2-13

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

$A\vec{x}=\vec{b} \Rightarrow LU\vec{x}=\vec{b}$ $\quad b_4=5$

$$L=\begin{bmatrix} 1 & & & \\ l_2 & 1 & & \\ & l_3 & 1 & \\ & & l_4 & 1 \end{bmatrix} \quad U=\begin{bmatrix} r_1 & -1 & & \\ & r_2 & -1 & \\ & & r_3 & -1 \\ & & & r_4 \end{bmatrix}$$

$Ly=b$

$Ux=y$

$r_1=\frac{2}{1}$ $\quad r_1 l_2=-1$ $\quad l_2=-\frac{1}{2}$

$-1l_2+r_2=2$ $\quad r_2=2+l_2=\frac{3}{2}$ $\quad l_3=\frac{-1}{r_2}=\frac{-2}{3}$

$-1l_3+r_3=2$ $\quad r_3=2+l_3=\frac{4}{3}$ $\quad l_4=\frac{-1}{r_3}=\frac{-3}{4}$

$-l_4+r_4=2$ $\quad r_4=2+l_4=\frac{5}{4}$

$y_1=b_1=0$ $\quad l_2y_1+y_2=b_2$ $\quad y_2=b_2-l_2y_1=0$ $\quad \vec{y}=\begin{bmatrix}0\\0\\0\\5\end{bmatrix}=\vec{b}$

$y_3=b_3-l_3y_2=0$ $\quad y_4=b_4-l_3\cdot y_3=5$

$r_4x_4=y_4$ $\quad x_4=y_4/r_4=\frac{5\cdot 4}{5}=4$ $\quad x_3r_3-x_4=0$ $\quad x_3=\frac{x_4}{r_3}=\frac{4\cdot 3}{4}=3$

$x_2=2$ $\quad x_1=1$ $\quad \vec{x}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

2-3 $A\vec{x}=\vec{b}$ $\quad A=LU$

$\vec{b}=\begin{bmatrix}2\\8\\4\\9\end{bmatrix}$ $\quad A=\begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 5 & 3 & -2 \\ -2 & -2 & 3 & 5 \\ 1 & 3 & 2 & 3 \end{bmatrix}$ $\quad A=LU$ $\quad \boxed{\text{Gauss}}$

$A|b \rightarrow G_1(A|b)=\begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 5 & 1 \\ 0 & 1 & 1 & 5 \end{bmatrix}\begin{bmatrix}2\\4\\8\\7\end{bmatrix} \rightarrow G_2G_1(A,b)=\begin{bmatrix} * & & & \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ & & 0 & 3 \end{bmatrix}\begin{bmatrix} \\ 4\\0\\3\end{bmatrix}$

$LU\cancel{x}=b \qquad (L^{-1}L)Ux=L^{-1}b \qquad Ly=b \qquad y=\begin{bmatrix}2\\4\\0\\3\end{bmatrix} \qquad Ux=y$

$$U=\left[\begin{array}{cccc|c}1&2&1&-2&\cancel{2}\\&1&1&2&\cancel{4}\\&&3&-3&\cancel{0}\\&&&3&\cancel{3}\end{array}\right] \qquad L=\begin{bmatrix}1&&&\\2&1&&\\-2&2&1&\\1&1&0&1\end{bmatrix}$$

$\boxed{\text{Doolittle}}$

$$L=\begin{bmatrix}1&&&\\l_{21}&1&&\\l_{31}&l_{32}&1&\\l_{41}&l_{42}&l_{43}&1\end{bmatrix} \qquad U=\begin{bmatrix}u_{11}&u_{12}&u_{13}&u_{14}\\&u_{22}&u_{23}&u_{24}\\&&u_{33}&u_{34}\\&&&u_{44}\end{bmatrix}$$

$u_{11}=1 \qquad \vec{l}_{i1}^{\,T}=[2\ \ -2\ \ 1] \qquad \cancel{u_{12}l_{21}+u_{22}}=u_{1j}^{T}=[2\ \ 1\ \ -2]$

$u_{12}l_{21}+u_{22}=5 \qquad u_{22}=5-4=1$

$u_{12}l_{31}+u_{22}l_{32}=-2 \qquad l_{32}=\frac{1}{u_{22}}\cdot(-2-2(-2))=+2$

$$L=\begin{bmatrix}1&&&\\2&1&&\\-2&2&1&\\1&1&0&1\end{bmatrix} \qquad U=\begin{bmatrix}1&2&1&-2\\&1&1&2\\&&3&-3\\&&&3\end{bmatrix} \qquad Ux=y$$

$x_4=\frac{y_4}{u_{44}}=1$

$x_3=(y_3-u_{34}x_4)/u_{33}=\frac{3}{3}=1$

$x_2=(y_2-u_{23}x_3-u_{24}x_4)/u_{22}=\frac{4-1-2}{1}=1$

$x_1=(y_1-u_{12}x_2-u_{13}x_3-u_{14}x_4)/u_{11}=2-2-1+2=1$

$\vec{x}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

3-3

3-3

A 对称正定 $A=L^TDL$ $L^TL=I$

$\|x\|_{A^{\frac{1}{2}}} = (Ax,x)^{\frac{1}{2}} = [L^T(\sqrt{D}Lx, \sqrt{D}Lx)]^{\frac{1}{2}} =$ $\|Lx\|_{\frac{1}{2}}$

$= \|\sqrt{D}Lx\|$

3-19

$Ax=b$ $A=L^TDL$

$(A+\delta A)(x+\delta x)=b$

$Ax+(\delta A)x+A(\delta x)+(\delta A)(\delta x)=b$

$(\delta A)x+A(\delta x)+(\delta A)(\delta x)=0$

$(\delta A)(x+\delta x)+A(\delta x)=0$

$(\delta A)(x+\delta x)=-A(\delta x)$

$\|A^{-1}\|=\frac{1}{|\lambda_n|}$ $\|A\|=|\lambda_1|$ $\|A^{-1}\|\cdot\|A\|=\left|\frac{\lambda_1}{\lambda_n}\right|$

$\|A^{-1}\|=\left|\frac{\lambda_1}{\lambda_n}\right|\frac{1}{\|A\|}$

$A^{-1}(\delta A)(x+\delta x)=\delta x$

$\|\delta x\|_2 \leqslant \|A^{-1}\|\,\|\delta A\|\,\|x+\delta x\| = \left|\frac{\lambda_1}{\lambda_n}\right|\frac{\|\delta A\|}{\|A\|}\|x+\delta x\|$

$$\frac{\|\delta x\|_2}{\|x+\delta x\|_2} \leqslant \left|\frac{\lambda_1}{\lambda_n}\right|\frac{\|\delta A\|_2}{\|A\|_2}$$

3-17 $A = D - L - U$

SOR $x^{k+1} = x^k + \omega D^{-1}[Lx^{k+1} + (U-D)x^k + b]$

$$x^{k+1} = H_\omega x^k + \omega (D-\omega L)^{-1} b$$

$$H_\omega = (D-\omega L)^{-1}[(1-\omega)D + \omega U] \quad 0<\omega\le 1$$

$$\lambda(D-\omega L)\vec{y} = [(1-\omega)D+\omega U]\vec{y}$$

$$\det\big[[(1-\omega)D+\omega U] - \lambda(D-\omega L)\big] = 0$$

$$\det[(1-\omega-\lambda)D + \omega U + \lambda\omega L] = 0$$

$$\det[(1-\lambda)D - \omega D + \omega U + \omega L + (\lambda-1)\omega L] = 0$$

$\det[(1-\lambda)(D-\omega L) - \omega A] = 0$ $\quad$ $\lambda \ge 1$ 时 $1-\lambda \le 0$

$\lambda - 1 \ge 0$

$\det[(\lambda-1)(D-\omega L) + \omega A] \ne 0$

$D-\omega L$ 与 A 均占优 则 $\quad$ 则 $\forall \lambda \ge 1$ 不是 H_ω 的特征根 $\lambda<1$ $\rho(H_\omega)<1$ SOR 收敛

3-4 $A=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$\|A\|_\infty = \max(3, 3) = 3$

$\|A\|_1 = \max(1, 5) = 5$

$A^T A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$

$\det(A^T A - \lambda I) = (5-\lambda)^2 - 4 = \lambda^2 - 10\lambda + 21$

$= (\lambda-7)(\lambda-3)$

$\|A\|_2 = \sqrt{7}$ $\quad \rho(A) = \sqrt{7}$

3-6 $x^{(k+1)} = (I - B^{-1}A) x^{(k)} - B^{-1} b \quad Ax=b$

$\rho((A-B)^T(A-B)) < \min\limits_{\sigma(BB^T)} |\lambda|$

$\min|\lambda| = \dfrac{1}{\rho^2(B^{-1})}$

$\rho((I-B^{-1}A)^T(I-B^{-1}A)) = \rho((B-A)(B^{-1})^T B^{-1}(B-A))$

$\leq \rho^2(B^{-1})\rho((A-B)^T(A-B))$

2-8(2)

列主元解 $\begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 7 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$

$$\xrightarrow{2\leftrightarrow 1} \left[\begin{array}{ccc|c} 3 & 4 & 7 & 6 \\ 2 & 3 & 5 & 5 \\ 1 & 3 & 3 & 5 \end{array}\right] \xrightarrow{2^{+}-1} \begin{bmatrix} 3 & 4 & 7 & 6 \\ 0 & \frac{1}{3} & \frac{1}{3} & 1 \\ 0 & \frac{5}{3} & \frac{2}{3} & 3 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 4 & 7 & 6 \\ & 5 & 2 & 9 \\ & 1 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 4 & 7 & 6 \\ 0 & 5 & 2 & 9 \\ 0 & 0 & \frac{3}{5} & \frac{6}{5} \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 3 & 4 & 7 & 6 \\ 0 & 5 & 2 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 4 & 7 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -4 \\ & 1 & & 1 \\ & & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$

2-9

Cholesky 与 LDL 解

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 3 \\ 2 & 10 & 1 & 6 \\ 2 & 1 & 2 & 2 \end{array}\right]$$

① $A = \tilde{L}\tilde{L}^T$

$$\tilde{L} = \begin{bmatrix} \overset{2}{l_{11}} & & \\ \overset{1}{l_{21}} & \overset{3}{l_{22}} & \\ \overset{1}{l_{31}} & \overset{0}{l_{32}} & \overset{1}{l_{33}} \end{bmatrix} \quad \tilde{L}^T = \begin{bmatrix} \overset{2}{l_{11}} & \overset{1}{l_{12}} & \overset{1}{l_{13}} \\ & \overset{3}{l_{22}} & \overset{0}{l_{23}} \\ & & \overset{1}{l_{33}} \end{bmatrix}$$

$l_{11} = 2 \quad l_{21} = 1 \quad l_{31} = 1$

$1 + l_{22}^2 = 10 \quad l_{22} = 3 \qquad 1 + 3l_{32} = 1 \quad l_{32} = 0$

$1 + l_{33}^2 = 2 \quad l_{33} = 1$

② $d_{11} = 4 \quad l_{11} = 1 \qquad l_{21} = \frac{1}{2} \quad l_{31} = \frac{1}{2}$

① $\tilde{L} = \begin{bmatrix} 2 & & \\ 1 & 3 & \\ 1 & 0 & 1 \end{bmatrix}$ ② $L = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ 1/2 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & & \\ & 9 & \\ & & 1 \end{bmatrix}$ $L^T = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ & 1 & 0 \\ & & 1 \end{bmatrix}$