※1 : Show that : if $\{x_n\}_{n\in\mathbb{N}}$ constructed by fixed point iteration, then $\lim_{n\to\infty} x_n = x$ and x is the fixed point of g.

pf:

(i) Given $\varepsilon > 0$, $\exists\, N > 0$ s.t. $(\lambda^N + \lambda^{N-1} + \cdots + \lambda + 1)\,|x_1 - x_0| + |x_0 - x| < \varepsilon$

$$n \ge N \Rightarrow |x_n - x|$$

$$= |x_n - x_{n-1} + x_{n-1} - x|$$

$$= \left| \sum_{i=1}^{n} (x_{n-i+1} - x_{n-i}) + x_0 - x \right|$$

$$\le \sum_{i=1}^{n-1} |\underbrace{x_{n-i+1} - x_{n-i}}_{g(x_{n-i}) - g(x_{n-i-1})}| + |x_0 - x| + |x_1 - x_0|$$

$$\le \lambda \sum_{i=1}^{n-2} |\underbrace{x_{n-i} - x_{n-i-1}}_{g(x_{n-i-1}) - g(x_{n-i-2})}| + (\lambda+1)|x_1 - x_0| + |x_0 - x|$$

$$\le \lambda^2 \sum_{i=1}^{n-3} |x_{n-i-1} - x_{n-i-2}| + (\lambda^2+\lambda+1)|x_1 - x_0| + |x_0 - x|$$

$$\cdots \le \lambda^n |x_1 - x_0| + (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda + 1)|x_1 - x_0| + |x_0 - x|$$

$$\begin{pmatrix} \because n \ge N \\ \therefore \lambda^n \le \lambda^N \end{pmatrix} \le (\lambda^N + \lambda^{N-1} + \lambda^{N-2} + \cdots + \lambda + 1)\,|x_1 - x_0| + |x_0 - x|$$

$$< \varepsilon$$

Hence, we have fixed point iteration $\{x_n\}_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{} x$.

(ii) Claim : x is the fixed point of g.

since $|x_{n+1} - g(x)| = |g(x_n) - g(x)|$

$$\le \lambda\,|x_n - x| < |x_n - x| \to 0$$

Thus, $\lim_{n\to\infty} x_n = g(x)$ and $\lim_{n\to\infty} x_n = x$

$\Rightarrow$ $g(x) = x$ i.e. x is the fixed point of g.

※2. Prove that if $\lim_{k\to\infty} x_k = x$, where $\{x_k\}_{k\in\mathbb{N}}$ is fixed point iteration seq., x is fixed point of g, then $\{x_k\}_{k\in\mathbb{N}}$ linearly converges to x.

pf: Claim: $\lim_{k\to\infty} \frac{|x_k - x|}{|x_{k-1} - x|^p} = c$, $c \le 1$, $p = 1$

Since $|x_k - x| = |g(x_{k-1}) - g(x)|$

$\le \lambda \, |x_{k-1} - x|$

$\left(\begin{array}{l} \text{if } |g'(x)| < 1 \\ \text{then } \frac{|g(x_1) - g(x_2)|}{|x_1 - x_2|} < 1 \\ \text{i.e. } \exists \underset{\in[0,1)}{\lambda} \text{ s.t. } |g(x_1) - g(x_2)| \le \lambda |x_2 - x_1| \end{array} \right)$

thus $\lim_{k\to\infty} \frac{|x_k - x|}{|x_{k-1} - x|} = \lim_{k\to\infty} \frac{\lambda |x_{k-1} - x|}{|x_{k-1} - x|} = \lambda < 1$, $(\lambda \in [0,1))$

So, $\{x_k\}_{k\in\mathbb{N}}$ linearly converges to x.