Functional Analysis

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Course	MSM421 Functional Analysis
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Prerequisites	MSM321 Complex Analysis and MSM411 Measure Theory
Learning Outcomes	Based on core analysis courses and linear algebra, this course builds further on the study of Banach and Hilbert spaces. The theory and techniques studied in this course support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas
Syllabus	• Normed linear spaces, Riesz lemma, characterization of finite dimensional spaces, Banach spaces. Operator norm, continuity and boundedness of linear maps on a normed linear space. (6) • Fundamental theorems: Hahn-Banach theorems, uniform boundedness principle, divergence of Fourier series, closed graph theorem, open mapping theorem and some applications. (8) • Dual spaces and adjoint of an operator: Duals of classical spaces, weak and weak* convergence, adjoint of an operator. (6) • Hilbert spaces: Inner product spaces, orthonormal set, Gram-Schmidt orthonormalization, Bessel's inequality, orthonormal basis, separable Hilbert spaces. Projection and Riesz representation theorems: Orthonormal complements, orthogonal projections, projection theorem, Riesz representation theorem. (10) • Bounded operators on Hilbert spaces: Adjoint, normal, unitary, self-adjoint operators, compact operators. (5) • Spectral theorem: Spectral theorem for compact self adjoint operators, statement of spectral theorem for bounded self adjoint operators. (5)

Lecture 01: Revision of Linear Algebra 08 Jan 2024 11:30

Recall (Linear Algebra)***:***

- Vector spaces, the axioms, and examples.
- Linear dependence of subsets of a vector spaces (both finite and infinite), basis is a maximal linear independent subset of a vector space.
- Every vector space has a basis (Zorn's Lemma), every basis of a vector space has the same cardinality, hence, dimension of a vector space defined as the cardinality of any of its basis is well defined.

Lecture 02: Introduction to Banach Spaces 10 Jan 2024 11:30

Definition* 1** (Normed Linear Space):***
Consider a vector space V over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : V \to [0, \infty)$ is said to be a norm if it satisfies the following axioms:

N1. $\|x\| = 0$ if and only if $x = 0$.

N2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in V$.

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then the vector space equipped with such a norm is called as a normed linear space.

Motivation: When a norm is defined on a vector space, it induces a topology (metric topology) on the vector space. With a topological structure on V, we can do further analysis in V such as limits of sequences, convergence, continuity of functions, etcetera.

Remark: If $(V, \|\cdot\|)$ is a normed linear space, then $d(x, y) = \|x - y\| : V \times V \to [0, \infty)$ is a metric on V:

M1. $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = \iff x = y$

M2. $d(x, y) = |-1|\|x - y\| = \|y - x\| = d(y, x)$

M3. $d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

Equipped with this metric, (V, d) is a metric space, and we can define an open subset of V as follows:
$U \subseteq V$ is said to be open if for all x in U, there exists an $r > 0$ such that

$$B(x, r) := \{y \in V \mid d(x, y) < r\} \subseteq U$$

The collection of these open subsets defines a topology on V.

Recall:

- In a normed linear space $(V, \|\cdot\|)$, if a sequence x_n converges to x and another sequence y_n converges to y, then the sequence $x_n + y_n$ converges to $x + y$.
- If α_n is a sequnce in $\mathbb{F}$ that converges to α and x_n is a sequence in V that converges to x, then the sequence $\alpha_n x_n$ converges to αx.

- Therefore, addition and scalar multiplication are continuous maps in the topological spaces induced by a norm.

Motivation: Now that we have a metric space structure on V, we can talk about completeness (every Cauchy sequence converges).

Definition* 2** (Banach Spaces):***
A normed linear space $(V, \|\cdot\|)$ is said to be a Banach Space if and only if (V, d) is a complete metric space. That is, every Cauchy sequence in V converges in the metric space (V, d).

Recall:

- **Hölder's inequality:**
  Let $p, p*$ be real numbers such that $\frac{1}{p}+\frac{1}{p*}=1$. Let $x, y \in \mathbb{R}^n$. Then the following inequality holds:
  $$|\langle x, y\rangle| = \sum_{i=1}^{n} \leq \|x\|_p \cdot \|y\|_{p*}$$
  Notice that when $p = 2$, we have $|\langle x, y\rangle| \leq \|x\|_2 \cdot \|y\|_2$.
- **Minkowski's inequality:**
  For any $1 \leq p < \infty$, and $x, y \in \mathbb{R}^n$, we have the following inequality:
  $$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$
- **Jensen's Inequality**:
  For any pair of real numbers $x, y \in \mathbb{R}$, the following inequality holds:
  $$|a+b|^p \leq 2^{p-1}(a^p + b^p)$$

Example:

1. $V = \mathbb{R}^n$, with the Euclidean norm $\|x\| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$ is a Banach space.
2. For $1 \leq p < \infty$, and $V = \mathbb{R}^N$, define $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$, then $(V, \|\cdot\|_p)$ is a Banach space.

   Proof. First we need to prove that $\|\cdot\|_p$ is a norm on $\mathbb{R}^N$. That follows from Minkowski's inequality.
   Then we need to prove the completeness of $(V, \|\cdot\|_p)$. Suppose $x^{(n)}$ is a Cauchy sequence in V, then for every $\epsilon > 0$, there exists a natural number t such that
   $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ whenever $m, n > t$
   $\implies \left(\sum_{i=1}^{n} \left|x_i^{(n)} - x_i^{(m)}\right|^p\right)^{\frac{1}{p}} < \epsilon$ whenever $m, n > t$
   $\implies |x_i^{(n)} - x_i^{(m)}| < \epsilon$ whenever $m, n > t$
   $\implies x_i^{(n)}$ is a Cauchy sequence in $\mathbb{R}$, hence convergent (since $\mathbb{R}$ is complete), to a real number, say x_i.
   Construct $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$. We claim that $x^{(n)}$ converges to x in $\mathbb{R}^N$.
   We have $|x_i^{(n)} - x_i^{(m)}| < \epsilon$ whenever $m, n > t$ for each i. Now, fix n and take $m \to \infty$. That implies $|x_i^{(n)} - x_i| < \epsilon$ whenever $n > t$ for each i and any ϵ. Therefore, with appropriate ϵ, the finite sum $\left(\sum_{i=1}^{n} \left|x_i^{(n)} - x_i\right|^p\right)^{\frac{1}{p}} < \epsilon$

Hence, for every $\epsilon > 0$, there is a suitable natural number t such that $\|x^{(n)} - x\| < \epsilon$ whenever $n > t$. Hence every Cauchy sequence in $(\mathbb{R}^N, \|\cdot\|_p)$ converges.
Hence $(\mathbb{R}^N, \|\cdot\|_p)$ is a Banach space. $\square$

3. **$\boldsymbol{\ell_p}$ spaces**
For $1 \leq p < \infty$, define a collection of sequences as follows:

$$\ell_p := \left\{ x = (x_1, x_2, \cdots) \mid x_i \in \mathbb{F}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

Now, define a norm on ℓ_p as $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. It is easy to verify that this is indeed a norm on ℓ_p. Then $(\ell_p, \|\cdot\|_p)$ is a Banach space.

Proof. **Claim:** ℓ_p is a vector space over $\mathbb{F}$.
If $x, y \in \ell_p$, then from the Jensen's Inequality, we have
$|x_i + y_i|^p \leq (|x_i| + |y_i|)^p \leq 2^{p-1} (|x_i|^p + |y_i|^p)$
Since $x, y \in \ell_p$, the right hand side is finite, hence $|x_i + y_i|^p$ is finite, and hence $x + y \in \ell_p$.
The set is trivially closed under scalar multiplication. Hence ℓ_p is a vector space over $\mathbb{F}$.

Claim: $(\ell_p, \|\cdot\|_p)$ is a Banach space.
Let $x^{(n)}$ be a Cauchy sequence in ℓ_p, then for every $\epsilon > 0$, there exists a natural number t such that $\|x^{(n)} - x^{(m)}\| < \epsilon$ whenever $m, n > t$.
$\implies \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p$ whenever $m, n > t$.
$\implies |x_i^{(n)} - x_i^{(m)}| < \epsilon$ for each i whenever $m, n > t$.
Hence, $x_i^{(n)}$ is a Cauchy sequence of real numbers. Due to completeness of $\mathbb{R}$, it must converge, and it converges to, say x_i.
Construct a sequence $x = (x_1, x_2, ...)$ using the above x_i.

Claim: $x \in \ell_p$.
Now, every Cauchy sequence is bounded. Therefore, $x^{(n)}$ is bounded. Hence, there exists a $c \in \mathbb{R}^+$ such that $\sum_{i=1}^{\infty} |x_i^{(n)}|^p \leq c$ for each n.
Hence, it is also true that $\sum_{i=1}^{k} |x_i^{(n)}|^p \leq c$ for each n.
Takeing the limit as $n \to \infty$ for this finite sum, we obtain that $\sum_{i=1}^{k} |x_i|^p \leq c$.
Since the right side is independent of k, it follows that as $k \to \infty$, we obtain $\sum_{i=1}^{\infty} |x_i|^p \leq c$.
Hence $x \in \ell_p$.

Claim: $x^{(n)}$ converges to x.
We have $\|x^{(n)} - x^{(m)}\| < \epsilon$ whenever $m, n > t$.
That is, $\left(\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p\right)^{\frac{1}{p}} < \epsilon$ whenever $m, n > t$.
Fix $n > t$, and take the limit $m \to \infty$.
Then the inequality becomes $\left(\sum_{i=1}^{\infty} |x_i^{(n)} - x_i|^p\right)^{\frac{1}{p}} < \epsilon$
Hence $\|x^{(n)} - x\| < \epsilon$ for all $n > t$, hence $x^{(n)}$ converges to x.

Hence $(\ell_p, \|\cdot\|_p)$ is a Banach space. $\square$

References

[1] R. Bhatia. *Notes on Functional Analysis*. 1st ed. Hindustan Book Agency, 2009.

[2] S. Kesavan. *Functional Analysis*. 2nd ed. Springer, 2023.