1.3 The variety X_m and its bijective normalization

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **07.07.2017**

REMARK. In fact, since W is a finite Coxeter group, a celebrated result of Chevalley says that the algebra $\mathbf{C}[\mathfrak{h}]^W$ is not only a finitely generated $\mathbf{C}$-algebra but actually a free (=polynomial) algebra. Namely, it is of the form $\mathbf{C}[q_1, \dots, q_n]$, where the q_i are homogeneous polynomials of some degrees d_i. Furthermore, if we denote by H the subspace of $\mathbf{C}[\mathfrak{h}]$ of harmonic polynomials, i.e. of polynomials killed by W-invariant differential operators with constant coefficients without constant term, then the multiplication map

$$\mathbf{C}[\mathfrak{h}]^W \otimes H \to \mathbf{C}[\mathfrak{h}]$$

is an isomorphism of $\mathbf{C}[\mathfrak{h}]^W$- and of W-modules. In particular, $\mathbf{C}[\mathfrak{h}]$ is a free $\mathbf{C}[\mathfrak{h}]^W$-module of rank $|W|$.

1.3 THE VARIETY X_m AND ITS BIJECTIVE NORMALIZATION

Using Proposition 1.3, we can define the irreducible affine variety $X_m = \mathrm{Spec}(Q_m)$. The inclusion $Q_m \subset \mathbf{C}[\mathfrak{h}]$ induces a morphism

$$\pi \colon \mathfrak{h} \to X_m ,$$

which again by Proposition 1.3 is birational and surjective. (Notice that in particular this implies that X_m is singular for all $m \neq 0$.)

In fact, not only is π birational, but a stronger result is true.

PROPOSITION 1.4 (Berest, see [BEG]). *π is a bijection.*

Proof. By the above remarks, we only have to show that π is injective. In order to achieve this, we need to prove that quasi-invariants separate points of $\mathfrak{h}$, i.e. that if $z, y \in \mathfrak{h}$ and $z \neq y$, then there exists $p \in Q_m$ such that $p(z) \neq p(y)$. This is obtained in the following way. Let $W_z \subset W$ be the stabilizer of z and choose $f \in \mathbf{C}[\mathfrak{h}]$ such that $f(z) \neq 0$, $f(y) = 0$. Set

$$p(x) = \prod_{s\in\Sigma\,, sz\neq z} \alpha_s(x)^{2m_s+1} \prod_{w\in W_z} f(wx) .$$

We claim that $p(x) \in Q_m$. Indeed, let $s \in \Sigma$ and assume that $s(z) \neq z$.

We have by definition $p(x) = \alpha_s(x)^{2m_s+1}\tilde{p}(x)$, with $\tilde{p}(x)$ a polynomial. So

$$p(x) - p(sx) = \alpha_s(x)^{2m_s+1}\tilde{p}(x) - \alpha_s(sx)^{2m_s+1}\tilde{p}(sx) = \alpha_s(x)^{2m_s+1}(\tilde{p}(x) + \tilde{p}(sx)) .$$

If on the other hand, $sz = z$, i.e. $s \in W_z$, then s preserves the set $W \setminus W_z$, and hence preserves $\prod_{s\in\Sigma\cap(W\setminus W_z)} \alpha_s(x)^{2m_s+1}$ (as it acts by -1 on the products $\prod_{s\in\Sigma} \alpha_s(x)^{2m_s+1}$ and $\prod_{s\in\Sigma\cap W_z} \alpha_s(x)^{2m_s+1}$). Since $\prod_{w\in W_z} f(wx)$ is

W_z-invariant, we deduce that $p(x)-p(sx)=0$, so that in this case $p(x)-p(sx)$ also is divisible by $\alpha_s(x)^{2m_s+1}$.

To conclude, notice that $p(z)\neq 0$. Indeed, for a reflection s, α_s vanishes exactly on the fixed points of s, so that $\prod_{s\in\Sigma, sz\neq z}\alpha_s(z)^{2m_s+1}\neq 0$. Also for all $w\in W_z$ $f(wz)=f(z)\neq 0$. On the other hand, it is clear that $p(y)=0$. $\square$

EXAMPLE 1.5. Take $W=\mathbf{Z}/2$. As we have already seen, Q_m has a basis given by the monomials $\{x^{2i}\mid i\geq 0\}\cup\{x^{2i+1}\mid i\geq m\}$. From this we deduce that setting $z=x^2$ and $y=x^{2m+1}$, $Q_m=\mathbf{C}[y,z]/(y^2-z^{2m+1})=\mathbf{C}[K]$, where K is the plane curve with a cusp at the origin, given by the equation $y^2=z^{2m+1}$. The map $\pi\colon\mathbf{C}\to K$ is given by $\pi(t)=(t^{2m+1},t^2)$, which is clearly bijective.

1.4 FURTHER PROPERTIES OF X_m

Let us get to some deeper properties of quasi-invariants. Let X be an irreducible affine variety over $\mathbf{C}$ and $A=\mathbf{C}[X]$. Recall that, by the Noether Normalization Lemma, there exist $f_1,\dots,f_n\in\mathbf{C}[X]$ which are algebraically independent over $\mathbf{C}$ and such that $\mathbf{C}[X]$ is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$. This means that we have a finite morphism of X onto an affine space.

DEFINITION 1.6. A (and X) is said to be *Cohen-Macaulay* if there exist $f_1,\dots,f_n$ as above, with the property that $\mathbf{C}[X]$ is a locally free module over $\mathbf{C}[f_1,\dots,f_n]$. (Notice that by the Quillen-Suslin theorem, this is equivalent to saying that A is a free module.)

REMARK. If A is Cohen-Macaulay, then for any $f_1,\dots,f_n$ which are algebraically independent over $\mathbf{C}$ and such that A is a finite module over the polynomial ring $\mathbf{C}[f_1,\dots,f_n]$, we have that A is a locally free $\mathbf{C}[f_1,\dots,f_n]$-module, see [Eis], Corollary 18.17.

THEOREM 1.7 ([EG2], [BEG], conjectured in [FV]). *Q_m is Cohen-Macaulay.*

Notice that, using Chevalley's result that $\mathbf{C}[\mathfrak{h}]^W$ is a polynomial ring, it will suffice, in order to prove Theorem 1.7, to prove:

THEOREM 1.8 ([EG2, BEG], conjectured in [FV]). *Q_m is a free $\mathbf{C}[\mathfrak{h}]^W$-module.*