Matrix Decompositions

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1 Matrix Phylogeny

- For non-square matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n} m \neq n$ the SVD always exist.
- For square matrices, the determinant informs us whether a matrix is invertible.
- If a $n \times n$ matrix consists of n linearly independent eigenvectors, then the matrix is non-defective and an eigendecomposition exists. We know that repeated eigenvalues may result in defective matrices, which cannot be diagonalized.
- Non-singular and non-defective matrices are not the same. A rotation matrix will be invertible but not diagonizable in the real numbers.
- A non-defective square matrix $\boldsymbol{A}$ is normal if the condition $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T$ holds. Additionally, $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$ holds, then $\boldsymbol{A}$ is called orthogonal. The set of orthogonal matrices is a subset of regular/invertible matrices and satisfies $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$.
- Normal matrices have subset called the symmetric matrices which satisfy $\boldsymbol{S} = \boldsymbol{S}^{-1}$. Symmetric matrices have only real eigenvalues. A subset of the symmetric matrices consists of positive definite matrices $\boldsymbol{P}$ that satisfy the condition of $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}^T > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$. In this case a unique Cholesky decomposition exists. Positive definite matrices have only positive eigenvalues are always invertible.
- Another subset of symmetric matrices consists of diagonal matrices $\boldsymbol{D}$. Diagonal matrices are closed under addition and mulitplication, but do not necessarily form a group. A special diagonal matric is the identity matrix $\boldsymbol{I}$.