Linear Algebra

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1 Linear Independence

According to vector space definition, any addition and scaling by some factor with elements within the space results with a new element that is also within this space. Finding a set of vectors that can represent every other vector within this space is called *basis*.

Definition - Linear Combination
Consider a vector space V and a finite number of vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in V$. Then, every $\boldsymbol{v} \in V$ of the form

$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V \tag{1}$$

with $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ is a *Linear Combination* of the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$.

The $\mathbf{0}$-vector can always be represented as linear combination if all scaling factors λ_i are 0. The more interesting way to construct the $\mathbf{0}$-vector is with factors that are no 0, which is called the non-trivial way.

Definition - Linear (In)dependence
Let us consider a vector space V with $k \in \mathbb{N}$ and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in V$. If there is a non-trivial linear combination, such that $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ are linearly dependent. If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$ the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ are linearly independent.

Linear independence is one of the most important concepts of linear algebra. Removing any vector of the basis, means that certain vectors cannot be represented anymore.

Some important properties:

- k vectors are either linearly dependent of linearly independent. There is no third option.
- If at least on of the vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ is $\mathbf{0}$ then they are linearly dependent. The same holds if two vectors are identical.
- The vectors $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \dots, k\}, k \geqslant 2$, are linearly dependent if and only if (at least) one of them is a linear combination of the others. In particular, if one vector is a multiple of another vector, i.e. $\boldsymbol{x}_i = \lambda \boldsymbol{x}_j, \lambda \in \mathbb{R}$ then the set $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_k : \boldsymbol{x}_i \neq \mathbf{0}, i = 1, \dots, k\}$ is linearly dependent.
- A practical way of checking whether vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in V$ are linearly independent is to use Gaussian elimination: Write all vectors as columns of a matrix $\boldsymbol{A}$ and perform Gaussian elimination until the matrix is in row echelon form (reduced row-echelon form is unnecessary here):
  - The pivot columns indicate the vectors, which are linearly independent of the vectors on the left. Note that there is an ordering of vectors when the matrix is built.

- The non-pivot columns can be represented as linear combinations of the pivot columns on their left. For instance, the row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{2}$$

  tells us that the first and third columns are pivot columns. The second column is a non-pivot column because it is three times the first column.

All column vectors are linearly independent if an only if all columns are pivot columns. If there is at least on non-pivot column, the columns (and, therefore, the corresponding vectors) are linearly dependent.