Differential Geometry

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Course	MSM423 Differential Geometry
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Prerequisites	MAT413 Analysis on Manifolds
Learning Outcomes	• Understanding the classical interpretation of various curvatures of a surface and their relation to geodesics. • Understanding the local and global geometry of smooth manifolds and smooth vector bundles.
Syllabus	• Gauss curvature, Gauss curvature formula in terms of first and second fundamental forms. Intrinsic property of the Gauss curvature. (6) • Covariant derivative of a vector field along a curve; Relation between covariant derivative and total curvature of a curve; A geodesic as a curve with vanishing covariant derivative. (6) • Manifolds: Definition, examples, Tangent vector space at a point, Basis of the tangent vector space. Smooth functions on a manifold, maps between Manifolds. Differential of a map. (6) • Sub-manifolds; Regular value theorem. Lie groups, examples; Submersion, Immersion and Embeddings. (6) • Smooth vector bundles, smooth sections, Dual bundles, existence of local sections. (5) • Tangent bundles; Smooth vector fields; Lie bracket of smooth vector fields; Co-tangent bundles; Differential 1-forms. (5) • Differential p-forms. Orientation. Exterior derivative. Closed and exact forms. Integration of a p-form on a p-dim sub manifold. Stokes theorem. (6)

Lecture 01: Introduction to Curves and Surfaces 05 Jan 2024 10:30

Definition* 1** (Curves):***
Let I be an open interval in $\mathbb{R}$. A smooth parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$.
The image $\gamma(I) \subseteq \mathbb{R}^n$ is called trace of the curve γ.

Example: $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$, defined as $\gamma_1(t) = (\cos(t), \sin(t))$ and $\gamma_2(t) = (\sin(3t), \cos(3t))$ have the same trace, but they are different curves.

Observe: Consider any circle in $\mathbb{R}^2$, it is the set of all points in $\mathbb{R}^2$ satisfying a certain quadratic equation.
It can also be viewed as a trace of some curve.

Definition* 2** (Level Set):***
Let $U \subseteq \mathbb{R}^n$ be any open set, and $f : U \to \mathbb{R}$ be any function.
Then for a given constant $c \in \mathbb{R}$, the level set is defined as

$$L_c(f) := \{X \in U \mid f(X) = c\} \subseteq U \subseteq \mathbb{R}^n$$

Example: Let $U = \mathbb{R}^2$, and $f(x, y) = x^2 + y^2$, then
$L_0(f) = (0, 0)$, a point,
$L_1(f) = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, a circle in $\mathbb{R}^2$,
$L_{-1}(f) = \varnothing$.

Definition* 3** (Graph):***
Let A and B be any sets. Let $f : A \to B$ be a function, then the graph of f is the function $G_f : A \to A \times B$, given by

$$G_f(x) = (x, f(x)) \in A \times B$$

Note: Notice that a graph is also a curve, and if f is a smooth function, then G_f is a smooth curve.

Theorem* 1** (Implicit Function Theorem (2D Case)):***
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defining a curve $F(x, y) = c$.
Let (x_0, y_0) be a point on the curve.

- If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, then there exists a neighborhood around (x_0, y_0) where we can write $x = g(y)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $x = g(y)$ in that neighborhood.
- If $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then there exists a neighborhood around (x_0, y_0) where we can write $y = f(x)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $y = f(x)$ in that neighborhood.

Example: Consider $S^1 = L_1(x^2 + y^2) = \{(x, y) \mid x^2 + y^2 = 1\}$
Here, $F(x, y) = x^2 + y^2$, and $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$.
Therefore, for any point (x_0, y_0) in S^1, where such that $x_0 \neq 0$, there exists a neighborhood in which $x = g(y)$.

Note: It is worthwhile to notice that a graph is always a curve. Implicit function theorem gives a condition on when a curve (level set in particular) can be seen as the graph of a curve.

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Observe: Consider the level set of $F(x, y) = x^2 + y^2$, $L_1(f) = S^1$. As seen before, since $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$, for any point (x_0, y_0) in S^1, where $x_0 \neq 0$, we obtain a neighborhood in S^1 such that the curve (level set) can be written as a graph of a function $y = g(x)$.
Let S^{1+} denote $(\mathbb{R} \times \mathbb{R}^+) \cap S^1$, the upper half of the unit circle. Now, for a point in S^{1+}, the corresponding $g(x)$ is $\sqrt{1-x^2}$.
Now the graph of g, say G_g is a function from $(-1, 1)$ to S^{1+}, and we can easily talk about continuity, differentiablity of G_g.
However, notice that G_g is a surjection, and an inverse exists, $G_g^{-1} : S^{1+} \to (-1, 1)$.
To talk about continuity of G_g^{-1} we need to define a topology on S^{1+}, and the subspace topology is the most obvious choice.
This G_g^{-1} happens to be continuous when we consider the subspace topology on S^{1+}. Further, open subsets of S^{1+} homeomorphic to the open subsets of $\mathbb{R}$, that means that this topological space is locally homeomorphic to $\mathbb{R}$, but the same does not extend to the whole space, that is, there is no homeomorphism from S^{1+} to $\mathbb{R}$.

Note: Notice that we cannot talk about differentiablity of S^{1+} in the above immediately, as differentiability requires some sort of vector space structure.

Example: Consider $S_3^1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
Doing the same analysis on this set as before using the Implicit function theorem, we obtain that that after removing the appropriate critical points, and giving a topology to whatever remains, we obtain patches obtain open sets in whatever remains that are homeomorphic to open subsets of $\mathbb{R}^2$, hence the topological subspace obtained after removing the critical points is locally homeomorphic to $\mathbb{R}^2$.

Definition* 4** (Tangent to a smooth curve at a point):***
Let I be an open interval in $\mathbb{R}$, and $\gamma : I \to \mathbb{R}^n$ be a smooth curve given by $\gamma(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then for a point $t \in I$, the tangent of the curve at t is defined as

$$\gamma'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$$

Definition* 5** (Regular Curve):***
A smooth curve $\gamma : I \to \mathbb{R}^n$ is said to be a regular curve if $\gamma'(t) \neq 0 \ \forall t \in I$.

Definition* 6** (Diffeomorphism):***
Let A and B be two manifolds, and a function $f : A \to B$ is said to be a diffeomorphism if f is bijective, differentiable, and the inverse $f^{-1} : B \to A$ is differentiable as well.

Note: In this course, by smooth, we mean a C^∞ function.

Definition 7 (Reparametrization of a Curve)**:**
Let J, I be open intervals of $\mathbb{R}$, let $\gamma : I \to \mathbb{R}^n$ be a curve, and let $\phi : J \to I$ be a diffeomorphism. Then $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ is a curve on J called as reparametrization of γ.

Observe: If β is a reparametrization of γ, and the setup is as above, then $\beta(s) = \gamma \circ \phi(s)$ for each $s \in J$. Then

$$\beta'(s) = \gamma'(\phi(s)) \cdot \phi'(s)$$

Now, ϕ is a diffeomorphism and $(\phi \circ \phi^{-1})' = 1$ and by chain rule, we have $(\phi^{-1})' = \frac{1}{\phi'}$. And since $(\phi^{-1})'$ exists throughout I, ϕ' cannot be 0 on J, hence $\gamma'(s) \neq 0$.
Hence if γ is regular, then so is β, hence regularity is preserved under a diffeomorphism.

Definition 8 (Arc Length of a Curve)**:**
Let $\gamma : I \to \mathbb{R}^n$ be a regular parametrized curve. For fixed $t_1, t_2 \in I$, the arc length between t_1 and t_2 is given by

$$\begin{aligned} L_\gamma(t_1, t_2) &= \int_{t_1}^{t_2} ||\gamma'(t)||\, dt \\ &= \int_{t_1}^{t_2} \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + \cdots + (x_n'(t))^2}\, dt \end{aligned}$$

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Theorem 2 (Inverse Function Theorem (a particular case))**:**
Let $U \subseteq \mathbb{R}^n$, and $f : U \to \mathbb{R}$ be a smooth function. Let $x \in U$ be such that for the Jacobian $D(f)(x)$ of f at x, the determinant is $\det[D(f)(x)] \neq 0$, then there exists a neighborhood W of x in U such that the restriction $f|_W : W \to f(W)$ is a diffeomorphism.

Note: Notice that this theorem gives a local property. That is, with the given hypothesis, we can only really claim that there exists a neighborhood upon which the restriction is a diffeomorphism. However, if the Jacobian of f has non-zero determinant at each x in U, then we cannot claim that f is a diffeomorphsim.

Example: $U = \mathbb{R}\backslash\{0\}$ and $f : U \to \mathbb{R}$ given by $f(x) = x^2$ satisfies the hypothesis, yet is not a diffeomorphism from U to $f(U)$, it is not even injective.

Observe: However, if U happens to be a connected, and f is smooth, then the above hypothesis is sufficient to claim that f is a diffeomorphism from U to $f(U)$.

Definition 9 (Unit Speed Reparametrization)**:**
For the intervals I, J, let $\gamma : I \to \mathbb{R}^n$ be a curve, let $\phi : J \to I$ be a diffeomorphism such that the reparametrization $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ has the property that $|\beta'(s)| = 1$ for each $s \in J$. Then the reparametrization β is said to be unit speed reparametrization.

Note: Notice that when a curve is reparametrized, the trace/image of the curve remains the same.

Note (Condition for a reparametrization to be unit speed)**:** If $\beta = \gamma \circ \phi$ is a reparametrization of the curve γ with the diffeomorphism ϕ as in the above setting, then

$$\begin{aligned} |\beta'(s)| = 1 &\implies |\gamma'(\phi(s))||\phi'(s)| = 1 \\ &\implies \boxed{|\phi'(s)| = \frac{1}{|\gamma'(\phi(s))|}} \end{aligned}$$

Definition* 10** (Arc Length Function):***
Let I be an interval, and $t_0 \in I$ be a fixed element. Let γ be a regular curve defined on I. Then the Arc Length function $L_\gamma : I \to \mathbb{R}$ is defined as

$$L_\gamma(t) = \int_{t_0}^{t} |\gamma'(t)|\, dt$$

Property (Properties of L_γ)**:**

1. $L'_\gamma(t) = |\gamma'(t)|$ (It is differentiable).
2. Since $\gamma(t)$ is regular, $\gamma'(t)$ is nowhere 0, and hence $|\gamma'(t)|$ is continuous, hence $L'_\gamma(t)$ is continuous and consequently, smooth.
3. $L'_\gamma(t) > 0$ on the interval I, that implies that L_γ is diffeomorphism from I to $L_\gamma(I) := J$.
4. Let $S_\gamma = L_\gamma^{-1} : J \to I$ be the inverse of L_γ (exists because γ is a diffeomorphism, also note that S_γ is a diffeomorphism).
   Consider the reparametrization $\beta = \gamma \circ S_\gamma$, it follows that

   $$|\beta'(s)| = |\gamma'(S_\gamma(s))||S'_\gamma(s)| \qquad (*)$$

   From the chain rule, we have

   $$S'_\gamma(s) = \frac{1}{L'_\gamma(S_\gamma(s))} = \frac{1}{\gamma'(S_\gamma(s))} \qquad (**)$$

   $(*)$ and $(**)$ imply that $|\beta'(s)| = 1$ for all $s \in J$, hence β is a unit speed reparametrization.

Theorem* 3*:
Given an interval I, and any regular curve $\gamma : I \to \mathbb{R}^n$, there exists an interval J and a diffeomorphism $\phi : J \to I$ such that the reparametrization $\beta = \gamma \circ \phi$ is a unit speed parametrization.

Proof. The proof follows from 4. in the above properties of L_γ, where
$J = L_\gamma(I)$, and $\phi = L_\gamma^{-1} = S_\gamma$ □

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Example: Let $\gamma : I \to \mathbb{R}^n$ be a constant speed curve, that is $\gamma'(t) = c\ \forall t \in I$. Then it is easy to reparametrize it to a unit speed curve as follows:

$$\phi : cI \to \mathbb{R}^n \text{ as } \phi(s) = s/c.$$

Note: Notice that a Jacobian of a function from $\gamma : I \to \mathbb{R}^n$ is a linear transformation, and can be represented with a $1 \times n$ matrix. Further, each point in I, defines such a linear transformation given by the Jacobian at that point. If a curve is regular, that is $\gamma'(t) \neq 0 \ \forall t \in I$, then the Jacobian is non-zero at each point, hence at each point, we can define a linear space spanned by the span of the Jacobian. This will be relevant later.

Motivation: Now we do an analysis of smooth curves that are parametrized by arc length. That is, the smooth curves that are reparametrized using arc length functions. As seen before, these curves are unit speed, and are smooth.

Definition* 11** (Curvature of a curve parametrized by arc length):***
Let $\gamma : I \to \mathbb{R}^3$ be a curve parametrized by arc length (then it is a unit speed curve smooth as seen before) in $\mathbb{R}^3$. Then for a $t \in I$, we define $|\gamma''(t)| = \kappa(t)$ as the curvature of γ at t.

Property:

i. Since γ is a unit speed curve, we have $|\gamma'(t)|^2 = \gamma'(t) \cdot \gamma'(t) = 1$. That implies that $2\gamma'(t) \cdot \gamma''(t) = 0$, hence $\gamma''(t)$ is perpendicular to $\gamma'(t)$ for all $t \in I$.

ii. Now, if $\hat{n}(t) = \frac{\gamma''(t)}{\|\gamma''(t)\|}$ is the unit vector in the direction of t, then $\gamma''(t) = \kappa(t)\hat{n}(t)$.

iii. If $\kappa(t) = 0 \ \forall t \in I$, then $\gamma''(t) = 0 \ \forall t \in I$, then $\gamma(t) = at + b$ for some $a, b \in \mathbb{R}^3$, that is, a straight line in $\mathbb{R}^3$.

iv. If γ is smooth and $|\gamma''(t)| \neq 0$ at a point t, then we denote $\gamma'(t)$ at that point as $\hat{t}(t)$. Notice that in such case, $\hat{t}(t)$ and $\hat{n}(t)$ are tangent and normal to the curve at $\gamma(t)$ respectively, hence they define a subspace (a plane) of $\mathbb{R}^3$.

Definition* 12** (Osculating plane):***
Let $\hat{t}(t), \hat{n}(t)$ be as defined as above, then the plane spanned by them is called as the Osculating plane of the curve at $\gamma(t)$. Notice that such a pair of vector and such a plane is defined for each point on the curve γ.

Definition* 13** (Bivector):***
Let the conditions be the same as above, then the vector $\hat{b}(t) = \hat{t}(t) \times \hat{n}(t)$ is called as the Bivector at t.

Remark:

i. $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$ forms an Orthnormal basis of $\mathbb{R}^3$.

ii. $\hat{b}(t)$ is smooth.

iii. If $\hat{b}'(t) = 0 \ \forall t \in I$, then the osculating plane at each point of the curve is the same, that means the whole curve lies on a plane in $\mathbb{R}^3$.

Note: Notice that $\hat{b}(t)$ is a cross product of two unit vectors, hence it is a unit vector.
Therefore $|\hat{b}(t)|^2 = \hat{b}(t) \cdot \hat{b}(t) = 1$. Which implies that $2\hat{b}(t) \cdot \hat{'}b(t) = 0$. Hence $\hat{b}'(t)$ is perpendicular to $\hat{b}(t)$. Hence $\hat{b}'(t) \in \text{span}\big(\hat{t}(t), \hat{n}(t)\big)$.

Definition* 14** (Torsion function):***
Let the conditions be the same as above. The function defined as $\tau(t) = \hat{b}'(t) \cdot \hat{n}(t)\ \forall t \in I$ is called a torsion function.

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Note: The value $\kappa(t)$ gives the curvature of γ, that is, it is denotes the straightness of γ at $\gamma(t)$. The value of $\tau(t)$ gives the rate of change of osculating plane at $\gamma(t)$, that is, it roughly denotes how fast the osculating plane at a point t is changing when t is varied.

Observe:

$$\begin{aligned}
& \hat{b}(t) = \hat{t}(t) \times \hat{n}(t) \\
\implies & \hat{b}'(t) = \hat{t}'(t) \times \hat{n}(t) + \hat{t}(t) \times \hat{n}'(t) \\
\implies & \hat{b}'(t) = \gamma''(t) \times \hat{n}(t) + \hat{t}(t) \times \hat{n}'(t) \qquad (\because \hat{t}(t) = \gamma'(t)) \\
\implies & \hat{b}'(t) = 0 + \hat{t}(t) \times \hat{n}'(t) \qquad (\because \gamma''(t) \text{ is parallel to } \hat{n}(t)) \\
\therefore\ & \hat{b}'(t) = \hat{t}(t) \times \hat{n}'(t)
\end{aligned}$$

Therefore, $\hat{b}'(t)$ is perpendicular to $\hat{t}(t)$. Also as $\hat{b}(t)$ is a unit vector, $\hat{b}'(t)$ is perpendicular to $\hat{b}(t)$ as well.
Now since $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$ is an orthonromal basis, and $\hat{b}'(t)$ is perpendicular to both $\hat{b}(t)$ and $\hat{t}(t)$, it must be along the span of $\hat{n}(t)$.
And by the way we defined torsion, we have

$$\hat{b}'(t) = \tau(t) \cdot \hat{n}(t)$$

Note: So far, we have $\hat{t}'(t) = \kappa(t) \cdot \hat{n}(t)$ and $\hat{b}'(t) = \tau(t) \cdot \hat{n}(t)$.

Observe: Notice that $\hat{n}'(t)$ is perpendicular to $\hat{n}(t)$, hence

$$\hat{n}'(t) \in \operatorname{span}\{\hat{t}(t), \hat{b}(t)\}.$$

Now, we have

$$\begin{aligned}
& \hat{n}(t) = \hat{b}(t) \times \hat{t}(t) \\
\implies & \hat{n}'(t) = \hat{b}'(t) \times \hat{t}(t) + \hat{b}(t) \times \hat{t}'(t) \\
\implies & \hat{n}'(t) = (\tau(t) \cdot \hat{n}(t)) \times \hat{t}(t) + \hat{b}(t) \times (\kappa(t) \cdot \hat{n}(t)) \\
\implies & \hat{n}'(t) = -\tau(t) \cdot \hat{b}(t) - \kappa(t) \cdot \hat{t}(t)
\end{aligned}$$

Theorem* 4** (Frenet-Seret formulae):***
From the above observations, we have the set of equations called Frenet-Seret formulae, given by

$$\begin{aligned}
\hat{t}'(t) &= \kappa(t) \cdot \hat{n}(t) \\
\hat{n}'(t) &= -\tau(t) \cdot \hat{b}(t) - \kappa(t) \cdot \hat{t}(t) \\
\hat{b}'(t) &= \tau(t) \cdot \hat{n}(t)
\end{aligned}$$

or, in a more concise way, we have

$$\begin{bmatrix} \hat{t}'(t) \\ \hat{n}'(t) \\ \hat{b}'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{t}(t) \\ \hat{n}(t) \\ \hat{b}(t) \end{bmatrix}$$

Observe: Observe that the set of differential equations in the Frenet-Seret formulae involves a Skew-Symmetric matrix, and also, when we have a non-planar curve, the corresponding $\tau(t)$ and $\kappa(t)$ are non-zero, so usually this is assumed.

Motivation:

- Given a smooth unit-speed regular curve, we can always find this set of equations (Frenet-Seret), by finding curvature and torsion functions.
- Given two functions $\kappa(t) : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$, can we find a regular unit-speed curve $\gamma(t)$ using the Frenet-Seret equations such that its curvature and torsion coincide with κ and τ respectively?
- Also, if such a curve exists, is it unique?
- Notice that given two such functions, and any initial condition on $\hat{t}, \hat{n}, \hat{b}$, we obtain a set of differential equations.

References

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[3] L. W. Tu. *An Introduction to Manifolds*. 2nd ed. Springer, 2011.