Linear Algebra

March 24, 2023

1 Matrices

Definition - Matrix
A matrix $\mathbf{A}$ is a $m \cdot n$-tuple of elements $a_{i,j}$ where m is the number of rows, n is the number of columns and $a_{i,j}$ is an element in this matrix. It can be expressed as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad a_{ij} \in \mathbb{R} \tag{1}$$

Rows are $(1, n)$-matrices and columns are $(m, 1)$-matrices, also called row/column vectors.

2 Addition and Multiplication

Defintion - Addition of 2 matrices

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{21} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}, \quad \in \mathbb{R}^{m \times n} \tag{2}$$

Obviously, the operation is commutative $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
Definition - Multiplication of 2 matrices
Matrix multiplication has a restriction, where the number of columns of the left factor has to be the same as the number of rows of the right factor.

$$\mathbf{A}_{m \times n} \mathbf{B}_{n \times k} = \mathbf{C}_{m \times k} \tag{3}$$

where each element in $\mathbf{C}$ is calculated as follows

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \dots, m, \quad j = 1, \dots, k. \tag{4}$$

An element c_{ij} is effectively calculated as sum of the products of the elements of the i-th row of $\mathbf{A}$ and elements of the j-th column of $\mathbf{B}$ (dot product). The restriction of matching row and column number implies that multiplication is not commutative.
Matrix multiplication is associative

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \tag{5}$$

and distributive

$$\begin{aligned} (\mathbf{A} + \mathbf{B})\mathbf{C} &= \mathbf{AC} + \mathbf{BC} \\ \mathbf{A}(\mathbf{C} + \mathbf{D}) &= \mathbf{AC} + \mathbf{AD} \end{aligned} \tag{6}$$

Multiplication is not an element-wise operation like addition. Element-wise multiplication is known as *Hadamard product* (just as addition, the matrices need to have the same dimensions).

Definition - Identity Matrix

$$\mathbf{I}_n := \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{7}$$

$\mathbf{I}_n$ is a matrix where each diagonal element is 1 while all other elements are 0. It's the neutral element for multiplication

$$\mathbf{A}_{n\times n}\mathbf{I}_n = \mathbf{A}_{n\times n} = \mathbf{I}_n\mathbf{A}_{n\times n} \tag{8}$$

3 Inverse and Transpose

Definition - Inverse
If $\mathbf{AB} = \mathbf{I}_n = \mathbf{BA}$, then $\mathbf{B} = \mathbf{A}^{-1}$ is called the inverse of $\mathbf{A}$.

Not every matrix is invertible, therefore we distinguish between *regular/invertible/nonsingular* and *singular/noninvertible* matrices. When the matrix inverse exists, it is unique.

Definition - Transpose
For a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ the matrix $\mathbf{B} \in \mathbb{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is the transpose of $\mathbf{A}$. Generally, we write $\mathbf{B} = \mathbf{A}^T$.
The transpose can be imagined as the mirrored version along the diagonal.
Some important properties:

$$\begin{aligned} \mathbf{AA}^{-1} &= \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} \\ (\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\ (\mathbf{A}+\mathbf{B})^{-1} &\neq \mathbf{A}^{-1} + \mathbf{B}^{-1} \\ (\mathbf{A}^T)^T &= \mathbf{A} \\ (\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\ (\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \end{aligned} \tag{9}$$

Defintion - Symmetric Matrix
A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$.
Only square matrices (same number of rows and columns) can be symmetric.
Sum of symmetric matrices is also symmetric, but not the product.

4 Multiplication by scalar

Multiplying a scalar with matrix is defined as

$$\lambda\mathbf{A} = \mathbf{K}, \quad K_{ij} = \lambda a_{ij} \tag{10}$$

This operation is associative and distributive. The scalar can basically moved around freely.

5 Compact Representation of Systems of Linear Equations

A simple linear equation system

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 1 \\ 4x_1 - 3x_2 - 7x_3 &= 8 \\ 9x_1 + 5x_2 - 3x_3 &= 2 \end{aligned} \tag{11}$$

can be expressed a matrix multiplication $\mathbf{Ax} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}}_{\mathbf{b}} \tag{12}$$

Solving such a system will be done in the next chapter.