2-13

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

$A\vec{x} = \vec{b} \Rightarrow LU\vec{x} = \vec{b}$ $\quad b_4 = 5$

$Ly = b$

$Ux = y$

$$L = \begin{bmatrix} 1 & & & \\ l_2 & 1 & & \\ & l_3 & 1 & \\ & & l_4 & 1 \end{bmatrix} \quad U = \begin{bmatrix} r_1 & -1 & & \\ & r_2 & -1 & \\ & & r_3 & -1 \\ & & & r_4 \end{bmatrix}$$

$r_1 = \frac{2}{1}$ $\quad r_1 l_2 = -1$ $\quad l_2 = -\frac{1}{2}$

$-1 l_2 + r_2 = 2$ $\quad r_2 = 2 + l_2 = \frac{3}{2}$ $\quad l_3 = \frac{-1}{r_2} = \frac{-2}{3}$

$-1 l_3 + r_3 = 2$ $\quad r_3 = 2 + l_3 = \frac{4}{3}$ $\quad l_4 = \frac{-1}{r_3} = \frac{-3}{4}$

$-l_4 + r_4 = 2$ $\quad r_4 = 2 + l_4 = \frac{5}{4}$

$y_1 = b_1 = 0$ $\quad l_2 y_1 + y_2 = b_2$ $\quad y_2 = b_2 - l_2 y_1 = 0$ $\quad \vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \vec{b}$

$y_3 = b_3 - l_3 y_2 = 0$ $\quad y_4 = b_4 - l_3 \cdot y_3 = 5$

$r_4 x_4 = y_4$ $\quad x_4 = y_4 / r_4 = \frac{5 \cdot 4}{5} = 4$ $\quad x_3 r_3 - x_4 = 0$ $\quad x_3 = \frac{x_4}{r_3} = \frac{4 \cdot 3}{4} = 3$

$x_2 = 2$ $\quad x_1 = 1$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

2-3 $A\vec{x} = \vec{b}$ $\quad A = LU$

$\vec{b} = \begin{bmatrix} 2 \\ 8 \\ 4 \\ 9 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 5 & 3 & -2 \\ -2 & -2 & 3 & 5 \\ 1 & 3 & 2 & 3 \end{bmatrix}$ $\quad A = LU$ $\quad \boxed{\text{Gauss}}$

$A|b \rightarrow G_1(A|b) = \begin{bmatrix} 1 & 2 & 1 & -2 \\ & 1 & 1 & 2 \\ 0 & 2 & 5 & 1 \\ & 1 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 8 \\ 7 \end{bmatrix} \rightarrow G_2 G_1 (A, b) = \begin{bmatrix} * & & & \\ & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ & & 0 & 3 \end{bmatrix}\begin{bmatrix} \\ 4 \\ 0 \\ 3 \end{bmatrix}$

$LU\cancel{x}=b \qquad (L^{-1}L)Ux=L^{-1}b \qquad Ly=b \qquad y=\begin{bmatrix}2\\4\\0\\3\end{bmatrix} \qquad Ux=y$

$$U=\begin{bmatrix}1&2&1&-2\\&1&1&2\\&&3&-3\\&&&3\end{bmatrix}\cancel{\begin{bmatrix}2\\4\\0\\3\end{bmatrix}} \qquad L=\begin{bmatrix}1&&&\\2&1&&\\-2&2&1&\\1&1&0&1\end{bmatrix}$$

$\boxed{\text{Doolittle}}$

$$L=\begin{bmatrix}1&&&\\l_{21}&1&&\\l_{31}&l_{32}&1&\\l_{41}&l_{42}&l_{43}&1\end{bmatrix} \qquad U=\begin{bmatrix}u_{11}&u_{12}&u_{13}&u_{14}\\&u_{22}&u_{23}&u_{24}\\&&u_{33}&u_{34}\\&&&u_{44}\end{bmatrix}$$

$u_{11}=1 \qquad \vec{l}_{i1}^{\,T}=[2\ -2\ 1] \qquad \cancel{u_{12}l_{21}+u_{22}}= u_{1i}^{\,T}=[2\ 1\ -2]$

$u_{12}l_{21}+u_{22}=5 \qquad u_{22}=5-4=1$

$u_{12}l_{31}+u_{22}l_{32}=-2 \qquad l_{32}=\frac{1}{u_{22}}\cdot(-2-2(-2))=+2$

$$L=\begin{bmatrix}1&&&\\2&1&&\\-2&2&1&\\1&1&0&1\end{bmatrix} \qquad U=\begin{bmatrix}1&2&1&-2\\&1&1&2\\&&3&-3\\&&&3\end{bmatrix} \qquad Ux=y$$

$x_4=\frac{y_4}{u_{44}}=1$

$x_3=(y_3-u_{34}x_4)/u_{33}=\frac{3}{3}=1$

$x_2=(y_2-u_{23}x_3-u_{24}x_4)/u_{22}=\frac{4-1-2}{1}=1$

$x_1=(y_1-u_{12}x_2-u_{13}x_3-u_{14}x_4)/u_{11}=2-2-1+2=1 \qquad \vec{x}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

3-3

3-3

A 对称正定 $A=L^TDL$ $\quad L^TL=I$

$\|x\|_{A^{\frac{1}{2}}}=(Ax,x)^{\frac{1}{2}}=\left[L^T(\sqrt{D}Lx,\sqrt{D}Lx)\right]^{\frac{1}{2}}=\quad \|Lx\|_{\frac{1}{2}}$

$=\|\sqrt{D}Lx\|$

3-19

$Ax=b \quad A=L^TDL$

$(A+\delta A)(x+\delta x)=b$

$Ax+(\delta A)x+A(\delta x)+(\delta A)(\delta x)=b$

$(\delta A)x+A(\delta x)+(\delta A)(\delta x)=0$

$(\delta A)(x+\delta x)+A(\delta x)=0$

$(\delta A)(x+\delta x)=-A(\delta x)$

$\|A^{-1}\|=\frac{1}{|\lambda_n|} \quad \|A\|=\|\lambda_1\| \quad \|A^{-1}\|\cdot\|A\|=\left|\frac{\lambda_1}{\lambda_n}\right|$

$\|A^{-1}\|=\left|\frac{\lambda_1}{\lambda_n}\right|\frac{1}{\|A\|}$

$A^{-1}(\delta A)(x+\delta x)=\delta x$

$\|\delta x\|_2\leqslant\|A^{-1}\|\,\|\delta A\|\,\|x+\delta x\|=\left|\frac{\lambda_1}{\lambda_n}\right|\frac{\|\delta A\|}{\|A\|}\|x+\delta x\|$

$\frac{\|\delta x\|_2}{\|x+\delta x\|_2}\leqslant\left|\frac{\lambda_1}{\lambda_n}\right|\frac{\|\delta A\|_2}{\|A\|_2}$

3-17 $A=D-L-U$

SOR $x^{k+1}=x^{k}+\omega D^{-1}[Lx^{k+1}+(U-D)x^{k}+b]$

$$x^{k+1}=H_\omega x^{k}+\omega(D-\omega L)^{-1}b$$

$$H_\omega=(D-\omega L)^{-1}[(1-\omega)D+\omega U] \quad 0<\omega\le 1$$

$$\lambda(D-\omega L)\vec{y}=[(1-\omega)D+\omega U]\vec{y}$$

$$\det[[(1-\omega)D+\omega U]-\lambda(D-\omega L)]=0$$

$$\det[(1-\omega-\lambda)D+\omega U+\lambda\omega L]=0$$

$$\det[(1-\lambda)D-\omega D+\omega U+\omega L+(\lambda-1)\omega L]=0$$

$$\det[(1-\lambda)(D-\omega L)-\omega A]=0$$

$\lambda\ge 1$ 时 $1-\lambda\le 0$

$\lambda-1\ge 0$

$$\det[(\lambda-1)(D-\omega L)+\omega A]\ne 0$$

$D-\omega L$ 与 A 均占优 则 则 $\forall\lambda\ge 1$ 不是 H_ω 的特征根 $\lambda<1$ $\rho(H_\omega)<1$ SOR 收敛

3-4 $A=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$\|A\|_\infty = \max(3, 3) = 3$

$\|A\|_1 = \max(1, 5) = 5$

$A^TA = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$

$\det(A^TA - \lambda I) = (5-\lambda)^2 - 4 = \lambda^2 - 10\lambda + 21$

$= (\lambda - 7)(\lambda - 3)$

$\|A\|_2 = \sqrt{7}$ $\quad \rho(A) = \sqrt{7}$

3-6 $x^{(k+1)} = (I - B^{-1}A)x^{(k)} - B^{-1}b$ $\quad Ax = b$

$\rho((A-B)^T(A-B)) < \min_{\sigma(BB^T)} |\lambda|$

$\min|\lambda| = \dfrac{1}{\rho^2(B^{-1})}$

$\rho((I - B^{-1}A)^T(I - B^{-1}A)) = \rho((B-A)(B^{-1})^T B((B-A)$

$\leq \rho^2(B^{-1})\rho((A-B)^T(A-B))$