(1) Assume that $f : [a,b] \to \mathbb{R}$ is a twice continuously differentiable function with $x^* \in (a,b)$ such that $f(x^*) = 0$ and $f'(x^*) \neq 0$. Please show that there exists a $\delta > 0$ such that the sequence of $\{x_n\}$ generated by Newton's method converges to x^* when $x_0 \in (x^* - \delta, x^* + \delta)$.

pf: the root of f $\Leftrightarrow$ the fixed point of g

($f(x^*) = 0 \Leftrightarrow g(x^*) = x^*$)

where $g(x) := x - \dfrac{f(x)}{f'(x)}$ ($\because \{x_n\}$ generated by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, \dots$)

Claim: g is contractive on $[x^* - \delta, x^* + \delta]$.

1° Claim: $|g(x_1) - g(x_2)| \le \lambda |x_1 - x_2| \Leftrightarrow |g'(x_3)| < 1 \quad \forall x_3 \in (a,b)$

consider $g'(x) = 1 - \dfrac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \dfrac{f(x) f''(x)}{(f'(x))^2}$

$\xRightarrow[\text{root of } f]{x^* \text{ is}} g'(x^*) = \dfrac{f(x^*) f''(x^*)}{(f'(x^*))^2} = 0$

Since f, f', f'' are conti. then we have g, g' are conti.

take $\varepsilon = 1$, $\exists \delta > 0$ s.t.

$|x - x^*| < \delta \Rightarrow |g'(x) - \underbrace{g'(x^*)}_{=0}| = |g'(x)| < \varepsilon = 1$

That is, for $x \in (x^* - \delta, x^* + \delta)$, $|g'(x)| < 1$

2° Claim: $g : [x^* - \delta, x^* + \delta] \to [x^* - \delta, x^* + \delta]$

$\forall x \in [x^* - \delta, x^* + \delta]$,

$|g(x) - x^*| = |g(x) - g(x^*)|$

$= |g'(c)(x - x^*)|$ with c between x and x^*

$< |x - x^*| < \delta$

i.e. $x^* - \delta < g(x) < x^* + \delta$

Hence, g is contractive on $[x^* - \delta, x^* + \delta]$,

by Fixed-point iteration Thm, for any $x_0 \in [x^* - \delta, x^* + \delta]$
the seq. $\{x_n\} \xrightarrow{n\to\infty} x^*$

(1)' Show that Newton's iterative method is quardratic convergence.

pf: by Taylor's expansion at x_n,

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2}(x - x_n)^2 + R_n$$

$$\overset{x^* \text{ is}}{\underset{\text{root of } f}{\Longrightarrow}} \quad 0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(x_n)}{2}(x^* - x_n)^2$$

$$\Rightarrow \quad \frac{f(x_n) + f'(x_n)(x^* - x_n)}{f'(x_n)} = -\frac{f''(x_n)}{2f'(x_n)}(x^* - x_n)^2$$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Rightarrow \quad x^* - x_{n+1} = -\frac{f''(x_n)}{2f'(x_n)}(x^* - x_n)^2$$

$$\Rightarrow \quad \frac{|x^* - x_{n+1}|}{|x^* - x_n|^2} = \left|\frac{f''(x_n)}{2f'(x_n)}\right|$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{|x^* - x_{n+1}|}{|x^* - x_n|^2} = \left|\frac{f''(x^*)}{2f'(x^*)}\right| \quad \text{where } 0 \le \left|\frac{f''(x^*)}{2f'(x^*)}\right| < \infty$$

i.e. $\{x_n\}$ is quardratic convergence.