Functional Analysis

Ojas G Bhagavath

Course	MSM421 Functional Analysis
Instructor	Prof. Rajeev Bhaskaran
Prerequisites	MSM321 Complex Analysis and MSM411 Measure Theory
Learning Outcomes	Based on core analysis courses and linear algebra, this course builds further on the study of Banach and Hilbert spaces. The theory and techniques studied in this course support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas
Syllabus	• Normed linear spaces, Riesz lemma, characterization of finite dimensional spaces, Banach spaces. Operator norm, continuity and boundedness of linear maps on a normed linear space. (6) • Fundamental theorems: Hahn-Banach theorems, uniform boundedness principle, divergence of Fourier series, closed graph theorem, open mapping theorem and some applications. (8) • Dual spaces and adjoint of an operator: Duals of classical spaces, weak and weak* convergence, adjoint of an operator. (6) • Hilbert spaces: Inner product spaces, orthonormal set, Gram-Schmidt orthonormalization, Bessel's inequality, orthonormal basis, separable Hilbert spaces. Projection and Riesz representation theorems: Orthonormal complements, orthogonal projections, projection theorem, Riesz representation theorem. (10) • Bounded operators on Hilbert spaces: Adjoint, normal, unitary, self-adjoint operators, compact operators. (5) • Spectral theorem: Spectral theorem for compact self adjoint operators, statement of spectral theorem for bounded self adjoint operators. (5)

Lecture 01: Revision of Linear Algebra 08 Jan 2024 11:30

Recall (Linear Algebra)***:***

- Vector spaces, the axioms, and examples.
- Linear dependence of subsets of a vector spaces (both finite and infinite), basis is a maximal linear independent subset of a vector space.
- Every vector space has a basis (Zorn's Lemma), every basis of a vector space has the same cardinality, hence, dimension of a vector space defined as the cardinality of any of its basis is well defined.

Lecture 02: Introduction to Banach Spaces 10 Jan 2024 11:30

Definition* 1** (Normed Linear Space):***
Consider a vector space V over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : V \to [0, \infty)$ is said to be a norm if it satisfies the following axioms:

N1. $\|x\| = 0$ if and only if $x = 0$.

N2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in V$.

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then the vector space equipped with such a norm is called as a normed linear space.

Motivation: When a norm is defined on a vector space, it induces a topology (metric topology) on the vector space. With a topological structure on V, we can do further analysis in V such as limits of sequences, convergence, continuity of functions, etcetera.

Remark: If $(V, \|\cdot\|)$ is a normed linear space, then $d(x, y) = \|x - y\| : V \times V \to [0, \infty)$ is a metric on V:

M1. $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = \iff x = y$

M2. $d(x, y) = |-1|\|x - y\| = \|y - x\| = d(y, x)$

M3. $d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

Equipped with this metric, (V, d) is a metric space, and we can define an open subset of V as follows:
$U \subseteq V$ is said to be open if for all x in U, there exists an $r > 0$ such that

$$B(x, r) := \{y \in V \mid d(x, y) < r\} \subseteq U$$

The collection of these open subsets defines a topology on V.

Recall:

- In a normed linear space $(V, \|\cdot\|)$, if a sequence x_n converges to x and another sequence y_n converges to y, then the sequence $x_n + y_n$ converges to $x + y$.
- If α_n is a sequnce in $\mathbb{F}$ that converges to α and x_n is a sequence in V that converges to x, then the sequence $\alpha_n x_n$ converges to αx.

- Therefore, addition and scalar multiplication are continuous maps in the topological spaces induced by a norm.

Motivation: Now that we have a metric space structure on V, we can talk about completeness (every Cauchy sequence converges).

> ***Definition* 2** (Banach Spaces)***:***
> A normed linear space $(V, \|\cdot\|)$ is said to be a Banach Space if and only if (V, d) is a complete metric space. That is, every Cauchy sequence in V converges in the metric space (V, d).

Recall:

- **Hölder's inequality:**
  Let $p, p* \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p*} = 1$ (such a pair of real numbers is called **conjugate exponents** of each other). For a $p \in [1, \infty)$, define a function from $\mathbb{R}^N$ to $\mathbb{R}$ as:

$$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}$$

  and for $p = \infty$, define another a function from $\mathbb{R}^N$ to $\mathbb{R}$ as:

$$\|x\|_\infty = \max_{1 \leq i \leq N} (x_i)$$

  Let $x = (x_1, \cdots, x_N), y = (y_1, \cdots, y_N) \in \mathbb{R}^N$.
  Then the following inequality holds:

$$\sum_{i=1}^{N} |x_i y_i| \leq \|x\|_p \|y\|_{p*}$$

- **Minkowski's inequality:**
  For any $p \in [1, \infty]$, and $x, y \in \mathbb{R}^N$, we have the following inequality:

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

- **Jensen's Inequality**:
  For any pair of real numbers $x, y \in \mathbb{R}$, the following inequality holds:

$$|a + b|^p \leq 2^{p-1}(a^p + b^p)$$

Example (p-norm on $\mathbb{R}^N$)**:** For any $p \in [1, \infty]$, $(\mathbb{R}^N, \|\cdot\|_p)$ is a Banach space.

Proof. First we need to prove that $\|\cdot\|_p$ is a norm on $\mathbb{R}^N$.
This follows from the Minkowski's inequality.

Note: This is usually called as p-norm on $\mathbb{R}^N$, and the special case when $p = 2$ is called the Euclidean norm.

Then we need prove the completeness of $(\mathbb{R}^N, \|\cdot\|_p)$. First we prove for $p \in [1, \infty)$ case.
Suppose $x^{(n)}$ is a Cauchy sequence in the NLS. Then for each $\epsilon > 0$, there exists a postive integer t such that $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ whenever $m, n > t$. That is, $m, n > t$ implies that

$$\sum_{i=1}^{N} \left|x_i^{(n)} - x_i^{(m)}\right|^p < \epsilon^p$$

Hence $\left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon$ whenever $m, n > t$. That means $x_i^{(n)}$ is a Cauchy sequence in $\mathbb{R}$, hence converges (since $\mathbb{R}$ is complete) to a point, say $x_i \in \mathbb{R}$.
Let $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, where each x_i are defined as above. We claim that $x^{(n)}$ converges to x.
We have the inequality $|x_i^{(n)} - x_i^{(m)}| < \epsilon^p$ whenever $m, n > t$.
Fix n, and take the limit as $m \to \infty$, we obtain $|x_i^{(n)} - x_i| \leq \epsilon^p$.
Therefore, after an appropriate adjustment, we obtain an ϵ', such that for any $\epsilon > 0$, there exists a postive integer t such that $n > t$ implies that the finite sum satisfies the inequality

$$\left(\sum_{i=1}^{N} \left|x_i^{(n)} - x_i\right|^p\right)^{\frac{1}{p}} = \|x^{(n)} - x\|_p < \epsilon'$$

Hence $x^{(n)}$ converges to x. Therefore, every Cauchy sequence in $(\mathbb{R}^N, \|\cdot\|_p)$ converges, hence it is a Banach space when $p \in [1, \infty)$.
When $p = \infty$, as convergence and Cauchy criterion hold if and only if they hold componentwise in $\mathbb{R}^N$, $(\mathbb{R}^N, \|\cdot\|_\infty)$ is Banach as well. □

Note: Due to completeness of $\mathbb{C}$, for any $p \in [1, \infty)$, $(\mathbb{C}, \|\cdot\|_p)$ is a Banach space as well.

Example ($\boldsymbol{\ell_p}$ spaces): For $1 \leq p < \infty$, define a collection of sequences as follows:

$$\ell_p := \left\{ x = (x_1, x_2, \cdots) \mid x_i \in \mathbb{F}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

Now, define a function from ℓ_p to $\mathbb{F}$ as $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Then $(\ell_p, \|\cdot\|_p)$ is a Banach space.

Proof. The proof proceeds in several steps:

1. **$\boldsymbol{\ell_p}$ is a vector space.**

   Proof. Let $x, y \in \ell_p$, then from the Jensen's inequality, we have

   $$|x_i + y_i|^p \leq (|x_i| + |y_i|)^p \leq 2^{p-1}(|x_i|^p + |y_i|^p) < \infty$$

   and for any $\alpha \in \mathbb{F}$ and $x \in \ell_p$, αx clearly belongs to ℓ_p.
   Hence ℓ_p is a subspace of the space of all sequences in $\mathbb{F}$. □

2. **$\|\cdot\|_p$ is a norm on $\boldsymbol{\ell_p}$.**

   Proof. The first two axioms of a norm are trivial to prove. We need to show the triangle inequality.
   Let $x, y \in \ell_p$, then for any postive integer N, from Minkowski's Inequality, we have

   $$\sum_{i=1}^{N} |x_i + y_i|^p \leq \left[\left(\sum_{i=1}^{N} |x_i|^p\right) + \left(\sum_{i=1}^{N} |y_i|^p\right)\right]^p \leq [\|x\|_p + \|y\|_p]^p < \infty$$

   Since N is arbitrary, and the right side is independent of the choice of N, it follows that

   $$\sum_{i=1}^{\infty} |x_i + y_i|^p \leq [\|x\|_p + \|y\|_p]^p < \infty$$

   Hence the triangle inequality holds, and hence $\|\cdot\|_p$ is a norm on ℓ_p. □

3. **($\ell_p, \|\cdot\|_p$) is a complete space.**

Proof. Let $x^{(n)}$ be a Cauchy sequence in the NLS, that is, for each $\epsilon > 0$, there exists a postive integer t such that $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ whever $m, n > t$. That is, $m, n > t$ implies that

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}| < \epsilon^p$$

Hence $\left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon$ whenever $m, n > t$, hence $x_i^{(n)}$ is a Cauchy sequence in $\mathbb{R}$, hence converges (since $\mathbb{R}$ is complete) to a point, say $x_i \in \mathbb{R}$.
Construct a sequence $x = (x_1, x_2 \cdots x_n, \cdots)$ where x_i is defined as above.
We need to show that $x \in \ell_p$. Since $x^{(n)}$ is a Cauchy sequence, it is bounded, hence there exists a constant $C > 0$ such that

$$\|x^{(n)}\|_p^p = \sum_{i=1}^{\infty} \left|x_i^{(n)}\right|^p \leq C, \ \forall n.$$

Let k be any fixed positive integer. Then,

$$\sum_{i=1}^{k} \left|x_i^{(n)}\right|^p \leq C$$

which implies that

$$\sum_{i=1}^{k} |x_i|^p \leq C.$$

Since k is arbitrary and the right side does not depend on it, it follows that

$$\sum_{i=1}^{\infty} |x_i|^p \leq C < \infty.$$

Hence $x \in \ell_p$.
Now we need to show that $x^{(n)}$ converges to x. From before, whenever $m, n > t$, we have

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Hence, for any positive integer k, and whenever $m, n > t$, we have

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Fix $n > t$, and take the limit $m \to \infty$, we obtain

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i|^p < \epsilon^p.$$

Since k is arbitrary and the right side does not depend on it, hence whenever $n > t$, we have

$$\|x^{(n)} - x\|_p \leq \epsilon.$$

That is, $x^{(n)}$ converges to x. Hence every Cauchy sequence in $(\ell_p, \|\|_p)$ converges, hence it is Banach. $\square$

Example: Consider the set of sequences

$$\ell_\infty := \left\{ x = (x_i) \mid x_i \in \mathbb{F}, \sup_{1 \le i < \infty} |x_i| < \infty \right\},$$

the set of all bounded sequences in $\mathbb{F}$. Clearly it is a vector space under componentwise addition and scalar multiplication. Also define the function

$$\|x\|_\infty = \sup_{1 \le i < \infty} |x_i|.$$

Then $\|x\|_\infty$ is a norm on ℓ_∞ and $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space.

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Example: Define the set of functions

$$C[0,1] := \{ f \mid f : [0,1] \to \mathbb{F} \text{ is a continuous function} \}.$$

Clearly, it is a vector space under the operations of addition of functions and scalar multiplication. Define a function from $C[0,1]$ to $\mathbb{F}$ as

$$\|f\|_\infty = \sup_t |f(t)|.$$

Then $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

Proof. The first two axioms of norm hold trivially. Now, for any $t \in [0,1]$, due to $|\cdot|$ being a norm on $\mathbb{F}$, it follows that

$$|f(t) + g(t)| \le |f(t)| + |g(t)|.$$

If the supremum of $|f(t) + g(t)|$ occurs at t_0, then

$$\|f + g\|_\infty = |f(t_0) + g(t_0)| \le f(t_0) + g(t_0) \le \|f\|_\infty + \|g\|_\infty.$$

Hence $\|\cdot\|_\infty$ is a norm on $C[0,1]$.
Now to show that the NLS $(C[0,1], \|\cdot\|_\infty)$ is complete, consider a Cauchy sequence f_n in $C[0,1]$. that is, for all $\epsilon > 0$, there exists a positive integer $p > 0$ such that $m, n > p$ implies that $\|f_n - f_m\| < \epsilon$. That is, whenever $m, n > p$ it implies that

$$\sup_t |f_n(t) - f_m(t)| < \epsilon.$$

Hence $|f_n(t) - f_m(t)| < \epsilon$ for each t, therefore, for a fixed t, the sequence $f_n(t)$ is Cauchy in $\mathbb{R}$, and hence converges (since $\mathbb{F}$ is complete) to a point, say $f(t)$.
Define a function $f : [0,1] \to \mathbb{F}$ as $f(t)$ as above. Now we claim that $f \in C[0,1]$. Notice the following:

$$|f(t) - f(s)| \le |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)|.$$

Since f_n converges to f, for any $\epsilon > 0$, there exists a sufficiently large n, such that $|f_n(t) - f(t)| < \epsilon$ and $|f_n(s) - f(s)| < \epsilon$. Now, since f_n is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$|s-t|<\delta$ implies that $|f_n(t)-f_n(s)|<\epsilon$, hence we have, for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|s-t|<\delta \implies |f(s)-f(t)|<3\epsilon.$$

Hence f is continuous and belongs to $C[0,1]$.
Now, we from the Cauchy criterion, we have, for each $\epsilon>0$, a positive integer p such that $m,n>p$ implies that

$$\sup_t |f_n(t)-f_m(t)|<\epsilon.$$

Fixing $n>p$, and taking the limit $m\to\infty$, we obtain $|f_n(t)-f(t)|\leq\epsilon$ for independent of t. Hence for a suitable for any $\epsilon>0$, there exists a suitable p such that $\|f_n-f\|_\infty<\epsilon$ whenever $n>p$. Hence f_n converges to f in the NLS.
Hence $(C[0,1],\|\cdot\|_\infty)$ is a Banach space. □

Recall:
Linear transformations; matrix representation of linear transformations; examples of linear transformations between finite dimensional vector spaces, between infinite dimensional vector spaces.

Note: For a function between normed linear spaces, since a normed linear space has a metric induced by the norm, we can talk about sequential definition, $\epsilon-\delta$ definition of continuity. And with the topological structure on the normed linear spaces, we can talk about the inverse image of an open (or closed) set being open (or closed) set definition of continuity. And all of those are equivalent.

Definition* 3** (Continuous Linear Transformation):***
A continuous linear transformation is what it says on the tin, a linear transformation between two normed linear spaces over the same field, that is also continuous (we can talk about continuity because a normed linear space has an induced topology).

Proposition* 1*:
Let $(V,\|\cdot\|_v)$ and $(W,\|\cdot\|_w)$ be normed linear spaces over the same field $\mathbb{F}$, then the following are equivalent.

i. T is continuous.

ii. T is continuous at 0.

iii. There exists a positive real number k such that $\|T(x)\|_w\leq k\|x\|_v\ \forall x\in V$.

iv. $T\left(\overline{B}_v(0,1)\right)\subseteq\overline{B}_w(0,r)$ for some $r>0$.

Proof. i. $\implies$ ii. is obvious.
Suppose ii. is true. Then for any sequence x_n converging to x in V, the sequence x_n-x converges to 0. Due to continuity of T at 0, the sequence $T(x_n-x)$ in W must converge to $T(0)=0$. That implies that $T(x_n)$ converges to $T(x)$ in W. Hence we have ii. $\implies$ i.
Suppose ii. is true. Then for every $\epsilon>0$, we have a $\delta>0$ such that

$$\|x\|_v<\delta \implies \|T(x)\|_w<\epsilon.$$

In particular, for $\epsilon=1$, we obtain a δ_1 satisfying the above. Consider $k=\frac{2}{\delta_1}$. For any $x\in V\backslash\{0\}$, define

$$y=\frac{\delta_1}{2}\frac{x}{\|x\|_v}.$$

Then $\|y\|_v = \frac{\delta_1}{2} < \delta$, and hence $\|T(y)\|_w < \epsilon$. That is,

$$\frac{\delta_1}{2\|x\|_v}\|T(x)\|_w < 1.$$

Upon rearrangement, we obtain

$$\|T(x)\|_w < \frac{2}{\delta_1}\|x\|_v = k\|x\|_v,$$

hence we have ii. $\implies$ iii.
iii. $\implies$ ii. is obvious from the $\epsilon - \delta$ definition of continuity.
iii $\iff$ iv. is obvious after substituting $k = r$. $\square$

Example (Operator Norm)**:** Let V and W be two normed linear spaces over the same field. On the collection of all linear operators from V to W, define an operation as follows: If T is a linear operator from V to W, then

$$\|T\|_{op} := \sup_{x \in \overline{B}_v(0,1)} \|T(x)\|_w = \sup\left\{\|T(x)\|_w : \|x\|_v \le 1\right\}.$$

Consider the set of operators given by

$$\mathcal{L}(V, W) := \{T \mid T \text{ is a continuous linear operator from } V \text{ to } W\}.$$

Then $(\mathcal{L}(V, W), \|\cdot\|_{op})$ is a normed linear space, and the norm $\|\cdot\|_{op}$ is called as the operator norm.

Proof. $\|\cdot\|_{op}$ being a norm on $\mathcal{L}(V, W)$ follows from follows from $\|\cdot\|_w$ being a norm on W. $\square$

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***Proposition* 2:**
Let V and W be normed linear spaces over the same field $\mathbb{F}$. Let $T \in \mathcal{L}(V, W)$. Consider the following quanitites

$$\begin{aligned}
\|T\|_{op} &:= \sup\left\{\|T(x)\|_w : \|x\|_v \le 1\right\} \\
\alpha &:= \sup\left\{\|T(x)\|_w : \|x\|_v = 1\right\} \\
\beta &:= \sup\left\{\frac{\|T(x)\|_w}{\|x\|_v} : x \in V\backslash\{0\}\right\} \\
\gamma &:= \inf\left\{k > 0 : \|T(x)\|_w \le k\|x\|_v \ \forall x \in V\right\}.
\end{aligned}$$

Then

$$\alpha = \beta = \gamma = \|T\|_{op}.$$

Proof. For every x such that $\|x\|_v = 1$, we have $\|T(x)\|_w = \frac{\|T(x)\|_w}{\|x\|_v} \le \beta$. Hence we have $\alpha \le \beta$. Also notice that

$$\begin{aligned}
&\|T(x)\|_w \le \alpha \quad \text{whenever } \|x\|_v = 1 \\
\implies &\left\|T\left(\frac{y}{\|y\|_v}\right)\right\|_w \le \alpha \quad \forall y \in V\backslash\{0\} \\
\implies &\frac{\|T(y)\|_w}{\|y\|_v} \le \beta \quad \forall y \in V\backslash\{0\} \\
\implies &\beta \le \alpha.
\end{aligned}$$

For any $k > 0$ such that $\|T(x)\|_w \leq k\|x\|_v \ \forall x \in V$, then it follows that $\frac{\|T(x)\|_w}{\|x\|_v} \leq k$. Then it is immediate that $\beta \leq k$. Hence we have $\beta \leq \gamma$.
We also have $\|T(x)\|_w \leq \beta\|x\|_v \ \forall x \in V$, hence by definition, $\gamma \leq \beta$.
Clearly, we have $\alpha \leq \|T\|_{op}$ by definition. Now if for any $k > 0$ such that $\|T(x)\|_w \leq \|x\|_v$ holds, for all $x \in V$, we have the particular case of $\|T(x)\|_w \leq k$ when $\|x\|_v \leq 1$, hence we have $\|T\|_{op} \leq k$, hence we have $\|T\|_{op} \leq \gamma$.
Hence we have

$$\alpha = \beta = \gamma = \|T\|_{op}.$$

□

Note: Notice from iv. point of *Proposition* 1 that a linear operator is continuos if and only if it takes bounded subsets to bounded subsets. So, the collection $\mathcal{L}(V, W)$ is also called as the set of bounded linear operators. And it is sufficient to check boundedness of bounded subsets to claim that an operator is continuous.

***Corollary* 1:**
If V and W are normed linear space over the same field, and if $T : V \to W$ is a continuos linear operator, then for any $x \in V$, we have

$$\|T(x)\|_w \leq \|T\|_{op}\|x\|_v$$

***Proposition* 3:**
If $(W, \|\cdot\|_w)$ is a Banach space, then $(\mathcal{L}(V, W), \|\cdot\|_{op})$ is a Banach space.

Proof. Suppose T_n is a Cauchy sequence in $(\mathcal{L}(V, W), \|\cdot\|_{op})$. Then for every $\epsilon > 0$, there exists a positive integer t such that $\|T_n - T_m\|_{op} < \epsilon$ whenever $m, n > t$. Now, for any $x \in V$, we have

$$\|T_n(x) - T_m(x)\|_w \leq \|T_n - T_m\|_{op}\|x\|_v$$

hence if follows that $T_n(x)$ is Cauchy in W, and (since W is complete), it converges to a point, say $T(x)$. Clearly T hence defined is linear. Now we need to show the boundedness of T.
Since T_n is Cauchy, it is bounded, hence $\|T_n\|_{op} < M$ for some $M > 0$.
Now, for any $x \in V$, we have

$$\|T_n(x)\|_w \leq \|T_n\|_{op}\|x\|_v \leq M\|x\|_v$$

Passing the limit as $n \to \infty$, we obtain that $\|T(x)\|_w \leq M\|x\|_v$. Hence $T \in \mathcal{L}(V, W)$.
Suppose x be vector in the closed unit ball. Then combined with the Cauchy criteria, whenever $m, n > t$, we have the following:

$$\begin{aligned} \|T_n(x) - T_m(x)\|_w &\leq \|T_n - T_m\|_{op}\|x\|_v \\ . &< \|T_n - T_m\|_{op} \\ &< \epsilon \end{aligned}$$

Fixing $n > t$ and taking $m \to \infty$, we obtain $\|T_n - T\|_{op} < \epsilon$. Hence T_n converges to T in $\mathcal{L}(V, W)$. □

Definition* 4** (Dual Space):***
The collection of all continuous linear functionals on a normed linear space, equipped with operator norm forms a Banach space, called as the dual space. That is

$$V^* = \mathcal{L}(V, \mathbb{F}) \text{ is called the dual space.}$$

Definition* 5** (Banach Algebra):***
Let V be a Banach space, then $\mathcal{L}(V) := \mathcal{L}(V, V)$ defines is a Banach space, and has a third operation of composition, hence forms an algebra.

Note: If V is a Banach space and $T_1, T_2 \in \mathcal{L}(V)$, then

$$\|T_1 T_2\|_{op} \le \|T_1\|_{op} \|T_2\|_{op}$$

Definition* 6** (Contraction):***
Let V be a normed linear space, linear operator $T \in \mathcal{L}(V)$ is said to be a contraction if $\|T\|_{op} \le 1$.

Example: Consider the set of sequences ℓ_2. For all $x = (x_i) \in \ell_2$, define a map T as follows:

$$T(x) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots\right).$$

Notice that

$$\|T(x)\|_2^2 = \sum_{i=1}^{\infty} \left|\frac{x_i}{i}\right|^2 \le \sum_{i=1}^{\infty} |x_i|^2 = \|x\|^2 < \infty.$$

This is a contraction.

Example: Let $1 \le p \le \infty$, let p^* be the conjugate exponent of p. Let $x \in \ell_p$ and $y \in \ell_{p*}$. Then define $T_y : \ell_p \to \mathbb{R}$ as:

$$T_y(x) := \sum_{i=1}^{\infty} x_i y_i$$

Then

$$\begin{aligned} |T_y(x)| &= \left|\sum_{i=1}^{\infty} x_i y_i\right| \\ &\le \|x\|_p \|y\|_{p*} \quad \text{(Hölder's inequality)} \\ &= k\|x\|_p \qquad \text{(Since } y \text{ is fixed)} \end{aligned}$$

Thus, T_y is a bounded linear functional on ℓ_p, and

$$\|T_y\|_{op} \le \|y\|_{p*}.$$

We shall later see that this inequality is infact an equality.

References

[1] R. Bhatia. *Notes on Functional Analysis*. 1st ed. Hindustan Book Agency, 2009.

[2] S. Kesavan. *Functional Analysis*. 2nd ed. Springer, 2023.