Hw 10

※1 : $(Y_1, Y_2, Y_3, Y_4) \sim \text{multnomial}\ (\underset{P_1}{0.5-0.5\theta},\ \underset{P_2}{0.25\theta},\ \underset{P_3}{0.25\theta},\ \underset{P_4}{0.5})$

$$f(y \mid p) = \binom{n}{y_1\ y_2\ y_3\ y_4} p_1^{y_1} p_2^{y_2} p_3^{y_3} p_4^{y_4}$$

$$= h(y) \prod_{i=1}^{4} p_i^{y_i}$$

$$= h(y) \exp\left(\sum_{i=1}^{4} y_i \log p_i \right)$$

$$\Rightarrow \log f(y \mid p) = \log h(y) + \sum_{i=1}^{4} y_i \log p_i \quad , \quad p = (p_1, p_2, p_3, p_4)$$

$$\Rightarrow \log f(y \mid \theta) = \log h(y) + y_1 \log \tfrac{1}{2} - \tfrac{\theta}{2} + y_2 \log \tfrac{\theta}{4} + y_3 \log \tfrac{\theta}{4} + y_4 \log \tfrac{1}{2}$$

E-step : $Q(\theta \mid \theta^{(n)}) = E_{\theta^{(n)}}\left(\log f(y \mid \theta) \mid \underset{\sim}{x} \right)$, (observed) $\underset{\sim}{x} = (y_1, y_2, y_3+y_4) = (38, 34, 125)$

$$= E_{\theta^{(n)}}(\log h(y) \mid \underset{\sim}{x}) + 38 \cdot \log \tfrac{1}{2} - \tfrac{\theta}{2} + 34 \cdot \log \tfrac{\theta}{4}$$

$$+ E_{\theta^{(n)}}\left(y_3 \log \tfrac{\theta}{4} \mid \underset{\sim}{x} \right) + E_{\theta^{(n)}}\left(y_4 \log \tfrac{1}{2} \mid \underset{\sim}{x} \right)$$

M-step : $\frac{d}{d\theta} Q(\theta \mid \theta^{(n)}) = 38 \cdot \frac{-\frac{1}{2}}{\frac{1}{2} - \frac{\theta}{2}} + 34 \cdot \frac{4}{\theta} \cdot \frac{1}{4} + E_{\theta^{(n)}}(y_3 \mid \underset{\sim}{x}) \frac{4}{\theta} \cdot \frac{1}{4} = 0$

$$\Rightarrow \frac{-38}{1-\theta} + \frac{34}{\theta} + \frac{1}{\theta} E_{\theta^{(n)}}\left(y_3 \mid (y_1, y_2, y_3+y_4) \right) \quad \text{———} \circledast$$

note : $(Y_1, Y_2, Y_3, Y_4) \sim \text{multnomial}\ (n, p_1, p_2, p_3, p_4)$

$$\Rightarrow (Y_3, Y_4) \mid Y_1 = y_1, Y_2 = y_2 \sim \text{multnomial}\ \left(n - y_1 - y_2, \frac{p_3}{1-p_1-p_2}, \frac{p_4}{1-p_1-p_2} \right)$$

$$\Rightarrow Y_3 \mid Y_1 = 38, Y_2 = 34, Y_3 + Y_4 = 125 \sim \text{Bin}\left(125, \frac{p_3}{p_3+p_4} \right)$$

$$\circledast \Rightarrow \frac{-38}{1-\theta} + \frac{34}{\theta} + \frac{1}{\theta} \cdot 125 \cdot \frac{\frac{\theta^{(n)}}{4}}{\frac{1}{2} + \frac{\theta^{(n)}}{4}} = 0$$

$$\Rightarrow -38\theta + 34(1-\theta) + (1-\theta)\, 125 \cdot \frac{\theta^{(n)}}{2+\theta^{(n)}} = 0$$

$$\Rightarrow -38\theta - 34\theta - 125 \frac{\theta^{(n)}}{2+\theta^{(n)}} \theta + 34 + 125 \frac{\theta^{(n)}}{2+\theta^{(n)}} = 0$$

$$\Rightarrow \left(72 + \frac{125\,\theta^{(n)}}{2+\theta^{(n)}} \right) \theta = 34 + \frac{125\,\theta^{(n)}}{2+\theta^{(n)}}$$

$$\Rightarrow \quad \theta^{(n+1)} = \frac{34 + \frac{125\,\theta^{(n)}}{2+\theta^{(n)}}}{72 + \frac{125\,\theta^{(n)}}{2+\theta^{(n)}}}$$

Simulation

Use EM algorithm to estimate θ with initial value $\theta_0 = 0.5$.

$\hookrightarrow$ After 8 iterations, satisfying stopping condition:

$$|\theta^{(n+1)} - \theta^{(n)}| < 10^{-7},$$

we have $\hat{\theta} = 0.6268214$

※2 :

(i) Since a window replying she have zero dependent children may be two reason.

① She did not have any children.

② She has, but the children grows up.

(ii) Let $X_1, X_2, X_3, \cdots, X_n \sim f(x;p) = p f_1(x) + (1-p) f_2(x)$

where $f_1(x) = 1$ if $x = 0$. and $f_2(x) = \frac{\mu^x}{x!} e^{-\mu}$, $x = 0, 1, \cdots$

Observed data :

$\underset{\sim}{x} = (x_1, x_2, \cdots, x_{4079}) = (0, 0, \cdots, 0, 1, 1, \cdots, 2, 2, \cdots, 3, 4, 5, 6)$

where the numbers of $X = 0$ is 3062

" $X = 1$ is 587

" $X = 2$ is 284

" $X = 3$ is 103

" $X = 4$ is 33

" $X = 5$ is 4

" $X = 6$ is 2.

latent variable :

let $Z_{ij} = \begin{cases} 1 & \text{if } x_i \text{ come from } f_j \\ 0 & \text{o.w.} \end{cases}$, $i = 1, 2, \cdots, n$; $j = 1, 2$

jpdf of $(\underset{\sim}{x}, \underset{\sim}{z})$:

$$f(\underset{\sim}{x}, \underset{\sim}{z} \mid p) = \prod_{i=1}^{n} \left[\left(p f_1(x_i) \right)^{z_{i1}} \left((1-p) f_2(x_i) \right)^{z_{i2}} \right]$$

$$\Rightarrow \text{loglikelihood} : \sum_{i=1}^{n} \left(Z_{i1} \log p f_1(x_i) + Z_{i2} \log (1-p) f_2(x_i) \right)$$

$$= \sum_{i=1}^{n} Z_{i1} \log p + \sum_{i=1}^{n} Z_{i1} \log f_1(x_i) + \sum_{i=1}^{n} Z_{i2} \log(1-p) + \sum_{i=1}^{n} Z_{i2} \log f_2(x_i) \quad \text{——} \circledast$$

E-step:

note. $E_{(p,\mu)^{(m)}} (Z_{ij} \mid \underset{\sim}{x})$

$$= 1 \cdot P_{(p,\mu)^{(m)}} (Z_{ij} = 1 \mid \underset{\sim}{x}) + 0 \cdot P_{(p,\mu)^{(m)}} (Z_{ij} = 0 \mid \underset{\sim}{x})$$

$$= Z_{ij}^{(m)}$$

where
$$Z_{i1}^{(m)} = \frac{p^{(m)} f_1(x_i)}{p^{(m)} f_1(x_i) + (1-p^{(m)}) f_2(x_i)}$$
$$Z_{i2}^{(m)} = \frac{(1-p^{(m)}) f_2(x_i)}{p^{(m)} f_1(x_i) + (1-p^{(m)}) f_2(x_i)} \qquad \left(Z_{i1}^{(m)} + Z_{i2}^{(m)} = 1 \right)$$

$$\Rightarrow Q(p,\mu \mid (p,\mu)^{(m)}) = E_{(p,\mu)^{(m)}} (\circledast \mid \underset{\sim}{x})$$

$$= \sum_{i=1}^{n} Z_{i1}^{(m)} \log p + \sum_{i=1}^{n} Z_{i1}^{(m)} \log f_1(x_i) + \sum_{i=1}^{n} Z_{i2}^{(m)} \log(1-p) + \sum_{i=1}^{n} Z_{i2}^{(m)} \log f_2(x_i)$$

M-step: $\frac{\partial}{\partial p} Q(p,\mu \mid (p,\mu)^{(m)}) = \frac{\sum_{i=1}^{n} Z_{i1}^{(m)}}{p} + \frac{-\sum_{i=1}^{n} Z_{i2}^{(m)}}{(1-p)} = 0$

$$\Rightarrow (1-p) \sum_{i=1}^{n} Z_{i1}^{(m)} - p \sum_{i=1}^{n} Z_{i2}^{(m)} = 0$$

$$\Rightarrow p \left(\sum_{i=1}^{n} Z_{i1}^{(m)} + \sum_{i=1}^{n} Z_{i2}^{(m)} \right) = \sum_{i=1}^{n} Z_{i1}^{(m)}$$

$$\Rightarrow p^{(m+1)} = \frac{\sum_{i=1}^{n} Z_{i1}^{(m)}}{\sum_{i=1}^{n} (Z_{i1}^{(m)} + Z_{i2}^{(m)})} = \frac{1}{n} \sum_{i=1}^{n} Z_{i1}^{(m)}$$

$$\frac{\partial}{\partial \mu} Q(p,\mu \mid (p,\mu)^{(m)}) = \frac{\partial}{\partial \mu} \left(\sum_{i=1}^{n} Z_{i2}^{(m)} x_i \log \mu - \sum_{i=1}^{n} Z_{i2}^{(m)} \mu \right) = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} Z_{i2}^{(m)} x_i}{\mu} = \sum_{i=1}^{n} Z_{i2}^{(m)}$$

$$\Rightarrow \mu^{(m+1)} = \frac{\sum_{i=1}^{n} x_i Z_{i2}^{(m)}}{\sum_{i=1}^{n} Z_{i2}^{(m)}}$$

Simulation

- consider x_i, $i = 1, \dots, 4079$ with probability p comes from $f_1(x)$ and with probability $(1-p)$ comes from $f_2(x)$

  where $f_1(x) = 1$, $x = 0$ and $f_2(x) = \frac{\mu^x}{x!} e^{-\mu}$, $x = 0, 1, 2, \dots$

- Use EM method, estimate (p, μ) with initial value

  $\theta_0 = (p_0, \mu_0) = (0.75, 0.4)$

  After 36 iterations, satisfying stopping condition:

  $$|p^{(n+1)} - p^{(n)}| < 10^{-7} \quad \text{and} \quad |\mu^{(n+1)} - \mu^{(n)}| < 10^{-7}.$$

  we get $\hat{\theta} = (\hat{p}, \hat{\mu}) = (0.6150565, 1.0378387)$