The Lagrange dynamics formulation provides a means of doing the equation of motion from a scalar function called "Lagrangian", which is defined as the difference between the kinetic and potential energies of a mechanical system:

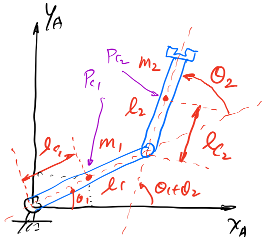
$$\underset{1\times1}{\mathcal{L}}(\Theta,\dot{\Theta}) = \underset{1\times1}{k}(\Theta,\dot{\Theta}) - \underset{1\times1}{u}(\Theta)$$

The equation of motion (dynamic equation) for the robot manipulator can be expressed as:

$$\frac{d}{dt}\cdot\left.\frac{\partial \mathcal{L}_{1\times1}(\Theta_{n\times1},\dot{\Theta}_{n\times1})}{\partial \dot{\Theta}_{n\times1}}\right|_{n\times1} - \left.\frac{\partial \mathcal{L}_{1\times1}(\Theta_{n\times1},\dot{\Theta}_{n\times1})}{\partial \Theta_{n\times1}}\right|_{n\times1} = \tau_{n\times1}$$

where $\tau \in R^{n\times1}$ is the vector of the joint torques/forces (rotational joint / prismatic joint).

Example: The equation of motion of the 2 DOF planar robot.



Solution: Because it's planar robot, then

$${}^{C_1}I_1 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & I_1 \end{bmatrix}, \quad {}^{C_2}I_2 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & I_2 \end{bmatrix}$$

* Kinetic energy: $k_i = \frac{1}{2} m_i \cdot v_{c_i}^T \cdot v_{c_i} + \frac{1}{2}\, {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i$

$\Rightarrow v_{c_i} = \dfrac{dP_{c_i}}{dt}$, ω_i can be obtained via velocity propagation.

$\Rightarrow v_{c_1} = \dfrac{dP_{c_1}}{dt} = J_{v_{c_1}} \cdot \dot{\Theta}$, $v_{c_2} = \dfrac{dP_{c_2}}{dt} = J_{v_{c_2}} \cdot \dot{\Theta}$

Since $P_{c_1} = \begin{bmatrix} l_{c_1}\cdot\cos\theta_1 \\ l_{c_1}\cdot\sin\theta_1 \end{bmatrix} \Rightarrow v_{c_1} = \dfrac{dP_{c_1}}{dt} = \underbrace{\begin{bmatrix} -l_{c_1}\cdot S_1 & 0 \\ l_{c_1}\cdot C_1 & 0 \end{bmatrix}}_{J_{v_{c_1}}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

and $P_{c_2} = \begin{bmatrix} l_1C_1 + l_{c_2}\cdot C_2 \\ l_1S_1 + l_{c_2}\cdot S_2 \end{bmatrix} \Rightarrow v_{c_2} = \dfrac{dP_{c_2}}{dt} = \underbrace{\begin{bmatrix} -l_1S_1 - l_{c_2}S_{12} & -l_{c_2}\cdot S_{12} \\ l_1C_1 + l_{c_2}\cdot S_{12} & l_{c_2}\cdot C_{12} \end{bmatrix}}_{J_{v_{c_2}}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

${}^1\omega_1 = \dot{\theta}_1 \cdot {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$, ${}^2\omega_2 = (\dot{\theta}_1 + \dot{\theta}_2)\cdot {}^2\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1+\dot{\theta}_2 \end{bmatrix}$

So the kinetic and potential energies of the robot are:

$$\begin{aligned}
k_1 &= \frac{1}{2} m_1 \cdot v_{c_1}^T \cdot v_{c_1} + \frac{1}{2}\, {}^1\omega_1^T \cdot {}^{C_1}I_1 \cdot {}^1\omega_1 \\
&= \frac{1}{2} m_1 \cdot [J_{v_{c_1}}\cdot\dot{\Theta}]^T \cdot J_{v_{c_1}}\cdot\dot{\Theta} + \frac{1}{2}[0 \ 0 \ \dot{\theta}_1]\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & I_1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\
&= \frac{1}{2} m_1 \cdot \dot{\Theta}^T \cdot J_{v_{c_1}}^T \cdot J_{v_{c_1}} \cdot \dot{\Theta} + \frac{1}{2} I_1 \dot{\theta}_1^2 \\
&= \frac{1}{2} m_1 l_{c_1}^2 \cdot \dot{\theta}_1^2 + \frac{1}{2} I_1 \cdot \dot{\theta}_1^2
\end{aligned}$$

Similarly, we have $k_2 = \frac{1}{2} m_2 v_{c_2}^T \cdot v_{c_2} + \frac{1}{2}\, {}^2\omega_2^T \cdot {}^{C_i}I_2 \cdot {}^2\omega_2$

$$\Rightarrow k_2 = \frac{1}{2} m_2 \left[(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cdot C_2)\dot{\Theta}_1 + 2(l_{c_2}^2 + l_1 l_{c_2} C_2)\dot{\theta}_1\dot{\theta}_2 + l_{c_2}^2 \dot{\theta}_2^2 \right] + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

The total kinetic energy of the robot manipulator is

$$k = k_1 + k_2$$

The potential energy is giving as

$$\mathcal{U}_i = -m_i \cdot {}^0g^T \cdot {}^0P_{c_i} + \mathcal{U}_{ref_i}$$

$$\Rightarrow \mathcal{U}_1 = m_1 \cdot g \cdot l_{c_1} \cdot S_1\theta_1 \qquad \left({}^0g^T = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \right)$$

and $\mathcal{U}_2 = m_2 g \cdot [l_1 S_1 + l_{c_2} \cdot S_{12}]$

The total potential energy of the robot manipulator is

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$$

Then the Lagrangian of the robot manipulator is given as

$$\mathcal{L}(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - \mathcal{U}(\Theta)$$

and then the Lagrangian formulation of robot manipulator dynamics is obtained as

$$\frac{d}{dt} \cdot \frac{\partial \mathcal{L}(\Theta, \dot{\Theta})}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}(\Theta, \dot{\Theta})}{\partial \Theta} = \tau$$

$$\Rightarrow \frac{d}{dt} \left[\frac{\partial k(\Theta, \dot{\Theta})}{\partial \dot{\Theta}} - \frac{\partial \mathcal{U}(\Theta)}{\partial \dot{\Theta}} \right] - \frac{\partial k(\Theta, \dot{\Theta})}{\partial \Theta} + \frac{\partial \mathcal{U}(\Theta)}{\partial \Theta} = \tau$$

Then the Lagrangian formulation of robot manipulator is

dynamics

$$\frac{d}{dt} \cdot \frac{\partial k(\Theta, \dot{\Theta})}{\partial \dot{\Theta}} - \frac{\partial k(\Theta, \dot{\Theta})}{\partial \Theta} + \frac{\partial \mathcal{U}(\Theta)}{\partial \Theta} = \tau$$

1st 2nd 3rd

Then

$$\frac{\partial k(\Theta, \dot{\Theta})}{\partial \dot{\Theta}} = \begin{bmatrix} \frac{\partial k}{\partial \dot{\theta}_1} \\ \frac{\partial k}{\partial \dot{\theta}_2} \end{bmatrix} = \begin{bmatrix} m_1 l_{c_1}^2 \dot{\theta}_1 + I_1 \cdot \dot{\theta}_1 + m_2(l_1^2 + l_{c_1}^2 + 2 l_1 l_{c_1}) \dot{\theta}_1 + m_2(l_{c_2}^2 + l_1 l_{c_2} \cdot C_2)\dot{\theta}_2 + I_1(\dot{\theta}_1 + \dot{\theta}_2) \\ m_2[l_{c_2}^2 + l_1 l_{c_2} \cdot C_2]\dot{\theta}_1 + m_2 l_{c_2}^2 \dot{\theta}_2 + I_2(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

time derivative:

$$\frac{d}{dt} \cdot \frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} [m_1 l_1^2 + I_1 + m_2(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cdot C_2) + I_2]\ddot{\theta}_1 + [m_2(l_{c_2}^2 + l_1 l_{c_2} \cdot C_2) + I_2]\ddot{\theta}_2 - 2 m_2 l_1 l_{c_2} \cdot S_2 \cdot \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_{c_2} \cdot S_1 \cdot \dot{\theta}_1 \cdot \dot{\theta}_2 \\ [m_2(l_{c_2}^2 + l_1 l_{c_2} \cdot C_2) + I_2]\ddot{\theta}_1 + (m_2 l_{c_2}^2 + I_2)\ddot{\theta}_2 - m_2 l_1 l_{c_2} \cdot S_1 \dot{\theta}_1 \cdot \dot{\theta}_2 \end{bmatrix}$$

1st

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} \frac{\partial k}{\partial \theta_1} \\ \frac{\partial k}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -m_2 l_1 l_{c_2} \cdot S_2 \dot{\theta}_1^2 - m_2 l_1 l_{c_2} S_2 \cdot \dot{\theta}_1 \cdot \dot{\theta}_2 \end{bmatrix}$$

2nd

$$\frac{\partial \mathcal{U}}{\partial \Theta} = \begin{bmatrix} \frac{\partial \mathcal{U}}{\partial \theta_1} \\ \frac{\partial \mathcal{U}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} (m_1 l_{c_1} + m_2 l_1) \cdot g \cdot C_1 + m_2 l_{c_2} \cdot g \cdot C_{12} \\ m_2 l_{c_2} \cdot g \cdot C_{12} \end{bmatrix}$$

3rd

Then the dynamic eqn. is $\frac{d}{dt} \cdot \frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial \mathcal{U}}{\partial \Theta} = \tau$!

Express in matrix form:

1st 2nd 3rd

$$\underset{n \times n}{M(\Theta)} \underset{n \times 1}{\ddot{\Theta}} + \underset{n \times 1}{V(\Theta, \dot{\Theta})} + \underset{n \times 1}{G(\Theta)} = \underset{n \times 1}{\tau}$$

where

$$M(\Theta) = \begin{bmatrix} m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cdot C_2) + I_1 + I_2 & m_2(l_{c_2}^2 + l_1 l_{c_2} \cdot C_2) + I_2 \\ m_2(l_{c_2}^2 + l_1 l_{c_2} \cdot C_2) + I_2 & m_2 l_{c_2}^2 + I_2 \end{bmatrix}_{2\times 2}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -2 m_2 l_1 l_{c_2} \cdot S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_{c_2} S_2 \dot{\theta}_2^2 \\ m_2 l_1 l_{c_2} \cdot S_2 \cdot \dot{\theta}_1^2 \end{bmatrix}_{2\times 1}$$

$$G(\Theta) = \begin{bmatrix} m_1 g \cdot l_{c_1} + m_2 g (l_1 C_1 + l_{c_2} \cdot C_2) \\ m_2 g \cdot l_{c_2} \cdot C_2 \end{bmatrix}_{2\times 1}$$

and $$\ddot{\Theta} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix}_{n\times 1}$$

So the robot manipulator dynamics equation is a non-linear matrix differential equation.

Where:

$M(\Theta) \in R^{n\times n}$ is the mass matrix. It's symmetric, positive definite

$V(\Theta, \dot{\Theta}) \in R^{n\times 1}$ is the vector of Contrifugal and Coriolis terms.

$G(\Theta) \in R^{n\times 1}$ is the vector of gravitational terms.

In addition to the following STANDARD form:

$$\boxed{M(\Theta) \cdot \ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) = \tau}$$

there are other alternative expressions that we may use:

① $$\underset{n\times n}{M(\Theta)} \cdot \underset{n\times 1}{\ddot{\Theta}} + \underset{n\times n}{V(\Theta, \dot{\Theta})} \cdot \underset{n\times 1}{\dot{\Theta}} + \underset{n\times 1}{G(\Theta)} = \underset{n\times 1}{\tau}$$

② $$\underset{n\times n}{M(\Theta)} \cdot \underset{n\times 1}{\ddot{\Theta}} + \underset{n\times n}{C(\Theta)} \cdot \underset{n\times 1}{\dot{\Theta}^2} + \underset{n\times n}{B(\Theta)} \cdot \underset{n\times 1}{\dot{\Theta}\dot{\Theta}} + \underset{n\times 1}{G(\Theta)} = \underset{n\times 1}{\tau}$$

Example: Our previous example of 2DOF robot has

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -2 m_2 l_1 l_{c_2} \cdot S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_{c_2} S_2 \dot{\theta}_2^2 \\ m_2 l_1 l_{c_2} \cdot S_2 \cdot \dot{\theta}_1^2 \end{bmatrix}_{2\times 1}$$

① $$\Rightarrow V(\Theta, \dot{\Theta}) = \begin{bmatrix} -2 m_2 l_1 l_{c_2} S_2 \cdot \dot{\theta}_2 & -m_2 l_1 l_{c_2} \cdot S_2 \cdot \dot{\theta}_2 \\ m_2 l_1 l_{c_2} \cdot S_2 \cdot \dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

② $$\Rightarrow \underset{2\times 1}{V(\Theta, \dot{\Theta})} = \underbrace{\begin{bmatrix} 0 & -m_2 l_1 l_{c_2} \cdot S_2 \\ m_2 l_1 l_{c_2} S_2 & 0 \end{bmatrix}_{2\times 2}}_{C} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}_{2\times 1} + \underbrace{\begin{bmatrix} -2 m_2 l_1 l_{c_2} \cdot S_2 \\ 0 \end{bmatrix}_{2\times 1}}_{B} \left[\dot{\theta}_1 \dot{\theta}_2\right]_{1\times 1}$$

* Regressor Formulation of Robot Dynamics

Robot dynamics: $\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$

can be linearized in terms of a vector of the robot manipulator dynamic parameters.

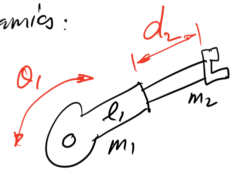
$$\underbrace{\underset{n\times1}{M(\Theta)}\,\underset{n\times1}{\ddot{\Theta}}}_{n\times1} + \underset{n\times1}{V(\Theta,\dot{\Theta})} + \underset{n\times1}{G(\Theta)} = \overset{\text{fix}}{\boxed{\underbrace{\underset{n\times r}{Y(\Theta,\dot{\Theta},\ddot{\Theta})}\cdot\underset{r\times1}{\varphi}}_{n\times1}}}$$

where $Y(\Theta,\dot{\Theta},\ddot{\Theta}) \in R^{n\times r}$ is the manipulator regressor matrix

$\varphi \in R^{r\times 1}$ is a vector of robot dynamic parameters.

Example: 2DoF RP robot dynamics:

$$I_1 = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & I_{zz_1} \end{bmatrix} \quad I_2 = \begin{bmatrix} * & * & * \\ \Delta & * & * \\ * & * & I_{zz_2} \end{bmatrix}$$



$$\Rightarrow \tau_1 = m_1 l_1^2 \ddot{\theta}_1 + I_{zz_1}\ddot{\theta}_1 + I_{zz_2}\ddot{\theta}_1 + m_2 d_2^2 \ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + m_1 l_1 g c_1 + m_2 d_2 g c_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g s_1$$

The regressor formulation is

$$\tau = \underset{2\times1}{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}} = \underbrace{\underset{2\times4}{\begin{bmatrix} l_1^2\ddot{\theta}_1 + l_1 g c_1 & d_2^2\ddot{\theta}_1 + 2d_2\dot{\theta}_1\dot{d}_2 + d_2 g c_1 & \ddot{\theta}_1 & \ddot{\theta}_1 \\ 0 & \ddot{d}_2 - d_2\dot{\theta}_1^2 + g s_1 & 0 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} m_1 \\ m_2 \\ I_{zz_1} \\ I_{zz_2} \end{bmatrix}_{r=4}}}_{2\times1}$$

where

$$Y = \begin{bmatrix} l_1^2\ddot{\theta}_1 + l_1 g c_1 & d_2^2\ddot{\theta}_1 + 2d_2\dot{\theta}_1\dot{d}_2 + d_2 g c_1 & \ddot{\theta}_1 & \ddot{\theta}_1 \\ 0 & \ddot{d}_2 - d_2\dot{\theta}_1^2 + g s_1 & & \end{bmatrix}, \quad \varphi = \begin{bmatrix} m_1 \\ m_2 \\ I_{zz_1} \\ I_{zz_2} \end{bmatrix}$$