Linear Algebra

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1 Systems of Linear Equations

Example - No Solution

$$\begin{array}{rcrcrcll} x_1 & + & x_2 & + & x_3 & = & 1 & \quad (1) \\ x_1 & - & x_2 & + & 2x_3 & = & 2 & \quad (2) \\ 2x_1 & & & + & 3x_3 & = & 3 & \quad (3) \end{array}$$

Let's add rows (1) and (2), which results in a new equation

$$\begin{array}{rcrcll} 2x_1 & + & 3x_3 & = & 5 & \quad (1)+(2) \\ 2x_1 & + & 3x_3 & = & 3 & \quad (3) \end{array}$$

No solutions since there's a contradiction.

Example - Unique Solution

$$\begin{array}{rcrcrcll} x_1 & + & x_2 & + & x_3 & = & 3 & \quad (1) \\ x_1 & - & x_2 & + & 2x_3 & = & 2 & \quad (2) \\ & & x_2 & + & x_3 & = & 2 & \quad (3) \end{array}$$

Let's subtract (3) from (1) and add (1) to (2), so we get

$$\begin{array}{rcrcrcll} x_1 & & & & & = & 1 & \quad (1-3) \\ 2x_1 & & & + & 3x_3 & = & 5 & \quad (1+2) \\ & & x_2 & + & x_3 & = & 2 & \quad (3) \end{array}$$

It follows that $x_1 = 1$ and $x_3 = 1$, therefore $x_2 = 1$. We have a unique solution for this equation system.

Example - Infinite Solutions

$$\begin{array}{rcrcrcll} x_1 & + & x_2 & + & x_3 & = & 3 & \quad (1) \\ x_1 & - & x_2 & + & 2x_3 & = & 2 & \quad (2) \\ 2x_1 & & & + & 3x_3 & = & 5 & \quad (3) \end{array}$$

Without any transformation we see that $(1)+(2) = (3)$, so we can omit (3). Let's add (1) and (2) and subtract (2) from (1), so we get

$$\begin{array}{rcrcrcll} 2x_1 & & & + & 3x_3 & = & 5 & \quad (1+2) \\ & & 2x_2 & - & x_3 & = & 1 & \quad (1-2) \end{array}$$

Since x_3 is a free variable we can assign it to anything that satisfies the equations e.g. $x_3 = a \in \mathbf{R}$. so our final solution is

$$\begin{aligned} x_1 &= \frac{5}{2} - \frac{3}{2}a \\ x_2 &= \frac{1}{2} + \frac{1}{2}a \\ x_3 &= a \end{aligned}$$

2 Linear Equations as Matrices

The equations above can be expressed as matrices

$$
\begin{aligned}
x_1 &+ x_2 &+ x_3 &= 3 \quad (1)\\
x_1 &- x_2 &+ 2x_3 &= 2 \quad (2)\\
& x_2 &+ x_3 &= 2 \quad (3)
\end{aligned}
$$

can be expressed as the following matrix

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}
$$

More on matrices in the next chapter.