Matrix Decompositions

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1 Eigenvalues and Eigenvectors

Definition
Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an *eigenvalue* of $\boldsymbol{A}$ and $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ is the corresponding *eigenvector* of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{1}$$

This equation (Eq. 1) is also called the *eigenvalue equation.*

The following statements are equivalent:

- λ is an eigenvalue of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
- There exists an $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ with $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, or equivalently, $(\boldsymbol{A} - \lambda \boldsymbol{I}_n)\boldsymbol{x} = \boldsymbol{0}$ can be solved non-trivially, i.e., $\boldsymbol{x} \neq \boldsymbol{0}$.
- $rk(\boldsymbol{A} - \lambda \boldsymbol{I}_n) < n$.
- $det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0$.

Definition - Collinearity and Codirection
Two vectors that point in the same direction are called *codirected.* Two vectors are *collinear* if they point in the same or opposite direction.

All vectors that are collinear to $\boldsymbol{x}$ are also eigenvectors of $\boldsymbol{A}$. More generally

$$\boldsymbol{A}(c\boldsymbol{x}) = c\boldsymbol{A}\boldsymbol{x} = c\lambda \boldsymbol{x} = \lambda(c\boldsymbol{x}) \tag{2}$$

for any $c \in \mathbb{R} \setminus \{0\}$.

Theorem
$\lambda \in \mathbb{R}$ is an eigenvalue of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ if and only if λ is a root of the characteristic polynomial $p_{\boldsymbol{A}}(\lambda)$ of $\boldsymbol{A}$.

Definition
Let a square matrix $\boldsymbol{A}$ have an eigenvalue λ_i. The algebraic multiplicity of λ_i is the number of times the root appears in the characteristic polynomial.

Definition - Eigenspace and Eigenspectrum
For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of $\boldsymbol{A}$ associated with an eigenvalue λ spans a subspace of $\mathbb{R}^n$, which is called an *eigenspace* of $\boldsymbol{A}$ with respect to λ and is denoted by E_λ. The set of all eigenvalues of $\boldsymbol{A}$ is called the *eigenspectrum*, or just *spectrum*, of $\boldsymbol{A}$.

If λ is an eigenvalue of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, then the corresponding eigenspace E_λ is the solution space of the homogeneous system of linear equations $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$.

Useful properties of eigenvalues and eigenvectors:

- A matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$ possess the same eigenvalues, but not necessarily the same eigenvectors.
- The eigenspace E_λ is the null-space of $\boldsymbol{A} - \lambda \boldsymbol{I}$ since

$$\begin{aligned} \boldsymbol{Ax} = \lambda \boldsymbol{x} &\iff \boldsymbol{Ax} - \lambda \boldsymbol{x} = \boldsymbol{0} \\ &\iff (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \iff \boldsymbol{x} \in ker(\boldsymbol{A} - \lambda \boldsymbol{I}) \end{aligned} \tag{3}$$

- Similar matrices possess the same eigenvalues. Therefore, a linear mapping Φ has eigenvalues are independent of the choice of basis of its transformation matrix. This makes eigenvalues, together with the determinant and the trace, key characteristic parameters of a linear mapping as they are all invariant under basis change.
- Symmetric, positive definite matrices always have positive real eigenvalues.

Definition
Let λ_i be an eigenvalue of a square matrix $\boldsymbol{A}$. Then the *geometric multiplicity* of λ_i is the number of linearly independent eigenvectors associated with λ_i. In other words, it is the dimensionality of the eigenspace spanned by the eignevectors associated with λ_i.

Theorem
The eigenvectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ of a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ with n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ are linearly independent.
This theorem states that eigenvectors of a matrix with n distinct eigenvalues form a basis of $\mathbb{R}^n$.

Definition
A square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is defective if it possess fewer than n linearly independent eigenvectors.
A defective matrix cannot have n distinct eigenvalues, as distinct eigenvalues have linearly independent eigenvectors.

Theorem
Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we can always obtain a symmetric, positive semidefinite matrix $\boldsymbol{S} \in \mathbb{R}^{n \times n}$ by defining

$$\boldsymbol{S} := \boldsymbol{A}^T \boldsymbol{A} \tag{4}$$

If $rk(\boldsymbol{A}) = n$, then $\boldsymbol{S} := \boldsymbol{A}^T \boldsymbol{A}$ is symmetric, positive definite.

Theorem - Spectral Theorem
If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is symmetric, there exists an orthonormal basis of the corresponding vector space V consisting of eigenvectors of $\boldsymbol{A}$ and each eigenvalue is real.
This theorem implicates that the eigendecomposition of a symmetric matrix $\boldsymbol{A}$ exists, and that we can find a ONB of eigenvectors so that $\boldsymbol{A} = \boldsymbol{PDP}^T$, where $\boldsymbol{D}$ is diagonal and the columns of $\boldsymbol{P}$ contain the eigenvectors.

Theorem
The determinant of a matrix $\boldsymbol{A}^{n \times n}$ is the product of its eigenvalues

$$det(\boldsymbol{A}) = \prod_{i=1}^{n} \lambda_i \tag{5}$$

where $\lambda_i \in \mathbb{C}$ are (possibly repeated) eigenvalues of $\boldsymbol{A}$.

Theorem
The trace of a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the sum of its eigenvalues

$$tr(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i \tag{6}$$

where $\lambda_i \in \mathbb{C}$ are (possibly repeated) eigenvalues of $\boldsymbol{A}$.