o parametric density estimation

Assume $X_1, X_2, \dots \sim f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E(ISE) = E\int \left(f(x;\hat{\theta}_{MLE}) - f(x;\theta)\right)^2 dx \quad \text{———} (*)$$

with $\hat{\theta}_{MLE} = (\bar{X}_n, S_n^2)$, $\theta = (\mu, \sigma^2)$

where $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

given x, consider $\left(f(x;\hat{\theta}_{MLE}) - f(x;\theta)\right)$

$$= f'(\theta_*)\,(\hat{\theta}_{MLE} - \theta)$$

where $f' = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot \frac{-1}{\sigma^2}(x-\mu)$

$= \frac{1}{\sqrt{2\pi}\sigma^3}(x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, let $v = x-\mu$

$= \frac{1}{\sqrt{2\pi}\sigma^3} v e^{-\frac{v^2}{2\sigma^2}}$

Back to $(*)$, $= \int E\left(f(\hat{\theta}_{MLE}) - f(\theta)\right)^2 dx$

$= \int E\left[f'(\theta_*)(\hat{\theta}_{MLE} - \theta)\right]^2 dx$

$\leq \int E\left[M(\hat{\theta}_{MLE} - \theta)\right]^2 dx \quad \text{———} (**)$

Since $f'(\theta_*) \propto v e^{-\frac{v^2}{2\sigma^2}}$ with $v = x-\mu$

and $\lim_{v\to\infty} \frac{v}{e^{-\frac{v}{2\sigma^2}}} = 0$,

there exist $M > 0$ s.t. $|f'(\theta_*)| \leq M$, $\forall \theta_*$

So, $(**) \leq \int E(\hat{\theta}_{MLE} - \theta)^2 dx$

$= Var\,\hat{\theta}_{MLE} + Bias(\hat{\theta}_{MLE})^2 \quad \text{———} (***)$

$$\Rightarrow \quad Bias(\hat{\theta}_{MLE})^2 = \left((E\bar{X}_n - \mu)^2 \ , \ (ES_n^2 - \sigma^2)^2 \right)$$

$$= (\ 0 \ , \ 0 \ )$$

$$Var(\hat{\theta}_{MLE}) = \left(Var(\bar{X}_n) \ , \ \underbrace{Var(S_n^2)}_{= \frac{\sigma^4}{(n-1)^2} Var\left(\frac{(n-1)S_n^2}{\sigma^2}\right) = \frac{2\sigma^4}{(n-1)}} \right)$$

$$= \left(\frac{\sigma^2}{n} \ , \ \frac{2\sigma^4}{n-1} \right)$$

$$= \left(O(n^{-1}) \ , \ O(n^{-1}) \right)$$

Hence, $E(ISE) = O(n^{-1})$