Chapter 5 Jacobian: Velocities and Static Forces (1st analysis)

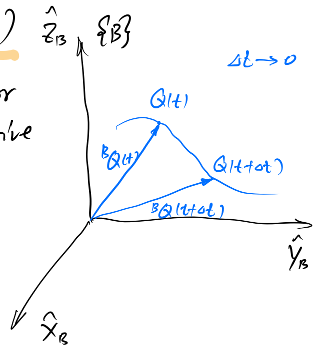
Problem to be studied:

(1) Velocity of the end-effector via Jacobian

(2) Static force and moments applied at the e.e. via Jacobian

△ Differentiation of a vector: linear and angular velocities

a. linear velocity of a frame (link)



Let ${}^BQ$ be a time-varying position vector w.r.t. $\{B\}$. It's first-order time derivative is denoted by:

$$ {}^BV_Q \triangleq \frac{d}{dt}\left({}^BQ(t)\right) $$

$$ = \lim_{\Delta t \to 0} \frac{{}^BQ(t+\Delta t) - {}^BQ(t)}{\Delta t} $$

Note that ${}^BV_Q$ is the linear velocity of the point Q and it is a vector. If one wants to represent ${}^BV_Q$ in another frame, say $\{A\}$, and denote it by ${}^A({}^BV_Q)$ and if no relative rotation between $\{B\}$ and $\{A\}$, then

$$ {}^A({}^BV_Q) = {}^A\left(\frac{d}{dt}{}^BQ(t)\right) = \frac{d}{dt}\left({}^A_BR \cdot {}^BQ(t)\right) $$

$$ = \frac{d}{dt}\left({}^A_BR\right) \cdot {}^BQ(t) + {}^A_BR \cdot \frac{d}{dt}\left({}^BQ(t)\right) $$

$$ \Rightarrow \quad {}^A({}^BV_Q) = {}^A_BR \cdot {}^BV_Q \quad (\text{no relative motion}) $$

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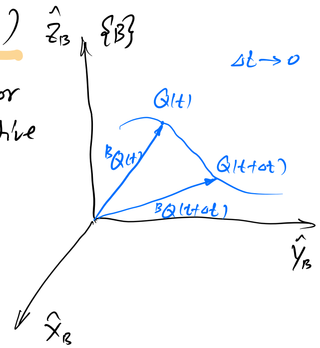

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Notation: If Q is the origin of a frame, say $\{C\}$, which is moving, then we use $\boxed{{}^{U}v_c}$ to denote the linear velocity of the origin of $\{C\}$ w.r.t. $\{U\}$. Therefore ${}^{A}v_c$ describes the linear velocity of the origin of $\{C\}$ w.r.t. $\{A\}$, i.e.,

$$\boxed{{}^{A}v_c = {}^{A}({}^{U}v_c)}$$

b. Angular Velocity of a frame

Angular velocity of a frame describes the rotational motion of the frame. We use a vector to describe the angular velocity of a frame, where

- The direction of the vector represents the instantaneous rotation axis
- The magnitude of the vector represents the rotational speed.

Notation:

- ${}^{A}\Omega_B$ denotes the angular velocity of $\{B\}$ w.r.t. $\{A\}$.
- $\omega_c = {}^{U}\omega_c$ is defined as the angular velocity of $\{C\}$ w.r.t. $\{U\}$.

B. Linear and Angular Velocities of $\{B\}$ w.r.t. $\{A\}$

a. linear velocity

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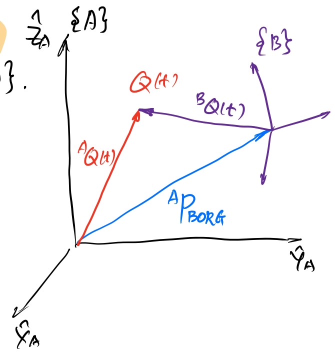
B. Linear and Angular Velocities of $\{B\}$ w.r.t. $\{A\}$

a. linear velocity

Assume that there is a linear motion (translational motion) of $\{B\}$ w.r.t. $\{A\}$.

From the illustration, it follows that



$$^AQ = {}^AP_{BORG} + {}^A(^BQ(t))$$

$$= {}^AP_{BORG} + {}^A_BR \cdot {}^BQ(t)$$

Differentiating the above eqn.

$$^AV_Q = {}^AV_{BORG} + {}^A_BR \cdot {}^BV_Q \quad ({}^A_BR = \text{constant}, \text{ no rotation})$$

b. Rotational Velocity

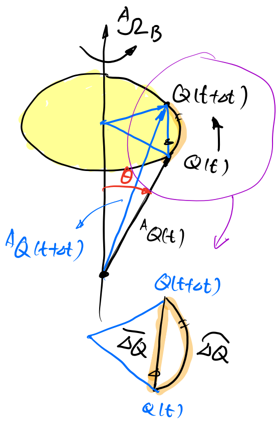
Assume that

1° No linear velocity of $\{B\}$ w.r.t. $\{A\}$ (rotation only)

2° There is a rotational velocity of $\{B\}$ w.r.t. $\{A\}$, i.e. ${}^A_BR$ is time-varying.

3° Point Q is fixed in $\{B\}$

Let compute ^AV_Q - rotational velocity of Q w.r.t. $\{A\}$



From the illustration, we have

$\Delta Q \perp {}^A\Omega_B$ and $\Delta Q \perp {}^AQ$

$$\Rightarrow \overline{\Delta Q} \underset{\Delta t \to 0}{\cong} \widehat{\Delta Q} = |{}^AQ| \cdot \sin\theta \cdot |{}^A\Omega_B| \cdot \Delta t$$

$\Downarrow$

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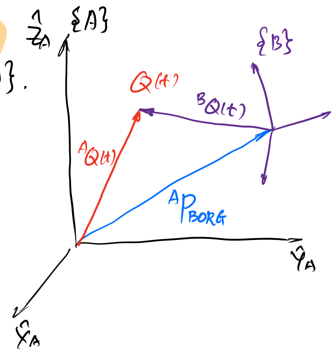

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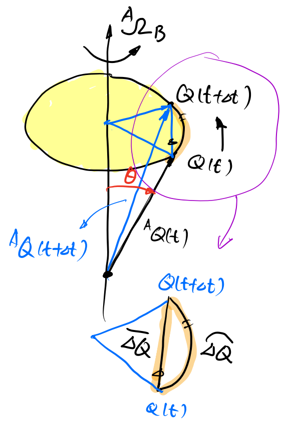

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$$\Rightarrow \lim_{\Delta t \to 0} \frac{\overline{\Delta Q}}{\Delta t} \triangleq {}^{A}V_Q = |{}^{A}Q| \cdot \sin\theta \cdot |{}^{A}\Omega_B| = {}^{A}\Omega_B \times {}^{A}Q$$

$$\Rightarrow \underbrace{{}^{A}V_Q}_{\text{rotational velocity of } Q \text{ w.r.t. } \{A\}} = \underbrace{{}^{A}\Omega_B}_{\text{angular velocity of } \{B\} \text{ w.r.t. } \{A\}} \times \underbrace{{}^{A}Q}_{\text{position vector of } Q \text{ w.r.t. } \{A\}}$$

If assumption 3° is removed, then point Q is no longer fixed w.r.t. $\{B\}$, then

$$\boxed{{}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}Q + {}^{A}_{B}R \cdot {}^{B}V_Q}$$

C. Motion of the link of a manipulator

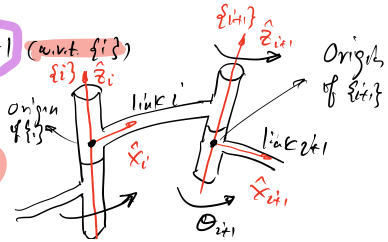
Notation:

- v_i – linear velocity of the origin of $\{i\}$ w.r.t. $\{U\}$
- ω_i – angular velocity of $\{i\}$ w.r.t. $\{U\}$
- ${}^{i}v_i$ – v_i expressed in $\{i\}$
- ${}^{i}\omega_i$ – ω_i expressed in $\{i\}$

* Velocity Propagation from link i to link $i+1$

Case I. If joint $i+1$ is rotational

(1) Angular velocity of link $i+1$ (w.r.t. $\{i\}$)



Angular velocity of link $i+1$

= rotational velocity of $\{i\}$ ①

+ angular velocity of link $i+1$ due to

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\overline{\Delta Q}}{\Delta t} \triangleq {}^{A}V_Q = |{}^{A}Q| \cdot \sin\theta \cdot |{}^{A}\Omega_B| = {}^{A}\Omega_B \times {}^{A}Q$$

$$\Rightarrow \underbrace{{}^{A}V_Q}_{\text{rotational velocity of } Q \text{ w.r.t. } \{A\}} = \underbrace{{}^{A}\Omega_B}_{\text{angular velocity of } \{B\} \text{ w.r.t. } \{A\}} \times \underbrace{{}^{A}Q}_{\text{position vector of } Q \text{ w.r.t. } \{A\}}$$

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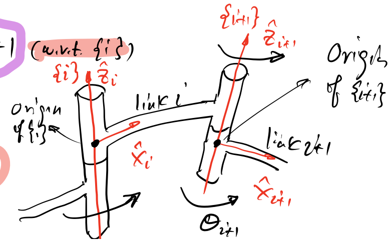

Angular velocity of link $i+1$

= rotational velocity of $\{i\}$ ①

+ angular velocity of link $i+1$ due to

the rotation of link i+1 axis ②

$$^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{z}_{i+1}$$

Multiply the above eqn. from both sides by ${}^{i+1}_{i}R$:

$$^{i+1}_{i}R \cdot {}^{i}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \underbrace{{}^{i+1}_{i}R \cdot {}^{i}_{i+1}R}_{I} \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{z}_{i+1}$$

$$\Rightarrow \; {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{z}_{i+1}$$

(2) linear velocity of the origin of {i+1}

linear velocity of the origin of {i+1}

= linear velocity of the origin of {i}

+ linear velocity of the origin of {i+1} due to the rotation of {i}

$$^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

multiply ${}^{i+1}_{i}R$ on both sides of the above eqn.:

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}v_i + {}^{i+1}_{i}R \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right)$$

Case II. If the joint i+1 is prismatic (translational)

Then $\dot{\theta}_{i+1} = 0_{3\times 1}$

and $^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i$

Further note that {i+1} will move along ${}^{i+1}\hat{z}_{i+1}$ direction

the rotation of link i+1 axis ②

$$^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{z}_{i+1}$$

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linear velocity of the origin of {i+1}

= linear velocity of the origin of {i}

+ linear velocity of the origin of {i+1} due to the rotation of {i}

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at a speed of $\dot{d}_{i+1}$, so the linear velocity of the origin of $\{i+1\}$ is

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}p_{i+1} \right. + \left. \dot{d}_{i+1} \cdot {}^{i+1}\hat{z}_{i+1} \right) $$

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