19 March 2001

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 281 (2001) 161–167

www.elsevier.nl/locate/pla

Bounds for trajectories of the Lorenz equations: an illustration of how to choose Liapunov functions

Peter Swinnerton-Dyer

Isaac Newton Institute, 20 Clarkson Road, Cambridge CB3 0EH, UK

Received 27 October 2000; received in revised form 31 January 2001; accepted 31 January 2001
Communicated by C.R. Doering

Abstract

This Letter uses Liapunov's method, as modified in Dynam. Stabil. Syst. 15 (2000) 3, to obtain bounds for trajectories of the Lorenz system. It is also intended as a guide to the efficient choice of Liapunov functions.

Keywords: Liapunov; Lorenz

In this Letter I obtain information about the trajectories of the Lorenz equations

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y, \\ \dot{Y} &= (r - Z)X - Y, \\ \dot{Z} &= -bZ + XY, \end{aligned} \tag{1}$$

as $t \to \infty$, where b, r, σ are positive constants. One reason for writing this Letter is that not all the existing literature is to be relied on. But a more important reason is to show that the process of choosing an efficient Liapunov function is not a black art, as is widely believed; using the modification introduced in [4] it is susceptible to logical analysis. However, the reader need not be acquainted with the somewhat delicate arguments of [4]; they are not needed to validate the results of this Letter, but only to show that the results of Examples 2–6 are as strong as those which would be obtained by applying traditional methods to the same choices of Liapunov functions.

The calculations needed in this Letter are undemanding. I plan to write a further paper on the Lorenz equations, showing the extent to which one can get stronger results by working harder. But it is only for artificial examples that any methods currently known will yield best possible results. Even in Example 7 below, though the conclusion is the strongest possible, it actually holds for a larger range of values of the parameters.

Recall that if $r \leqslant 1$ the only fixed point of (1) is the origin, and it is globally stable; the well-known proof of this is reproduced as Example 1. Except in that example, we shall always suppose that $r > 1$, in which case system (1) has three fixed points: the origin, which has a stable manifold of dimension 2, and the points $P_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ each of which has an unstable manifold of dimension 2 if

$$\sigma > b + 1, \qquad r(\sigma - b - 1) > \sigma(\sigma + b + 3),$$

and is stable otherwise.

E-mail address: hpfs100@newton.cam.ac.uk (P. Swinnerton-Dyer).

PII: S0375-9601(01)00109-8

For any system of autonomous first-order differential equations, one says that V is a Liapunov function for the value c if $W = \dot{V} \leqslant 0$ at all points of $V = c$, and a strict Liapunov function for c if $W < 0$ there. (Throughout this Letter, $W = \dot{V}$.) A particularly important case is when V is Liapunov or strict Liapunov for all $c \geqslant c_0$; if so, we denote the set $\{V \leqslant c_0\}$ by Ω_V or occasionally Ω_{V,c_0}. In this case, subject to some abuse of language in respect of behaviour at infinity, each trajectory either enters the set Ω_V and never thereafter leaves it or tends to the set given by $\{W = 0\}$. It is usually easy to investigate the second alternative, and in practice it is only likely to occur for trajectories which tend to a fixed point. It should be noted that the set $\{W = 0\}$ need not lie in Ω_V; see for instance Example 7, where V is Liapunov for all c. However, if V is strict Liapunov for all $V > c_0$ then the set $\{W = 0\}$ must lie in Ω_V; and if it is strict Liapunov for all $V \geqslant c_0$ then every trajectory eventually enters Ω_V and never thereafter leaves it.

It is convenient to call a set $\mathfrak{R}$ a *trapping region* for a system if each trajectory either enters the set $\mathfrak{R}$ and never thereafter leaves it or tends to the boundary of $\mathfrak{R}$. Thus Ω_{V,c_0} is a trapping region if it is bounded (or all trajectories are uniformly bounded for other reasons) and V is strict Liapunov for all $c > c_0$. The justification for introducing this definition is that the intersection of any collection of trapping regions is a trapping region. In particular, if V, c_0 depend on one or more parameters then (subject to boundedness) we can take $\mathfrak{R}$ to be the intersection of all the relevant Ω_{V,c_0}. (See Example 4, which also illustrates that not all Ω_{V,c_0} need to be bounded even if $\mathfrak{R}$ itself is to be bounded.) There may be advantages in making $\mathfrak{R}$ smaller, but it is a matter of judgement to what extent it is worth while expending extra effort to do so; note also that making $\mathfrak{R}$ smaller usually involves making its description more complicated.

Not all systems admit trapping regions, even if they are physically realistic; for example, most conservative or reversible systems do not. Clearly it would be a waste of time to attempt to apply Liapunov's method to such a system. For systems which are amenable to Liapunov's method, there can still be complications over behaviour at infinity; but these cannot happen if we know that all trajectories are eventually uniformly bounded. For physically realistic systems to which it is sensible to apply Liapunov's method, uniform boundedness will usually be true; and if it is true it will usually be easy to prove. If this turns out not to be so, one should at least ask oneself whether the approximation process by which the equations were derived is inappropriate when some of the variables are large. A proof of boundedness should usually be the first step in applying Liapunov's method. For this it is enough to exhibit a function V_0 such that $V_0(P) \to +\infty$ and $W_0(P) \to -\infty$ as the point P tends to infinity. (See, for instance, Example 2.)

Most of this Letter consists of examples, showing the conclusions that can be drawn from particular choices of V, W and the motivations for these choices; in each case the subscripts of V, W correspond to the number of the example. One expects that assertions typically only hold for a certain range of values of the parameters; hence there needs to be a lot of case-splitting according to inequalities between the parameters. This is a phenomenon common to all approaches; it is not confined to the methods used in this Letter. The end of an example is denoted by the end-of-proof sign $\square$. For $r > 1$, the most important conclusions, obtained in Examples 3 and 7, are that if $2\sigma \leqslant b$ every trajectory tends to a fixed point; and if $r > 1$ and $2\sigma > b$ then every trajectory either tends to the origin or eventually enters $2\sigma Z \geqslant X^2$ and never thereafter leaves this region. The latter statement was first correctly proved in [2].

For given V, the standard method of testing whether V is a Liapunov function for c is to find the maximum of W on $\{V = c\}$. Calculations of this kind become rapidly more onerous as the degrees of V and W increase, and for this reason most authors have confined themselves to the case when V and W are inhomogeneous quadratic; even then, the published literature contains a remarkable number of errors, especially when the set $\{V = c\}$ is unbounded. But if instead one requires for some $\lambda \geqslant 0$ that $W + \lambda(V - c_0)$ is negative semidefinite or negative definite in the entire space (which is much easier to test for), then V is Liapunov or strict Liapunov respectively for all $c \geqslant c_0$. It might appear that this is a stronger condition than Liapunov's, and this is true in general; but the main result of [4] is that when V, W are inhomogeneous quadratic the two criteria are equivalent. Once we use this new criterion, we are really only considering the single value c_0 of c. If, as happens from Example 3 onwards, we wish the set $\{V = c_0\}$ to contain one of the fixed points, it is natural to absorb c_0 into V and therefore to take $c = 0$.

Note that the case when $W + \lambda(V - c_0)$ is only negative semidefinite yields some of the strongest conclusions. These need not be the limits of cases where $W + \lambda(V - c_0)$ is negative definite; see for instance Examples 3 and 7.

One would always like to choose V to be as simple as possible; one way of doing this is to restrict the degrees of V, W. (An important alternative will be found after Example 6.) For system (1), the condition that V, W have degrees at most d is equivalent to V being inhomogeneous of degree at most d with the terms which have exact degree d being dependent only on X and $Y^2 + Z^2$. System (1) also possesses a time-preserving symmetry, given by

$$(x, y, z, t) \mapsto (-x, -y, z, t).$$

It is frequently sensible to restrict attention to those V which are invariant under the symmetry; in this case W will have the same property.

Example 1. The case $r \leqslant 1$ is straightforward; choose, for example,

$$V_1 = X^2 + \sigma\bigl(Y^2 + Z^2\bigr),$$
$$W_1 = -2\sigma\bigl\{X^2 - (1 + r)XY + Y^2\bigr\} - 2bZ^2 \leqslant 0.$$

Thus V_1 is non-increasing on any trajectory, and all trajectories are bounded. If $r < 1$ then $W_1 = 0$ only at the origin; if $r = 1$ then $W_1 = 0$ if and only if $X = Y,\ Z = 0$, but the only trajectory which lies in this set is the origin. Thus in either case every trajectory tends to the origin. □

Henceforth we shall always assume that $r > 1$. In Examples 2–5 we illustrate cases where V is chosen to be symmetric and such that V, W are inhomogeneous quadratic. This implies that

$$V = a_1X^2 + a_2\bigl(Y^2 + Z^2\bigr) + 2a_3Z \tag{2}$$

for some constants a_1, a_2, a_3. Thus

$$\begin{aligned} W + \lambda(V - c) = {} & -a_1(2\sigma - \lambda)X^2 - a_2(2 - \lambda)Y^2 \\ & + 2XY(\sigma a_1 + ra_2 + a_3) \\ & - a_2(2b - \lambda)Z^2 - 2a_3(b - \lambda)Z - \lambda c. \end{aligned}$$

When $\lambda > 0$ the right-hand side is negative semidefinite if and only if

$$a_1(2\sigma - \lambda) \geqslant 0, \qquad a_2(2 - \lambda) \geqslant 0, \tag{3}$$
$$(\sigma a_1 + ra_2 + a_3)^2 \leqslant a_1a_2(2\sigma - \lambda)(2 - \lambda), \tag{4}$$
$$a_2(2b - \lambda) \geqslant 0, \qquad c \geqslant 0,$$
$$\lambda ca_2(2b - \lambda) \geqslant a_3^2(b - \lambda)^2. \tag{5}$$

Example 2. If we wish to prove a boundedness result, then we need $a_1 > 0$, $a_2 > 0$. For fixed a_i, λ we make Ω_V smaller by making c smaller; so to satisfy the last two inequalities (5) we should take

$$c = a_3^2(b - \lambda)^2/a_2\lambda(2b - \lambda).$$

The simplest (though not the most powerful) way of satisfying (4) is to choose

$$a_3 = -\sigma a_1 - ra_2;$$

now the only constraint on λ is $0 < \lambda < \min(2, 2b, 2\sigma)$. So Ω_V is

$$a_1X^2 + a_2Y^2 + a_2\bigl(Z - (a_1\sigma + a_2r)/a_2\bigr)^2 \leqslant \frac{b^2(a_1\sigma + a_2r)^2}{\lambda a_2(2b - \lambda)}. \tag{6}$$

For given a_1, a_2 the optimum choice of λ is $\lambda = \min(2\sigma, b, 2)$. Rather than choose a particular ratio a_1/a_2, we can demand that X, Y, Z are such that (6) holds for all positive a_1, a_2. This condition can be written

$$\begin{aligned} & a_1^2\left(\frac{\sigma^2(b - \lambda)^2}{\lambda(2b - \lambda)}\right) \\ & \quad + a_1a_2\left(\frac{2\sigma r(b - \lambda)^2}{\lambda(2b - \lambda)} + 2\sigma Z - X^2\right) \\ & \quad + a_2^2\left(\frac{b^2r^2}{\lambda(2b - \lambda)} - Y^2 - (Z - r)^2\right) \geqslant 0, \end{aligned}$$

so that we must certainly have

$$Y^2 + (Z - r)^2 \leqslant \frac{b^2r^2}{\lambda(2b - \lambda)}, \tag{7}$$

where λ is as above. According to [1], this result was first found by Trève (unpublished). We shall see in the next example that the trapping region lies in $\{X^2 \leqslant 2\sigma Z\}$, and within this region we need no condition other than (7) for (6) to hold. It might appear that (7) is simply the limiting case $a_1 = 0$; but if $a_1 = 0$ we no longer need $2\sigma \geqslant \lambda$, which was only imposed to ensure that the first inequality (3) was satisfied. Thus we can take $\lambda = \min(b, 2)$; but note that if $b \leqslant 2$ this result is contained in (9), so it is only of interest when $b > 2$ and $\lambda = 2$. This illustrates that a limiting case can be discontinuously stronger than the examples of which it is the limit. We could get a stronger but more complicated result if we allow a_3 to run through the interval given by (4). We leave the ugly details to any reader who is interested; a simpler version of the same idea can be found in Examples 4 and 5. □

We can also use V of the form (2) with Ω_V unbounded. An important special case is when the boundary of Ω_V passes through the origin. (If we retain symmetry, we do not have enough free parameters to make it pass through one and therefore both of the other fixed points; for what happens when we drop symmetry, see Example 6.) We remind the reader that from now on c will be absorbed into V, so that we take $c = 0$ in (5). We have now two alternative recipes, according to whether V is pure quadratic or the linear part of V is an eigenform of the system derived from (1) by linearizing at the origin and $-\lambda$ is the corresponding eigenvalue. We first consider the latter case. Using a more complicated choice of V, we shall see in Example 7 that if $b \geqslant 2\sigma$ every trajectory tends to one of the three fixed points; until that example we can therefore assume that $2\sigma > b$.

Examples 3 and 4. If we choose the relevant eigenvector to be $(0, 0, 1)$, then we must take $\lambda = b$ and the most general V to consider is still (2); now

$$W + bV = a_1X^2(b - 2\sigma) + 2XY(a_1\sigma + a_2r + a_3) + a_2Y^2(b - 2) - 2a_2bZ^2.$$

This last expression is negative semidefinite if and only if

$$a_1(b - 2\sigma) \leqslant 0, \qquad a_2(b - 2) \leqslant 0, \qquad a_2 \geqslant 0,$$
$$(a_1\sigma + a_2r + a_3)^2 \leqslant a_1a_2(b - 2\sigma)(b - 2). \tag{8}$$

Since $2\sigma > b$ we must have $a_1 \geqslant 0$. Also $a_2 = 0$ or $b \leqslant 2$, and if $a_2 = 0$ then $a_3 = -a_1\sigma$. The choice $a_2 = 0$ is particularly simple; we have

$$V_3 = X^2 - 2\sigma Z, \qquad W_3 + bV_3 = (b - 2\sigma)X^2,$$

and Ω_V is $\{X^2 - 2\sigma Z \leqslant 0\}$. In particular, Ω_V lies entirely in $Z \geqslant 0$. This result has already been obtained by Giacomini and Neukirch [2], using the concept of semipermeable surfaces; this is closely related to Liapunov's method. This Ω_V is unbounded; but we can obtain boundedness by intersecting it with the Ω_V of Example 2.

If we instead assume that $b \leqslant 2$ conditions (8) reduce to

$$a_1 \geqslant 0, \qquad a_2 \geqslant 0,$$
$$(a_1\sigma + a_2r + a_3)^2 \leqslant a_1a_2(2 - b)(2\sigma - b).$$

(If a_1, a_2 are both strictly positive, this example will give another proof of boundedness.) We could of course choose any set of values of a_1, a_2, a_3 which satisfy these conditions; but we can obtain a stronger result without too much effort by taking Ω_V to be the intersection of all the sets $\{V \leqslant 0\}$ for a_1, a_2, a_3 satisfying these conditions. By the result in the previous paragraph, we need only consider trajectories lying in $X^2 \leqslant 2\sigma Z$. For fixed a_1, a_2 we wish to make Ω_V as small as possible, so we should take a_3 to be as large as possible; if a_3 is necessarily negative (as is the case here) this means that we should take the absolute value of a_3 to be as small as possible. Thus we choose

$$V_4 = a_1X^2 + a_2\big(Y^2 + Z^2\big) - 2Z\Big(a_1\sigma + a_2r - \sqrt{a_1a_2(2 - b)(2\sigma - b)}\Big).$$

The corresponding Ω is the set of points for which $V \leqslant 0$ for all positive a_1, a_2; so it is given by

$$f_1 \leqslant 0, \qquad f_2 \leqslant 0,$$
$$g = f_1f_2 - Z^2(2 - b)(2\sigma - b) \geqslant 0, \tag{9}$$

where $f_1 = X^2 - 2\sigma Z$, $f_2 = Y^2 + Z^2 - 2rZ$. This region is strictly smaller than the region obtained in the previous paragraph, but it is more complicated. I am indebted to a referee for pointing out that the surface $g = 0$ consists of two sheets which touch at the origin; the closed one defines an ovoid, the closure of whose interior is just region (6). However, this result is only valid for $b \leqslant 2$. It is worth remarking that, contrary to what one might expect, g itself is not a Liapunov function. □

There is another eigenvector at the origin whose associated eigenvalue is negative, and we could make use of it. But the algebra is predictably more complicated, and the results turn out to be less interesting.

Example 5. The other case in which the boundary of Ω_V passes through the origin is when V is homogeneous quadratic. Now the value of λ is not constrained, and

$$V_5 = a_1 X^2 + a_2\big(Y^2 + Z^2\big),$$
$$W_5 = -2\sigma a_1 X^2 + 2(\sigma a_1 + r a_2)XY - 2a_2 Y^2 - 2b a_2 Z^2;$$

so $W_5 + \lambda V_5$ is negative semidefinite if and only if

$$a_1(2\sigma - \lambda) \geqslant 0, \qquad a_2(2 - \lambda) \geqslant 0,$$
$$a_2(2b - \lambda) \geqslant 0, \tag{10}$$
$$a_1 a_2(2\sigma - \lambda)(2 - \lambda) \geqslant (\sigma a_1 + r a_2)^2. \tag{11}$$

To take a_1, a_2 both negative would yield only a trivial conclusion; and if we could take them both positive we would be able to prove that all trajectories tend to the origin, which is false. We have therefore two alternatives:

$$a_1 > 0, \qquad a_2 < 0, \qquad 2\sigma \geqslant \lambda \geqslant \max(2b, 2), \tag{12}$$
$$a_1 < 0, \qquad a_2 > 0, \qquad 2\sigma \leqslant \lambda \leqslant \min(2b, 2). \tag{13}$$

To get the smallest Ω_V for (12) we should make $|a_1/a_2|$ as large as possible; for (13) we should make it as small as possible. Thus in either case we should take $\lambda = 1 + \sigma$ if this is allowed, and $\lambda = 2b$ otherwise; we then take $a_1/a_2 = -\mu$, where μ is the larger root of

$$(\sigma\mu - r)^2 + \mu(2\sigma - \lambda)(2 - \lambda) = 0 \tag{14}$$

for (12) and the smaller root for (13). Almost all of this can already be found in [2], though the arguments there are considerably more complicated. □

Example 6. We could also consider V for which the boundary of Ω_V passes through one of the other fixed points — say P_+. We retain the condition that V, W are inhomogeneous quadratic, but we must drop symmetry. If we require the linear part of V at P_+ to be an eigenvector of the linearization of the system about P_+, the resulting algebra will clearly be very unattractive because it involves the solution of a cubic equation; so we consider the case when V is homogeneous quadratic about P_+. Write

$$\rho = \sqrt{b(r-1)}, \qquad \xi = X - \rho, \qquad \eta = Y - \rho,$$
$$\zeta = Z - r + 1,$$

so that P_+ is $(\rho, \rho, r-1)$; then the most general V which fits the specifications above is $V = a_1\xi^2 + a_2(\eta^2 + \zeta^2)$, for which

$$W + \lambda V = -(2\sigma - \lambda)a_1\xi^2 - (2 - \lambda)a_2\eta^2 - (2b - \lambda)a_2\zeta^2 + 2\xi\eta\big(a_1\sigma + a_2(2r-1)\big) - 2a_2\rho\xi\zeta.$$

This is negative semidefinite if and only if either

$$a_2(2 - \lambda) > 0, \qquad a_2(2b - \lambda) > 0,$$
$$a_1(2\sigma - \lambda) - \frac{(a_1\sigma + a_2(2r-1))^2}{a_2(2-\lambda)} - \frac{a_2\rho^2}{2b - \lambda} \geqslant 0,$$

or

$$\lambda = 2, \qquad a_1\sigma + a_2(2r - 1) = 0,$$
$$a_2(2b - 2) > 0, \qquad a_1(2\sigma - 2) - \frac{a_2\rho^2}{2b - 2} \geqslant 0.$$

We confine ourselves to the second alternative, on the grounds that it is the simpler. We now require

$$\frac{4(\sigma - 1)(b - 1)}{\sigma b} + \frac{r - 1}{2r - 1} \leqslant 0,$$

and the conclusion which we draw is that Ω_V is given by

$$(2r - 1)\xi^2 \geqslant \sigma\big(\eta^2 + \zeta^2\big) \quad \text{if } \sigma > 1 > b,$$
$$(2r - 1)\xi^2 \leqslant \sigma\big(\eta^2 + \zeta^2\big) \quad \text{if } \sigma < 1 < b.$$

Similar results hold for P_-. □

In the canonical case $r = 28$, $\sigma = 10$, $b = 8/3$ the only results in this Letter which are relevant are those of (7), Example 3 and case (12) of Example 5. All these can already be found in [2], and the reader who wishes to see pictures is therefore advised to consult that paper. Each of these conditions only involves X and Y through their squares. This is because in deriving them we have used functions V with the same property — which follows unavoidably from the requirements that V, W are inhomogeneous quadratic and are fixed by the symmetry of (1). It follows that the trapping region $\mathfrak{R}$ which we have obtained is invariant when we change the sign of either X or Y.

Since the set of fixed points and the strange attractor (when it exists) do not have this property, there is scope for considerable improvement. In terms of the methodology of this Letter, there are two alternative ways to make progress, and in particular to obtain a trapping region which does not have this extra and undesirable symmetry. One is to drop the symmetry property on V; the other is to retain the symmetry but allow V and W to be of weight 4 in the sense defined below. In a subsequent paper I hope to show that each of these does provide substantial improvements.

It is an inconvenient consequence of the results in [4] that if one uses V such that V, W are inhomogeneous quadratic then the only results which one can prove are those which assert that some connected region $\mathfrak{R}$ is a trapping region. In particular, the closure of such an $\mathfrak{R}$ must contain all fixed points. Thus, for example, one can only exhibit a basin of attraction for an attracting fixed point P by means of such a V if it is the only fixed point of the system. It is therefore natural to ask what is the next simplest family of V. For this, we introduce the *weight* of a polynomial, defined as follows. The weight of a monomial $CX^\ell Y^m Z^n$ is $\ell + 2m + 2n$, and the weight of a polynomial is the largest of the weights of its component monomials. (Not all systems admit a useful weight function, and some admit more than one, but if such functions exist they are easy to find.) This definition is chosen so that the weights of $\dot{X}, \dot{Y}, \dot{Z}$ exceed by 1 the weights of X, Y, Z, respectively. Thus differentiation with respect to t at worst increases the weight of a polynomial by 1. An easy calculation shows that a necessary and sufficient condition for differentiation not to increase the weight of a polynomial $F(X, Y, Z)$ is that its terms of highest weight can be written in terms of $(X^2 - 2\sigma Z)$ and $(Y^2 + Z^2)$. The simplest V which it is useful to consider for system (1) are those for which V, W have weight 2. If one also requires symmetry, these are just the multiples of $X^2 - 2\sigma Z$, and we have seen in Example 3 that this choice gives an interesting result. The next simplest are those for which V, W have degree 2. The symmetric V of this kind are just those given by (2). After them come those for which V, W have weight at most 4; if we retain symmetry, this relaxation extends the space of allowable V by the two extra generators XY and $(X^2 - 2\sigma Z)^2$. Then come the V which V, W have weight at most 6 and degree at most 4; however, we shall not use this last family in this Letter.

Example 7. Finally we ask in what circumstances we can easily prove that each trajectory tends to one of the three fixed points. To prove this, we need to exhibit a V such that $W \leqslant 0$ everywhere and the set on which $W = 0$ contains no complete trajectories other than the fixed points. We know that it is futile to require V, W to be of degree at most 2; so we examine the case when they are both of weight 4. Since we need $W \leqslant 0$ everywhere, and $W = 0$ automatically at each fixed point, the linear terms in W at each fixed point must vanish. If we also impose symmetry, then we must take

$$W_7 = -B_1\big(X^2 - bZ\big)^2 - B_2(X - Y)^2 \tag{15}$$

with B_1, B_2 non-negative. This is a very strong constraint on V. It cannot contain any linear terms because these would give rise to non-zero linear terms in W, and it costs us nothing to require that its constant term vanishes. Moreover, expression (15) has weight 4, and we noted above the constraint which this imposes on the terms of weight 4 in V. Using symmetry, we see that V must be a linear combination of

$$X^2,\ XY,\ \big(X^2 - 2\sigma Z\big)^2,\ \big(Y^2 + Z^2\big);$$

and the fact that the expansion of V about either of the fixed points $P_\pm$ cannot contain any linear terms (though it may contain a constant term) is now enough to determine V up to multiplication by a constant. It turns out that we must choose

$$\begin{aligned} V_7 &= X^4 - 4\sigma X^2 Z + 2\sigma b Z^2 \\ &\quad + 2(2\sigma - b)\big((r-1)X^2 - \sigma(X - Y)^2\big), \\ W_7 &= -4\sigma\big(X^2 - bZ\big)^2 \\ &\quad + 4\sigma(\sigma + 1)(2\sigma - b)(X - Y)^2. \end{aligned}$$

This W is negative semidefinite if and only if $b \geqslant 2\sigma$.

It remains to consider the trajectories which lie entirely in $\{W = 0\}$. Even when $b = 2\sigma$ these satisfy $X^2 - bZ = 0$. Differentiating and using (1) we obtain

$$-2\sigma X^2 + (2\sigma - b)XY + b^2 Z = 0$$

whence $X = 0$ or $X = Y$ or $2\sigma = b$. Differentiating again, each of the first two possibilities gives only fixed points. If $2\sigma = b$ then

$$V = \big(X^2 - bZ\big)^2, \qquad \dot{V} = -2bV,$$

so that $\{W=0\}$ is $\{X^2-bZ=0\}$ and is a union of complete trajectories. On this surface system (1) reduces to

$$\dot{X}=-\sigma X+\sigma Y,$$
$$\dot{Y}=\left(r-X^2/2\sigma\right)X-Y. \tag{16}$$

This motion is area-contracting, so by Bendixson's criterion it can have no closed orbits — not even homoclinic ones. Since every trajectory is bounded, every trajectory on the surface tends to one of the fixed points; and since $V\to 0$ the same result follows for all trajectories in the original space. Thus if $b\geqslant 2\sigma$ each trajectory tends to one of the fixed points. Leonov [3] claims to have already proved this result, but I have only seen an American translation of his paper, and I have been unable to follow his argument in the form in which it is presented there.

The reader will note that although the set consisting of the three fixed points is a trapping region when $b\geqslant 2\sigma$, it cannot be expressed as an intersection of sets Ω_V; for any Ω_V will contain the unstable manifold at P_0.

The same calculation also yields domains of attraction for P_+ and P_-. For let $\mathfrak{R}$ be the open set $\{V<0\}$ which contains P_+ and P_- and has O on its boundary. I claim that the connected components of $\mathfrak{R}$ which contain P_+ and P_- are distinct. For if not, let $\mathfrak{R}_0$ be the connected component which contains them both. Since the intersections of $\mathfrak{R}_0$ with the domains of attraction of P_+ and P_- are open and disjoint, there is a point Q in $\mathfrak{R}_0$ which lies in neither domain of attraction. Thus the trajectory through Q must tend to O; but now $V(Q)\geqslant V(O)\geqslant 0$ because $W\leqslant 0$ everywhere, and this is a contradiction. Hence these two connected components lie in the corresponding domains of attraction. □

References

[1] C.R. Doering, J.D. Gibbon, Dynam. Stabil. Syst. 10 (1995) 255; C.R. Doering, J.D. Gibbon, Dynam. Stabil. Syst. 13 (1998) 299, Corrigendum.
[2] H. Giacomini, S. Neukirch, Phys. Lett. A 227 (1997) 309.
[3] G.A. Leonov, Prikl. Mat. Mekh. 47 (1983) 861, transl. Appl. Math. Mech. 47 (1983) 690.
[4] P. Swinnerton-Dyer, Dynam. Stabil. Syst. 15 (2000) 3.