Analytic Geometry

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1 Inner Products of Functions

The concept of an inner product can be generalized to vectors with an infinite number of entries (countably infinite) and also continuous-valued functions (un-countably infinite). Then the sum over individual components of vectors turns into an integral. For example, given two functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ can be defined as the definite integral

$$\langle u, v \rangle := \int_a^b u(x)v(x)dx \tag{1}$$

for lower and upper limits $a, b < \infty$, respectively.

To make the preceding inner product mathematically precise, we need to take care of measures and the definition of integrals, leading to the definition of a Hilbert space. Furthermore, unlike inner products on finite-dimensional vectors, inner products on functions may diverge (have infinite value).

Example - Inner Product of Functions
I we choose $u = \sin(x)$ and $v = \cos(x)$, the integrand $f(x) = u(x)v(x)$ of Eq. 1 can easily be visualized. We see that this function is odd, i.e., $f(-x) = -f(x)$. Therefore, the integral with limits $a = -\pi, b = \pi$ of this product evaluates to 0. Therefore, sin and cos are orthogonal functions.