Mathematical Models and Numerical Methods of Computational Neuroscience
計算神経科学の数学モデルと数値手法

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Skill Pills+

Agenda

- Brief introduction of Differential Equations
- Neuronal Models
- Numerical Method to Simulate Neuronal Models
- How Network of Neurons Performs Computation
- How the Network can be trained

Ordinary Differential Equations

- Fundamental tool in theoretical Science (Not only in Physics or Computational Neuroscience)
- The differential equation gives the dependence of derivatives

$$\boxed{F\left[\frac{d^n x(t)}{dt^n}, \ldots, \frac{dx(t)}{dt}, C\right] = 0}$$

Interpretation of Differential Equations

- The first differential equation I have ever seen is the definition of velocity:

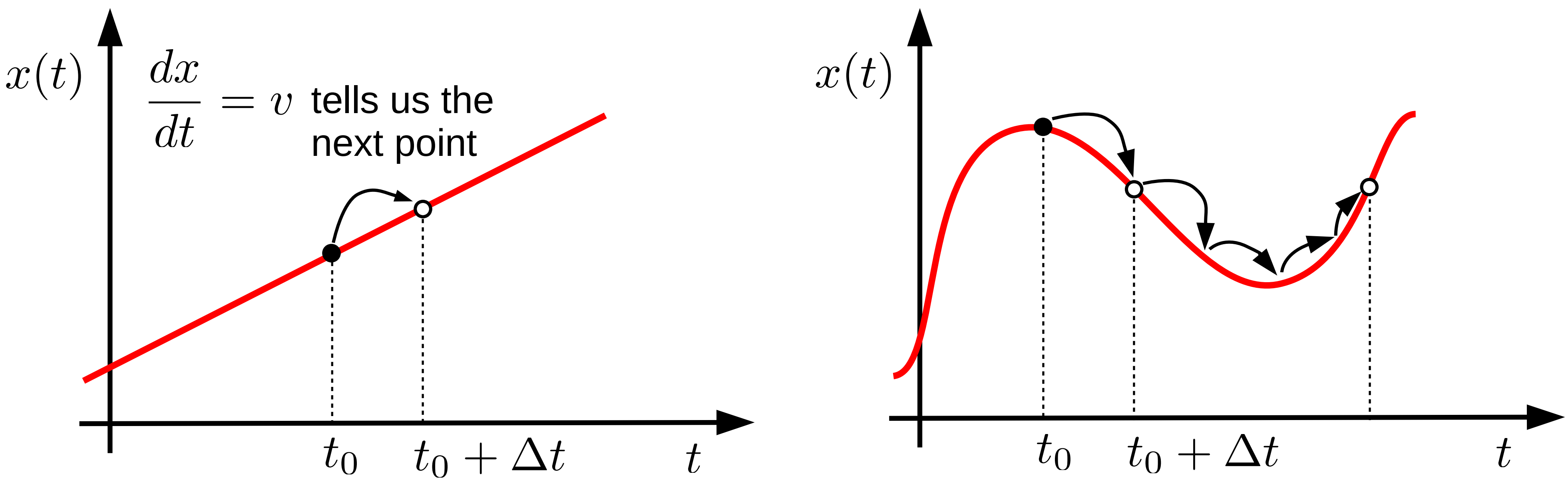
$$\boxed{\frac{dx(t)}{dt} = v}$$

- By definition,

$$\boxed{\frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \approx \frac{\Delta x}{\Delta t}}$$

Interpretation of Differential Equations

- By knowing the slopes, one may trace the whole curve!



Examples of Models and Theories

- Simple Harmonic Oscillator

Acceleration

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0$$

Returning Force

- Heat Equation

Rate of Heat

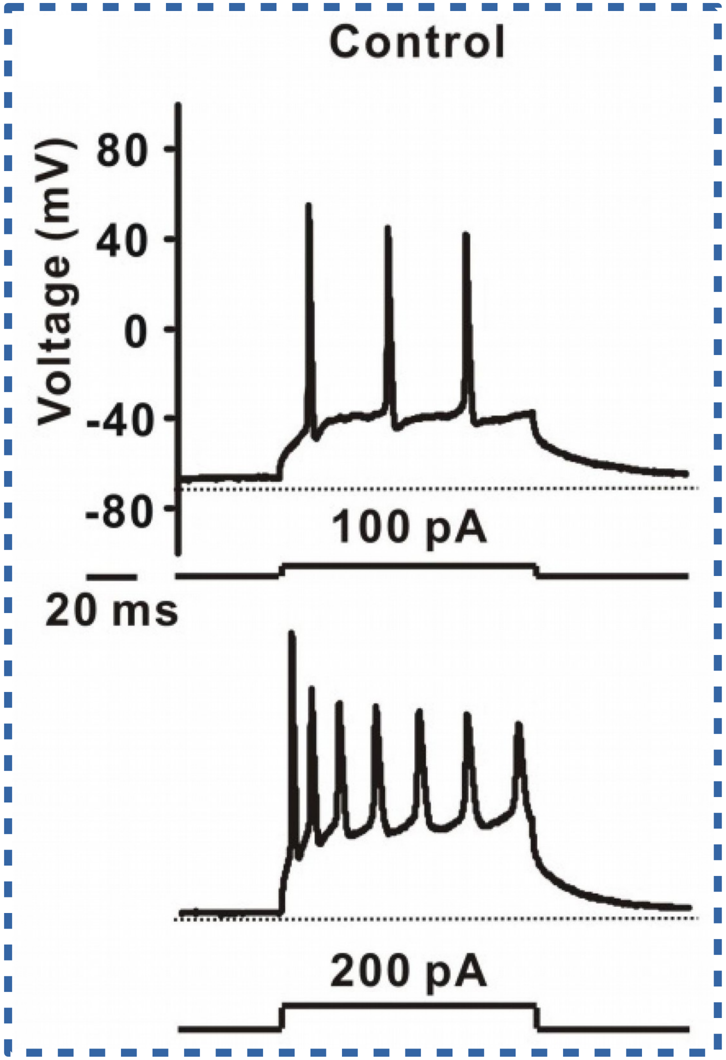
$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}$$

Influx of Heat (by Divergence Theorem)

To Model a Neuron …

- Neurons can be excited by current injection
- If the <u>membrane potential</u> is larger than some thresholds, an action potential will be triggered

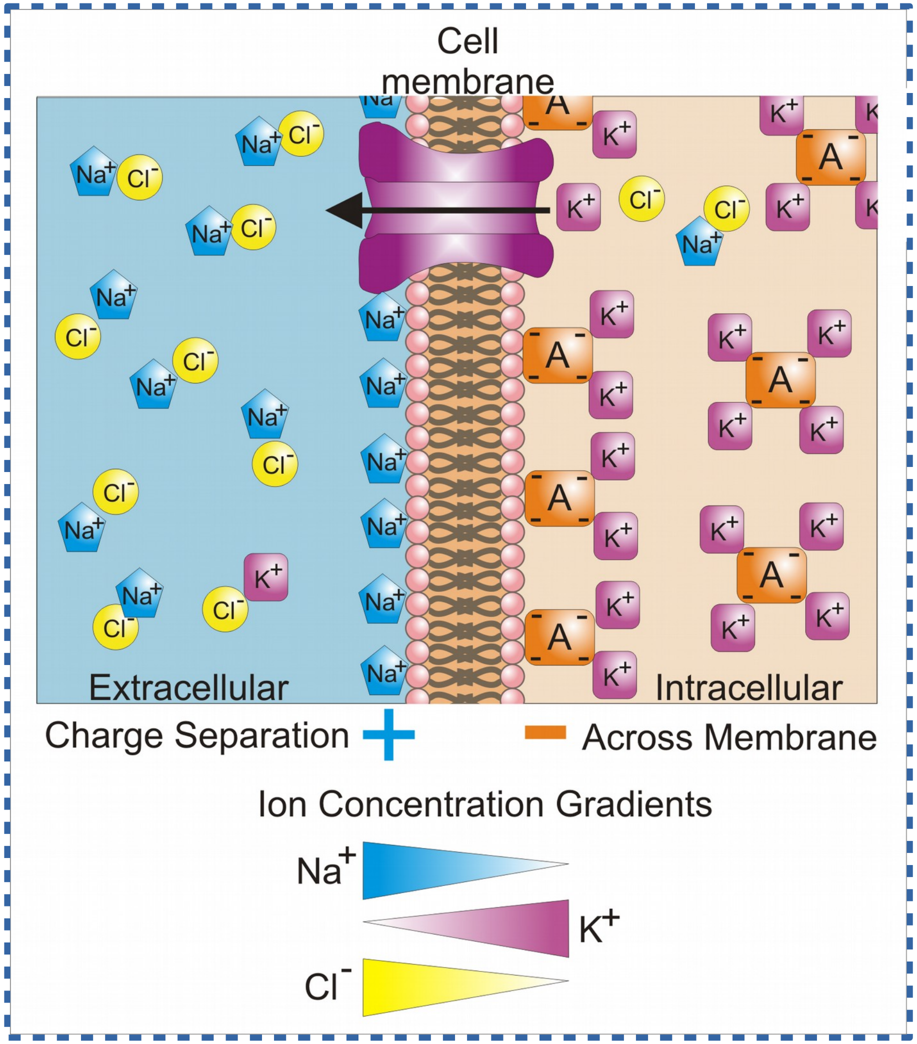
A. D. Reyes, Nature (2019)



Z. Lin et al, Front. Pharmacol. (2019)

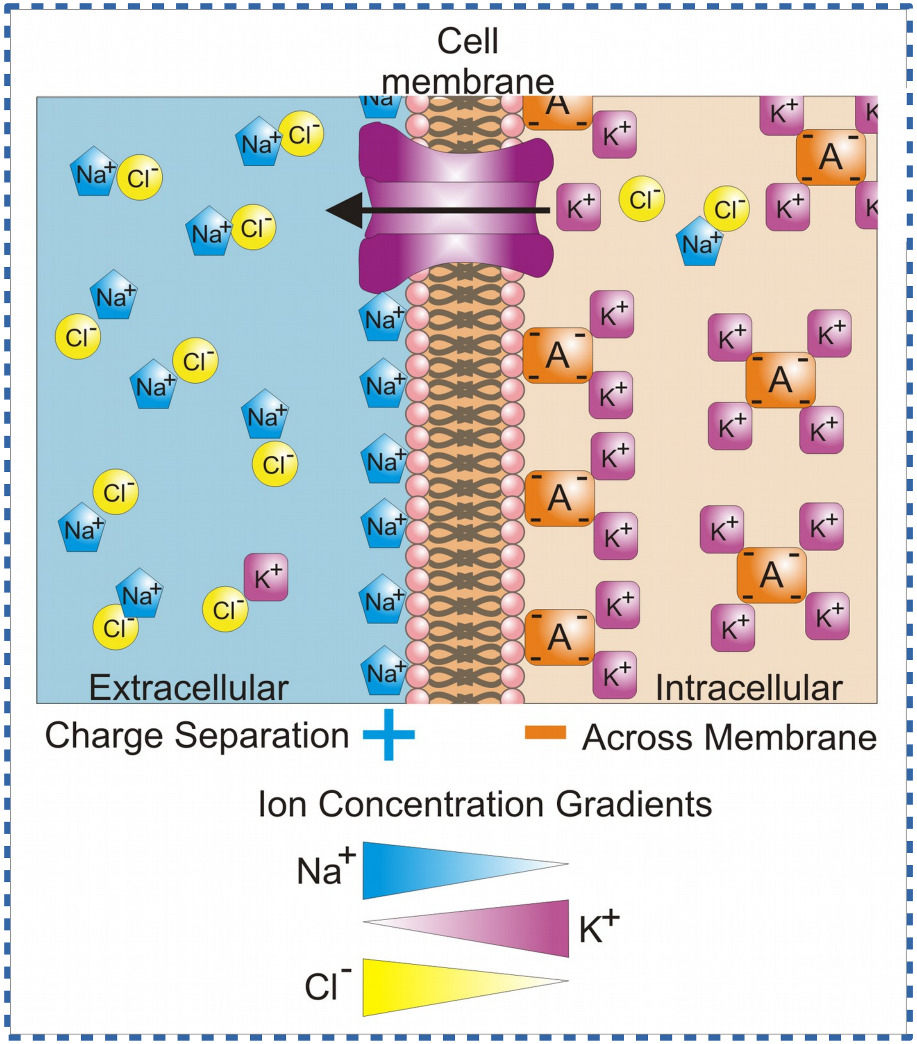
Membrane Potential

- The voltage mentioned in the previous slide is the membrane potential
- Membrane potential means the potential difference across the neuron membrane



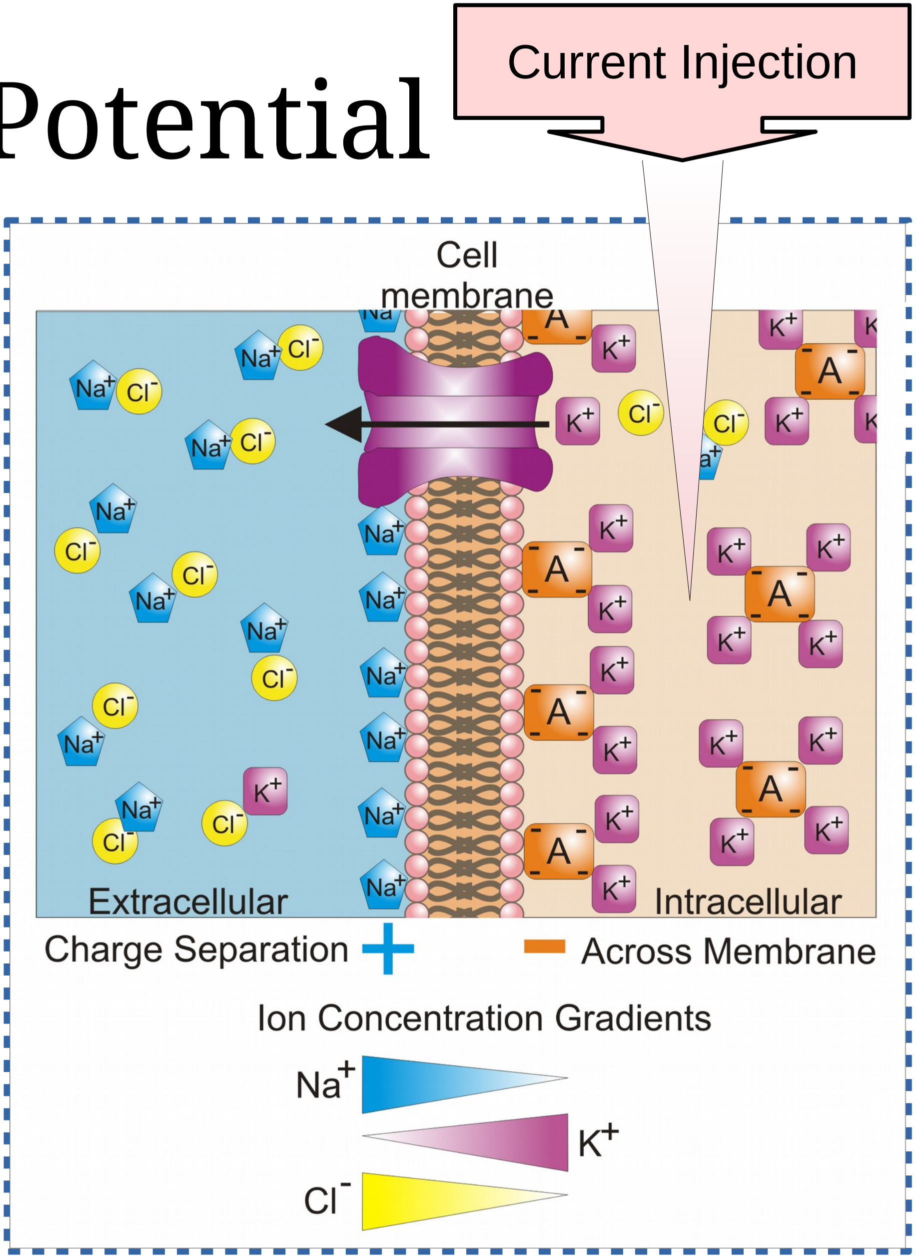
Membrane Potential

- The membrane potential can be controlled by concentrations of ions
- Living cells are actively maintain the potential difference by their ion bumps
- The typical resting potential difference is -65 mV



Membrane Potential

- By injecting current, the membrane potential can be changed
- As the membrane potential large than the threshold, the neuron gives a spike



For More Knowledge about Membrane Potential

Membrane potential

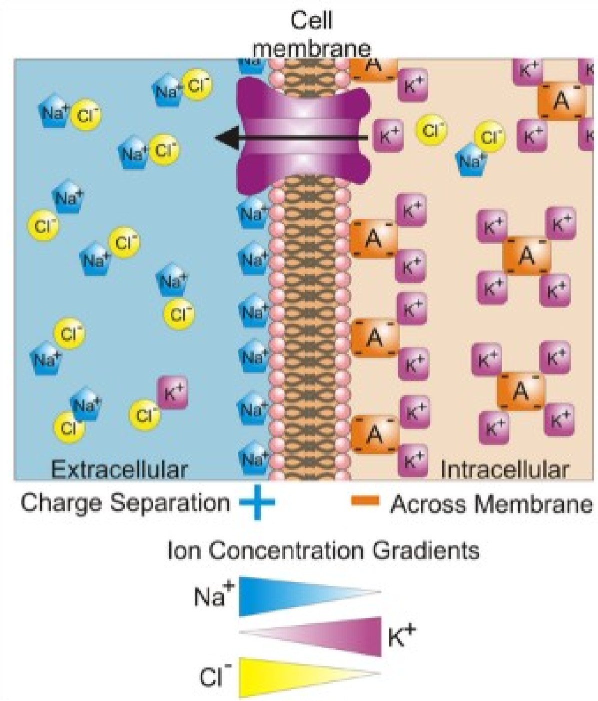
From Wikipedia, the free encyclopedia

Membrane potential (also **transmembrane potential** or **membrane voltage**) is the difference in electric potential between the interior and the exterior of a biological cell. With respect to the exterior of the cell, typical values of membrane potential, normally given in units of millivolts and denoted as mV, ranges from –40 mV to –80 mV.

All animal cells are surrounded by a membrane composed of a lipid bilayer with proteins embedded in it. The membrane serves as both an insulator and a diffusion barrier to the movement of ions. Transmembrane proteins, also known as ion transporter or ion pump proteins, actively push ions across the membrane and establish concentration gradients across the membrane, and ion channels allow ions to move across the membrane down those concentration gradients. Ion pumps and ion channels are electrically equivalent to a set of batteries and resistors inserted in the membrane, and therefore create a voltage between the two sides of the membrane.

Almost all plasma membranes have an electrical potential across them, with the inside usually negative with respect to the outside.[1] The membrane potential has two basic functions. First, it allows a cell to function as a battery, providing power to operate a variety of "molecular devices" embedded in the membrane. Second, in electrically **excitable cells** such as neurons and muscle cells, it is used for transmitting signals between different parts of a cell. Signals are generated by opening or closing of ion channels at one point in the membrane, producing a local change in the membrane potential. This change in the electric field can be quickly affected by either adjacent or more distant ion channels in the membrane. Those ion channels can then open or close as a result of the potential change, reproducing the signal.

In non-excitable cells, and in excitable cells in their baseline states, the membrane potential is held at a



Differences in the concentrations of ions on opposite sides of a cellular membrane lead to a voltage called the **membrane potential**. Typical values of membrane potential are in the range –40 mV to –70 mV. Many ions have a

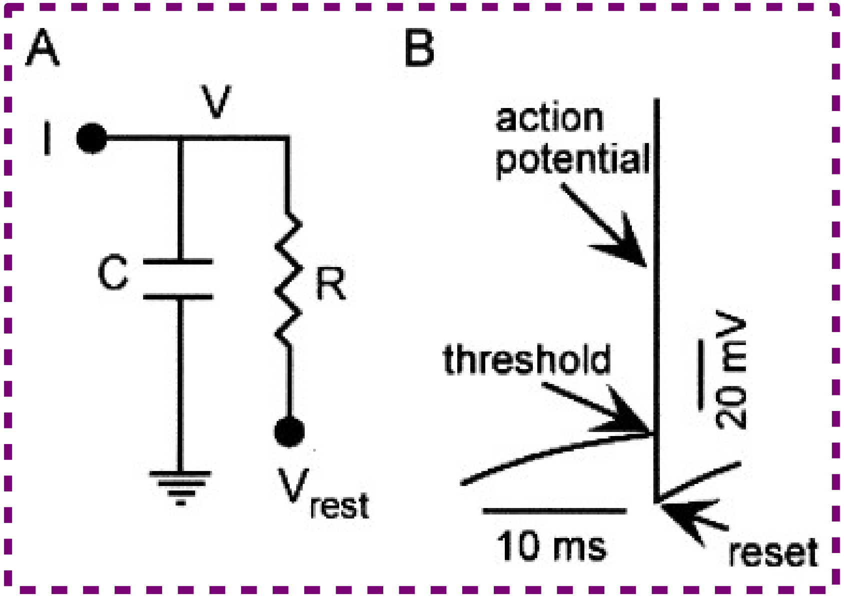
Membrane Works Like ...

- By injecting current, the potential across membrane changes
- As the potential across the threshold, the neuron fires
- It works like <u>a capacitor</u> with a small break-down voltage

Leaky Integrate-and-Fire (LIF) Model

- In 1907, Lapicque proposed a simple model for neurons

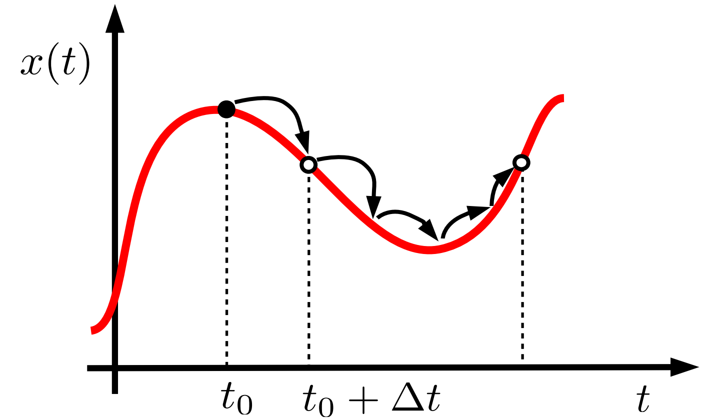
$$I(t) - \frac{V_{\mathrm{m}}(t) - V_{\mathrm{rest}}}{R_{\mathrm{m}}} = C_{\mathrm{m}} \frac{dV_{\mathrm{m}}(t)}{dt}$$



L. F. Abbott, Brain Research Bulletin (1999)

First Simulation (Numerically Solve the Differential Equation)

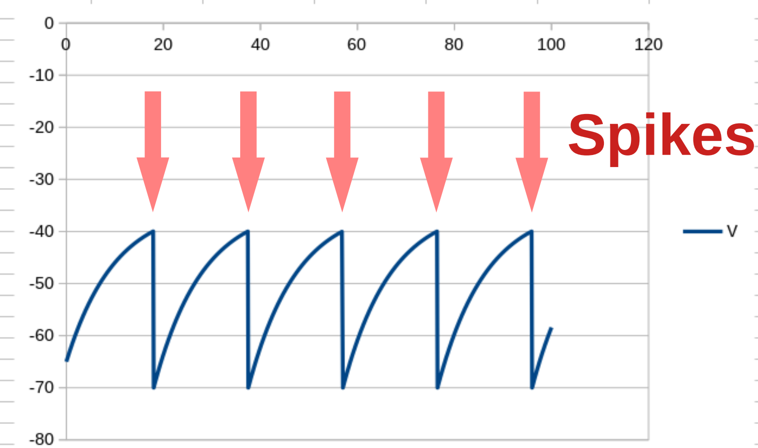
- By calculating derivatives at t_0, one can estimate the voltage at $t_0+\Delta t$



$$V_{\text{rest}} = -65mV$$
$$V_{\text{threshold}} = -40mV$$
$$V_{\text{reset}} = -70mV$$
$$IR = 30mV$$
$$\tau = RC = 10ms$$

E5 =IF(E4>B2,B3,E4+0.1*F4)

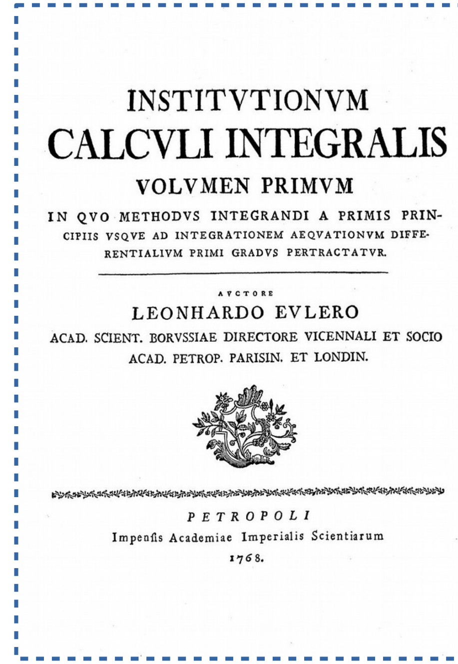
	A	B	C	D	E	F
1	V_rest	-65		t	V	dVdt
2	V_thre	-40		0	-65	3
3	V_reset	-70		0.1	-64.7	2.97
4	I*R	30		0.2	-64.403	2.9403
5	tau	10		0.3	-64.10897	2.910897
6				0.4	-63.8178803	2.88178803
7				0.5	-63.5297015	2.85297015
8				0.6	-63.24440448	2.824440448
9				0.7	-62.96196044	2.796196044
10				0.8	-62.68234083	2.768234083
11				0.9	-62.40551742	2.740551742
12				1	-62.13146225	2.713146225
13				1.1	-61.86014763	2.686014763
14				1.2	-61.59154615	2.659154615
15				1.3	-61.32563069	2.632563069
16				1.4	-61.06237438	2.606237438
17				1.5	-60.80175064	2.580175064
18				1.6	-60.54373313	2.554373313
19				1.7	-60.2882958	2.52882958
20				1.8	-60.03541284	2.503541284
21				1.9	-59.78505872	2.478505872
22				2	-59.53720813	2.453720813
23				2.1	-59.29183605	2.429183605
24				2.2	-59.04891769	2.404891769
25				2.3	-58.80842851	2.380842851
26				2.4	-58.57034422	2.357034422



Euler Method

- Demonstrated by Euler in 1768
- Easy to understand, but the error could be big

$$\frac{dy(x)}{dx} = x$$


INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.
AVCTORE
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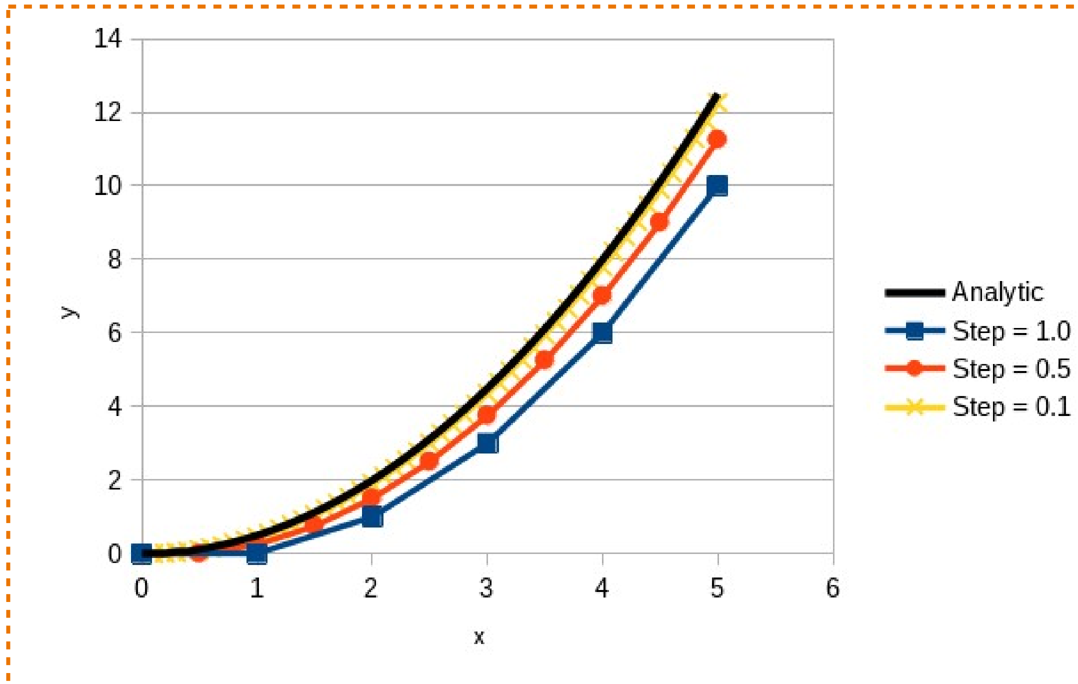
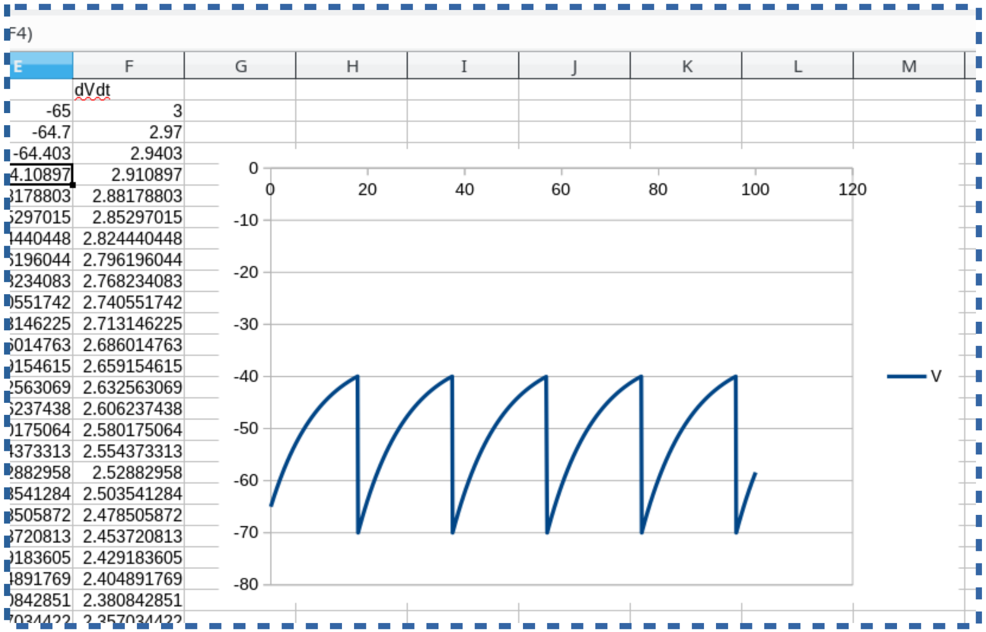
1768.



The Simulation Features ...

- Good
  - Simple
  - More input current, more spikes
  - Capture some neuronal features
- Bad
  - The stepping process will not be accurate
  - The trace of membrane potential does not look like that in experiments

Result Revisit



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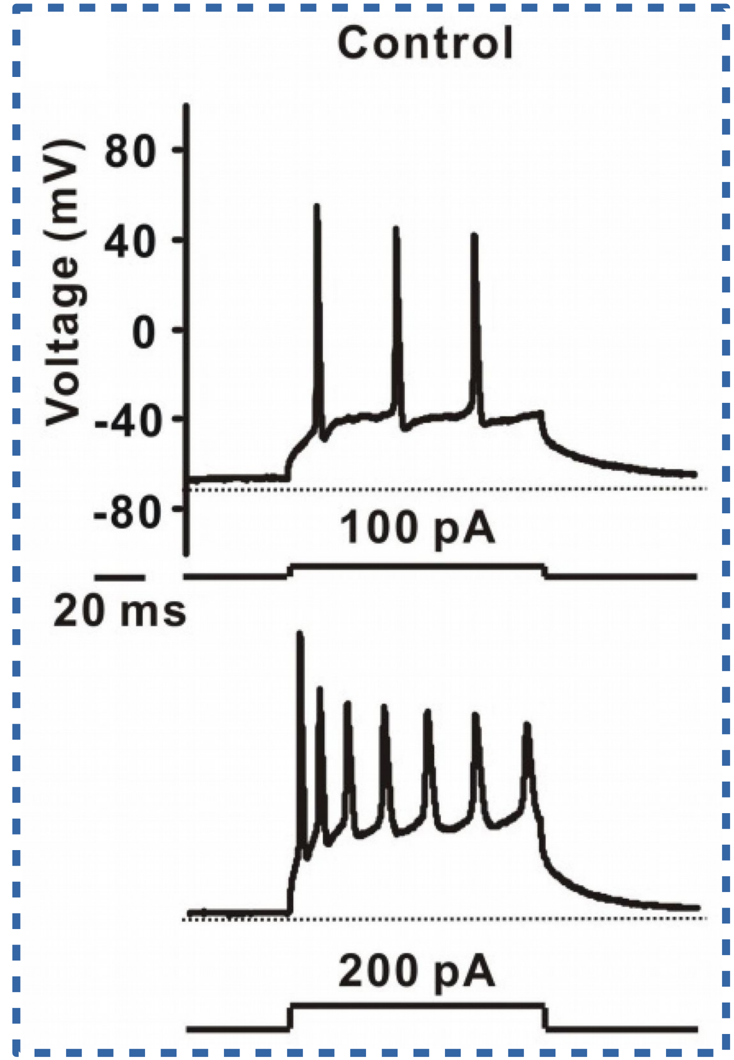


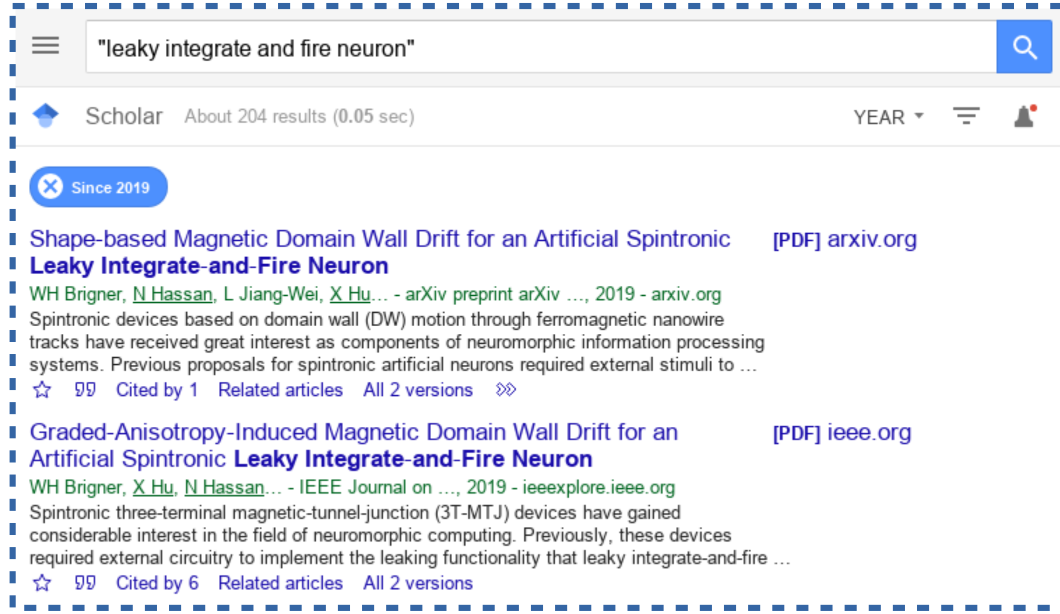
Z. Lin et al, Front. Pharmacol. (2019)

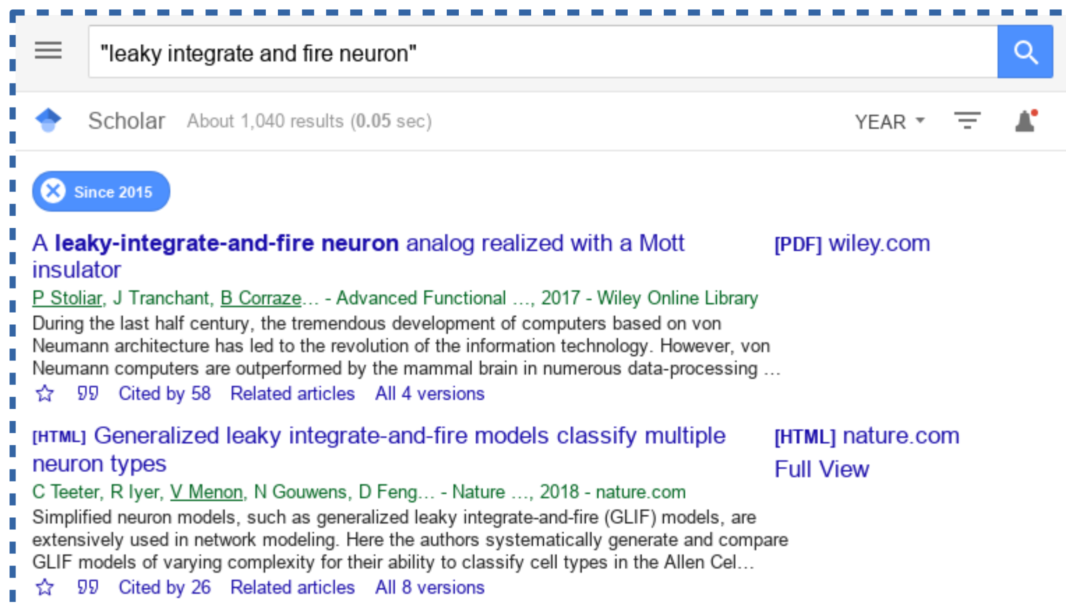
LIF Model is Still Valuable

- LIF neurons can be excited, and the spiking rate is proportional to the input current
- Useful enough for investigations on neural computing
- Cheap computational cost

Studies Involving LIF Model

- Last 10 Months, 204 papers
- Last 4.83 Years, 1040 papers
- There are ~200 paper talking about “leaky integrate and fire neuron” every year





Other Neuronal Models

- Hodgkin–Huxley model
  - Proposed by Hodgkin and Huxley in 1952
    - They received the 1963 Nobel Prize in Physiology or Medicine for this work
  - The input current charges up the membrane capacitor
  - But there are other current influenced by the change in membrane potential

Hodgkin–Huxley model

$$I = C_m \frac{dV_m}{dt} + g_{\mathrm{K}} n^4 \left(V_m - V_{\mathrm{K}}\right) + g_{\mathrm{Na}} m^3 \left(V_m - V_{\mathrm{Na}}\right) + g_l \left(V_m - V_l\right)$$

$$\frac{dn}{dt} = \alpha_n \left(V_m\right)\left(1 - n\right) + \beta_n \left(V_m\right) n$$

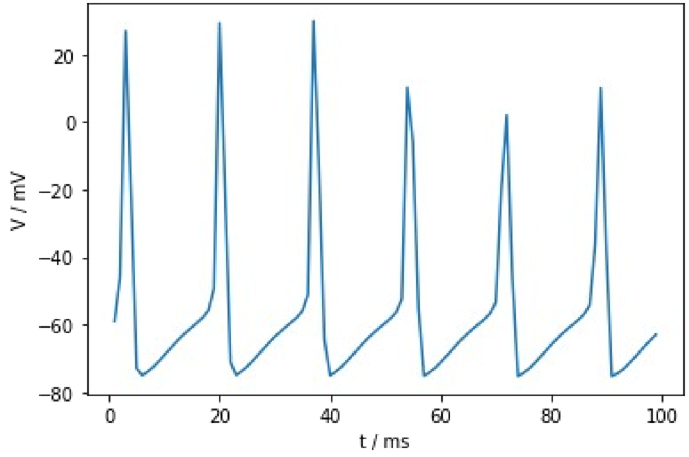
$$\frac{dm}{dt} = \alpha_m \left(V_m\right)\left(1 - m\right) + \beta_m \left(V_m\right) m$$

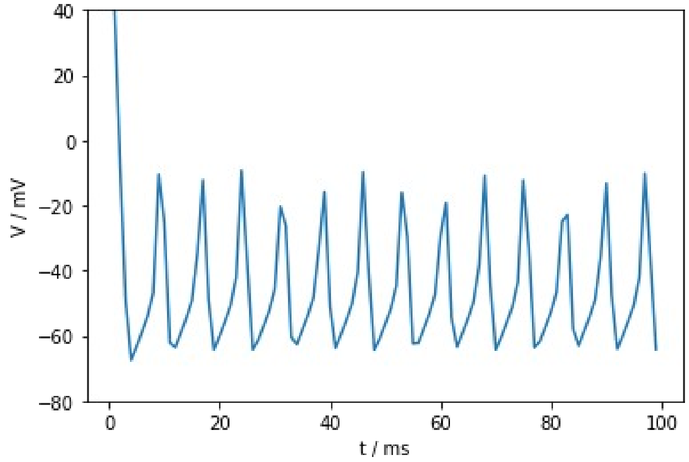
$$\frac{dh}{dt} = \alpha_h \left(V_m\right)\left(1 - h\right) + \beta_h \left(V_m\right) h$$

- There are other equations omitted here
- This model consider also ion currents involving potassium ion and calcium ion induced by the potential difference

Simulating HH Neuron

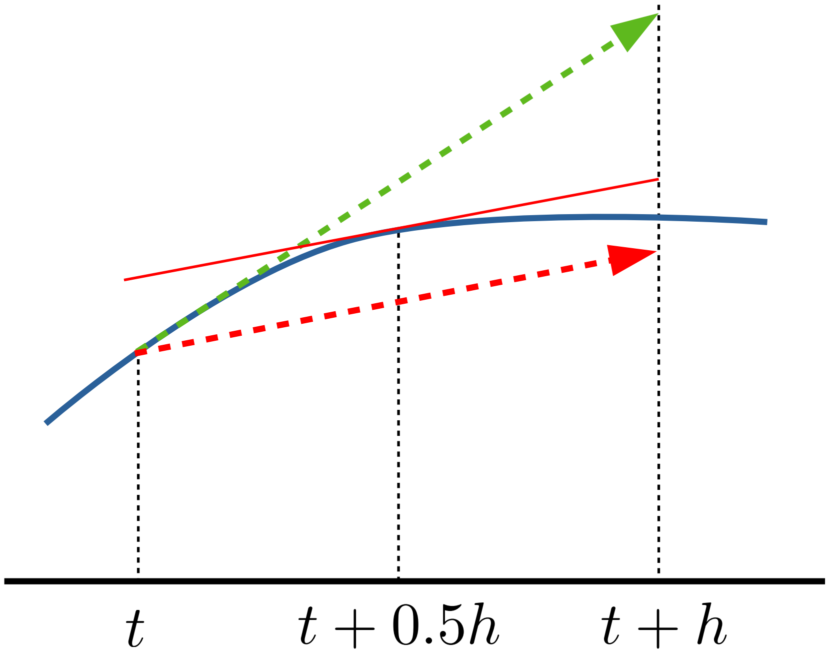
- HH Neurons show a more biological behavior
- But more complicated
  - Very difficult to implement in Excel
  - But you can do it easily in Nest
- Computationally Expensive





Improve Numerical Method

- Euler method has a relatively large error, is there alternatives ?
- There are extensions of Euler method
- The simplest one is the <u>mid-point method</u>



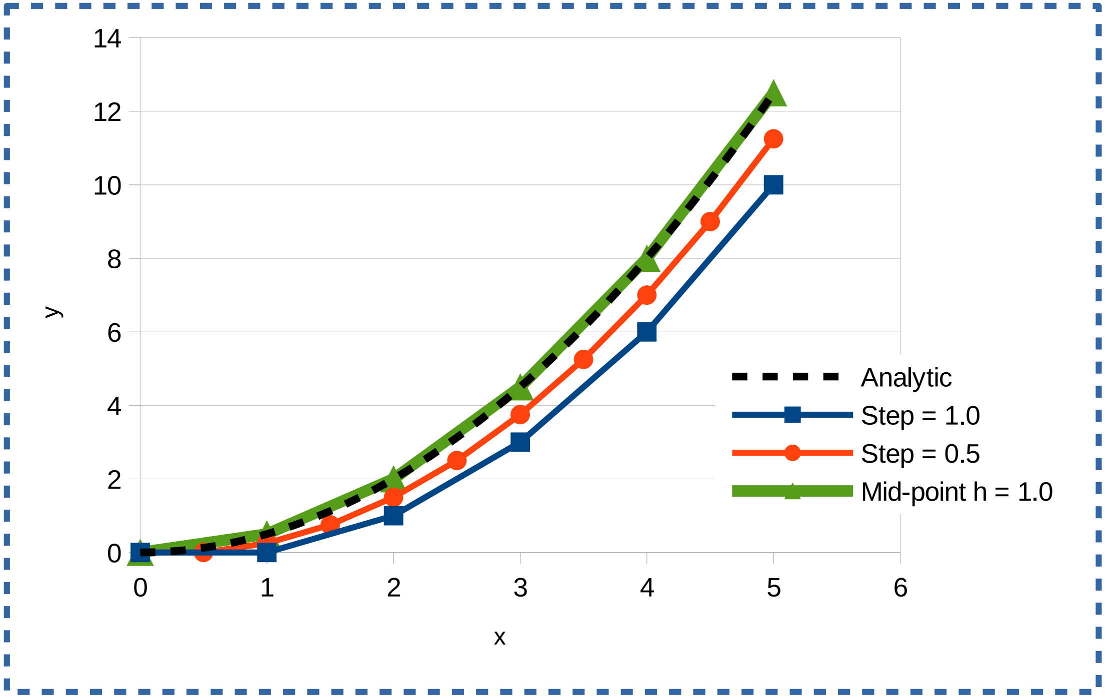
$$\frac{dx}{dt} = f(x,t)$$

$$x\left(t_{n+1}\right) = x\left(t_n\right) + hf\{x\left(t_n\right) + 0.5hf\left[x\left(t_n\right), t_n\right], t_n + 0.5h\}$$

Performance of Mid-point Method

- Can Euler method with a smaller step beats mid-point method?
- Mid-point method is surprisingly good

$$\frac{dy(x)}{dx} = x$$



Runge–Kutta methods

- Runge–Kutta methods is a family of method has the following form:

$$
\begin{aligned}
\frac{dx}{dt} &= f(t, x) \\
x_{n+1} &= x_n + h \sum_{i}^{s} b_i k_i \\
k_1 &= f(t_n, x_n) \\
k_2 &= f(t_n + c_2 h, x_n + h(a_{21} k_1)) \\
&\vdots \\
k_s &= f(t_n + c_s h, x_n + h(a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1}))
\end{aligned}
$$

No Worry. Use the Implementations Shipped with Libraries

Remarks of RK Methods

- Higher order integration takes care higher order curves
- Some algorithms enable error control
- Implementation is tedious
  - Use library to avoid mistake
- RK methods are not perfect
  - In some scenario, implicit method should be used (see Wikipedia)

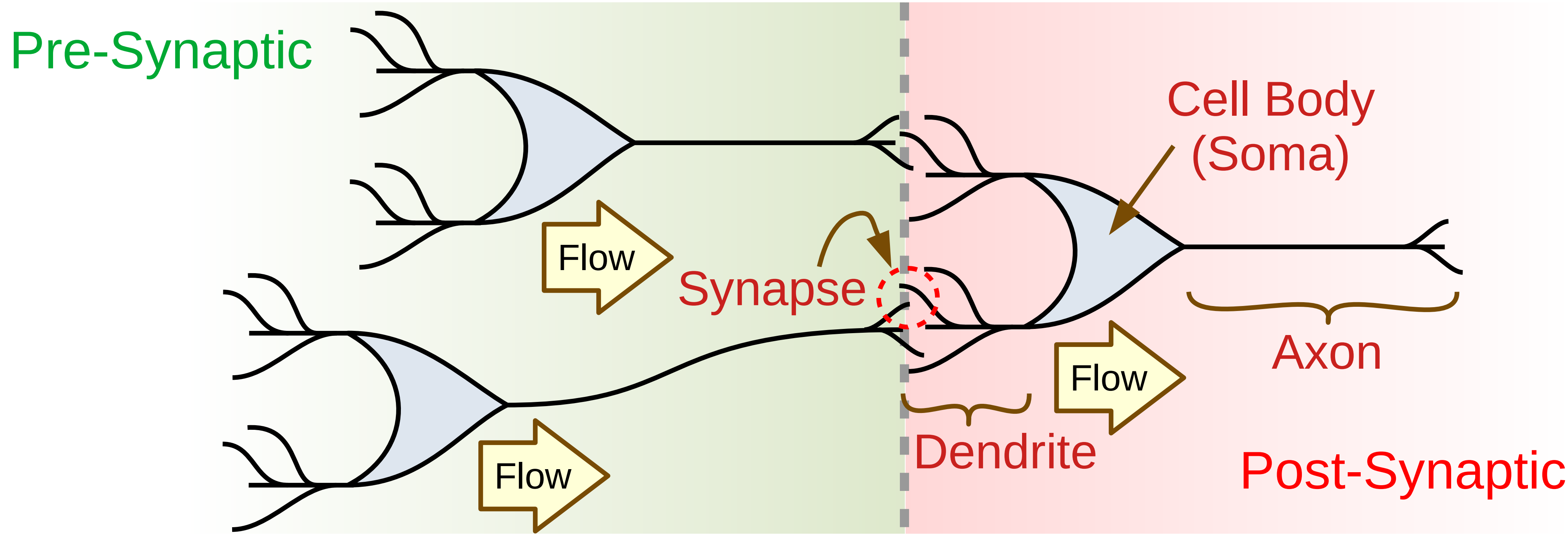
Summary 1

- Differential Equation
  - An important tool for Quantitative Science
- Numerical Method
  - Euler Method
  - Runge-Kutta Methods
- Neuronal Model
  - Leaky-Integrate-and-Fire Model
  - Hodgkin-Huxley model

Neuronal Computing

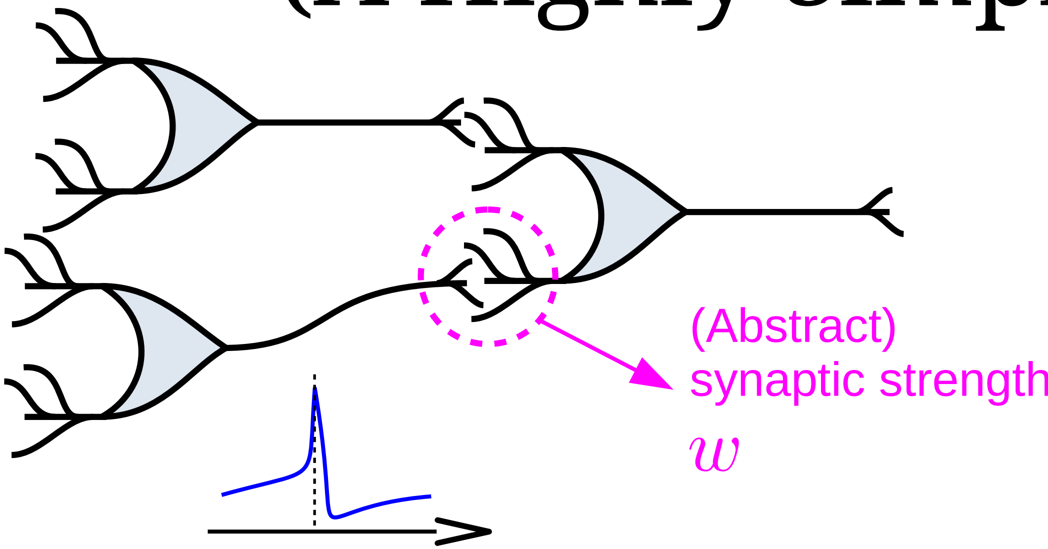
- Single neuron can not do much
  - More or less a sensor for input current
- Network of neurons can do more complicated computations
  - Connection between neuron are the key

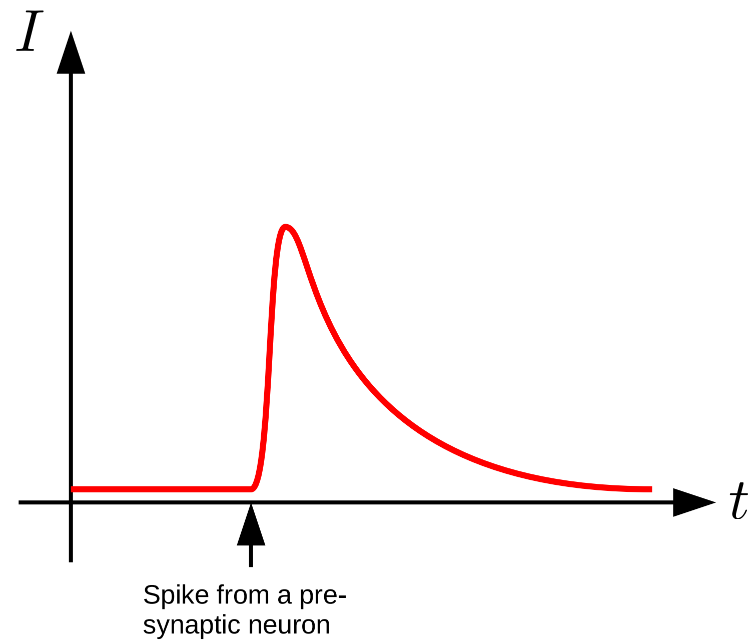
Network of Neurons

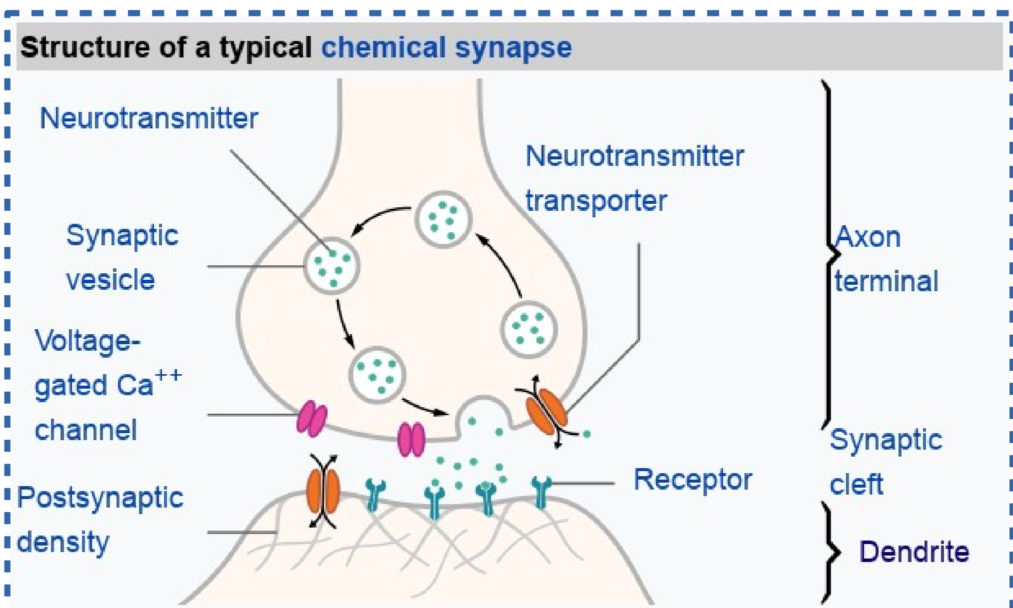


- Neurons integrate inputs from other neurons
  - Gathering information to “decide” firing or not firing
  - Connection between neurons is the key

How Synapse Work?
(A Highly Simplified Version)





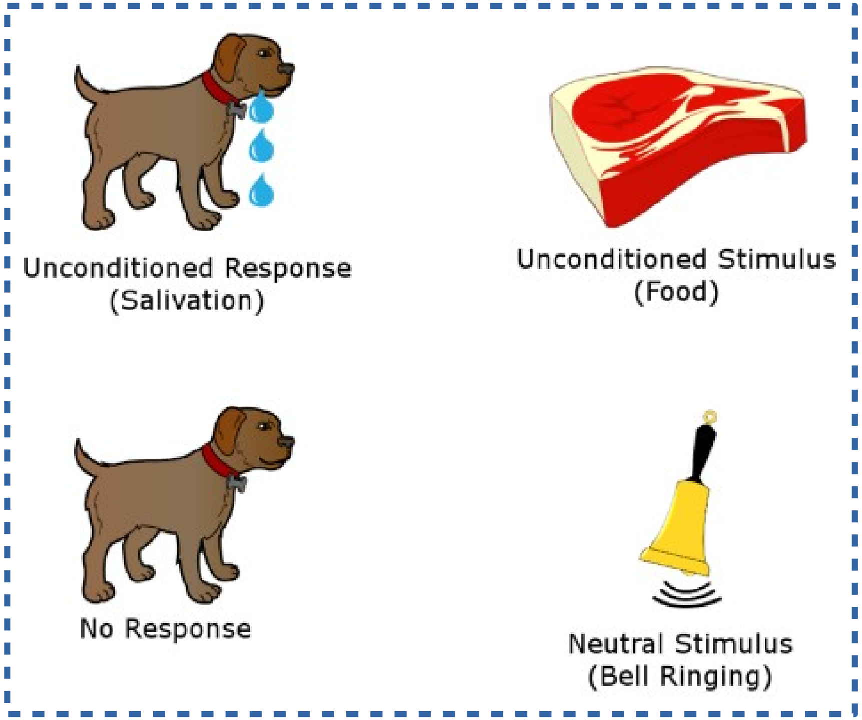


By Thomas Splettstoesser (www.scistyle.com) - Own work, CC BY-SA 4.0

$$\frac{dI}{dt} = -I(t) + w(t) \sum_{t_f} \delta(t - t_f)$$

Naive Analogy

- The Pavlov's experiment
- What if there is a small circuit in the dog's brain



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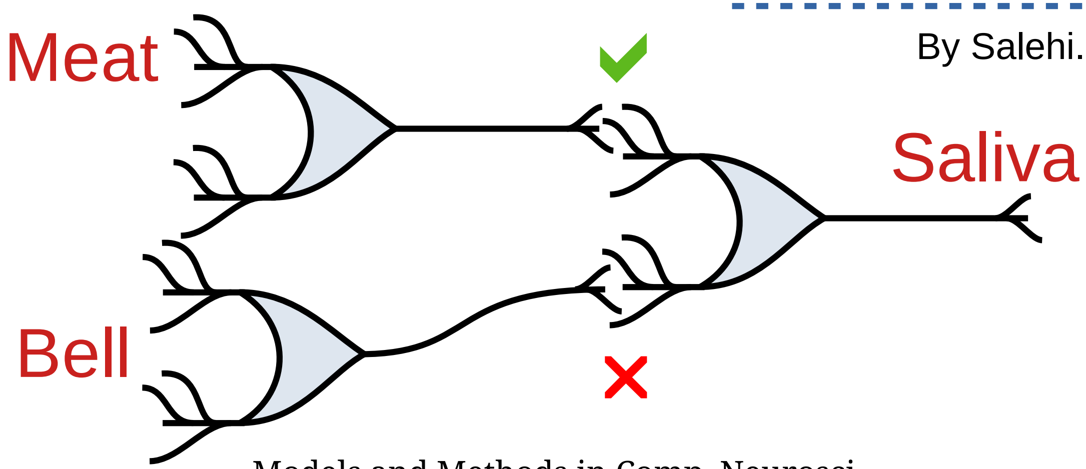
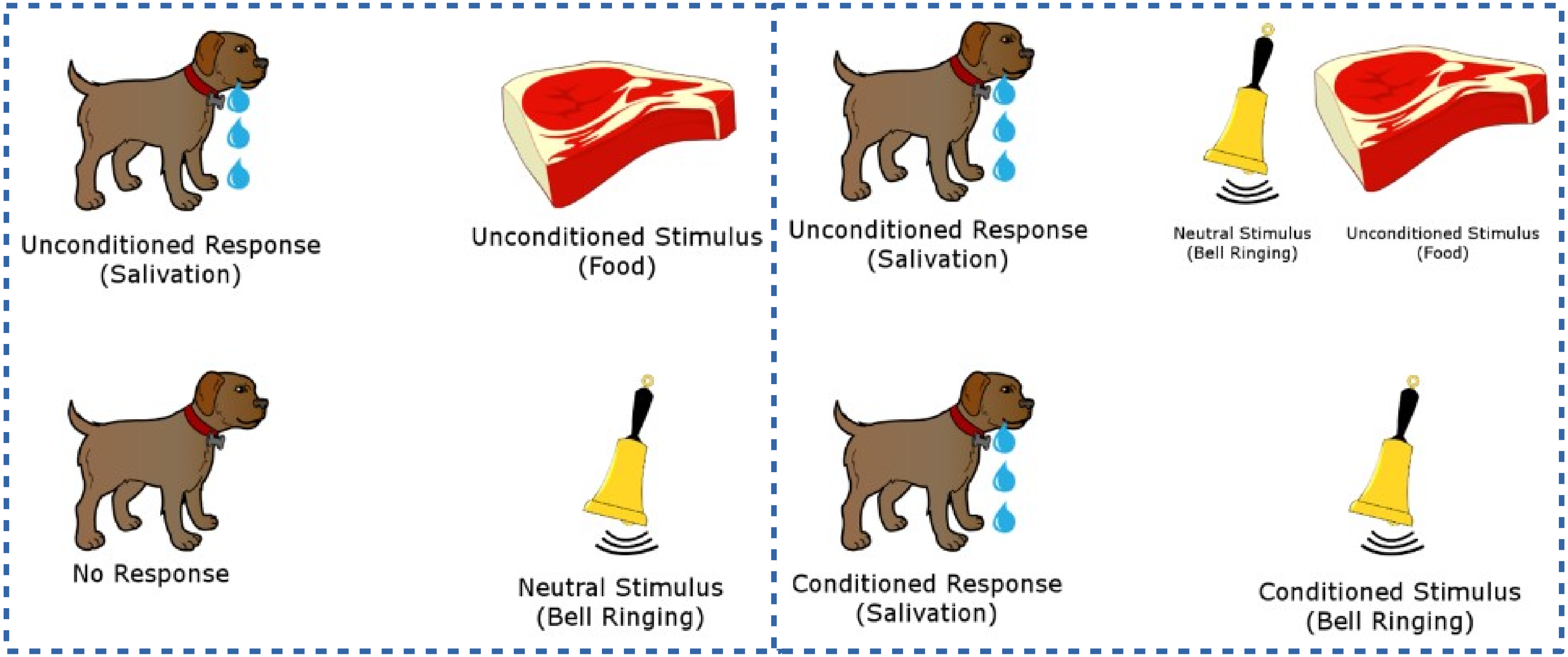
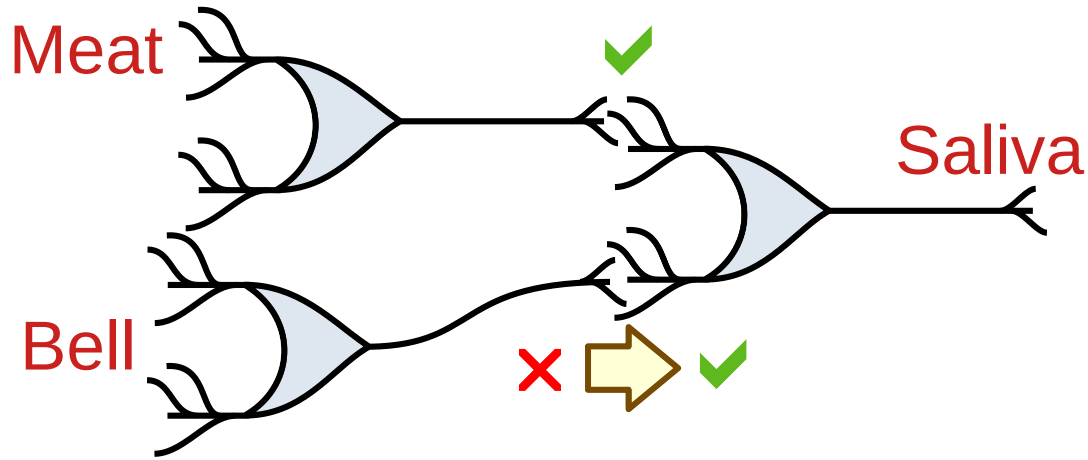


The Pavlov's Experiment

- But the bell condition can be trained

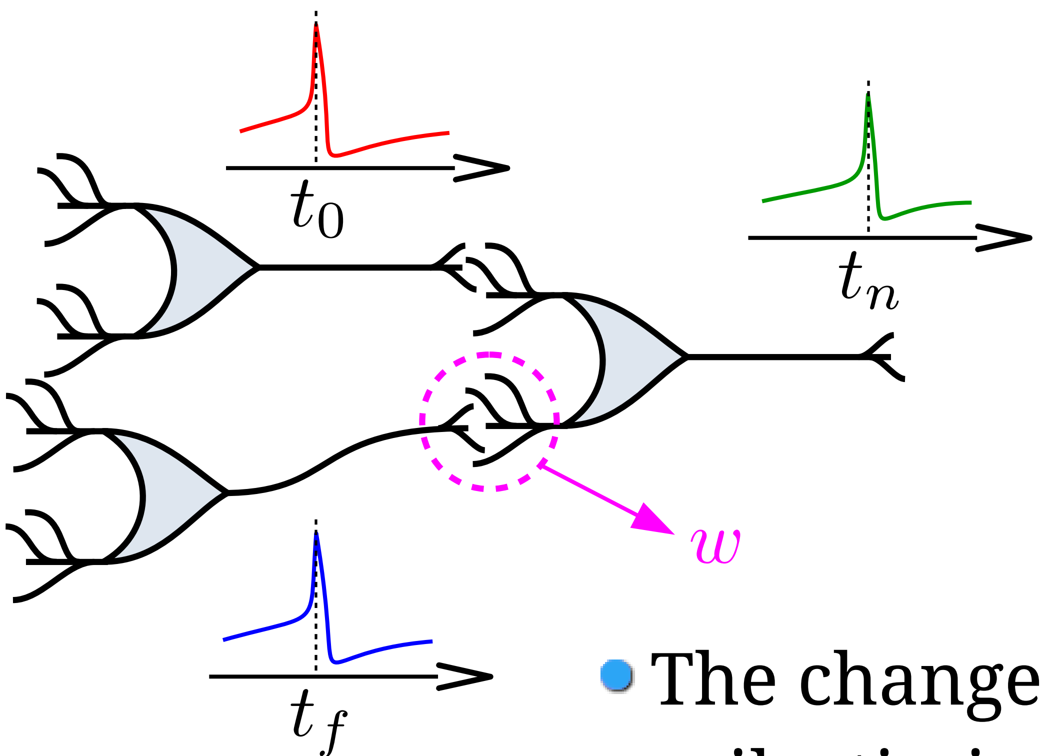


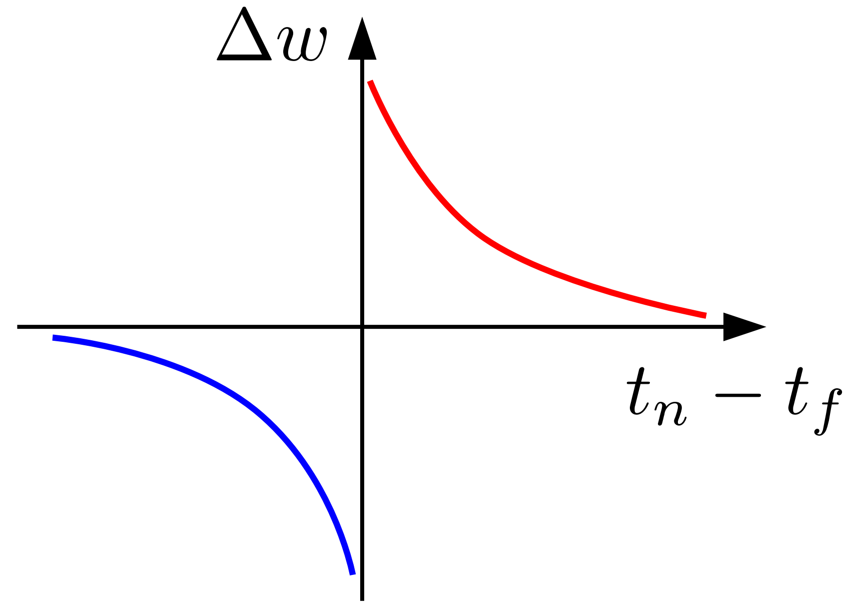
Neuronal Connections are Plastic



- In the naive example, the neuronal circuit should be updated
- In Neuroscience, this kind of update is called Long-term Plasticity
- One important process of plasticity is <u>Spike-Timing-Dependent Plasticity (STDP)</u>

Spike-Timing-Dependent Plasticity





- The change in w depends on the spike timings
- The change shown here is just a classical view

Online implementation of STDP models

- In order to perform STDP with differential equations

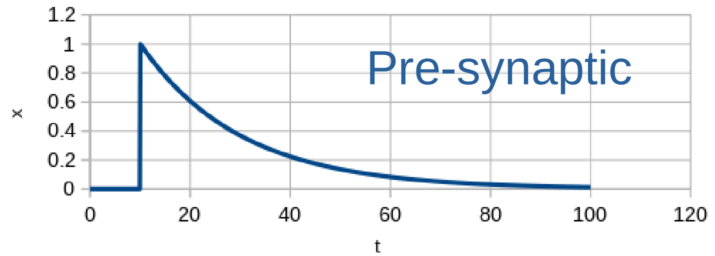
$$\left.\begin{aligned}\tau_+ \frac{dx}{dt} &= -x + a_+(x) \sum_{t_f} \delta(t - t_f) \\ \tau_- \frac{dy}{dt} &= -y + a_-(y) \sum_{t_n} \delta(t - t_n)\end{aligned}\right\} \text{To be explained in the following}$$

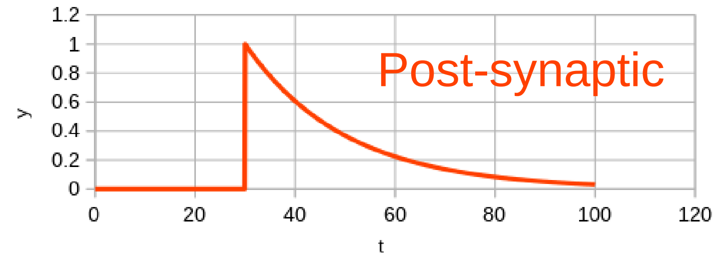
$$\frac{dw}{dt} = A_+(w)\, x(t) \sum_{t_n} \delta(t - t_n) - A_-(w)\, y(t) \sum_{t_f} \delta(t - t_f)$$

Simulation!

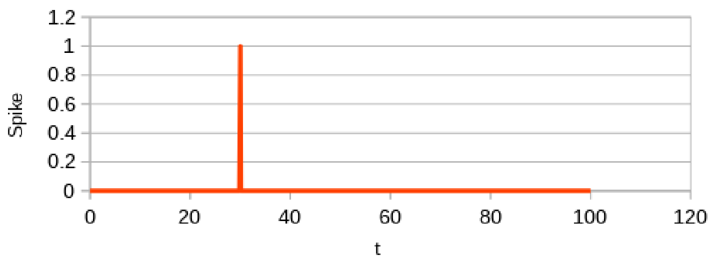
$$\tau_+ \frac{dx}{dt} = -x + a_+(x) \sum_{t_f} \delta(t - t_f)$$

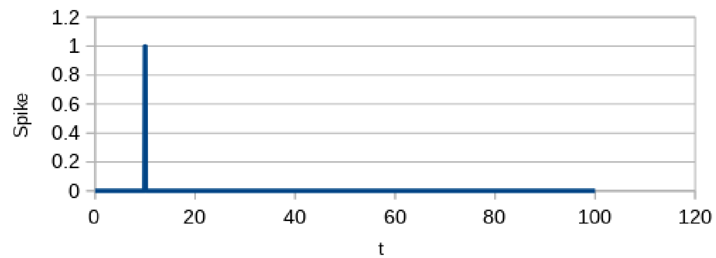
$$\tau_- \frac{dy}{dt} = -y + a_-(y) \sum_{t_n} \delta(t - t_n)$$



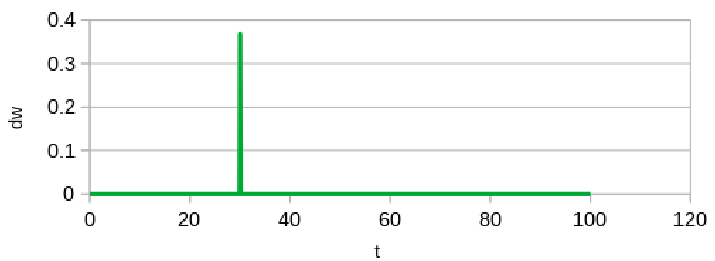


Both pre-synaptic spike and post-synaptic spike generate a decay curve in *x* and *y*



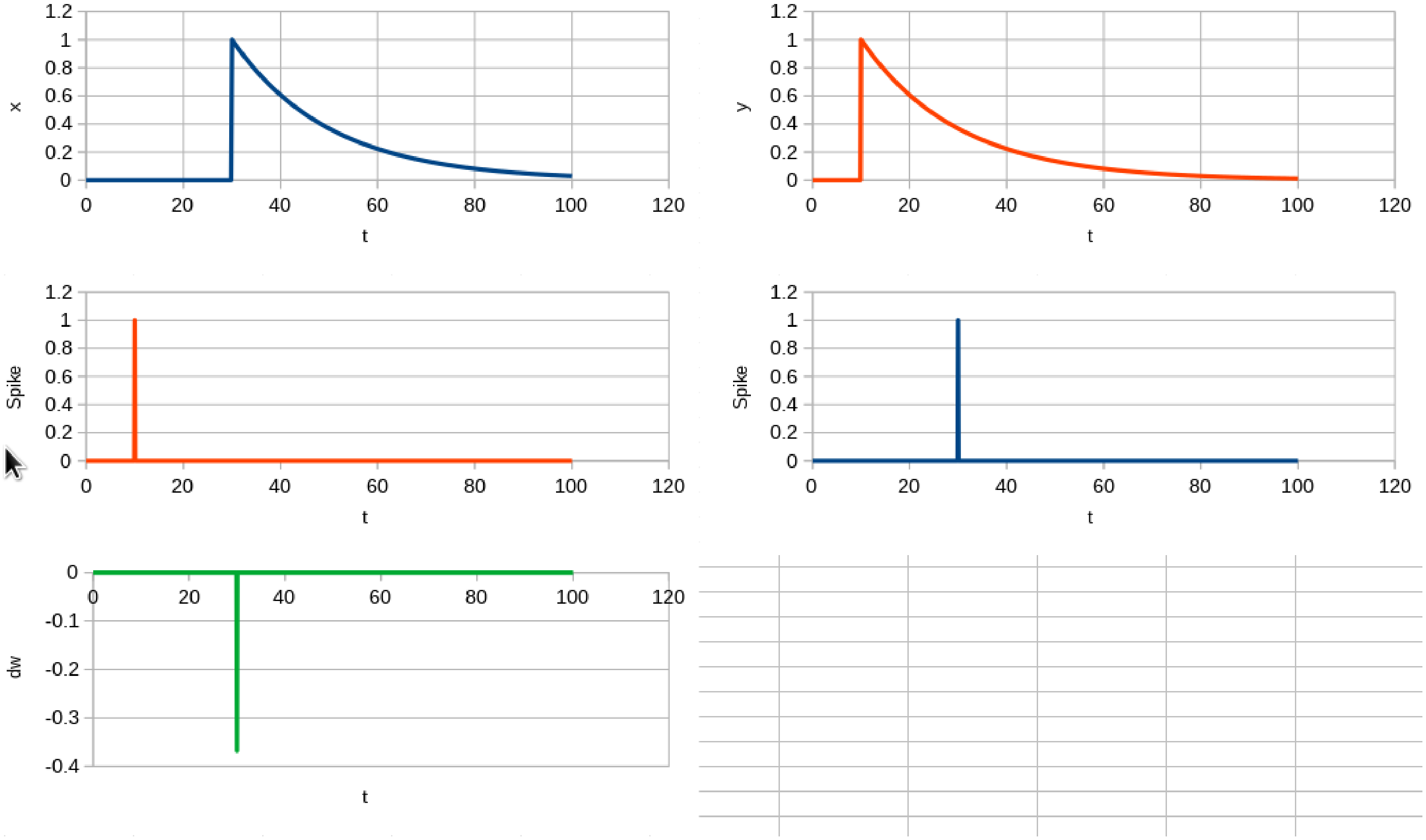


If x / y is compared with delta functions of corresponding to post-synaptic / pre-synaptic spike ...



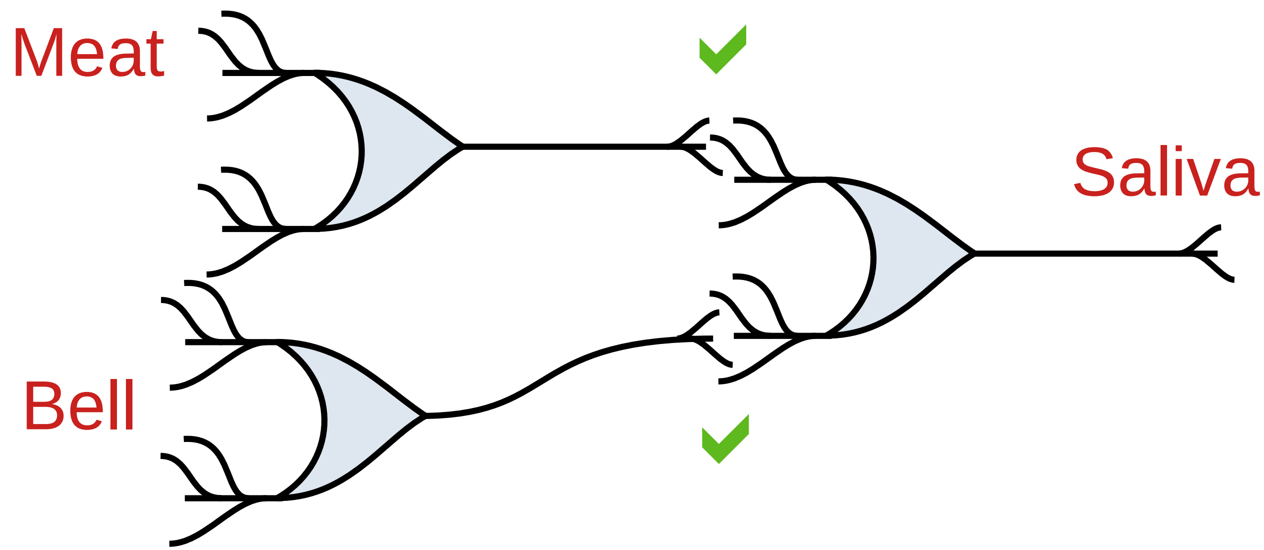
The change in w is then determined

If the Spike Order is Reversed, ...



Then, ...

- With appropriate firing sequence, the network can be trained,



Summary 2

- Network of neurons can perform more complicated computation
- Output of the computation can be trained
  - Output is determined by connections between neurons
- In real neuronal network, one important process changing connections between neurons is STDP
- STDP can be implemented using a group of differential equations

Take-Home Message

- Some neuronal behaviors can be mathematically modeled
- Network of neurons has a potential to do different computations
- Those network can be trained

Caution

- Don’t really do scientific simulation in spreadsheet

Thank you