Discrete Math Test 1 Practice Questions

1. Negate each of the following statements:
   a. $(P \wedge Q) \vee (\neg P \wedge R)$
   b. $P \rightarrow (Q \wedge R)$
   c. $\exists x: 2 \leq x < 4$
   d. Neither Butternut nor Squash are hungry.
   e. People who live in glass houses do not throw stones.
   f. If x is an integer, then x is either even or odd.
   g. x is even if, and only if, x^2 is even.
   h. x is prime only if x is odd.
   i. Whenever a function is differentiable, it is continuous.
   j. Continuity is a sufficient condition for boundedness.
   k. If f is not continuous at 1 and -1, then the group of invariants is an infinite cyclic group, a cyclic group of order 2, or the trivial group.
   l. For every rational number x, there exists an integer n that is greater than x.
   m. Every odd integer is nonzero.
   n. There exists x such that $g(x) > 0$.
   o. There are no even prime numbers.
   p. For every x there is a y such that $xy = 1$.
   q. There is a y such that $xy = 0$ for every x.
   r. If $x \neq 0$, then there exists y such that $xy = 1$.
   s. If $x \neq 0$, then $xy^2 \geq 0$ for all y.
   t. For all $\varepsilon > 0$, there exists $\delta > 0$ such that if x is a real number with $|x - 1| < \delta$, then $|x^2 - 1| < \varepsilon$.
   u. For all real numbers M, there exists a real number N such that $|f(n)| > M$ for all $n > N$.
   v. There is a unique solution to the equation $x^2 + 2x = 15$.
   w. For all real numbers x, if x is greater than a or equal to b, then x does not equal c.
   x. If 8 does not divide $x^2 - 1$, then x is even.
   y. There is a number n for which no other number is either less than n or equal to n.

2. Determine the truth value of the following statements:
   a. If 2 is even, then 3 is even.
   b. If 2 is even, then 3 is odd.
   c. If 2 is odd, then 3 is even.
   d. If 2 is odd, then 3 is odd.
   e. 1+1=2 if, and only if, 2+2=4.
   f. 1+1=3 if, and only if, 2+3=4.
   g. 0 > 1 if, and only if, 2 > 1.

3. Write these statements as conditional statements:
   a. It snows or it is not sunny.
   b. x is divisible by 3 or x is divisible by 5.
   c. Seven is an integer and seven is even.

4. Write the contrapositive, converse, and inverse of the statements:
   a. $(P \vee \neg Q) \to R$
   b. If it rains, then it pours.
   c. $2x + 1$ is odd whenever x is an integer.
   d. To run quickly, it is necessary to have long legs.
   e. A positive integer is prime only if it has no divisors other than 1 and itself.
   f. If x is a nonzero real number, then there exists a real number y such that $xy = 1$.
   g. For all positive real numbers x, there exists an integer n such that $\frac{1}{n} < x$.

5. Consider the statement form $P \to Q$.
   a. Write the negation of this statement form in as simple a form as possible.
   b. Write the negation of the contrapositive of this statement form in as simple a form as possible.
   c. Write the negation of the inverse of this statement form in as simple a form as possible.

6. Consider the statement form $\forall x, \big((P(x) \wedge Q(x)) \to R(x)\big)$.
   a. Write the contrapositive of the statement.
   b. Write the converse of the statement.
   c. Write the negation of the contrapositive of the statement.
   d. Write the negation of the converse of the statement.

7. Indicate which of these are tautologies, which are contradictions, and which are neither.
   a. $P \to (P \vee Q)$
   b. $Q \to (Q \to P)$
   c. $\neg(P \to Q) \to P$
   d. $P \to \neg(Q \wedge \neg P)$
   e. $\big(P \to (\neg R \vee Q)\big) \wedge R$
   f. $\big(P \wedge (P \to Q)\big) \to Q.$
   g. $\big(\neg Q \wedge (P \to Q)\big) \to \neg P$
   h. $(P \to Q) \leftrightarrow (P \to (Q \vee \neg P)$
   i. $P \to (Q \to R) \leftrightarrow (P \wedge Q) \to R$

8. Suppose $P \to Q$ is false. Determine the truth value:
   a. $\neg(P \wedge Q) \to Q$
   b. $\neg P \vee (P \leftrightarrow Q)$

9. Construct a statement form S using P, Q, and R such that $\big((P \wedge \neg Q) \to R\big) \leftrightarrow S$ is a tautology.

10. Consider the statement form $(P \vee \neg Q) \to (R \wedge Q)$.
    a. Write out the truth table for this form.
    b. Give a statement in English that is in this form.
    c. Write the negation of your English statement, and simplify the sentence as much as possible.

d. Write the contrapositive of your English statement, and simplify the sentence as much as possible.

11. Consider the statement: “If H and G/H are p-groups, then G is a p-group.”
    a. State the converse of this statement.
    b. State the contrapositive of this statement.
    c. Consider the following: “G is a p-group if, and only if, H and G/H are p-groups.” Write this in terms of your answers to the first two parts of this problem.

12. Put parentheses in $P \wedge Q \vee R$ to create a statement form with the given truth table:

P	Q	R	$P \wedge Q \vee R$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

13. We are on the island of knights (truth-tellers) and knaves (liars).
    a. Fig says “If I am a knight, then each person living on this island is a knight or a knave.” Can you determine whether Fig is a knight or a knave?
    b. Suppose Fig says “If I am a knight, then so is Maple.” Can you tell what Fig and Maple are?

14. We are on the island of knights (truth-tellers) and knaves (liars). You meet a native at a fork in the road. One branch of the road leads to the ruins you want to visit, while the other branch leads deep into the jungle. What is one yes-or-no question you can ask the native in order to determine which branch of road to take?

15. A horrendous crime was committed, and there are three suspects: Alice, Betty, and Charlotte.
    Alice says: “If Betty did not do it, then it was Charlotte.”
    Betty says: “Alice and Charlotte did it together or Charlotte did it alone.
    Charlotte says: “We all did it together.”
    a. If the police know that exactly one person committed the crime, and exactly one person is lying, who is guilty?
    b. As it turns out, exactly one person committed the crime and all the students are lying. Who is the guilty party?

16. Three professors walk into a coffee shop. The waiter asks: “Does everyone want coffee?” The first professor says “I don’t know.” The second professor says “I don’t know.” The third professor says “No, not everyone wants coffee.” Who will the waiter serve coffee to?

17. Find a (different) useful way to describe the following sets (your useful way could be a sketch):
    a. $\{x \in \mathbb{Z}: x^2 = 1\}$
    b. $\{x \in \mathbb{N}: x^2 = 1\}$
    c. $\{(x, y) \in \mathbb{R}^2: y = 0\}$
    d. $\{(x, y, z) \in \mathbb{R}^3: z = 0\}$
    e. $\{x \in \mathbb{Z}: x$ is even$\}$
    f. $\{(m, n): m, n \in \mathbb{Z}\}$
    g. $\{x \in \mathbb{N}: x \leq 0\}$
    h. $\{x \in \mathbb{Z}: x^2 > 0\}$.

18. What are the sets A and B described by the following statements?
    a. $\forall x, \big(x \in A \leftrightarrow \exists n: (n \in \mathbb{Z} \land x = 2n)\big)$.
    b. $\forall x, \big(x \in B \leftrightarrow \exists n: (n \in \mathbb{Z} \land x = 2n + 1)\big)$.

19. Write these statements using symbols. Then, negate them.
    a. For all x, it is the case that x is an integer.
    b. There exists an integer x such that $x > 0$.
    c. There is a rational number x such that $x^2 + 1 > 0$.
    d. For every real number x, there exists a real number y such that $x < y$.
    e. There is a real number y such that $x < y$ for all x.
    f. If x is a rational number, then $x^2 - \pi \neq 0$.
    g. A real number x satisfies $x^2 > 0$, if $x \neq 0$.
    h. If $x > 0$, then $x > 4$ or $x < 6$.
    i. There is a unique solution to the equation $x^2 + 2x = 15$.
    j. For all positive integers x, there exists a real number y such that for all real numbers z, we have $y = z^x$ or $z = y^x$.
    k. Every rational number is the product of two irrational numbers.
    l. There are integers m and n such that for each rational number x, we have $m < nx$ or $n < mx$.
    m. Every rational number is the solution of an equation $ax + b = 0$, where a and b are integers.
    n. All odd positive integers are prime.

20. Write these statements using symbols.
    a. For every x there is a y such that $x = 2y$.
    b. For every y, there is an x such that $x = 2y$.
    c. For every x and for every y, it is the case that $x = 2y$.
    d. There exists an x such that for some y the equality $x = 2y$ holds.
    e. There exists an x and a y such that $x = 2y$.

21. For the statements in the previous question,
    a. Which are true if the universe for x and y is $\mathbb{R}$?
    b. Which are true if the universe for x is $\mathbb{R}$ and the universe for y is $\mathbb{Z}$?

22. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the universe is $\mathbb{Z}$, which of these are true?
    a. $Q(0)$
    b. $Q(-1)$
    c. $Q(1)$
    d. $\exists x: Q(x)$
    e. $\forall x, Q(x)$.
    f. $\exists x: \neg Q(x)$.
    g. $\forall x, \neg Q(x)$.

23. Let $P(x, y)$ be the statement "$x + y = x - y$". If the universe for both variables is $\mathbb{Z}$, which of these are true?
    a. $\forall y, Q(1, y)$
    b. $\exists x: Q(x, 2)$
    c. $\exists x: \exists y: Q(x, y)$
    d. $\forall x, \exists y: Q(x, y)$
    e. $\exists y: \forall x, Q(x, y)$
    f. $\forall y, \exists x: Q(x, y)$
    g. $\forall x, \forall y, Q(x, y)$

24. Which of these are true?
    a. $\exists x \in \mathbb{R}: x^3 = -1$
    b. $\exists x \in \mathbb{R}: x^4 < x^2$
    c. $\forall x \in \mathbb{R}, ((-x)^2 = x^2$
    d. $\forall x \in \mathbb{R}, (2x > x)$
    e. $\forall n \in \mathbb{Z}, n^2 \geq 0$
    f. $\forall n \in \mathbb{Z}, n^2 \geq n$
    g. $\exists n \in \mathbb{Z}: n^2 = 2$
    h. $\exists n \in \mathbb{Z}: n^2 < 0$
    i. $\exists x \in \mathbb{R}: x^2 = 2$
    j. $\exists! \, x \in \mathbb{R}: x^2 = 2$
    k. $\exists! \, x \in \mathbb{Z}^+: x^2 = 1$.
    l. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} : n^2 < m$
    m. $\exists n \in \mathbb{Z}: \forall m \in \mathbb{Z}, n < m^2$
    n. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}: n + m = 0$.
    o. $\exists n \in \mathbb{Z}: \exists m \in \mathbb{Z}: n^2 + m^2 = 5$.
    p. $\exists n \in \mathbb{Z}: \forall m \in \mathbb{Z}, nm = m$
    q. $\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, \exists p \in \mathbb{Z}: p = (m + n)/2$
    r. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x^2 = y$
    s. $\forall x \in \mathbb{R}, \big(x \neq 0 \rightarrow \exists y: (xy = 1)\big)$.
    t. $\exists x \in \mathbb{R}: \forall x \in \mathbb{R}, (y \neq 0 \rightarrow xy = 1)$.
    u. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists p \in \mathbb{R}: p = (x + y)/2$

25. Provide a counterexample to show that the following statements are false:
    a. $\forall x \in \mathbb{Z}, x^2 > 0$.
    b. $\forall x \in \mathbb{R}, x^2 \neq 2$.

c. $\forall x \in \mathbb{N}, x^2 \neq x$.
d. $\forall x, y \in \mathbb{Z}, (x^2 = y^2 \rightarrow x = y)$.
e. $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}: y^2 = x$.
f. $\forall x, y \in \mathbb{Z}, x^2 \neq y^3$.

26. What are the truth values of these statements?
a. $\exists!\, x: P(x) \rightarrow \exists x: P(x)$
b. $\forall x, P(x) \rightarrow \exists!\, x: P(x)$
c. $\exists!\, x: \neg P(x) \rightarrow \neg\forall x, P(x)$

27. Consider the statement $\exists M: \big((M \in \mathbb{Z}) \wedge \forall x, (x^2 \leq M)\big)$.
a. Negate the statement.
b. Which statement is true, the original or the negation?

28. Consider the statement: $\forall x, \big(\exists y: (x^3 = y^2) \vee \forall z, (z^2 < 0 \rightarrow x^3 \neq z^2)\big)$.
a. Negate the statement.
b. Which statement is true, the original or the negation?

29. Determine which arguments are valid. Suppose x, y, and z are real numbers.
a. Everyone who loves math loves logic.
I don't love logic.
∴ I don't love math.
b. If Susie wears red pants, then I'll do a jig.
Susie is wearing green pants.
∴ I did not do a jig.
c. If l is a positive real number, then there exists a real number m such that $m > l$.
Every real number m is less than t.
∴ The real number t is not positive.
d. Every Discrete quiz is hard or my name is Bob.
My name is Aarav.
∴ Every Discrete quiz is hard.
e. Every math or computer science major takes Discrete Math.
Natasha is taking Discrete Math.
∴ Natasha is a math or computer science major.
f. There is a house on every street such that if that house is blue, the one next to it is black.
There is no blue house on my street.
∴ There is no black house on my street.
g. If $x > 5$, then $y < \frac{1}{5}$.
$y = 1$.
$\therefore x \leq 5$.
h. If $x > y$, then $x^2 > y^2$.
$x < y$.
$\therefore x^2 \leq y^2$.
i. If $y > x$ and $y > 0$, then $y > z$.

$y \le z$.
$\therefore y \le x$ or $y \le 0$.

j. If x is a positive real number, then x^2 is a positive real number.
$\therefore$ If a^2 is positive, where a is real, then a is positive.

k. $P \to (Q \vee R)$
$\neg(P \to Q)$
$\therefore R$

l. $(P \wedge Q) \to R$
$\neg P \vee \neg Q$
$\therefore \neg R$

m. $(P \wedge Q) \to R$
$P \vee Q$
$Q \to P$
$\therefore R$

n. $P \to (Q \vee R)$
$\neg Q \vee \neg R$
$\therefore \neg P \vee \neg R$

30. Convert the integers from binary to decimal:
    a. 00101110_2
    b. 1011011111000101_2

31. Convert from decimal to binary:
    a. 87
    b. 1311

32. Use 8-bit representations to compute the sums:
    a. $37 + (-109)$
    b. $120 + (-82)$
    c. $(-91) + (-16)$
    d. $93 + (-77)$
    e. $41 - 87$

33. Consider the following Boolean expressions. Draw a circuit with only one logic gate that is equivalent to each expression.
    a. $\neg P \wedge \neg(P \wedge Q \wedge R \wedge S)$
    b. $\neg P \wedge \neg(\neg P \wedge Q)$

34. Simplify the Boolean expressions as much as possible. Draw a circuit for your simplified expressions.
    a. $P \vee (P \wedge \neg Q \wedge R \wedge S)$
    b. $(P \vee R) \wedge (\neg P \vee Q) \wedge (R \vee Q)$
    c. $P \vee (P \vee (P \wedge Q))$

35. Do some binary computations:
    a. Add $110111011_2 + 1001010010_2$

    b. Subtract $10000001_2 - 111111_2$

    c. Subtract $110110001_2 - 1011101_2$

36. Write the input/output table for a half-adder. Write the related Boolean expressions for both outputs (the sum and the carry). Draw a circuit for your half-adder. Can you draw your circuit using only two logic gates?

37. What is the difference between a half-adder and a full-adder?

38. Why does a half-adder require two outputs? Why does a full-adder not require more than two outputs?

39. Write the input/output table for a full-adder. Write the related Boolean expressions for both outputs (the sum and the carry). Draw a circuit for your full-adder.

40. Given the input/output table, draw a circuit with as few gates as possible:

P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

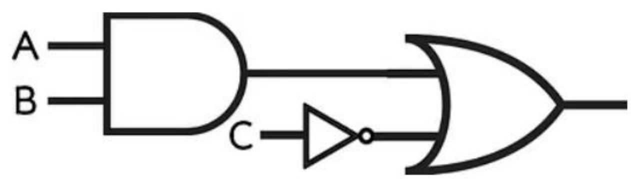
41. Given the input/output table, draw a circuit with as few gates as possible:

P	Q	R	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

42. Given the input/output table, draw a circuit with as few gates as possible:

P	Q	R	S
1	1	1	0
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

43. Write a Boolean expression that matches this digital logic circuit:



44. Write a Boolean expression that matches this digital logic circuit: