3: Apply Metropolis-Hasting algorithm with $g(\cdot|x) \sim U(x-\varepsilon, x+\varepsilon)$ to simulate data from $\pi(\cdot) \sim N(0,1)$.

(i)

- Proposal distribution:

$$g(y|x) \sim U(x-\varepsilon, x+\varepsilon)$$

$$\text{where} \quad g(y|x) = \begin{cases} \dfrac{1}{2\varepsilon} & , y \in (x-\varepsilon, x+\varepsilon) \\ 0 & , \quad o.w. \end{cases}$$

Acceptance probability:

$$\alpha(x,y) = \min\{ \frac{\pi(y)g(x|y)}{\pi(x)g(y|x)}, 1 \}$$

- Metropolis-Hasting algorithm:

Step 1: start with $X^{(0)} = x^{(0)}$ s.t. $\pi(x^{(0)}) > 0$.

Step 2: generate $y \sim g(\cdot|x^{(m)}) \sim U(x^{(m)}-\varepsilon, x^{(m)}+\varepsilon)$.

Step 3: compute

$$\alpha(x^{(m)},y) = \min\{ \frac{\pi(y)g(x^{(m)}|y)}{\pi(x^{(m)})g(y|x^{(m)})}, 1 \}$$

$$= \begin{cases} \exp\left(-\dfrac{y^2}{2} + \dfrac{x^{(m)\,2}}{2}\right) & , |x^{(m)}| \le |y| \\ 1 & , o.w. \end{cases}$$

Step 4: generate $u \sim U(0,1)$

If $u \le \alpha(x^{(m)},y)$, then set $x^{(m+1)} = y$.

Otherwise, set $x^{(m+1)} = x^{(m)}$.

Repeat Step 2~4, we have $x^{(1)}, x^{(2)}, \dots, x^{(n)} \sim N(0,1)$

- Simulation:

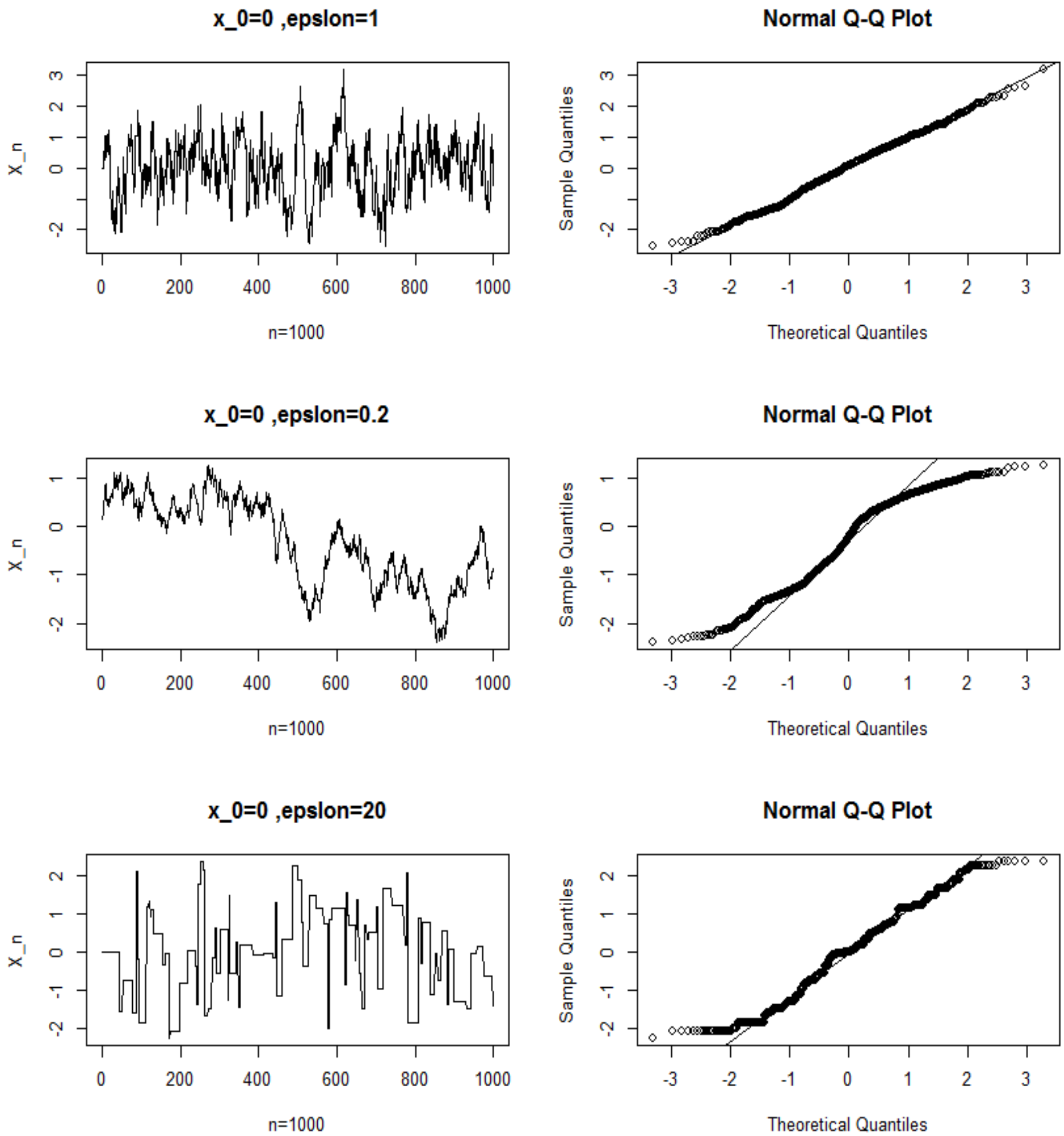
Case 1:

$x^{(0)} = 0$, $n = 1000$, $\varepsilon_1 = 1$, $\varepsilon_2 = 0.2$, $\varepsilon_3 = 20$.

我們發現不同的 ε 將造成生成的 $x^{(1)}, x^{(2)}, \dots, x^{(1000)}$不一定是$N(0,1)$。 (如下的右圖)

首先，把 $g(y|x^{(m)})$ 想成是給定現在狀態 $x^{(m)}$轉移到狀態 y 的機率，而 $\alpha(x^{(m)},y)$ 則是接受 ”狀態 $x^{(m)}$轉移到狀態 y ” 的機率。

因為 ε 會影響著 $y \sim g(\cdot|x^{(m)})$的可能值，所以如果 ε 大，則轉移前後的差異也相對的大；如果 ε 小，轉移前後的差異也相對的小。(如下的左圖)

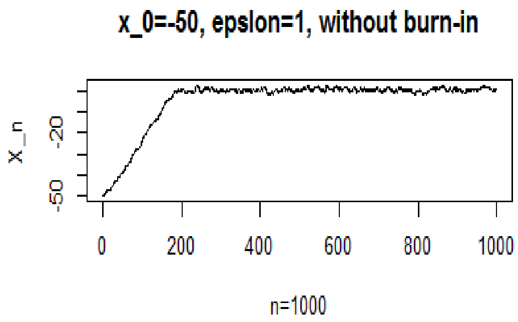


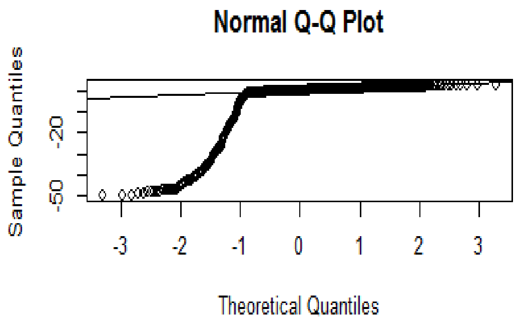
Case 2:

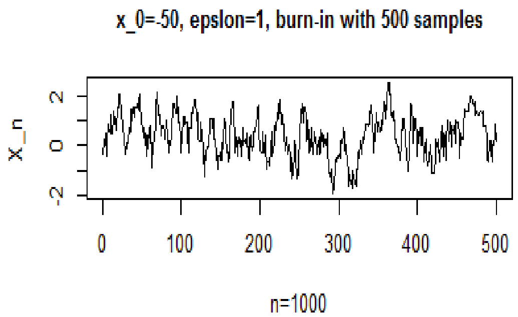
$x^{(0)} = -50$, $n = 1000$,$\varepsilon_1 = 1$,$\varepsilon_2 = 10$.

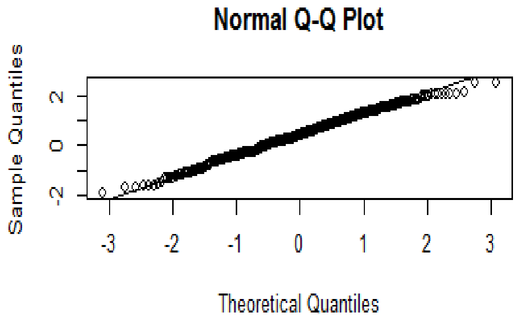
首先，起始狀態 $x^{(0)} = -50$，如果 $\varepsilon_1 = 1$ (上例中不算太大也不算太小)，則發現它需要一些步數才定進入服從 N(0,1) 的狀態。也就是說，如果我們起始點選的不夠好，此演算法需要所謂的暖機時間，才能夠穩定的生成我們想要的分佈。(如下圖上半部)

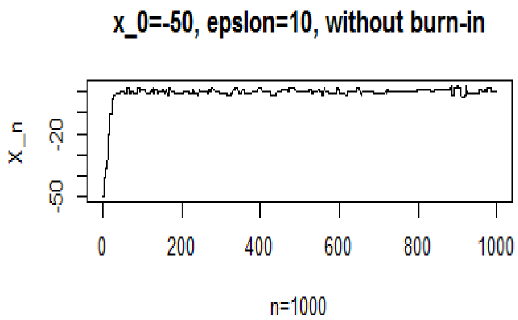
如果取 $\varepsilon_2 = 10$ ，雖然能夠減少暖機時間，但是，相對的會遇到 case 1 中 ε 大，轉移前後的差異也大，使得生成出來的不服從 N(0,1)。

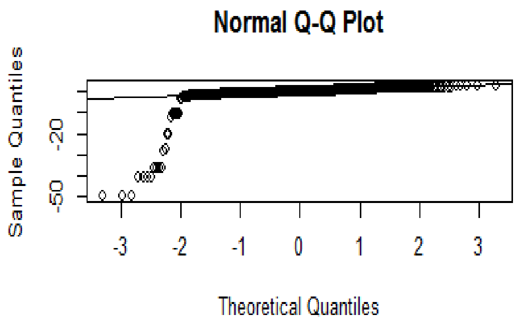

x_0=-50, epslon=1, without burn-in
X_n
n=1000

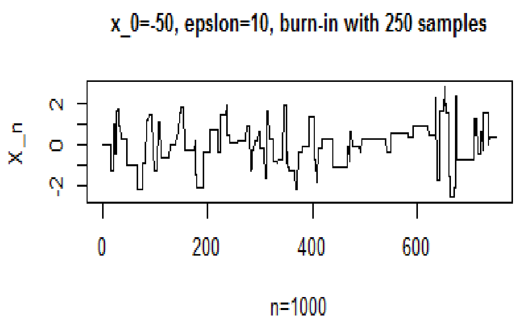

Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles

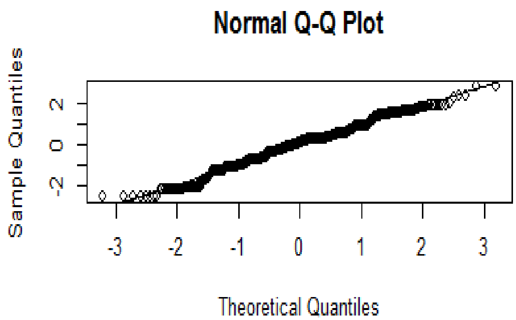

x_0=-50, epslon=1, burn-in with 500 samples
X_n
n=1000


Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles


x_0=-50, epslon=10, without burn-in
X_n
n=1000


Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles


x_0=-50, epslon=10, burn-in with 250 samples
X_n
n=1000


Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles

(ii)

To estimate $\mathrm{E}(\exp(\mathrm{Z}^{16}))$.

Note:

By Ergodic theorem,

$\{\mathrm{X_k}\}_{k=1}$ be an irreducible and aperiodic with stationary distribution $\pi(\cdot)$

Then,

$$\frac{1}{\mathrm{n}}\sum_{i=1}^{n}\exp(X_i^{16}) \overset{P}{\rightarrow} E_\pi(\exp(Z^{16}))$$

<u>Simulation:</u>

By (i), with $\mathrm{x}^{(0)} = 0$ $,\mathrm{n} = 1000$, $\varepsilon = 1$

Since $\frac{1}{\mathrm{n}}\sum_{i=1}^{n}\exp(X_i^{16}) = \infty$,

In this case, we can not estimate $\mathrm{E}(\exp(\mathrm{Z}^{16}))$.