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Monte Carlo Method

$$F_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)$$

Assume that we uniformly indep. draw a random sample from (a,b), i.e. $X_1, X_2, \dots, X_n, \dots \sim U(a,b)$

$$\Rightarrow E(F_n) = \frac{b-a}{n}\sum_{i=1}^{n} E(f(X_i))$$

$$= \frac{b-a}{n}\, n \cdot \int_a^b f(x)\,p(x)\,dx \quad , \text{w/ } p(x) = \frac{1}{b-a}$$

$$= \int_a^b f(x)\,dx$$

$$\Rightarrow Var(F_n) = Var\left(\frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\right)$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n}\frac{f(X_i)}{P(X_i)}\right)$$

$$= \frac{1}{n^2}\, n\, Var\left(\frac{f(X)}{P(X)}\right) = \frac{1}{n} Var\left(\frac{f(X)}{P(X)}\right) \xrightarrow[n\to\infty]{} 0.$$

- Let $\hat{I} = \frac{1}{n}\sum_{i=1}^{n}\frac{f(X_i)}{P(X_i)}$

Assume that we randomly sample $X_1, X_2, \dots, X_n, \dots$ iid. with $p(X)$,

We have $Var(\hat{I}) \to 0$ and $E(\hat{I}) = I$

i.e. $\hat{I} \xrightarrow{P} I$ as $n\to\infty$

Also, $|\hat{I} - I| = O(n^{-\frac{1}{2}})$

Note Quardract, 梯形 $|\hat{I}_T - I| = O(n^{-2})$

Simpson $|\hat{I}_S - I| = O(n^{-4})$

How to choose $p(x)$?

The Best is $p(x) \propto f(x)$ i.e. $\frac{f(x)}{p(x)} = c$, for c: constant

Since $Var\left(\frac{f(x)}{p(x)}\right) = Var(c) = 0$

But, the problem coming soon,

How to generate $X \sim p(X)$?

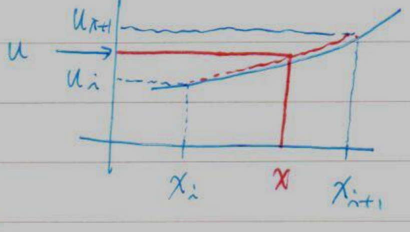
If F^{-1} is not available, but F is either available or can be approx.

Consider grid point $x_1, x_2, \cdots, x_K$

Evaluate $u_1 = F(x_1), \cdots, u_K = F(x_K)$

Sample $U \sim U(0,1)$, Find index i s.t. $u_i \le u \le u_{i+1}$

Then $X = \frac{u_{i+1} - u}{u_{i+1} - u_i} x_i + \frac{u - u_i}{u_{i+1} - u_i} x_{i+1}$



$F(x) = \int_0^x \frac{2}{3} t^{-\frac{1}{3}} dt = x$

$F^{-1}(x) = x$

$X = F^{-1}(U) \sim p_2(X)$