Part II. Robot Manipulator Dynamics (2nd-order Analysis)

Problem to be studied:

Task Planning → Given $\Theta, \dot{\Theta}, \ddot{\Theta}$ $\xrightarrow{\text{Dynamics}}$ Find the joint torques $\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$

$\xleftarrow{}$ Dynamic Simulation

+ animation

+ Control τ_c

A. Mass Distribution

* Inertia Tensor is used to characterize the mass distribution of a rigid body moving in 3D space, which can be thought of as a generalization of the scalar moment of an object.

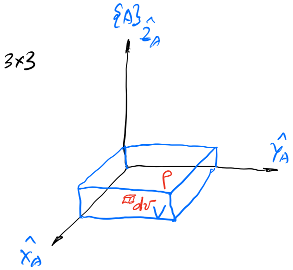
Inertia tensor is defined relative to a frame, $\{A\}$, as

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}_{3\times 3} $$

Let's look at the 3-D rigid body:

where dv - volumn element

ρ - material density



and then the inertia tensor elements are defined as follows:

$$ \left.\begin{aligned} I_{xx} &= \iiint_v (y^2+z^2)\cdot\rho\cdot dv \\ I_{yy} &= \iiint_v (x^2+z^2)\cdot\rho\cdot dv \\ I_{zz} &= \iiint_v (x^2+y^2)\cdot\rho\cdot dv \end{aligned}\right\} $$

mass moment of the inertia tensor

$$ \left.\begin{aligned} I_{xy} &= \iiint_v x\cdot y\cdot\rho\cdot dv \\ I_{xz} &= \iiint_v x\cdot z\cdot\rho\cdot dv \\ I_{yz} &= \iiint_v y\cdot z\cdot\rho\cdot dv \end{aligned}\right\} $$

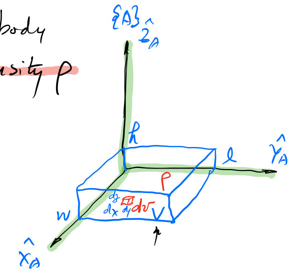
mass product of the inertia tensor

Because the inertia tensor is frame dependent, we may choose a frame such that $I_{xy} = I_{xz} = I_{yz} = 0$. Then the inertia tensor becomes diagonal:

$$ {}^{A}I = \begin{bmatrix} I_{xx} & & 0 \\ & I_{yy} & \\ 0 & & I_{zz} \end{bmatrix} = diag\{I_{xx}, I_{yy}, I_{zz}\} $$

The corresponding axes of the frame are called "principal axes". The mass moments are called "principal moments".

Example: Find the inertia tensor of the body on the right with uniform density ρ and w.r.t. $\{A\}$



Solution: We have $dv = dx\cdot dy\cdot dz$

and $m = \rho\cdot h\cdot w\cdot l = \rho\cdot V$

and $I_{xx} = \iiint_v (y^2+z^2)\cdot\rho\cdot dv = \int_0^h\int_0^l\int_0^w (y^2+z^2)\cdot\rho\cdot dx\, dy\, dz$

$$ = \frac{\overbrace{\rho\cdot w\cdot l\cdot h}^{m}}{3}\cdot[l^2+h^2] = \frac{m}{3}(l^2+h^2) $$

Rewrite the terms, we have

$I_{yy} = \frac{m}{3}(w^2+h^2)$, $I_{zz} = \frac{m}{3}(w^2+l^2)$

and $I_{xy} = \iiint_V x \cdot y \cdot \rho \cdot dv = \int_0^h \int_0^l \int_0^w x \cdot y \cdot \rho \cdot dx\,dy\,dz = \frac{m}{4} wl$

Rewrite the terms, we have

$$I_{xz} = \frac{m}{4} hw, \qquad I_{yz} = \frac{m}{4} \cdot hl$$

$$\Rightarrow \quad {}^A I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & -\frac{m}{4} wl & -\frac{m}{4} hw \\ -\frac{m}{4} wl & \frac{m}{3}(w^2+h^2) & -\frac{m}{4} hl \\ -\frac{m}{4} hw & -\frac{m}{4} hl & \frac{m}{3}(w^2+l^2) \end{bmatrix}$$

* Parallel Axis Theorem:

If frame {A} is obtained from {C} by a translation of

$${}^A P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

where the origin of {C} is the center of mass of the object, then

$$\underset{3\times3}{{}^A I} = \underset{3\times3}{{}^C I} + \underset{1\times1}{m}\Big[\underbrace{\underset{1\times3}{{}^A P_c^T} \cdot \underset{3\times1}{{}^A P_c}}_{1\times1} \underset{3\times3}{I_3} - \underbrace{\underset{3\times1}{{}^A P_c} \cdot \underset{1\times3}{{}^A P_c^T}}_{3\times3}\Big]_{3\times3} \quad \text{where } I_3 = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

$\Rightarrow {}^A I_{zz} = {}^C I_{zz} + m(x_c^2 + y_c^2) \Rightarrow {}^C I_{zz} = {}^A I_{zz} - m(x_c^2 + y_c^2)$

$\Rightarrow {}^A I_{xy} = {}^C I_{xy} - m x_c y_c \quad \Rightarrow {}^C I_{xy} = {}^A I_{xy} + m x_c y_c$

Example: {C} is the frame at the center of mass:

$${}^A \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} w \\ l \\ h \end{bmatrix}$$

Then ${}^C I_{zz} = {}^A I_{zz} - m\left(\frac{w^2}{4} + \frac{l^2}{4}\right) = \frac{m}{12}(w^2 + l^2)$. ${}^C I_{xx}$, ${}^C I_{yy}$.

${}^C I_{xy} = {}^A I_{xy} + m x_c y_c = -\frac{m}{4} wl + \frac{m}{4} w \cdot l = 0$, ${}^C I_{xz} = 0$, ${}^C I_{yz} = 0$

$$\Rightarrow \quad {}^C I = \begin{bmatrix} \frac{m}{12}(l^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(w^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2+h^2) \end{bmatrix}$$

$\Rightarrow \hat{x}_c, \hat{y}_c, \hat{z}_c$ are principal axes

I_{xx}, I_{yy}, I_{zz} are principal moments.

B. Lagrangian Formulation of Robot Manipulator Dynamics

* Newton-Euler formulation is a "force balance" and "torque balance" approach to robot manipulator dynamics.

iterative formula
Good for computing

* Lagrangian formulation is an "energy-based" approach to robot manipulator dynamics

closed-form formula, good for analysis

For the same manipulator, both approaches will give you the same equation of dynamics. We will limit our discussion to the Lagrangian formulation for the case of a serial-chain rigid-body robot manipulator with rigid links.

Since the Lagrangian dynamics is an energy-based approach, We will first define kinetic and potential energies.

The energy function is always a scalar function ?

The kinetic energy of the i-th link, k_i, can be expressed as

$$\Rightarrow \quad k_i = \underbrace{\frac{1}{2} m_i \cdot \underbrace{v_{c_i}^T \cdot v_{c_i}}_{1\times1}}_{} + \frac{1}{2}\, \underbrace{{}^i\omega_i^T \cdot {}^cI_i \cdot {}^i\omega_i}_{1\times1}$$

where v_{c_i} – linear velocity of the center of mass of link i

$\frac{1}{2} m_i \cdot v_{c_i}^T \cdot v_{c_i}$ – kinetic energy due to the linear velocity of the link's center of mass

$\frac{1}{2}\, {}^i\omega_i^T \cdot {}^cI_i \cdot {}^i\omega_i$ – kinetic energy due to the angular velocity of the link.

m_i – the mass of link i

${}^cI_i$ – the inertia tensor of link i w.r.t. $\{c\}$, which is located at the center of mass of link i.

${}^i\omega_i$ – is the angular velocity of link i

The total kinetic energy of the manipulator is the sum of the kinetic energies of all the links:

$$k = \sum_{i=1}^{n} k_i = k(\Theta, \dot{\Theta})$$

Since the v_{c_i} and ω_i are functions of Θ and $\dot{\Theta}$, the kinetic energy of a manipulator is a scalar function of Θ and $\dot{\Theta}$.

Since all energy functions must be positive,

$$\Rightarrow \quad \underset{1\times1}{k(\Theta, \dot{\Theta})} = \frac{1}{2} \underbrace{\underset{1\times n}{\dot{\Theta}^T} \underset{n\times n}{M(\Theta)} \cdot \underset{n\times1}{\dot{\Theta}}}_{1\times1} > 0 \qquad \underset{n\times n}{M(\Theta)} > 0$$

the matrix is P.D. positive definite.

The potential energy of the i-th link, u_i, can be expressed

$$\underset{1\times1}{u_i} = -\underset{1\times1}{m_i} \cdot \underbrace{\underset{1\times3}{{}^0g^T} \cdot \underset{3\times1}{{}^0P_{c_i}}}_{1\times1} + \underset{1\times1}{u_{ref}} \geq 0$$

where: ${}^0g \in R^{3\times1}$ is gravitational acceleration

${}^0P_{c_i} \in R^{3\times1}$ is the position vector locating the com of link i (Center of Mass)

$u_{ref} \in R^{1\times1}$ is a constant chosen such that $\min u_i = 0$.

The total potential energy stored in the robot manipulator is the sum of the potential energies of all links:

$$u = \sum_{i=1}^{n} u_i \geq 0$$

$\Rightarrow$ Potential energy of a robot manipulator is a scalar function of the joint position ONLY, i.e. $u(\Theta)$.