Linear Algebra

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1 Solving Systems of Linear Equations

Previously we saw how linear equations can be represented as matrices and vectors. Here we will see how to solve them.

2 Particular and General Solutions

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix} \tag{1}$$

This system has a particular and a general solution.
Setting $x_1 = 42$ and $x_2 = 8$, while ignoring x_3 and x_4 gives us a particular solution

$$\boldsymbol{b} = \begin{bmatrix} 42 \\ 8 \end{bmatrix} = 42 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 8 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 12 \end{bmatrix} \tag{2}$$

Other solutions can be found be setting the equation to $\mathbf{0}$ in a creative way. We can express the third and fourth columns as a combination of the first two columns, which lets us easily subtract them. Let's start with the third column.

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \lambda_1 (8\boldsymbol{c_1} + 2\boldsymbol{c_2} - \boldsymbol{c_3}) = \mathbf{0} \tag{3}$$

Since the third column is dependent on the first two we can scale the factors by $\lambda_1 \in \mathbb{R}$, so we get infinite solutions.
Similarly, we can express the fourth column

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} \right) = \lambda_2 (-4\boldsymbol{c_1} + 12\boldsymbol{c_2} - \boldsymbol{c_4}) = \mathbf{0} \tag{4}$$

Our final solution for this linear system of equations can be summarized as

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}. \tag{5}$$

Steps to find the solution:

1. Find particular solution $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

2. Find all solutions $\boldsymbol{Ax} = \boldsymbol{0}$

3. Combine solutions form step 1. and 2. to find general solution

This example was easy, generally finding solutions is harder, see next step.

3 Elementary Transformations

An equation system can be simplified by applying the following transformations:

- Exchange rows
- Multiplication of a row with a constant $\lambda \in \mathbb{R} \setminus \{0\}$
- Addition of two rows

The goal is to transform a matrix into *row-echelon* form (REF), which makes finding solutions easy.

Definition - Row-Echelon Form

- All rows that contain only zeros are at the bottom of the matrix; correspondingly, all rows that contain at least one nonzero element are on top of rows that contain only zeros.
- Looking at nonzero rows only, the first nonzero number from the left (also called the *pivot* or the *leading coefficient*) is always strictly to the right of the pivot of the row above it.

Pivot rows are represented by *basic variables* while non-pivot rows are represented by *free variables*.

Gaussian elimination is an algorithm that performs elementary transformations to bring a system of linear equations into reduced row-echelon form (every pivot is 1, basic variable only non-zero in column).

Example - Reduced Row Echelon Form

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \tag{6}$$

We have 5 variables, where 3 are basic ones, while the other 2 are free. First we need to find solutions for non-pivot columns to get $\boldsymbol{Ax} = \boldsymbol{0}$, which is straight forward when we use the *Minus-1 Trick*, as follows.

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \mathbf{1} & 3 & 0 & 0 & 3 \\ 0 & \mathbf{-1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 9 \\ 0 & 0 & 0 & \mathbf{1} & -4 \\ 0 & 0 & 0 & 0 & \mathbf{-1} \end{bmatrix} \tag{7}$$

Looking at the pivot columns gives us the final solutions

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 : \boldsymbol{x} = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\} \tag{8}$$

4 Calculating the Inverse

Calculating the matrix requires finding a matrix $\boldsymbol{X}$, that satisfies $\boldsymbol{AX} = \boldsymbol{I}$, so $\boldsymbol{X} = \boldsymbol{A}^{-1}$. This can be done with linear transformations, as seen previously, but instead of satisfying $\boldsymbol{Ax} = \boldsymbol{b}$, we need to satisfy $\boldsymbol{AX} = \boldsymbol{I}$. The following steps illustrate how Gaussian elimination is applied to get the inverse:

$$[\boldsymbol{A}|\boldsymbol{I}_n] \rightsquigarrow \cdots \rightsquigarrow [\boldsymbol{I}_n|\boldsymbol{A}^{-1}] \tag{9}$$

5 Algorithms for Solving a System of Linear Equations

Gaussian elimination, in cases where we have millions of variables, is very impractical. Other methods include indirect solutions such as Richardson method, Jacobi method, Gauß-Seidel method, successive over-relaxation method (stationary iterative methods) or conjugate gradients, generalized minimal residuals, biconjugate gradients (Krylov subspace methods).