

Algebraic Topology

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Course	MAT422 Algebraic Topology
Instructor	Dr. Viji Z. Thomas
Prerequisites	MAT312 Theory of Groups and Rings and MAT325 General Topology
Learning Outcomes	<ul style="list-style-type: none">• Understanding basic homotopy theory.• Familiarity with the language of categories to express various results in algebraic topology (in particular Van Kampen theorem).• Understanding the notions of simplicial and singular homologies, their homotopy invariances.• Understanding cohomology as a dual notion of homology.• Learning computational techniques for homologies and cohomologies and their applications.
Syllabus	<ul style="list-style-type: none">• Homotopy. Homotopy equivalence. Relative homotopy, Paths. Fundamental group. Induced homomorphism, Fundamental group of a product, Fundamental group of the circle, Homotopy lifting property. (7)• Some basic category theory (upto Natural transformations and push forward), Van Kampen theorem. (6)• Existence of covering spaces, and classification of covering spaces. (3)• Deck Transformations and Group actions, simplicial homology, singular homology, Homotopy invariance. (9)• Relative and reduced homology, long exact sequence of a pair. (3)• Mayer-Vietoris, Applications of Mayer Vietoris, Homology with coefficients etc. (4.5)• Cohomology, cup-product, Poincare Duality. (7.5)

Proposition 1:

Let (X_i, \mathcal{T}_i) be a collection of topological spaces, and Y be any set.

Let $f_i : X_i \rightarrow Y$ be a collection of functions.

Define $\mathcal{T} := \{U \subseteq Y \mid f_i^{-1}(U) \in \mathcal{T}_i \forall i\}$, then

1. \mathcal{T} is a topology on Y .
2. \mathcal{T} is the largest (finest) topology on Y such that each f_i is continuous.
3. If Z is any topological space and if $g : Y \rightarrow Z$ is any function, then g is continuous if and only if each $g \circ f_i$ is continuous.

Proof.

1. Trivial.
2. Each f_i is clearly continuous by the definition of continuity.
If \mathcal{T}' is any topology on Y such that each f_i is continuous, then for any $U \in \mathcal{T}'$, it follows that $f_i^{-1}(U) \in \mathcal{T}_i$ for each i , hence $U \in \mathcal{T}$ by definition, hence $\mathcal{T}' \subseteq \mathcal{T}$.
3. If g is continuous, then clearly $g \circ f_i$ is continuous as it is a composition of two continuous functions.
Conversely, if $g \circ f_i$ is continuous for each i , then for any open subset W of Z ,
 $(g \circ f_i)^{-1}(W) = f_i^{-1}(g^{-1}(W))$ is open in X_i for each i .
Hence $f_i^{-1}(g^{-1}(W)) \in \mathcal{T}_i$ for each i , hence $g^{-1}(W) \in \mathcal{T}$.
Therefore g is continuous.

□

Definition 1 (Final Topology):

Let (Y_i, \mathcal{T}_i) be a collection of topological spaces, and let (X, \mathcal{T}) be a topological space.

Let $f_i : Y_i \rightarrow X$ be a collection of functions. \mathcal{T} is said to be final topology on X with respect to $\{f_i\}$ if \mathcal{T} is the finest topology on X with respect to which each f_i is continuous. That is,

- \mathcal{T} makes each f_i continuous.
- If \mathcal{T}' is any other topology on X that makes each f_i continuous, then $\mathcal{T}' \subseteq \mathcal{T}$

Theorem 1 (Universal Property of Final Topology):

Let $(X, \mathcal{T}), (Y_i, \mathcal{T}_i)$ be a collection of topological spaces, and let $f_i : Y_i \rightarrow X$ be a collection of functions. Then

1. Let (Z, \mathcal{U}) be a topological space, and let $g : X \rightarrow Z$ be a function. If \mathcal{T} is the final topology on X with respect to $\{f_i\}$, then g is continuous if and only if $g \circ f_i$ is continuous for each i .
2. If for each topological space (Z, \mathcal{U}) and each function $g : X \rightarrow Z$, it holds that g is continuous if and only if $g \circ f_i$ is continuous for each i , then \mathcal{T} has to be the final topology on X with respect to $\{f_i\}$.

Proof.

□

Definition 2 (Quotient Topology):

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces, a function $f : X \rightarrow Y$ is said to be a quotient map if f is surjective and \mathcal{U} is the largest topology on Y such that f is continuous. Then \mathcal{U} is said to be the quotient topology on Y with respect to f .

Remark: Quotient topology is the final topology with respect to a single function f .

If (X, \mathcal{T}_X) is a topological space and Y is a set, and $f : X \rightarrow Y$ is a surjective function then the quotient topology on Y with respect to f is defined as

$$\mathcal{T}_Y := \{U \subseteq Y \mid f^{-1}(U) \in \mathcal{T}_X\}$$

Theorem 2:

Let $f(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ be a continuous open surjection, then \mathcal{T}_Y is the quotient topology on Y with respect to f .

Theorem 3:

Let $f(X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ be a continuous closed surjection, then \mathcal{T}_Y is the quotient topology on Y with respect to f .

References

- [1] J. R. Munkres. *Topology*. 2nd ed. Prentice Hall, 2000.
- [2] E. H. Spanier. *Algebraic Topology*. 1st ed. Springer, 1966.