Differential Geometry

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Course	MSM423 Differential Geometry
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Prerequisites	MAT413 Analysis on Manifolds
Learning Outcomes	• Understanding the classical interpretation of various curvatures of a surface and their relation to geodesics. • Understanding the local and global geometry of smooth manifolds and smooth vector bundles.
Syllabus	• Gauss curvature, Gauss curvature formula in terms of first and second fundamental forms. Intrinsic property of the Gauss curvature. (6) • Covariant derivative of a vector field along a curve; Relation between covariant derivative and total curvature of a curve; A geodesic as a curve with vanishing covariant derivative. (6) • Manifolds: Definition, examples, Tangent vector space at a point, Basis of the tangent vector space. Smooth functions on a manifold, maps between Manifolds. Differential of a map. (6) • Sub-manifolds; Regular value theorem. Lie groups, examples; Submersion, Immersion and Embeddings. (6) • Smooth vector bundles, smooth sections, Dual bundles, existence of local sections. (5) • Tangent bundles; Smooth vector fields; Lie bracket of smooth vector fields; Co-tangent bundles; Differential 1-forms. (5) • Differential p-forms. Orientation. Exterior derivative. Closed and exact forms. Integration of a p-form on a p-dim sub manifold. Stokes theorem. (6)

Definition* 1** (Curves):***
Let I be an open interval in $\mathbb{R}$. A smooth parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$.
The image $\gamma(I) \subseteq \mathbb{R}^n$ is called trace of the curve γ.

Example: $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$, defined as $\gamma_1(t) = (\cos(t), \sin(t))$ and $\gamma_2(t) = (\sin(3t), \cos(3t))$ have the same trace, but they are different curves.

Observe: Consider any circle in $\mathbb{R}^2$, it is the set of all points in $\mathbb{R}^2$ satisfying a certain quadratic equation.
It can also be viewed as a trace of some curve.

Definition* 2** (Level Set):***
Let $U \subseteq \mathbb{R}^n$ be any open set, and $f : U \to \mathbb{R}$ be any function.
Then for a given constant $c \in \mathbb{R}$, the level set is defined as

$$L_c(f) := \{X \in U \mid f(X) = c\} \subseteq U \subseteq \mathbb{R}^n$$

Example: Let $U = \mathbb{R}^2$, and $f(x, y) = x^2 + y^2$, then
$L_0(f) = (0, 0)$, a point,
$L_1(f) = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, a circle in $\mathbb{R}^2$,
$L_{-1}(f) = \varnothing$.

Definition* 3** (Graph):***
Let A and B be any sets. Let $f : A \to B$ be a function, then the graph of f is the function $G_f : A \to A \times B$, given by

$$G_f(x) = (x, f(a)) \in A \times B$$

Note: Notice that a graph is also a curve, and if f is a smooth function, then G_f is a smooth curve.

Theorem* 1** (Implicit Function Theorem (2D Case)):***
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defining a curve $F(x, y) = c$.
Let (x_0, y_0) be a point on the curve.

- If $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$, then there exists a neighborhood around (x_0, y_0) where we can write $y = f(x)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $y = f(x)$ in that neighborhood.
- If $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then there exists a neighborhood around (x_0, y_0) where we can write $x = f(y)$ for a real valued function $f(x)$. That is, the curve $F(x, y) = c$ behaves like the graph of $x = g(y)$ in that neighborhood.

Example: Consider $S^1 = L_1(x^2 + y^2) = \{(x, y) \mid x^2 + y^2 = 1\}$
Here, $F(x, y) = x^2 + y^2$, and $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$.
Therefore, for any point (x_0, y_0) in S^1, where such that $x_0 \neq 0$, there exists a neighborhood in which $x = g(y)$.

Note: It is worthwhile to notice that a graph is always a curve. Implicit function theorem gives a condition on when a curve (level set in particular) can be seen as the graph of a curve.

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Observe: Consider the level set of $F(x, y) = x^2 + y^2$, $L_1(f) = S^1$. As seen before, since $\frac{\partial F}{\partial x} = 2x \neq 0$ when $x \neq 0$, for any point (x_0, y_0) in S^1, where $x_0 \neq 0$, we obtain a neighborhood in S^1 such that the curve (level set) can be written as a graph of a function $y = g(x)$.
Let S^{1+} denote $(\mathbb{R} \times \mathbb{R}^+) \cap S^1$, the upper half of the unit circle. Now, for a point in S^{1+}, the corresponding $g(x)$ is $\sqrt{1 - x^2}$.
Now the graph of g, say G_g is a function from $(-1, 1)$ to S^{1+}, and we can easily talk about continuity, differentiablity of G_g.
However, notice that G_g is a surjection, and an inverse exists, $G_g^{-1} : S^{1+} \to (-1, 1)$.
To talk about continuity of G_g^{-1} we need to define a topology on S^{1+}, and the subspace topology is the most obvious choice.
This G_g^{-1} happens to be continuous when we consider the subspace topology on S^{1+}. Further, open subsets of S^{1+} homeomorphic to the open subsets of $\mathbb{R}$, that means that this topological space is locally homeomorphic to $\mathbb{R}$, but the same does not extend to the whole space, that is, there is no homeomorphism from S^{1+} to $\mathbb{R}$.

Note: Notice that we cannot talk about differentiablity of S^{1+} in the above immediately, as differentiability requires some sort of vector space structure.

Example: Consider $S_3^1 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
Doing the same analysis on this set as before using the Implicit function theorem, we obtain that that after removing the appropriate critical points, and giving a topology to whatever remains, we obtain patches obtain open sets in whatever remains that are homeomorphic to open subsets of $\mathbb{R}^2$, hence the topological subspace obtained after removing the critical points is locally homeomorphic to $\mathbb{R}^2$.

> ***Definition* 4** (Tangent to a smooth curve at a point)***:***
> Let I be an open interval in $\mathbb{R}$, and $\gamma : I \to \mathbb{R}^n$ be a smooth curve given by $\gamma(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then for a point $t \in I$, the tangent of the curve at t is defined as
>
> $$\gamma'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$$

> ***Definition* 5** (Regular Curve)***:***
> A smooth curve $\gamma : I \to \mathbb{R}^n$ is said to be a regular curve if $\gamma'(t) \neq 0 \ \forall t \in I$.

> ***Definition* 6** (Diffeomorphism)***:***
> Let A and B be two manifolds, and a function $f : A \to B$ is said to be a diffeomorphism if f is bijective, differentiable, and the inverse $f^{-1} : B \to A$ is differentiable as well.

Note: In this course, by smooth, we mean a C^∞ function.

Definition 7 (Reparametrization of a Curve)**:**
Let J, I be open intervals of $\mathbb{R}$, let $\gamma : I \to \mathbb{R}^n$ be a curve, and let $\phi : J \to I$ be a diffeomorphism. Then $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ is a curve on J called as reparametrization of γ.

Observe: If β is a reparametrization of γ, and the setup is as above, then $\beta(s) = \gamma \circ \phi(s)$ for each $s \in J$. Then

$$\beta'(s) = \gamma'(\phi(s)) \cdot \phi'(s)$$

Now, ϕ is a diffeomorphism and $(\phi \circ \phi^{-1})' = 1$ and by chain rule, we have $(\phi^{-1})' = \frac{1}{\phi'}$. And since $(\phi^{-1})'$ exists throughout I, ϕ' cannot be 0 on J, hence $\gamma'(s) \neq 0$.
Hence if γ is regular, then so is β, hence regularity is preserved under a diffeomorphism.

Definition 8 (Arc Length of a Curve)**:**
Let $\gamma : I \to \mathbb{R}^n$ be a regular parametrized curve. For fixed $t_1, t_2 \in I$, the arc length between t_1 and t_2 is given by

$$\begin{aligned} L_\gamma(t_1, t_2) &= \int_{t_1}^{t_2} ||\gamma'(t)||\, dt \\ &= \int_{t_1}^{t_2} \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + \cdots + (x_n'(t))^2}\, dt \end{aligned}$$

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Theorem 2 (Inverse Function Theorem (a particular case))**:**
Let $U \subseteq \mathbb{R}^n$, and $f : U \to \mathbb{R}$ be a smooth function. Let $x \in U$ be such that for the Jacobian $D(f)(x)$ of f at x, the determinant is $\det[D(f)(x)] \neq 0$, then there exists a neighborhood W of x in U such that the restriction $f|_W : W \to f(W)$ is a diffeomorphism.

Note: Notice that this theorem gives a local property. That is, with the given hypothesis, we can only really claim that there exists a neighborhood upon which the restriction is a diffeomorphism. However, if the Jacobian of f has non-zero determinant at each x in U, then we cannot claim that f is a diffeomorphsim.

Example: $U = \mathbb{R}\backslash\{0\}$ and $f : U \to \mathbb{R}$ given by $f(x) = x^2$ satisfies the hypothesis, yet is not a diffeomorphism from U to $f(U)$, it is not even injective.

Observe: However, if U happens to be a connected, and f is smooth, then the above hypothesis is sufficient to claim that f is a diffeomorphism from U to $f(U)$.

Definition 9 (Unit Speed Reparametrization)**:**
For the intervals I, J, let $\gamma : I \to \mathbb{R}^n$ be a curve, let $\phi : J \to I$ be a diffeomorphism such that the reparametrization $\beta = \gamma \circ \phi : J \to \mathbb{R}^n$ has the property that $|\beta'(s)| = 1$ for each $s \in J$. Then the reparametrization β is said to be unit speed reparametrization.

Note: Notice that when a curve is reparametrized, the trace/image of the curve remains the same.

Note (Condition for a reparametrization to be unit speed)**:** If $\beta = \gamma \circ \phi$ is a reparametrization of the curve γ with the diffeomorphism ϕ as in the above setting, then

$$\begin{aligned}|\beta'(s)| = 1 &\implies |\gamma'(\phi(s))||\phi'(s)| = 1 \\ &\implies \boxed{|\phi'(s)| = \frac{1}{|\gamma'(\phi(s))|}}\end{aligned}$$

Definition* 10** (Arc Length Function):***
Let I be an interval, and $t_0 \in I$ be a fixed element. Let γ be a regular curve defined on I. Then the Arc Length function $L_\gamma : I \to \mathbb{R}$ is defined as

$$L_\gamma(t) = \int_{t_0}^{t} |\gamma'(t)|\, dt$$

Property (Properties of L_γ)**:**

1. $L'_\gamma(t) = |\gamma'(t)|$ (It is differentiable).
2. Since $\gamma(t)$ is regular, $\gamma'(t)$ is nowhere 0, and hence $|\gamma'(t)|$ is continuous, hence $L'_\gamma(t)$ is continuous and consequently, smooth.
3. $L'_\gamma(t) > 0$ on the interval I, that implies that L_γ is diffeomorphism from I to $L_\gamma(I) := J$.
4. Let $S_\gamma = L_\gamma^{-1} : J \to I$ be the inverse of L_γ (exists because γ is a diffeomorphism, also note that S_γ is a diffeomorphism).
   Consider the reparametrization $\beta = \gamma \circ S_\gamma$, it follows that

   $$|\beta'(s)| = |\gamma'(S_\gamma(s))||S'_\gamma(s)| \qquad (*)$$

   From the chain rule, we have

   $$S'_\gamma(s) = \frac{1}{L'_\gamma(S'_\gamma(s))} = \frac{1}{\gamma'(S_\gamma(s))} \qquad (**)$$

   $(*)$ and $(**)$ imply that $|\beta'(s)| = 1$ for all $s \in J$, hence β is a unit speed reparametrization.

Theorem* 3*:
Given an interval I, and any regular curve $\gamma : I \to \mathbb{R}^n$, there exists an interval J and a diffeomorphism $\phi : J \to I$ such that the reparametrization $\beta = \gamma \circ \phi$ is a unit speed parametrization.

Proof. The proof follows from 4. in the above properties of L_γ, where
$J = L_\gamma(I)$, and $\phi = L_\gamma^{-1} = S_\gamma$ □

References

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[2] J. M. Lee. *Introduction to Smooth Manifolds*. 2nd ed. Springer, 2013.

[3] L. W. Tu. *An Introduction to Manifolds*. 2nd ed. Springer, 2011.