

# Differential Geometry

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Course	MSM423 Differential Geometry
Instructor	Dr. Saikat Chatterjee
Prerequisites	MAT413 Analysis on Manifolds
Learning Outcomes	<ul style="list-style-type: none"> <li>• Understanding the classical interpretation of various curvatures of a surface and their relation to geodesics.</li> <li>• Understanding the local and global geometry of smooth manifolds and smooth vector bundles.</li> </ul>
Syllabus	<ul style="list-style-type: none"> <li>• Gauss curvature, Gauss curvature formula in terms of first and second fundamental forms. Intrinsic property of the Gauss curvature. (6)</li> <li>• Covariant derivative of a vector field along a curve; Relation between covariant derivative and total curvature of a curve; A geodesic as a curve with vanishing covariant derivative. (6)</li> <li>• Manifolds: Definition, examples, Tangent vector space at a point, Basis of the tangent vector space. Smooth functions on a manifold, maps between Manifolds. Differential of a map. (6)</li> <li>• Sub-manifolds; Regular value theorem. Lie groups, examples; Submersion, Immersion and Embeddings. (6)</li> <li>• Smooth vector bundles, smooth sections, Dual bundles, existence of local sections. (5)</li> <li>• Tangent bundles; Smooth vector fields; Lie bracket of smooth vector fields; Co-tangent bundles; Differential 1-forms. (5)</li> <li>• Differential p-forms. Orientation. Exterior derivative. Closed and exact forms. Integration of a p-form on a p-dim sub manifold. Stokes theorem. (6)</li> </ul>

**Definition 1** (Curves):

Let  $I$  be an open interval in  $\mathbb{R}$ . A smooth parametrized curve in  $\mathbb{R}^n$  is a smooth map  $\gamma : I \rightarrow \mathbb{R}^n$ .

The image  $\gamma(I) \subseteq \mathbb{R}^n$  is called trace of the curve  $\gamma$ .

**Example:**  $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ , defined as  $\gamma_1(t) = (\cos(t), \sin(t))$  and  $\gamma_2(t) = (\sin(3t), \cos(3t))$  have the same trace, but they are different curves.

**Observe:** Consider any circle in  $\mathbb{R}^2$ , it is the set of all points in  $\mathbb{R}^2$  satisfying a certain quadratic equation.

It can also be viewed as a trace of some curve.

**Definition 2** (Level Set):

Let  $U \subseteq \mathbb{R}^n$  be any open set, and  $f : U \rightarrow \mathbb{R}$  be any function.

Then for a given constant  $c \in \mathbb{R}$ , the level set is defined as

$$L_c(f) := \{X \in U \mid f(X) = c\} \subseteq U \subseteq \mathbb{R}^n$$

**Example:** Let  $U = \mathbb{R}^2$ , and  $f(x, y) = x^2 + y^2$ , then

$L_0(f) = (0, 0)$ , a point,

$L_1(f) = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ , a circle in  $\mathbb{R}^2$ ,

$L_{-1}(f) = \emptyset$ .

**Definition 3** (Graph):

Let  $A$  and  $B$  be any sets. Let  $f : A \rightarrow B$  be a function, then the graph of  $f$  is the function  $G_f : A \rightarrow A \times B$ , given by

$$G_f(x) = (x, f(x)) \in A \times B$$

**Note:** Notice that a graph is also a curve, and if  $f$  is a smooth function, then  $G_f$  is a smooth curve.

**Theorem 1** (Implicit Function Theorem (2D Case)):

Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function defining a curve  $F(x, y) = c$ .

Let  $(x_0, y_0)$  be a point on the curve.

- If  $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$ , then there exists a neighborhood around  $(x_0, y_0)$  where we can write  $x = g(y)$  for a real valued function  $f(x)$ . That is, the curve  $F(x, y) = c$  behaves like the graph of  $x = g(y)$  in that neighborhood.
- If  $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ , then there exists a neighborhood around  $(x_0, y_0)$  where we can write  $y = f(x)$  for a real valued function  $f(x)$ . That is, the curve  $F(x, y) = c$  behaves like the graph of  $y = f(x)$  in that neighborhood.

**Example:** Consider  $S^1 = L_1(x^2 + y^2) = \{(x, y) \mid x^2 + y^2 = 1\}$

Here,  $F(x, y) = x^2 + y^2$ , and  $\frac{\partial F}{\partial x} = 2x \neq 0$  when  $x \neq 0$ .

Therefore, for any point  $(x_0, y_0)$  in  $S^1$ , where such that  $x_0 \neq 0$ , there exists a neighborhood in which  $x = g(y)$ .

**Note:** It is worthwhile to notice that a graph is always a curve. Implicit function theorem gives a condition on when a curve (level set in particular) can be seen as the graph of a curve.

## Lecture 02: Curves and Surfaces

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**Observe:** Consider the level set of  $F(x, y) = x^2 + y^2$ ,  $L_1(f) = S^1$ . As seen before, since  $\frac{\partial F}{\partial x} = 2x \neq 0$  when  $x \neq 0$ , for any point  $(x_0, y_0)$  in  $S^1$ , where  $x_0 \neq 0$ , we obtain a neighborhood in  $S^1$  such that the curve (level set) can be written as a graph of a function  $y = g(x)$ .

Let  $S^{1+}$  denote  $(\mathbb{R} \times \mathbb{R}^+) \cap S^1$ , the upper half of the unit circle. Now, for a point in  $S^{1+}$ , the corresponding  $g(x)$  is  $\sqrt{1 - x^2}$ .

Now the graph of  $g$ , say  $G_g$  is a function from  $(-1, 1)$  to  $S^{1+}$ , and we can easily talk about continuity, differentiability of  $G_g$ .

However, notice that  $G_g$  is a surjection, and an inverse exists,  $G_g^{-1} : S^{1+} \rightarrow (-1, 1)$ .

To talk about continuity of  $G_g^{-1}$  we need to define a topology on  $S^{1+}$ , and the subspace topology is the most obvious choice.

This  $G_g^{-1}$  happens to be continuous when we consider the subspace topology on  $S^{1+}$ . Further, open subsets of  $S^{1+}$  homeomorphic to the open subsets of  $\mathbb{R}$ , that means that this topological space is locally homeomorphic to  $\mathbb{R}$ , but the same does not extend to the whole space, that is, there is no homeomorphism from  $S^{1+}$  to  $\mathbb{R}$ .

**Note:** Notice that we cannot talk about differentiability of  $S^{1+}$  in the above immediately, as differentiability requires some sort of vector space structure.

**Example:** Consider  $S_3^1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

Doing the same analysis on this set as before using the Implicit function theorem, we obtain that after removing the appropriate critical points, and giving a topology to whatever remains, we obtain patches obtain open sets in whatever remains that are homeomorphic to open subsets of  $\mathbb{R}^2$ , hence the topological subspace obtained after removing the critical points is locally homeomorphic to  $\mathbb{R}^2$ .

**Definition 4** (Tangent to a smooth curve at a point):

Let  $I$  be an open interval in  $\mathbb{R}$ , and  $\gamma : I \rightarrow \mathbb{R}^n$  be a smooth curve given by  $\gamma(t) = (x_1(t), x_2(t), \dots, x_n(t))$ , then for a point  $t \in I$ , the tangent of the curve at  $t$  is defined as

$$\gamma'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$$

**Definition 5** (Regular Curve):

A smooth curve  $\gamma : I \rightarrow \mathbb{R}^n$  is said to be a regular curve if  $\gamma'(t) \neq 0 \forall t \in I$ .

**Definition 6** (Diffeomorphism):

Let  $A$  and  $B$  be two manifolds, and a function  $f : A \rightarrow B$  is said to be a diffeomorphism if  $f$  is bijective, differentiable, and the inverse  $f^{-1} : B \rightarrow A$  is differentiable as well.

**Note:** In this course, by smooth, we mean a  $C^\infty$  function.

**Definition 7** (Reparametrization of a Curve):

Let  $J, I$  be open intervals of  $\mathbb{R}$ , let  $\gamma : I \rightarrow \mathbb{R}^n$  be a curve, and let  $\phi : J \rightarrow I$  be a diffeomorphism. Then  $\beta = \gamma \circ \phi : J \rightarrow \mathbb{R}^n$  is a curve on  $J$  called as reparametrization of  $\gamma$ .

**Observe:** If  $\beta$  is a reparametrization of  $\gamma$ , and the setup is as above, then  $\beta(s) = \gamma(\phi(s))$  for each  $s \in J$ . Then

$$\beta'(s) = \gamma'(\phi(s)) \cdot \phi'(s)$$

Now,  $\phi$  is a diffeomorphism and  $(\phi \circ \phi^{-1})' = 1$  and by chain rule, we have  $(\phi^{-1})' = \frac{1}{\phi'}$ . And since  $(\phi^{-1})'$  exists throughout  $I$ ,  $\phi'$  cannot be 0 on  $J$ , hence  $\gamma'(s) \neq 0$ .

Hence if  $\gamma$  is regular, then so is  $\beta$ , hence regularity is preserved under a diffeomorphism.

**Definition 8** (Arc Length of a Curve):

Let  $\gamma : I \rightarrow \mathbb{R}^n$  be a regular parametrized curve. For fixed  $t_1, t_2 \in I$ , the arc length between  $t_1$  and  $t_2$  is given by

$$\begin{aligned} L_\gamma(t_1, t_2) &= \int_{t_1}^{t_2} \|\gamma'(t)\| dt \\ &= \int_{t_1}^{t_2} \sqrt{(x'_1(t))^2 + (x'_2(t))^2 + \cdots + (x'_n(t))^2} dt \end{aligned}$$

**Lecture 03: Reparametrizations****10 Jan 2024 10:30****Theorem 2** (Inverse Function Theorem (a particular case)):

Let  $U \subseteq \mathbb{R}^n$ , and  $f : U \rightarrow \mathbb{R}^n$  be a smooth function. Let  $x \in U$  be such that for the Jacobian  $D(f)(x)$  of  $f$  at  $x$ , the determinant is  $\det[D(f)(x)] \neq 0$ , then there exists a neighborhood  $W$  of  $x$  in  $U$  such that the restriction  $f|_W : W \rightarrow f(W)$  is a diffeomorphism.

**Note:** Notice that this theorem gives a local property. That is, with the given hypothesis, we can only really claim that there exists a neighborhood upon which the restriction is a diffeomorphism. However, if the Jacobian of  $f$  has non-zero determinant at each  $x$  in  $U$ , then we cannot claim that  $f$  is a diffeomorphism.

**Example:**  $U = \mathbb{R} \setminus \{0\}$  and  $f : U \rightarrow \mathbb{R}$  given by  $f(x) = x^2$  satisfies the hypothesis, yet is not a diffeomorphism from  $U$  to  $f(U)$ , it is not even injective.

**Observe:** However, if  $U$  happens to be a connected, and  $f$  is smooth, then the above hypothesis is sufficient to claim that  $f$  is a diffeomorphism from  $U$  to  $f(U)$ .

**Definition 9** (Unit Speed Reparametrization):

For the intervals  $I, J$ , let  $\gamma : I \rightarrow \mathbb{R}^n$  be a curve, let  $\phi : J \rightarrow I$  be a diffeomorphism such that the reparametrization  $\beta = \gamma \circ \phi : J \rightarrow \mathbb{R}^n$  has the property that  $|\beta'(s)| = 1$  for each  $s \in J$ . Then the reparametrization  $\beta$  is said to be unit speed reparametrization.

**Note:** Notice that when a curve is reparametrized, the trace/image of the curve remains the same.

**Note** (Condition for a reparametrization to be unit speed): If  $\beta = \gamma \circ \phi$  is a reparametrization of the curve  $\gamma$  with the diffeomorphism  $\phi$  as in the above setting, then

$$\begin{aligned} |\beta'(s)| = 1 &\implies |\gamma'(\phi(s))||\phi'(s)| = 1 \\ &\implies |\phi'(s)| = \frac{1}{|\gamma'(\phi(s))|} \end{aligned}$$

**Definition 10** (Arc Length Function):

Let  $I$  be an interval, and  $t_0 \in I$  be a fixed element. Let  $\gamma$  be a regular curve defined on  $I$ . Then the Arc Length function  $L_\gamma : I \rightarrow \mathbb{R}$  is defined as

$$L_\gamma(t) = \int_{t_0}^t |\gamma'(t)| dt$$

**Property** (Properties of  $L_\gamma$ ):

1.  $L'_\gamma(t) = |\gamma'(t)|$  (It is differentiable).
2. Since  $\gamma(t)$  is regular,  $\gamma'(t)$  is nowhere 0, and hence  $|\gamma'(t)|$  is continuous, hence  $L'_\gamma(t)$  is continuous and consequently, smooth.
3.  $L'_\gamma(t) > 0$  on the interval  $I$ , that implies that  $L_\gamma$  is diffeomorphism from  $I$  to  $L_\gamma(I) := J$ .
4. Let  $S_\gamma = L_\gamma^{-1} : J \rightarrow I$  be the inverse of  $L_\gamma$  (exists because  $\gamma$  is a diffeomorphism, also note that  $S_\gamma$  is a diffeomorphism).

Consider the reparametrization  $\beta = \gamma \circ S_\gamma$ , it follows that

$$|\beta'(s)| = |\gamma'(S_\gamma(s))||S'_\gamma(s)| \quad (*)$$

From the chain rule, we have

$$S'_\gamma(s) = \frac{1}{L'_\gamma(S_\gamma(s))} = \frac{1}{\gamma'(S_\gamma(s))} \quad (**)$$

(\*) and (\*\*) imply that  $|\beta'(s)| = 1$  for all  $s \in J$ , hence  $\beta$  is a unit speed reparametrization.

**Theorem 3:**

Given an interval  $I$ , and any regular curve  $\gamma : I \rightarrow \mathbb{R}^n$ , there exists an interval  $J$  and a diffeomorphism  $\phi : J \rightarrow I$  such that the reparametrization  $\beta = \gamma \circ \phi$  is a unit speed parametrization.

*Proof.* The proof follows from 4. in the above properties of  $L_\gamma$ , where  $J = L_\gamma(I)$ , and  $\phi = L_\gamma^{-1} = S_\gamma$  □

## Lecture 04: Further analysis of Curves

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**Example:** Let  $\gamma : I \rightarrow \mathbb{R}^n$  be a constant speed curve, that is  $\gamma'(t) = c \forall t \in I$ . Then it is easy to reparametrize it to a unit speed curve as follows:

$$\phi : cI \rightarrow \mathbb{R}^n \text{ as } \phi(s) = s/c.$$

**Note:** Notice that a Jacobian of a function from  $\gamma : I \rightarrow \mathbb{R}^n$  is a linear transformation, and can be represented with a  $1 \times n$  matrix. Further, each point in  $I$ , defines such a linear transformation given by the Jacobian at that point. If a curve is regular, that is  $\gamma'(t) \neq 0 \forall t \in I$ , then the Jacobian is non-zero at each point, hence at each point, we can define a linear space spanned by the span of the Jacobian. This will be relevant later.

**Motivation:** Now we do an analysis of smooth curves that are parametrized by arc length. That is, the smooth curves that are reparametrized using arc length functions. As seen before, these curves are unit speed, and are smooth.

**Definition 11** (Curvature of a curve parametrized by arc length):

Let  $\gamma : I \rightarrow \mathbb{R}^3$  be a curve parametrized by arc length (then it is a unit speed curve smooth as seen before) in  $\mathbb{R}^3$ . Then for a  $t \in I$ , we define  $|\gamma''(t)| = \kappa(t)$  as the curvature of  $\gamma$  at  $t$ .

**Property:**

- i. Since  $\gamma$  is a unit speed curve, we have  $|\gamma'(t)|^2 = \gamma'(t) \cdot \gamma'(t) = 1$ . That implies that  $2\gamma'(t) \cdot \gamma''(t) = 0$ , hence  $\gamma''(t)$  is perpendicular to  $\gamma'(t)$  for all  $t \in I$ .
- ii. Now, if  $\hat{n}(t) = \frac{\gamma''(t)}{\|\gamma''(t)\|}$  is the unit vector in the direction of  $t$ , then  $\gamma''(t) = \kappa(t)\hat{n}(t)$ .
- iii. If  $\kappa(t) = 0 \forall t \in I$ , then  $\gamma''(t) = 0 \forall t \in I$ , then  $\gamma(t) = at + b$  for some  $a, b \in \mathbb{R}^3$ , that is, a straight line in  $\mathbb{R}^3$ .
- iv. If  $\gamma$  is smooth and  $|\gamma''(t)| \neq 0$  at a point  $t$ , then we denote  $\gamma'(t)$  at that point as  $\hat{t}(t)$ . Notice that in such case,  $\hat{t}(t)$  and  $\hat{n}(t)$  are tangent and normal to the curve at  $\gamma(t)$  respectively, hence they define a subspace (a plane) of  $\mathbb{R}^3$ .

**Definition 12** (Osculating plane):

Let  $\hat{t}(t), \hat{n}(t)$  be as defined as above, then the plane spanned by them is called as the Osculating plane of the curve at  $\gamma(t)$ . Notice that such a pair of vector and such a plane is defined for each point on the curve  $\gamma$ .

**Definition 13** (Bivector):

Let the conditions be the same as above, then the vector  $\hat{b}(t) = \hat{t}(t) \times \hat{n}(t)$  is called as the Bivector at  $t$ .

**Remark:**

- i.  $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$  forms an Orthonormal basis of  $\mathbb{R}^3$ .
- ii.  $\hat{b}(t)$  is smooth.
- iii. If  $\hat{b}'(t) = 0 \forall t \in I$ , then the osculating plane at each point of the curve is the same, that means the whole curve lies on a plane in  $\mathbb{R}^3$ .

**Note:** Notice that  $\hat{b}(t)$  is a cross product of two unit vectors, hence it is a unit vector. Therefore  $|\hat{b}(t)|^2 = \hat{b}(t) \cdot \hat{b}(t) = 1$ . Which implies that  $2\hat{b}(t) \cdot \hat{b}'(t) = 0$ . Hence  $\hat{b}'(t)$  is perpendicular to  $\hat{b}(t)$ . Hence  $\hat{b}'(t) \in \text{span}(\hat{t}(t), \hat{n}(t))$ .

**Definition 14** (Torsion function):

Let the conditions be the same as above. The function defined as  $\tau(t) = \hat{b}'(t) \cdot \hat{n}(t) \forall t \in I$  is called a torsion function.

## Lecture 05: Frenet–Serret formulae

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**Note:** The value  $\kappa(t)$  gives the curvature of  $\gamma$ , that is, it denotes the straightness of  $\gamma$  at  $\gamma(t)$ . The value of  $\tau(t)$  gives the rate of change of osculating plane at  $\gamma(t)$ , that is, it roughly denotes how fast the osculating plane at a point  $t$  is changing when  $t$  is varied.

**Observe:**

$$\begin{aligned} \hat{b}(t) &= \hat{t}(t) \times \hat{n}(t) \\ \implies \hat{b}'(t) &= \hat{t}'(t) \times \hat{n}(t) + \hat{t}(t) \times \hat{n}'(t) \\ \implies \hat{b}'(t) &= \gamma''(t) \times \hat{n}(t) + \hat{t}(t) \times \hat{n}'(t) \quad (\because \hat{t}(t) = \gamma'(t)) \\ \implies \hat{b}'(t) &= 0 + \hat{t}(t) \times \hat{n}'(t) \quad (\because \gamma''(t) \text{ is parallel to } \hat{n}(t)) \\ \therefore \hat{b}'(t) &= \hat{t}(t) \times \hat{n}'(t) \end{aligned}$$

Therefore,  $\hat{b}'(t)$  is perpendicular to  $\hat{t}(t)$ . Also as  $\hat{b}(t)$  is a unit vector,  $\hat{b}'(t)$  is perpendicular to  $\hat{b}(t)$  as well.

Now since  $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$  is an orthonormal basis, and  $\hat{b}'(t)$  is perpendicular to both  $\hat{b}(t)$  and  $\hat{t}(t)$ , it must be along the span of  $\hat{n}(t)$ .

And by the way we defined torsion, we have

$$\hat{b}'(t) = \tau(t) \cdot \hat{n}(t)$$

**Note:** So far, we have  $\hat{t}'(t) = \kappa(t) \cdot \hat{n}(t)$  and  $\hat{b}'(t) = \tau(t) \cdot \hat{n}(t)$ .

**Observe:** Notice that  $\hat{n}'(t)$  is perpendicular to  $\hat{n}(t)$ , hence

$$\hat{n}'(t) \in \text{span}\{\hat{t}(t), \hat{b}(t)\}.$$

Now, we have

$$\begin{aligned} \hat{n}(t) &= \hat{b}(t) \times \hat{t}(t) \\ \implies \hat{n}'(t) &= \hat{b}'(t) \times \hat{t}(t) + \hat{b}(t) \times \hat{t}'(t) \\ \implies \hat{n}'(t) &= (\tau(t) \cdot \hat{n}(t)) \times \hat{t}(t) + \hat{b}(t) \times (\kappa(t) \cdot \hat{n}(t)) \\ \implies \hat{n}'(t) &= -\tau(t) \cdot \hat{b}(t) - \kappa(t) \cdot \hat{t}(t) \end{aligned}$$

**Theorem 4** (Frenet-Serret formulae):

From the above observations, we have the set of equations called Frenet-Serret formulae, given by

$$\begin{aligned} \hat{t}'(t) &= \kappa(t) \cdot \hat{n}(t) \\ \hat{n}'(t) &= -\tau(t) \cdot \hat{b}(t) - \kappa(t) \cdot \hat{t}(t) \\ \hat{b}'(t) &= \tau(t) \cdot \hat{n}(t) \end{aligned}$$

or, in a more concise way, we have

$$\begin{bmatrix} \hat{t}'(t) \\ \hat{n}'(t) \\ \hat{b}'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & -\tau(t) \\ 0 & \tau(t) & 0 \end{bmatrix} \begin{bmatrix} \hat{t}(t) \\ \hat{n}(t) \\ \hat{b}(t) \end{bmatrix}$$

**Observe:** Observe that the set of differential equations in the Frenet-Serret formulae involves a Skew-Symmetric matrix, and also, when we have a non-planar curve, the corresponding  $\tau(t)$  and  $\kappa(t)$  are non-zero, so usually this is assumed.

**Motivation:**

- Given a smooth unit-speed regular curve, we can always find this set of equations (Frenet-Serret), by finding curvature and torsion functions.
- Given two functions  $\kappa(t) : I \rightarrow \mathbb{R}^+$  and  $\tau : I \rightarrow \mathbb{R}$ , can we find a regular unit-speed curve  $\gamma(t)$  using the Frenet-Serret equations such that its curvature and torsion coincide with  $\kappa$  and  $\tau$  respectively?
- Also, if such a curve exists, is it unique?
- Notice that given two such functions, and any initial condition on  $\hat{t}, \hat{n}, \hat{b}$ , we obtain a set of differential equations.

**Lecture 06: Parametrized Surfaces in  $\mathbb{R}^3$**

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**Definition 15** (Rigid Transformation):

Let  $S$  be a subset of  $\mathbb{R}^n$ , then a rigid transformation  $T$  on  $S$  is function such that

$$T(x) = \rho(x) + c \quad \forall x \in S,$$

where  $\rho$  is a special orthogonal transformation (orthogonal with determinant 1, preserves orientation), and  $c \in \mathbb{R}^n$  is a constant.

**Theorem 5:**

Let  $\kappa : I \rightarrow \mathbb{R}^+$  and  $\tau : I \rightarrow \mathbb{R}$ , be two smooth functions. Then

- There exists a unit speed parametrized curve  $\gamma : I \rightarrow \mathbb{R}^3$  such that the curvature and torsion functions of  $\gamma$  are  $\kappa$  and  $\tau$  respectively.
- If  $\gamma_1$  and  $\gamma_2$  are any two curves whose curvature and torsion coincide respectively with the given  $\kappa$  and  $\tau$ , then they are related to each other by a rigid transformation. That is,

$$\gamma_1(t) = \rho \circ \gamma_2(t) + c \quad \forall t \in I.$$

In other words, the curve corresponding to the given curvature and torsion functions is unique upto a rigid transformation.

**Definition 16** (Parametrized Surfaces):

Let  $U$  be an open subset of  $\mathbb{R}^2$ . A smooth injective map  $\sigma : U \rightarrow \mathbb{R}^n$  is said to be a parametrized surface in  $\mathbb{R}^n$ .

**Note:** Regularity in curves meant that the tangent to the curve at each point was a straight line, and the Jacobian had rank 1. A surface  $\sigma : U \rightarrow \mathbb{R}^3$ , given by

$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v)),$$



is said to be regular if the Jacobian

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

has rank 2 at each point in  $U$ . That is, the set of vectors  $\left\{ \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial v} \right\}$  (they are basically columns of the Jacobian matrix above) is linearly independent.

**Definition 17** (Tangent Space of a Surface at a Point):

Let  $\sigma : U \rightarrow \mathbb{R}^3$  be a regular surface, and let  $p \in \sigma(U) = S$  be a point on the surface. Then due to  $\sigma$  being an injection, there exists a unique  $(x_0, y_0) \in U$  such that  $\sigma(x_0, y_0) = p$ . Then the tangent to the surface  $S$  at the point  $p$  is defined as

$$T_p(S) := \text{span} \left\{ \frac{\partial \sigma}{\partial u}(x_0, y_0), \frac{\partial \sigma}{\partial v}(x_0, y_0) \right\}.$$

Notice that this is 2 dimensional subspace of  $\mathbb{R}^3$  as  $\sigma$  is regular.

**Example:** Consider  $U \subseteq \mathbb{R}^2$  be open, and  $\sigma : U \rightarrow \mathbb{R}^3$  given by  $\sigma(u, v) = (u, v, 0)$ . It is clearly regular, as the Jacobian has rank 2 and observe that since  $U$  is homeomorphic to  $\sigma(U)$ , every open subset in  $\mathbb{R}^2$  can be embedded this way as a surface in  $\mathbb{R}^3$ . Also, for any such set,  $T_p(S)$  will be plane, homeomorphic to  $\mathbb{R}^2$ .

**Example:** Let  $\sigma : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be an regular parametrized surface, then for every point  $\sigma(u, v) = p$ , we have the tangent plane  $T_p(S)$ , and a linear map  $\sigma_* : \mathbb{R}^2 \rightarrow T_p(S)$  given by

$$\sigma_*(a, b) = a\sigma_u + b\sigma_v.$$

**Example** (Spherical Coordinates): Consider the unit sphere in  $\mathbb{R}^3$ . Let  $(x, y, z)$  be a point on the sphere. Let  $\theta$  be the angle between the vector  $(x, y, 0)$  and the  $x$ -axis (called Azimuthal angle). Let  $\phi$  be the angle between the vector  $(x, y, z)$  and the  $z$ -axis. Then each point on the sphere can be uniquely represented using appropriate  $\theta$  and  $\phi$  as

$$(x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

## References

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