Linear Algebra

March 31, 2023

1 Linear Mappings

This chapter introduces linear mappings, which are transformations that produces another vector in the same vector space e.g. $\Phi : V \to W$ where

$$\Phi(\boldsymbol{x} + \boldsymbol{y}) = \Phi(\boldsymbol{x}) + (\boldsymbol{y}) \tag{1}$$

$$\Phi(\lambda \boldsymbol{x}) = \lambda \Phi(\boldsymbol{x}) \tag{2}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $\lambda \in \mathbb{R}$

Definition - Linear Mapping
For vector spaces V, W, a mapping $\Phi : V \to W$ is called a *linear mapping* (or *vector space homomorphism/linear transformation* if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in V \forall \lambda, \psi \in \mathbb{R} : \Phi(\lambda \boldsymbol{x} + \psi \boldsymbol{y}) = \lambda \Phi(\boldsymbol{x}) + \psi \Phi(\boldsymbol{y}) \tag{3}$$

Linear mappings can be represented as matrices.

Definition - Injective, Surjective, Bijective
Consider a mapping $\Phi : \mathcal{V} \to \mathcal{W}$, where $\mathcal{V}, \mathcal{W}$ can be arbitrary sets. Then Φ is called

- *Injective* if $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \Phi(\boldsymbol{x}) = \Phi(\boldsymbol{y}) \implies \boldsymbol{x} = \boldsymbol{y}$.
- *Surjective* if $\Phi(\mathcal{V}) = \mathcal{W}$.
- *Bijective* if it is injective and surjective.

These definitions introduce special cases of linear mappings:

- *Isomorphism*: $\Phi : V \to W$ linear and bijective
- *Endomorphism*: $\Phi : V \to V$ linear
- *Automorphism*: $\Phi : V \to V$ linear and bijective
- We define $id_V : V \to V, \boldsymbol{x} \mapsto \boldsymbol{x}$ as the *identity mapping* or *identity automorphism* in V.

Theorem
Finite-dimensional vector spaces V and W are isomorphic if and only if $dim(V) = dim(W)$.

This theorem gives us the justification to treat $\mathbb{R}^{m \times n}$ and $\mathbb{R}^{mn}$ the same, as their dimensions are mn, and there exists a linear, bijective mapping that transforms one into the other (programming examples: single loop with mn entries instead of double loop with m (first loop) and n (second loop) entries).

2 Matrix Representation of Linear Mappings

Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$. Consider $\mathcal{B} = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ as basis of a n-dimensional vector space V. If we want to emphasize on the importance of the order of the basis vectors we write

$$B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n) \tag{4}$$

and call it the *ordered basis* of V.

Definition - Coordinates
Consider a vector space V and an ordered basis $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ of V. For any $\boldsymbol{x} \in V$ we obtain a unique representation (linear combination)

$$\boldsymbol{x} = \alpha_1 \boldsymbol{b}_1 + \cdots + \alpha_n \boldsymbol{b}_n \tag{5}$$

of $\boldsymbol{x}$ with respect to B. Then $\alpha_1, \dots, \alpha_n$ are the coordinates of $\boldsymbol{x}$ with respect to B, and the vector

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \tag{6}$$

is the *coordinate vector/coordinate representation* of $\boldsymbol{x}$ with respect to the ordered basis B.

A basis is effectively a coordinate system e.g. Cartesian coordinate in $\mathbb{R}^k, k \in \{1, 2, 3\}$. Different bases have different coordinates for same data point!

Definition - Transformation Matrix
Consider vector spaces V, W with corresponding (ordered) bases $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m)$. Moreover, we consider a linear mapping $\Phi : V \to W$. For $j \in \{1, \dots, n\}$,

$$\Phi(\boldsymbol{b}_j) = \alpha_{1j} \boldsymbol{c}_1 + \cdots + \alpha_{mj} \boldsymbol{c}_m = \sum_{i=1}^{m} \alpha_{ij} \boldsymbol{c}_i \tag{7}$$

is the unique representation of $\Phi(\boldsymbol{b}_j)$ with respect to C. Then, we call the $m \times n$-matrix $\boldsymbol{A}_\Phi$, whose elements are given by

$$A_\Phi(i, j) = \alpha_{ij} \tag{8}$$

the *transformation matrix* of Φ (with respect to the ordered bases B of V and C of W).

The coordinates of $\Phi(\boldsymbol{b}_j)$ with respect to the ordered basis C of W are j-th column of $\boldsymbol{A}_\Phi$.

Transformation matrices can be used to transform objects in many different ways e.g. rotations, translations, scaling etc.

3 Basis Change

Transformation matrices change if we change the basis of our vector space.

Theorem - Basis Change
For a linear mapping $\Phi : V \to W$, ordered bases

$$B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n), \quad \tilde{B} = (\tilde{\boldsymbol{b}}_1, \dots, \tilde{\boldsymbol{b}}_n) \tag{9}$$

of V and

$$C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_m), \quad \tilde{C} = (\tilde{\boldsymbol{c}}_1, \dots, \tilde{\boldsymbol{c}}_m) \tag{10}$$

of W, and a transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to B and C, the corresponding transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ with respect to the bases $\tilde{B}$ and $\tilde{C}$ is given as

$$\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{T}^{-1}\boldsymbol{A}_\Phi \boldsymbol{S} \tag{11}$$

Here, $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ is the transformation matrix of id_V that maps coordinates with respect to $\tilde{B}$ onto coordinates with respect to B, and $\boldsymbol{T} \in \mathbb{R}^{m\times m}$ is the transformation matrix of id_W that maps coordinates with respect to $\tilde{C}$ onto coordinates with respect to C.

Proof

$$\tilde{\boldsymbol{b}}_j = s_{1j}\boldsymbol{b}_1 + \cdots + s_{nj}\boldsymbol{b}_n = \sum_{i=1}^{n} s_{ij}\boldsymbol{b}_i, \quad j = 1, \ldots, n \tag{12}$$

$$\tilde{\boldsymbol{c}}_k = t_{1k}\boldsymbol{c}_1 + \cdots + t_{mk}\boldsymbol{c}_m = \sum_{l=1}^{m} t_{lk}\boldsymbol{c}_l, \quad k = 1, \ldots, m \tag{13}$$

These transformation imply that there exist transformation matrix $\boldsymbol{S} = ((s_{ij})) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{T} = ((t_{lk})) \in \mathbb{R}^{m\times m}$. If we consider the transformation $\Phi(\tilde{\boldsymbol{b}}_j)$ we get

$$\Phi(\tilde{\boldsymbol{b}}_j) = \sum_{k=1}^{m} \underbrace{\tilde{a}_{kj}\tilde{c}_k}_{\in W} = \sum_{k=1}^{m} \tilde{a}_{kl} \sum_{l=1}^{m} t_{lk}\boldsymbol{c}_l = \sum_{l=1}^{m}\left(\sum_{k=1}^{m} t_{lk}\tilde{a}_{kj}\right)\boldsymbol{c}_l \tag{14}$$

where we first expressed the new basis vectors $\tilde{\boldsymbol{c}}_k \in W$ as linear combinations of the basis vectors $\boldsymbol{c}_l \in W$ and then swapped the order of summation.
Alternatively, when we express the $\tilde{\boldsymbol{b}}_j \in V$ as linear combinations of $\boldsymbol{b}_j \in V$, we arrive at

$$\Phi(\tilde{\boldsymbol{b}}_j) = \Phi\left(\sum_{i=1}^{n} s_{ij}\boldsymbol{b}_i\right) = \sum_{i=1}^{n} s_{ij}\Phi(\boldsymbol{b}_i) = \sum_{i=1}^{n} s_{ij}\sum_{l=1}^{m} a_{li}\boldsymbol{c}_l = \sum_{l=1}^{m}\left(\sum_{i=1}^{n} a_{li}s_{ij}\right)\boldsymbol{c}_l, \quad j = 1, \ldots, n \tag{15}$$

where we exploited the linearity of Φ. Comparing the last two equations it follows for all $j = 1, \ldots, n$ and $l = 1, \ldots, m$ that

$$\sum_{k=1}^{m} t_{lk}\tilde{a}_{kj} = \sum_{i=1}^{n} a_{li}s_{ij} \tag{16}$$

and, therefore,

$$\boldsymbol{T}\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{A}_\Phi\boldsymbol{S} \in \mathbb{R}^{m\times n} \tag{17}$$

such that

$$\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{T}^{-1}\boldsymbol{A}_\Phi \boldsymbol{S} \tag{18}$$

which concludes the proof.

Definition - Equivalence
Two matrices $\boldsymbol{A}, \tilde{\boldsymbol{A}} \in \mathbb{R}^{m\times n}$ are *equivalent* if there exist regular matrices $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{T} \in \mathbb{R}^{m\times m}$, such that $\tilde{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{S}$.

Definition - Similarity
Two matrices $\boldsymbol{A}, \tilde{\boldsymbol{A}} \in \mathbb{R}^{n\times n}$ are *similar* if there exists a regular $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ with $\tilde{\boldsymbol{A}} = \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$.

Similar matrices are always equivalent, but necessarily vice versa.

Concept of basis change will be important to find simple transformation matrices (diagonal form) and for data compression.