1.

(i)a.利用 Quadrature integrtion,求 f_z(z) 積分從 0 到 0.7 ,

用 R 算的真實值為 0.872516089908

	|真實值-I.hat|<0.0001 ,所需要到樣本數
Rectangle(矩形法)	1670
Trapezoidal(梯型法)	44
Simpson(辛普森)	6

b.利用 Monte Carlo method with U(0,0.7), 求 f_z(z) 積分從 0 到 0.7 .

由於 Monte Carlo 估計量本身為隨機變數，故每次|真實值-I.hat|<0.0001 ,所需要到樣本數都不一樣。

對於不同的重複生成次數，建構 95%的信賴區間:

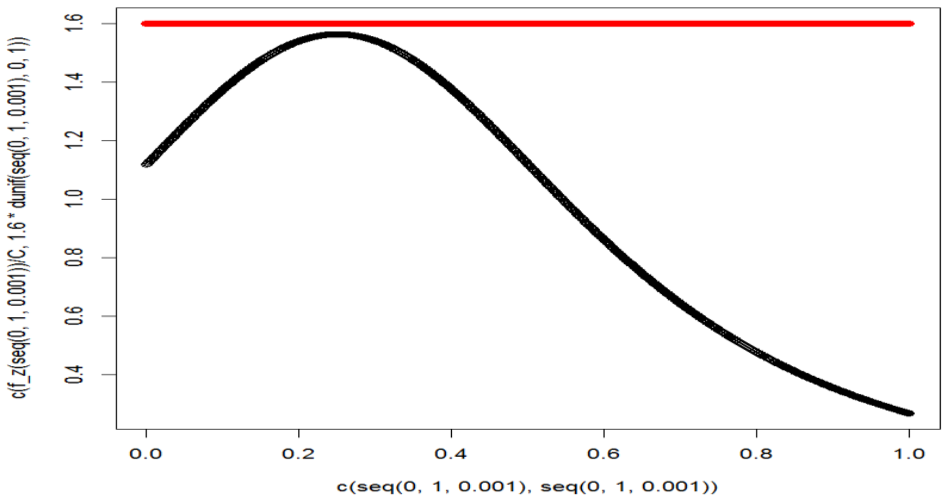
重複生成次數	樣本平均	95%信賴區間
1670	0.8726629	[0.8714788 0.8738471]
44	0.8762858	[0.8688807 0.8836909]
6	0.8668569	[0.8501588 0.8835549]

(ii)利用 empirical CDF 去估計 F_z(0.7)=P(Z < 0.7) 的值,

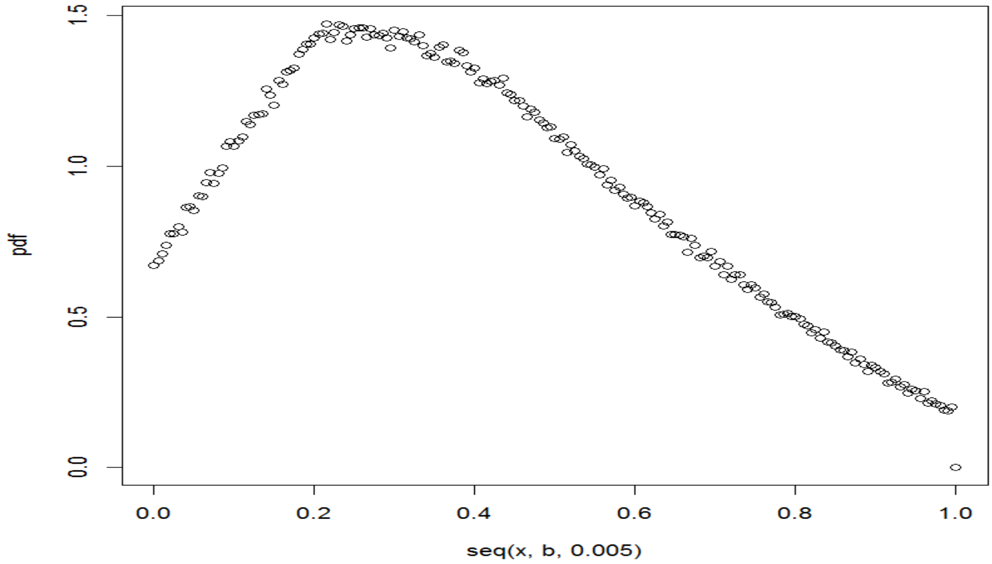
1. sample $z1, z2, \dots, zn$ iid from f_z
2. $\mathrm{F\_n}(0.7) = 1/\mathrm{n} \sum_{i=0}^{n} I(Z_i \leq 0.7)$

首先利用 Rejection sampling 生成$z1, z2, \dots, zn$

選擇 g ~ U(0,1)作為 envolope , e(x)=c*g(x) ,取 c=1.6 畫圖如下



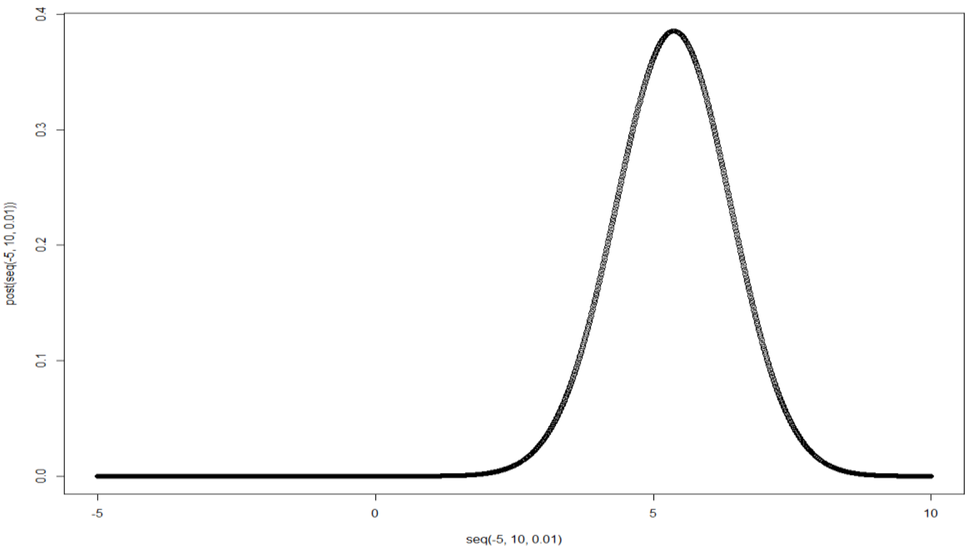
(紅色為 envolope ; 黑色為 f_z)



(此為生成後z1, z2, …, zn 利用 empirical pdf 估計 f_z)

由於$\mathrm{F_n}(0.7) = 1/\mathrm{n} \sum_{i=0}^{n} I(Z_i \leq 0.7)$ 為一個估計量，可以得到在樣本數為10^5 時估計值為 0.87254 , bias 為 2.391009e-05.

2.估計 mean and variance of posterior density ,
由於此處無法理論推得 posterior density 屬於某個特定分配 ,
模擬 theta=5 ,X ~N(5,1) , 先畫出 posterior density ,



(posterior density 給定 theta=5 ,X ~N(5,1)的畫圖)

因

$$
\begin{aligned}
\mathrm{E}(\theta|\mathrm{X}) &= \int_{-\infty}^{\infty} \theta p(\theta|X) d\theta \\
&= \int_{-\infty}^{\infty} \theta \frac{p(X|\theta)p(\theta)}{p(X)} d\theta
\end{aligned}
$$

我們生成 x1,x2,…,xn ~N(5,1) ,取 n=10^4 ,利用 Simpson method 算出積分值作為一個估計值 ,得到對於 mean 及 variance 的樣本估計為

I_bar=4.528903 , I_var=1.213188 , bias=0.471097

發現利用樣本平均和樣本方差估計 posterior mean and variance ,不是一個不偏的估計量.

(附上程式碼部分)

1.

```
f_z<-function(z){ (1+( (z-0.25)/0.5 )^2 )^-1.5}
C<-integrate(f_z,0,1)$value
I_true<-integrate(f_z,0,0.7)$value /C

plot(seq(0,1,0.001),f_z(seq(0,1,0.001))/C )
########################
#Quadrature integration
a<-0 ;b<-0.7 ;n<-100
h<-seq(a,b,(b-a)/n)
#Rectangle rule
I_upper<-0;I_lower<-0
for(i in 1:n){
   I_upper<-I_upper + f_z(h[i+1])*(b-a)/n/C
   I_lower<-I_lower + f_z(h[i])*(b-a)/n/C
}

a<-0 ;b<-0.7 ;n<-0
I_upper<-0
while(abs(I_upper-I_true)>=10^-4){
   cat("\n","Now is",n,"-th iteration ,bias is",abs(I_upper-I_true),"\n")
   n<-n+1
   h<-seq(a,b,(b-a)/n)
   I_upper<-0
   for(i in 1:n){    I_upper<-I_upper + f_z(h[i+1])*(b-a)/n/C }
```

```
}
cat("\n","The sample size is",n," ,bias is",abs(I_upper-I_true),"\n")

a<-0 ;b<-0.7 ;n<-0
I_lower<-0
while(abs(I_lower-I_true)>=10^-4){
   cat("\n","Now is",n,"-th iteration ,bias is",abs(I_lower-I_true),"\n")
   n<-n+1
   h<-seq(a,b,(b-a)/n)
   I_lower<-0
   for(i in 1:n){    I_lower<-I_lower + f_z(h[i])*(b-a)/n/C }
}
cat("\n","The sample size is",n," ,bias is",abs(I_lower-I_true),"\n")

#Trapezoidal rule
I_T<-0
for(i in 1:n ){    I_T<-I_T+( f_z(h[i])+f_z(h[i+1]) )/2    }
I_T<-I_T*(b-a)/n/C

a<-0 ;b<-0.7 ;n<-0
I_T<-0
while(abs(I_T-I_true)>=10^-4){
   cat("\n","Now is",n,"-th iteration ,bias is",abs(I_T-I_true),"\n")
   n<-n+1
   h<-seq(a,b,(b-a)/n)
   I_T<-0
   for(i in 1:n){    I_T<-I_T+( f_z(h[i])+f_z(h[i+1]) )/2 }
   I_T<-I_T*(b-a)/n/C
}
cat("\n","The sample size is",n," ,bias is",abs(I_T-I_true),"\n")

#Simpsons rule
if(n%%2==0){
   z1<-0;z2<-0
   for(i in 2:(n/2)){      z1<-z1+2*f_z(h[2*i-1])    }
   for(i in 1:(n/2)){      z2<-z2+4*f_z(h[2*i])    }
   I_S<-( f_z(h[1])+f_z(h[n+1])+z1+z2 )*(b-a)/n/3/C
   rm(z1,z2)
```

```
} else { print("n need to be even") }

a<-0 ;b<-0.7 ;n<-0
I_S<-0
while(abs(I_S-I_true)>=10^-4){
   cat("\n","Now is",n,"-th iteration ,bias is",abs(I_S-I_true),"\n")
   n<-n+2
   h<-seq(a,b,(b-a)/n)
   I_S<-0
   z1<-0;z2<-0
   for(i in 2:(n/2)){      z1<-z1+2*f_z(h[2*i-1])    }
   for(i in 1:(n/2)){      z2<-z2+4*f_z(h[2*i])    }
   I_S<-( f_z(h[1])+f_z(h[n+1])+z1+z2 )*(b-a)/n/3/C
};rm(z1,z2)
cat("\n","The sample size is",n," ,bias is",abs(I_S-I_true),"\n")

#Monte Carlo method with U(0,0.7)
G<-function(z,n){
   z<-c(z,rep(0,n))
   for(i in 1:n){z[i+1]<-(16807*z[i])%%(2^31-1)}
   u<-z[-1]/(2^31-1)
   return(u)
}
confi.of.I_M<-function(n,N,alpha){
   #n=100 ;seed<-1;N=50 ;alpha=0.05
   x<-rep(0,N) ;seed<-1
   for(i in 1:N){
      u<-rep(0,n)
      for(j in 1:n){
         u[j]<-f_z(0.7*G(seed,1))/(10/7)
         seed<-G(seed,1)*(2^31-1) }
      x[i]<-x[i]+sum(u)/n/C                        #I_M<-sum(u)/n/C
   }

   I.bar<-sum(x)/N
   I.sd<-sd(x)
   cat("The 95% confidence interval for estimating I is [",
         I.bar-I.sd/sqrt(N)*qnorm(1-alpha),"",
```

```
          I.bar+I.sd/sqrt(N)*qnorm(1-alpha),"]","\n",I.bar)
}
confi.of.I_M(6,50,0.05)
seed<-2
I_M<-0 ;n<-0
while(abs(I_M-I_true)>=10^-4){
  cat("\n","Now is",n,"-th iteration ,bias is",abs(I_M-I_true),"\n")
  n<-n+1
  u<-rep(0,n)
  for(j in 1:n){
    u[j]<-f_z(0.7*G(seed,1))/(10/7)
    seed<-G(seed,1)*(2^31-1) }
  I_M<-sum(u)/n/C
}
cat("\n","The sample size is",n," ,bias is",abs(I_M-I_true),"\n")

#Acce. Rejection
windows();plot(x=c(seq(0,1,0.001),seq(0,1,0.001)),y=c(f_z(seq(0,1,0.001))/
C,1.6*dunif(seq(0,1,0.001),0,1)),
               col =c(rep(1,1001),rep(2,1001)) )
c<-1.6
envo<-function(x,c){   f_z(x)/C / c }
acc.rej.exp<-function(n,c=1.6){
  u1<-runif(n,0,1)
  Y<-u1
  u2<-runif(n,0,1)
  X<-rep(0,n)
  N<-length(which(u2 <= envo(Y,c) ))
  for(i in 1:N){    X[i]<-Y[which(u2 <=envo(Y,c))][i]     }     #exp(-(Y-
1)^2/2)
  while(N<n){
    uu1<-runif(n-N,0,1)
    Y<-uu1
    uu2<-runif(n-N,0,1)
    if(length(which(uu2<=envo(Y,c)))>0){
      for(i in 1:length(which(uu2<=envo(Y,c)))){ X[N+i]<-
Y[which(uu2<=envo(Y,c))][i] }}
    N<-N+length(which(uu2 <= envo(Y,c)))
```

```
    }
    return(X)
}
acc.rej.exp(1)
PDF<-function(a,b,n,band){
    #a=0;b=1;n=1000;band=0.2
    x<-a
    pdf<-rep(0,length(seq(x,b,0.005)))
    j<-1
    while(a<=b){
        I<-0
        for(i in 1:n){ if( abs(a-acc.rej.exp(1))<=band){ I<-I+1    }    }
        pdf[j]<-I/(2*n*band)
        a<-a+0.005
        j<-j+1
    }
    windows();plot(seq(x,b,0.005),pdf)
    return(pdf)    }
PDF(0,1,5000,0.2)
# F(0.7)
cdf<-0 ;n<-1
while(abs(cdf-I_true)>=10^-4){
    cat("\n","bias is",abs(cdf-I_true),"\n")
    if(n<10^5){n<-n*10}
    I<-0
    for(i in 1:n){ if(acc.rej.exp(1)<=0.7){ I<-I+1    }    }
    cdf<-I/n
}
cat("\n","The sample size is",n," ,bias is",abs(cdf-I_true),"\n")
```

2.

```
px.theta<-function(x,theta){ 1/sqrt(2*pi)*exp(-(x-theta)^2/2) }
p.theta<-function(theta){ 1/(pi*(1+theta^2)) }
px<-function(theta){ px.theta(x,theta)*p.theta(theta) }
#plot(seq(-10,10,0.01),px(seq(-10,10,0.01)))    #plot p(x)
post<-function(theta){ px.theta(x,theta)*p.theta(theta)/I_Spx }
f<-function(theta){    theta*post(theta) }        #theta*p(theta|x)
#plot p(theta|x)
theta<-5 ;x<-rnorm(1,theta,1)
```

```
a<-theta-10 ;b<-theta+10 ;n<-100
h<-seq(a,b,(b-a)/n)
if(n%%2==0){
   z1<-0;z2<-0
   for(i in 2:(n/2)){      z1<-z1+2*px(h[2*i-1])    }
   for(i in 1:(n/2)){      z2<-z2+4*px(h[2*i])    }
   I_Spx<-( px(h[1])+px(h[n+1])+z1+z2 )*(b-a)/n/3
   rm(z1,z2)
} else { print("n need to be even") }
plot(seq(-5,10,0.01),post(seq(-5,10,0.01)))

#estmator with theta=5
N=10000 ;theta=5
y<-rep(0,N)
for(j in 1:N){
   x<-rnorm(1,theta,1)
   a<-theta-10 ;b<-theta+10 ;n<-100
   h<-seq(a,b,(b-a)/n)
   if(n%%2==0){
      z1<-0;z2<-0
      for(i in 2:(n/2)){      z1<-z1+2*px(h[2*i-1])    }
      for(i in 1:(n/2)){      z2<-z2+4*px(h[2*i])    }
      I_Spx<-( px(h[1])+px(h[n+1])+z1+z2 )*(b-a)/n/3
      rm(z1,z2)
   } else { print("n need to be even") }
   if(n%%2==0){
      z1<-0;z2<-0
      for(i in 2:(n/2)){      z1<-z1+2*f(h[2*i-1])    }
      for(i in 1:(n/2)){      z2<-z2+4*f(h[2*i])    }
      I_Sposterior<-( f(h[1])+f(h[n+1])+z1+z2 )*(b-a)/n/3
      rm(z1,z2)
   } else { print("n need to be even") }
   y[j]<-I_Sposterior
};rm(i,I_Spx,I_Sposterior)
I_bar<-sum(y)/N
I_var<-(sum(y^2)-N*I_bar^2)/(N-1)
list(theta=theta,I_bar=I_bar,I_var=I_var)
```

```
#use simpsons to est px's integral
a<--5 ;b<-10 ;n<-100
h<-seq(a,b,(b-a)/n)
if(n%%2==0){
   z1<-0;z2<-0
   for(i in 2:(n/2)){      z1<-z1+2*px(h[2*i-1])    }
   for(i in 1:(n/2)){      z2<-z2+4*px(h[2*i])    }
   I_Spx<-( px(h[1])+px(h[n+1])+z1+z2 )*(b-a)/n/3
   rm(z1,z2)
} else { print("n need to be even") }                  #integrate(px,-Inf,Inf)
#use simpsons to est [theta*p(theta|x)]'s integral
a<--5 ;b<-10 ;n<-1000
h<-seq(a,b,(b-a)/n)
if(n%%2==0){
   z1<-0;z2<-0
   for(i in 2:(n/2)){      z1<-z1+2*f(h[2*i-1])    }
   for(i in 1:(n/2)){      z2<-z2+4*f(h[2*i])    }
   I_Sposterior<-( f(h[1])+f(h[n+1])+z1+z2 )*(b-a)/n/3
   rm(z1,z2)
 } else { print("n need to be even") }              #integrate(f,-Inf,Inf)
```