{n-1}R\cdot\left[{}^{n-1}_{n-2}R\cdot{}^{n-2}\omega_{n-2} + \dot\theta_{n-1}\cdot{}^{n-1}\hat Z_{n-1}\right] + {}^0_nr_3\cdot\dot\theta_n$$

$$= {}^0_{n-2}R\cdot{}^{n-2}\omega_{n-2} + {}^0_{n-1}r_3\cdot\dot\theta_{n-1} + {}^0_nr_3\cdot\dot\theta_n$$

$$\vdots$$

$$\Rightarrow \underset{3\times1}{{}^0\omega_n} = \underbrace{\underset{3\times1}{{}^0_1r_3}\cdot\underset{1\times1}{\dot\theta_1}}_{3\times1} + \underbrace{\underset{3\times1}{{}^0_2r_3}\cdot\underset{1\times1}{\dot\theta_2}}_{3\times1} + \cdots + \underbrace{\underset{3\times1}{{}^0_nr_3}\cdot\underset{1\times1}{\dot\theta_n}}_{3\times1}$$

$$\Rightarrow {}^0\omega_n = \begin{bmatrix}{}^0_1r_3 & {}^0_2r_3 & \cdots & {}^0_nr_3\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \vdots\\ \dot\theta_6\end{bmatrix}$$

$$\Rightarrow {}^0\Omega_n = {}^0J_\omega(\Theta)\cdot\dot\Theta$$

$$\Rightarrow \underset{3\times n}{{}^0J_\omega(\Theta)} = \overbrace{\begin{bmatrix}\underset{3\times1}{{}^0_1r_3} & \underset{3\times1}{{}^0_2r_3} & \cdots & \underset{3\times1}{{}^0_nr_3}\end{bmatrix}}^{n}_{3\times n}$$

where ${}^0_ir_3$ is obtained from the 3rd column of ${}^0_iR$.

So the procedure becomes that find ${}^0_1R, {}^0_2R, \cdots {}^0_nR$, and then take the 3rd columns of ${}^0_1R \cdots {}^0_nR$ to form ${}^0J_\omega$.

Back to our example, find ${}^0_1R$ and ${}^1_2R$ first, and then ${}^0_2R = {}^0_1R\cdot{}^1_2R$ and the third columns are:

$${}^0_1r_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ and } {}^0_2r_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow {}^0J_\omega(\Theta) = \begin{bmatrix}0&0\\0&0\\1&1\end{bmatrix} \xrightarrow{\text{planar}} {}^0J_\omega = [1\ \ 1].$$

* Changing Jacobian Reference Frames

Given a Jacobian in {B}, i.e.,

$$\begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix} = {}^B V = {}^B J(\Theta)\cdot\dot{\Theta} \quad . \quad \text{Find } {}^A J(\Theta)$$

Note that a 6x1 Cartesian velocity vector given in {B} is described relative to {A} by the transformation:

$$\underbrace{{}^A v = {}^A_B R\,{}^B v, \quad {}^A\omega = {}^A_B R\cdot{}^B\omega}, \quad \text{then}$$

$$\begin{bmatrix} {}^A v \\ {}^A \omega \end{bmatrix}_{6\times1} = {}^A V_{6\times1} = \begin{bmatrix} {}^A_B R_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & {}^A_B R_{3\times3} \end{bmatrix}_{6\times6} \begin{bmatrix} {}^B v \\ {}^B \omega \end{bmatrix}_{6\times1}$$

$$\Rightarrow \quad {}^A J(\Theta)_{6\times6} = \underbrace{\begin{bmatrix} {}^A_B R_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & {}^A_B R_{3\times3} \end{bmatrix}_{6\times6} \cdot {}^B J(\Theta)_{6\times6}}_{6\times6}$$

E. Singularity

Consider an n dof robot manipulator and assume that the e.e. velocity as a vector with a dimension m ($m \le 6$)

Jacobian that relates $\dot{\Theta}$ to ${}^0V$ is an $m\times n$ matrix:

$${}^0V_{m\times1} = {}^0J(\Theta)_{m\times n}\cdot\dot{\Theta}_{n\times1} \qquad \text{linear transformation}$$

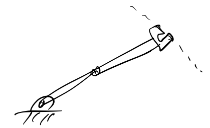
$$\Rightarrow \quad \dot{\Theta} = {}^0J^{-1}(\Theta)\cdot{}^0V$$

Given ${}^0V$ to find $\dot{\Theta}$ $\Rightarrow$ ${}^0J^{-1}$ exists? $\Rightarrow$ $|{}^0J| \neq 0$?

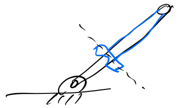
* If ${}^0J$ is a square matrix, the above question becomes weather ${}^0J$ is invertible:

$$\dot{\Theta} = {}^0J^{-1}\cdot{}^0V \quad \Rightarrow \quad {}^0J \text{ is invertible} \Leftrightarrow |{}^0J| = 0?$$

* Workspace Singularities: Singularities that occur when the robot manipulator is fully stretched out or fully folded back on itself such that the e.e. is near or at the boundary of the workspace.



fully stretched out



fully folded back.

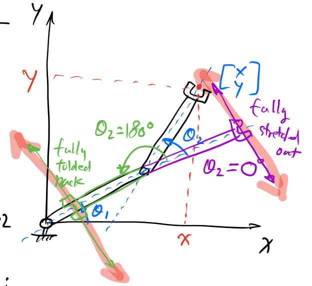
* Workspace Interior Singularities: Singularities that occur away from the workspace boundary and generally are caused by two or more robot joint axis lining up.

When $|{}^0J| = 0$, then the robot manipulator is said in a singularity configuration and the robot velocity in joint space cannot be obtainted/calculated.

$\cancel{\dot{\Theta} = {}^0J^{-1}\cdot{}^0V}$ Does not exist

Example: The 2 dof planar robot

The Jacobian is as follows:



$$^0J_v(\Theta) = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ l_1C_1 + l_2C_{12} & l_2C_{12} \end{bmatrix}_{2\times 2}$$

The determinent of the Jacobian:

$$\det {}^0J_v = |{}^0J_v| = \begin{vmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ l_1C_1 + l_2C_{12} & l_2C_{12} \end{vmatrix} = l_1 l_2 \cdot S_2 = 0$$

$$\underset{\neq 0}{l_1}\,\underset{\neq 0}{l_2} \cdot \sin\theta_2 = 0 \Rightarrow \sin\theta_2 = 0 \Rightarrow \theta_2 = \begin{cases} 0° \\ 180° \end{cases}$$

From the illustration, we can see that the robot manipulator is in workspace singularity when $\theta_2 = 0°$ & $\theta_2 = 180°$, which put the manipulator into singularity.

$\Rightarrow$ Singularities happen at the boundary, so this is workspace singularity.

$$\Rightarrow \text{If } \theta_2 = \begin{cases} 0° \\ 180° \end{cases} \rightarrow {}^0v = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = k \cdot \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}$$ direction

In this case, the manipulator only moves at the tangent direction at the boundary.

IF. Jacobians in force domain

Let $\mathcal{F}$ be a 6×1 force/torque (moment) vector with

$$\mathcal{F}_{6\times 1} = \begin{bmatrix} f_{3\times 1} \\ \hdashline n_{3\times 1} \end{bmatrix}_{6\times 1}$$

where f is the force along the linear velocity and n is the moment (torque) associated with angular velocity. $f_{3\times 1}$ is the force applying to the environment by the robot. and $n_{3\times 1}$ is the torque applying to the environment by robot e.e. so that we have

$$\underset{\text{joint space}}{\tau_{n\times 1}} = {}^0J^T(\Theta) \cdot \underset{\text{Cartesian space}}{{}^0\mathcal{F}_{6\times 1}}$$

where $\tau_{n\times 1} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}$ is the vector of torques at each joint in joint space and ${}^0\mathcal{F}_{6\times 1}$ is the generalized forces/torques at the e.e. of the manipulator in Cartesian space.

In summary,

$${}^0V_{6\times 1} = {}^0J(\Theta)_{6\times n}\, \dot{\Theta}_{n\times 1}$$

$$\tau_{n\times 1} = {}^0J^T(\Theta)\, {}^0\mathcal{F}_{6\times 1}$$

joint space

Cartesian space