where $\gamma = \sup \{\|x\| \mid x \in \Phi^n(G)\}$. Thus,

$$d(T^n(E),\ T^n(F)) \le (\log\ \mu)\left(1 - \frac{\log\ (1+(\mu-1)\sigma/\gamma}{\log\ \mu}\right) \le \lambda d(E,\ F)$$

where

$$\lambda = \sup_{1<\mu<\infty} \left(1 - \frac{\log\ (1+(\mu-1)\sigma/\gamma}{\log\ \mu}\right)$$

$$= \sup_{0<s<\infty} \left(1 - \frac{\log\ (1+s\sigma/\gamma)}{\log\ (1+s)}\right).$$

By a simple calculus argument (see, [3, Lemma 3.1]) this last expression equals $1 - \sigma/\gamma < 1$ and the proof is complete. ■

Remark 2.8: The numbers σ and γ are clearly determined solely by the form of the sets G and Ω and by the matrices Φ and Γ. Thus, the same is true for $\lambda = 1 - \sigma/\gamma$. To obtain a somewhat more explicit upper bound for λ we may use the fact that $\gamma = \sup \{\|\Phi^n g_k\| \mid k = 1, 2, \cdots, N\}$ where g_k, $k = 1, \cdots, N$ are the vertices of G, together with an estimate from below for σ which can be obtained for example by the following procedure. Choose any orthonormal basis $x_1, x_2, \cdots, x_n$ of R^n. For each $j = 1, 2, \cdots, n\rho(x_j, H) = \rho(-x_j, H) = \alpha_j > 0$ since there is a linear combination and therefore also a convex combination of elements of H equaling αx_j for some nonzero α. Then clearly $\sigma \ge 1/n \min_{j=1}^n \alpha_j$.

We are now ready to obtain the integer $t(\epsilon)$ with properties as promised above.

Corollary 2.9: Given any $\epsilon > 0$ the inclusions (1.3) and (1.4) hold for all $t \ge t(\epsilon)$ where $t(\epsilon) = kn$ and k is the smallest integer dominating the expression

$$1 + \frac{1}{\log\left(\frac{1}{\lambda}\right)} \log\left[\frac{d(X_n,\ X_{2n})}{(1-\lambda)\ \log\ (1/(1-\epsilon))}\right]. \tag{2.10}$$

Proof: It suffices of course to show that for all $t \ge t(\epsilon)$ we have $(1 - \epsilon)X_{\max} \subseteq X_t$, or equivalently $(1 - \epsilon)X_s \subseteq X_t$ for all $s \ge t \ge t(\epsilon)$. This follows if $(1 - \epsilon)\rho(x, X_s) \le \rho(x, X_t)$ for all $x \in \Sigma$, i.e.,

$$d(X_t,\ X_s) \le \log\ (1/(1-\epsilon)) \qquad \text{for all } s \le t \le t(\epsilon). \tag{2.11}$$

(Note that since $t(\epsilon) \ge n$, X_t and X_s both contain full n-dimensional neighborhoods of the origin, cf. [1, Lemma 4.3].)

Let $\alpha_1 = d(X_n, X_{2n})$. Then, by Theorem 2.3, $d(X_{2n}, X_{sn}) = d(T^n(X_n), T^n(X_{2n})) \le \lambda\alpha_1$. Similarly, by repeated applications of the theorem we obtain that, for any integer $j \ge 1$, $d(X_{jn}, X_{(j+1)n}) \le \lambda^{j-1}\alpha_1$.

Therefore, for any $s \ge t \ge t(\epsilon) = kn$

$$d(X_t,\ X_s) = \sup_{x\in\Sigma} \log\ [\rho(x,\ X_s)/\rho(x,\ X_t)]$$

$$\le \sup_{x\in\Sigma} \log\ [\rho(x,\ X_{qn})/\rho(x,\ X_{kn})]$$

where q is any integer such that $qn \ge s$.

$$\le \sum_{j=k}^{q-1} \sup_{x\in\Sigma} \log\ [\rho(x,\ X_{(j-1)n})/\rho(x,\ X_{jn})] \le \sum_{j=k}^{q-1} \lambda^{j-1}\alpha_1 \le \frac{\lambda^{k-1}\alpha_1}{1-\lambda}.$$

By (2.10) this last expression is dominated by $\log (1/(1 - \epsilon))$ proving (2.11). ■

References

[1] P. O. Gutman and M. Cwikel, "Admissible sets and feedback control for discrete time linear dynamic systems with bounded controls and states," in *Proc. 23rd IEEE Conf. Decision Contr.*, Las Vegas, NV, Dec. 12-14, 1984, paper FP9; also *IEEE Trans. Automat. Contr.*, submitted for publication.

[2] ——, "An algorithm to find maximal state constraint sets for discrete time linear dynamical systems with bounded control and states," in *Proc. 14th IEEE Convention*, Tel Aviv, Israel, Mar. 26-29, 1985; also *IEEE Trans. Automat. Contr.*, submitted for publication.

[3] M. Cwikel and P.-O. Gutman, "Convergence of an algorithm to find maximal state constraint sets for discrete time linear dynamical systems with bounded controls and states," in *Proc. 24th IEEE Conf. Decision Contr.*, Fort Lauderdale, FL, Dec. 11-13, 1985.

[1] E. Kreyszig, *Introductory Functional Analysis with Applications*. New York: Wiley, 1978.

On Time Scaling for Nonlinear Systems: Application to Linearization

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Abstract—**In this note, we propose a method to analyze systems in a time scale which is varied depending on the state such as $dt/d\tau = s(x)$ (where t and τ are the actual time scale and that of new one, respectively, and $s(x)$ is the function which we call time scaling function). Analysis of the system in the new time scale τ enables us to investigate the intrinsic structure of the system. A linearization problem in the new time scale is formulated as wide-sense feedback equivalence and is solved. It is also shown that the time scaling function which makes the system linear is derived as the solution of differential equations.**

I. Introduction

When we analyze a continuous true system, the time scale t is thought to be already given. However, as far as considering the stability of the system or system trajectory, we need not be constrained by the actual time t and we can use any time scale τ as far as τ does not reverse itself.

Hollerbach used a new time scale r, which is different from the actual time t, for the trajectory planning of robots [1], but his work is restricted to robots and r is introduced as a function of t [i.e., $r(t)$], which is not appropriate for system analysis because $r(t)$ must be defined for each trajectory and each trajectory must be analyzed in each time scale. To avoid this difficulty, he considers only the case of $r = ct$ (c: constant value) in his application.

In this note, we introduce the new time τ using a time scaling function $\infty > s(x) > 0$ as $dt/d\tau = s(x)$. A time scale of this type is convenient for system analysis because the state differential equation can be easily rewritten in the time scale τ and ordinary methods for analysis can be applied for the rewritten system.

On the other hand, linearization is one of the most important problems in nonlinear system theory because a great deal of control strategy, which is highly developed in linear system theory, can be applied for linearized system. Brockett [2], Su [3], Hunt *et al.* [4], and Jakubczyk [5] have solved this problem successfully using the concept of feedback equivalence.

In this note, we are concerned with linearization in a new time scale τ, and show that there exist nonlinear systems which cannot be linearized in ordinary time scale t but can be linearized in a new time scale τ. Since the new time scale τ does not go backward against the actual time t, controllers which stabilize the system written in the new time scale τ also stabilize the actual system. So it is meaningful, in the sense of stability, to be concerned with the linearization in the new time scale.

We will define wide-sense feedback equivalence in order to study the linearization in some time scale. In this case, the time scaling function is derived as the solution of differential equations.

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Throughout this note, we consider the single input system

$$\frac{dx}{dt}=f(x)+g(x)u \tag{1}$$

where $x \in M = R^n$, and $u \in R$ are state and input. In (1), $f(x)$ and $g(x)$ are C^∞ vector fields on M, and we assume $f(0) = 0$.

II. New Time Scale

In this section, new time scale τ which depends on state x is introduced and then the state equation of the system is rewritten in this time scale.

New time scale τ is defined using a continuous function $\infty > s(x) > 0$ as

$$\frac{dt}{d\tau}=s(x), \tag{2-a}$$

$$\tau|_{t_0}=\tau_0, \tag{2-b}$$

where t is actual time. The function $s(x)$ is called *time scaling function.*

Theorem 1: System (1) is rewritten in time scale τ as

$$\frac{dx}{d\tau}=s(x)f(x)+g(x)\mu, \tag{3}$$

where

$$u=\frac{1}{s(x)}\mu.$$

Proof: Equation (3) is obvious from the relation

$$\frac{dx}{d\tau}=\frac{dx}{dt}\frac{dt}{d\tau}.$$

Also u is well-defined because $s(x) \neq 0$. □

It has usually been considered that the drift term of the system ($f(x)$) cannot be manipulated directly, but using this theorem the drift term can be altered. Consequently, we can analyze the structure of the system more precisely; in fact, the concept of feedback equivalence will be extended in the following section.

We must emphasize the importance of the restriction $\infty > s(x) > 0$. This means that the new time τ is a strictly monotone increasing function with respect to the actual time t. In other words, the new time τ never goes backward against the actual time t. This fact guarantees that the stability and the state trajectory of the system is preserved by transformation of time scale. So, it is worth analyzing systems in a new time scale.

In order to preserve the smoothness of the vector fields, the time scaling function $s(x)$ is required to be C^∞ with respect to x in the following section. But in other cases, $s(x)$ need possess only a sufficient number of continuous partial derivatives with respect to x.

III. Feedback Equivalence

In this section, we are concerned with the linearization in a new time scale τ. First, we review the work of Su [3], and derive a lemma concerned with ordinary feedback equivalence.

Definition 2 [3]: The system (1) is *feedback equivalent* in a neighborhood of the origin to a controllable linear system if there exist some neighborhood of the origin $N \subset M$ and a C^∞, one to one mapping

$$\Psi : N\times R \to R^n \times R \tag{4}$$
$$x\times u \to y \times v$$

such that the system is expressed in $R^n \times R$ as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v.$$

Su has given the next lemma.

Lemma 3 [3]: The system (1) is feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if

a) $\{g, ad_f g, \cdots, ad_f^{n-1} g\}(x)$ span $T_x M$, and
b) $\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive in a neighborhood of the origin.

Condition b) in this lemma requires us to check $(n-1)(n-2)/2$ Lie brackets $[ad_f^i g, ad_f^j g]$ $(0 \leqq i < j \leqq n-2)$, but the next lemma ensures that it is sufficent to check only $n-2$ Lie brackets.

Lemma 4: Suppose $\{g, ad_f g, \cdots, ad_f^{n-1} g\}(x)$ span $T_x M$ on an open set, then the next three conditions are equivalent on that open set.

a) $\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive.

b) There exist C^∞ functions $\theta_i^{(k-1,k)}$ $(k = 1, 2, \cdots, n-2;\ i = 0, 1, \cdots, k)$ such that

$$[ad_f^{k-1} g,\ ad_f^k g] = \sum_{i=0}^{k} \theta_i^{(k-1,k)} ad_f^i g, \tag{5}$$

for

$$k = 1, 2, \cdots, n-2.$$

c) $\{g, ad_f g, \cdots, ad_f^k g\}$ is involutive for $k = 1, 2, \cdots, n-2$.

Proof a) → c): This is proven by induction. For $k = n-2$, $\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive from condition a). Assume that $\{g, ad_f g, \cdots, ad_f^k g\}$ is involutive, then there exist C^∞ functions $\theta_i^{(j,r)}(x)$ such that

$$[ad_f^j g,\ ad_f^r g] = \sum_{i=0}^{k} \theta_i^{(j,r)} ad_f^i g \qquad (j,\ r \leqq k). \tag{6}$$

On the other hand, from the property of Lie bracket

$$\begin{aligned}[ad_f^j g,\ ad_f^r g] &= [ad_f^j g,\ [f,\ ad_f^{r-1} g]] \\ &= -[f,\ [ad_f^{r-1} g,\ ad_f^j g]] - [ad_f^{r-1} g,\ [ad_f^j g,\ f]] \\ &= [f,\ [ad_f^j g,\ ad_f^{r-1} g]] + [ad_f^{r-1} g,\ ad_f^{j+1} g].\end{aligned} \tag{7}$$

If $j \leqq k-1$ and $r \leqq k$, then (7) is rewritten using (6) as

$$\begin{aligned}[ad_f^j g,\ ad_f^r g] &= \sum_{i=0}^{k} [f,\ \theta_i^{(j,r-1)} ad_f^i g] + \sum_{i=0}^{k} \theta_i^{(r-1,j+1)} ad_f^i g \\ &= \{\theta_0^{(r-1,j+1)} + L_f \theta_0^{(j,r-1)}\} g \\ &\quad + \sum_{i=1}^{k} \{\theta_{i-1}^{(j,r-1)} + L_f \theta_i^{(j,r-1)} + \theta_i^{(r-1,j+1)}\} ad_f^i g \\ &\quad + \theta_k^{(j,r-1)} ad_f^{k+1} g.\end{aligned} \tag{8}$$

Comparing (8) to (6), we conclude $\theta_k^{(j,r-1)} = 0$ $(j \leqq k-1, r-1 \leqq k-1)$ because $\{g, ad_f g, \cdots, ad_f^{k+1} g\}$ are independent for $k \leqq n-2$. This means $\{g, ad_f g, \cdots, ad_f^{k-1} g\}$ is also involutive.

b) → a): It follows directly from (7).

c) → b): Obvious. □

Condition c) in this Lemma 4 is used as the condition for the linearization by Jakubczyk.

Now, we will consider the linearization in a new time scale τ.

Definition 5: A system is *wide-sense feedback equivalent* in a neighborhood of the origin to a controllable linear system if there exists a C^∞ time scaling function $\infty > s(x) > 0$ such that the system

$$\frac{dx}{d\tau}=s(x)f(x)+g(x)\mu, \tag{9}$$

is feedback equivalent in a neighborhood of the origin to a controllable linear system in the time scale τ.

Obviously, the system which is wide-sense feedback equivalent to a linear system need not be linearizable in the actual time scale t. It is meaningful to consider wide-sense feedback equivalence because a transformation of time scale preserves stability. This fact ensures that the feedback laws which stabilize the system (9) also stabilize the system (1). The following lemma is obvious from the definition.

Lemma 6: The system (1) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if there exist C^∞ functions $\infty > s(x) > 0$ and $\delta_i^k(x)$ ($k = 1, 2, \cdots, n-2$; $i = 0, 1, \cdots, k$) such that

a) $\{g, ad_{sf}g, \cdots, ad_{sf}^{n-1}g\}(x)$ span T_xM and

b) $[ad_{sf}^{k-1}g, ad_{sf}^k g] = \sum_{i=0}^{k} \delta_i^k ad_{sf}^i g$

$$\text{for } k=1, 2, \cdots, n-2 \text{ in a neighborhood of the origin.} \tag{10}$$

The main problem of this section is how to find such $s(x)$. To solve this problem, we must examine the conditions in Lemma 6. At first, we show the property of $ad_{sf}^i g$.

Lemma 7: There exist $\xi_j^i(x)$ ($j = 0, 1, \cdots, i$) and $\xi_f^i(x)$, which are C^∞ functions consisting of x, $s(x)$, and partial derivatives of $s(x)$, such that

$$ad_{sf}^i g = \sum_{j=0}^{i} \xi_j^i ad_f^j g + \xi_f^i f. \tag{11}$$

Also $\xi_i^i = \{s(x)\}^i$.

Proof: Obviously, it is satisfied when $i = 0$. Assume that it is satisfied when $i = k$, then

$$\begin{aligned} ad_{sf}^{k+1}g &= [sf, ad_{sf}^k g] \\ &= (sL_f\xi_0^k)g \\ &\quad + \sum_{j=1}^{k} (s\xi_{j-1}^k + sL_f\xi_j^k) ad_f^j g \\ &\quad + s\xi_k^k ad_f^{k-1} g \\ &\quad + \left[sL_f\xi_f^k - \xi_f^k L_f s - \sum_{j=0}^{k} \{\xi_j^k L_{(ad_f^j g)} s\} \right] f. \end{aligned} \tag{12}$$

This means that there exist C^∞ functions ξ_j^{k+1} ($j = 0, 1, \cdots, k+1$) and ξ_f^{k+1} as mentioned in this lemma. □

From this lemma and condition a) in Lemma 6, it is easily shown that "if the system (1) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system, then $\{f, g, ad_f g, \cdots, ad_f^{n-1}g\}(x)$ span T_xM in a neighborhood of the origin."

From now on, we assume that $\{f, g, ad_f g, \cdots, ad_f^{n-1}g\}(x)$ span T_xM in a neighborhood of the origin. The next lemma ensures that there *does not* exist a set of C^∞ functions $\{\bar{\xi}_j^i, \bar{\xi}_f^i\}$ which is different from $\{\xi_j^i, \xi_f^i\}$ and satisfy

$$ad_{sf}^i g = \sum_{j=0}^{i} \bar{\xi}_j^i ad_f^j g + \bar{\xi}_f^i f$$

for $i = 0, 1, \cdots, n-2$.

Lemma 8: If the set of vector fields $\{f, g, ad_f g, \cdots, ad_f^{n-1}g\}(x)$ span T_xM on some open set $N \subset M$, then there *does not* exist an open subset $U \subset N$ such that there exist nonzero C^∞ functions $\psi_i(x)$ ($i = 0, 1, \cdots, n-2$) and $\psi_f(x)$ which satisfy

$$\psi_f f + \sum_{i=0}^{n-2} \psi_i ad_f^i g = 0 \tag{13}$$

on U.

Proof: If such U exists, then there exists an open subset $V \subset U$ and integer $k (0 \leqq k \leqq n-2)$ such that $\psi_k \neq 0$ for all $x \in V$ (i.e., ψ_k^{-1} is also C^∞ function on V) and

$$\psi_f f + \sum_{i=0}^{k} \psi_i ad_f^i g = 0 \tag{14}$$

on V. So, $ad_f^k(g)$ can be rewritten as

$$ad_f^k g = -\psi_k^{-1}\psi_f f - \sum_{i=0}^{k-1} \psi_k^{-1}\psi_i ad_f^i g \tag{15}$$

where $\psi_k^{-1}\psi_f$ and $\psi_k^{-1}\psi_i$ are C^∞ functions. Using this relation

$$\begin{aligned} ad_f^{k+1}g &= [f, ad_f^k g] \\ &= -[f, \psi_k^{-1}\psi_f f] - \sum_{i=0}^{k-1} [f, \psi_k^{-1}\psi_i ad_f^i g] \\ &= -\{L_f(\psi_k^{-1}\psi_f)\} f - \{L_f(\psi_k^{-1}\psi_0)\} g \\ &\quad - \sum_{i=1}^{k-1} \{\psi_k^{-1}\psi_{i-1} + L_f(\psi_k^{-1}\psi_i)\} ad_f^i g \\ &\quad - \psi_k^{-1}\psi_{k-1} ad_f^k g \\ &= \{\psi_k^{-2}\psi_{k-1}\psi_f - L_f(\psi_k^{-1}\psi_f)\} f \\ &\quad + \{\psi_k^{-2}\psi_{k-1}\psi_0 - L_f(\psi_k^{-1}\psi_0)\} g \\ &\quad + \sum_{i=1}^{k-1} \{\psi_k^{-2}\psi_{k-1}\psi_i - \psi_k^{-1}\psi_{i-1} - L_f(\psi_k^{-1}\psi_i)\} ad_f^i g. \end{aligned} \tag{16}$$

Calculate $ad_f^{k+2}g, ad_f^{k+3}g, \cdots$, and $ad_f^{n-1}g$ similarly, we can easily find that

$$\begin{aligned} &\text{span } \{f, g, ad_f g, \cdots, ad_f^{n-1}g\}(x) \\ &\qquad = \text{span } \{f, g, ad_f g, \cdots, ad_f^{k-1}g](x) \end{aligned} \tag{17}$$

for all $x \in V$. This contradicts the assumption. □

Next we examine the condition b) in Lemma 6. We have the following lemma.

Lemma 9: If the system (1) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system, then there exist unique C^∞ functions $\eta_i^{(j,r)}(x)$ and $\eta_f^{(j,r)}(x)$ ($j, r \leqq k$, $i = 0, 1, \cdots, q$) such that

$$[ad_f^j g, ad_f^r g] = \sum_{i=0}^{q} \eta_i^{(j,r)} ad_f^i g + \eta_f^{(j,r)} f, \tag{18}$$

$$q = \begin{cases} k+1 & (2 \leqq k) \\ k & (k \leqq 1), \end{cases} \tag{19}$$

for $k = 0, 1, \cdots, n-3$ (if $n \geqq 4$) or $k = 0, 1, \cdots, n-2$ (if $n \leqq 3$).

Proof of this lemma is analogous to that of Lemma 4, so we omit it.

From ξ in Lemma 7 and η in Lemma 9, we can derive the differential equations which $s(x)$ must satisfy. Using (11) and (18), we have

$$\begin{aligned} &[ad_{sf}^{k-1}g, ad_{sf}^k g] \\ &= \sum_{r=0}^{k+1} \Bigg\{ \sum_{i=0}^{k-1} \sum_{j=0}^{k} \xi_i^{k-1}\xi_j^k \eta_r^{(i,j)} + \sum_{i=0}^{k-1} \xi_i^{k-1} L_{(ad_f^i g)} \xi_r^k \\ &\quad - \sum_{j=0}^{k} \xi_j^k L_{(ad_f^j g)} \xi_r^{k-1} - \xi_{r-1}^{k-1}\xi_f^k - \xi_f^k L_f \xi_r^{k-1} \\ &\quad + \xi_f^{k-1}\xi_{r-1}^k + \xi_f^{k-1} L_f \xi_r^k \Bigg\} ad_f^r g \\ &\quad + \Bigg\{ \sum_{i=0}^{k-1} \xi_i^{k-1} L_{(ad_f^i g)} \xi_f^k \\ &\quad - \sum_{j=0}^{k} \xi_j^k L_{(ad_f^j g)} \xi_f^{k-1} \\ &\quad - \xi_f^k L_f \xi_f^{k-1} + \xi_f^{k-1} L_f \xi_f^k \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=0}^{k} \xi_i^{k-1}\xi_j^k \eta_f^{(i,j)} \Bigg\} f \\ &\overset{def}{=} \sum_{r=0}^{k+1} \zeta_r^k ad_f^r g + \zeta_f^k \end{aligned} \tag{20}$$

where ζ are C^∞ functions consisting of x, s, partial derivative of s and η. (We assume that $\xi_j^i = 0$ for $j > i$ or $j < 0$.) On the other hand, condition b) in Lemma 6 can be rewritten as

$$[ad_{sf}^{k-1}g,\ ad_{sf}^k g] = \sum_{r=0}^{k} \delta_r^k ad_{sf}^r g$$
$$= \sum_{r=0}^{k} \sum_{i=0}^{r} \delta_r^k \xi_i^r ad_f^i g + \sum_{r=0}^{k} \delta_r^k \xi_f^r f$$
$$= \sum_{i=0}^{k} \sum_{r=i}^{k} \delta_r^k \xi_i^r ad_f^i g + \sum_{r=0}^{k} \delta_r^k \xi_f^r f. \tag{21}$$

Compare (20) to (21), keeping the uniqueness of η and ξ in mind, it is easily concluded that $\infty > s(x) > 0$ and δ are solutions of differential equations

$$\zeta_{k+1}^k = 0, \tag{22a}$$

$$\zeta_i^k = \sum_{r=i}^{k} \delta_r^k \xi_i^r \qquad (i = 0,\ 1,\ \cdots,\ k), \tag{22b}$$

$$\zeta_f^k = \sum_{r=0}^{k} \delta_r^k \xi_f^r, \tag{22c}$$

for $k = 1, 2, \cdots, n-3$ (if $n \geqq 4$) or $k = 1, 2, \cdots, n-2$ (if $n \leqq 3$).

Until now, we have investigated necessary conditions for wide-sense feedback equivalence to a linear system. These conditions become sufficient conditions by adding other conditions.

Theorem 10: The system (1) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if

a) there exist C^∞ functions $\eta_i^{(j,r)}$ and $\eta_f^{(j,r)}$ [$j, r \leqq k$; $i = 0, 1, \cdots, q$; q is the same as that of (19)] which satisfy (18) for $k = 0, 1, \cdots, n-3$ (if $n \geqq 4$) or $k = 0, 1, \cdots, n-2$ (if $n \leqq 3$),

b) differential equations (22) have C^∞ solutions $\infty > s(x) > 0$, δ_i^k for $k = 1, 2, \cdots, n-3$ (if $n \geqq 4$) or $k = 1, 2, \cdots, n-2$ (if $n \leqq 3$),

c) in the case of $n \geqq 4$, there exist C^∞ functions δ_i^{n-2} ($i = 0, 1, \cdots, n-2$) which satisfy

$$[ad_{sf}^{n-3}g,\ ad_{sf}^{n-2}g] = \sum_{i=0}^{n-2} \delta_i^{n-2} ad_{sf}^i g, \tag{23}$$

where $s(x)$ is the solution of condition b), and

d) $\{g, ad_{sf}g, \cdots, ad_{sf}^{n-1}g\}(x)$ span T_xM in a neighborhood of the origin.

Obviously, $n \leqq 3$ are special cases. We have the following corollaries.

Corollary 11: The system (1) ($n = 3$) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if:

1) there exist C^∞ functions $\eta_0^{(0,1)}$, $\eta_1^{(0,1)}$, and $\eta_f^{(0,1)}$ such that

$$[g,\ ad_f g] = \eta_0^{(0,1)} g + \eta_1^{(0,1)} ad_f g + \eta_f^{(0,1)} f, \tag{24}$$

2) there exists C^∞ solution $\infty > s(x) > 0$ for

$$L_g^2 s - \frac{2}{s}(L_g s)^2 - \eta_1^{(0,1)} L_g s - s\eta_f^{(0,1)} = 0, \tag{25}$$

3) $\{g, ad_{sf}g, ad_{sf}^2 g\}(x)$ span T_xM in a neighborhood of the origin.

Corollary 12: The system (1) ($n = 2$) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if there exists $\infty > s(x) > 0$ such that $\{g, ad_{sf}g\}(x)$ span T_xM in a neighborhood of the origin.

Corollary 13: The system (1) ($n = 1$) is wide-sense feedback equivalent in a neighborhood of the origin to a controllable linear system if and only if $g(x) \neq 0$ in a neighborhood of the origin.

IV. Example

Consider the system

$$\frac{dx}{dt} = f(x) + g(x)u$$

where $x = (x_1, x_2, x_3)^T$, $f(x) = (x_2 e^{x_3}, x_3 e^{x_3}, 0)^T$ and $g(x) = (0, 0, 1)^T$. Then,

$$[f,\ g] = -\begin{bmatrix} x_2 e^{x_3} \\ (x_3+1)e^{x_3} \\ 0 \end{bmatrix},$$

$$[g,\ ad_f g] = -\begin{bmatrix} x_2 e^{x_3} \\ (x_3+2)e^{x_3} \\ 0 \end{bmatrix}$$
$$= f + 2ad_f g \notin \text{span } \{g,\ ad_f g\}.$$

This system is not feedback equivalent to a linear system, but it is wide-sense feedback equivalent to a linear system if there exists C^∞ solution $\infty > s(x) > 0$ which satisfies

$$L_g^2 s - \frac{2}{s}(L_g s)^2 - 2L_g s - s = \frac{\partial^2 s}{\partial x_3^2} - \frac{2}{s}\left[\frac{\partial s}{\partial x_3}\right]^2 - 2\frac{\partial s}{\partial x_3} - s = 0.$$

Since $s(x) = e^{-x_3}$ satisfies this differential equation, the system is wide-sense feedback equivalent to a linear system. In fact, using time scaling function $dt/d\tau = e^{-x_3}$, the system is rewritten as

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v,$$

$$v = e^{x_3} u$$

and this system is a linear system in τ time scale.

VII. Conclusion

In this note, we propose a method to analyze a system using a new time scale which is different from the actual one, and we apply this method to solve a linearization problem under the new time scale. This method may be helpful when other structural concepts of nonlinear systems are to be investigated.

References

[1] J. M. Hollerbach, "Dynamic scaling of manipulator trajectories," *Trans. ASME*, vol. 106, no. 5, pp. 102–106, 1984.

[2] R. W. Brockett, "Feedback invariants for nonlinear systems," *IFAC Congress*, Helsinki, 1978.

[3] R. Su, "On the linear equivalents of nonlinear systems," *Syst. Contr. Lett.*, vol. 2, no. 1, pp. 48–52, 1982.

[4] L. R. Hunt, R. Su, and G. Meyer, "Design for multi-input nonlinear systems," in *Differential Geometric Control Theory*, R. Brockett *et al.* Eds. New York: Birkhauser, 1982, pp. 268–298.

[5] B. Jakubczyk and W. Respondek, "On linearization of control systems," *Bull. Acad. Pol. Sci. Ser. Sci. Math.*, vol. XXVIII, pp. 517–522, 1980.

Robust Linear Filtering for Multivariable Stationary Time Series

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Abstract—**We consider the problem of asymptotic noncausal linear filtering for multivariable second-order stationary time series, under spectral uncertainty in both the signal and the noise processes. The spectral uncertainty is modeled by ϵ-contaminated and p-point classes.**

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