SDM273: Intelligent Sensing and Signal Processing

3: Sensor Characteristics

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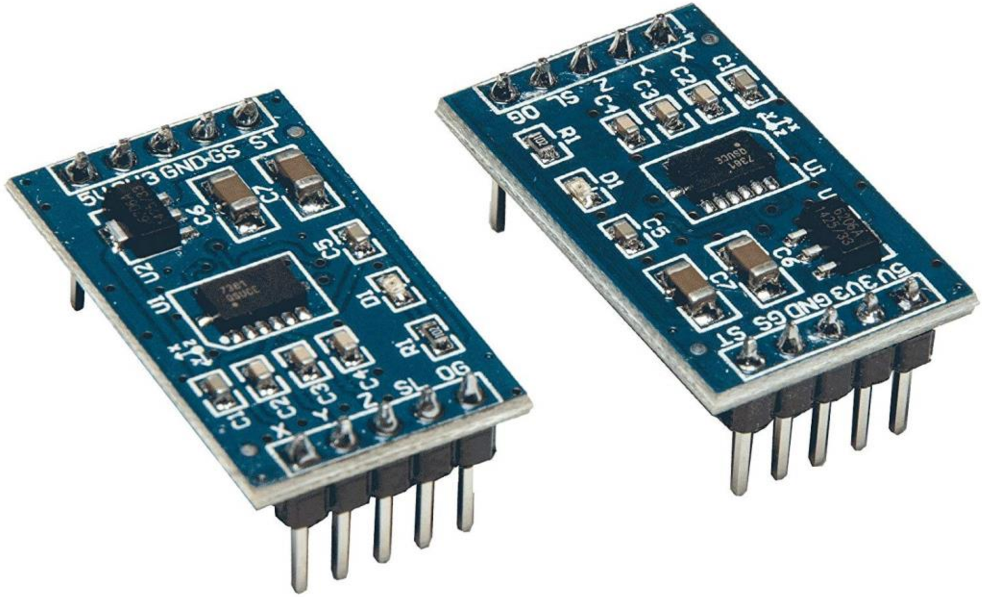
Static Characteristics of Sensors

- **Transfer function**
- **Calibration**
- **Static characteristics of sensors**
  - Span
  - Resolution
  - Hysteresis
  - Nonlinearity
  - Saturation
  - Accuracy and Precision

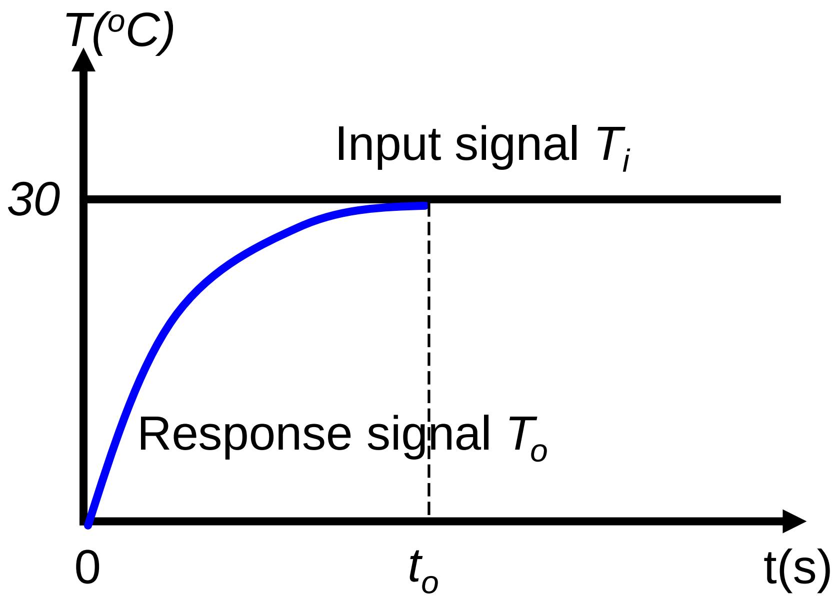


Dynamic Characteristics

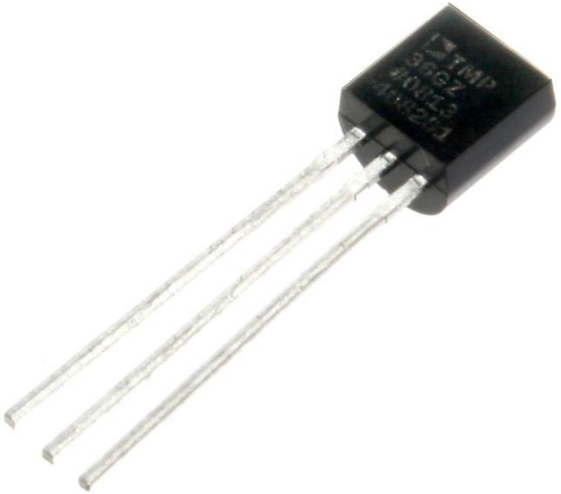
In dynamic measurement, when the input quantity changes rapidly, if the sensor can immediately respond to the changing input quantity, this sensor is with ideal dynamic characteristics. However, in practice, except for components with proportional characteristics, sensors generally do not have such characteristics, such as elastic components, inertial components or damping components(阻尼元件).

Case Study: *When a mercury thermometer at ambient temperature (0° C) was quickly placed in water at a constant 30° C, the change of the mercury column was observed.*

It can be seen that due to the existence of the heat capacity, the mercury column does not immediately reach the magnitude of the input signal, but a certain time delay t0 has passed.



Example of Thermal Sensor



- •High accuracy solid state temperature sensor
- •Voltage Input: 2.7V to 5.5V DC
- •Output range: 0.1V (-40°C) to 2.0V (150°C)
- •Temperature range: -40°C to 150°C
- •±2°C accuracy

ANALOG DEVICES

Low Voltage Temperature Sensors

Data Sheet **TMP35/TMP36/TMP37**

FEATURES

Low voltage operation (2.7 V to 5.5 V)
Calibrated directly in °C
10 mV/°C scale factor (20 mV/°C on TMP37)
±2°C accuracy over temperature (typ)
±0.5°C linearity (typ)
Stable with large capacitive loads
Specified −40°C to +125°C, operation to +150°C
Less than 50 μA quiescent current
Shutdown current 0.5 μA max
Low self-heating
Qualified for automotive applications

APPLICATIONS

Environmental control systems
Thermal protection
Industrial process control
Fire alarms
Power system monitors
CPU thermal management

GENERAL DESCRIPTION

The TMP35/TMP36/TMP37 are low voltage, precision centigrade temperature sensors. They provide a voltage output that is linearly proportional to the Celsius (centigrade) temperature. The TMP35/TMP36/TMP37 do not require any external calibration to provide typical accuracies of ±1°C at +25°C

FUNCTIONAL BLOCK DIAGRAM

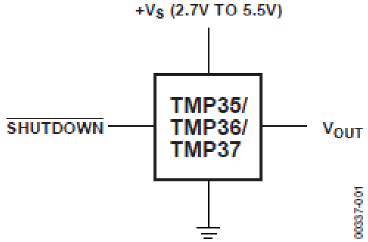


Figure 1.

PIN CONFIGURATIONS

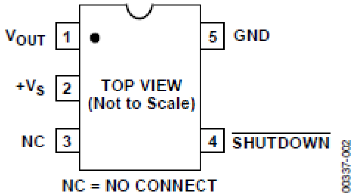


Figure 2. RJ-5 (SOT-23)

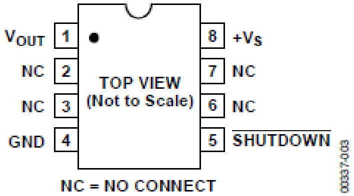
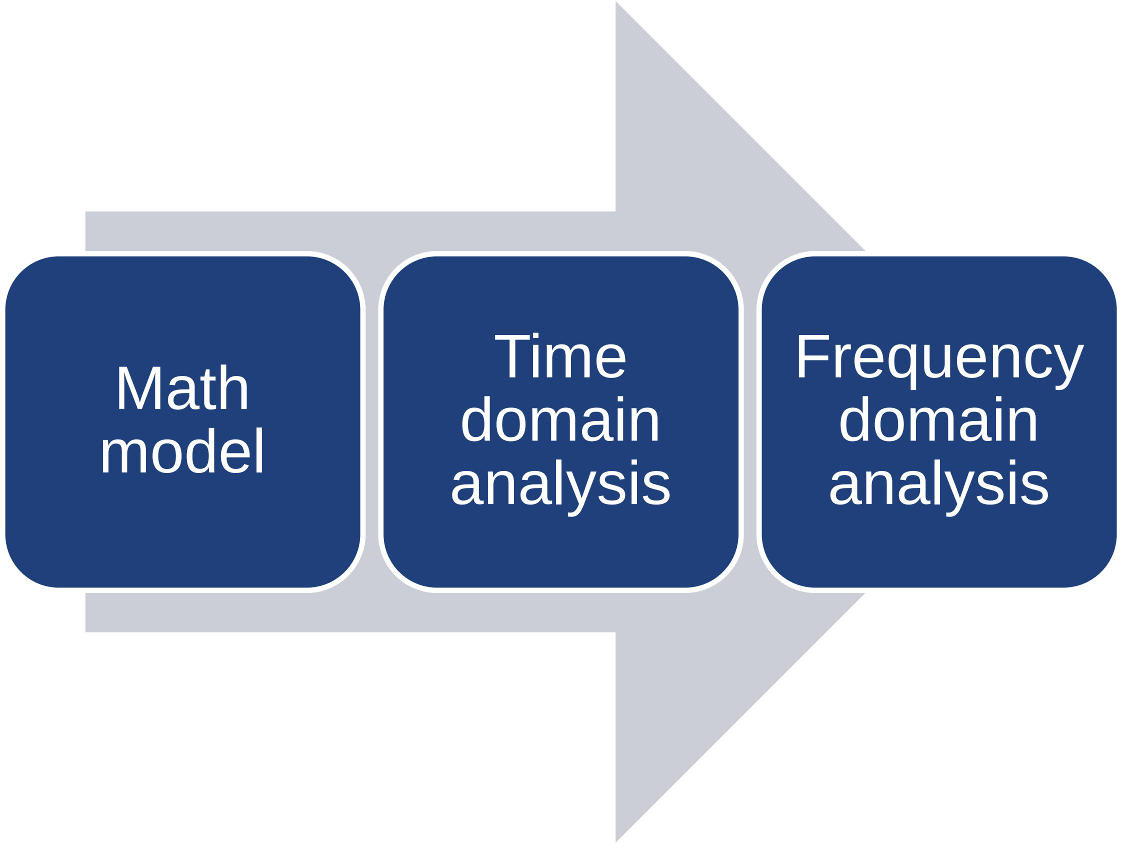


Figure 3. R-8 (SOIC_N)

Dynamic Characteristics

- **The sensor's response to a variable input is different from that exhibited when the input signals are constant (the latter is described by the static characteristics)**
- **The reason for dynamic characteristics is the presence of energy-storing elements (储能元件):**
  - Inertial: masses, inductances (电感)
  - Capacitances (电容): electrical, thermal
  - Question: is resistor (电阻) an energy-storing element?

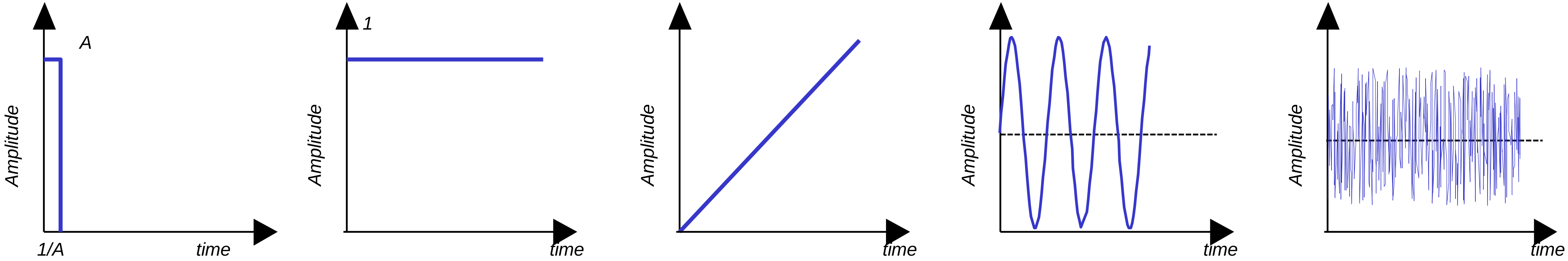
A Method in Control Engineering



- **Differential Equation Model (time domain)**
- **Transfer Function model (frequency domain)**

Methods for Studying Dynamic Characteristics of Sensors

- **Dynamic characteristics are determined by analyzing the response of the sensor to a family of variable input waveforms:**



Question: what are these responses called, respectively ?

Methods for Studying the Dynamic Characteristics

- **Time domain**
  - Transient inputs → y(t) expression, typical input → Dynamic characteristics
  - Input signal
    - Step signal, Ramp signal, Impulse signal
  - Indicator
    - Time constant, rise time, response time, overshoot
- **Frequency domain**
  - Periodic inputs → Frequency response function → Amplitude response, Phase response → dynamic characteristics
  - Input signal
    - Sinusoidal signal
  - Indicator
    - Bandwidth

Dynamic models

- Ideal dynamic characteristics of the sensor: when the input quantity changes with time, the output quantity can immediately change without distortion.
- In fact: there are elastic, inertial, and damping components, which are related to the input, the changing speed of the input, the system sampling rate, etc.
- Linear time-invariant system theory is commonly used in engineering to describe the dynamic characteristics of sensors
- Representing the relationship between sensor output and input using linear constant differential equations (linear stationary systems)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

Transfer Function

Differential Equations for a Linear Steady-State System

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+a_{n-2}y^{(n-2)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t)$$

$$=b_mr^{(m)}(t)+b_{m-1}r^{(m-1)}(t)+b_{m-2}r^{(m-2)}(t)+\cdots+b_1\dot{r}(t)+b_0r(t)$$

At the initial state

$$r^{(i)}(0)=0 \qquad i=0,1,2,\cdots,m-1$$

$$y^{(i)}(0)=0 \qquad i=0,1,2,\cdots,n-1$$

The Laplace Transform (review)

Definition

Complex variable $s = \sigma + j\omega$

$$f(t) \longrightarrow L[f(t)] = F(s) = \int_0^{+\infty} f(t)e^{-st}dt$$

$f(t) = 0 \text{ for } t < 0$

Laplace Transform of f(t) in s-domain

signal

$$f(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \xrightarrow{\text{Laplace Transform}} F(s) = \int_0^{+\infty} f(t)e^{-st}dt = \int_0^{+\infty} (1)e^{-st}dt$$

$$= \frac{1}{-s}e^{-st}\Big|_0^{+\infty} = \boxed{\frac{1}{s}}$$

Time Function $f(t)$ $f(t)=\mathscr{L}^{-1}\{F(s)\}$	Laplace Transform of $f(t)$ $F(s)=\mathscr{L}\{f(t)\}$	
1	$\frac{1}{s}$	$s>0$
t (unit-ramp function)	$\frac{1}{s^2}$	$s>0$
t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$s>0$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$s>0$
$t^n g(t)$, for $n=1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$	
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$s>\lvert\omega\rvert$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$s>\lvert\omega\rvert$
$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$ Scale property	
$e^{at}g(t)$	$G(s-a)$ Shift property	
$e^{at}t^n$, for $n=1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	$s>a$
te^{-t}	$\frac{1}{(s+1)^2}$	$s>-1$
$1-e^{-t/T}$	$\frac{1}{s(1+Ts)}$	$s>-\frac{1}{T}$
$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$	$s>a$
$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$	$s>a$
$u(t)$	$\frac{1}{s}$	$s>0$
$u(t-a)$	$\frac{e^{-as}}{s}$	$s>0$
$u(t-a)\cdot g(t-a)$	$e^{-as}G(s)$ Time-displacement theorem	
$g'(t)$	$sG(s)-g(0)$	
$g''(t)$	$s^2\cdot G(s)-s\cdot g(0)-g'(0)$	
$g^{(n)}(t)$	$s^n\cdot G(s)-s^{n-1}\cdot g(0)-s^{n-2}\cdot g'(0)-\ldots-g^{n-1}(0)$	
$\int_0^t g(t)\ dt$	$\frac{G(s)}{s}$	
$\int g(t)\ dt$	$\frac{G(s)}{s}+\frac{1}{s}\left\{\int g(t)\ dt\right\}_{t=0}$	

The Laplace Transform (review)

- **The Laplace transform of a time signal y(t) is denoted by**

$$\boldsymbol{L[y(t)] = Y(s)}$$

  - The ***s*** variable is a complex number ***s=σ+jω***
    - The real component ***σ*** defines the real exponential behavior
    - The imaginary component defines the frequency of oscillatory behavior

- **The fundamental relationship is the one that concerns the transformation of differentiation**

$$L\left[\frac{d}{dt}y(t)\right] = sY(s) - y(0)$$

Properties of the unilateral Laplace transform

	Time domain	s domain	Comment
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$	Can be proved using basic rules of integration.
Frequency-domain derivative	$tf(t)$	$-F'(s)$	F' is the first derivative of F with respect to s.
Frequency-domain general derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, nth derivative of $F(s)$.
Derivative	$f'(t)$	$sF(s)-f(0^+)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second derivative	$f''(t)$	$s^2F(s)-sf(0^+)-f'(0^+)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General derivative	$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+)$	f is assumed to be n-times differentiable, with nth derivative of exponential type. Follows by mathematical induction.
Frequency-domain integration	$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)\,d\sigma$	This is deduced using the nature of frequency differentiation and conditional convergence.
Time-domain integration	$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\frac{1}{s}F(s)$	$u(t)$ is the Heaviside step function and $(u*f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the Heaviside step function
Time scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$a>0$
Multiplication	$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)\,d\sigma$	The integration is done along the vertical line Re(σ) = c that lies entirely within the region of convergence of F.[21]
Convolution	$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	
Complex conjugation	$f^*(t)$	$F^*(s^*)$	
Cross-correlation	$f(t)\star g(t)$	$F^*(-s^*)\cdot G(s)$	
Periodic function	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	$f(t)$ is a periodic function of period T so that $f(t)=f(t+T)$, for all $t\geq 0$. This is the result of the time shifting property and the geometric series.

The Laplace Transform (review)

- **Applying the Laplace transform to the sensor model yields**

$$G(s)=\frac{Y(s)}{R(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+b_{m-2}s^{m-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1 s+a_0}$$

  - G(s) is called the transfer function of the sensor

$$Y(s)=G(s)R(s)$$

- **The position of the poles of G(s) -zeros of the denominator- in the s-plane determines the dynamic behavior of the sensor**

Transfer Function

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

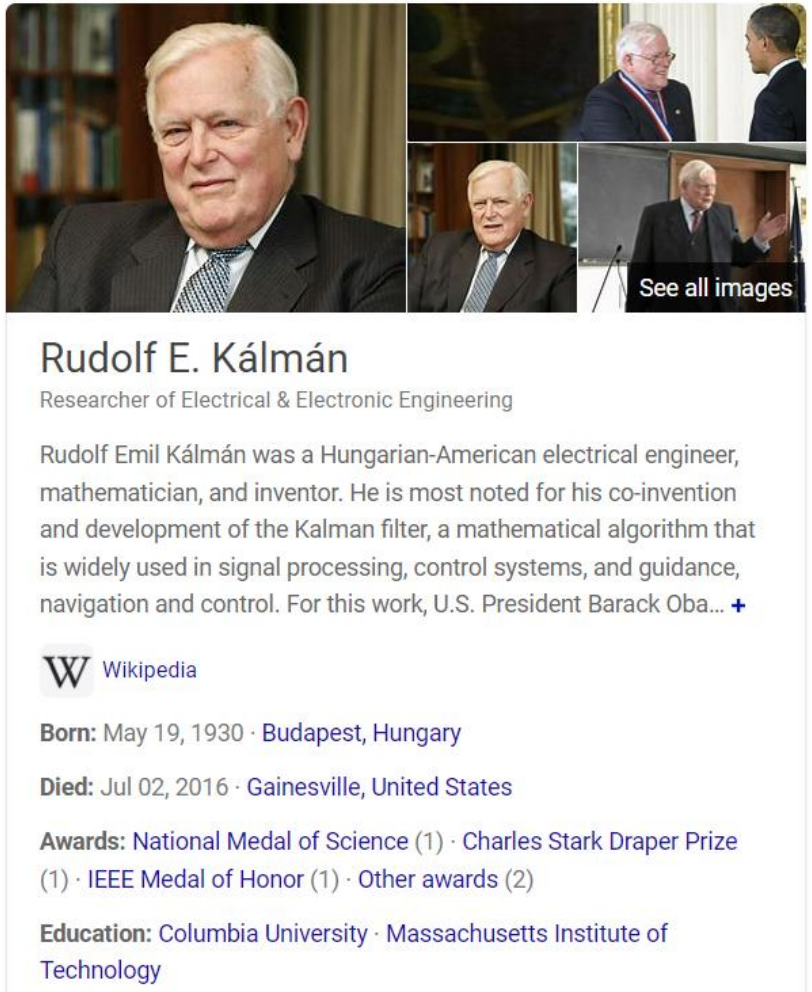
Linear stationary systems

$$G(s) = \frac{Y(s)}{R(s)}$$

$$Y(s) = G(s)R(s)$$

- **Transfer function**
  - A mathematical model of the system and represents the inherent properties of the system. The transfer function is independent of the input and output of the system.
  - The transfer function cannot indicate the physical characteristics and structure of the system, so different systems may have the same transfer function.

A Digression: Some stories on Kalman, a founder of state-space approach to control system analysis and design



Transfer Function

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0}$$

- R (s) = 0 characteristic equation of system → characteristic root
- The characteristic equation determines the dynamic characteristics of the system
- The highest order of s in R (s) is the order of the system

When s=0 $G(0) = \frac{b_0}{a_0} = K$ System amplification factor or gain

- From the perspective of the differential equation, this is equivalent to all the derivative terms being zero. K ——The ratio of output to input when the system is in static state

Zeros and Poles

$$G(s)=\frac{Y(s)}{R(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+b_{m-2}s^{m-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1 s+a_0}$$

$$G(s)=\frac{b_0(s-z_1)(s-z_2)\ldots(s-z_m)}{a_0(s-p_1)(s-p_2)\ldots(s-p_n)}$$

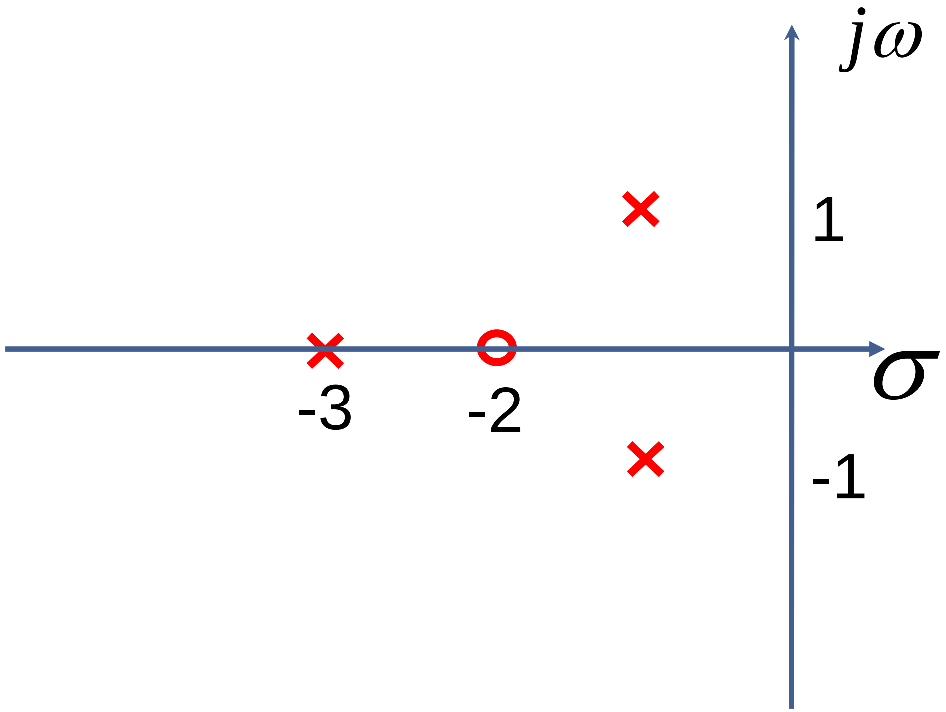
- *Zeros:* $Y(s)=b0(s-z_1)(s-z_2)\ldots(s-z_m)=0$, $s=z_i(i=1, 2, \ldots, m)$
- *Poles:* $R(s)=a_0(s-p_1)(s-p_2)\ldots(s-p_n)=0, s=pj(j=1, 2, \ldots, n)$
  - Poles and **Zeros** of a **transfer function** are the frequencies for which the value of the denominator and numerator of **transfer function** becomes **zero** respectively. The values of the poles and the **zeros** of a system determine whether the system is stable, and how well the system performs

Distribution of Zeros and Poles

- Zeros and Poles distribution of the transfer function:
  - A graph that represents the zeros and poles of a transfer function on a complex plane.

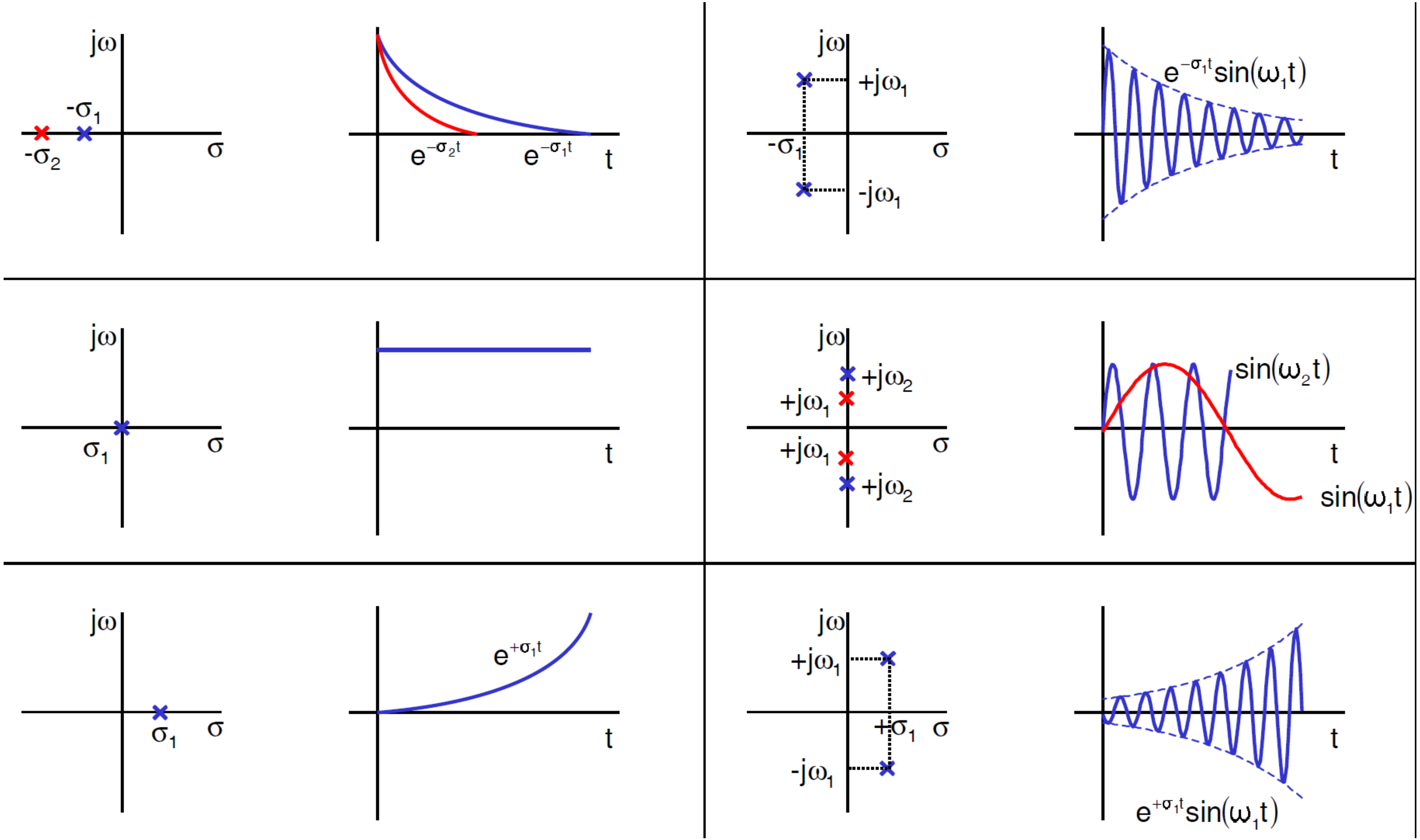
- Zero point is represented by "**o**"

- Pole is indicated by "**×**"



$$G(s)=\frac{s+2}{(s+3)(s^2+2s+2)}$$

Pole Location and Dynamic Behavior



To Sum Up

- **Characteristic relationship (differential equation):**

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m r}{dt^m} + b_{m-1}\frac{d^{m-1}r}{dt^{m-1}} + \cdots + b_1\frac{d\mathrm{r}}{dt} + b_0 r$$

- **Laplace transform (set the initial value of the time derivative of each order to 0):**

$$Y(s)(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0) = R(s)(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)$$

- **Transfer function (external reflection of internal structural parameters, only related to system structural parameters):**

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

- **Transfer Function:**

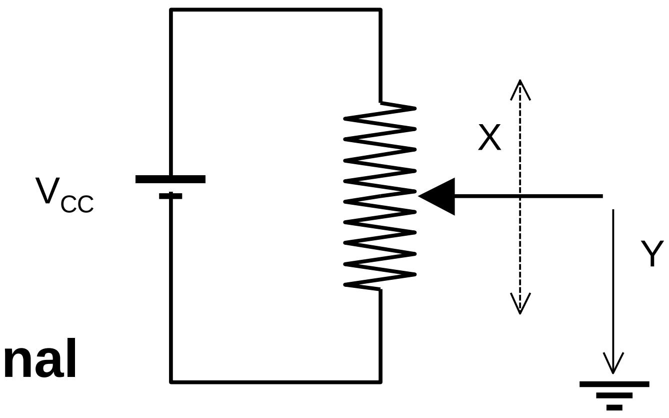
$$G(s) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(s)}{R(s)}$$

Zero-order Sensors

- Input and output are related by an equation of the type

$$G(s) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(s)}{X(s)} = K$$

  - Zero-order is the desirable response of a sensor
    - No delays
    - Infinite bandwidth
    - The sensor only changes the amplitude of the input signal



- Zero-order systems do not include energy-storing elements
- Example of a zero-order sensor
  - A potentiometer (电位器, 是可变电阻器的一种) used to measure linear and rotary displacements
    - This model would not work for fast-varying displacements

First-order Sensors

- Input and output are related by an equation of the type

$$G(s) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(s)}{X(s)} = \frac{K}{\mathrm{T}s + 1}$$

- First-order sensors have one element that stores energy and one that dissipates it
  - where T——time constant
  - K——static sensitivity
  - Since the sensitivity K is constant in the linear sensor, K only plays a role of increasing the output by K times in the dynamic characteristic analysis.

First-order Sensors

$$G(s) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(s)}{X(s)} = \frac{K}{\mathrm{T}s + 1}$$

K = 1 is used in the following discussion.

- **Step response**
  - When the input signal is a **Unit Step Signal**,

$$x(t) = 1(t)$$

  - the Laplace transform of the output signal is:

$$Y(s) = \Phi(s)X(s) = \frac{1}{Ts+1} \cdot \frac{1}{s} = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

First-order Sensor- Inverse Laplace Transform

The inverse Laplace transform gives:

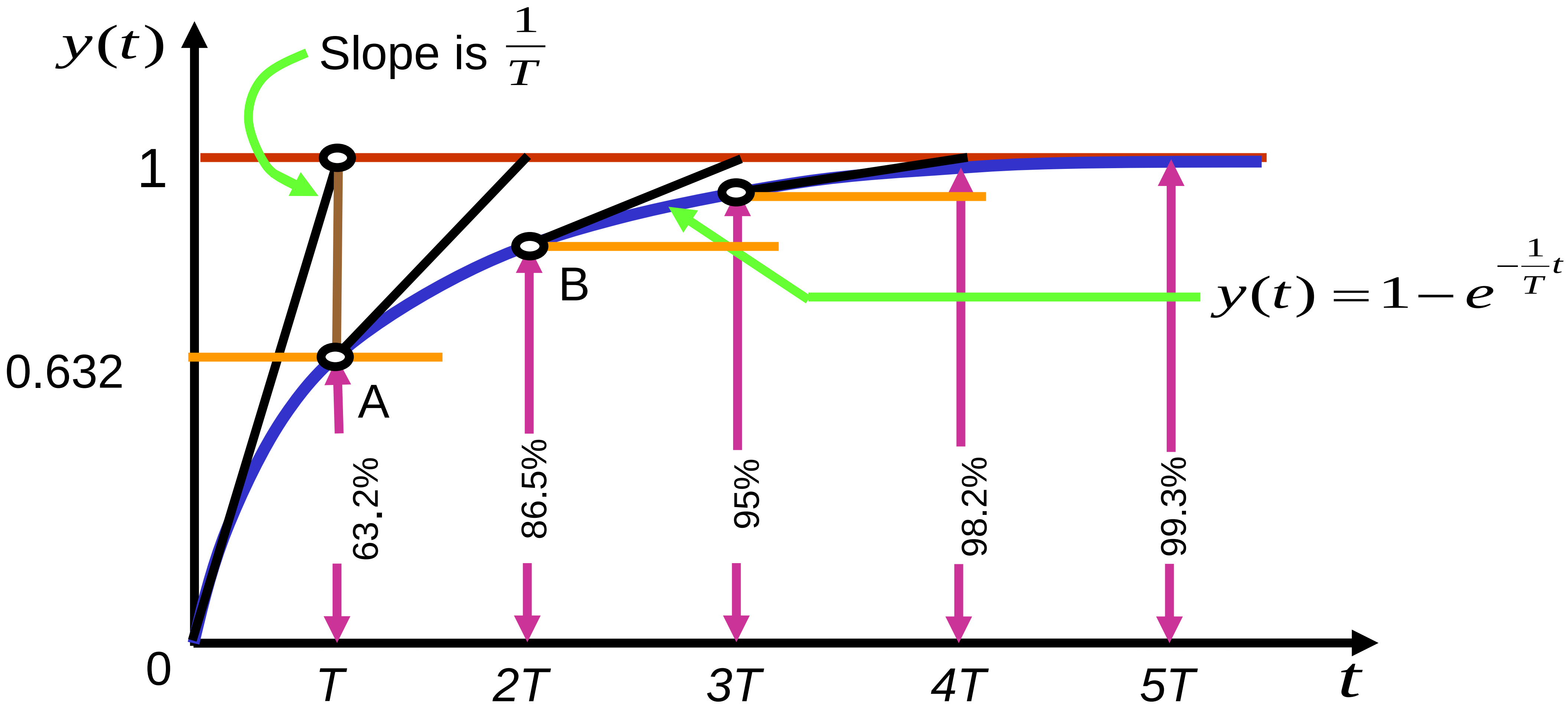
$$y(t) = L^{-1}[Y(s)] = 1 - e^{-\frac{1}{T}t}$$

$$y(t) = 1 - e^{-\frac{1}{T}t}, \qquad (t \geq 0)$$

Static response

Dynamic response

First-order Sensor- Inverse Laplace Transform



- The smaller the time constant τ, the better the first-order sensor

First-order Sensor- Inverse Laplace Transform

- It is generally believed that the dynamic process can be considered to be over when the difference between its value and its steady state value is less than a certain number
  - Transition time of **Unit Step Response** of first-order system:

$$t_s = 3T \quad \text{———} \quad 5\% \text{ error}$$

$$t_s = 4T \quad \text{———} \quad 2\% \text{ error}$$

  - Methods to improve the speed of the unit step response of first-order systems:
    - Reduce the time constant ***T***

System Identification of First order system

- Often it is not possible or practical to obtain a system's transfer function analytically
- The system's step response can lead to a representation even though the inner construction is not known
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated
- If we can identify T and K from laboratory testing we can obtain the transfer function of the system

Second-order Sensors

- **Input and output are related by a second-order differential equation**

$$G(s) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(s)}{X(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

- **We can express this second-order transfer function as**

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with $K = \dfrac{1}{a_0}$, $\zeta = \dfrac{a_1}{2\sqrt{a_0 a_1}}$, $\omega_n = \sqrt{\dfrac{a_0}{a_2}}$

Where

- **K is the static gain**
- **ζ is known as the damping coefficient**
- **ω_n is known as the natural frequency**

Step Response of Second-order Sensors

- **When applying a Unit Step Signal, its Laplace transform is**

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

- **The response of the second-order system under a Unit Step Signal as**

$$y(t) = 1 - e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \quad t \geq 0$$

Steady state component

Decide decay speed

Transient component

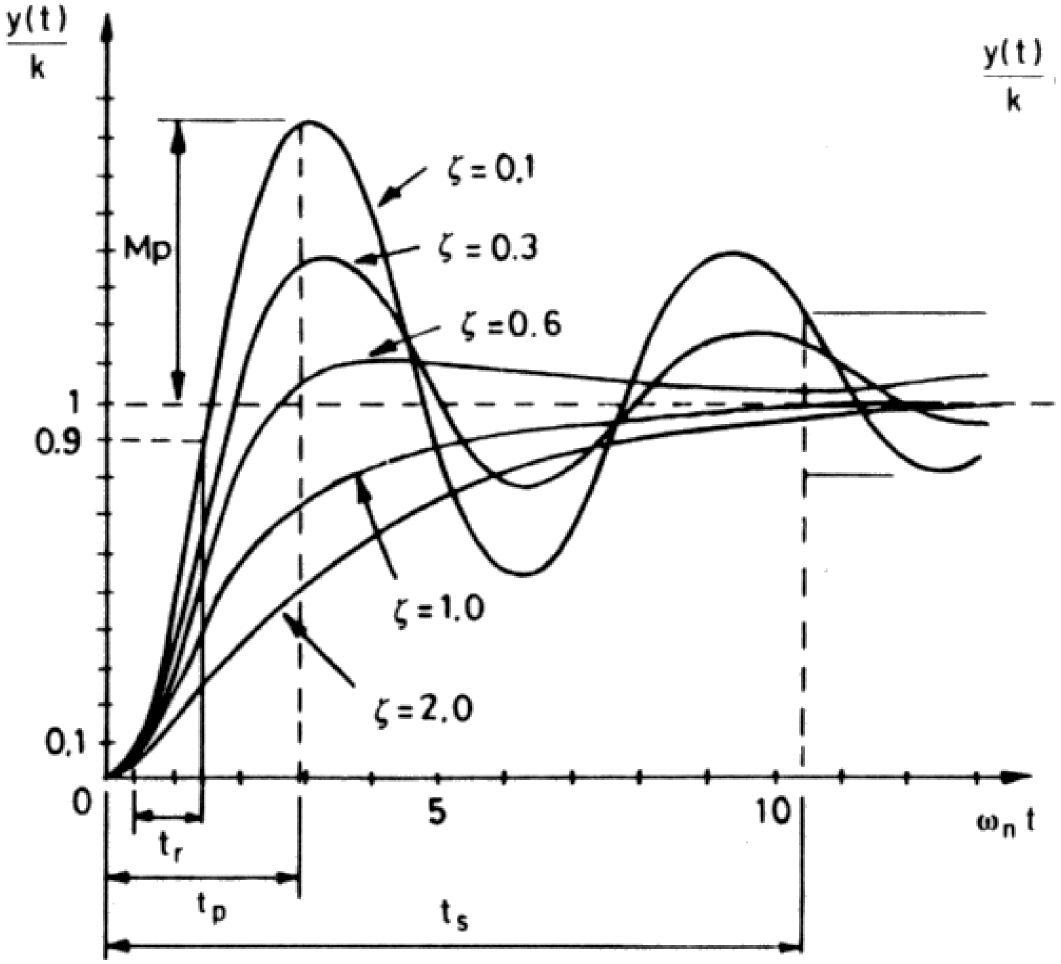
$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

Performance of Second-order Sensors

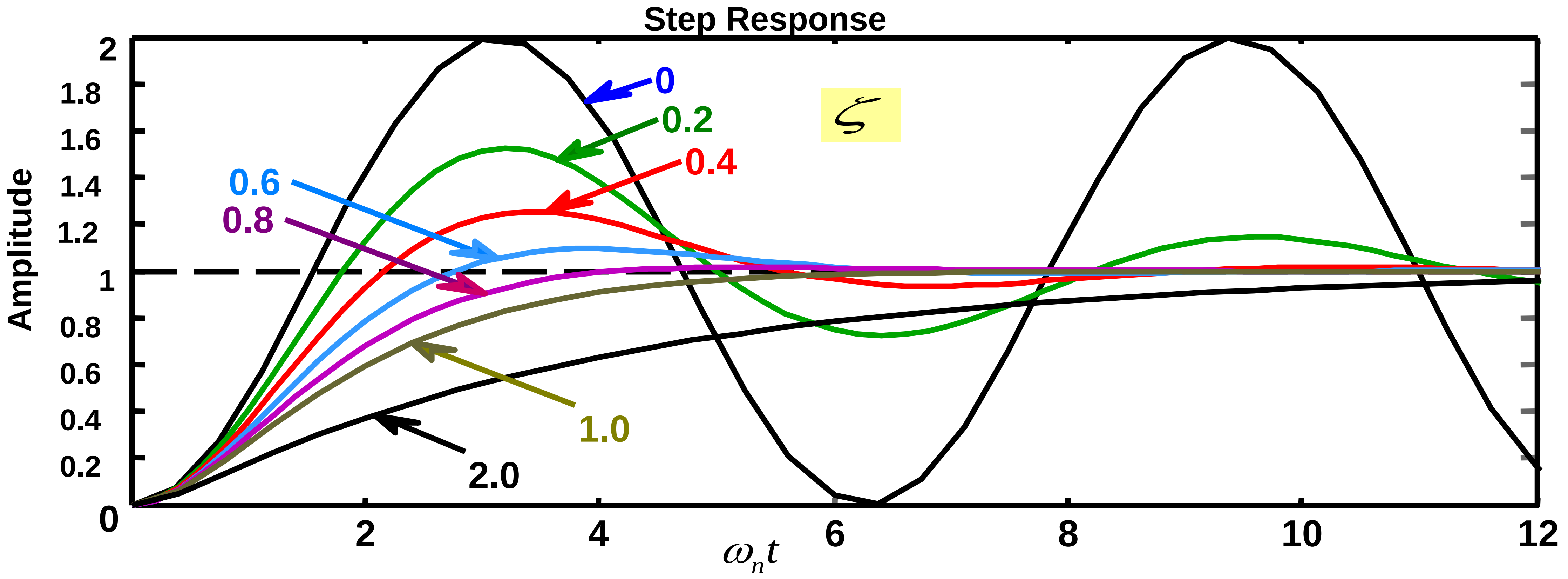
- The dynamic process is also called the transient process, which refers to the response process of the output of the system from the initial state to the final state under the stimulation of the input signal
- Performance depends on
  - Response time
  - Smoothness
- **Unit Step Function Signal** is commonly used as an input to measure the dynamic response

Second-order Step Response

- Response types
  - Underdamped (ζ <1)
  - Critically damped (ζ =1)
  - Overdamped (ζ >1)
- Response parameters
  - Rise time (t_r)
  - Peak overshoot (M_p)
  - Time to peak (t_p)
  - Settling time (t_s)



Second-order Step Response



- The response depends greatly on damping ratio ζ and the natural frequency ω_n
- In practical use, in order to have a short rise time and a small overshoot, generally, sensors are designed to be underdamped, and the damping ratio ζ is generally between 0.6 and 0.8.

Another digression: Some more stories on Kalman

A new approach to linear filtering and prediction problems
RE **Kalman** - 1960 - asmedigitalcollection.asme.org
The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the “state-transition” method of analysis of dynamic …
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Work [edit]

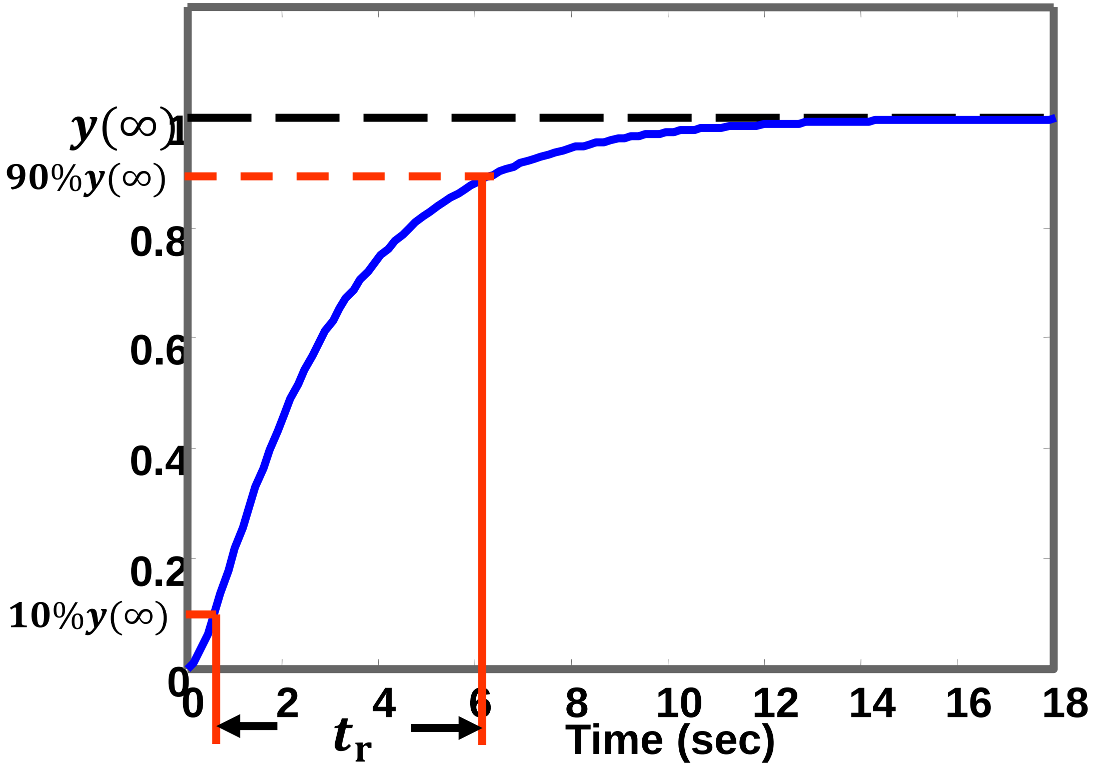
Kálmán was an electrical engineer by his undergraduate and graduate education at M.I.T. and Columbia University, and he was noted for his co-invention of the Kalman filter (or Kalman-Bucy Filter), which is a mathematical technique widely used in the digital computers of control systems, navigation systems, avionics, and outer-space vehicles to extract a signal from a long sequence of noisy or incomplete measurements, usually those done by electronic and gyroscopic systems.

Kálmán's ideas on filtering were initially met with vast skepticism, so much so that he was forced to do the first publication of his results in mechanical engineering, rather than in electrical engineering or systems engineering. Kálmán had more success in presenting his ideas, however, while visiting Stanley F. Schmidt at the NASA Ames Research Center in 1960. This led to the use of Kálmán filters during the Apollo program, and furthermore, in the NASA Space Shuttle, in Navy submarines, and in unmanned aerospace vehicles and weapons, such as cruise missiles.[citation needed]

Kálmán published several seminal papers during the sixties, which rigorously established what is now known as the state-space representation of dynamical systems. He introduced the formal definition of a system, the notions of controllability and observability, eventually leading to the Kalman decomposition. Kálmán also gave groundbreakin control and provided, in his joint work with J. E. Bertram, a comprehensive and insightful exposure of stability theory with B. L. Ho on the minimal realization problem, providing the well known Ho-Kalman algorithm.

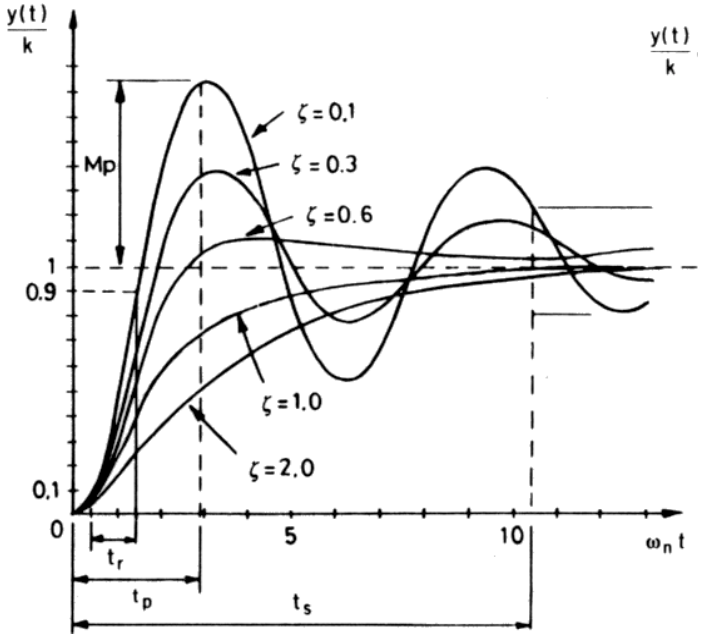
Once Again

- Rise time
  - The time required to reach the steady state value for the first time
  - Reflects the rapidity of the system
  - For overdamped systems, the time required to increase from 10% to 90% of the steady state value can be used



Once Again

- Peak time
  - Time to reach first peak
- Overshoot
  - refers to an output exceeding its final, steady-state value



$$\boldsymbol{\sigma}_{\mathbf{p}} = \frac{\boldsymbol{y}(\boldsymbol{t}_{\mathbf{p}}) - \boldsymbol{y}(\infty)}{\boldsymbol{y}(\infty)} \times \mathbf{100\%}$$

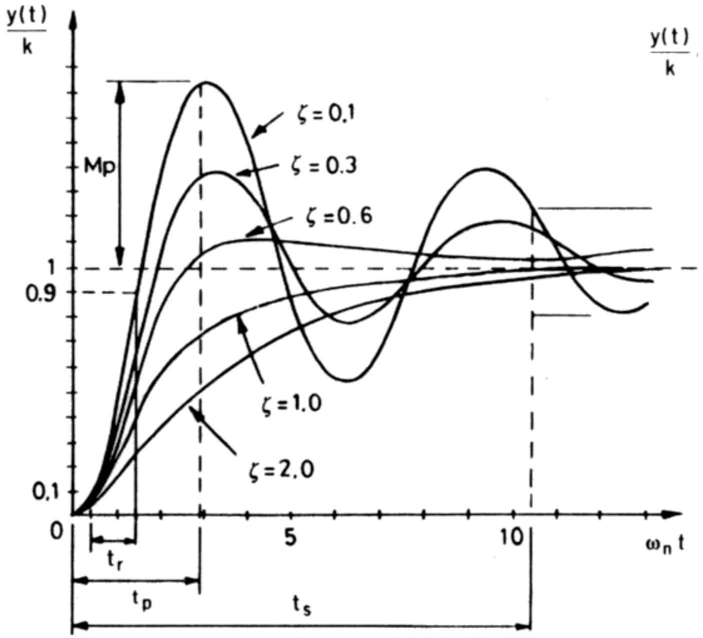
  - Overshoot reflects the smoothness of the dynamic process of the system
  - No overshoot for first-order and overdamped second-order systems!

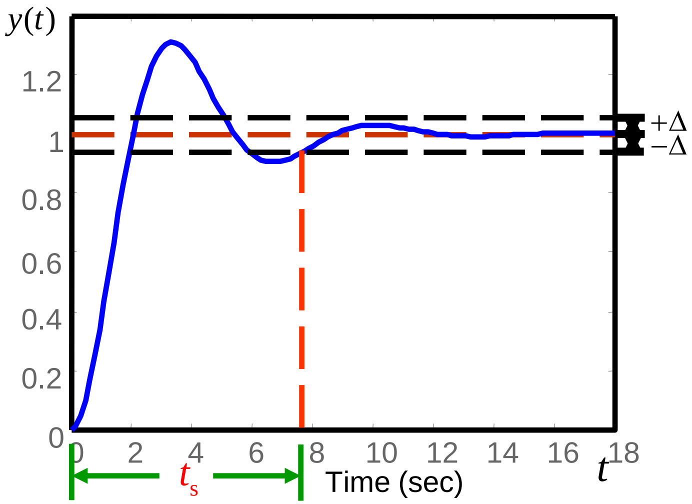
Once Again

- **Settling time (t_s)**
  - **The minimum time required for a unit step response to reach and maintain a steady state value within ± 5% or ± 2%**

$$\left| y(t) - y(\infty) \right| \le \Delta y(\infty) \qquad t \ge t_s$$

  - **After $t > t_s$, the dynamic process can be considered to be over**





Calculation of Rise Time

- **For underdamped second-order sensors**

$$y(t_r) = 1 - e^{-\zeta\omega_n t_r}\frac{1}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r + \theta) = 1$$

$$\sin(\omega_d t_r + \theta) = 0$$

$$\theta = \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\omega_d t_r + \theta = \pi$$

$$\boxed{t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\theta}{\omega_n\sqrt{1-\zeta^2}}}$$

- When the damping coefficient is constant, the rise time is related to the undamped natural frequency

Calculation of Peak Time

- **For underdamped second-order sensors**

$$y(t) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}\sin(\omega_\mathrm{d} t + \theta)$$

- **Derivative with respect to time,** $\left.\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right|_{t=t_\mathrm{p}} = 0$

$$\boxed{t_\mathrm{p} = \frac{\pi}{\omega_\mathrm{d}} = \frac{\pi}{\omega_\mathrm{n}\sqrt{1-\zeta^2}}}$$

$$y(t) = 1 - e^{-\zeta w_n t} \frac{1}{\sqrt{1-\zeta^2}} \sin(w_d t + \theta)$$

$$\frac{dy(t)}{dt} = \zeta w_n e^{-\zeta w_n t} \cdot \frac{1}{\sqrt{1-\zeta^2}} \sin(w_d t + \theta)$$

$$- w_d \cdot e^{-\zeta w_n t} \cdot \frac{1}{\sqrt{1-\zeta^2}} \cos(w_d t + \theta)$$

$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \left[\zeta w_n \sin(w_d t + \theta) - w_d \cos(w_d t + \theta) \right]$$

$$\left.\frac{dy(t)}{dt}\right|_{t=t_p} = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t_p} \left[\zeta w_n \sin(w_d \cdot t_p + \theta) - w_d \cdot \cos(w_d t_p + \theta) \right] = 0$$

$$\Rightarrow \quad \zeta w_n \sin(w_d t_p + \theta) - w_d \cdot \cos(w_d t_p + \theta) = 0$$

$$\tan(w_d t_p + \theta) = \frac{w_d}{\zeta w_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\theta$$

$$w_d = w_n \sqrt{1-\zeta^2}$$

$$\cancel{t_p = \frac{\arctan\left(\frac{w_d}{\zeta w_n}\right) - \theta}{w_d}}$$

$$w_d t_p + \theta = k\pi + \theta, \quad k \text{取} 1.$$

$$t_p = \frac{\pi}{w_d} = \frac{\pi}{w_n \sqrt{1-\zeta^2}}$$

Calculation of Overshoot

- **For underdamped second-order sensors**

$$y(t) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}\sin(\omega_\mathrm{d} t + \theta)$$

$$\boldsymbol{\sigma}_\mathbf{p} = \frac{\boldsymbol{y}(\boldsymbol{t}_\mathbf{p}) - \boldsymbol{y}(\infty)}{\boldsymbol{y}(\infty)} \times \mathbf{100\%} \quad = -\boldsymbol{e}^{-\boldsymbol{\zeta}\boldsymbol{\omega}_{\boldsymbol{n}}\boldsymbol{t}_\mathbf{p}}\frac{\mathbf{1}}{\sqrt{\mathbf{1}-\boldsymbol{\zeta}^\mathbf{2}}}\mathbf{sin}(\boldsymbol{\omega}_\mathbf{d}\boldsymbol{t}_\mathbf{p} + \boldsymbol{\theta})$$

$$= \boldsymbol{e}^{-\frac{\boldsymbol{\zeta}}{\sqrt{\mathbf{1}-\boldsymbol{\zeta}^\mathbf{2}}}\boldsymbol{\pi}}$$

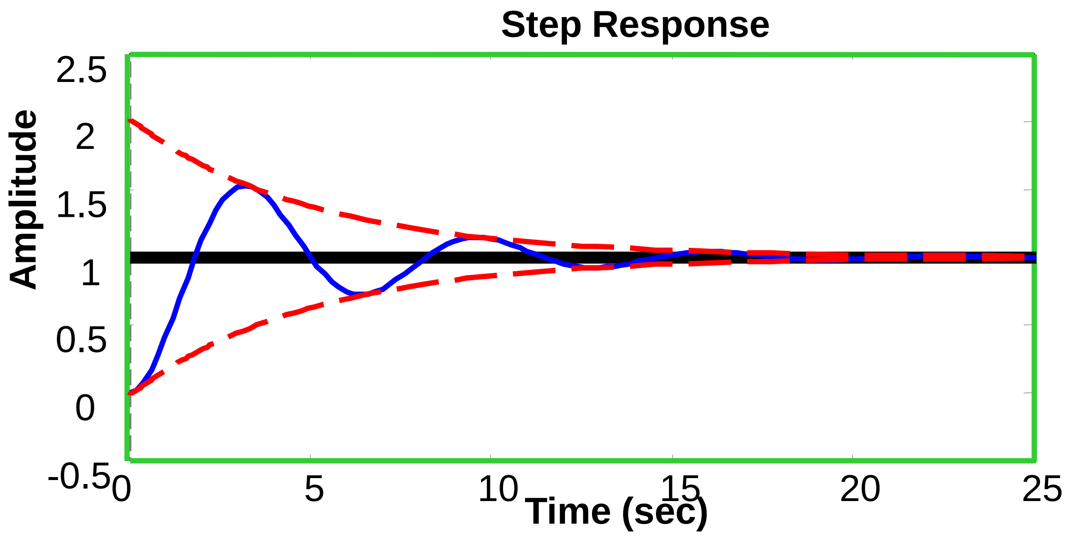
- **The overshoot is only related to the damping coefficient and has nothing to do with the undamped natural frequency**
- **The smaller the damping coefficient, the bigger the overshoot**

Calculation of Settling Time

- **For underdamped second-order sensors**

$$y(t) = 1 - e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathrm{d}} t + \theta)$$

- **The equation for the envelope** $\bar{y}(t) = 1 \pm e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}$



Calculation of Settling Time

$$|y(t) - y(\infty)| \le \Delta y(\infty) \qquad t \ge t_s$$

$$y(\infty) = 1$$

$$\left| e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \right| \le \Delta \qquad t \ge t_s$$

- **Calculation based on the envelope curve**

$$\left| e^{-\zeta\omega_n t} \frac{1}{\sqrt{1-\zeta^2}} \right| \le \Delta \qquad t \ge t_s$$

Calculation of Settling Time

$$e^{-\zeta\omega_n t_s}\frac{1}{\sqrt{1-\zeta^2}}=\Delta$$

$$\boxed{t_s=\frac{1}{\zeta\omega_n}\ln\frac{1}{\Delta\sqrt{1-\zeta^2}}}$$

- When $\Delta=0.02$

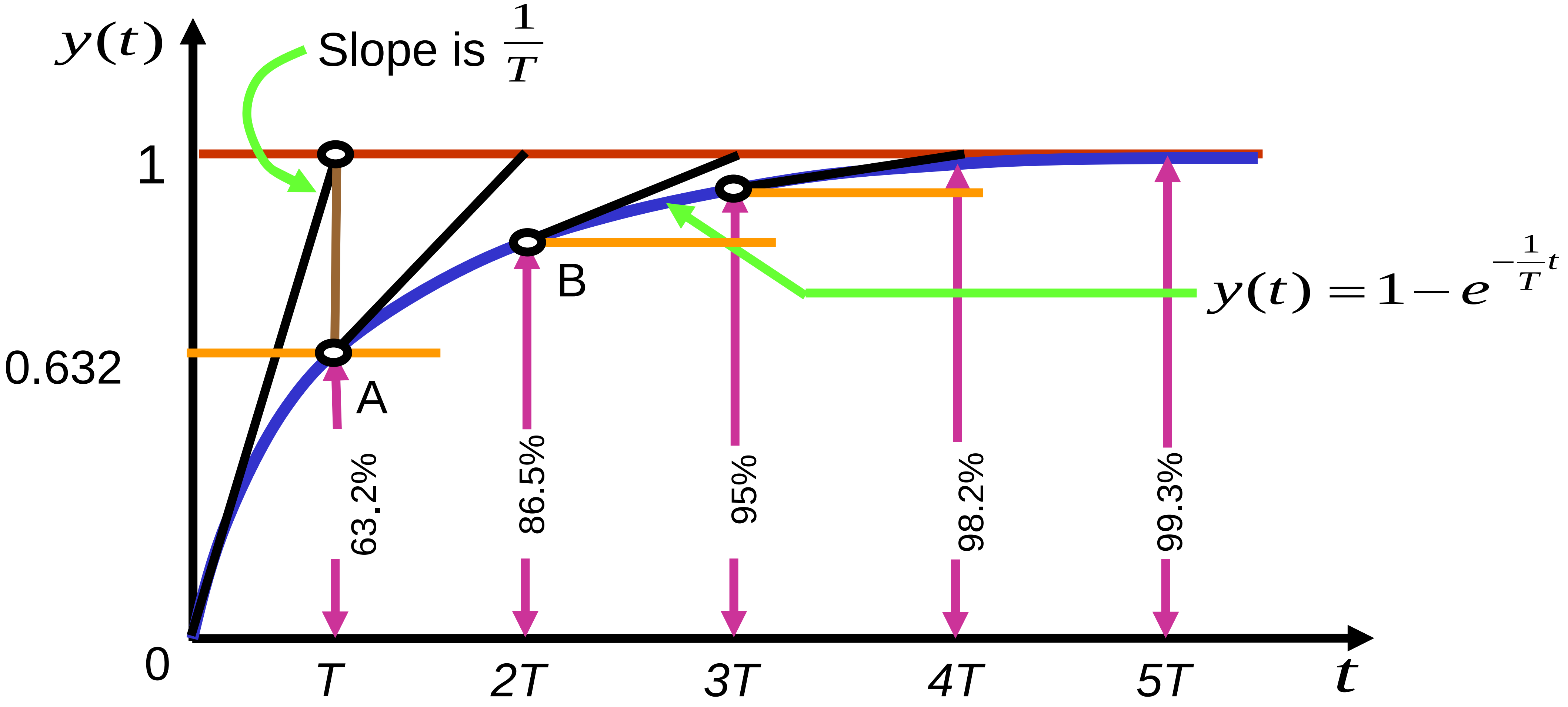
$$t_s\approx\frac{1}{\zeta\omega_n}\left(4+\ln\frac{1}{\sqrt{1-\zeta^2}}\right)$$

- When $\Delta=0.05$

$$t_s\approx\frac{1}{\zeta\omega_n}\left(3+\ln\frac{1}{\sqrt{1-\zeta^2}}\right)$$

Summary

- First-order Sensor

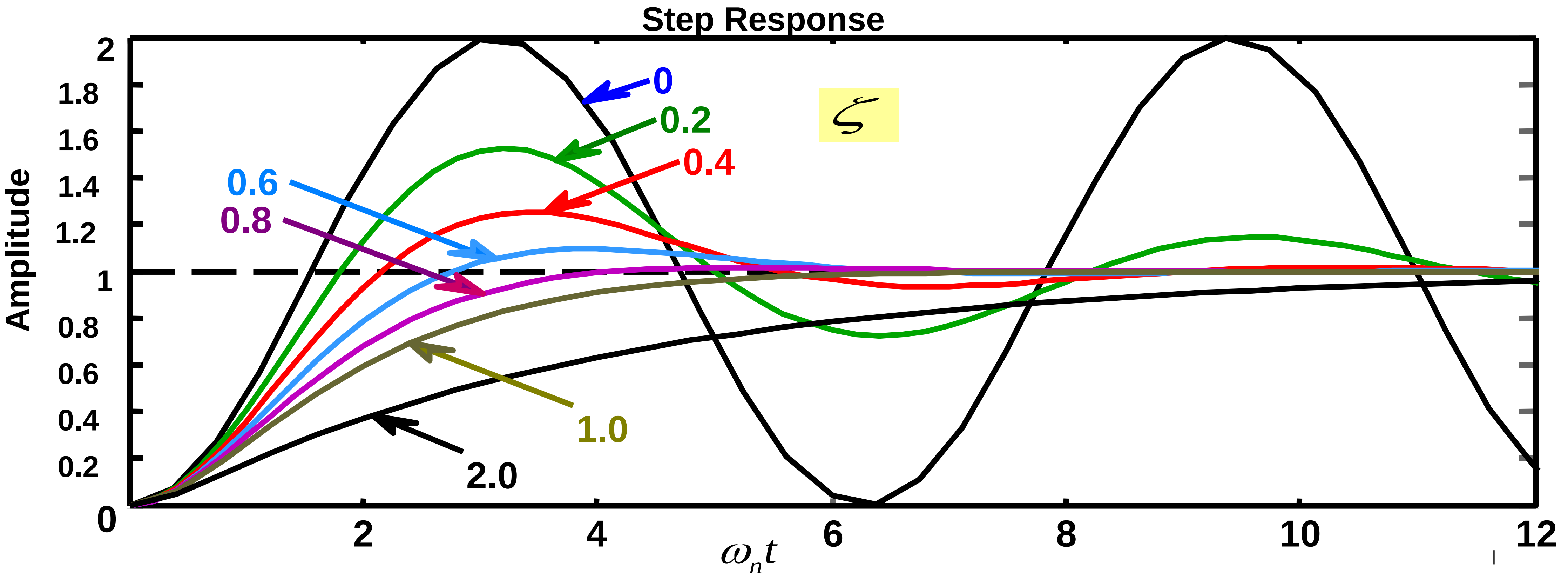


- The smaller the time constant τ, the better the first-order sensor

Summary

- Second-order Sensor

$$y(t)=S_n\left[1-\frac{e^{-w_n\zeta t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,w_n t+\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]A_0$$



Summary

- Performance depends on
  - Response time

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\theta}{\omega_n\sqrt{1-\zeta^2}}$$

    - Increasing the undamped natural frequency can reduce the response time
  - Smoothness

$$\boldsymbol{\sigma}_{\mathbf{p}} = \frac{\boldsymbol{y}(\boldsymbol{t}_{\mathbf{p}}) - \boldsymbol{y}(\infty)}{\boldsymbol{y}(\infty)} \times \mathbf{100\%} = \boldsymbol{e}^{-\frac{\boldsymbol{\zeta}}{\sqrt{\mathbf{1}-\boldsymbol{\zeta}^{\mathbf{2}}}}\boldsymbol{\pi}}$$

    - The increase of Damping Coefficient can increase the stability of the system and reduce the amount of overshoot and the number of oscillations
- Unit step function signal is commonly used as an input to measure the dynamic response