Algebraic Topology

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Course	MAT422 Algebraic Topology
Instructor	Dr. Viji Z. Thomas
Prerequisites	MAT312 Theory of Groups and Rings and MAT325 General Topology
Learning Outcomes	• Understanding basic homotopy theory. • Familiarity with the language of categories to express various results in algebraic topology (in particular Van Kampen theorem). • Understanding the notions of simplicial and singular homologies, their homotopy invariances. • Understanding cohomology as a dual notion of homology. • Learning computational techniques for homologies and cohomologies and their applications.
Syllabus	• Homotopy. Homotopy equivalence. Relative homotopy, Paths. Fundamental group. Induced homomorphism, Fundamental group of a product, Fundamental group of the circle, Homotopy lifting property. (7) • Some basic category theory (upto Natural transformations and push forward), Van Kampen theorem. (6) • Existence of covering spaces, and classification of covering spaces. (3) • Deck Transformations and Group actions, simplicial homology, singular homology, Homotopy invariance. (9) • Relative and reduced homology, long exact sequence of a pair. (3) • Mayer-Vietoris, Applications of Mayer Vietoris, Homology with coefficients etc. (4.5) • Cohomology, cup-product, Poincare Duality. (7.5)

Lecture 01: Final Topology, and its Universal Property — 04 Jan 2024 10:00

***Proposition* 1:**
Let $(X_i, \mathcal{T}_i)$ be a collection of topological spaces, and Y be any set.
Let $f_i : X_i \to Y$ be a collection of functions.
Define $\mathcal{T} := \{U \subseteq Y \mid f_i^{-1}(U) \in \mathcal{T}_i \ \forall i\}$, then

1. $\mathcal{T}$ is a topolgy on Y.
2. $\mathcal{T}$ is the largest (finest) topology on Y such that each f_i is continuous.
3. If Z is any topological space and if $g : Y \to Z$ is any function, then g is continuous if and only if each $g \circ f_i$ is continuous.

Proof.

1. Trivial.
2. Each f_i is clearly continuous by the definitoin of continuity.
   If $\mathcal{T}$' is any topology on Y such that each f_i is continuous, then for any $U \in \mathcal{T}'$, it follows that $f_i^{-1}(U) \in \mathcal{T}_i$ for each i, hence $U \in \mathcal{T}$by definition, hence $\mathcal{T}' \subseteq \mathcal{T}$.
3. If g is continuous, then clearly $g \circ f_i$ is continuous as it is a composition of two continuous functions.
   Conversely, if $g \circ f_i$ is continous for each i, then for any open subset W of Z,
   $(g \circ f_i)^{-1} = f_i^{-1} \circ g^{-1}(W)$ is open in X_i for each i.
   Hence $f_i^{-1}(g^{-1}(W)) \in \mathcal{T}_i$ for each i, hence $g^{-1}(W) \in \mathcal{T}$.
   Therefore g is continuous.

□

***Definition* 1** (Final Topology)**:**
Let $(Y_i, \mathcal{T}_i)$ be a collection of topological space, and let $(X, \mathcal{T})$ be a topological space.
Let $f_i : Y_i \to X$ be a collection of functions. $\mathcal{T}$ is said to be final topology on X with respect to $\{f_i\}$ if $\mathcal{T}$ is the finest topology on X with respect to which each f_i is continous. That is,

- $\mathcal{T}$ makes each f_i continuous.
- If $\mathcal{T}'$ is any other topology on X that makes each f_i continuous, then $\mathcal{T}' \subseteq \mathcal{T}$

***Theorem* 1** (Universal Property of Final Topology)**:**
Let $(X, \mathcal{T}), (Y_i, \mathcal{T}_i)$ be a collection of topological spaces, and let $f_i : Y_i \to X$ be a collection of functions. Then

1. Let $(Z, \mathcal{U})$ be a topological space, and let $g : X \to Z$ be a function. If $\mathcal{T}$ is the final topology on X with respect to $\{f_i\}$, then g is continous if and only if $g \circ f_i$ is continuous for each i.
2. If for each topological space $(Z, \mathcal{U})$ and each function $g : X \to Z$, it holds that g is continuous if and only if $g \circ f_i$ is continous for each i, then $\mathcal{T}$ has to be the final topology on X with respect to $\{f_i\}$.

Proof.

1. Already proven before.
2. Take $Y = X$ and g to be the identity map, then each f_i is continuous. Let $\mathcal{T}$ be any topology on X such that each f_i is continuous, take $g(x) = x$, then

□

***Definition* 2** (Quotient Topology)**:**
Let $(X, \mathcal{T}$ and $(Y, \mathcal{U})$ be two topological spaces, a function $f : X \to Y$ is said to be a quotient map if f is surjective and $\mathcal{U}$ is the largest topology on Y such that f is continuous. Then $\mathcal{U}$ is said to be the quotient topology on Y with respect to f.

Remark: Quotient topology is the final topology with respect to a single function f.
If $(X, \mathcal{T}_X)$ is a topological space and Y is a set, and $f : X \to Y$ is a surjective function then the quotient topology on Y with respect to x is defined as

$$\mathcal{T}_Y := \{U \subseteq Y \mid f^{-1}(U) \in \mathcal{T}_X\}$$

Theorem 2:
Let $f(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous open surjection, then $\mathcal{T}_Y$ is the quotient topology on Y with respect to f.

Theorem 3:
Let $f(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous closed surjection, then $\mathcal{T}_Y$ is the quotient topology on Y with respect to f.

Lecture 02: Basics of Homotopy Theory — 09 Jan 2024 10:00

Definition* 3** (Sum Topology):***
Let $\{X_i, \mathcal{T}_i \mid i \in I\}$ be a collection of topological spaces that are pairwise disjoint. Then consider $X = \sqcup_{i \in I} X_i$. For each $i \in I$, we have the inclusion $j_i : X_i \to X$. The final topology on X with respect to the set of functions $\{j_i \mid i \in I\}$ is called as the sum topology on X.

Note: Notice that $X_i \subseteq X$ are both open and closed in X.

Definition* 4** (Quotient Topology with respect to an Equivalence Relation):***
Let X be a topological space. Let $\sim$ be an equivalence relation on X. Let

$$Y = \frac{X}{\sim} := \{[x] \mid x \in X\}$$

be the quotient set (collection of all equivalence classes). Then there exists the natural canonical surjction

$$q : X \to \frac{X}{\sim} \text{ given by } q(x) = [x] \; \forall x \in X.$$

Then the final topoology on Y with respect to the funcion q is said to be the quotient topology on Y. That is, $U \subseteq Y$ is open if and only if $q^{-1}(Y)$ is open in X.

References

[1] J. R. Munkres. *Topology*. 2nd ed. Prentice Hall, 2000.

[2] E. H. Spanier. *Algebraic Topology*. 1st ed. Springer, 1966.