※2: $X_1, X_2, \dots, X_n \overset{iid}{\sim} \pi(x \mid \theta)$, where $\pi(x \mid \theta) = \sum_{j=1}^{3} P_j \phi_{\sigma_j}(x-\mu_j)$

$$, \sum_{j=1}^{3} P_j = 1 \quad , \quad \phi_\sigma(x-\mu) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) \sim N(\mu, \sigma^2)$$

<u>latent variable</u>

let $Z_{ij} = \begin{cases} 1 & \text{if } x_i \text{ comes from } \phi_{\sigma_j}(x-\mu_j) \\ 0 & \text{o.w.} \end{cases}$ $\quad i = 1, \dots, n$, $j = 1, 2, 3$

s.t. $\quad Z_{n\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix}_{n\times 3}$

<u>j.p.d.f of complete data $(\underset{\sim}{x}, \underset{\sim}{z})$</u>

$$f(\underset{\sim}{x}, \underset{\sim}{z} \mid \theta) = \prod_{i=1}^{n}\left(\prod_{j=1}^{3}\left(P_j \phi_{\sigma_j}(x_i-\mu_j)\right)^{z_{ij}}\right) \quad , \quad \theta = (P_1, P_2, P_3, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3)$$

$\Rightarrow$ loglikelihood : $\log f(\underset{\sim}{x}, \underset{\sim}{z} \mid \theta)$

$$= \sum_{i=1}^{n}\left(Z_{i1}\log P_1 \phi_{\sigma_1}(x_i-\mu_1) + Z_{i2}\log P_2 \phi_{\sigma_2}(x_i-\mu_2) + Z_{i3}\log P_3 \phi_{\sigma_3}(x_i-\mu_3)\right)$$

<u>E-step :</u>

note that $\quad E_{\theta^{(m)}}(Z_{ij} \mid \underset{\sim}{x}) = P_{\theta^{(m)}}(Z_{ij} = 1 \mid \underset{\sim}{x})$

$$= \frac{P_j^{(m)} \phi_{\sigma_j^{(m)}}(x_i - \mu_j^{(m)})}{\sum_{j=1}^{3} P_j^{(m)} \phi_{\sigma_j^{(m)}}(x_i - \mu_j^{(m)})}$$

$$= Z_{ij}^{(m)} \quad , \quad i = 1, 2, \dots, n, \; j = 1, 2, 3$$

$$\Longrightarrow Q(\theta \mid \theta^{(m)}) = E_{\theta^{(m)}}\left(\log f(\underset{\sim}{x}, \underset{\sim}{z} \mid \theta) \mid \underset{\sim}{x}\right)$$

$$= \sum_{i=1}^{n} Z_{i1}^{(m)} \log P_1 \phi_{\sigma_1}(x_i - \mu_1) + \sum_{i=1}^{n} Z_{i2}^{(m)} \log P_2 \phi_{\sigma_2}(x_i - \mu_2) + \sum_{i=1}^{n} Z_{i3}^{(m)} \log P_3 \phi_{\sigma_3}(x_i - \mu_3)$$

where $\quad \phi_{\sigma_j}(x_i - \mu_j) = \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{(x_i-\mu_j)^2}{2\sigma_j^2}} \quad , \; j = 1, 2, 3$

M-step: $\theta^{(m+1)} = \underset{\theta}{\arg\max}\ Q(\theta \mid \theta^{(m)})$, $\theta = (p_1, p_2, p_3, \mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3)$

$$\Rightarrow \begin{cases} \dfrac{\partial Q(\theta \mid \theta^{(m)})}{\partial p_j} = 0 \\ \dfrac{\partial Q(\theta \mid \theta^{(m)})}{\partial \mu_j} = 0 \\ \dfrac{\partial Q(\theta \mid \theta^{(m)})}{\partial \sigma_j} = 0 \end{cases}, \quad j = 1, 2, 3.$$

① for μ_j, $j = 1, 2, 3$.

$$\frac{\partial Q(\theta \mid \theta^{(m)})}{\partial \mu_1} = \frac{\partial}{\partial \mu_1}\left(\sum_{i=1}^{n} z_{i1}^{(m)} \cdot \left(-\frac{(x_i-\mu_1)^2}{2\sigma_1^2}\right)\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} z_{i1}^{(m)} \frac{1}{2\sigma_1^2} \cdot 2 \cdot (x_i - \mu_1) = 0$$

$$\Rightarrow \mu_1^{(m+1)} = \frac{\sum_{i=1}^{n} z_{i1}^{(m)} x_i}{\sum_{i=1}^{n} z_{i1}^{(m)}}$$

Similarly, $\mu_2^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i2}^{(m)} x_i}{\sum_{i=1}^{n} z_{i2}^{(m)}}$, $\mu_3^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i3}^{(m)} x_i}{\sum_{i=1}^{n} z_{i3}^{(m)}}$

② for σ_j^2, $j = 1, 2, 3$.

$$\frac{\partial Q(\theta \mid \theta^{(m)})}{\partial \sigma_1} = \frac{\partial}{\partial \sigma_1}\left(\sum_{i=1}^{n} z_{i1}^{(m)} \left(\log \frac{1}{\sigma_1} + -\frac{(x_i-\mu_1)^2}{2\sigma_1^2}\right)\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} z_{i1}^{(m)} \left(\sigma_1 \cdot \frac{-1}{\sigma_1^2} + \frac{(x_i-\mu_1)^2}{\sigma_1^3} \right) = 0$$

$$\overset{\times \sigma_1^3}{\Rightarrow} \sum_{i=1}^{n} z_{i1}^{(m)} \left(-\sigma_1^2 + (x_i - \mu_1)^2 \right) = 0$$

$$\Rightarrow \sigma_1^{2\,(m+1)} = \frac{\sum_{i=1}^{n} z_{i1}^{(m)} (x_i - \mu_1^{(m+1)})^2}{\sum_{i=1}^{n} z_{i1}^{(m)}}$$

Similarly, $\sigma_2^{2\,(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i2}^{(m)} (x_i - \mu_2^{(m+1)})^2}{\sum_{i=1}^{n} z_{i2}^{(m)}}$

$$\sigma_3^{2\,(m+1)} = \frac{\sum_{i=1}^{n} z_{i3}^{(m)} (x_i - \mu_3^{(m+1)})^2}{\sum_{i=1}^{n} z_{i3}^{(m)}}$$

③ for P_j , $j = 1, 2, 3$.

note : $\max_{P_j} Q(\theta \mid \theta^{(m)})$ with $\sum_{j=1}^{3} P_j = 1$

$\overset{\Leftrightarrow}{\text{Largrange Multiple}}$ $\max_{P_j} Q(\theta \mid \theta^{(m)}) + \lambda\left(\sum_{j=1}^{3} P_j - 1\right)$ —— ⊛

Thus , $\dfrac{\partial (⊛)}{\partial P_1} = \dfrac{\sum_{i=1}^{n} z_{i1}^{(m)}}{P_1} + \lambda = 0$

$\Rightarrow P_1^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i1}^{(m)}}{-\lambda}$

Also , have $P_2^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i2}^{(m)}}{-\lambda}$, $P_3^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{i3}^{(m)}}{-\lambda}$

since $\sum_{j=1}^{3} P_j^{(m+1)} = 1$ $\Rightarrow \dfrac{\sum_{i=1}^{n}\left(z_{i1}^{(m)} + z_{i2}^{(m)} + z_{i3}^{(m)}\right)}{-\lambda} = 1$

$\Rightarrow \lambda = -n$

That is , $P_j^{(m+1)} = \dfrac{\sum_{i=1}^{n} z_{ij}^{(m)}}{n}$, $j = 1, 2, 3$.

Simulation :

true parameter $\theta_0 = (\underset{\sim}{P_0}, \underset{\sim}{\mu_0}, \underset{\sim}{\sigma_0^2})$

with $\underset{\sim}{P_0} = (P_1, P_2, P_3) = (0.2, 0.3, 0.5)$

$\underset{\sim}{\mu_0} = (\mu_1, \mu_2, \mu_3) = (-3, 0, 2)$

$\underset{\sim}{\sigma_0^2} = (\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.64, 0.36, 0.25)$

$\Rightarrow$ case 1 : $\theta^{(0)} = \theta_0$ (起始值即為真實參數值) , $n=2000$

$\Rightarrow \theta^{(1)} = (\underset{\sim}{P}^{(1)}, \underset{\sim}{\mu}^{(1)}, \underset{\sim}{\sigma}^{(1)^2})$

with $\underset{\sim}{P}^{(1)} = (0.2, 0.3, 0.4)$

$\underset{\sim}{\mu}^{(1)} = (0.4077481, 0.4077481, 0.4077481)$

$\underset{\sim}{\sigma}^{(1)^2} = (3.9921538, 3.9921538, 3.9921538)$

$\Rightarrow$ 接著 $\theta^{(2)}, \theta^{(3)}, \theta^{(4)}, \ldots$ 的值皆沒有改變.

case 2：設 $\theta^{(0)}$ 為其他值，但皆和 case 1 一樣。

$\theta^{(m+1)}$ 沒有透過 $\theta^{(m)}$ 更新。

即使將樣本數 $n=2000$ 增加 $n=20000$，結果也一樣。