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**5.8.6:** Let  $b_0, b_1, b_2, \dots$  be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 2$$

where  $C$  and  $D$  are real numbers. Show that for any choice of  $C$  and  $D$ ,

$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2$$

---

*Proof.* Let  $b_0, b_1, b_2, \dots$  be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 2$$

We must prove that  $b_k$  is true when  $b_n$  is plugged in with any choice of  $C$  and  $D$ :

$$\begin{aligned} b_k &= b_{k-1} + 6b_{k-2} \\ &= \left( C \cdot 3^{(k-1)} + D(-2)^{(k-1)} \right) + 6 \left( C \cdot 3^{(k-2)} + D(-2)^{(k-2)} \right) \\ &= C \cdot 3^{(k-1)} + D(-2)^{(k-1)} + 6C \cdot 3^{(k-2)} + 6D(-2)^{(k-2)} \\ &= C \left( 3^{(k-1)} + 2 \left( 3^{(k-1)} \right) \right) + D \left( (-2)^{(k-1)} - 3(-2)^{(k-1)} \right) \\ &= C \left( 3 \left( 3^{(k-1)} \right) \right) + D \left( -2(-2)^{(k-1)} \right) \\ &= C(3^k) + D(-2)^k \\ &= b_k \end{aligned}$$

Thus, we have proved that, for any choice of  $C$  and  $D$ ,

$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2$$

□

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**5.8.9:**

- Suppose a sequence of the form  $1, t, t^2, t^3, \dots, t^n \dots$  where  $t \neq 0$ , satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of  $t$ .
- Suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

$$\begin{aligned} b_k &= 7b_{k-1} - 10b_{k-2} \quad \text{for all integers } k \geq 2 \\ b_0 &= 2 \quad b_1 = 2 \end{aligned}$$

---

*Proof.* a. Suppose a sequence of the form  $1, t, t^2, t^3, \dots, t^n \dots$  where  $t \neq 0$ , satisfies

$$b_k = 7b_{k-1} - 10b_{k-2} \quad \text{for all integers } k \geq 2$$

Since we suppose the sequence is of form  $1, t, t^2, \dots, t^n, \dots$ , by the characteristic equation of the second-order linear homogeneous recurrence relation with constant coefficients,

$$\begin{aligned}t^2 - 7t + 10 &= 0 \\(t - 5)(t - 2) &= 0 \\t &= 2, 5\end{aligned}$$

Thus, the only possible values of  $t$  are 2 and 5.

b.  $b_0, b_1, b_2, \dots$  satisfies the equation

$$b_n = C \cdot 2^n + D \cdot 5^n \quad \text{for all integers } n \geq 0$$

for some constants  $C$  and  $D$  due to part (a) and the distinct roots theorem. We are given that  $b_0 = 2, b_1 = 2$ . Therefore,

$$\begin{cases} b_0 = C \cdot 2^0 + D \cdot 5^0 \\ b_1 = C \cdot 2^1 + D \cdot 5^1 \end{cases} \quad \begin{cases} 2 = C + D \\ 2 = 2C + 5D \end{cases}$$

Plugging in  $C = 2 - D$ ,

$$\begin{aligned}2(2 - D) + 5D &= 2 \\4 - 2D + 5D &= 2 \\4 + 3D &= 2 \\3D &= -2 \\D &= \frac{-2}{3}\end{aligned}$$

Thus,

$$b_n = C \cdot 2^n + D \cdot 5^n \quad \text{for all integers } n \geq 0$$

$$C = 2 - D$$

$$D = \frac{-2}{3}$$

$$C = \frac{8}{3}$$

$$b_n = \frac{8}{3} \cdot 2^n + \frac{-2}{3} \cdot 5^n \quad \text{for all integers } n \geq 0$$

□

---

**5.8.10:**

- Suppose a sequence of the form  $1, t, t^2, t^3, \dots, t^n \dots$  where  $t \neq 0$ , satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of  $t$ .
- Suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

$$c_k = c_{k-1} + 6c_{k-2} \quad \text{for all integers } k \geq 2$$
$$c_0 = 0 \quad c_1 = 3$$

---

*Proof.* a. Suppose a sequence of the form  $1, t, t^2, t^3, \dots, t^n \dots$  where  $t \neq 0$ , satisfies

$$c_k = c_{k-1} + 6c_{k-2} \quad \text{for all integers } k \geq 2$$

Since we suppose the sequence is of form  $1, t, t^2, \dots, t^n, \dots$ , by the characteristic equation of the second-order linear homogeneous recurrence relation with constant coefficients,

$$t^2 - t - 6 = 0$$
$$(t - 3)(t + 2) = 0$$
$$t = -2, 3$$

Thus, the only possible values of  $t$  are  $-2$  and  $3$ .

b.  $c_0, c_1, c_2, \dots$  satisfies the equation

$$c_n = C \cdot (-2)^n + D \cdot 3^n \quad \text{for all integers } n \geq 0$$

for some constants  $C$  and  $D$  due to part (a) and the distinct roots theorem. We are given that  $c_0 = 0, c_1 = 3$ . Therefore,

$$\begin{cases} c_0 = C \cdot (-2)^0 + D \cdot 3^0 \\ c_1 = C \cdot (-2)^1 + D \cdot 3^1 \end{cases} \quad \begin{cases} 0 = C + D \\ 3 = -2C + 3D \end{cases}$$

Plugging in  $C = -D$ ,

$$-2(-D) + 3D = 3$$
$$2D + 3D = 3$$
$$5D = 3$$
$$D = \frac{3}{5}$$

Thus,

$$c_n = C \cdot (-2)^n + D \cdot 3^n \quad \text{for all integers } n \geq 0$$

$$C = -D$$
$$D = \frac{3}{5}$$
$$C = \frac{-3}{5}$$
$$c_n = \frac{-3}{5} \cdot (-2)^n + \frac{3}{5} \cdot 3^n \quad \text{for all integers } n \geq 0$$

□

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**5.8.15:** Suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for the sequence.

$$t_k = 6t_{k-1} - 9t_{k-2} \quad \text{for all integers } k \geq 2$$

$$t_0 = 1 \quad t_1 = 3$$

---

*Proof.* Suppose a sequence of the form  $t_0, t_1, t_2, \dots$  satisfies

$$t_k = 6t_{k-1} - 9t_{k-2} \quad \text{for all integers } k \geq 2$$

Since we suppose the sequence is of form  $t_0, t_1, t_2, \dots$ , by the characteristic equation of the second-order linear homogeneous recurrence relation with constant coefficients,

$$\begin{aligned} y^2 - 6y + 9 &= 0 \\ y &= 3 \end{aligned}$$

for some number  $y$  that satisfies the Characteristic Equation. Thus, the only possible value of  $y$  is 3. Due to the single-root theorem,  $t_0, t_1, t_2, \dots$  satisfies the equation

$$t_n = Cr^n + Dnr^n \quad \text{for all integers } n \geq 0$$

for some constants  $C$  and  $D$  and real root  $r$ . We are given that  $t_0 = 1, t_1 = 3$ . Therefore,

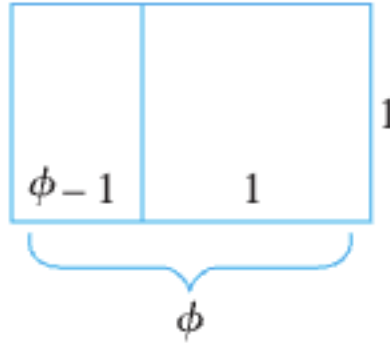
$$\begin{cases} t_0 = C \cdot 3^0 + D(0) \cdot 3^0 \\ t_1 = C \cdot 3^1 + D \cdot 3^1 \end{cases} \quad \begin{cases} 1 = C \\ 3 = 3C + 3D \end{cases} \quad \begin{cases} C = 1 \\ D = 0 \end{cases}$$

Hence,

$$t_n = 3^n \quad \text{for all integers } n \geq 0$$

□

**5.8.24:** The numbers  $\frac{1 + \sqrt{5}}{2}$  and  $\frac{1 - \sqrt{5}}{2}$  that appear in the explicit formula for the Fibonacci sequence are related to a quantity called the *golden ratio* in Greek mathematics. Consider a rectangle of length  $\phi$  units and height 1, where  $\phi > 1$ .



Divide the rectangle into a rectangle and a square as shown in the preceding diagram. The square is 1 unit on each side, and the rectangle has sides of length 1 and  $\phi - 1$ . The ancient Greeks considered the outer rectangle to be perfectly proportioned (saying that the lengths of its sides were in a *golden ratio* to each other) if the ratio of the length to the width of the outer rectangle equaled the ratio of the length to the width of the inner rectangle. That is,

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

- Show that  $\phi$  satisfies the following quadratic equation  $t^2 - t - 1 = 0$ .
- Find the two solutions of  $t^2 - t - 1 = 0$  and call them  $\phi_1$  and  $\phi_2$ .
- Express the explicit formula for the Fibonacci sequence in terms of  $\phi_1$  and  $\phi_2$ .

*Proof.* a.

$$\begin{aligned} \frac{\phi}{1} &= \frac{1}{\phi - 1} \\ \phi \cdot (\phi - 1) &= 1 \\ \phi^2 - \phi &= 1 \\ \phi^2 - \phi - 1 &= 0 \end{aligned}$$

which is in the form  $t^2 - t - 1 = 0$ . Hence,  $\phi$  satisfies the quadratic equation  $t^2 - t - 1 = 0$ .

b.

$$\begin{aligned}\phi_1, \phi_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{1+4}}{2} \\ &= \frac{1 + \sqrt{1+4}}{2}, \frac{1 - \sqrt{1+4}}{2} \\ &= \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \\ \phi_1 &= \frac{1 + \sqrt{5}}{2} \\ \phi_2 &= \frac{1 - \sqrt{5}}{2}\end{aligned}$$

- c. We know that the characteristic equation of the Fibonacci sequence has the solutions  $\phi_1, \phi_2$ . Due to the distinct-roots theorem, the Fibonacci sequence is given by the explicit formula:

$$F_n = C\phi_1^n + D\phi_2^n \quad \text{for all integers } n \geq 0$$

for some constants  $C$  and  $D$ . We are given that  $F_0 = 0, F_1 = 1$ . Therefore, when substituting for  $n = 0, 1$

$$\begin{cases} C + D = 1 \\ C\phi_1 + D\phi_2 = 1 \end{cases}$$

Plugging in  $C = 1 - D$ ,

$$\begin{aligned}(1 - D)\phi_1 + D\phi_2 &= 1 \\ D(\phi_2 - \phi_1) &= 1 - \phi_1 \\ D &= \frac{1 - \phi_1}{\phi_2 - \phi_1}\end{aligned}$$

Thus,

$$\begin{aligned}C &= 1 - D \\ C &= 1 - \frac{1 - \phi_1}{\phi_2 - \phi_1} \\ C &= \frac{\phi_2 - \phi_1}{\phi_2 - \phi_1} - \frac{1 - \phi_1}{\phi_2 - \phi_1} \\ C &= \frac{\phi_2 - 1}{\phi_2 - \phi_1}\end{aligned}$$

Therefore, the explicit formula for the Fibonacci Sequence in terms of  $\phi_1$  and  $\phi_2$  can be written as

$$F_n = \frac{\phi_2 - 1}{\phi_2 - \phi_1}(\phi_1^n) + \frac{1 - \phi_1}{\phi_2 - \phi_1}(\phi_2^n) \quad \text{for all integers } n \geq 0$$

□

---

**5.9.4b:** The set of arithmetic expressions over the real numbers can be defined recursively as follows:

I. BASE: Each real number  $r$  is an arithmetic expression.

II. RECURSION: If  $u$  and  $v$  are arithmetic expressions, then the following are also arithmetic expressions:

- a.  $(+u)$
- b.  $(-u)$
- c.  $(u + v)$
- d.  $(u - v)$
- e.  $(u \cdot v)$
- f.  $\left(\frac{u}{v}\right)$

III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

(Note that the *expression*  $\left(\frac{u}{v}\right)$  is legal even though the value of  $v$  may be 0.) Give derivations showing that each of the following is an arithmetic expression.

$$\left(\frac{(9 \cdot (6.1 + 2))}{((4 - 7) \cdot 6)}\right)$$

---

*Proof.* 1. According to BASE (I), numbers 9, 6.1, 2, 4, 7, and 6 are each arithmetic expressions.

2. Utilizing (1) and RECURSION II(c), the expression  $6.1 + 2$  qualifies as an arithmetic expression.

3. From (1), (2), and RECURSION II(e), the expression  $9 \cdot (6.1 + 2)$  is an arithmetic expression.

4. Using (1) and RECURSION II(d), the expression  $4 - 7$  is an arithmetic expression.

5. Applying (1), (4), and RECURSION II(e), the expression  $(4 - 7) \cdot 6$  is an arithmetic expression.

6. Combining (3), (5), and RECURSION II(f), the expression  $\frac{9 \cdot (6.1 + 2)}{(4 - 7) \cdot 6}$  is confirmed to be an arithmetic expression. □

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**5.9.6:** Define a set  $S$  recursively as follows:

I. BASE:  $a \in S$

II. RECURSION: If  $s \in S$ , then

- a.  $sa \in S$
- b.  $sb \in S$

III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

Use structural induction to prove that every string in  $S$  begins with an  $a$ .

---

*Proof.* The proof is conducted using structural induction to demonstrate that every string in  $S$  begins with an 'a'.

**Base Case:**

The base case involves the string 'a', as defined by BASE (I). Clearly, 'a' begins with an 'a', satisfying the property.

**Inductive Step:**

Assume for the recursion step that any string  $s \in S$  begins with 'a', a hypothesis based on the RECURSION rule (II). According to rules II(a) and II(b), if  $s \in S$ , then both 'sa' and 'sb' are in  $S$ . Given our hypothesis, 'sa' and 'sb' also begin with 'a'.

**Conclusion:**

Since both the base case and the inductive step hold, and no elements are in  $S$  other than those defined in BASE and RECURSION, we conclude that every string in  $S$  begins with an 'a'.  $\square$

---

**5.9.11:** Define a set  $S$  recursively as follows:

I. BASE:  $0 \in S$

II. RECURSION: If  $s \in S$ , then

a.  $s + 3 \in S$

b.  $s - 3 \in S$

III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

Use structural induction to prove that every integer in  $S$  is divisible by 3.

---

*Proof.* The proof employs structural induction to establish that every integer in  $S$  is divisible by 3.

**Base Case:**

The BASE (I) of  $S$  includes the integer 0. Since 0 is divisible by 3, it fulfills the property.

**Inductive Step:**

For the inductive step, assume that any integer  $s \in S$  is divisible by 3. This is based on the RECURSION rule (II). According to rules II(a) and II(b), if  $s \in S$ , then  $s + 3$  and  $s - 3$  also belong to  $S$ . If  $s$  is divisible by 3, it can be represented as  $s = 3k$  for some integer  $k$ . Therefore, both  $s + 3$  and  $s - 3$  are divisible by 3, as they can be expressed as  $3(k + 1)$  and  $3(k - 1)$ , respectively.

**Conclusion:**

Given that the base case is satisfied and the inductive step holds, and considering that no elements other than those derived through BASE and RECURSION are in  $S$ , it can be concluded that every integer in  $S$  is divisible by 3.  $\square$



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**5.9.16:** Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

---

*Proof.* Let  $S$  be the set of all strings of 0's and 1's for which all the 0's precede all the 1's. Here is the recursive definition for  $S$ :

- I. BASE:**  $\epsilon \in S$
- II. RECURSION:** If  $s \in S$ , then  $0s \in S$  and  $s1 \in S$
- III. RESTRICTION:** There are no elements of  $S$  other than those inferred from rules I and II.

□

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**5.9.18:** Give a recursive definition for the set of all strings of  $a$ 's and  $b$ 's that contain exactly one  $a$ .

---

*Proof.* Let  $S$  be the set of all strings of  $a$ 's and  $b$ 's that contain exactly one  $a$ . The following is a recursive definition of  $S$ :

- I. BASE:**  $a \in S$
- II. RECURSION:** If  $s \in S$ , then  $sb \in S$  and  $bs \in S$
- III. RESTRICTION:** There are no elements of  $S$  other than those inferred from rules I and II.

□

---

**6.1.6:** Let  $A = \{x \in \mathbf{Z} \mid x = 5a + 2 \text{ for some integer } a\}$ ,  $B = \{y \in \mathbf{Z} \mid y = 10b - 3 \text{ for some integer } b\}$ , and  $C = \{z \in \mathbf{Z} \mid z = 10c + 7 \text{ for some integer } c\}$ . Prove or disprove each of the following statements.

- a.  $A \subseteq B$
- b.  $B \subseteq A$
- c.  $B = C$

---

*Proof.* Let  $A = \{x \in \mathbf{Z} \mid x = 5a + 2 \text{ for some integer } a\}$ ,  $B = \{y \in \mathbf{Z} \mid y = 10b - 3 \text{ for some integer } b\}$ , and  $C = \{z \in \mathbf{Z} \mid z = 10c + 7 \text{ for some integer } c\}$ .

- a. Disprove  $A \subseteq B$  using a counterexample:  
Consider  $x = 2 \in A$  for  $a = 0$ . For  $x$  to be in  $B$ , we need  $10b - 3 = 2$ , leading to  $b = 0.5$ , a contradiction since  $b$  must be an integer. Therefore,  $A \not\subseteq B$ .
- b. Let  $x \in B$ , so  $x = 10b - 3$  for some integer  $b$ . Then  $x = 5(2b - 1) + 2$ , which means there exists an integer  $a = 2b - 1$  such that  $x \in A$ . Therefore,  $B \subseteq A$ .
- c. To show  $B \subseteq C$ , let  $x \in B$ . Then  $x = 10b - 3$  and setting  $c = b - 1$ , we get  $x = 10c + 7$ , thus  $x \in C$ . To show  $C \subseteq B$ , let  $x \in C$ . Then  $x = 10c + 7$  and setting  $b = c + 1$ , we get  $x = 10b - 3$ , thus  $x \in B$ . Hence,  $B = C$ .

□

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**6.1.20:** Let  $B_i = \{x \in \mathbf{R} \mid 0 \leq x \leq i\}$  for all integers  $i = 1, 2, 3, 4$ .

- a.  $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$
- b.  $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$
- c. Are  $B_1, B_2, B_3$ , and  $B_4$  mutually disjoint? Explain.

---

*Proof.* a.  $\{x \in \mathbf{R} \mid 0 \leq x \leq 4\}$

b.  $\{x \in \mathbf{R} \mid 0 \leq x \leq 1\}$

- c. The sets  $B_1, B_2, B_3$ , and  $B_4$  are not mutually disjoint. For instance, the number 1 is included in all these sets. More generally, if a number is in  $B_i$ , it will be in all  $B_j$  for  $j \geq i$ .

□

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**6.1.23:** Let  $V_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$  for all positive integers  $i$ .

- a.  $\bigcup_{i=1}^4 V_i = ?$
- b.  $\bigcap_{i=1}^4 V_i = ?$
- c. Are  $V_1, V_2, V_3, \dots$  mutually disjoint? Explain.

- d.  $\bigcup_{i=1}^n V_i = ?$
  - e.  $\bigcap_{i=1}^n V_i = ?$
  - f.  $\bigcup_{i=1}^{\infty} V_i = ?$
  - g.  $\bigcap_{i=1}^{\infty} V_i = ?$
- 

*Proof.* a.  $\bigcup_{i=1}^4 V_i = [-1, 1]$

b.  $\bigcap_{i=1}^4 V_i = [-\frac{1}{4}, \frac{1}{4}]$

c. The sets  $V_1, V_2, V_3, \dots$  are not mutually disjoint since they all contain the point 0.

d.  $\bigcup_{i=1}^n V_i = [-1, 1]$  for any positive integer  $n$ .

e.  $\bigcap_{i=1}^n V_i = \{0\}$  as  $n \rightarrow \infty$ .

f.  $\bigcup_{i=1}^{\infty} V_i = [-1, 1]$

g.  $\bigcap_{i=1}^{\infty} V_i = \{0\}$

□

---

**6.1.33:**

- a. Find  $\mathcal{P}(\emptyset)$ .
  - b. Find  $\mathcal{P}(\mathcal{P}(\emptyset))$ .
  - c. Find  $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ .
- 

*Proof.* a.

$$\mathcal{P}(\emptyset) = \{\emptyset\}$$

b.

$$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$

c.

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

□

---

**6.2.10:** Use an element argument to prove the statement. Assume that all sets are subsets of a universal set  $U$ . For all sets  $A, B$ , and  $C$ ,

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

---

*Proof.* Consider arbitrary sets  $A, B, C$ . The given statement is proven true by establishing two key sub-statements:

1.  $(A - B) \cap (C - B) \subseteq (A \cap C) - B$
2.  $(A \cap C) - B \subseteq (A - B) \cap (C - B)$

Firstly, for statement 1:

Let  $x$  be an element of  $(A - B) \cap (C - B)$ .

Since  $x$  is in the intersection, it must be in both  $A - B$  and  $C - B$ . Thus,  $x$  belongs to  $A$  and  $C$  but not to  $B$ . Consequently,  $x$  is an element of  $A \cap C$  and not in  $B$ , leading to  $x$  being in  $(A \cap C) - B$ . This confirms  $(A - B) \cap (C - B) \subseteq (A \cap C) - B$ .

Next, for statement 2:

Take  $x$  as an element of  $(A \cap C) - B$ .

From the set difference and intersection definitions,  $x$  is in both  $A$  and  $C$ , and not in  $B$ . Hence,  $x$  lies in both  $A - B$  and  $C - B$ . Therefore, by intersection properties,  $x$  is in  $(A - B) \cap (C - B)$ .

This confirms  $(A \cap C) - B \subseteq (A - B) \cap (C - B)$ . Since both sub-statements are proven, it follows that  $(A - B) \cap (C - B) = (A \cap C) - B$ , thereby validating the original statement.  $\square$

---

**6.2.14:** Use an element argument to prove the statement. Assume that all sets are subsets of a universal set  $U$ . For all sets  $A, B$ , and  $C$ , if  $A \subseteq B$  then  $A \cup C \subseteq B \cup C$ .

---

*Proof.* Assume sets  $A, B, C$  with  $A \subseteq B$ . Consider an arbitrary element  $x$ . Assume  $x \in A \cup C$ . According to the definition of union, this implies  $x \in A$  or  $x \in C$ .

**Case 1:**  $x \in A$ :

Given  $A \subseteq B$ , if  $x$  is in  $A$ , it must also be in  $B$ . Hence, by the nature of union,  $x$  is in  $B \cup C$ .

**Case 2:**  $x \in C$ :

Directly by the definition of union, if  $x$  is in  $C$ , it is necessarily in  $B \cup C$ . In both scenarios,  $x \in B \cup C$  is validated. Thus, it is concluded that  $A \cup C \subseteq B \cup C$ .  $\square$

---

**6.2.32:** Use the element method for proving a set equals the empty set to prove the statement. Assume that all sets are subsets of a universal set  $U$ . For all sets  $A, B$ , and  $C$ , if  $A \subseteq B$  and  $B \cap C = \emptyset$  then  $A \cap C = \emptyset$ .

---

*Proof.* To prove the statement, assume the contrary. Consider sets  $A, B, C$  such that  $A \subseteq B$  and  $B \cap C = \emptyset$ . The negation of the statement is  $A \cap C \neq \emptyset$ , implying the existence of an element  $x$  such that  $x \in A$  and  $x \in C$ .

Since  $x \in A$  and  $A \subseteq B$ , it follows that  $x \in B$ . However, if  $x \in C$  and  $B \cap C = \emptyset$ , this leads to a contradiction because  $x$  cannot be in both  $B$  and  $C$  if their intersection is empty.

This contradiction disproves the negation, thereby proving the original statement: If  $A \subseteq B$  and  $B \cap C = \emptyset$ , then  $A \cap C = \emptyset$ .  $\square$

---

**6.2.39:** Prove the statement. For all integers  $n \geq 1$ , if  $A_1, A_2, A_3, \dots$  and  $B$  are any sets, then

$$\bigcap_{i=1}^n (A_i - B) = \left( \bigcap_{i=1}^n A_i \right) - B.$$

---

*Proof.* Assume  $A_1, A_2, A_3, \dots$  and  $B$  are arbitrary sets. The statement is proven by showing:

1.  $[\bigcap_{i=1}^n (A_i - B)] \subseteq [\bigcap_{i=1}^n A_i] - B$
2.  $[\bigcap_{i=1}^n A_i] - B \subseteq [\bigcap_{i=1}^n (A_i - B)]$

First, to prove statement 1:

Let  $x$  be an element of  $\bigcap_{i=1}^n (A_i - B)$ . This means  $x$  is in each  $A_i - B$  for all  $i$ , so  $x$  is in every  $A_i$  and not in  $B$ . Therefore,  $x$  is in  $\bigcap_{i=1}^n A_i$  and, by set difference,  $x$  is in  $(\bigcap_{i=1}^n A_i) - B$ .

Next, to prove statement 2:

Consider  $x$  as an element of  $(\bigcap_{i=1}^n A_i) - B$ . By set difference,  $x$  is in  $\bigcap_{i=1}^n A_i$  and not in  $B$ . Hence,  $x$  is in each  $A_i$  and not in  $B$ , implying  $x$  is in every  $A_i - B$ , and therefore in  $\bigcap_{i=1}^n (A_i - B)$ .

As both sub-statements are confirmed, it follows that  $\bigcap_{i=1}^n (A_i - B) = (\bigcap_{i=1}^n A_i) - B$ , thus proving the original statement.  $\square$