Matrix Decompositions

April 10, 2023

1 Determinant and Trace

A determinant is a mathematical object in the analysis and solution of linear equation systems. They are only defined for square matrices e.g.:

$$det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \tag{1}$$

Theorem
For any square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ it holds that $\boldsymbol{A}$ is invertible if and only if $det(\boldsymbol{A}) \neq 0$.

For $n = 2$

$$det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{2}$$

and for $n = 3$

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =& a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ & - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \end{aligned} \tag{3}$$

A matrix $\boldsymbol{T}$ is called an *upper-triangular matrix* if $T_{ij} = 0$ for $i > j$, i.e., the matrix is zero below its diagonal. Similarly, a *lower-triangular matrix* can be defined if all its elements above the diagonal are zero. For a triangular matrix its determinant is the product of its diagonal elements

$$det(\boldsymbol{T}) = \prod_{i=1}^{n} T_i i \tag{4}$$

Determinants can be used as a measure of volume.

Computation of determinants for higher dimensions $n > 3$ can be achieved by recursively applying the Laplace expansion.

Theorem - Laplace Expansion
Consider a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Then, for all $j = 1, \dots, n$:

1. Expansion along column j

$$det(\boldsymbol{A}) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} det(A_{k,j}) \tag{5}$$

2. Expansion along row j

$$det(\boldsymbol{A}) = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} det(A_{j,k}) \tag{6}$$

Here $\boldsymbol{A}_{k,j} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the *submatrix* of $\boldsymbol{A}$ that we obtain when we delete row k and column j.

For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ the determinant exhibits the following properties:

- The determinant of a matrix product is the product of the corresponding determinants, $det(\boldsymbol{AB}) = det(\boldsymbol{A})det(\boldsymbol{B})$.
- Determinants are invariant to transposition, i.e., $det(\boldsymbol{A}) = det(\boldsymbol{A}^T)$.
- If $\boldsymbol{A}$ is regular (invertible), then $det(\boldsymbol{A}^{-1}) = \frac{1}{det(\boldsymbol{A})}$.
- Similar matrices possess the same determinant. Therefore, for a linear mapping $\Phi : V \to V$ all transformation matrices $\boldsymbol{A}_\Phi$ of Φ have the same determinant. Thus, the determinant is invariant to the choice of basis of a linear mapping.
- Adding a multiple of a column/row to another one does not change $det(\boldsymbol{A})$.
- Multiplication of a column/row with $\lambda \in \mathbb{R}$ scales $det(\boldsymbol{A})$ by λ. In particular, $det(\lambda\boldsymbol{A}) = \lambda^n det(\boldsymbol{A})$.
- Swapping two rows/columns changes the sign of $det(\boldsymbol{A})$.

Theorem
A matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has $det(\boldsymbol{A}) \neq 0$ if and only if $rk(\boldsymbol{A}) = n$. In other words, $\boldsymbol{A}$ is invertible if and only if it has full rank.

Definition
The *trace* of a square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is defined as

$$tr(\boldsymbol{A}) := \sum_{i=1}^{n} a_{ii} \tag{7}$$

i.e., the trace is the sum of the diagonal elements of $\boldsymbol{A}$.
The trace satisfies the following properties:

- $tr(\boldsymbol{A}+\boldsymbol{B}) = tr(\boldsymbol{A}) + tr(\boldsymbol{B})$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$.
- $tr(\alpha\boldsymbol{A}) = \alpha\ tr(\boldsymbol{A})$ for $\alpha \in \mathbb{R}$ and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$.
- $tr(\boldsymbol{I}_n) = n$
- $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$ for $\boldsymbol{A} \in \mathbb{R}^{n\times k}$ and $\boldsymbol{B} \in \mathbb{R}^{k\times n}$

The trace is invariant under cyclic permutations

$$tr(\boldsymbol{AKL}) = tr(\boldsymbol{KLA}) \tag{8}$$

for matrices $\boldsymbol{A} \in \mathbb{R}^{a\times k}, \boldsymbol{K} \in \mathbb{R}^{k\times l}, \boldsymbol{L} \in \mathbb{R}^{l\times a}$.

As a special case it holds that

$$tr(\boldsymbol{x}\boldsymbol{y}^{\boldsymbol{T}}) = tr(\boldsymbol{y}^T\boldsymbol{x}) = \boldsymbol{y}^T\boldsymbol{x} \in \mathbb{R} \tag{9}$$

Given a linear mapping $\Phi : V \to V$, which can also be represented with a transformation matrix $\boldsymbol{A}$ we know that a basis change results in a different transformation matrix $\boldsymbol{B}$. This new basis change can be represented as $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{AS}$ for a suitable matrix $\boldsymbol{S}$. The corresponding trace can be calculated as follows

$$tr(\boldsymbol{A}) = tr(\boldsymbol{S}^{-1}\boldsymbol{AS}) = tr(\boldsymbol{ASS}^{-1}) = tr(\boldsymbol{A}) \tag{10}$$

Hence, the matrix representations of linear mappings are basis dependent, but the trace of a linear mapping Φ is independent of the basis.

Definition - Characteristic Polynomial
For $\lambda \in \mathbb{R}$ and a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$

$$\begin{aligned} p_{\boldsymbol{A}}(\lambda) &:= det(\boldsymbol{A} - \lambda \boldsymbol{I}) \\ &= c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + (-1)^n \lambda^n \end{aligned} \tag{11}$$

$c_0, \ldots, c_{n-1} \in \mathbb{R}$, is the *characteristic polynomial* of $\boldsymbol{A}$. In particular

$$c_0 = det(\boldsymbol{A}) \tag{12}$$

$$c_{n-1} = (-1)^{n-1} tr(\boldsymbol{A}) \tag{13}$$

The characteristic polynomial will allow us to compute eigenvalues and eigenvectors.