Analytic Geometry

April 3, 2023

1 Angles and Orthogonality

Assume $\boldsymbol{x} \neq \mathbf{0}, \boldsymbol{y} \neq \mathbf{0}$, then

$$-1 \leqslant \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|} \leqslant 1 \tag{1}$$

Therefore, there exists a unique $\omega \in [0, \pi]$, with

$$\cos \omega = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|} \tag{2}$$

The number ω is the *angle* between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. It usually tells us how similar the orientation of two vectors is.

Definition - Orthogonality
Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are *orthogonal* if and only if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, and we write $\boldsymbol{x} \perp \boldsymbol{y}$. If additionally $\|\boldsymbol{x}\| = 1 = \|\boldsymbol{y}\|$, i.e., the vectors are unit vectors, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are *orthonormal*.

This implies that the $\mathbf{0}$-vector is orthogonal to every vector in the vector space.

Depending on the inner product when calculating orthogonality, the result may differ e.g. orthogonal in scenario with dot product vs. not-orthogonal in scenario with arbitrary inner product.

Definition - Orthogonal Matrix
A square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is an *orthogonal matrix* if and only if its columns are orthonormal so that

$$\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I} = \boldsymbol{A}^T\boldsymbol{A} \tag{3}$$

which implies that

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^T \tag{4}$$

i.e., the inverse is obtained by simply transposing the matrix.

Transformations by orthogonal matrices are special because the length of a vector $\boldsymbol{x}$ is not changed when transforming it using an orthogonal matrix $\boldsymbol{A}$. For the dot product, we obtain

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = (\boldsymbol{A}\boldsymbol{x})^T(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{I}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{x} = \|\boldsymbol{x}\|^2 \tag{5}$$

Moreover, the angle between any two vectors $\boldsymbol{x}, \boldsymbol{y}$, as measured by their inner product, is also unchanged when transforming both of them using an orthogonal matrix $\boldsymbol{A}$. Assuming the dot product as the inner product, the angle of the images $\boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{y}$ is given as

$$\cos \omega = \frac{(\boldsymbol{A}\boldsymbol{x})^T(\boldsymbol{A}\boldsymbol{y})}{\|\boldsymbol{A}\boldsymbol{x}\|\|\boldsymbol{A}\boldsymbol{y}\|} = \frac{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{y}}{\sqrt{\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}\boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{y}}} = \frac{\boldsymbol{x}^T\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} \tag{6}$$

which gives exactly the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$. This means that orthogonal matrices $\boldsymbol{A}$ with $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$ preserve both angles and distances. It turns out that orthogonal matrices define transformations that are rotations (with the possibility of flips).