Analytic Geometry

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1 Orthogonal Complement

This chapter focuses on the inspection of vector spaces that are orthogonal to each other, which is an important concept for dimensionality reduction from a geometric perspective.

Consider a D-dimensional vector space V and an M-dimensional subspace $U \subseteq V$. Then its *orthogonal complement* $U^{\perp}$ is a $(D-M)$-dimensional subspace of V and contains all vectors in V that are orthogonal to every vector in U. Furthermore, $U \cap U^{\perp} = \{\mathbf{0}\}$ so that any vector $\boldsymbol{x} \in V$ can be uniquely decomposed into

$$\boldsymbol{x} = \sum_{m=1}^{M} \lambda_m \boldsymbol{b}_m + \sum_{j=1}^{D-M} \psi_j \boldsymbol{b}_j^{\perp}, \quad \lambda_m, \psi_j \in \mathbb{R} \tag{1}$$

where $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_M)$ basis of U and $(\boldsymbol{b}_1^{\perp}, \ldots, \boldsymbol{b}_{D-M}^{\perp})$ is a basis of $U^{\perp}$.

Therefore, the orthogonal complement can also be used to describe a plane U (two-dimensional subspace) in a three-dimensional vector space. More specifically, the vector $\boldsymbol{w}$ with $\|\boldsymbol{w}\| = 1$, which is orthogonal to the plane U, is the basis vector of $U^{\perp}$. The vector $\boldsymbol{w}$ is called the *normal vector* of U.