

Lecture Notes

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1 Speaking Mathematically

1.1 Variables

1.2 The Language of Sets

1.3 The Language of Relations and Functions

2 The Logic of Compound Statements

2.1 Logical Form and Logical Equivalence

2.2 Conditional Statements

2.3 Valid and Invalid Arguments

2.4 Application: Digital Logic Circuits

2.5 Application: Number Systems and Circuits for Addition

3 The Logic of Quantified Statements

3.1 Predicates and Quantified Statements I

3.2 Predicates and Quantified Statements II

3.3 Statements with Multiple Quantifiers

3.4 Arguments with Quantified Statements

4 Elementary Number Theory and Methods of Proof

4.1 Direct Proof and Counterexample I: Introduction

4.2 Direct Proof and Counterexample II: Rational Numbers

4.3 Direct Proof and Counterexample III: Divisibility

4.4 Direct Proof and Counterexample IV: Division into Cases and the Quotient-Remainder Theorem

4.5 Direct Proof and Counterexample V: Floor and Ceiling

4.6 Indirect Argument: Contradiction and Contraposition

4.7 Indirect Argument: Two Classical Theorems

4.8 Application: Algorithms

5 Sequences, Mathematical Induction, and Recursion

5.1 Sequences

5.1.1 Definition

A function whose domain is either all of the integers between two given integers (**finite sequence**)

$$a_m, a_{m+1}, a_{m+2}, \dots, a_n$$

or all of the integers greater than or equal to a given integer (**infinite sequence**)

$$a_m, a_{m+1}, a_{m+2}, \dots$$

Each individual element a_k (read “ a sub k ”) is called a **term**. The k in a_k is called a **subscript** or **index**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k .

5.1.2 Alternating Sequence

A sequence whose terms alternate in sign.

$$a_1, -a_2, a_3, -a_4, \dots$$

Example:

$$c_j = (-1)^j \quad \text{for all integers } j \geq 0$$

This sequence will oscillate endlessly between 1 and -1 .

5.1.3 Finding an Explicit Formula to fit Given Initial Terms

Note: Any such formula is a guess, but it is very useful to be able to make such guesses.

Problem. Find an explicit formula for a sequence that has the following initial terms:

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots$$

Proof. Denote the general term of the sequence by a_k and suppose the first term of the sequence is a_1 . Then observe that the denominator of each term is a perfect square. Thus, the terms can be rewritten as

$$\frac{1}{1^2}, -\frac{1}{2^2}, \frac{1}{3^2}, -\frac{1}{4^2}, \frac{1}{5^2}, -\frac{1}{6^2}, \dots$$
$$a_1, a_2, a_3, a_4, a_5, a_6, \dots$$

Note that the denominator of each term equals the square of the subscript of that term, and that the numerator equals ± 1 . Hence

$$a_k = \frac{\pm 1}{k^2}$$

The ± 1 oscillates back and forth between $+1$ and -1 , meaning a_k is an alternating sequence. In this specific example, when k is odd, the numerator is $+1$ and when k is even, the numerator is -1 . Therefore, the explicit formula of a_k can be written as

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1$$

If we were to make the first term a_0 instead of a_1 , the formula would be

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0$$

□

5.1.4 Summation Notation

5.2 Mathematical Induction I

5.3 Mathematical Induction II

5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

5.5 Application: Correctness of Algorithms

5.6 Defining Sequences Recursively

5.7 Solving Recurrence Relations by Iteration

5.8 Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

5.9 General Recursive Definitions and Structural Induction

6 Set Theory

6.1 Set Theory: Definitions and the Element Method of Proof

6.2 Properties of Sets

6.3 Disproofs, Algebraic Proofs, and Boolean Algebras

6.4 Boolean Algebras, Russell's Paradox, and the Halting Problem