Analytic Geometry

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1 Inner Products

Introduction of intuitive geometrical concepts, such as lengths and angles in the vector space.

2 Dot Product

Also called the *scalar product/dot product* in $\mathbb{R}^n$, given by

$$\boldsymbol{x}^T \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \tag{1}$$

The dot product is just one example of an inner product, but is most commonly used.

3 General Inner Products

Definition
Let V be a vector space and $\Omega : V \times V \to \mathbb{R}$ be a bilinear mapping that takes two vectors and maps them onto a real number. Then

- Ω is called *symmetric* if $\Omega(\boldsymbol{x}, \boldsymbol{y}) = \Omega(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$, i.e., the order of the arguments does not matter.
- Ω is called *positive definite* if

$$\forall \boldsymbol{x} \in V \setminus \{\boldsymbol{0}\} : \Omega(\boldsymbol{x}, \boldsymbol{x}) > 0, \quad \Omega(\boldsymbol{0}, \boldsymbol{0}) = 0 \tag{2}$$

Definition
Let V be a vector space and $\Omega : V \times V \to \mathbb{R}$ be bilinear mapping that takes two vectors and maps them onto a real number. Then

- A positive definite, symmetric bilinear mapping $\Omega : V \times V \to \mathbb{R}$ is called an *inner product* on V. We typically write $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ instead of $\Omega(\boldsymbol{x}\boldsymbol{y})$.
- The pair $(V, \langle \cdot, \cdot \rangle)$ is called an *inner product space* or (real) *vector space with inner product.* If we use the dot product as defined in Eq. 1, we call $(V, \langle \cdot, \cdot \rangle)$ a *Euclidean vector space.*

4 Symmetric, Positive Definite Matrices

These kinds of matrices are very important for ML and they are defined via the inner product. The idea of symmetric positive semidefinite matrices is the key in the definition of kernels.
It generally holds that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \left\langle \sum_{i=1}^{n} \psi_i \boldsymbol{b}_i, \sum_{j=1}^{n} \lambda_i \boldsymbol{b}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_i \langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle \lambda_j = \hat{\boldsymbol{x}}^T \boldsymbol{A} \hat{\boldsymbol{y}} \tag{3}$$

where $A_{ij} := \langle \boldsymbol{b}_i, \boldsymbol{b}_j \rangle$ and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ are the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ with respect to the basis B. This implies that the inner product $\langle \cdot, \cdot \rangle$ is uniquely determined through $\boldsymbol{A}$. The symmetry of the inner product also means that $\boldsymbol{A}$ is symmetric. Furthermore, the positive definiteness of the inner product implies that

$$\forall \boldsymbol{x} \in V \setminus \{\boldsymbol{0}\} : \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0 \tag{4}$$

Definition - Symmetric, Positive Definite Matrix

A symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ that satisfies Eq. 4 is called *symmetric, positive definite*, or just *positive definite*. If only $\geqslant$ holds in Eq. 4, then $\boldsymbol{A}$ is called *symmetric, positive semidefinite.*

If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric, positive definite, then

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \hat{\boldsymbol{x}}^T \boldsymbol{A} \hat{\boldsymbol{y}} \tag{5}$$

defines an inner product with respect to an ordered basis B, where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are the coordinate representations of $\boldsymbol{x}, \boldsymbol{y} \in V$ with respect to B.

Theorem

For a real-valued, finite-dimensional vector space V and an ordered basis B of V, it holds that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is an inner product if and only if there exists a symmetric, positive definite matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ with

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \hat{\boldsymbol{x}}^T \boldsymbol{A} \hat{\boldsymbol{y}} \tag{6}$$

The following properties hold if $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric and positive definite:

- The null space (kernel) of $\boldsymbol{A}$ consists only of $\boldsymbol{0}$ because $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$. This implies that $\boldsymbol{A}\boldsymbol{x} \neq \boldsymbol{0}$ if $\boldsymbol{x} \neq \boldsymbol{0}$.
- The diagonal elements a_{ii} of $\boldsymbol{A}$ are positive because $a_{ii} = \boldsymbol{e}_i^T \boldsymbol{A} \boldsymbol{e}_i > 0$, where $\boldsymbol{e}_i$ is the i-th vector of the standard basis in $\mathbb{R}^n$.