置換 $\sigma \in \mathfrak{S}_n$ と順序組 $(a_1, \dots, a_n)$ に対し

$$\sigma(a_1, \dots, a_n) \coloneqq (a_{\sigma(1)}, \dots, a_{\sigma(n)}), \quad \sigma(a_i) \coloneqq a_{\sigma(i)} \tag{1}$$

と書くことにする。

$r \leq k+1$ で成立を仮定する。とくに

$$d\omega(X_0, X_1) = X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]) \tag{2}$$

$$\begin{aligned} d\eta(X_0, \dots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\eta(X_0, \dots, \widehat{X}_i, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k) \end{aligned} \tag{3}$$

が仮定される。示したいことは

$$d(\omega \wedge \eta)(X_0, \dots, X_{k+1}) = \sum_{i=0}^{k+1} (-1)^i X_i(\omega \wedge \eta(X_0, \dots, \widehat{X}_i, \dots, X_{k+1})) \tag{4}$$

$$+ \sum_{i<j} (-1)^{i+j} \omega \wedge \eta([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}) \tag{5}$$

である。左辺を変形すると

$$d(\omega \wedge \eta)(X_0, \dots, X_{k+1}) \tag{6}$$

$$= (d\omega \wedge \eta - \omega \wedge d\eta)(X_0, \dots, X_{k+1}) \tag{7}$$

$$= \frac{1}{2!k!} \sum_{\sigma \in \mathfrak{S}_{k+2}} (\operatorname{sgn} \sigma) d\omega \otimes \eta(\sigma(X_0, \dots, X_{k+1})) \tag{8}$$

$$- \frac{1}{1!(k+1)!} \sum_{\sigma \in \mathfrak{S}_{k+2}} (\operatorname{sgn} \sigma) \omega \otimes d\eta(\sigma(X_0, \dots, X_{k+1})) \tag{9}$$

$$= (\text{省略}) \tag{10}$$

$$= \sum_{i_0 < i_1} (-1)^{i_0 + i_1 - 1} d\omega(X_{i_0}, X_{i_1}) \eta(X_0, \dots, \widehat{X}_{i_0}, \dots, \widehat{X}_{i_1}, \dots, X_{k+1}) \tag{11}$$

$$- \sum_{i=0}^{k+1} (-1)^i \omega(X_i) d\eta(X_0, \dots, \widehat{X}_i, \dots, X_{k+1}) \tag{12}$$

を得る。帰納法の仮定 (式 (2) and (3)) とあわせて

$$(\text{式 } (11)) = \sum_{i_0 < i_1} (-1)^{i_0 + i_1 - 1} \{X_{i_0}(\omega(X_{i_1})) - X_{i_1}(\omega(X_{i_0})) - \omega([X_{i_0}, X_{i_1}])\} \eta(X_0, \dots, \widehat{X}_{i_0}, \dots, \widehat{X}_{i_1}, \dots, X_{k+1}) \tag{13}$$

$$(\text{式 } (12)) = \sum_{i=0}^{k+1} (-1)^{i+1} \omega(X_i) \left\{ \sum_{j<i} (-1)^j X_j(\eta(X_0, \dots, \widehat{X}_j, \dots, \widehat{X}_i, \dots, X_{k+1})) \right. \tag{14}$$

$$+\sum_{j>i}(-1)^{j-1}X_j(\eta(X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{15}$$

$$+\sum_{j<l<i}(-1)^{j+l}\eta([X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,X_{k+1}) \tag{16}$$

$$+\sum_{j<i<l}(-1)^{j+l-1}\eta([X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,X_{k+1}) \tag{17}$$

$$+\sum_{i<j<l}(-1)^{j+l-2}\eta([X_j,X_l],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1})\Bigg\} \tag{18}$$

となる。(式 (11)) + (式 (12)) の bracket を含まない項と含む項をそれぞれまとめる。まず

$$(\text{bracket を含まない項}) \tag{19}$$

$$=\sum_{i_0<i_1}(-1)^{i_0+i_1-1}(X_{i_0}(\omega(X_{i_1}))-X_{i_1}(\omega(X_{i_0})))\eta(X_0,\dots,\widehat{X}_{i_0},\dots,\widehat{X}_{i_1},\dots,X_{k+1})$$

$$+\sum_{i=0}^{k+1}(-1)^{i+1}\Bigg\{\sum_{j<i}(-1)^{j}X_j(\eta(X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_{k+1}))$$

$$+\sum_{j>i}(-1)^{j-1}X_j(\eta(X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}))\Bigg\} \tag{20}$$

である。これが (式 (4)) に一致してほしいから、(式 (4)) を変形してみると

$$(\text{式 } (4)) \tag{21}$$

$$=\sum_{i=0}^{k+1}(-1)^{i}\frac{1}{1!k!}\sum_{\sigma\in\mathfrak{S}_{k+1}}(\operatorname{sgn}\sigma)X_i(\omega\otimes\eta(\sigma(X_0,\dots,\widehat{X}_i,\dots,X_{k+1}))) \tag{22}$$

$$=\sum_{i=0}^{k+1}(-1)^{i}\frac{1}{1!k!}\sum_{j\neq i}\sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma(X_0)=X_j}}(\operatorname{sgn}\sigma)X_i(\omega\otimes\eta(\sigma(X_0,\dots,\widehat{X}_i,\dots,X_{k+1}))) \tag{23}$$

$$=\sum_{i=0}^{k+1}(-1)^{i}\frac{1}{1!k!}\Bigg\{\sum_{j<i}\sum_{\sigma'\in\mathfrak{S}_{k}}(\operatorname{sgn}\sigma')(-1)^{j}X_i(\omega\otimes\eta(X_j,\sigma'(X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_{k+1}))) \tag{24}$$

$$+\sum_{j>i}\sum_{\sigma'\in\mathfrak{S}_{k}}(\operatorname{sgn}\sigma')(-1)^{j-1}X_i(\omega\otimes\eta(X_j,\sigma'(X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})))\Bigg\} \tag{25}$$

$$=\sum_{i=0}^{k+1}(-1)^{i}\Bigg\{\sum_{j<i}(-1)^{j}X_i(\omega\otimes\eta(X_j,X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_{k+1}))$$

$$+\sum_{j>i}(-1)^{j-1}X_i(\omega\otimes\eta(X_j,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}))\Bigg\} \tag{26}$$

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$$=\sum_{j=0}^{k+1}(-1)^{j}\Bigg\{\sum_{i<j}(-1)^{i}X_j(\omega\otimes\eta(X_j,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}))$$

$$+\sum_{i>j}(-1)^{i-1}X_j(\omega\otimes\eta(X_i,X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_{k+1}))\Bigg\} \tag{27}$$

$$=\sum_{i=0}^{k+1}\Bigg\{\sum_{j>i}(-1)^{i+j}X_j(\omega\otimes\eta(X_j,X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}))$$

$$+\sum_{j<i}(-1)^{i+j-1}X_j(\omega\otimes\eta(X_i,X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,X_{k+1}))\Bigg\} \tag{28}$$

$$=(\text{式 }(20)) \tag{29}$$

となり、たしかに一致する。つぎに

$$(\text{bracket を含む項}) \tag{30}$$

$$=\sum_{i_0<i_1}(-1)^{i_0+i_1}\omega([X_{i_0},X_{i_1}])\eta(X_0,\dots,\widehat{X}_{i_0},\dots,\widehat{X}_{i_1},\dots,X_{k+1})$$

$$+\sum_{i=0}^{k+1}(-1)^{i+1}\omega(X_i)\Bigg\{\sum_{j<l<i}(-1)^{j+l}\eta([X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,X_{k+1})$$

$$+\sum_{j<i<l}(-1)^{j+l-1}\eta([X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,X_{k+1})$$

$$+\sum_{i<j<l}(-1)^{j+l-2}\eta([X_j,X_l],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1})\Bigg\} \tag{31}$$

である。これが (式 (5)) に一致してほしいから、(式 (5)) を変形してみると

$$(\text{式 }(5)) \tag{32}$$

$$=\sum_{i<j}(-1)^{i+j}\frac{1}{1!k!}\sum_{\sigma\in\mathfrak{S}_{k+1}}(\operatorname{sgn}\sigma)\omega\otimes\eta(\sigma([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{33}$$

$$=\sum_{i<j}(-1)^{i+j}\frac{1}{1!k!}\Bigg\{\sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma([X_i,X_j])=[X_i,X_j]}}(\operatorname{sgn}\sigma)\omega\otimes\eta(\sigma([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{34}$$

$$+\sum_{\substack{l\neq i\\ l\neq j}}\sum_{\substack{\sigma\in\mathfrak{S}_{k+1}\\ \sigma([X_i,X_j])=X_l}}(\operatorname{sgn}\sigma)\omega\otimes\eta(\sigma([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}))\Bigg\} \tag{35}$$

$$=\sum_{i<j}(-1)^{i+j}\frac{1}{1!k!}\Bigg\{\sum_{\sigma'\in\mathfrak{S}_k}(\operatorname{sgn}\sigma')\omega\otimes\eta([X_i,X_j],\sigma'(X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{36}$$

$$+\sum_{l<i<j}\sum_{\sigma'\in\mathfrak{S}_k}(\operatorname{sgn}\sigma')(-1)^{l+1}\omega\otimes\eta(X_l,\sigma'([X_i,X_j],X_0,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{37}$$

$$+\sum_{i<l<j}\sum_{\sigma'\in\mathfrak{S}_k}(\operatorname{sgn}\sigma')(-1)^{l}\omega\otimes\eta(X_l,\sigma'([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,\widehat{X}_j,\dots,X_{k+1})) \tag{38}$$

$$+\sum_{i<j<l}\sum_{\sigma'\in\mathfrak{S}_k}(\operatorname{sgn}\sigma')(-1)^{l-1}\omega\otimes\eta(X_l,\sigma'([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1}))\Bigg\} \tag{39}$$

$$
\begin{aligned}
&= \sum_{i<j}(-1)^{i+j}\Big\{\omega\otimes\eta([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}) && (40)\\
&\qquad + \sum_{l<i<j}(-1)^{l+1}\omega\otimes\eta(X_l,[X_i,X_j],X_0,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}) && (41)\\
&\qquad + \sum_{i<l<j}(-1)^{l}\omega\otimes\eta(X_l,[X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,\widehat{X}_j,\dots,X_{k+1}) && (42)\\
&\qquad + \sum_{i<j<l}(-1)^{l-1}\omega\otimes\eta(X_l,[X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1})\Big\} && (43)
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{j<l}(-1)^{j+l}\Big\{\omega\otimes\eta([X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1}) && (44)\\
&\qquad + \sum_{i<j<l}(-1)^{i+1}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1}) && (45)\\
&\qquad + \sum_{j<i<l}(-1)^{i}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,X_{k+1}) && (46)\\
&\qquad + \sum_{j<l<i}(-1)^{i-1}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,X_{k+1})\Big\} && (47)\\
&= \sum_{i_0<i_1}(-1)^{i_0+i_1}\omega\otimes\eta([X_{i_0},X_{i_1}],X_0,\dots,\widehat{X}_{i_0},\dots,\widehat{X}_{i_1},\dots,X_{k+1}) && (48)\\
&\qquad + \sum_{i=0}^{k+1}(-1)^{i}\Big\{\sum_{i<j<l}(-1)^{j+l+1}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,X_{k+1}) && (49)\\
&\qquad + \sum_{j<i<l}(-1)^{j+l}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_i,\dots,\widehat{X}_l,\dots,X_{k+1}) && (50)\\
&\qquad + \sum_{j<l<i}(-1)^{j+l-1}\omega\otimes\eta(X_i,[X_j,X_l],X_0,\dots,\widehat{X}_j,\dots,\widehat{X}_l,\dots,\widehat{X}_i,\dots,X_{k+1})\Big\} && (51)\\
&= \text{式 } (31) && (52)
\end{aligned}
$$

となり、たしかに一致する。