# Chapter 5
Sequences, Mathematical Induction, and Recursion

Ojas Chaturvedi

## Contents

# 5.1: Sequences

## Notes

- Sequence: a function whose domain is either all the integers between two given integers, or all the integers greater than or equal to a given integer.
  - Know subscript/index, initial and final term, infinite sequence, general/explicit formula
- Summation Notation:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

  where k is the index, m is the lower limit, and n is the upper limit.

- When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form. Expand the summation notation to first 3 or so, then put ellipsis and then variable form.
- Product Notation:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$$

- Properties of Summations and Products (aka Theorem 5.1.1)

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k) \tag{1}$$

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k) \tag{2}$$

$$(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k) \tag{3}$$

- When replacing a new variable into a summation or product, make sure to change the index variable to the new variable and the numbers by putting them into the equation of the new variable.
- Factorial: the quantity **n!** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

  and

$$0! = 1$$

  Recursive definition:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

- $n$ choose $r$: the number of subsets (therefore an integer) of size $r$ that can be chosen from a set of $n$ elements.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

  for all integers $n$ and $r$ with $0 \leq r \leq n$.

# 5.2: Mathematical Induction I

## Notes

- Principles of Mathematical Induction: Let $P(n)$ be a property that is defined for integers $n$, and let $a$ be a fixed integer. Suppose the following two statements are true:

  1. Basis Step: Show that $P(a)$ is true.
  2. Inductive Step: For all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.
     - To perform this step:
       (a) Suppose that $P(k)$ is true for an arbitrary integer $k \geq a$, which is called the inductive hypothesis.
       (b) Show that $P(k+1)$ is true.
     - Remember that you need to prove each side of the equation separately. Otherwise, the proof is invalid.
  3. Conclusion: Then $P(n)$ is true for all integers $n \geq a$.

- Steps of Proof by Mathematical Induction:

  1. State the theorem to be proved.
     - Let the property $P(n)$ be the equation: problem goes here
  2. Prove the basis step.
     - Show that $P(a)$ is true.
  3. State the inductive hypothesis.
     - Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:
  4. Prove the inductive step.
  5. State the conclusion.
     - Therefore the equation $P(k+1)$ is true [*as was to be shown*]. [*Since we have proved both the basis step and the inductive step, the conclusion follows by the principle of mathematical induction. Therefore the equation $P(n)$ is true for all integers $n \geq 1$.*]

- Sum of the first $n$ integers is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Geometric sum of the first $n$ integers is

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

# 5.3: Mathematical Induction II

## Different types of problems

**Problem Type: Divisibility Property.** For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

**Proof.** Let the property $P(n)$ be the sentence:

$$2^{2n} - 1 \text{ is divisible by } 3$$

First, we must prove that $P(0)$ is true (basis step).

$$2^{2 \cdot 0} - 1 \text{ is divisible by } 3$$

$$\begin{aligned} 2^{2(0)} - 1 &= 2^0 - 1 \\ &= 1 - 1 \\ &= 0 \\ &= 3 \cdot 0 \end{aligned}$$

Thus, $P(0)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 0$ (inductive hypothesis). That is,

$$2^{2k} - 1 \text{ is divisible by } 3$$

By the definition of divisibility, for some integer $r$,

$$2^{2k} - 1 = 3r$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$2^{2(k+1)} - 1 \text{ is divisible by } 3$$

The left-hand side of $P(k+1)$ is:

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 2^{2k} \cdot 2^2 - 1 \\ &\quad \text{by the product rule for exponents} \\ &= 4 \cdot 2^{2k} - 1 \\ &= 3 \cdot 2^{2k} + 2^{2k} - 1 \\ &= 3 \cdot 2^{2k} + 3r \\ &\quad \text{by substituting the inductive hypothesis} \\ &= 3(2^{2k} + r) \end{aligned}$$

$2^{2k} + r$ is an integer since it is the sum of products of integers, so $2^{2(k+1)} - 1$ can be written as $6m$ for some integer $m = (2^{2k} + r)$.
By the definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3, and thus, $P(k+1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. □

**Problem Type: Inequality.** For all integers $n \geq 3$, $2n + 1 < 2^n$.

**Proof.** Let the property $P(n)$ be the inequality:

$$2n + 1 < 2^n$$

First, we must prove that $P(3)$ is true (basis step).

$$\begin{aligned} 2(3) + 1 &< 2^3 \\ 7 &< 8 \end{aligned}$$

Thus, $P(3)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 3$ (inductive hypothesis). That is,

$$2k + 1 < 2^k$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$\begin{aligned} 2(k+1) + 1 &< 2^{k+1} \\ 2k + 3 &< 2^{k+1} \end{aligned}$$

The left-hand side of $P(k+1)$ is:

$$\begin{aligned} 2k + 3 &= 2k + 1 + 2 \\ &< 2^k + 2 \\ &\quad \text{by substitution of the inductive hypothesis} \\ &< 2^k + 2^k \\ &\quad \text{because } 2 < 2^k \text{ for all integers } k \geq 2 \\ &< 2 \cdot 2^k \\ &< 2^{k+1} \\ &\quad \text{by the product rule for exponents} \end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is less than the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. $\square$

**Problem Type: Property of a Sequence.** Define a sequence $a_1, a_2, a_3, \ldots$ as follows:

$$\begin{aligned} a_1 &= 2 \\ a_k &= 5a_{k-1} \qquad \text{for all integers } k \geq 2 \end{aligned}$$

a. Write the first four terms of the sequence.

b. It is claimed that for each integer $n \geq 0$, the $n$th term of the sequence has the same value as that given by the formula $2 \cdot 5^{n-1}$. In other words, the claim is that the terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$. Prove that this is true.

**Proof.**

a. $a_1 = 2$
$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = 10$
$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$
$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = 250$

b. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 2$ and $a_k = 5a_{k-1}$ for all integers $k \geq 2$. Let the property $P(n)$ be the equation:

$$a_n = 2 \cdot 5^{n-1}$$

First, we must prove that $P(1)$ is true (basis step).

$$a_1 = 2 \cdot 5^{1-1}$$

The left-hand side of $P(1)$ is

$$a_1 = 2$$

by the definition of $a_1, a_2, a_3, \ldots$

The right-hand side of $P(1)$ is

$$\begin{aligned} 2 \cdot 5^{1-1} &= 2 \cdot 5^0 \\ &= 2 \cdot 1 \\ &= 2 \end{aligned}$$

Thus, the left-hand side of $P(1)$ is equal to the right-hand side of $P(1)$, and $P(1)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 1$ (inductive hypothesis). That is,

$$a_k = 2 \cdot 5^{k-1}$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$a_{k+1} = 2 \cdot 5^{(k+1)-1}$$
$$a_{k+1} = 2 \cdot 5^k$$

The left-hand side of $P(k+1)$ is:

$$\begin{aligned} a_{k+1} &= 5a_k \\ &= 5(2 \cdot 5^{k-1}) \\ &= 2 \cdot 5^k \end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction.

$\square$

## Template for Mathematical Induction

Let the property $P(n)$ be the equation/sentence/inequality:

$$\{\text{problem goes here}\}$$

First, we must prove that $P(\{\text{smallest possible number goes here}\})$ is true (basis step).

$$\text{Show left-hand side} = \text{right-hand side of the equation}$$

Thus, $P(\{\text{smallest possible number goes here}\})$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq \{\text{smallest possible number goes here}\}$ (inductive hypothesis). That is,

$$\{\text{problem with k substituted goes here}\}$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$\{\text{problem with (k+1) substituted goes here}\}$$

The left-hand side of $P(k+1)$ is:

$$\{\text{work with reasoning goes here}\}$$

The right-hand side of $P(k+1)$ is:

$$\{\text{work with reasoning goes here}\}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction.