Analytic Geometry

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1 Rotations

Orthogonal transformations can also be used to rotate "objects".

A *rotation* is a linear mapping (more specifically, an automorphism of a Euclidean vector space) that rotates a plane by an angle θ about the origin, i.e., the origin is a fixed point. For a positive angle $\theta > 0$, by common convention, we rotate in a counterclockwise direction. A rotation matrix can defined as follows:

$$\boldsymbol{R} = \begin{bmatrix} -0.38 & -0.92 \\ 0.92 & -0.38 \end{bmatrix} \tag{1}$$

This matrix rotates an object on the origin by $\theta = 112.5°$.

2 Rotation in $\mathbb{R}^2$

Consider the standard basis

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2}$$

of $\mathbb{R}^2$, which defines the standard coordinate system in $\mathbb{R}^2$. We aim to rotate this coordinate system by an angle θ. Note that the rotated vectors are still linearly independent and, therefore, are a basis of $\mathbb{R}^2$. This means that the rotation performs a basis change.
Rotations Φ are linear mappings so that we can express them by a *rotation matrix* $\boldsymbol{R}(\theta)$. Trigonometry allows us to determine the coordinates of the rotated axes (images of Φ) with respect to the standard basis in $\mathbb{R}^2$. We obtain

$$\Phi(\boldsymbol{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \Phi(\boldsymbol{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{3}$$

Therefore, the rotation matrix that performs the basis change into the rotated coordinates $\mathbb{R}(\theta)$ is given as

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \Phi(\boldsymbol{e}_1) & \Phi(\boldsymbol{e}_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

3 Rotations in $\mathbb{R}^3$

In $\mathbb{R}^3$ we can rotate any two-dimensional plane about a one-dimensional axis. The easiest way to specify the general rotation matrix is to specify how the images fo the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are orthonormal to each other. We can then obtain a general rotation matrix $\boldsymbol{R}$ by combining the images of the standard basis.
To have a meaningful rotation angle, we have to define what "counterclockwise" means when we operate in more than two dimensions. We use the convention that a "counterclockwise" (planar) rotation about an axis refers to a rotation about an axis when we look at the axis "head on , from the end toward the origin". In $\mathbb{R}^3$, there are therefore three (planar) rotations about the three standard basis vectors:

- Rotation about the $\boldsymbol{e}_1$-axis

$$\boldsymbol{R}_1(\theta) = \begin{bmatrix}\Phi(\boldsymbol{e}_1) & \Phi(\boldsymbol{e}_2) & \Phi(\boldsymbol{e}_3)\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{bmatrix} \tag{5}$$

  Here, the $\boldsymbol{e}_1$ coordinate is fixed, and the counterclockwise rotation is performed in the $\boldsymbol{e}_2\boldsymbol{e}_e$ plane.

- Rotation about the $\boldsymbol{e}_2$-axis

$$\boldsymbol{R}_2(\theta) = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta\end{bmatrix} \tag{6}$$

  If we rotate the $\boldsymbol{e}_1\boldsymbol{e}_3$ plane about the $\boldsymbol{e}_2$ axis, we need to look at the $\boldsymbol{e}_2$ axis from its "tip" toward the origin.

- Rotation about the $\boldsymbol{e}_3$-axis

$$\boldsymbol{R}_3(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{7}$$

4 Rotations in n-Dimensions

The generalization of rotations from 2D and 3D to n-dimensional Euclidean vector spaces can be intuitively described as fixing $n-2$ dimensions and restrict the rotation to a two-dimensional plane in the n-dimensional space.

Definition - Givens Rotation
Let V be an n-dimensional Euclidean vector space and $\Phi : V \to V$ and automorphism with transformation matrix

$$\boldsymbol{R}_{ij}(\theta) := \begin{bmatrix}\boldsymbol{I}_{i-1} & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \cos\theta & \boldsymbol{0} & -\sin\theta & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{j-i-1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sin\theta & \boldsymbol{0} & \cos\theta & \boldsymbol{0}\\ \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0} & \boldsymbol{I}_{n-j}\end{bmatrix} \in \mathbb{R}^{n\times n} \tag{8}$$

for $1 \leqslant i < j \leqslant n$ and $\theta \in \mathbb{R}$. Then $\boldsymbol{R}_{ij}(\theta)$ is called a *Givens rotation*. Essentially, $\boldsymbol{R}_{ij}(\theta)$ is the identity matrix $\boldsymbol{I}_n$ with

$$r_{ii} = \cos\theta, \quad r_{ij} = -\sin\theta, \quad r_{ji} = \sin\theta, \quad r_{jj} = \cos\theta \tag{9}$$

In two dimensions (i.e., $n = 2$), we obtain a special case (as seen previously).

5 Properties of Rotations

- Rotations preserve distances, i.e., $\|\boldsymbol{x} - \boldsymbol{y}\| = \|\boldsymbol{R}_\theta(\boldsymbol{x}) - \boldsymbol{R}_\theta(\boldsymbol{y})\|$. In other words, rotations leave the distance between any two points unchanged after the transformation.

- Rotation preserve angles, i.e., the angle between $\boldsymbol{R}_\theta\boldsymbol{x}$ and $\boldsymbol{R}_\theta\boldsymbol{y}$ equals the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

- Rotations in three (or more) dimensions are generally not commutative. Therefore, the order in which rotations are applied is important, even if they rotate about the same point. Only in two dimensions vector rotations are commutative, such that $\boldsymbol{R}(\phi)\boldsymbol{R}(\theta) = \boldsymbol{R}(\theta)\boldsymbol{R}(\phi)$ for all $\phi, \theta \in [0, 2\pi)$. They form an Abelian group (with multiplication) only if they rotate about the same point (e.g., the origin).