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A note on Liapunov's method

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The purpose of this communication is to exhibit, in an important special case, a reformulation of Liapunov's method which makes the calculations much less onerous and from which one can draw interesting conclusions which do not obviously follow from the classical presentation.

Let $\dot{x}_\nu = f_\nu(x_1, \ldots, x_n)$ be a system of n first-order differential equations, let $V(x_1, \ldots, x_n)$ be a well-behaved function and write $W = (1/2)\,\mathrm{d}V/\mathrm{d}t$. We call V a Liapunov function for the value c if $W \leqslant 0$ at all points of $V = c$, and a strict Liapunov function for the value c if $W < 0$ there. If V is a strict Liapunov function for c then the trajectory through a point of $V = c$ crosses from $V > c$ to $V < c$ there; in other words, the region $V > c$ is one which trajectories can only leave, and the region $V < c$ is one which they can only enter. The same is often true if V is merely a Liapunov function for c, but in this case a more detailed analysis is needed. If the hypersurface $V = c$ is not connected, then similar results hold for each of its connected components.

In practice one normally expresses this differently. Let $\mathcal{C}$ be the set of values which V takes on the hypersurface $W = 0$, let c be any real number not in $\mathcal{C}$, and assume for simplicity that the hypersurface $V = c$ is connected. Then W has fixed sign on $V = c$, so that if $W < 0$ on $V = c$ then V is a strict Liapunov function for c, and if $W > 0$ on $V = c$ then $-V$ is a strict Liapunov function for $-c$. If $W = 0$ is bounded, then $\mathcal{C}$ can be found simply by finding the extremal points of V on $W = 0$; but if $W = 0$ is unbounded more complicated arguments are needed.

The calculation of $\mathcal{C}$ becomes rapidly more onerous as the degrees of V and W increase; and for this reason many workers have confined themselves to the case when V and W are inhomogeneous quadratic. It seems not to have been noticed that in this case there is a simpler way of deciding which V are Liapunov functions, and that this has implications which are not obvious from the classical Liapunov approach. Here I shall present this new approach in general terms. In a subsequent paper it will be applied to the Lorenz equations.

Lemma 1. Let $f(X_1, \ldots, X_N)$ be a real homogeneous quadratic form, let $\mathcal{S}$ be the sphere

$$\mathcal{S} : X_1^2 + \cdots + X_N^2 = 1 \tag{1}$$

and let $\mathcal{S}^+$ be the subset of $\mathcal{S}$ on which $f > 0$. Then $\mathcal{S}^+$ is either empty, pathwise connected, or the disjoint union of two pathwise connected sets which are images of each other in the origin. The same holds for the subset of $\mathcal{S}$ on which $f \geqslant 0$.

Proof. By means of an orthogonal transformation we can take f into diagonal form; so assume that $f = \sum \lambda_\nu X_\nu^2$ where $\lambda_\nu > 0$ for $\nu \leqslant N_1$, $\lambda_\nu = 0$ for $N_1 < \nu \leqslant N_2$ and $\lambda_\nu < 0$ for $N > N_2$. If $N_1 = 0$ then $\mathcal{S}^+$ is empty, and if $N_1 = N$ then $\mathcal{S}^+ = \mathcal{S}$; so we can assume that $0 < N_1 < N$. Now let P be a point of $\mathcal{S}^+$. We first continuously decrease $|X_{N_1+1}|, \ldots, |X_N|$ until they reach 0, at the same time increasing $|X_1|$ so that (1) continues to hold. This process continuously increases f, so it takes place entirely with $\mathcal{S}^+$, and it connects P to a point such that $X_1^2 + \cdots + X_{N_1}^2 = 1$ and $X_{N_1+1} = \cdots = X_N = 0$. These equations determine a set which can be identified with a sphere in N_1 dimensions which lies entirely in $\mathcal{S}^+$; and this set obviously satisfies the conclusions of Lemma 1. Explicitly, it is pathwise connected if $N_1 > 1$ and consists of two points which are reflections of each other in the origin if $N_1 = 1$. For the final sentence, we argue in the same way but with N_2 in place of N_1. $\square$

Lemma 2. Let f, g be homogeneous quadratic forms in $X_1, \ldots, X_N$ and let $\mathcal{S}$ be as in (1). Then $f < 0$ at all points of $g = 0$ other than the origin if and only if $f + \lambda g$ is negative definite for some real λ.

Proof. ‘If’ is trivial, so we assume that $f < 0$ at all points of $g = 0$ other than the origin. Clearly this happens if and only if $f < 0$ at all points of $\mathcal{S} \cap \{g = 0\}$; the usefulness of this remark is that the latter set is compact. If g is definite or semi-definite then after a linear transformation of variables and possibly writing $-g$ for g we can assume that g is a negative definite form in $X_1, \ldots, X_M$ for some $M \leqslant N$. Now the condition on f is that it is negative definite on $X_1 = \cdots = X_M = 0$, so by absorbing multiples of $X_1, \ldots, X_M$ into each of $X_{M+1}, \ldots, X_N$ we can suppose that

$$f = f_1(X_1, \ldots, X_M) + f_2(X_{M+1}, \ldots, X_N)$$

where f_2 is negative definite. If λ is positive and large enough, then $f_1 + \lambda g$ is a negative definite quadratic form in $X_1, \ldots, X_M$, which proves Lemma 2 in this case. Thus henceforth we can assume that g is indefinite.

Let h be any homogeneous quadratic form in $X_1, \ldots, X_N$ such that

$$h < 0 \text{ at all points of } g = 0 \text{ other than the origin} \tag{2}$$

I claim that $\mathrm{S} \cap \{h \geqslant 0\}$ lies either entirely in $g > 0$ or entirely in $g < 0$. For if not, there would be points P_1, P_2 on $\mathcal{S}$ such that

$$h(P_1) \geqslant 0, g(P_1) > 0 \quad \text{and} \quad h(P_2) \geqslant 0, g(P_2) < 0$$

Using the last sentence of Lemma 1 with h for f, we see that P_1 is pathwise connected in $\mathcal{S} \cap \{h \geqslant 0\}$ to at least one of P_2 or $-P_2$. Any such path would contain a point at which $g = 0$, and this contradicts (2).

Changing the sign of g if necessary, we can now assume that $\mathcal{S} \cap \{f \geqslant 0\}$ lies entirely in $g < 0$. Thus $f < 0$ throughout the compact set $\mathcal{S} \cap \{g \geqslant 0\}$. The function g/f is continuous there and takes some strictly negative value because g is indefinite, so it attains its minimum $-m_1$ and $m_1 > 0$. Hence $g + m_1 f \leqslant 0$ in $\mathcal{S} \cap \{g \geqslant 0\}$, and it does take the value 0 there; moreover, by hypothesis this cannot happen at a point where $g = 0$. Now apply the second paragraph of this proof with $h = g + m_1 f$; since there is a point of $\mathcal{S} \cap \{g > 0\}$ with $h = 0$, the whole set $\mathcal{S} \cap \{h \geqslant 0\}$ must lie in $g > 0$

and therefore $g+m_1 f<0$ throughout $\mathcal{S}\cap\{g\leqslant 0\}$. This last set is again compact, so let $-m_2$ with $m_2>0$ be the maximum of $g+m_1 f$ there and let $m_3>0$ be an upper bound for f there. If we write $\lambda^{-1}=m_1+(1/2)m_2 m_3^{-1}$ we have

$$g+\lambda^{-1}f\leqslant g+m_1 f+(1/2)m_2<0 \text{ in } \mathcal{S}\cap\{g\leqslant 0\}$$

and we already know that

$$g+\lambda^{-1}f<g+m_1 f\leqslant 0 \text{ in } \mathcal{S}\cap\{g\geqslant 0\}$$

Thus $f+\lambda\varepsilon<0$ throughout $\mathcal{S}$. □

If one tries to weaken both the hypothesis and the conclusion of Lemma 2 in the obvious way, then the example $f=X_1X_2$, $g=X_1^2$ shows that an escape clause is needed.

Corollary 1. Let f,g be homogeneous quadratic forms in $X_1,\dots,X_N$ with g indefinite and let $\mathcal{S}$ be as in (1). Then $f\leqslant 0$ at all points of $g=0$ if and only if $f+\lambda g$ is negative semidefinite for some real λ.

Proof. Again 'if' is trivial, so let $f\leqslant 0$ for all points of $\mathcal{S}\cap\{g=0\}$. We can assume that there is a point P_0 on $\mathcal{S}$ with $f(P_0)>0$, for otherwise f itself would be negative semidefinite. Since $g(P_0)\neq 0$, by changing the sign of g if necessary we can suppose that $g(P_0)<0$. Write

$$f_\varepsilon=f-\varepsilon(X_1^2+\cdots+X_N^2)$$

where we require $\varepsilon<(1/2)f(P_0)$. Applying Lemma 2 to f_ε and g, we find that there exists λ_y such that $f_\varepsilon+\lambda_\varepsilon g$ is negative definite; and by considering the value of this function at P_0 we see that $\lambda_\varepsilon>0$.

We can choose a sequence of values of ε tending to 0 such that the corresponding λ_ε tend either to a finite limit λ or to infinity. In the former case, letting $\varepsilon\to 0$ in the statement

$$f_\varepsilon(P)+\lambda_\varepsilon g(P)<0 \text{ for } P \text{ in } \mathcal{S}$$

we find that $f+\lambda g\leqslant 0$ for each P in $\mathcal{S}$, as desired. In the latter case

$$\lambda_\varepsilon^{-1}f_\varepsilon(P)+g(P)<0 \text{ for } P \text{ in } \mathcal{S}$$

and letting $\varepsilon\to 0$ gives $g\leqslant 0$ for each P in $\mathcal{S}$, contrary to hypothesis. □

In Liapunov's method one considers a system of differential equations

$$\dot{x}_\nu=f_\nu(x_1,\dots,x_n) \text{ for } \nu=1,\dots,n \tag{3}$$

and a function $V(x_1,\dots,x_n)$, and one writes

$$W=\frac{1}{2}\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1}{2}\sum_{\nu=1}^{n}f_\nu\frac{\partial V}{\partial x_\nu}$$

We shall assume that V is inhomogeneous quadratic, and the results which follow are only likely to be useful when the f_ν are at worst inhomogeneous quadratic. The simplest case is as follows.

Theorem 1. In the notation above, suppose that V,W are both inhomogeneous quadratic and that c_0 is a real number such that $V-c_0$ is indefinite. Then $W\leqslant 0$ at all points where $V = c_0$ if and only if $W + \lambda(V - c_0)$ is negative semidefinite for some λ.

Proof. Let f, g be the homogeneous quadratic forms in $x_0, \ldots, x_n$ obtained by multiplying the monomials in $W, V - c_0$ respectively by appropriate powers of x_0. Then $f \leqslant 0$ at all points where $g = 0$ except perhaps those at which $x_0 = 0$. Since g is indefinite, any point on $x_0 = 0$ with $g = 0$ is the limit of a sequence of points with $g = 0$ and $x_0 \neq 0$; hence by continuity $f \leqslant 0$ at such a point. Applying Corollary 1 to Lemma 2 to f, g gives Theorem 1. □

The calculation involved in testing whether there is a value of λ such that $W + \lambda(V - c_0)$ is negative semidefinite is usually less tedious than that involved in finding the range of values of W on $V = c_0$. It is the 'if' clause, which is trivial, that one uses in the explicit applications; the importance of the 'only if' clause is that it shows that under the hypotheses of Theorem 1 the use of the condition that $W + \lambda(V - c_0)$ is negative semidefinite gives results just as strong as those of the classical method.

It should be emphasised that for practical applications the case when $W + \lambda(V - c_0)$ is only negative semidefinite is an important one. Not only are the calculations usually simpler, but it can also happen that the conclusions are substantially stronger; for pairs V, c_0 with this property are by no means always the limit of pairs V, c_0 for which $W + \lambda(V - c_0)$ is negative definite. However, the user must not forget that while the usual necessary and sufficient conditions for a quadratic form to be negative definite are straightforward, there are no correspondingly simple conditions for it to be negative semidefinite. In practice this causes no difficulty, but it may be necessary to argue on an *ad hoc* basis. Moreover, while in the definite case the values of λ will fill an interval, in the semidefinite case the value of λ may be uniquely determined.

There is another advantage of this method over the classical Liapunov one, in that in the criterion that $W + \lambda(V - c_0)$ is negative semidefinite it is easy to vary c_0. Suppose first that $\lambda > 0$, which we shall shortly see is the usual case; then $W < 0$ at every point where $V > c_0$. In other words, each trajectory either has V monotone decreasing and $V \to c_0$ as $t \to +\infty$ or it enters $V \leqslant c_0$ at some finite time and never thereafter leaves it; moreover, in the former case if the trajectory is bounded it must tend to the point set given by $V = c_0$ and $W = 0$. This result also makes clear the limitations involved in using functions V of this kind. All invariant sets of the system (3), including fixed points and periodic orbits, must lie in $V \leqslant c_0$. Thus for example if the system (3) has a fixed point P_1 which is a local attractor, any proof that P_1 is a local attractor using a Liapunov function V of this kind will also prove that it is a global attractor; so one appears to be helpless if the latter fact is not true. If instead $\lambda < 0$ then $W < 0$ at every point where $V < c_0$, and indeed $W < \lambda(c_0 - c_1)$ at every point where $V < c_1 < c_0$. Thus any trajectory which enters $V < c_0$ has $V \to -\infty$ as $t \to \infty$, so that the trajectory must tend to infinity. In the particularly simple case when $\lambda = 0$, V is monotone non-increasing and all trajectories tend either to the point set where $W = 0$ or to infinity—and the latter can only happen if the quadratic part of V is not positive definite.

In general, either there are no suitable functions V for which V and W are both inhomogeneous quadratic or one can choose V to be dependent on a number of parameters; these parameters are of course additional to any which may appear in the original equations. (If the f_ν are inhomogeneous quadratic, then the condition that $W = (1/2)\dot{V}$ should be quadratic is a strong constraint on the quadratic terms in V, but imposes no constraint at all on its linear terms.) Two special cases are of particular interest.

First, suppose that P_1 is a fixed point of (3); then it will be important to study those pairs V, c_0 for which $V(P_1) = c_0$, so that P_1 is on the boundary of the repelling region. Without loss of generality we can take P_1 to be the origin. Thus the f_ν have no constant terms, whence the same is true of W. Since we wish $V - c_0$ to have no constant term, the same must be true of $W + \lambda(V - c_0)$; and since this is to be negative semidefinite, it must be homogeneous quadratic. This may sometimes be a further constraint on V, but it is equally likely to determine the choice of λ. It can even sometimes happen that we can take V to be homogeneous quadratic, so that $c_0 = 0$ and the set which all trajectories eventually enter is a cone with the origin as vertex. If this set contains no point (other than the origin) of the stable manifold at the origin, it follows that there can be no homoclinic or heteroclinic orbit terminating at the origin, and that any trajectory either tends to the origin or is eventually bounded away from the origin. This happens for example with the Lorenz equations for certain values of the parameters; see Swinnerton-Dyer (2000).

Second, suppose that one of the f_ν, say f_n, is linear. If we write down the most general inhomogeneous quadratic V such that W is also inhomogeneous quadratic, then V will contain a term $2bx_n$ with no constraint on b. Write $-L = W + \lambda(V - c_0)$; the simplest way to require L to be positive semidefinite is to require the homogeneous quadratic part of L to be positive definite and the determinant of L to be non-negative. To fix ideas, we shall again assume that $\lambda > 0$. The first of these conditions does not involve b or c_0, and the second one has the form

$$\lambda c_0 D_1 - D_2 b^2 \geqslant \text{ expression linear in } b \text{ and independent of } c_0 \qquad (4)$$

If L^* denotes the homogeneous quadratic part of L, then D_1 is the determinant of L^* and is therefore forced to be positive. D_2 can be identified with the determinant of the restriction of L^* to a certain hyperplane, and is therefore also positive; but the reader may find it simpler to note that if $D_2 < 0$ we could satisfy (4) for any fixed c_0 by choosing b large of either sign, so that every point of the space would lie in a repelling region $V > c_0$. Regarding λ and all the parameters in V other than b as fixed, for any b we determine c_0 by the condition that there is equality in (4). In this way we obtain a one-parameter family of repelling regions $V > c_0$, each given by an inequality of the form

$$(D_2/\lambda D_1)b^2 + \text{expression linear in } b < 0$$

here of course the expression linear in b involves the x_ν as well as λ and the other parameters, and it is inhomogeneous quadratic in b and the x_ν together. The union of all these repelling regions as b varies is itself a repelling region, given by an inequality which is straightforward to write down in any particular case; and this inequality is itself quadratic in the x_ν.

The ideas in the two previous paragraphs can of course be combined.

There is a strengthening of this method which is frequently available; it is equally available with Liapunov's method, but with that method the algebra involved is unattractive. Suppose we already know, perhaps by a previous application of these ideas, that Φ is a Liapunov function for each $c \geqslant 0$. Then, broadly speaking, we are only interested in the behaviour of trajectories in the region $\Phi \leqslant 0$; and we know that once a trajectory enters this region it never leaves it. If V, W and Φ are all inhomogeneous quadratic and V is such that for some λ and some $\mu \geqslant 0$ the function $W + \lambda(V - c_0) - \mu\Phi$ is negative semidefinite, then $W \leqslant 0$ at all points where $V = c_0$ and $\Phi \leqslant 0$. From this we can draw for trajectories in $\Phi \leqslant 0$ the same conclusions as we have already drawn for all trajectories from Theorem 1. But the fact that we have an extra parameter μ at our disposal may mean that we can now use functions V which do not satisfy the conditions of Theorem 1.

A variant of these techniques is sometimes useful. It can happen that while V is inhomogeneous quadratic, W is not; but nevertheless there is a polynomial ϕ such that $W + \phi V$ is inhomogeneous quadratic. Assume also that V is indefinite. Then as in the proof of Theorem 1, $W \leqslant 0$ at all points where $V = 0$ if and only if $W + (\lambda + \phi)V$ is negative semidefinite for some λ. This is a less powerful result, in that the effects of replacing V by $V - c$ are not straightforward; on the other hand, it is applicable in circumstances in which Theorem 1 is not.

The referee has suggested that I should give an example, to show how the process works. In such circumstances one should investigate a system provided by someone else, rather than invent one's own; so I shall consider the system

$$\dot{x} = yz, \quad \dot{y} = x - y, \quad \dot{z} = 1 - xy$$

which is the second of Sprott's list (Sprott 1994) of simple and interesting systems. (The first is a special case of the Nosé equations, and appears not to admit any Liapunov functions of this kind.) This system has fixed points as $(1,1,0)$ and $(-1,-1,0)$ and a trajectory $x = y = 0$ which tends to infinity. It also has a symmetry, obtained by reversing the signs of x and y. For $\dot{V}$ to be quadratic, V must have the form

$$V = a_1(x^2 + z^2) + a_2 y^2 + 2a_3 x + 2a_4 y + 2a_5 z + a_6 \tag{5}$$

where the presence of a_6 enables us to require $c_0 = 0$. In view of the number of parameters in this expression, we need some rule for picking out the most interesting V.

One plausible rule would be to require the surface $V = 0$ to pass through the two fixed points. Since also $\dot{V} = 0$ at those points, for $\dot{V} + \lambda V$ to be negative semidefinite it must be homogeneous quadratic in the two variables $x - y$ and z. This is an easy condition to handle, because it is linear in the a_i; indeed it is equivalent to

$$a_3 = a_4 = a_1 + \lambda a_5 = a_1 + a_2 + a_6 = 0, \quad \lambda a_1 = (\lambda - 2)a_2 = a_5 - a_2$$

With the right choice of signs, this gives

$$V = (\lambda - 2)(x^2 + z^2) + \lambda y^2 + 2\lambda(\lambda - 1)z - 2(\lambda - 1)$$

where $\lambda = 1.353\,21\ldots$ is the unique real root of $\lambda^2(\lambda - 1) + \lambda - 2 = 0$. Now

$$\dot{V} + \lambda V = -\lambda(2 - \lambda)\{(x - y)^2 + z^2\}$$

which is indeed negative semidefinite. It follows that any trajectory either tends to infinity or to one of the two fixed points or else eventually enters $V \leqslant 0$ and thereafter remains there.

We can obtain a stronger result, though at the price of heavier algebra. If we merely require the surface $V = 0$ to pass through the fixed point $(1,1,0)$ then $\dot{V} + \lambda V$ must be homogeneous quadratic in $x - 1$, $y - 1$ and z. If V is given by

(5), this requires

$$\lambda(a_1 + a_3) = (2 - \lambda)(a_2 + a_4) = -\lambda^2 a_5, \quad a_6 = a_1 + a_2 - 4\lambda^{-1} a_5$$

with the same value of λ as before. Hence the most general V which we need to consider is given by

$$\varepsilon V = (\lambda - 2 - \mu_1)(x^2 + z^2) + (\lambda - \mu_2)y^2 \\ + 2\mu_1 x + 2\mu_2 y + 2\lambda(\lambda - 1)z - 2(\lambda - 1) - \mu_1 - \mu_2 \tag{6}$$

where $\varepsilon = \pm 1$. Writing $X = x - 1$, $Y = y - 1$ this gives

$$\varepsilon(\dot{V} + \lambda V) = \lambda(\lambda - 2 - \mu_1)(X^2 + z^2) + (\lambda - 2)(\lambda - \mu_2)Y^2 \\ - 2(\lambda^2 - 2\lambda + \mu_2)XY + 2\mu_1 Yz \tag{7}$$

The right-hand side is negative semidefinite if and only if $\varepsilon\lambda(\lambda - 2 - \mu_1) < 0$ and

$$\varepsilon \begin{vmatrix} \lambda(\lambda - 2 - \mu_1) & -(\lambda^2 - 2\lambda + \mu_2) & 0 \\ -(\lambda^2 - 2\lambda + \mu_2) & (\lambda - 2)(\lambda - \mu_2) & \mu_1 \\ 0 & \mu_1 & \lambda(\lambda - 2 - \mu_1) \end{vmatrix} \geqslant 0$$

and since the determinant has $\lambda(\lambda - 2 - \mu_1)$ as a factor, the second condition reduces to

$$\mu_1^2 + \lambda(2 - \lambda)\mu_1\mu_2 + \mu_2^2 - \lambda^2(2 - \lambda)(\mu_1 + \mu_2) \leqslant 0 \tag{8}$$

If $\lambda - 2 - \mu_1$ is small then the quadratic form (7) cannot be semidefinite, so (8) fixes the sign of $\lambda - 2 - \mu_1$; and it must fix it to be negative because (8) permits $\mu_1 = \mu_2 = 0$. Hence the first condition reduces to $\varepsilon = +1$.

Let $\mathcal{R}(\mu_1, \mu_2)$ denote the set in the x, y, z space given by $V > 0$, where V is given by (6) with $\varepsilon = 1$. Then every trajectory either tends to infinity or to one of the fixed points or else eventually leaves $\mathcal{R}(\mu_1, \mu_2)$ and never thereafter re-enters it. The same property holds for $\mathcal{R} = \cup\mathcal{R}(\mu_1, \mu_2)$, where the union is taken over all pairs μ_1, μ_2 satisfying (8); it even holds for $\mathcal{R} \cup \mathcal{R}^*$ where $\mathcal{R}^*$ is obtained from $\mathcal{R}$ be means of the symmetry.

The simplest way to obtain $\mathcal{R}$ is as follows. Clearly $\mathcal{R}$ is also the set of points such that $V \geqslant 0$ for some μ_1, μ_2 in the interior of the ellipse (8); and since $\partial V/\partial\mu_1$ and $\partial V/\partial\mu_2$ both vanish only at the fixed point (1,1,0), this is the same as saying that $V > 0$ for some μ_1, μ_2 in the interior of the ellipse. Since for fixed x, y, z and fixed $c > 0$ the equation $V = c$ defines a straight line, $\mathcal{R}$ is also the set of points such that V takes a positive value at some point on the boundary of the ellipse. To parametrize the boundary, set $\mu_1 = t_1\mu$, $\mu_2 = t_2\mu$; thus the general point of the boundary is given by

$$\mu_1 = \frac{\lambda^2(2 - \lambda)t_1(t_1 + t_2)}{t_1^2 + \lambda(2 - \lambda)t_1 t_2 + t_2^2} \quad \mu_2 = \frac{\lambda^2(2 - \lambda)t_2(t_1 + t_2)}{t_1^2 + \lambda(2 - \lambda)t_1 t_2 + t_2^2}$$

Substituting into the formula for V and multiplying by the common denominator (which is positive definite) we see that the condition for (x, y, z) to lie in $\mathcal{R}$ is that

$$t_1^2\{A - \lambda^2(2 - \lambda)\mathrm{B}\} + t_1 t_2 \lambda(2 - \lambda)\{A - \lambda(B + C)\} + t_2^2\{A - \lambda^2(2 - \lambda)C\}$$

should take a positive value for some t_1, t_2, where

$$A = (\lambda - 2)(x^2 + z^2) + \lambda y^2 + 2\lambda(\lambda - 1)z - 2(\lambda - 1)$$

$$B = (x-1)^2 + z^2, \quad C = (y-1)^2$$

If $A - \lambda^2(2-\lambda)B > 0$ then $A > 0$ so that (x, y, z) is in $\mathcal{R}(0,0)$. Hence the points additional to $\mathcal{R}(0,0)$ which we obtain are just those at which

$$\lambda^2(2-\lambda)^2\{A - \lambda(B+C)\}^2 > \{A - \lambda^2(2-\lambda)B\}\{A - \lambda^2(2-\lambda)C\}$$

and this reduces to

$$\lambda^4(2-\lambda)(B-C)^2 > 5A^2 + 2(2-3\lambda)A(B+C)$$

The reader will note that this improves on the earlier region $\mathcal{R}(0,0)$ even near the other fixed point $(-1,-1,0)$, despite the fact that the surface $V = 0$ only passes through that fixed point when $\mu_1 = \mu_2 = 0$. Presumably an even stronger result could be obtained by taking V in the general form (5), but the improvement is unlikely to be worth the effort involved.

References

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