

## 1.3 The variety $X_m$ and its bijective normalization

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REMARK. In fact, since  $W$  is a finite Coxeter group, a celebrated result of Chevalley says that the algebra  $\mathbf{C}[\mathfrak{h}]^W$  is not only a finitely generated  $\mathbf{C}$ -algebra but actually a free (=polynomial) algebra. Namely, it is of the form  $\mathbf{C}[q_1, \dots, q_n]$ , where the  $q_i$  are homogeneous polynomials of some degrees  $d_i$ . Furthermore, if we denote by  $H$  the subspace of  $\mathbf{C}[\mathfrak{h}]$  of harmonic polynomials, i.e. of polynomials killed by  $W$ -invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \rightarrow \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of  $\mathbf{C}[\mathfrak{h}]^W$ - and of  $W$ -modules. In particular,  $\mathbf{C}[\mathfrak{h}]$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module of rank  $|W|$ .

### 1.3 THE VARIETY $X_m$ AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety  $X_m = \text{Spec}(Q_m)$ . The inclusion  $Q_m \subset \mathbf{C}[\mathfrak{h}]$  induces a morphism

$$\pi: \mathfrak{h} \rightarrow X_m,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that  $X_m$  is singular for all  $m \neq 0$ .)

In fact, not only is  $\pi$  birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]).  *$\pi$  is a bijection.*

*Proof.* By the above remarks, we only have to show that  $\pi$  is injective. In order to achieve this, we need to prove that quasi-invariants separate points of  $\mathfrak{h}$ , i.e. that if  $z, y \in \mathfrak{h}$  and  $z \neq y$ , then there exists  $p \in Q_m$  such that  $p(z) \neq p(y)$ . This is obtained in the following way. Let  $W_z \subset W$  be the stabilizer of  $z$  and choose  $f \in \mathbf{C}[\mathfrak{h}]$  such that  $f(z) \neq 0$ ,  $f(y) = 0$ . Set

$$p(x) = \prod_{s \in \Sigma, sz \neq z} \alpha_s(x)^{2m_s+1} \prod_{w \in W_z} f(wx).$$

We claim that  $p(x) \in Q_m$ . Indeed, let  $s \in \Sigma$  and assume that  $s(z) \neq z$ .

We have by definition  $p(x) = \alpha_s(x)^{2m_s+1} \tilde{p}(x)$ , with  $\tilde{p}(x)$  a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1} \tilde{p}(x) - \alpha_s(sx)^{2m_s+1} \tilde{p}(sx) = \alpha_s(x)^{2m_s+1} (\tilde{p}(x) + \tilde{p}(sx)).$$

If on the other hand,  $sz = z$ , i.e.  $s \in W_z$ , then  $s$  preserves the set  $W \setminus W_z$ , and hence preserves  $\prod_{s \in \Sigma \cap (W \setminus W_z)} \alpha_s(x)^{2m_s+1}$  (as it acts by  $-1$  on the products  $\prod_{s \in \Sigma} \alpha_s(x)^{2m_s+1}$  and  $\prod_{s \in \Sigma \cap W_z} \alpha_s(x)^{2m_s+1}$ ). Since  $\prod_{w \in W_z} f(wx)$  is

$W_z$ -invariant, we deduce that  $p(x) - p(sx) = 0$ , so that in this case  $p(x) - p(sx)$  also is divisible by  $\alpha_s(x)^{2m_s+1}$ .

To conclude, notice that  $p(z) \neq 0$ . Indeed, for a reflection  $s$ ,  $\alpha_s$  vanishes exactly on the fixed points of  $s$ , so that  $\prod_{s \in \Sigma, sz \neq z} \alpha_s(z)^{2m_s+1} \neq 0$ . Also for all  $w \in W_z$   $f(wz) = f(z) \neq 0$ . On the other hand, it is clear that  $p(y) = 0$ .  $\square$

EXAMPLE 1.5. Take  $W = \mathbf{Z}/2$ . As we have already seen,  $Q_m$  has a basis given by the monomials  $\{x^{2i} \mid i \geq 0\} \cup \{x^{2i+1} \mid i \geq m\}$ . From this we deduce that setting  $z = x^2$  and  $y = x^{2m+1}$ ,  $Q_m = \mathbf{C}[y, z]/(y^2 - z^{2m+1}) = \mathbf{C}[K]$ , where  $K$  is the plane curve with a cusp at the origin, given by the equation  $y^2 = z^{2m+1}$ . The map  $\pi: \mathbf{C} \rightarrow K$  is given by  $\pi(t) = (t^{2m+1}, t^2)$ , which is clearly bijective.

#### 1.4 FURTHER PROPERTIES OF $X_m$

Let us get to some deeper properties of quasi-invariants. Let  $X$  be an irreducible affine variety over  $\mathbf{C}$  and  $A = \mathbf{C}[X]$ . Recall that, by the Noether Normalization Lemma, there exist  $f_1, \dots, f_n \in \mathbf{C}[X]$  which are algebraically independent over  $\mathbf{C}$  and such that  $\mathbf{C}[X]$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ . This means that we have a finite morphism of  $X$  onto an affine space.

DEFINITION 1.6.  $A$  (and  $X$ ) is said to be *Cohen-Macaulay* if there exist  $f_1, \dots, f_n$  as above, with the property that  $\mathbf{C}[X]$  is a locally free module over  $\mathbf{C}[f_1, \dots, f_n]$ . (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that  $A$  is a free module.)

REMARK. If  $A$  is Cohen-Macaulay, then for any  $f_1, \dots, f_n$  which are algebraically independent over  $\mathbf{C}$  and such that  $A$  is a finite module over the polynomial ring  $\mathbf{C}[f_1, \dots, f_n]$ , we have that  $A$  is a locally free  $\mathbf{C}[f_1, \dots, f_n]$ -module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]).  $Q_m$  is Cohen-Macaulay.

Notice that, using Chevalley's result that  $\mathbf{C}[\mathfrak{h}]^W$  is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]).  $Q_m$  is a free  $\mathbf{C}[\mathfrak{h}]^W$ -module.