

# Functional Analysis

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Course	MSM421 Functional Analysis
Instructor	Prof. Rajeev Bhaskaran
Prerequisites	MSM321 Complex Analysis and MSM411 Measure Theory
Learning Outcomes	Based on core analysis courses and linear algebra, this course builds further on the study of Banach and Hilbert spaces. The theory and techniques studied in this course support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas
Syllabus	<ul style="list-style-type: none"><li>• Normed linear spaces, Riesz lemma, characterization of finite dimensional spaces, Banach spaces. Operator norm, continuity and boundedness of linear maps on a normed linear space. (6)</li><li>• Fundamental theorems: Hahn-Banach theorems, uniform boundedness principle, divergence of Fourier series, closed graph theorem, open mapping theorem and some applications. (8)</li><li>• Dual spaces and adjoint of an operator: Duals of classical spaces, weak and weak* convergence, adjoint of an operator. (6)</li><li>• Hilbert spaces: Inner product spaces, orthonormal set, Gram-Schmidt orthonormalization, Bessel's inequality, orthonormal basis, separable Hilbert spaces. Projection and Riesz representation theorems: Orthonormal complements, orthogonal projections, projection theorem, Riesz representation theorem. (10)</li><li>• Bounded operators on Hilbert spaces: Adjoint, normal, unitary, self-adjoint operators, compact operators. (5)</li><li>• Spectral theorem: Spectral theorem for compact self adjoint operators, statement of spectral theorem for bounded self adjoint operators. (5)</li></ul>

**Recall** (Linear Algebra):

- Vector spaces, the axioms, and examples.
- Linear dependence of subsets of a vector spaces (both finite and infinite), basis is a maximal linear independent subset of a vector space.
- Every vector space has a basis (Zorn's Lemma), every basis of a vector space has the same cardinality, hence, dimension of a vector space defined as the cardinality of any of its basis is well defined.

## Lecture 02: Introduction to Banach Spaces

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**Definition 1** (Normed Linear Space):

Consider a vector space  $V$  over a field  $\mathbb{F}$  (either  $\mathbb{R}$  or  $\mathbb{C}$ ). A function  $\|\cdot\| : V \rightarrow [0, \infty)$  is said to be a norm if it satisfies the following axioms:

- N1.  $\|x\| = 0$  if and only if  $x = 0$ .
- N2.  $\|\alpha x\| = |\alpha| \|x\|$  for all  $\alpha \in \mathbb{F}$  and  $x \in V$ .
- N3.  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in V$ .

Then the vector space equipped with such a norm is called as a normed linear space.

**Motivation:** When a norm is defined on a vector space, it induces a topology (metric topology) on the vector space. With a topological structure on  $V$ , we can do further analysis in  $V$  such as limits of sequences, convergence, continuity of functions, etcetera.

**Remark:** If  $(V, \|\cdot\|)$  is a normed linear space, then  $d(x, y) = \|x - y\| : V \times V \rightarrow [0, \infty)$  is a metric on  $V$ :

- M1.  $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$
- M2.  $d(x, y) = \|-1\| \|x - y\| = \|y - x\| = d(y, x)$
- M3.  $d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

Equipped with this metric,  $(V, d)$  is a metric space, and we can define an open subset of  $V$  as follows:

$U \subseteq V$  is said to be open if for all  $x$  in  $U$ , there exists an  $r > 0$  such that

$$B(x, r) := \{y \in V \mid d(x, y) < r\} \subseteq U$$

The collection of these open subsets defines a topology on  $V$ .

**Recall:**

- In a normed linear space  $(V, \|\cdot\|)$ , if a sequence  $x_n$  converges to  $x$  and another sequence  $y_n$  converges to  $y$ , then the sequence  $x_n + y_n$  converges to  $x + y$ .
- If  $\alpha_n$  is a sequence in  $\mathbb{F}$  that converges to  $\alpha$  and  $x_n$  is a sequence in  $V$  that converges to  $x$ , then the sequence  $\alpha_n x_n$  converges to  $\alpha x$ .

- Therefore, addition and scalar multiplication are continuous maps in the topological spaces induced by a norm.

**Motivation:** Now that we have a metric space structure on  $V$ , we can talk about completeness (every Cauchy sequence converges).

**Definition 2** (Banach Spaces):

A normed linear space  $(V, \|\cdot\|)$  is said to be a Banach Space if and only if  $(V, d)$  is a complete metric space. That is, every Cauchy sequence in  $V$  converges in the metric space  $(V, d)$ .

**Recall:**

- **Hölder's inequality:**

Let  $p, p^* \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{p^*} = 1$  (such a pair of real numbers is called **conjugate exponents** of each other). For a  $p \in [1, \infty)$ , define a function from  $\mathbb{R}^N$  to  $\mathbb{R}$  as:

$$\|x\|_p = \left( \sum_{i=1}^N |x_i|^p \right)^{\frac{1}{p}}$$

and for  $p = \infty$ , define another a function from  $\mathbb{R}^N$  to  $\mathbb{R}$  as:

$$\|x\|_\infty = \max_{1 \leq i \leq N} (x_i)$$

Let  $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in \mathbb{R}^N$ .

Then the following inequality holds:

$$\sum_{i=1}^N |x_i y_i| \leq \|x\|_p \|y\|_{p^*}$$

- **Minkowski's inequality:**

For any  $p \in [1, \infty]$ , and  $x, y \in \mathbb{R}^N$ , we have the following inequality:

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

- **Jensen's Inequality:**

For any pair of real numbers  $x, y \in \mathbb{R}$ , the following inequality holds:

$$|a + b|^p \leq 2^{p-1}(a^p + b^p)$$

**Example** ( $p$ -norm on  $\mathbb{R}^N$ ): For any  $p \in [1, \infty]$ ,  $(\mathbb{R}^N, \|\cdot\|_p)$  is a Banach space.

*Proof.* First we need to prove that  $\|\cdot\|_p$  is a norm on  $\mathbb{R}^N$ .

This follows from the [Minkowski's inequality](#).

**Note:** This is usually called as  $p$ -norm on  $\mathbb{R}^N$ , and the special case when  $p = 2$  is called the Euclidean norm.

Then we need to prove the completeness of  $(\mathbb{R}^N, \|\cdot\|_p)$ . First we prove for  $p \in [1, \infty)$  case.

Suppose  $x^{(n)}$  is a Cauchy sequence in the NLS. Then for each  $\epsilon > 0$ , there exists a positive integer  $t$  such that  $\|x^{(n)} - x^{(m)}\|_p < \epsilon$  whenever  $m, n > t$ . That is,  $m, n > t$  implies that

$$\sum_{i=1}^N \left| x_i^{(n)} - x_i^{(m)} \right|^p < \epsilon^p$$

Hence  $\left| x_i^{(n)} - x_i^{(m)} \right| < \epsilon$  whenever  $m, n > t$ . That means  $x_i^{(n)}$  is a Cauchy sequence in  $\mathbb{R}$ , hence converges (since  $\mathbb{R}$  is complete) to a point, say  $x_i \in \mathbb{R}$ .

Let  $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ , where each  $x_i$  are defined as above. We claim that  $x^{(n)}$  converges to  $x$ .

We have the inequality  $|x_i^{(n)} - x_i^{(m)}| < \epsilon^p$  whenever  $m, n > t$ .

Fix  $n$ , and take the limit as  $m \rightarrow \infty$ , we obtain  $|x_i^{(n)} - x_i| \leq \epsilon^p$ .

Therefore, after an appropriate adjustment, we obtain an  $\epsilon'$ , such that for any  $\epsilon > 0$ , there exists a positive integer  $t$  such that  $n > t$  implies that the finite sum satisfies the inequality

$$\left( \sum_{i=1}^N |x_i^{(n)} - x_i|^p \right)^{\frac{1}{p}} = \|x^{(n)} - x\|_p < \epsilon'$$

Hence  $x^{(n)}$  converges to  $x$ . Therefore, every Cauchy sequence in  $(\mathbb{R}^N, \|\cdot\|_p)$  converges, hence it is a Banach space when  $p \in [1, \infty)$ .

When  $p = \infty$ , as convergence and Cauchy criterion hold if and only if they hold componentwise in  $\mathbb{R}^N$ ,  $(\mathbb{R}^N, \|\cdot\|_\infty)$  is Banach as well.  $\square$

**Note:** Due to completeness of  $\mathbb{C}$ , for any  $p \in [1, \infty)$ ,  $(\mathbb{C}, \|\cdot\|_p)$  is a Banach space as well.

**Example ( $\ell_p$  spaces):** For  $1 \leq p < \infty$ , define a collection of sequences as follows:

$$\ell_p := \left\{ x = (x_1, x_2, \dots) \mid x_i \in \mathbb{F}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

Now, define a function from  $\ell_p$  to  $\mathbb{F}$  as  $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$ . Then  $(\ell_p, \|\cdot\|_p)$  is a Banach space.

*Proof.* The proof proceeds in several steps:

1.  $\ell_p$  is a vector space.

*Proof.* Let  $x, y \in \ell_p$ , then from the Jensen's inequality, we have

$$|x_i + y_i|^p \leq (|x_i| + |y_i|)^p \leq 2^{p-1}(|x_i|^p + |y_i|^p) < \infty$$

and for any  $\alpha \in \mathbb{F}$  and  $x \in \ell_p$ ,  $\alpha x$  clearly belongs to  $\ell_p$ .

Hence  $\ell_p$  is a subspace of the space of all sequences in  $\mathbb{F}$ .  $\square$

2.  $\|\cdot\|_p$  is a norm on  $\ell_p$ .

*Proof.* The first two axioms of a norm are trivial to prove. We need to show the triangle inequality.

Let  $x, y \in \ell_p$ , then for any positive integer  $N$ , from [Minkowski's Inequality](#), we have

$$\sum_{i=1}^N |x_i + y_i|^p \leq \left[ \left( \sum_{i=1}^N |x_i|^p \right) + \left( \sum_{i=1}^N |y_i|^p \right) \right]^p \leq [\|x\|_p^p + \|y\|_p^p]^p < \infty$$

Since  $N$  is arbitrary, and the right side is independent of the choice of  $N$ , it follows that

$$\sum_{i=1}^{\infty} |x_i + y_i|^p \leq [\|x\|_p^p + \|y\|_p^p]^p < \infty$$

Hence the triangle inequality holds, and hence  $\|\cdot\|_p$  is a norm on  $\ell_p$ .  $\square$

3.  $(\ell_p, \|\cdot\|_p)$  is a complete space.

*Proof.* Let  $x^{(n)}$  be a Cauchy sequence in the NLS, that is, for each  $\epsilon > 0$ , there exists a positive integer  $t$  such that  $\|x^{(n)} - x^{(m)}\|_p < \epsilon$  whenever  $m, n > t$ . That is,  $m, n > t$  implies that

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}| < \epsilon^p$$

Hence  $|x_i^{(n)} - x_i^{(m)}| < \epsilon$  whenever  $m, n > t$ , hence  $x_i^{(n)}$  is a Cauchy sequence in  $\mathbb{R}$ , hence converges (since  $\mathbb{R}$  is complete) to a point, say  $x_i \in \mathbb{R}$ .

Construct a sequence  $x = (x_1, x_2, \dots, x_n, \dots)$  where  $x_i$  is defined as above.

We need to show that  $x \in \ell_p$ . Since  $x^{(n)}$  is a Cauchy sequence, it is bounded, hence there exists a constant  $C > 0$  such that

$$\|x^{(n)}\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)}|^p \leq C, \quad \forall n.$$

Let  $k$  be any fixed positive integer. Then,

$$\sum_{i=1}^k |x_i^{(n)}|^p \leq C$$

which implies that

$$\sum_{i=1}^k |x_i|^p \leq C.$$

Since  $k$  is arbitrary and the right side does not depend on it, it follows that

$$\sum_{i=1}^{\infty} |x_i|^p \leq C < \infty.$$

Hence  $x \in \ell_p$ .

Now we need to show that  $x^{(n)}$  converges to  $x$ . From before, whenever  $m, n > t$ , we have

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Hence, for any positive integer  $k$ , and whenever  $m, n > t$ , we have

$$\sum_{i=1}^k |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Fix  $n > t$ , and take the limit  $m \rightarrow \infty$ , we obtain

$$\sum_{i=1}^k |x_i^{(n)} - x_i|^p < \epsilon^p.$$

Since  $k$  is arbitrary and the right side does not depend on it, hence whenever  $n > t$ , we have

$$\|x^{(n)} - x\|_p \leq \epsilon.$$

That is,  $x^{(n)}$  converges to  $x$ . Hence every Cauchy sequence in  $(\ell_p, \|\cdot\|_p)$  converges, hence it is Banach.  $\square$

**Example:** Consider the set of sequences

$$\ell_\infty := \left\{ x = (x_i) \mid x_i \in \mathbb{F}, \sup_{1 \leq i < \infty} |x_i| < \infty \right\},$$

the set of all bounded sequences in  $\mathbb{F}$ . Clearly it is a vector space under componentwise addition and scalar multiplication. Also define the function

$$\|x\|_\infty = \sup_{1 \leq i < \infty} |x_i|.$$

Then  $\|x\|_\infty$  is a norm on  $\ell_\infty$  and  $(\ell_\infty, \|\cdot\|_\infty)$  is a Banach space.

## Lecture 03: Continuous Linear Operators

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**Example:** Define the set of functions

$$C[0, 1] := \{f \mid f : [0, 1] \rightarrow \mathbb{F} \text{ is a continuous function}\}.$$

Clearly, it is a vector space under the operations of addition of functions and scalar multiplication. Define a function from  $C[0, 1]$  to  $\mathbb{F}$  as

$$\|f\|_\infty = \sup_t |f(t)|.$$

Then  $(C[0, 1], \|\cdot\|_\infty)$  is a Banach space.

*Proof.* The first two axioms of norm hold trivially. Now, for any  $t \in [0, 1]$ , due to  $|\cdot|$  being a norm on  $\mathbb{F}$ , it follows that

$$|f(t) + g(t)| \leq |f(t)| + |g(t)|.$$

If the supremum of  $|f(t) + g(t)|$  occurs at  $t_0$ , then

$$\|f + g\|_\infty = |f(t_0) + g(t_0)| \leq |f(t_0)| + |g(t_0)| \leq \|f\|_\infty + \|g\|_\infty.$$

Hence  $\|\cdot\|_\infty$  is a norm on  $C[0, 1]$ .

Now to show that the NLS  $(C[0, 1], \|\cdot\|_\infty)$  is complete, consider a Cauchy sequence  $f_n$  in  $C[0, 1]$ . that is, for all  $\epsilon > 0$ , there exists a positive integer  $p > 0$  such that  $m, n > p$  implies that  $\|f_n - f_m\| < \epsilon$ . That is, whenever  $m, n > p$  it implies that

$$\sup_t |f_n(t) - f_m(t)| < \epsilon.$$

Hence  $|f_n(t) - f_m(t)| < \epsilon$  for each  $t$ , therefore, for a fixed  $t$ , the sequence  $f_n(t)$  is Cauchy in  $\mathbb{R}$ , and hence converges (since  $\mathbb{F}$  is complete) to a point, say  $f(t)$ .

Define a function  $f : [0, 1] \rightarrow \mathbb{F}$  as  $f(t)$  as above. Now we claim that  $f \in C[0, 1]$ . Notice the following:

$$|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)|.$$

Since  $f_n$  converges to  $f$ , for any  $\epsilon > 0$ , there exists a sufficiently large  $n$ , such that  $|f_n(t) - f(t)| < \epsilon$  and  $|f_n(s) - f(s)| < \epsilon$ . Now, since  $f_n$  is continuous, for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$|s - t| < \delta$  implies that  $|f_n(t) - f_n(s)| < \epsilon$ , hence we have, for any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|s - t| < \delta \implies |f(s) - f(t)| < 3\epsilon.$$

Hence  $f$  is continuous and belongs to  $C[0, 1]$ .

Now, we from the Cauchy criterion, we have, for each  $\epsilon > 0$ , a positive integer  $p$  such that  $m, n > p$  implies that

$$\sup_t |f_n(t) - f_m(t)| < \epsilon.$$

Fixing  $n > p$ , and taking the limit  $m \rightarrow \infty$ , we obtain  $|f_n(t) - f(t)| \leq \epsilon$  for independent of  $t$ . Hence for a suitable for any  $\epsilon > 0$ , there exists a suitable  $p$  such that  $\|f_n - f\|_\infty < \epsilon$  whenever  $n > p$ . Hence  $f_n$  converges to  $f$  in the NLS.

Hence  $(C[0, 1], \|\cdot\|_\infty)$  is a Banach space.  $\square$

### **Recall:**

Linear transformations; matrix representation of linear transformations; examples of linear transformations between finite dimensional vector spaces, between infinite dimensional vector spaces.

**Note:** For a function between normed linear spaces, since a normed linear space has a metric induced by the norm, we can talk about sequential definition,  $\epsilon - \delta$  definition of continuity. And with the topological structure on the normed linear spaces, we can talk about the inverse image of an open (or closed) set being open (or closed) set definition of continuity. And all of those are equivalent.

### **Definition 3** (Continuous Linear Transformation):

A continuous linear transformation is what it says on the tin, a linear transformation between two normed linear spaces over the same field, that is also continuous (we can talk about continuity because a normed linear space has an induced topology).

### **Proposition 1:**

Let  $(V, \|\cdot\|_v)$  and  $(W, \|\cdot\|_w)$  be normed linear spaces over the same field  $\mathbb{F}$ , then the following are equivalent.

- i.  $T$  is continuous.
- ii.  $T$  is continuous at 0.
- iii. There exists a positive real number  $k$  such that  $\|T(x)\|_w \leq k\|x\|_v \ \forall x \in V$ .
- iv.  $T(\overline{B}_v(0, 1)) \subseteq \overline{B}_w(0, r)$  for some  $r > 0$ .

*Proof.* i.  $\implies$  ii. is obvious.

Suppose ii. is true. Then for any sequence  $x_n$  converging to  $x$  in  $V$ , the sequence  $x_n - x$  converges to 0. Due to continuity of  $T$  at 0, the sequence  $T(x_n - x)$  in  $W$  must converge to  $T(0) = 0$ . That implies that  $T(x_n)$  converges to  $T(x)$  in  $W$ . Hence we have ii.  $\implies$  i.

Suppose ii. is true. Then for every  $\epsilon > 0$ , we have a  $\delta > 0$  such that

$$\|x\|_v < \delta \implies \|T(x)\|_w < \epsilon.$$

In particular, for  $\epsilon = 1$ , we obtain a  $\delta_1$  satisfying the above. Consider  $k = \frac{2}{\delta_1}$ . For any  $x \in V \setminus \{0\}$ , define

$$y = \frac{\delta_1}{2} \frac{x}{\|x\|_v}.$$

Then  $\|y\|_v = \frac{\delta_1}{2} < \delta$ , and hence  $\|T(y)\|_w < \epsilon$ . That is,

$$\frac{\delta_1}{2\|x\|_v} \|T(x)\|_w < 1.$$

Upon rearrangement, we obtain

$$\|T(x)\|_w < \frac{2}{\delta_1} \|x\|_v = k\|x\|_v,$$

hence we have ii.  $\implies$  iii.

iii.  $\implies$  ii. is obvious from the  $\epsilon - \delta$  definition of continuity.

iii  $\iff$  iv. is obvious after substituting  $k = r$ . □

**Example (Operator Norm):** Let  $V$  and  $W$  be two normed linear spaces over the same field. On the collection of all linear operators from  $V$  to  $W$ , define an operation as follows: If  $T$  is a linear operator from  $V$  to  $W$ , then

$$\|T\|_{op} := \sup_{x \in \overline{B}_v(0,1)} \|T(x)\|_w = \sup \{ \|T(x)\|_w : \|x\|_v \leq 1 \}.$$

Consider the set of operators given by

$$\mathcal{L}(V, W) := \{T \mid T \text{ is a linear operator from } V \text{ to } W \text{ and } \|T\|_{op} < \infty\}.$$

Then  $(\mathcal{L}(V, W), \|\cdot\|_{op})$  is a normed linear space, and the norm  $\|\cdot\|_{op}$  is called as the operator norm.

## Lecture 04: Operator Norm and Stuff

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### **Proposition 2:**

Let  $V$  and  $W$  be normed linear spaces over the same field  $\mathbb{F}$ . Let  $T \in \mathcal{L}(V, W)$ . Consider the following quantities

$$\begin{aligned} \|T\|_{op} &:= \sup \{ \|T(x)\|_w : \|x\|_v \leq 1 \} \\ \alpha &:= \sup \{ \|T(x)\|_w : \|x\|_v = 1 \} \\ \beta &:= \sup \left\{ \frac{\|T(x)\|_w}{\|x\|_v} : x \in V \setminus \{0\} \right\} \\ \gamma &:= \inf \{ k : \|T(x)\|_w \leq k\|x\|_v \ \forall x \in V \}. \end{aligned}$$

Then  $\alpha = \beta = \gamma = \|T\|_{op}$ .



## References

- [1] R. Bhatia. *Notes on Functional Analysis*. 1st ed. Hindustan Book Agency, 2009.
- [2] S. Kesavan. *Functional Analysis*. 2nd ed. Springer, 2023.