# Differential Geometry

**Ojas G Bhagavath**

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| Course | MSM423 Differential Geometry |
| Instructor | Dr. Saikat Chatterjee |
| Prerequisites | MAT413 Analysis on Manifolds |
| Learning Outcomes | • Understanding the classical interpretation of various curvatures of a surface and their relation to geodesics.<br>• Understanding the local and global geometry of smooth manifolds and smooth vector bundles. |
| Syllabus | • Gauss curvature, Gauss curvature formula in terms of first and second fundamental forms. Intrinsic property of the Gauss curvature. (6)<br>• Covariant derivative of a vector field along a curve; Relation between covariant derivative and total curvature of a curve; A geodesic as a curve with vanishing covariant derivative. (6)<br>• Manifolds: Definition, examples, Tangent vector space at a point, Basis of the tangent vector space. Smooth functions on a manifold, maps between Manifolds. Differential of a map. (6)<br>• Sub-manifolds; Regular value theorem. Lie groups, examples; Submersion, Immersion and Embeddings. (6)<br>• Smooth vector bundles, smooth sections, Dual bundles, existence of local sections. (5)<br>• Tangent bundles; Smooth vector fields; Lie bracket of smooth vector fields; Co-tangent bundles; Differential 1-forms. (5)<br>• Differential p-forms. Orientation. Exterior derivative. Closed and exact forms. Integration of a p-form on a p-dim sub manifold. Stokes theorem. (6) |

## Lecture 01: Introduction to Curves and Surfaces 05 Jan 2024 10:30

***Definition* 1** (Curves)**:**
Let $I$ be an open interval in $\mathbb{R}$. A smooth parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$.
The image $\gamma(I) \subseteq \mathbb{R}^n$ is called trace of the curve $\gamma$.

**Example:** $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$, defined as $\gamma_1(t) = (\cos(t), \sin(t))$ and $\gamma_2(t) = (\sin(3t), \cos(3t))$ have the same trace, but they are different curves.

**Observe:** Consider any circle in $\mathbb{R}^2$, it is the set of all points in $\mathbb{R}^2$ satisfying a certain quadratic equation.
It can also be viewed as a trace of some curve.

***Definition* 2** (Level Set)**:**
Let $U \subseteq \mathbb{R}^n$ be any open set, and $f : U \to \mathbb{R}$ be any function.
Then for a given constant $c \in \mathbb{R}$, the level set is defined as

$$L_c(f) := \{X \in U \mid f(X) = c\} \subseteq U \subseteq \mathbb{R}^n$$

**Example:** Let $U = \mathbb{R}^2$, and $f(x, y) = x^2 + y^2$, then
$L_0(f) = (0, 0)$, a point,
$L_1(f) = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, a circle in $\mathbb{R}^2$,
$L_{-1}(f) = \varnothing$.

***Definition* 3** (Graph)**:**
Let $A$ and $B$ be any sets. Let $f : A \to B$ be a function, then the graph of $f$ is the function $G_f : A \to A \times B$, given by

$$G_f(x) = (x, f(x)) \in A \times B$$

**Note:** Notice that a graph is also a curve, and if $f$ is a smooth function, then $G_f$ is a smooth curve.

***Theorem* 1** (Implicit Function Theorem (2D Case))**:**
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defining a curve $F(x, y) = c$.
Let $(x_0, y_0)$ be a point on the curve.

- If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, then there exists a neighborhood around $(x_0, y_0)$ where we can write $x = g(y)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $x = g(y)$ in that neighborhood.
- If $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then there exists a neighborhood around $(x_0, y_0)$ where we can write $y = f(x)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $y = f(x)$ in that neighborhood.

**Example:** Consider $S^1 = L_1(x^2 + y^2) = \{(x, y) \mid x^2 + y^2 = 1\}$
Here, $F(x, y) = x^2 + y^2$, and $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$.
Therefore, for any point $(x_0, y_0)$ in $S^1$, where such that $x_0 \neq 0$, there exists a neighborhood in which $x = g(y)$.

**Note:** It is worthwhile to notice that a graph is always a curve. Implicit function theorem gives a condition on when a curve (level set in particular) can be seen as the graph of a curve.

## Lecture 02: Curves and Surfaces 08 Jan 2024 10:30

**Observe:** Consider the level set of $F(x, y) = x^2 + y^2$, $L_1(f) = S^1$. As seen before, since $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$, for any point $(x_0, y_0)$ in $S^1$, where $x_0 \neq 0$, we obtain a neighborhood in $S^1$ such that the curve (level set) can be written as a graph of a function $y = g(x)$.
Let $S^{1+}$ denote $(\mathbb{R} \times \mathbb{R}^+) \cap S^1$, the upper half of the unit circle. Now, for a point in $S^{1+}$, the corresponding $g(x)$ is $\sqrt{1 - x^2}$.
Now the graph of $g$, say $G_g$ is a function from $(-1, 1)$ to $S^{1+}$, and we can easily talk about continuity, differentiablity of $G_g$.
However, notice that $G_g$ is a surjection, and an inverse exists, $G_g^{-1} : S^{1+} \to (-1, 1)$.
To talk about continuity of $G_g^{-1}$ we need to define a topology on $S^{1+}$, and the subspace topology is the most obvious choice.
This $G_g^{-1}$ happens to be continuous when we consider the subspace topology on $S^{1+}$. Further, open subsets of $S^{1+}$ homeomorphic to the open subsets of $\mathbb{R}$, that means that this topological space is locally homeomorphic to $\mathbb{R}$, but the same does not extend to the whole space, that is, there is no homeomorphism from $S^{1+}$ to $\mathbb{R}$.

**Note:** Notice that we cannot talk about differentiablity of $S^{1+}$ in the above immediately, as differentiability requires some sort of vector space structure.

**Example:** Consider $S_3^1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
Doing the same analysis on this set as before using the Implicit function theorem, we obtain that that after removing the appropriate critical points, and giving a topology to whatever remains, we obtain patches obtain open sets in whatever remains that are homeomorphic to open subsets of $\mathbb{R}^2$, hence the topological subspace obtained after removing the critical points is locally homeomorphic to $\mathbb{R}^2$.

***Definition* 4** (Tangent to a smooth curve at a point)**:**
Let $I$ be an open interval in $\mathbb{R}$, and $\gamma : I \to \mathbb{R}^n$ be a smooth curve given by $\gamma(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then for a point $t \in I$, the tangent of the curve at $t$ is defined as

$$\gamma'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$$

***Definition* 5** (Regular Curve)**:**
A smooth curve $\gamma : I \to \mathbb{R}^n$ is said to be a regular curve if $\gamma'(t) \neq 0$ $\forall t \in I$.

***Definition* 6** (Diffeomorphism)**:**
Let $A$ and $B$ be two manifolds, and a function $f : A \to B$ is said to be a diffeomorphism if $f$ is bijective, differentiable, and the inverse $f^{-1} : B \to A$ is differentiable as well.

**Note:** In this course, by smooth, we mean a $C^\infty$ function.

**Definition 7** (Reparametrization of a Curve)**:**
Let $J, I$ be open intervals of $\mathbb{R}$, let $\gamma : I \to \mathbb{R}^n$ be a curve, and let $\phi : J \to I$ be a diffeomorphism. Then $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ is a curve on $J$ called as reparametrization of $\gamma$.

**Observe:** If $\beta$ is a reparametrization of $\gamma$, and the setup is as above, then $\beta(s) = \gamma \circ \phi(s)$ for each $s \in J$. Then

$$\beta'(s) = \gamma'(\phi(s)) \cdot \phi'(s)$$

Now, $\phi$ is a diffeomorphism and $(\phi \circ \phi^{-1})' = 1$ and by chain rule, we have $(\phi^{-1})' = \frac{1}{\phi'}$. And since $(\phi^{-1})'$ exists throughout $I$, $\phi'$ cannot be 0 on $J$, hence $\gamma'(s) \neq 0$.
Hence if $\gamma$ is regular, then so is $\beta$, hence regularity is preserved under a diffeomorphism.

**Definition 8** (Arc Length of a Curve)**:**
Let $\gamma : I \to \mathbb{R}^n$ be a regular parametrized curve. For fixed $t_1, t_2 \in I$, the arc length between $t_1$ and $t_2$ is given by

$$\begin{aligned} L_\gamma(t_1, t_2) &= \int_{t_1}^{t_2} ||\gamma'(t)||\, dt \\ &= \int_{t_1}^{t_2} \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + \cdots + (x_n'(t))^2}\, dt \end{aligned}$$

## Lecture 03: Reparametrizations — 10 Jan 2024 10:30

**Theorem 2** (Inverse Function Theorem (a particular case))**:**
Let $U \subseteq \mathbb{R}^n$, and $f : U \to \mathbb{R}$ be a smooth function. Let $x \in U$ be such that for the Jacobian $D(f)(x)$ of $f$ at $x$, the determinant is $\det[D(f)(x)] \neq 0$, then there exists a neighborhood $W$ of $x$ in $U$ such that the restriction $f|_W : W \to f(W)$ is a diffeomorphism.

**Note:** Notice that this theorem gives a local property. That is, with the given hypothesis, we can only really claim that there exists a neighborhood upon which the restriction is a diffeomorphism. However, if the Jacobian of $f$ has non-zero determinant at each $x$ in $U$, then we cannot claim that $f$ is a diffeomorphsim.

**Example:** $U = \mathbb{R}\backslash\{0\}$ and $f : U \to \mathbb{R}$ given by $f(x) = x^2$ satisfies the hypothesis, yet is not a diffeomorphism from $U$ to $f(U)$, it is not even injective.

**Observe:** However, if $U$ happens to be a connected, and $f$ is smooth, then the above hypothesis is sufficient to claim that $f$ is a diffeomorphism from $U$ to $f(U)$.

**Definition 9** (Unit Speed Reparametrization)**:**
For the intervals $I, J$, let $\gamma : I \to \mathbb{R}^n$ be a curve, let $\phi : J \to I$ be a diffeomorphism such that the reparametrization $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ has the property that $|\beta'(s)| = 1$ for each $s \in J$. Then the reparametrization $\beta$ is said to be unit speed reparametrization.

**Note:** Notice that when a curve is reparametrized, the trace/image of the curve remains the same.

**Note** (Condition for a reparametrization to be unit speed)**:** If $\beta = \gamma \circ \phi$ is a reparametrization of the curve $\gamma$ with the diffeomorphism $\phi$ as in the above setting, then

$$
\begin{aligned}
|\beta'(s)| = 1 &\implies |\gamma'(\phi(s))||\phi'(s)| = 1 \\
&\implies \boxed{|\phi'(s)| = \frac{1}{|\gamma'(\phi(s))|}}
\end{aligned}
$$

***Definition* 10** (Arc Length Function)***:***
Let $I$ be an interval, and $t_0 \in I$ be a fixed element. Let $\gamma$ be a regular curve defined on $I$. Then the Arc Length function $L_\gamma : I \to \mathbb{R}$ is defined as

$$L_\gamma(t) = \int_{t_0}^{t} |\gamma'(t)|\, dt$$

**Property** (Properties of $L_\gamma$)**:**

1. $L'_\gamma(t) = |\gamma'(t)|$ (It is differentiable).
2. Since $\gamma(t)$ is regular, $\gamma'(t)$ is nowhere 0, and hence $|\gamma'(t)|$ is continuous, hence $L'_\gamma(t)$ is continuous and consequently, smooth.
3. $L'_\gamma(t) > 0$ on the interval $I$, that implies that $L_\gamma$ is diffeomorphism from $I$ to $L_\gamma(I) := J$.
4. Let $S_\gamma = L_\gamma^{-1} : J \to I$ be the inverse of $L_\gamma$ (exists because $\gamma$ is a diffeomorphism, also note that $S_\gamma$ is a diffeomorphism).
   Consider the reparametrization $\beta = \gamma \circ S_\gamma$, it follows that

   $$|\beta'(s)| = |\gamma'(S_\gamma(s))||S'_\gamma(s)| \qquad (*)$$

   From the chain rule, we have

   $$S'_\gamma(s) = \frac{1}{L'_\gamma(S_\gamma(s))} = \frac{1}{\gamma'(S_\gamma(s))} \qquad (**)$$

   $(*)$ and $(**)$ imply that $|\beta'(s)| = 1$ for all $s \in J$, hence $\beta$ is a unit speed reparametrization.

***Theorem* 3*:***
Given an interval $I$, and any regular curve $\gamma : I \to \mathbb{R}^n$, there exists an interval $J$ and a diffeomorphism $\phi : J \to I$ such that the reparametrization $\beta = \gamma \circ \phi$ is a unit speed parametrization.

*Proof.* The proof follows from 4. in the above properties of $L_\gamma$, where
$J = L_\gamma(I)$, and $\phi = L_\gamma^{-1} = S_\gamma$ □

# References

[1] M. P. Do Carmo. *Differential Geometry of Curves and Surfaces*. 2nd ed. Dover Publications, 2016.

[2] J. M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Springer, 2013.

[3] L. W. Tu. *An Introduction to Manifolds*. 2nd ed. Springer, 2011.