Linear Algebra

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1 Affine Spaces

This chapter is about spaces that are offset from the origin, or in other words, spaces that are no longer subspaces.

Definition - Affine Subspace
Let V be a vector space, $\boldsymbol{x}_0 \in V$ and $U \subseteq V$ a subspace. Then the subset

$$\begin{aligned} L = \boldsymbol{x}_0 + U &:= \{\boldsymbol{x}_0 + \boldsymbol{u} : \boldsymbol{u} \in U\} \\ &= \{\boldsymbol{v} \in V | \exists \boldsymbol{u} \in U : \boldsymbol{v} = \boldsymbol{x}_0 + \boldsymbol{u}\} \subseteq V \end{aligned} \tag{1}$$

is called *affine subspace* of *linear manifold* of V. U is called *direction* or *direction space*, and $\boldsymbol{x}_0$ is called *support point*.

Important to note is, that the according to the definition to the affine subspace, it does not include $\boldsymbol{0}$ if $\boldsymbol{x}_0 \notin U$. Therefore, an affine subspace is not a (linear) subspace (vector subspace) of V for $\boldsymbol{x}_0 \notin U$.

Examples of affin subspace are points, lines, and planes in $\mathbb{R}^3$, which do not (necessarily) go through the origin.

in $\mathbb{R}^n$, the $(n-1)$-dimensional affine subspaces are called hyperplanes, and the corresponding parametric equation is $\boldsymbol{y} = \boldsymbol{x}_0 + \sum_{i=1}^{n-1} \lambda_i \boldsymbol{b}_i$, where $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-1}$ form a basis of an $(n-1)$-dimensional subspace U of $\mathbb{R}^n$. This means that a hyperplane is defined by a support point $\boldsymbol{x}_0$ and $(n-1)$ linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-1}$ that span the direction space. In $\mathbb{R}^2$, a line is also a hyperplane. In $\mathbb{R}^3$, a plane is also a hyperplane.

2 Affine Mappings

Definition - Affine Mapping
For two vector spaces V, W, a linear mapping $\Phi : V \to W$, and $\boldsymbol{a} \in W$, the mapping

$$\phi : V \to W \tag{2}$$

$$\boldsymbol{x} \mapsto \boldsymbol{a} + \Phi(\boldsymbol{x}) \tag{3}$$

is an *affine mapping* from V to W. The vector $\boldsymbol{a}$ is called *translation vector* of ϕ.

Properties:

- Every affine mapping $\phi : V \to W$ is also the composition of a linear mapping $\Phi : V \to W$ and a translation $\tau : W \to W$ in W, such that $\phi = \tau \circ \Phi$. The mappings Φ and τ are uniquely determined.
- The composition $\phi' \circ \phi$ of affine mappings $\phi : V \to W$, $\phi' : W \to X$ is affine.

- Affine mappings keep the geometric structure invariant. They also preserve the dimension and parallelism.