Functional Analysis

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Course	MSM421 Functional Analysis
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Prerequisites	MSM321 Complex Analysis and MSM411 Measure Theory
Learning Outcomes	Based on core analysis courses and linear algebra, this course builds further on the study of Banach and Hilbert spaces. The theory and techniques studied in this course support, in a variety of ways, many advanced courses, in particular in analysis and partial differential equations, as well as having applications in mathematical physics and other areas
Syllabus	• Normed linear spaces, Riesz lemma, characterization of finite dimensional spaces, Banach spaces. Operator norm, continuity and boundedness of linear maps on a normed linear space. (6) • Fundamental theorems: Hahn-Banach theorems, uniform boundedness principle, divergence of Fourier series, closed graph theorem, open mapping theorem and some applications. (8) • Dual spaces and adjoint of an operator: Duals of classical spaces, weak and weak* convergence, adjoint of an operator. (6) • Hilbert spaces: Inner product spaces, orthonormal set, Gram-Schmidt orthonormalization, Bessel's inequality, orthonormal basis, separable Hilbert spaces. Projection and Riesz representation theorems: Orthonormal complements, orthogonal projections, projection theorem, Riesz representation theorem. (10) • Bounded operators on Hilbert spaces: Adjoint, normal, unitary, self-adjoint operators, compact operators. (5) • Spectral theorem: Spectral theorem for compact self adjoint operators, statement of spectral theorem for bounded self adjoint operators. (5)

Lecture 01: Revision of Linear Algebra 08 Jan 2024 11:30

Recall (Linear Algebra)***:***

- Vector spaces, the axioms, and examples.
- Linear dependence of subsets of a vector spaces (both finite and infinite), basis is a maximal linear independent subset of a vector space.
- Every vector space has a basis (Zorn's Lemma), every basis of a vector space has the same cardinality, hence, dimension of a vector space defined as the cardinality of any of its basis is well defined.

Lecture 02: Introduction to Banach Spaces 10 Jan 2024 11:30

Definition* 1** (Normed Linear Space):***
Consider a vector space V over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : V \to [0, \infty)$ is said to be a norm if it satisfies the following axioms:

N1. $\|x\| = 0$ if and only if $x = 0$.

N2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and $x \in V$.

N3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then the vector space equipped with such a norm is called as a normed linear space.

Motivation: When a norm is defined on a vector space, it induces a topology (metric topology) on the vector space. With a topological structure on V, we can do further analysis in V such as limits of sequences, convergence, continuity of functions, etcetera.

Remark: If $(V, \|\cdot\|)$ is a normed linear space, then $d(x, y) = \|x - y\| : V \times V \to [0, \infty)$ is a metric on V:

M1. $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = \iff x = y$

M2. $d(x, y) = |-1|\|x - y\| = \|y - x\| = d(y, x)$

M3. $d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$

Equipped with this metric, (V, d) is a metric space, and we can define an open subset of V as follows:
$U \subseteq V$ is said to be open if for all x in U, there exists an $r > 0$ such that

$$B(x, r) := \{y \in V \mid d(x, y) < r\} \subseteq U$$

The collection of these open subsets defines a topology on V.

Recall:

- In a normed linear space $(V, \|\cdot\|)$, if a sequence x_n converges to x and another sequence y_n converges to y, then the sequence $x_n + y_n$ converges to $x + y$.
- If α_n is a sequnce in $\mathbb{F}$ that converges to α and x_n is a sequence in V that converges to x, then the sequence $\alpha_n x_n$ converges to αx.

- Therefore, addition and scalar multiplication are continuous maps in the topological spaces induced by a norm.

Motivation: Now that we have a metric space structure on V, we can talk about completeness (every Cauchy sequence converges).

> ***Definition* 2** (Banach Spaces)***:***
> A normed linear space $(V, \|\cdot\|)$ is said to be a Banach Space if and only if (V, d) is a complete metric space. That is, every Cauchy sequence in V converges in the metric space (V, d).

Recall:

- **Hölder's inequality:**
  Let $p, p* \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p*} = 1$ (such a pair of real numbers is called **conjugate exponents** of each other). For a $p \in [1, \infty)$, define a function from $\mathbb{R}^N$ to $\mathbb{R}$ as:
  $$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}}$$
  and for $p = \infty$, define another a function from $\mathbb{R}^N$ to $\mathbb{R}$ as:
  $$\|x\|_\infty = \max_{1 \le i \le N} (x_i)$$
  Let $x = (x_1, \cdots, x_N), y = (y_1, \cdots, y_N) \in \mathbb{R}^N$.
  Then the following inequality holds:
  $$\sum_{i=1}^{N} |x_i y_i| \le \|x\|_p \|y\|_{p*}$$

- **Minkowski's inequality:**
  For any $p \in [1, \infty]$, and $x, y \in \mathbb{R}^N$, we have the following inequality:
  $$\|x + y\|_p \le \|x\|_p + \|y\|_p$$

- **Jensen's Inequality**:
  For any pair of real numbers $x, y \in \mathbb{R}$, the following inequality holds:
  $$|a + b|^p \le 2^{p-1}(a^p + b^p)$$

Example (p-norm on $\mathbb{R}^N$)**:** For any $p \in [1, \infty]$, $(\mathbb{R}^N, \|\cdot\|_p)$ is a Banach space.

Proof. First we need to prove that $\|\cdot\|_p$ is a norm on $\mathbb{R}^N$.
This follows from the Minkowski's inequality.

Note: This is usually called as p-norm on $\mathbb{R}^N$, and the special case when $p = 2$ is called the Euclidean norm.

Then we need prove the completeness of $(\mathbb{R}^N, \|\cdot\|_p)$. First we prove for $p \in [1, \infty)$ case.
Suppose $x^{(n)}$ is a Cauchy sequence in the NLS. Then for each $\epsilon > 0$, there exists a natural number t such that $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ whenever $m, n > t$. That is, $m, n > t$ implies that
$$\sum_{i=1}^{N} \left| x_i^{(n)} - x_i^{(m)} \right|^p < \epsilon^p$$

Hence $\left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon$ whenever $m, n > t$. That means $x_i^{(n)}$ is a Cauchy sequence in $\mathbb{R}$, hence converges (since $\mathbb{R}$ is complete) to a point, say $x_i \in \mathbb{R}$.
Let $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, where each x_i are defined as above. We claim that $x^{(n)}$ converges to x.
We have the inequality $|x_i^{(n)} - x_i^{(m)}| < \epsilon^p$ whenever $m, n > t$.
Fix n, and take the limit as $m \to \infty$, we obtain $|x_i^{(n)} - x_i| \leq \epsilon^p$.
Therefore, after an appropriate adjustment, we obtain an ϵ', such that for any $\epsilon > 0$, there exists a natural number t such that $n > t$ implies that the finite sum satisfies the inequality

$$\left(\sum_{i=1}^{N} \left|x_i^{(n)} - x_i\right|^p\right)^{\frac{1}{p}} = \|x^{(n)} - x\|_p < \epsilon'$$

Hence $x^{(n)}$ converges to x. Therefore, every Cauchy sequence in $(\mathbb{R}^N, \|\cdot\|_p)$ converges, hence it is a Banach space when $p \in [1, \infty)$.
When $p = \infty$, as convergence and Cauchy criterion hold if and only if they hold componentwise in $\mathbb{R}^N$, $(\mathbb{R}^N, \|\cdot\|_\infty)$ is Banach as well. □

Note: Due to completeness of $\mathbb{C}$, for any $p \in [1, \infty)$, $(\mathbb{C}, \|\cdot\|_p)$ is a Banach space as well.

Example ($\boldsymbol{\ell_p}$ spaces): For $1 \leq p < \infty$, define a collection of sequences as follows:

$$\ell_p := \left\{x = (x_1, x_2, \cdots) \mid x_i \in \mathbb{F}, \sum_{i=1}^{\infty} |x_i|^p < \infty\right\}$$

Now, define a function from ℓ_p to $\mathbb{F}$ as $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$. Then $(\ell_p, \|\cdot\|_p)$ is a Banach space.

Proof. The proof proceeds in several steps:

1. **$\boldsymbol{\ell_p}$ is a vector space.**

   Proof. Let $x, y \in \ell_p$, then from the Jensen's inequality, we have

   $$|x_i + y_i|^p \leq (|x_i| + |y_i|)^p \leq 2^{p-1}(|x_i|^p + |y_i|^p) < \infty$$

   and for any $\alpha \in \mathbb{F}$ and $x \in \ell_p$, αx clearly belongs to ℓ_p.
   Hence ℓ_p is a subspace of the space of all sequences in $\mathbb{F}$. □

2. **$\|\cdot\|_p$ is a norm on $\boldsymbol{\ell_p}$.**

   Proof. The first two axioms of a norm are trivial to prove. We need to show the triangle inequality.
   Let $x, y \in \ell_p$, then for any natural number N, from Minkowski's Inequality, we have

   $$\sum_{i=1}^{N} |x_i + y_i|^p \leq \left[\left(\sum_{i=1}^{N} |x_i|^p\right) + \left(\sum_{i=1}^{N} |y_i|^p\right)\right]^p \leq [\|x\|_p + \|y\|_p]^p < \infty$$

   Since N is arbitrary, and the right side is independent of the choice of N, it follows that

   $$\sum_{i=1}^{\infty} |x_i + y_i|^p \leq [\|x\|_p + \|y\|_p]^p < \infty$$

   Hence the triangle inequality holds, and hence $\|\cdot\|_p$ is a norm on ℓ_p. □

3. **$(\ell_p, \|\cdot\|_p)$ is a complete space.**

Proof. Let $x^{(n)}$ be a Cauchy sequence in the NLS, that is, for each $\epsilon > 0$, there exists a natural number t such that $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ whever $m, n > t$. That is, $m, n > t$ implies that

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}| < \epsilon^p$$

Hence $\left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon$ whenever $m, n > t$, hence $x_i^{(n)}$ is a Cauchy sequence in $\mathbb{R}$, hence converges (since $\mathbb{R}$ is complete) to a point, say $x_i \in \mathbb{R}$.
Construct a sequence $x = (x_1, x_2 \cdots x_n, \cdots)$ where x_i is defined as above.
We need to show that $x \in \ell_p$. Since $x^{(n)}$ is a Cauchy sequence, it is bounded, hence there exists a constant $C > 0$ such that

$$\|x^{(n)}\|_p^p = \sum_{i=1}^{\infty} \left|x_i^{(n)}\right|^p \leq C, \ \forall n.$$

Let k be any fixed positive integer. Then,

$$\sum_{i=1}^{k} \left|x_i^{(n)}\right|^p \leq C$$

which implies that

$$\sum_{i=1}^{k} |x_i|^p \leq C.$$

Since k is arbitrary and the right side does not depend on it, it follows that

$$\sum_{i=1}^{\infty} |x_i|^p \leq C < \infty.$$

Hence $x \in \ell_p$.
Now we need to show that $x^{(n)}$ converges to x. From before, whenever $m, n > t$, we have

$$\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Hence, for any positive integer k, and whenever $m, n > t$, we have

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i^{(m)}|^p < \epsilon^p.$$

Fix $n > t$, and take the limit $m \to \infty$, we obtain

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i|^p < \epsilon^p.$$

Since k is arbitrary and the right side does not depend on it, hence whenever $n > t$, we have

$$\|x^{(n)} - x\|_p \leq \epsilon.$$

That is, $x^{(n)}$ converges to x. Hence every Cauchy sequence in $(\ell_p, \|\|_p)$ converges, hence it is Banach. $\square$

Example: Consider the set of sequences

$$\ell_\infty := \left\{ x = (x_i) \mid x_i \in \mathbb{F}, \sup_{1 \le i < \infty} |x_i| < \infty \right\},$$

the set of all bounded sequences in $\mathbb{F}$. Clearly it is a vector space under componentwise addition and scalar multiplication. Also define the function

$$\|x\|_\infty = \sup_{1 \le i < \infty} |x_i|.$$

Then $\|x\|_\infty$ is a norm on ℓ_∞ and $(\ell_\infty, \|\cdot\|_\infty)$ is a Banach space.

References

[1] R. Bhatia. *Notes on Functional Analysis*. 1st ed. Hindustan Book Agency, 2009.

[2] S. Kesavan. *Functional Analysis*. 2nd ed. Springer, 2023.