MATH143C: Homework 2

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Unfinished problems: 3.1 17, 21

Exercise Set 3.1

1

A: $f(x) = \cos x$

$$\begin{aligned}
P_1(x) &= f(0)\frac{x-0.3}{-0.3} + f(0.3)\frac{x}{0.3-0} \\
P_1(0.45) &= 0.933005 \\
|Error_{P_1}| &= 0.032558 \\
P_2(x) &= f(0)\frac{(x-0.3)(x-0.9)}{(0-0.3)(0-0.9)} + f(0.3)\frac{(x-0)(x-0.9)}{(0.3-0)(0.3-0.9)} + f(0.91)\frac{(x-0)(x-0.3)}{(0.9-0)(0.9-0.3)} \\
P_2(0.45) &= 0.902455 \\
|Error_{P_2}| &= 0.002008
\end{aligned}$$

B: $f(x) = ln(x+1)$

$$\begin{aligned}
P_1(x) &= f(0)\frac{x-0.3}{-0.3} + f(0.3)\frac{x}{0.3-0} \\
P_1(0.45) &= 0.393546 \\
|Error_{P_1}| &= 0.021983 \\
P_2(x) &= f(0)\frac{(x-0.3)(x-0.9)}{(0-0.3)(0-0.9)} + f(0.3)\frac{(x-0)(x-0.9)}{(0.3-0)(0.3-0.9)} + f(0.9)\frac{(x-0)(x-0.3)}{(0.9-0)(0.9-0.3)} \\
P_2(0.45) &= 0.375392 \\
|Error_{P_2}| &= 0.003828
\end{aligned}$$

3B

Error for 1A:

$$\begin{aligned}
R_1(\xi, x) &= \frac{f^2(\xi)}{2}(x)(x-0.3) \\
R_1(0.3, 0.45) &= 0.045558 \\
R_2(\xi, x) &= \frac{f^3(\xi)}{3!}(x)(x-0.3)(x-0.9) \\
R_2(0.9, 0.45) &= 0.012452
\end{aligned}$$

Error for 1B:

$$\begin{aligned}
R_1(\xi, x) &= \frac{f^2(\xi)}{2}(x)(x-0.3) \\
R_1(0, 0.45) &= 0.03375 \\
R_2(\xi, x) &= \frac{f^3(\xi)}{3!}(x)(x-0.3)(x-0.9) \\
R_2(0, 0.45) &= 0.005063
\end{aligned}$$

5

Using $[x_2, x_4]$ for P_1, and $[x_1, x_3, x_4]$ for P_2

B

$$P_1(-\frac{1}{3}) = 0.3505$$
$$P_2(-\frac{1}{3}) = 0.162944$$
$$P_3(-\frac{1}{3}) = 0.174519$$

D

$$P_1(0.9) = 0.443312$$
$$P_2(0.9) = 0.436628$$
$$P_3(0.9) = 0.441985$$

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Error for 5B

$$\begin{aligned}
n = 1 \rightarrow & \frac{f^2(\xi)}{2!}(x)(x+0.5) \\
&= \frac{6\xi + 8.002}{2}(x)(x+0.5) \\
&= (3\xi + 4.001)(x)(x+0.5) \\
\left(\xi = 0, x = -\frac{1}{3}\right) \rightarrow &= 0.222277 \\
n = 2 \rightarrow & \frac{f^3(\xi)}{3!}(x)(x+0.25)(x+0.75) \\
&= \frac{6}{3!}(x)(x+0.25)(x+0.75) \\
&= (x)(x+0.25)(x+0.75) \\
\left(\xi = 0, x = -\frac{1}{3}\right) \rightarrow &= 0.011574
\end{aligned}$$

Error for 5D

$$\begin{aligned}
n = 1 \rightarrow & \frac{f^2(\xi)}{2!}(x-0.7)(x-1) \\
&= \frac{(x-1)(x-0.7)\left(-e^{2\xi}\sin\left(e^{\xi}-2\right)+e^{\xi}\cos\left(e^{\xi}-2\right)\right)}{2} \\
(\xi = 1, x = 0.9) \rightarrow &= 0.028160 \\
n = 2 \rightarrow & \frac{f^3(\xi)}{3!}(x-0.6)(x-0.8)(x-1) \\
&= \frac{(x-1)(x-0.8)(x-0.6)\left(-e^{3\xi}\cos\left(e^{\xi}-2\right)-3e^{2\xi}\sin\left(e^{\xi}-2\right)+e^{\xi}\cos\left(e^{\xi}-2\right)\right)}{6} \\
(\xi = 1, x = 0.9) \rightarrow &= 0.013832
\end{aligned}$$

9

$$y = 4.25$$

10

$$\begin{aligned}
f(x) &= \sqrt{x-x^2} \\
P_2(x) &= y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_2)(x_0-x_1)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_2)}{(x_2-x_0)(x_0-x_1)} \\
\text{Given: } & x_0 = 0, x_2 = 1, x = 0.5 : \\
f(0.5) &= \sqrt{0.25} = \pm 0.5 \\
P_2(x) &= 0\frac{(0.5-x_1)(0.5-1)}{(0-1)(0-x_1)} + \left(\sqrt{x_1-x_1^2}\right)\frac{(0.5-0)(0.5-1)}{(x_1-0)(x_1-1)} + 0\frac{(0.5-0)(0.5-1)}{(1-0)(0-x_1)} \\
&= \left(\sqrt{x_1-x_1^2}\right)\frac{-0.25}{x_1^2-x_1} = \frac{-0.25}{\sqrt{x_1-x_1^2}}
\end{aligned}$$

Solving for $f(0.5) - P_2(0.5) = -0.25$:

$$\begin{aligned}
\pm 0.5 - \frac{-0.25}{\sqrt{x_1-x_1^2}} &= -0.25 \\
\frac{-0.25}{\sqrt{x_1-x_1^2}} &= \{0.75, -0.25\} \\
-\frac{1}{\sqrt{x_1-x_1^2}} &= \{3, -1\} \\
\left\{\frac{1}{9}, 1\right\} &= x_1 - x_1^2 \\
\text{Using } \frac{1}{9} : x_1 &= \{0.127322003750035, 0.872677996249965\} \\
\text{Using } 1 : x_1 &= \left\{\frac{1}{2} - \frac{\sqrt{3}i}{2}, \frac{1}{2} + \frac{\sqrt{3}i}{2}\right\}
\end{aligned}$$

The largest real value between $(0, 1)$ for $x_1 = 0.872678$

13D

$$
\begin{aligned}
P_3(x) &= \frac{1.216316x\,(x-1)\,(x-0.5)}{0.25\,(0.25-1)\,(0.25-0.5)} + \frac{1.357008x\,(x-1)\,(x-0.25)}{0.5\,(0.5-1)\,(0.5-0.25)} \\
&+ \frac{1.381773x\,(x-0.5)\,(x-0.25)}{1\,(1-0.5)\,(1-0.25)} + \frac{1.0\,(x-1)\,(x-0.5)\,(x-0.25)}{(-1)\,(-0.5)\,(0.25)} \\
&= 25.948083x\,(x-1)\,(x-0.5) - 21.712130x\,(x-1)\,(x-0.25) \\
&+ 3.684729x\,(x-0.5)\,(x-0.25) - 8.0\,(x-1)\,(x-0.5)\,(x-0.25) \\
R_3(x) &= \frac{f^4(\xi)}{4!}\,(x-1)\,(x-0.5)\,(x-0.25)\,(x) \\
&= \frac{\sin(\xi)+\cos(\xi)}{24}\,(x-1)\,(x-0.5)\,(x-0.25)\,(x) \\
\left(\xi = \frac{\pi}{4}, x = 0.8316\right) \rightarrow &= 0.001591
\end{aligned}
$$

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Interpolating polynomial

$$
P_5(x) = -\frac{10089x^5}{4000000} + \frac{6001123x^4}{240000} - \frac{2379665339x^3}{24000} + \frac{471801682097x^2}{2400} - \frac{116923918291129x}{600} + 77269170756852
$$

Estimates

1. 1950: 192539
2. 1975: 215525
3. 2014: 306214
4. 2020: 266165

The estimated 1950 from the interpolating polynomial was off by more than 40%, while the 2014 figure was off by approximately 3%. I would be skeptical about the estimates that I found in the interpolating polynomial.

Exercise Set 3.2

1B

$$
\begin{aligned}
f(x) = -1.333333x\,&(-2.0\,(-x-0.75)\,(1.43875x+0.694625) - 2.0\,(0.18825x+0.069375)\,(x+0.25)) \\
&- 1.333333\,(-x-0.75)\,(-2.0x\,(1.43875x+0.694625) - 2.0\,(-x-0.5)\,(3.06425x+1.101)) \\
f\left(-\frac{1}{3}\right) &= 0.174519
\end{aligned}
$$

3

A

$$\begin{aligned} f(x) &= 3^x \\ (x_0, x_1, x_2, x_3, x_4) &= (-2, -1, 0, 1, 2) \\ P(\sqrt{3}) &= 6.780246 \\ f(\sqrt{3}) &= 6.704992 \\ \text{Error (Part C)} &= 6.780246 - 6.704992 = 0.075254 \end{aligned}$$

B

$$\begin{aligned} f(x) &= \sqrt{x} \\ (x_0, x_1, x_2, x_3, x_4) &= (0, 1, 2, 4, 5) \\ P(\sqrt{3}) &= 1.330337 \\ f(\sqrt{3}) &= 1.316074 \\ \text{Error (Part C)} &= 1.330337 - 1.316074 = 0.014263 \end{aligned}$$

5

$$\begin{aligned} P_2 &= 4.0 \\ P_{1,2} &= 3.2 \\ P_{0,1,2} &= 3.08 \end{aligned}$$

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$$\begin{aligned} x &= [0.3, 0.4, 0.5, 0.6] \\ y = x - e^{-x} &= [-0.440818,\ -0.27032,\ -0.106531,\ 0.051188] \\ f^{-1}(0) &= 0.567143 \end{aligned}$$

3.3

1B

$$\begin{aligned} P_1 &= -0.1769446 + 1.9069687(x - 0.6) \\ P_1(0.9) &= 0.395146 \\ P_2 &= -0.1769446 + 1.9069687(x - 0.6) + 0.959224(x - 0.7)(x - 0.6) \\ P_2(0.9) &= 0.452700 \\ P_3 &= -0.1769446 + 1.9069687(x - 0.6) + 0.959224(x - 0.7)(x - 0.6) + (-1.785741)(x - 0.8)(x - 0.7)(x - 0.6) \\ P_3(0.9) &= 0.441985 \end{aligned}$$

3B

Using x_1, x_2 for P_1, and x_1, x_2, x_3 for P_2

$$\begin{aligned}
P_1 &= 2.905876x - 0.574574 \\
P_1(0.25) &= 0.151895 \\
P_2 &= 2.905876x - 2.438209(x-0.2)(x-0.1) - 0.574574 \\
P_2(0.25) &= 0.133608 \\
P_3 &= 3.365129x - 0.473152(x-0.3)(x-0.2)(x-0.1) - 2.296264(x-0.2)(x-0.1) - 0.957012 \\
P_3(0.25) &= -0.132775
\end{aligned}$$

5B

Using x_1, x_2 for P_1, and x_1, x_2, x_3 for P_2

$$\begin{aligned}
P_1 &= 2.905876x - 0.574574 \\
P_1(0.25) &= 0.151895 \\
P_2 &= 2.905876x - 2.296264\,(x-0.3)\,(x-0.2) - 0.865162 \\
P_2(0.25) &= -0.132952 \\
P_3 &= 2.418235x - 0.473152\,(x-0.4)\,(x-0.3)\,(x-0.2) - 2.438209\,(x-0.4)\,(x-0.3) - 0.718869 \\
P_3(0.25) &= -0.132775
\end{aligned}$$

8

A

$P_4(x) = 0.063016x\,(x-0.6)\,(x-0.3)\,(x-0.1) + 0.215x\,(x-0.3)\,(x-0.1) + 0.5725x\,(x-0.1) + 1.0517x - 6.0$

B

$$\begin{aligned}
P_5(x) = 0.014159x\,(x-1)\,(x-0.6)\,(x-0.3)\,(x-0.1) + 0.063016x\,(x-0.6)\,(x-0.3) \\
(x-0.1) + 0.215x\,(x-0.3)\,(x-0.1) + 0.5725x\,(x-0.1) + 1.0517x - 6.0
\end{aligned}$$

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$$\begin{aligned}
f[x_0, x_1] &= 5.0 \\
f[x_0] &= 1.0 \\
f[x_1] &= 3.0
\end{aligned}$$

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A

$$P_4(0.75) = 72.86 \rightarrow 1 \text{ minute, } 12.86 \text{ seconds}$$

The actual time was 1:13, so this estimate is extremely close.

B

$$\frac{d}{dx}P_4(1.25) = 89.72\frac{\text{seconds}}{\text{mile}} \approx 40.12 \text{ miles per hour}$$

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	x	P(x)	Q(x)	Actual
0	-2.0	-1	-1	-1
1	-1.0	3	3	3
2	0.0	1	1	1
3	1.0	-1	-1	-1
4	2.0	3	3	3

$P(x)$ and $Q(x)$ are the same function - when you reduce the two equations they are equivalent. Thus, $P(x)$ does not violate the uniqueness property of interpolating polynomials.

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$$\begin{aligned}
f[x_2] &= a_2\left(-x_0+x_2\right)\left(-x_1+x_2\right)+f[x_0]+f[x_1]\left(-x_0+x_2\right)\\
-f[x_0]-f[x_1]\left(-x_0+x_2\right)+f[x_2] &= a_2\left(-x_0+x_2\right)\left(-x_1+x_2\right)\\
\frac{-f[x_0]-f[x_1]\left(-x_0+x_2\right)+f[x_2]}{\left(-x_0+x_2\right)\left(-x_1+x_2\right)} &= a_2\\
-\frac{f[x_1]}{-x_1+x_2}+\frac{-f[x_0]+f[x_2]}{\left(-x_0+x_2\right)\left(-x_1+x_2\right)} &= a_2\\
f[x_0,x_1,x_2] &= a_2
\end{aligned}$$

22

$$\begin{aligned}
f(x) = P_{n+1}(x) &= P_n(x)+f[x_0,x_1,...,x_n,x](x-x_0)(x-x_1)...(x-x_n)\\
&\text{Substituting } f(x):\\
P_n(x)+\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n) &= P_n(x)+f[x_0,x_1,...,x_n,x](x-x_0)(x-x_1)...(x-x_n)\\
\frac{f^{n+1}(\xi)}{(n+1)!} &= f[x_0,x_1,...,x_n,x]
\end{aligned}$$

Exercise Set 8.3

1A / 3A

$$\begin{aligned}
(x_0,x_1,x_2) &= \left(-\frac{3}{\sqrt{2}},0,\frac{3}{\sqrt{2}}\right)\\
P_3(x) &= 0.532042x\left(x+0.866025\right)+0.66901x+1.0\\
f(x)-P(x) &= \frac{e^{\xi}}{3!}*\frac{1}{2^2} = 0.1132617
\end{aligned}$$

Exercise Set 3.4

1C

$$\begin{aligned}P_5(x) = -0.024751 + 0.751(x+0.5) + 2.751\left(x+0.5\right)^2 + \left(x+0.25\right)\left(x+0.5\right)^2 \\ - 7.105427 * 10^{-15}\left(x+0.25\right)^2\left(x+0.5\right)^2 + 2.131628 * 10^{-14} x\left(x+0.25\right)^2\left(x+0.5\right)^2\end{aligned}$$

5

A / B

$$\begin{aligned}H_2(x) &= 0.29552 + 0.95534(x-0.3) + -0.142\left(x-0.3\right)^2 + -1.05\left(x-0.32\right)\left(x-0.3\right)^2 \\ &\quad + 20.77778\left(x-0.32\right)^2\left(x-0.3\right)^2 + -436.29630\left(x-0.35\right)\left(x-0.32\right)^2\left(x-0.3\right)^2 \\ H_2(0.34) &= 0.33349 \\ \sin 0.34 &= 0.33349 \\ R_2(0.34) &= 0.000003\end{aligned}$$

With 5 digit rounding, the error bound exceeds the actual error of 0.

C

$$\begin{aligned}H_7(x) &= 0.29552 + 0.95534(x-0.3) + -0.142\left(x-0.3\right)^2 + -1.05\left(x-0.32\right)\left(x-0.3\right)^2 \\ &\quad - 32.77778\left(x-0.32\right)^2\left(x-0.3\right)^2 + 17574.07407\left(x-0.33\right)\left(x-0.32\right)^2\left(x-0.3\right)^2 \\ &\quad - 744814.81482\left(x-0.33\right)^2\left(x-0.32\right)^2\left(x-0.3\right)^2 + 29455555.55570\left(x-0.35\right)\left(x-0.33\right)^2\left(x-0.32\right)^2\left(x-0.3\right)^2 \\ H_7(0.34) &= 0.33349 \\ \sin 0.34 &= 0.33349 \\ R_7(0.34) &= 3.75264 * 10^{-19}\end{aligned}$$

9

$$\begin{aligned}H_9(x) &= 75x + -0.031111x^2\left(x-3\right)^2 + -0.006444x^2\left(x-5\right)\left(x-3\right)^2 + 0.002264x^2\left(x-5\right)^2\left(x-3\right)^2 \\ &\quad - 0.000913x^2\left(x-8\right)\left(x-5\right)^2\left(x-3\right)^2 + 0.000131x^2\left(x-8\right)^2\left(x-5\right)^2\left(x-3\right)^2 \\ &\quad - 2.022363 * 10^{-5}x^2\left(x-13\right)\left(x-8\right)^2\left(x-5\right)^2\left(x-3\right)^2 \\ H_9(10) &= 742.50 \\ H_9'(10) &= 48.38\end{aligned}$$

The car surpasses 55 miles per hour at approximately 5.65147 seconds.

10

The given divided difference table provides the coefficients to write the Hermite polynomial using the Newton's Divided Difference form of the interpolating polynomial.

$$a_0 = f[z_0] = f(x_0)$$
$$a_1 = f[z_0, z_1] = f'(x_0)$$
$$a_2 = f[z_0, z_1, z_2] = \frac{f[z_1, z_2](x - z_0) - f[z_0, z_1](x - z_2)}{z_2 - z_1}$$
$$a_3 = f[z_0, z_1, z_2, z_3] = \frac{f[z_1, z_2, z_3](x - z_0) - f[z_0, z_1, z_2](x - z_3)}{z_3 - z_0}$$

Using these coefficients:

$$H_3(x) = f[z_0] + f[z_0, z_1](x - x_0) + f[z_0, z_1, z_2](x - x_0)^2 + f[z_0, z_1, z_2, z_3](x - x_0)^2(x - x_1)$$

Exercise Set 3.5

3C

i	a	b	c	d
0	-0.02475000	1.03237500	-0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	-6.50200000

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$$3x^2 - 2 = -b - 2c\,(x - 1) - 3d\,(x - 1)^2$$
$$6x = -2c - 6d\,(x - 1)$$
$$d = -1$$
$$c = -3$$
$$b = -1$$

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17