Formulation of Dynamics in Cartesian Space

Recall:

1° Given X, find Θ $\Longrightarrow$ Inverse Kinematics

2° Given $\dot{X}$, find $\dot{\Theta}$ $\Longrightarrow$ Inverse Jacobian

$$\dot{\Theta} = J^{-1}(\Theta)\cdot\dot{X}$$

3° Given $\ddot{X}$, find $\ddot{\Theta}$ $\Longrightarrow$?

Since $\dot{X} = J(\Theta)\cdot\dot{\Theta}$, then $\frac{d}{dt}$ on both sides:

$$\frac{d}{dt}(\dot{X}) = \frac{d}{dt}\left(J(\Theta)\cdot\dot{\Theta}\right)$$

$$\Rightarrow \ddot{X} = \dot{J}(\Theta)\cdot\dot{\Theta} + J(\Theta)\cdot\ddot{\Theta}$$

$$\ddot{X} - \dot{J}(\Theta)\cdot\dot{\Theta} = J(\Theta)\cdot\ddot{\Theta}, \quad \times\ J^{-1}(\Theta) \text{ from left:}$$

$$\Rightarrow \ddot{\Theta} = J^{-1}(\Theta)\left[\ddot{X} - \dot{J}(\Theta)\cdot\dot{\Theta}\right]$$

The total kinetic energy of the manipulator is:

$$\frac{1}{2}\dot{\Theta}^T M_\Theta(\Theta)\cdot\dot{\Theta} \quad \text{in Joint space}$$

|| for the same robot

and $\frac{1}{2}\dot{X}^T M_x(\Theta)\cdot\dot{X}$ in Cartesian space

$$\Rightarrow \cancel{\frac{1}{2}}\dot{\Theta}^T M_\Theta(\Theta)\dot{\Theta} = \cancel{\frac{1}{2}}\dot{X}^T M_x(\Theta)\cdot\dot{X}$$

$$\Rightarrow \left[J^{-1}(\Theta)\cdot\dot{X}\right]^T\cdot M_\Theta(\Theta)\cdot J^{-1}(\Theta)\cdot\dot{X} = \dot{X}^T M_x(\Theta)\cdot\dot{X}$$

$$\Rightarrow \cancel{\dot{X}^T}\cdot J^{-T}(\Theta)\cdot M_\Theta(\Theta)\cdot J^{-1}(\Theta)\cdot\cancel{\dot{X}} = \cancel{\dot{X}^T} M_x(\Theta)\cdot\cancel{\dot{X}}$$

$$\Rightarrow M_x(\Theta) = J^{-T}(\Theta)\cdot M_\Theta(\Theta)\cdot J^{-1}(\Theta) \quad *$$

We can try to express the dynamic equation in Cartesian space:

$$M_x(\Theta)\cdot\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta) = \mathcal{F} \quad (\tau = J^T\mathcal{F})$$

$$J^{-T}\cdot M_\Theta(\Theta)\cdot J^{-1}\left[\dot{J}\dot{\Theta} + J\ddot{\Theta}\right] + V_x(\Theta,\dot{\Theta}) + G_x(\Theta) = \mathcal{F}$$

$$\xRightarrow{\times J^T} M_\Theta(\Theta)\cdot J^{-1}\left[\dot{J}\dot{\Theta} + J\ddot{\Theta}\right] + J^T V_x(\Theta,\dot{\Theta}) + J^T G_x(\Theta) = J^T\mathcal{F} = \tau$$

$$\Rightarrow M_\Theta(\Theta)\ddot{\Theta} + M_\Theta(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^T V_x + J^T G_x = \tau$$

Compare: $M_\Theta(\Theta)\ddot{\Theta} + V_\Theta(\Theta,\dot{\Theta}) + G_\Theta(\Theta) = \tau$

Then

$$V_\Theta(\Theta,\dot{\Theta}) = M_\Theta(\Theta)J^{-1}\dot{J}\dot{\Theta} + J^T V_x(\Theta,\dot{\Theta}) \quad **$$

$$G_\Theta(\Theta,\dot{\Theta}) = J^T G_x(\Theta) \quad ***$$

Then the Cartesian space dynamic equation is

$$M_x(\Theta)\ddot{X} + V_x(\Theta,\dot{\Theta}) + G_x(\Theta) = \mathcal{F}$$

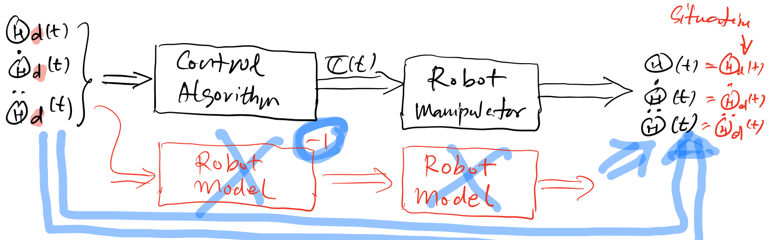
Part III Control of Robot Manipulators

The Control Problem: Given desired trajectory, $\Theta_d(t)$, $\dot{\Theta}_d(t)$ and $\ddot{\Theta}_d(t)$ in joint space or $X_d(t)$, $\dot{X}_d(t)$ and $\ddot{X}_d(t)$ in Cartesian space, find the required joint torques or forces $\tau(t)$ or $\mathcal{F}(t)$ that will cause the desired dynamic motion:

$$\Theta_d, \dot{\Theta}_d, \ddot{\Theta}_d \text{ or } X_d, \dot{X}_d, \ddot{X}_d \xrightarrow[\text{algorithm}]{\text{Control}} \tau(t) \text{ or } \mathcal{F}(t)$$

cause

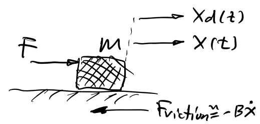
A. Feedback and closed-loop control



Since the robot dynamic model is always a simplified approximation of the true dynamics of the real robot manipulator, so there are unmodeled dynamic model components of the real robot manipulator that do not present in the dynamic model equation!

Example: The dynamic model of the example system is:

$$m\ddot{x} + B\dot{x} = F, \quad x(0) = 0, \ \dot{x}(0) = 0$$



Assume $X_d(t)$ is given, we need to make sure, $x(t) \xrightarrow{t\to\infty} X_d(t)$.

Let's use the model inverse as the control algorithm:

$$F(t) = \hat{m}\ddot{X}_d(t) + \hat{B}\dot{X}_d(t)$$

where $\hat{m}$ and $\hat{B}$ are the (estimated) model parameters of m and B. Apply the above (model inverse) control algorithm to the system:

$$\underbrace{m\ddot{x} + B\dot{x}}_{\text{real system}} = \underbrace{\hat{m}\ddot{X}_d(t) + \hat{B}\dot{X}_d(t)}_{\text{model}}$$

$$\Rightarrow \quad m\ddot{x} - m\ddot{X}_d + B\dot{x} - B\dot{X}_d = \hat{m}\ddot{X}_d - m\ddot{X}_d + \hat{B}\dot{X}_d - B\dot{X}_d$$

$$\Rightarrow \quad m(\ddot{x} - \ddot{X}_d) + B(\dot{x} - \dot{X}_d) = (\hat{m} - m)\ddot{X}_d + (\hat{B} - B)\dot{X}_d$$

Define error $e(t) = x(t) - X_d(t)$, $\dot{e} = \dot{x} - \dot{X}_d$, $\ddot{e} = \ddot{x} - \ddot{X}_d$,

Then $m\ddot{e} + B\dot{e} = (\hat{m} - m)\ddot{X}_d + (\hat{B} - B)\dot{X}_d$

This equation in control theory is called "error dynamics"

① If we have precise dynamic model: $\hat{m} = m$, $\hat{B} = B$, then the error dynamics becomes

$$m\ddot{e} + B\dot{e} = 0 \Rightarrow e(t) \xrightarrow{t\to\infty} 0 \Rightarrow x(t) \xrightarrow{t\to\infty} X_d(t)$$

Control work!

② If we do not have precise dynamic model,

$\hat{m} \neq m$, $\hat{B} \neq B$, i.e., $\hat{m} = m$, $\hat{B} - B = 0.1 \neq 0$, and $x_d(t) = \frac{t^2}{2}$, $m=1$, $B=1$, $\dot{x}_d = t$, then the error dynamic equation is

$$m\ddot{e} + B\dot{e} = 0 + 0.1\cdot t \neq 0$$

⇒ $m\ddot{e} + B\dot{e} = 0.1\cdot t$ has a solution as follows:

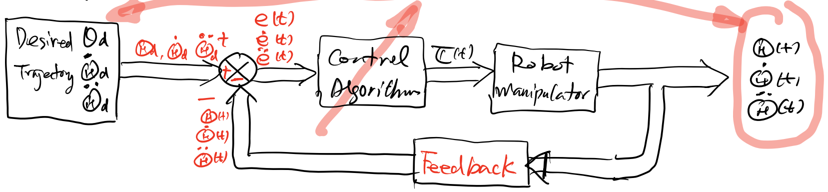
$$e(t) = 0.1\cdot\int_0^t (1-e^{-(t-\tau)})\cdot\tau\, d\tau = \underbrace{0.05t^2}_{t\to\infty:\ \to\infty} - \underbrace{0.1t}_{t\to\infty:\ \to\infty} - \underbrace{e^{-t}}_{t\to\infty:\ \to 0} \to \infty$$

$$e(t) \xrightarrow{t\to\infty} \infty$$

⇒ We cannot use dynamic model inverse as the control algorithm if the dynamic model is not precisely and exactly known.

* The control algorithm:

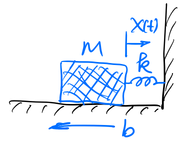
The key issue is designing a practically useful control algorithm using feedback:



objective: $\left.\begin{matrix}\Theta\\ \dot\Theta\\ \ddot\Theta\end{matrix}\right\} \longrightarrow \begin{matrix}\Theta_d\\ \dot\Theta_d\\ \ddot\Theta_d\end{matrix}$

IB Second-Order Linear Systems

Example: Mass-Spring-Damper System



The dynamic model of the system is

$$m\ddot{x}(t) + b\dot{x}(t) + k\,x(t) = 0$$

Taking Laplace transform on both sides with zero initial conditions, then

$$(ms^2 + bs + k)\,X(s) = 0$$

The characteristic equation of the above system is

$$ms^2 + bs + k = 0 \longrightarrow \text{second-order system}$$

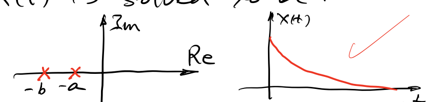
Solve it ⇒ $\boxed{S_{1,2} = -\frac{b}{2m} \pm \frac{\sqrt{b^2-4mk}}{2m}}$ (poles of the system, roots of the C.E.)

where $S_{1,2}$ are the roots of the characteristic equation (CE) and also called the poles of the system, which are complex numbers in the most general case. The dynamic behaviour of the system is determined by the LOCATIONS of the system poles (roots of CE).

Case I: Real and distinct (unequal) poles: $b^2-4mk>0$, $S_1 \neq S_2$

Then the solution for $X(t)$ is solved to be:

$$x(t) = C_1 e^{S_1 t} + C_2 e^{S_2 t}$$

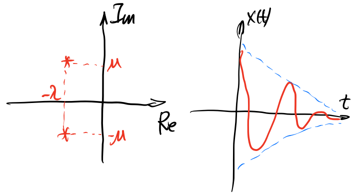


If both S_1 and S_2 are negative (located on the left-half S-plane) then $x(t) = C_1 e^{-at} + C_2 e^{-bt} \xrightarrow{t\to\infty} 0$ ⇒ Stable?

Case II. Complex poles : $b^2-4mk<0 \Rightarrow s_{1,2}=\lambda \pm i\mu$

Then the solution for $x(t)$ is solved to be :

$$x(t) = c \cdot e^{\lambda t} \cdot \cos(\mu t - \delta)$$

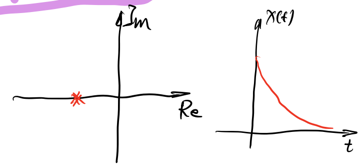


If both s_1 and s_2 are located on the left-half of the s-plane as shown here, then

$$x(t) = c \cdot e^{-at} \cdot \cos(bt-\delta) \xrightarrow{t\to\infty} 0 \Rightarrow \text{Stable!}$$

$e^{-at} \to 0$

Case III. Real repeated poles : $b^2-4mk=0$, $s_{1,2}=-\frac{b}{2m}$



If s_1 and s_2 are located on the left-half of the s-plane as shown here,

Then $x(t)=(C_1+C_2t)\cdot e^{st}$

$$= (C_1+C_2t)\cdot e^{-at} \xrightarrow{t\to\infty} 0 \Rightarrow \text{Stable!}$$

$C_2t \to \infty$ slower, $e^{-at} \to 0$ faster

In this case of $b^2-4mk=0$, the system response is called critically damped.

In summary,