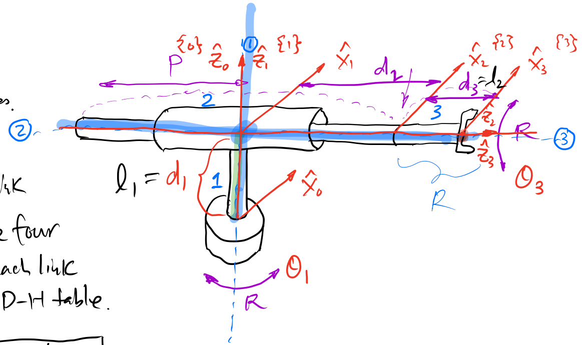
Example. A 2R-1P non-planar robot arm as shown.
Find the D-H table for the robot arm.

Step #1: Number the links and axes.

Step #2: Attach frames to each link

Step #3: Find the four parameters for each link and form the D-H table.



link	α_{i-1}	a_{i-1}	d_i	θ_i
1	0°	0	d_1	θ_1
2	90°	0	d_2	0°
3	0°	0	d_3	θ_3

R ← θ_1 is variable, d_1 is fixed
P ← θ_2 is fixed $= 0$, d_2 is variable
R ← θ_3 is variable, d_3 is fixed.

Using the ${}^{i-1}_{i}T$ formula, we have

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -d_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & 0 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and then

$$ {}^{0}_{3}T = {}^{0}_{1}T \cdot \underbrace{{}^{1}_{2}T \cdot {}^{2}_{3}T}_{{}^{1}_{3}T} = \begin{bmatrix} c\theta_1 \cdot c\theta_3 & -c\theta_1 \cdot s\theta_3 & s\theta_1 & s\theta_1 \cdot l_2 + s\theta_1 \cdot d_2 \\ s\theta_1 \cdot c\theta_3 & -s\theta_1 \cdot s\theta_3 & -c\theta_1 & -c\theta_1 \cdot l_2 - c\theta_1 \cdot d_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

${}^{0}_{3}R$ ${}^{0}P_{3ORG}$

where $l_1 = d_1$ and $l_2 = d_3$

$$ \Rightarrow \quad {}^{0}_{3}R = \begin{bmatrix} c\theta_1 \cdot c\theta_3 & -c\theta_1 \cdot s\theta_3 & s\theta_1 \\ s\theta_1 \cdot c\theta_3 & -s\theta_1 \cdot s\theta_3 & -c\theta_1 \\ s\theta_3 & c\theta_3 & 0 \end{bmatrix} \quad \text{Orientation} $$

$$ \text{and} \quad {}^{0}P_{3ORG} = \begin{bmatrix} s\theta_1 \cdot l_2 + s\theta_1 \cdot d_2 \\ -c\theta_1 \cdot l_2 - c\theta_1 \cdot d_2 \\ 0 \end{bmatrix} \begin{matrix} {}^{0}p_x \leftarrow \\ {}^{0}p_y \leftarrow \\ {}^{0}p_z \leftarrow \end{matrix} \quad \mathbb{P}. $$

This is the kinematics: $\Theta \longrightarrow \mathbb{P}$

Chapter 4 Inverse Kinematics

In joint space

$$ \Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \underset{\text{Inverse Kinematics}}{\overset{\text{Kinematics } T}{\rightleftarrows}} T = \begin{bmatrix} R & P \\ 0_{1\times 3} & 1 \end{bmatrix} $$

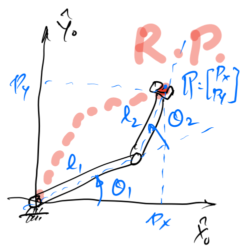
Joint Space Cartesian Space

Inverse Kinematics deals with the situation of giving position and orientation of the E.E. w.r.t. the base frame, i.e., ${}^{0}_{tip}T$, find the corresponding joint positions Θ?

Given

A. Solvability

a. Nonlinear Nature of the problem



Example: The 2DOF planar robot

Inverse kinematic problem:

Given $\mathbb{P} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$, find $\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = ?$

Solution:

$$P_y = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \quad (1)$$

$$P_x = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \quad (2)$$

$$\Rightarrow \quad P_y - l_1 \sin\theta_1 = l_2 \sin(\theta_1+\theta_2) \quad (3)$$

$$P_x - l_1 \cos\theta_1 = l_2 \cos(\theta_1+\theta_2) \quad (4)$$

$(3)^2 + (4)^2$:

$$(P_y - l_1 S\theta_1)^2 + (P_x - l_1 C\theta_1)^2 = l_2 \Big[\overbrace{S^2\theta_{12} + C^2\theta_{12}}^{1} \Big] = l_2$$

$$\Rightarrow \text{let} \quad P_x \cdot C\theta_1 + P_y \cdot S\theta_1 = \frac{P_x^2 + P_y^2 + l_1^2 - l_2^2}{2 l_1} \triangleq q \quad (5)$$

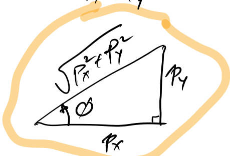
To solve this type of question, we have two approachs.

#1 Approach (Geometric Approach)

$$(5) \Rightarrow \frac{P_x}{\sqrt{P_x^2+P_y^2}} \cos\theta_1 + \frac{P_y}{\sqrt{P_x^2+P_y^2}} \sin\theta_1 = \frac{q}{\sqrt{P_x^2+P_y^2}} \triangleq k \quad (6)$$

let's construct a right triangle:

then $\cos\phi = \dfrac{P_x}{\sqrt{P_x^2+P_y^2}}$, $\sin\phi = \dfrac{P_y}{\sqrt{P_x^2+P_y^2}}$



$$(6) \Rightarrow \boxed{\cos\phi \cdot \cos\theta_1 + \sin\phi \cdot \sin\theta_1} = k \quad (7)$$

$$\Rightarrow \cos(\phi - \theta_1) = \cos(\theta_1 - \phi) = k$$

$$\Rightarrow \phi - \theta_1 = \cos^{-1} k \Rightarrow \theta_1 = \phi - \cancel{\cos^{-1} k}$$

But this type of solution ($\cos^{-1} k$) is not recommended

Instead we solve eqn. (7) in the following way:

Since $\cos(\theta_1 - \phi) = k$

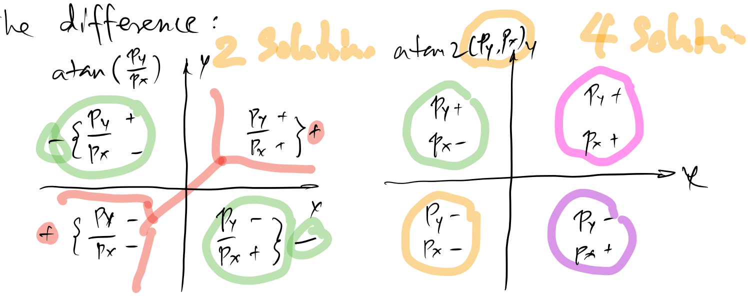
then $\sin(\theta_1 - \phi) = \pm\sqrt{1-k^2}$ $\quad (\cos^2\theta + \sin^2\theta = 1)$

$$\Rightarrow \tan(\theta_1 - \phi) = \frac{\sin(\theta_1-\phi)}{\cos(\theta_1-\phi)} = \frac{\pm\sqrt{1-k^2}}{k}$$

$$\Rightarrow \theta_1 = \phi + \tan^{-1}\frac{\pm\sqrt{1-k^2}}{k}$$ not the best

Note in MATLAB, there are two functions for $\tan^{-1}$: atan and atan2. What is the difference?

The difference:



Therefore, the best solution is

$$\frac{\sin(\theta_1-\phi)}{\cos(\theta_1-\phi)} = \tan(\theta_1-\phi) = \frac{\pm\sqrt{1-k^2}}{k}$$

$$\Rightarrow \quad \theta_1-\phi = \operatorname{atan2}(\pm\sqrt{1-k^2}, k)$$

$$\Rightarrow \quad \theta_1 = \phi + \underline{\operatorname{atan2}(\pm\sqrt{1-k^2}, k)}$$

It gives the solution in exactly the quadrant.

$\Rightarrow$ Going back to original equation to find θ_2.

#2 Approach (Algebraic Approach)

Using mathematical substitutions:

$$\cos\theta_1 = \frac{1-\tan^2(\frac{\theta_1}{2})}{1+\tan^2(\frac{\theta_1}{2})}, \quad \sin\theta_1 = \frac{2\cdot\tan(\frac{\theta_1}{2})}{1+\tan^2(\frac{\theta_1}{2})}$$

$$(5) \Rightarrow \quad p_x\cdot\frac{1-\tan^2(\frac{\theta_1}{2})}{1+\tan^2(\frac{\theta_1}{2})} + p_y\cdot\frac{2\tan(\frac{\theta_1}{2})}{1+\tan^2(\frac{\theta_1}{2})} = q.$$

$$\Rightarrow \quad \tan\frac{\theta_1}{2} = f(p_x, p_y, q) \Rightarrow \theta_1 \Rightarrow \theta_2 \ ?$$

Solutions

B. Existence of Solutions

- Concept of workspace: The workspace of a robot is a part of the Cartesian space that can be reached by the robot e.e.

- Two types of workspace:
  * Dextrous workspace: workspace that can be reached by the robot e.e. with any orientation.
  * Reachable Workspace: workspace that can be reached by the robot e.e. with at least one orientation.

Obviously, dextrous workspace $\subseteq$ reachable workspace.

Conclusions:

* If the position of the robot e.e. is required to be outside of the reachable workspace, no solution exists.
* If the positions are inside the dextrous workspace, there are infinitely many solutions.
* If the positions are inside the reachable workspace, at least one solution exists.

C. Pieper's Theorem:

For any 6 DOF robot manipulator a closed-form solution for its inverse kinematic problems exist if there are three neighboring joint axes intersecting at one point.