MATH143C: Homework 2

Oisin Coveney

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Unfinished problems: 3.1 17, 21

Exercise Set 3.1

1

A: $f(x) = \cos x$

$$\begin{aligned}
P_1(x) &= f(0)\frac{x-0.3}{-0.3} + f(0.3)\frac{x}{0.3-0} \\
P_1(0.45) &= 0.933005 \\
|Error_{P_1}| &= 0.032558 \\
P_2(x) &= f(0)\frac{(x-0.3)(x-0.9)}{(0-0.3)(0-0.9)} + f(0.3)\frac{(x-0)(x-0.9)}{(0.3-0)(0.3-0.9)} + f(0.91)\frac{(x-0)(x-0.3)}{(0.9-0)(0.9-0.3)} \\
P_2(0.45) &= 0.902455 \\
|Error_{P_2}| &= 0.002008
\end{aligned}$$

B: $f(x) = ln(x+1)$

$$\begin{aligned}
P_1(x) &= f(0)\frac{x-0.3}{-0.3} + f(0.3)\frac{x}{0.3-0} \\
P_1(0.45) &= 0.393546 \\
|Error_{P_1}| &= 0.021983 \\
P_2(x) &= f(0)\frac{(x-0.3)(x-0.9)}{(0-0.3)(0-0.9)} + f(0.3)\frac{(x-0)(x-0.9)}{(0.3-0)(0.3-0.9)} + f(0.9)\frac{(x-0)(x-0.3)}{(0.9-0)(0.9-0.3)} \\
P_2(0.45) &= 0.375392 \\
|Error_{P_2}| &= 0.003828
\end{aligned}$$

3B

Error for 1A:

$$\begin{aligned}
R_1(\xi, x) &= \frac{f^2(\xi)}{2}(x)(x-0.3) \\
R_1(0.3, 0.45) &= 0.045558 \\
R_2(\xi, x) &= \frac{f^3(\xi)}{3!}(x)(x-0.3)(x-0.9) \\
R_2(0.9, 0.45) &= 0.012452
\end{aligned}$$

Error for 1B:

$$\begin{aligned}
R_1(\xi, x) &= \frac{f^2(\xi)}{2}(x)(x-0.3) \\
R_1(0, 0.45) &= 0.03375 \\
R_2(\xi, x) &= \frac{f^3(\xi)}{3!}(x)(x-0.3)(x-0.9) \\
R_2(0, 0.45) &= 0.005063
\end{aligned}$$

5

Using $[x_2, x_4]$ for P_1, and $[x_1, x_3, x_4]$ for P_2

B

$$P_1(-\frac{1}{3}) = 0.3505$$
$$P_2(-\frac{1}{3}) = 0.162944$$
$$P_3(-\frac{1}{3}) = 0.174519$$

D

$$P_1(0.9) = 0.443312$$
$$P_2(0.9) = 0.436628$$
$$P_3(0.9) = 0.441985$$

7

Error for 5B

$$\begin{aligned}
n = 1 \rightarrow & \frac{f^2(\xi)}{2!}(x)(x+0.5) \\
&= \frac{6\xi + 8.002}{2}(x)(x+0.5) \\
&= (3\xi + 4.001)(x)(x+0.5) \\
\left(\xi = 0, x = -\frac{1}{3}\right) \rightarrow &= 0.222277 \\
n = 2 \rightarrow & \frac{f^3(\xi)}{3!}(x)(x+0.25)(x+0.75) \\
&= \frac{6}{3!}(x)(x+0.25)(x+0.75) \\
&= (x)(x+0.25)(x+0.75) \\
\left(\xi = 0, x = -\frac{1}{3}\right) \rightarrow &= 0.011574
\end{aligned}$$

Error for 5D

$$
\begin{aligned}
n = 1 \rightarrow & \frac{f^2(\xi)}{2!}(x-0.7)(x-1) \\
& = \frac{(x-1)(x-0.7)\left(-e^{2\xi}\sin\left(e^{\xi}-2\right)+e^{\xi}\cos\left(e^{\xi}-2\right)\right)}{2} \\
(\xi = 1, x = 0.9) \rightarrow & = 0.028160 \\
n = 2 \rightarrow & \frac{f^3(\xi)}{3!}(x-0.6)(x-0.8)(x-1) \\
& = \frac{(x-1)(x-0.8)(x-0.6)\left(-e^{3\xi}\cos\left(e^{\xi}-2\right)-3e^{2\xi}\sin\left(e^{\xi}-2\right)+e^{\xi}\cos\left(e^{\xi}-2\right)\right)}{6} \\
(\xi = 1, x = 0.9) \rightarrow & = 0.013832
\end{aligned}
$$

9

$$y = 4.25$$

10

$$
\begin{aligned}
f(x) &= \sqrt{x-x^2} \\
P_2(x) &= y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_2)(x_0-x_1)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_2)}{(x_2-x_0)(x_0-x_1)} \\
\text{Given: } & x_0 = 0, x_2 = 1, x = 0.5: \\
f(0.5) &= \sqrt{0.25} = \pm 0.5 \\
P_2(x) &= 0\frac{(0.5-x_1)(0.5-1)}{(0-1)(0-x_1)} + \left(\sqrt{x_1-x_1^2}\right)\frac{(0.5-0)(0.5-1)}{(x_1-0)(x_1-1)} + 0\frac{(0.5-0)(0.5-1)}{(1-0)(0-x_1)} \\
&= \left(\sqrt{x_1-x_1^2}\right)\frac{-0.25}{x_1^2-x_1} = \frac{-0.25}{\sqrt{x_1-x_1^2}}
\end{aligned}
$$

Solving for $f(0.5) - P_2(0.5) = -0.25$:

$$
\begin{aligned}
\pm 0.5 - \frac{-0.25}{\sqrt{x_1-x_1^2}} &= -0.25 \\
\frac{-0.25}{\sqrt{x_1-x_1^2}} &= \{0.75, -0.25\} \\
-\frac{1}{\sqrt{x_1-x_1^2}} &= \{3, -1\} \\
\left\{\frac{1}{9}, 1\right\} &= x_1 - x_1^2 \\
\text{Using } \frac{1}{9}: x_1 &= \{0.127322003750035, 0.872677996249965\} \\
\text{Using } 1: x_1 &= \left\{\frac{1}{2} - \frac{\sqrt{3}i}{2}, \frac{1}{2} + \frac{\sqrt{3}i}{2}\right\}
\end{aligned}
$$

The largest real value between $(0, 1)$ for $x_1 = 0.872678$

13D

$$
\begin{aligned}
P_3(x) &= \frac{1.216316x\,(x-1)\,(x-0.5)}{0.25\,(0.25-1)\,(0.25-0.5)} + \frac{1.357008x\,(x-1)\,(x-0.25)}{0.5\,(0.5-1)\,(0.5-0.25)} \\
&+ \frac{1.381773x\,(x-0.5)\,(x-0.25)}{1\,(1-0.5)\,(1-0.25)} + \frac{1.0\,(x-1)\,(x-0.5)\,(x-0.25)}{(-1)\,(-0.5)\,(0.25)} \\
&= 25.948083x\,(x-1)\,(x-0.5) - 21.712130x\,(x-1)\,(x-0.25) \\
&+ 3.684729x\,(x-0.5)\,(x-0.25) - 8.0\,(x-1)\,(x-0.5)\,(x-0.25) \\
R_3(x) &= \frac{f^4(\xi)}{4!}\,(x-1)\,(x-0.5)\,(x-0.25)\,(x) \\
&= \frac{\sin(\xi)+\cos(\xi)}{24}\,(x-1)\,(x-0.5)\,(x-0.25)\,(x) \\
\left(\xi = \frac{\pi}{4}, x = 0.8316\right) \rightarrow &= 0.001591
\end{aligned}
$$

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19

Interpolating polynomial

$$P_5(x) = -\frac{10089x^5}{4000000} + \frac{6001123x^4}{240000} - \frac{2379665339x^3}{24000} + \frac{471801682097x^2}{2400} - \frac{116923918291129x}{600} + 77269170756852$$

Estimates

1. 1950: 192539
2. 1975: 215525
3. 2014: 306214
4. 2020: 266165

The estimated 1950 from the interpolating polynomial was off by more than 40%, while the 2014 figure was off by approximately 3%. I would be skeptical about the estimates that I found in the interpolating polynomial.

Exercise Set 3.2

1B

$$
\begin{aligned}
f(x) = -1.333333x\,&(-2.0\,(-x-0.75)\,(1.43875x+0.694625) - 2.0\,(0.18825x+0.069375)\,(x+0.25)) \\
&- 1.333333\,(-x-0.75)\,(-2.0x\,(1.43875x+0.694625) - 2.0\,(-x-0.5)\,(3.06425x+1.101)) \\
f\left(-\frac{1}{3}\right) &= 0.174519
\end{aligned}
$$

3

A

$$\begin{aligned}
f(x) &= 3^x \\
(x_0, x_1, x_2, x_3, x_4) &= (-2, -1, 0, 1, 2) \\
P(\sqrt{3}) &= 6.780246 \\
f(\sqrt{3}) &= 6.704992 \\
\text{Error (Part C)} &= 6.780246 - 6.704992 = 0.075254
\end{aligned}$$

B

$$\begin{aligned}
f(x) &= \sqrt{x} \\
(x_0, x_1, x_2, x_3, x_4) &= (0, 1, 2, 4, 5) \\
P(\sqrt{3}) &= 1.330337 \\
f(\sqrt{3}) &= 1.316074 \\
\text{Error (Part C)} &= 1.330337 - 1.316074 = 0.014263
\end{aligned}$$

5

$$\begin{aligned}
P_2 &= 4.0 \\
P_{1,2} &= 3.2 \\
P_{0,1,2} &= 3.08
\end{aligned}$$

12

$$\begin{aligned}
x &= [0.3, 0.4, 0.5, 0.6] \\
y = x - e^{-x} &= [-0.440818,\ -0.27032,\ -0.106531,\ 0.051188] \\
f^{-1}(0) &= 0.567143
\end{aligned}$$

3.3

1B

$$\begin{aligned}
P_1 &= -0.1769446 + 1.9069687(x - 0.6) \\
P_1(0.9) &= 0.395146 \\
P_2 &= -0.1769446 + 1.9069687(x - 0.6) + 0.959224(x - 0.7)(x - 0.6) \\
P_2(0.9) &= 0.452700 \\
P_3 &= -0.1769446 + 1.9069687(x - 0.6) + 0.959224(x - 0.7)(x - 0.6) + (-1.785741)(x - 0.8)(x - 0.7)(x - 0.6) \\
P_3(0.9) &= 0.441985
\end{aligned}$$

3B

Using x_1, x_2 for P_1, and x_1, x_2, x_3 for P_2

$$\begin{aligned}
P_1 &= 2.905876x - 0.574574 \\
P_1(0.25) &= 0.151895 \\
P_2 &= 2.905876x - 2.438209(x-0.2)(x-0.1) - 0.574574 \\
P_2(0.25) &= 0.133608 \\
P_3 &= 3.365129x - 0.473152(x-0.3)(x-0.2)(x-0.1) - 2.296264(x-0.2)(x-0.1) - 0.957012 \\
P_3(0.25) &= -0.132775
\end{aligned}$$

5B

Using x_1, x_2 for P_1, and x_1, x_2, x_3 for P_2

$$\begin{aligned}
P_1 &= 2.905876x - 0.574574 \\
P_1(0.25) &= 0.151895 \\
P_2 &= 2.905876x - 2.296264\left(x-0.3\right)\left(x-0.2\right) - 0.865162 \\
P_2(0.25) &= -0.132952 \\
P_3 &= 2.418235x - 0.473152\left(x-0.4\right)\left(x-0.3\right)\left(x-0.2\right) - 2.438209\left(x-0.4\right)\left(x-0.3\right) - 0.718869 \\
P_3(0.25) &= -0.132775
\end{aligned}$$

8

A

$P_4(x) = 0.063016x\left(x-0.6\right)\left(x-0.3\right)\left(x-0.1\right) + 0.215x\left(x-0.3\right)\left(x-0.1\right) + 0.5725x\left(x-0.1\right) + 1.0517x - 6.0$

B

$$\begin{aligned}
P_5(x) = 0.014159x\left(x-1\right)\left(x-0.6\right)\left(x-0.3\right)\left(x-0.1\right) + 0.063016x\left(x-0.6\right)\left(x-0.3\right) \\
\left(x-0.1\right) + 0.215x\left(x-0.3\right)\left(x-0.1\right) + 0.5725x\left(x-0.1\right) + 1.0517x - 6.0
\end{aligned}$$

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$$\begin{aligned}
f[x_0, x_1] &= 5.0 \\
f[x_0] &= 1.0 \\
f[x_1] &= 3.0
\end{aligned}$$

18

A

$$P_4(0.75) = 72.86 \rightarrow 1 \text{ minute, } 12.86 \text{ seconds}$$

The actual time was 1:13, so this estimate is extremely close.

B

$$\frac{d}{dx} P_4(1.25) = 89.72 \frac{\text{seconds}}{\text{mile}} \approx 40.12 \text{ miles per hour}$$

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	x	P(x)	Q(x)	Actual
0	-2.0	-1	-1	-1
1	-1.0	3	3	3
2	0.0	1	1	1
3	1.0	-1	-1	-1
4	2.0	3	3	3

$P(x)$ and $Q(x)$ are the same function - when you reduce the two equations they are equivalent. Thus, $P(x)$ does not violate the uniqueness property of interpolating polynomials.

21

$$\begin{aligned}
f[x_2] &= a_2\left(-x_0+x_2\right)\left(-x_1+x_2\right)+f[x_0]+f[x_1]\left(-x_0+x_2\right)\\
-f[x_0]-f[x_1]\left(-x_0+x_2\right)+f[x_2] &= a_2\left(-x_0+x_2\right)\left(-x_1+x_2\right)\\
\frac{-f[x_0]-f[x_1]\left(-x_0+x_2\right)+f[x_2]}{\left(-x_0+x_2\right)\left(-x_1+x_2\right)} &= a_2\\
-\frac{f[x_1]}{-x_1+x_2}+\frac{-f[x_0]+f[x_2]}{\left(-x_0+x_2\right)\left(-x_1+x_2\right)} &= a_2\\
f[x_0,x_1,x_2] &= a_2
\end{aligned}$$

22

$$\begin{aligned}
f(x) = P_{n+1}(x) &= P_n(x)+f[x_0,x_1,...,x_n,x](x-x_0)(x-x_1)...(x-x_n)\\
\text{Substituting } f(x): &\\
P_n(x)+\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n) &= P_n(x)+f[x_0,x_1,...,x_n,x](x-x_0)(x-x_1)...(x-x_n)\\
\frac{f^{n+1}(\xi)}{(n+1)!} &= f[x_0,x_1,...,x_n,x]
\end{aligned}$$

Exercise Set 8.3

1A / 3A

$$\begin{aligned}
(x_0,x_1,x_2) &= \left(-\frac{3}{\sqrt{2}},0,\frac{3}{\sqrt{2}}\right)\\
P_3(x) &= 0.532042x\left(x+0.866025\right)+0.66901x+1.0\\
f(x)-P(x) &= \frac{e^{\xi}}{3!}*\frac{1}{2^2} = 0.1132617
\end{aligned}$$

Exercise Set 3.4

1C

$$\begin{aligned}P_5(x) = &-0.024751 + 0.751(x+0.5) + 2.751\,(x+0.5)^2 + (x+0.25)\,(x+0.5)^2 \\ &- 7.105427 * 10^{-15}\,(x+0.25)^2\,(x+0.5)^2 + 2.131628 * 10^{-14} x\,(x+0.25)^2\,(x+0.5)^2\end{aligned}$$

5

A / B

$$\begin{aligned}H_2(x) &= 0.29552 + 0.95534(x-0.3) + -0.142\,(x-0.3)^2 + -1.05\,(x-0.32)\,(x-0.3)^2 \\ &\quad + 20.77778\,(x-0.32)^2\,(x-0.3)^2 + -436.29630\,(x-0.35)\,(x-0.32)^2\,(x-0.3)^2 \\ H_2(0.34) &= 0.33349 \\ \sin 0.34 &= 0.33349 \\ R_2(0.34) &= 0.000003\end{aligned}$$

With 5 digit rounding, the error bound exceeds the actual error of 0.

C

$$\begin{aligned}H_7(x) &= 0.29552 + 0.95534(x-0.3) + -0.142\,(x-0.3)^2 + -1.05\,(x-0.32)\,(x-0.3)^2 \\ &\quad - 32.77778\,(x-0.32)^2\,(x-0.3)^2 + 17574.07407\,(x-0.33)\,(x-0.32)^2\,(x-0.3)^2 \\ &\quad - 744814.81482\,(x-0.33)^2\,(x-0.32)^2\,(x-0.3)^2 + 29455555.55570\,(x-0.35)\,(x-0.33)^2\,(x-0.32)^2\,(x-0.3)^2 \\ H_7(0.34) &= 0.33349 \\ \sin 0.34 &= 0.33349 \\ R_7(0.34) &= 3.75264 * 10^{-19}\end{aligned}$$

9

$$\begin{aligned}H_9(x) &= 75x + -0.031111x^2\,(x-3)^2 + -0.006444x^2\,(x-5)\,(x-3)^2 + 0.002264x^2\,(x-5)^2\,(x-3)^2 \\ &\quad - 0.000913x^2\,(x-8)\,(x-5)^2\,(x-3)^2 + 0.000131x^2\,(x-8)^2\,(x-5)^2\,(x-3)^2 \\ &\quad - 2.022363 * 10^{-5} x^2\,(x-13)\,(x-8)^2\,(x-5)^2\,(x-3)^2 \\ H_9(10) &= 742.50 \\ H_9'(10) &= 48.38\end{aligned}$$

The car surpasses 55 miles per hour at approximately 5.65147 seconds.

10

The given divided difference table provides the coefficients to write the Hermite polynomial using the Newton's Divided Difference form of the interpolating polynomial.

$$
\begin{aligned}
a_0 &= f[z_0] = f(x_0) \\
a_1 &= f[z_0, z_1] = f'(x_0) \\
a_2 &= f[z_0, z_1, z_2] = \frac{f[z_1, z_2](x - z_0) - f[z_0, z_1](x - z_2)}{z_2 - z_1} \\
a_3 &= f[z_0, z_1, z_2, z_3] = \frac{f[z_1, z_2, z_3](x - z_0) - f[z_0, z_1, z_2](x - z_3)}{z_3 - z_0}
\end{aligned}
$$

Using these coefficients:

$$H_3(x) = f[z_0] + f[z_0, z_1](x - x_0) + f[z_0, z_1, z_2](x - x_0)^2 + f[z_0, z_1, z_2, z_3](x - x_0)^2(x - x_1)$$

Exercise Set 3.5

3C

i	a	b	c	d
0	-0.02475000	1.03237500	-0.00000000	6.50200000
1	0.33493750	2.25150000	4.87650000	-6.50200000

11

$$
\begin{aligned}
3x^2 - 2 &= -b - 2c\,(x-1) - 3d\,(x-1)^2 \\
6x &= -2c - 6d\,(x-1) \\
d &= -1 \\
c &= -3 \\
b &= -1
\end{aligned}
$$

15

$$
\begin{aligned}
S_0 &= 2.103418x + 1.0 \\
S_1 &= 2.324637x + 0.988939 \\
\int_0^{0.05} S_0 + \int_{0.05}^{0.1} S_1 &= 0.110794 \\
\int_0^{0.1} e^x &= 0.110701 \\
|Error| &= 9.223578 * 10^{-5}
\end{aligned}
$$

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Range	a	b	c	d
(0,0.25)	1.00	-0.76	-0.00	-6.63
(0.25,0.5)	0.71	-2.00	-4.97	6.63
(0.5,0.75)	0.00	-3.24	-0.00	6.63
(0.75,1)	-0.71	-2.00	4.97	-6.63

$$\int_0^1 S(x) = -0.493792$$
$$\int_0^1 f(x) = 0$$
$$|Error| = 0.493792$$

$$S'(0.5) = f'(0.5) = 0$$
$$|Error| = 0$$

$$S''(0.5) = 0$$
$$f''(0.5) = -\frac{\pi}{2}$$
$$|Error| = -\frac{\pi}{2}$$

Exercise Set 3.6

3A

	0	1	2	3
a	1.00	1.50	15.00	-11.50
b	6.00	-14.25	19.50	-9.25

Exercise Set 4.1

1

A

$\begin{bmatrix} 0.85 & 0.80 \end{bmatrix}$

B

$\begin{bmatrix} 3.71 & 3.15 \end{bmatrix}$

3

A

$$\begin{bmatrix} x & |Error| & Bound \\ 0.000000 & 0.294000 & 0.300000 \\ 0.200000 & 0.284597 & 0.277860 \\ 0.400000 & 0.259175 & 0.250818 \end{bmatrix}$$

B

$$\begin{bmatrix} x & |Error| & Bound \\ 0.000000 & 0.294000 & 0.300000 \\ 0.200000 & 0.284597 & 0.277860 \\ 0.400000 & 0.259175 & 0.250818 \end{bmatrix}$$

13

Using the five point midpoint formula:

$$f'(x) = \frac{1}{12}\left(f(x-2h) - 8(f(x-h)) + 8(f(x+h)) - f(x+h)\right)$$
$$f'(3) = 0.22585$$
$$|Error| = \frac{f^5(\xi) * (1)^4}{30} = 0.766667$$

15

A

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = 0.852$$
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0.796$$

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} = 0.852$$
$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0.796$$

B

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = 3.707$$
$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = 3.153$$

$$\frac{f(x_0)-f(x_1)}{x_0-x_1} = 3.707$$
$$\frac{f(x_1)-f(x_2)}{x_1-x_2} = 3.153$$

18A

$$\begin{aligned}
f(x) &= -0.339961x^4 + 0.428736x^3 - 0.770008x^2 + 0.072283x + 0.993323 \\
f'(x) &= -1.359844x^3 + 1.286209x^2 - 1.540016x + 0.072283 \\
f'(0.4) &= -0.424960 \\
f''(x) &= -4.079531x^2 + 2.572419x - 1.540016 \\
f''(0.4) &= -1.163773
\end{aligned}$$

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$$\begin{aligned}
f'(x_0) = &A\left(f(x_0) + (-h)f'(x_0) + \frac{f''(x_0)(-h)^2}{2!} + \frac{f'''(x_0)(-h)^3}{3!} + \frac{f^4(x_0)(-h)^4}{4!} + \frac{f^5(\xi)(-h)^5}{5!}\right) \\
&+ B\left(f(x_0)\right) \\
&+ C\left(f(x_0) + ((h))f'(x_0) + \frac{f''(x_0)(h)^2}{2!} + \frac{f'''(x_0)(h)^3}{3!} + \frac{f^4(x_0)(h)^4}{4!} + \frac{f^5(\xi)(h)^5}{5!}\right) \\
&+ D\left(f(x_0) + ((2h))f'(x_0) + \frac{f''(x_0)(2h)^2}{2!} + \frac{f'''(x_0)(2h)^3}{3!} + \frac{f^4(x_0)(2h)^4}{4!} + \frac{f^5(\xi)(2h)^5}{5!}\right) \\
&+ E\left(f(x_0) + ((3h))f'(x_0) + \frac{f''(x_0)(3h)^2}{2!} + \frac{f'''(x_0)(3h)^3}{3!} + \frac{f^4(x_0)(3h)^4}{4!} + \frac{f^5(\xi)(3h)^5}{5!}\right) \\
&= (A+B+C+D+E)f(x_0) \\
&+ (-hA + hC + 2hD + 3hE)(f'(x_0)) \\
&+ (0.5(h^2)A + 0.5(h^2)C + 2h^2D + 4.5h^2E) \\
&+ \frac{1}{6}(-h^3A + h^3C + 8h^3D + 27h^3) \\
&+ \frac{1}{24}(h^4A + h^4C + 16h^4D + 81h^4E) \\
&\text{Using } h = 1: \\
(A,B,C,D,E) &= (-0.25, -0.83333333, 1.5, -0.5, 0.08333333)
\end{aligned}$$

The polynomial:

$$f'(x_0) = -\frac{0.25}{h}(h+x_0) - \frac{0.83333333}{h}(x_0) + \frac{1.5}{h}(-h+x_0) - \frac{0.5}{h}(-2h+x_0) + \frac{0.08333333}{h}(-3h+x_0) + O(h^4)$$

25

$$
\begin{aligned}
f'(0.4) &= -1.25f(0.2) - 4.16666665f(0.4) + 7.5f(0.6) - 2.5f(0.8) + 0.41666665f(1.0) \\
&= -0.424984 \\
f'(0.8) &= -1.25f(0.6) - 4.16666665f(0.8) + 7.5f(1.0) - 2.5f(1.2) + 0.41666665f(1.4) \\
&= -1.032772
\end{aligned}
$$

29

$$
\begin{aligned}
e(h) &= \frac{\epsilon}{h} + \frac{h^2}{6}M \\
e'(h) &= \frac{Mh}{3} - \frac{\epsilon}{h^2} \\
\text{Using } e'(h) &= 0 \\
h &= \sqrt[3]{\frac{3\epsilon}{M}}
\end{aligned}
$$

If $e'(h) < 0, h < \sqrt[3]{\frac{3\epsilon}{M}}$, and if $e'(h) > 0, h > \sqrt[3]{\frac{3\epsilon}{M}}$. Thus, there is a minimum at $h = \sqrt[3]{\frac{3\epsilon}{M}}$.

27

Using $x = 420$ and $f(x) = x^4$:

n	$f'_n(x)$	$f(x)$	Absolute Error
0	297412081.000000000000000	296352000	1060081.000000000000000
1	296457856.800999999046326	296352000	105856.800999999046326
2	296362584.168000996112823	296352000	10584.168000996112823
3	296353058.401679992675781	296352000	1058.401679992675781
4	296352105.840016782283783	296352000	105.840016782283783
5	296352010.584000170230865	296352000	10.584000170230865
6	296352001.058399975299835	296352000	1.058399975299835
7	296352000.105839967727661	296352000	0.105839967727661
8	296352000.010583996772766	296352000	0.010583996772766
9	296352000.001058399677277	296352000	0.001058399677277
10	296352000.000105857849121	296352000	0.000105857849121
11	296352000.000010609626770	296352000	0.000010609626770
12	296352000.000001072883606	296352000	0.000001072883606
13	296352000.000000119209290	296352000	0.000000119209290
14	296352000	296352000	0
15	296352000	296352000	0
16	296352000	296352000	0
17	296352000	296352000	0
18	296352000	296352000	0
19	296352000	296352000	0
20	296352000	296352000	0

As n increases, $f'_n(x)$ comes closer to the actual value of $f(x)$.

4.2

1

A

Output from program:

$$\left[\begin{array}{c|ccc} h & N_1 & N_2 & N_3 \\ \hline h & 0.841181 & 0.000000 & 0.000000 \\ \frac{h}{2} & 0.911608 & 0.982035 & 0.000000 \\ \frac{h}{4} & 0.953102 & 0.994596 & 1.007157 \end{array}\right]$$

$$N_3(h) = 1.007157$$

C

$$\begin{bmatrix} h & N_1 & N_2 & N_3 \\ \hline h & 2.290365 & 0.000000 & 0.000000 \\ \frac{h}{2} & 2.305264 & 2.305501 & 0.000000 \\ \frac{h}{4} & 2.295243 & 2.295084 & 2.295074 \end{bmatrix}$$

$$N_3(h) = 2.250282$$

5

$$\begin{bmatrix} h & N_1 & N_2 & N_3 & N_4 \\ \hline h & 1.570796 & 0.000000 & 0.000000 & 0.000000 \\ \frac{h}{2} & 1.896119 & 1.901283 & 0.000000 & 0.000000 \\ \frac{h}{4} & 1.974232 & 1.975472 & 1.975544 & 0.000000 \\ \frac{h}{8} & 1.993570 & 1.993877 & 1.993895 & 1.993896 \end{bmatrix}$$

$$N_4(h) = 1.993896$$

8

To remove fluff from the calculations, let's define some variables:

$$A = \frac{1}{2} f''(x_0)$$
$$B = \frac{1}{6} f^3(x_0)$$
$$D(a) = f(x_0 + a) - f(x_0)$$

$$\mathrm{N}_1(h) = -Ah - Bh^2 + \mathcal{O}(h^3) + \frac{D(h)}{h}$$
$$\mathrm{N}_1(2h) = -2Ah - 4Bh^2 + \mathcal{O}(h^3) + \frac{D(2h)}{2h}$$
$$\mathrm{N}_2(h) = 2 * \mathrm{N}_1(h) - \mathrm{N}_1(2h) = 2Bh^2 + \mathcal{O}(h^3) + \frac{2D(h)}{h} - \frac{D(2h)}{2h}$$
$$\mathrm{N}_2(2h) = 8Bh^2 + \mathcal{O}(h^3) + \frac{D(2h)}{h} - \frac{D(4h)}{4h}$$
$$\mathrm{N}_3(h) = 4 * \mathrm{N}_2(h) - \mathrm{N}_2(2h) = \mathcal{O}(h^3) + \frac{32D(h) - 12D(2h) + D(4h)}{12h}$$

Substituting $D(a)$ back into the equation:

$$\mathrm{N}_3(h) = \mathcal{O}(h^3) + \frac{-21f(x) + 32f(h+x) - 12f(2h+x) + f(4h+x)}{12h} + \mathcal{O}(h^3)$$

9

$$
\begin{aligned}
N_1(h) &= K_1 h + K_2 h^2 + D(h) \\
N_1(h/3) &= \frac{K_1 h}{3} + \frac{K_2 h^2}{9} + D\left(\frac{h}{3}\right) \\
N_1(h/9) &= \frac{K_1 h}{9} + \frac{K_2 h^2}{81} + D\left(\frac{h}{9}\right) \\
N_2(h) &= -\frac{K_2 h^2}{3} + \frac{3D\left(\frac{h}{3}\right)}{2} - \frac{D(h)}{2} \\
N_2(h/3) &= -\frac{K_2 h^2}{27} + \frac{3D\left(\frac{h}{9}\right)}{2} - \frac{D\left(\frac{h}{3}\right)}{2} \\
N_3(h) &= \frac{27D\left(\frac{h}{9}\right)}{16} - \frac{3D\left(\frac{h}{3}\right)}{4} + \frac{D(h)}{16}
\end{aligned}
$$

11

A / B / C

$$
\begin{aligned}
N_1(h) &= K_1 h + K_2 h^2 + K_3 h^3 + (h+1)^{\frac{1}{h}} \\
N_1(0.04) &= 0.04K_1 + 0.0016K_2 + 6.4 \cdot 10^{-5} K_3 + 2.665836 \\
N_2(h) &= -\frac{K_2 h^2}{2} - \frac{3K_3 h^3}{4} + 2\left(\frac{h}{2} + 1\right)^{\frac{2}{h}} - (h+1)^{\frac{1}{h}} \\
N_2(0.04) &= -0.0008K_2 - 4.8 \cdot 10^{-5} K_3 + 2.71734 \\
N_3(h) &= \frac{K_3 h^3}{8} + \frac{8\left(\frac{h}{4}+1\right)^{\frac{4}{h}}}{3} - 2\left(\frac{h}{2} + 1\right)^{\frac{2}{h}} + \frac{(h+1)^{\frac{1}{h}}}{3} \\
N_3(0.04) &= 8.0 \cdot 10^{-6} K_3 + 2.718273 \\
e &= 2.718282
\end{aligned}
$$

The assumption in B seems to hold up. The absolute error between N_3 and the exact answer is approximately 0.000009.

Exercise Set 4.3

1 / 3

Using the trapezoidal rule:

$$\frac{h}{2}\left(f(a) + f(a+h)\right)$$

A

$$X = [0.5, 1], Y = [0.0625, 1] \rightarrow 0.265625$$

$$\text{Error Bound} = 0.125000000000000$$

$$\text{Actual Error} = 0.071875$$

C

$$X = [1, 1.5], Y = [0.0, 0.912296] \to 0.228074$$
$$\text{Error Bound} = 0.0396971897522534$$
$$\text{Actual Error} = 0.03581476557804644$$

H

$$X = [0, 0.785398], Y = [0.0, 10.550724] \to 4.143260$$
$$\text{Error Bound} = 2.12980904832081$$
$$\text{Actual Error} = 1.5546310226869071$$

19

i	x^i	$\int_{-1}^{1} f(x)$	$f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$
0	1	2	2
1	x	0	0
2	x^2	2/3	2/3
3	x^3	0	0
4	x^4	2/5	2/9

At x^4, the values diverge. Thus, the degree of precision is 3.

21

i	$x^i = f(x)$	$c_0 f(-1) + c_1 f(1) + c_2(1)$	$\int_{-1}^{1} f(x)$
0	1	$c_0 + c_1 + c_2$	2
1	x	$-c_0 + c_2$	0
2	x^2	$c_0 + c_2$	2/3

Solving for the unknowns:

$$\begin{aligned} c_0 + c_1 + c_2 - 2 &= 0 \\ -c_0 + c_2 &= 0 \\ c_0 + c_2 - \frac{2}{3} &= 0 \\ (c_0, c_1, c_2) &= \left(\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right) \end{aligned}$$

26

Substituting (x_0, x_1, x_2) for (x_0, x_0+h, x_0+2h)

i	$f(x) = x^{i+1}$	$\int_{x_0}^{x_2} f(x)\,dx$	$a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2)$
0	x	$-\frac{x_0^2}{2} + \frac{(2h+x_0)^2}{2}$	$a_0 x_0 + a_1 (h + x_0) + a_2 (2h + x_0)$
1	x^2	$-\frac{x_0^3}{3} + \frac{(2h+x_0)^3}{3}$	$a_0 x_0^2 + a_1 (h + x_0)^2 + a_2 (2h + x_0)^2$
2	x^3	$-\frac{x_0^4}{4} + \frac{(2h+x_0)^4}{4}$	$a_0 x_0^3 + a_1 (h + x_0)^3 + a_2 (2h + x_0)^3$

$$(a_0, a_1, a_2) = \left(\frac{h}{3}, \frac{4h}{3}, \frac{h}{3}\right)$$

Using the new equation:

$$\int_{x_0}^{x_2} x^4\,dx = f^4(\xi)k + \frac{hx_0^4}{3} + \frac{4hx_1^4}{3} + \frac{hx_2^4}{3}$$

$$-\frac{x_0^5}{5} + \frac{x_2^5}{5} = 24k + \frac{hx_0^4}{3} + \frac{4hx_1^4}{3} + \frac{hx_2^4}{3}$$

Substituting (x_0, x_1, x_2) for (x_0, x_0+h, x_0+2h)

$$\Rightarrow -\frac{x_0^5}{5} + \frac{(2h+x_0)^5}{5} = \frac{hx_0^4}{3} + \frac{4h(h+x_0)^4}{3} + \frac{h(2h+x_0)^4}{3} + 24k$$

Solving for k reveals:

$$k = -\frac{h^5}{90}$$

4.4

1 / 3

	$f(x)$	$Q1$	$Q3$
A	$x\log(x)$	0.639900	0.636310
C	$\frac{2}{x^2+4}$	0.784241	0.785398
E	$e^{2x}\sin(3x)$	−13.575979	−14.183342

7B

$$\int_0^2 x^2 ln(x^2+1) \approx 3.109337$$

9

Plugging in the given (X, Y) data gives use the equation:

$$4\alpha + \frac{3h}{2} + 0.5 = 0$$

Substituting $h = 0.25$ and solving for α gives:

$$\alpha = 0.21875$$

11A

$$f(\xi) = e^{2\xi}\sin(3\xi)$$
$$f''(\xi) = (-5\sin(3\xi) + 12\cos(3\xi))\, e^{2\xi}$$
$$\frac{(b-a)h^2}{12} f''(\xi) = \frac{h^2(-5\sin(3\xi) + 12\cos(3\xi))\, e^{2\xi}}{6}$$
$$\frac{h^2(-5\sin(6) + 12\cos(6))\, e^4}{6} < 0.0001$$
$$h < 0.000922296$$
$$n \geq \frac{b-a}{h} = \frac{2}{h} = \frac{2}{0.000922296} = 2168.5$$
$$h < 0.000922296, n \geq 2169$$

13

A

$$f(\xi) = \frac{1}{\xi + 4}$$
$$f''(\xi) = \frac{2}{(\xi+4)^3}$$
$$\frac{(b-a)h^2}{12} f''(\xi) = \frac{h^2}{3(\xi+4)^3}$$
$$\frac{h^2}{192} < 1.0 \cdot 10^{-5}$$
$$h < 0.0438178$$
$$n \geq \frac{b-a}{h} = \frac{2}{h} = \frac{2}{0.0438178} = 45.6435$$
$$h < 0.0438178, n \geq 46$$
$$\int_0^2 \frac{1}{x+4} dx \approx 0.4054708$$

B

$$\frac{-(b-a)h^4 f^4(\xi)}{180} = \frac{4\left|\frac{h^4}{(\xi+4)^5}\right|}{15}$$
$$\frac{\left|h^4\right|}{3840} < 1.0 \cdot 10^{-5}$$
$$h < 0.4426727$$
$$n \geq \frac{2}{h} = 4.51801$$
$$\int_0^2 \frac{1}{x+4} dx \approx 0.4054714$$

21

Using the trapezoidal composite rule:

$$n = 9600, h = 0.005, \int_0^{48} \sqrt{1 + \cos x^2}\, dx \approx 58.470469$$

23

The given error term for the composite Simpson's rule is:

$$E(f) = -\frac{h^5}{90}\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)$$

If we look at $\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)(2h)$, we see that the Δ_x term is $2h$, and the y term is $f^4(\xi_j)$. Thus, we see that $\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)(2h)$ is a Riemann sum for $\int_a^b f^4(x)dx$.

We can manipulate the error term to contain $\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)(2h)$:

$$\begin{aligned}
-\frac{h^5}{90}\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j) &= -\frac{1}{2h} * \frac{h^5}{90}\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)(2h) \\
&= -\frac{h^4}{180}\sum_{j=1}^{\frac{n}{2}} f^4(\xi_j)(2h) \\
&\approx -\frac{h^4}{180}\int_a^b f^4(x)dx \\
&= -\frac{h^4}{180}\left[f'''(b) - f'''(a)\right]
\end{aligned}$$

24

A

$$\begin{aligned}
E(f) &= \frac{h^3}{12}\sum_{j=1}^{\frac{n}{2}} f''(\xi_j) \\
&= \frac{h^2}{24}\sum_{j=1}^{\frac{n}{2}} f''(\xi_j)(2h) \\
&\approx \frac{h^2}{24}\int_a^b f''(x)dx \\
&= \frac{h^2}{24}\left[f(b) - f(a)\right]
\end{aligned}$$

B

$$\begin{aligned}
E(f) &= \frac{h^3}{6}\sum_{j=1}^{\frac{n}{2}} f''(\xi_j) \\
&= \frac{h^2}{12}\sum_{j=1}^{\frac{n}{2}} f''(\xi_j)(2h) \\
&\approx \frac{h^2}{12}\int_a^b f''(x)dx \\
&= \frac{h^2}{12}\left[f(b) - f(a)\right]
\end{aligned}$$

17

Using the trapezoidal method:

$$x(t) = \frac{3\sqrt{-(t-2)(t+2)}}{2}$$
$$y(t) = \frac{2\sqrt{-(t-3)(t+3)}}{3}$$
$$f(t) = y(t)x'(t) = \frac{t\sqrt{-(t-3)(t+3)}\sqrt{-(t-2)(t+2)}}{(t-2)(t+2)}$$
$$\int_0^{2\pi} f(t)dt \approx -17.359446$$

4.5

1 / 2 / 3

	a	b	$R_{3,3}$	$R_{4,4}$
$1/3A$	1	1.500000	0.192259	0.192259
$1/3C$	0	0.350000	−0.176820	−0.176820
$1/3E$	0	0.785398	2.587968	2.588627
$2D$	e	$2e$	0.526816	0.526594

6D

Steps	Step Size	Results					
1	2.718282	0.647654					
2	1.359141	0.560996	0.532111				
4	0.679570	0.535609	0.527146	0.526816			
8	0.339785	0.528876	0.526632	0.526597	0.526594		
16	0.169893	0.527163	0.526592	0.526589	0.526589	0.526589	
32	0.084946	0.526733	0.526589	0.526589	0.526589	0.526589	0.526589

7

$R_{x,y}$	1	2	3
1	11.38920		
2	11.48940	11.52280	
3	11.51570	11.52447	11.52458

11

$$
\begin{aligned}
5.33333333333333 &= 1.33333333333333R_{2,1} - 2.66666666666667 \\
R_{2,1} &= 6.0 \\
4.62222222222222 &= 1.06666666666667R_{3,2} - 0.355555555555556 \\
R_{3,2} &= 4.66666666666667 \\
4.66666666666667 &= 1.33333333333333R_{3,1} - 2.0 \\
R_{3,1} &= 5.0
\end{aligned}
$$

15

$$\int_0^{48} \sqrt{\cos^2(x) + 1} \approx 58.470469$$

17

$$
\begin{aligned}
R_{k,2} &= R_{k,1} + \frac{1}{3}(R_{k,1} - R_{k-1,1}) \\
&= \frac{2}{3}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right] - \frac{2}{6}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(2a + (4i-2)h_k)\right] \\
&= \frac{1}{3}\left[R_{k-1,1} + 2h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a + h_{k-1}\left(i - \frac{1}{2}\right)\right)\right] \\
&= \frac{1}{3}\left(\left[\frac{(f(a)+f(b))h_{k-1}}{2} + h_{k-1}\sum_{i=1}^{2^{k-2}-1} f(a + ih_{k-1})\right] + \left[2h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a + h_{k-1}\left(i - \frac{1}{2}\right)\right)\right]\right) \\
&= \frac{1}{3}\left(\left[f(a)+f(b))h_k + 2h_k\sum_{i=1}^{2^{k-2}-1} f(a + 2ih_k)\right] + \left[4h_k\sum_{i=1}^{2^{k-2}} f\left(a + h_k\left(2i - 1\right)\right)\right]\right) \\
&= \frac{h_k}{3}\left[f(a) + f(b) + 2\sum_{i=1}^{2^{k-2}-1} f(a + 2ih_k) + 4\sum_{i=1}^{2^{k-2}} f\left(a + h_k\left(2i - 1\right)\right)\right]
\end{aligned}
$$

Exercise Set 4.7

1 / 3 / 7

	a	b	$f(x)$	$n=2$	$n=3$	$n=7$
A	1	1.500000	$x^2\log(x)$	0.192269	0.192259	0.192259
C	0	0.350000	$\frac{2}{x^2-4}$	−0.176819	−0.176820	−0.176820
D	0	0.785398	$x^2\sin(x)$	0.089263	0.088754	0.088755

9

$$
\begin{aligned}
f(x, y) &= 4x^2 + 9y^2 - 36 \\
x(t) &= 3\cos(t) \\
y(t) &= 2\sin(t) \\
f(t) &= \sqrt{9\sin^2(t) + 4\cos^2(t)} \int_0^{\pi/2} f(t)dt && \approx 3.966355 |Error| = 0.222642
\end{aligned}
$$

11

Using x_0 to x_3 as $f(x)$ yields the equations:

$$
\begin{aligned}
a + b &= 2 \\
-a + b + c + d &= 0 \\
a + b - 2c + 2d &= \frac{2}{3} \\
-a + b + 3c + 3d &= 0 \\
(a, b, c, d) &= \left(1,\ 1,\ \frac{1}{3},\ -\frac{1}{3}\right)
\end{aligned}
$$

12

$$
\begin{aligned}
a + b + c &= 2 \\
-a + c + d + e &= 0 \\
a + c - 2d + 2e &= \frac{2}{3} \\
-a + c + 3d + 3e &= 0 \\
a + c - 4d + 4e &= \frac{2}{5} \\
(a, b, c, d, e) &= \left(\frac{7}{15},\ \frac{16}{15},\ \frac{7}{15},\ \frac{1}{15},\ -\frac{1}{15}\right)
\end{aligned}
$$

Exercise Set 4.8

1A

0.3115733

3A

0.3115733 - the results from both algorithms were the same.

11A

$$
\begin{aligned}
\text{Approximation } &= 5.204036265137036 \\
\text{Exact } &= 5.206446553838018 \\
|\text{Error}| &= 0.0024102887009815888
\end{aligned}
$$

13A

$$\begin{aligned} \text{Approximation} &= 5.206446548172842 \\ \text{Exact} &= 5.206446553838018 \\ |\text{Error}| &= 5.6651758839620925 * 10^{-9} \end{aligned}$$

Exercise Set 4.9

1A

3AC

4A

5

Exercise Set 8.3

1A

3A

9