Linear Algebra

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1 Vector Spaces

Introduction to spaces of vectors, which are basically sets with vectors as elements which are defined via certain rules.

2 Groups

Definition - Group
Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a group if the following hold:

1. Closure of $\mathcal{G}$ under $\otimes : \forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. Associativity: $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. Neutral element: $\exists e \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. Inverse element: $\forall x \in \mathcal{G} \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$, where e is the neutral element. We often write x^{-1} to denote the inverse element of x.
5. Additionally Abelian group if commutative.

Examples - Groups

- $(\mathbb{Z}, +)$ is an Abelian group.
- $(\mathbb{N}_0, +)$ is not a group.
- ...

Definition - General Linear Group
The set of regular (invertible) matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a group with respect to matrix multiplication and is called general linear group $GL(n, \mathbb{R})$. However, since matrix multiplication is not commutative, the group is not Abelian.

3 Vector Spaces

Similar to groups, but additionally to an inner operation (addition), it is also defined by an outer operation (multiplication).

Definition - Vector Space
A real-values vector space $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

$$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{1}$$

$$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V} \tag{2}$$

where

1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity, outer operation
3. Associativity, outer operation
4. Neutral element, outer operation

Generally, if vectors are mentioned x is column vector and x^T is a row vector.

4 Vector Subspaces

Definition - Vector Subspace
Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{U} \subseteq \mathcal{V} \neq \emptyset$. Then $U = (\mathcal{U}, +, \cdot)$ is called vector subspace of V (or linear subspace) if U is a vector space with the vector space operations $+$ and $\cdot$ restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$. We write $U \subseteq V$ to denote a subspace U of V.
If $\mathcal{U} \subseteq \mathcal{V}$ and V is a vector space, then U naturally inherits many properties directly from V because they hold for all $\boldsymbol{x} \in \mathcal{V}$, and in particular for all $\boldsymbol{x} \in \mathcal{U} \subseteq \mathcal{V}$. This includes the Abelian group properties, the distributivity, the associativity and the neutral element. To determine whether $(\mathcal{U}, +, \cdot)$ is a subspace of V we still do need to show

1. $\mathcal{U} \neq \emptyset$, in particular: $\mathbf{0} \in \mathcal{U}$
2. Closure of U with respect to inner and outer operations.