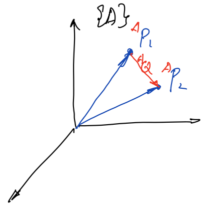
The Homogeneous Transformation :

$$^A_BT \triangleq \left[\begin{array}{c|c} ^A_BR & ^AP_{BORG} \\ \hline 0\,0\,0 & 1 \end{array}\right]$$

where ^A_BR is the rotation matrix and $^AP_{BORG}$ is the position vector pointing from the origin of $\{A\}$ to $\{B\}$, and

$\left[\begin{array}{c} ^AP_{BORG} \\ \hline 1 \end{array}\right]$ is called the augmented position vector.



Operators

a. Translation Operator

Given $\{A\}$, we want to move point P_1 to P_2.

$\Rightarrow$ $^AP_2 = {}^AP_1 + {}^AQ$

In other words, moving $^AP_1$ to $^AP_2$ is equivalent to adding vector ^AQ to $^AP_1$. Since $\left[\begin{array}{c} ^AP_2 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} I_3 & ^AQ \\ \hline 0\,0\,0 & 1 \end{array}\right]\left[\begin{array}{c} ^AP_1 \\ \hline 1 \end{array}\right]$, then the translation from $^AP_1$ to $^AP_2$ can be accomplished by the matrix operator :

$$\boxed{\left[\begin{array}{c|c} I_3 & ^AQ \\ \hline 0\,0\,0 & 1 \end{array}\right]} \triangleq \boxed{\text{TRANS}}(^A\hat{Q}, |Q|)$$

$\Rightarrow$ Then the translation operator is defined as

$$\boxed{\text{TRANS}(^A\hat{Q}, |Q|) = \left[\begin{array}{c|c} I_3 & ^AQ \\ \hline 0\,0\,0 & 1 \end{array}\right]}$$

where $^AQ = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$, $^A\hat{Q}$ = unit vector along ^AQ, $|Q|$ = length of ^AQ.

Then $^AP_2 = {}^AP_1 + {}^AQ \Rightarrow \left[\begin{array}{c} ^AP_2 \\ \hline 1 \end{array}\right] = \text{TRANS}(^A\hat{Q}, |Q|) \cdot \left[\begin{array}{c} ^AP_1 \\ \hline 1 \end{array}\right]$.

b. Rotation Operator

Given $\{A\}$, we rotate a vector $^AP_1$ about a vector $\hat{k}$ by θ degree and call the resulting vector $^AP_2$. Denote this action as an operator $\text{ROT}(\hat{k}, \theta)$, then we have

$$\left[\begin{array}{c} ^AP_2 \\ \hline 1 \end{array}\right] = \text{ROT}(\hat{k}, \theta) \cdot \left[\begin{array}{c} ^AP_1 \\ \hline 1 \end{array}\right]$$

It is expected that we can find a rotation matrix, denoted by R, such that $^AP_2 = R \cdot {}^AP_1$. Hence the rotation operator $\text{ROT}(\hat{k}, \theta)$ can be written as

$$\text{ROT}(\hat{k}, \theta) = \left[\begin{array}{c|c} R_{3\times3} & 0_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right]_{4\times4}$$

Question : How to find $R \longleftrightarrow \hat{k}$? arbitrary ?

Special case : Rotations around the three principle axes :

$$\text{ROT}(\hat{x}, \theta) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{ROT}(\hat{y}, \theta) = \left[\begin{array}{ccc|c} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\text{ROT}(\hat{z}, \theta) = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

C. General Operator

Given $\{A\}$ and vector ${}^{A}P_1$. Let T be an operator that rotates the vector ${}^{A}P_1$ about a rotating axis by θ via R and then translate it via Q to get a new vector ${}^{A}P_2$:

$$\begin{bmatrix} {}^{A}P_2 \\ \hline 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ \hline 1 \end{bmatrix} \quad \text{where} \quad T \triangleq \left[\begin{array}{c|c} R & Q \\ \hline 0\,0\,0 & 1 \end{array}\right]_{4\times4} \quad \text{is defined}$$

as the general operator.

Remark: The two operations (R & Q) DO NOT commute!

(i) If you do rotation first and translation second:

$$\begin{bmatrix} {}^{A}P_2 \\ \hline 1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} I_{3\times3} & Q \\ \hline 0\,0\,0 & 1 \end{array}\right]}_{2^{nd}} \underbrace{\left[\begin{array}{c|c} R_{3\times3} & O_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right]}_{1^{st}} \begin{bmatrix} {}^{A}P_1 \\ \hline 1 \end{bmatrix}$$

$$\Downarrow$$

$$\left[\begin{array}{c|c} I_{3\times3} & Q_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] \left[\begin{array}{c|c} R_{3\times3} & O_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R_{3\times3} & Q_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] = T$$

Same as the homogeneous Transformation

(ii) If you do translation first and rotation second:

$$\left[\begin{array}{c|c} R_{3\times3} & O_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] \left[\begin{array}{c|c} I_3 & Q_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & RQ \\ \hline 0\,0\,0 & 1 \end{array}\right] \neq T$$

D. Three Interpretations of the Homogeneous Transformation:

a. It describes a frame $\{B\}$ w.r.t. a reference frame $\{A\}$:

$${}^{A}_{B}T_{4\times4} = \left[\begin{array}{c|c} {}^{A}_{B}R_{3\times3} & {}^{A}Q_{3\times1} \\ \hline 0\,0\,0 & 1 \end{array}\right] \quad {}^{A}Q = {}^{A}P_{BORG}$$

where R – 3×3 rotation matrix indicating the orientation relationship between $\{A\}$ and $\{B\}$, ${}^{A}_{B}R$.

Q – ${}^{A}P_{BORG}$, the 3×1 position vector relating the origins of $\{A\}$ & $\{B\}$.

b. It is a transform mapping that maps ${}^{B}P$ to ${}^{A}P$:

$$\begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix}$$

c. It is a transform operator that moves ${}^{A}P_1$ to ${}^{A}P_2$ when ${}^{A}P_2$ is obtained by rotating ${}^{A}P_1$ by R and then translated by Q:

$$\begin{bmatrix} {}^{A}P_2 \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} R & Q \\ \hline 0\,0\,0 & 1 \end{array}\right] \cdot \begin{bmatrix} {}^{A}P_1 \\ \hline 1 \end{bmatrix}$$

E. Two Remarks on Rotation matrix R

\#1. There are only three independent parameters in a rotation matrix R!

Since rotation matrix is proper-orthogonal matrix:

- Proper : $\det(R) = +1$.
- Orthogonal : all columns of R are mutually orthogonal (perpendicular) and have unit length.

* Cayley's Formulation : For orthogonal matrices, if they are proper, say, R, then there exists a skew-symmetric matrix S, such that

$$\boxed{R_{3\times3} = (\mathbb{I}_{3\times3} - S_{3\times3})^{-1} \cdot (\mathbb{I}_{3\times3} + S_{3\times3})}$$

where $\mathbb{I}_3$ is the 3x3 identity matrix and S is a skew-symmetric matrix, i.e., it satisfies the condition of $\boxed{S = -S^T}$.

A skew-symmetric matrix of dimension 3 is specified by a three-parameter (S_x, S_y, S_z) matrix as

$$S = \begin{bmatrix} 0 & -S_z & S_y \\ S_z & 0 & -S_x \\ -S_y & S_x & 0 \end{bmatrix} = -S^T$$

$\Rightarrow$ S is skew-symmetric !

S has three elements $\Rightarrow$ R has three independent elements S_x, S_y, S_z !

$$R \longrightarrow S_x, S_y, S_z$$

*2 Rotation operators do not commute :

$$R_1 \cdot R_2 \neq R_2 \cdot R_1 \quad \text{in general}$$

F. Transformation Arithmetics :

Given $\{B\}$ in a reference frame $\{A\}$ and $\{C\}$ in reference to $\{B\}$, then

$${}^{A}_{C}T = {}^{A}_{B}T \cdot {}^{B}_{C}T \iff \text{compound arithmetics}$$

Find the inverse of transformation :

$$({}^{A}_{B}T)^{-1} = {}^{B}_{A}T$$

$$\Rightarrow \quad {}^{B}_{A}T = ({}^{A}_{B}T)^{-1} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ 0_{1\times3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}P_{AORG} \\ 0_{1\times3} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ 0_{1\times3} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} {}^{A}_{B}R^T & -{}^{A}_{B}R^T \cdot {}^{A}P_{BORG} \\ 0_{1\times3} & 1 \end{bmatrix}$$

Since $\begin{bmatrix} {}^{B}({}^{A}P_{BORG}) \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P_{BORG} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}P_{AORG} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_{BORG} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

From the above, we have

$$^B_AR\,^AP_{BORG} + {}^BP_{AORG} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

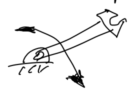
$$\Leftrightarrow \boxed{^BP_{AORG} = -\,^B_AR \cdot {}^AP_{BORG}}$$

Chapter 3 Kinematics

Given joint displacement, the position and orientation of the end-effector (e.e.) of a robot arm in reference to the base. This relationship is described by kinematics.

Link Parameters – Denavit-Hartenberg (D-H) Notation.

A robot manipulator is usually an open mechanical linkage formed by several rigid links that are connected in a chain by joints. The joint types:

- Rotational or revolute joint 
- Translational or prismatic 

* Denavit-Hartenberg Notation (D-H Notation).

A convention for the description of robot manipulator linkage. It is a convention for the assignment of a set of frames that are attached to the links of a robot arm. There are three steps to obtain link parameters according to the D-H notation:

Step #1 Number the links and joint axes from the base to the tip of the robot manipulator.

Step #2 Attach link frames (coordinates) to each link as follows:

* Z_{i-1}-axis coincides with axis $i-1$ (joint $i-1$)

* X_{i-1}-axis coincides with the common normal between the joint axis $i-1$ and the joint axis i.

* Y_{i-1}-axis can be obtained by using the right hand rule.