Analytic Geometry

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1 Norms

Definition - Norm
A *Norm* on a vector space V is a function

$$\|\cdot\| : V \to \mathbb{R} \tag{1}$$

$$\boldsymbol{x} \mapsto \|\boldsymbol{x}\| \tag{2}$$

which assigns each vector $\boldsymbol{x}$ its *length* $\|\boldsymbol{x}\| \in \mathbb{R}$, such that for all $\lambda \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$ the following hold:

- *Absolutely homogeneous*: $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$
- *Triangle inequality*: $\|\boldsymbol{x} + \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- *Positive definite*: $\|\boldsymbol{x}\| \geqslant 0$ and $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$

Example - Manhattan Norm
The *Manhattan norm* on $\mathbb{R}^n$ is defined for $\boldsymbol{x} \in \mathbb{R}^n$ as

$$\|\boldsymbol{x}\|_1 := \sum_{i=1}^{n} |x_i| \tag{3}$$

where $|\cdot|$ is the absolute value. The Manhattan norm is also called the ℓ_1 norm.

Example - Euclidean Norm
The *Euclidean norm* of $\boldsymbol{x} \in \mathbb{R}^n$ is defined as

$$\|\boldsymbol{x}\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\boldsymbol{x}^T \boldsymbol{x}} \tag{4}$$

and computes the *Euclidean distance* of $\boldsymbol{x}$ from the origin. The Euclidean norm is also called the ℓ_2 norm.