

Capstone: Discrete Math

Homework 8

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Due: November 2, 2023

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Homework: 4.8

Problem 15. Use the Euclidean algorithm to hand-calculate the greatest common divisors of 832 and 10933.

Proof. ENTER HERE

□

Problem 16. Use the Euclidean algorithm to hand-calculate the greatest common divisors of 4131 and 2431.

Proof. Enter proof here.

□

Problem 19. Prove that for all positive integers a and b , $a|b$ if, and only if, $\gcd(a, b) = a$. (Note that to prove “ A if, and only if, B ,” you need to prove “if A then B ” and “if B then A .”)

Proof. Enter proof here.

□

Problem 20.

- Prove that if a and b are integers, not both zero, and $d = \gcd(a, b)$, then a/d and b/d are integers with no common divisor that is greater than one.
- Write an algorithm that accepts the numerator and denominator of a fraction as input and produces as output the numerator and denominator of that fraction written in lowest terms. (The algorithm may call upon the Euclidean algorithm as needed.)

Proof.

- Enter here.
- Enter here.

□

Homework: 5.1

Problem 44. Write using summation or product notation:

$$(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$$

Proof.

$$\sum_{n=1}^5 ((n^3 - 1) \cdot (-1)^{n-1})$$

□

Problem 45. Write using summation or product notation:

$$(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$$

Proof.

$$\prod_{n=2}^4 (n^2 - 1)$$

□

Problem 46. Write using summation or product notation:

$$\frac{2}{3 \cdot 4} - \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} - \frac{5}{6 \cdot 7} + \frac{6}{7 \cdot 8}$$

Proof.

$$\sum_{n=2}^6 \left(\left(\frac{n}{(n+1) \cdot (n+2)} \right) \cdot (-1)^n \right)$$

□

Problem 77.

- Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.
- Prove that $n! + k$ is divisible by k , for all integers $n \geq 2$ and $k = 2, 3, \dots, n$.

Proof.

- a. Let n be any integer greater than or equal to 2.

By definition of factorial,

$$n! = n \cdot (n-1) \cdots 2 \cdot 1$$

Since 2 will always be a factor of $n!$, $n!$ can be written as $2k$ for some integer k . Thus,

$$\begin{aligned} n! + 2 &= 2k + 2 \\ &= 2(k+1) \end{aligned}$$

$k+1$ is an integer, so $n! + 2$ can be written as $2m$ for some integer m .

Therefore, $n! + 2$ is divisible by 2 by definition of divisibility.

- b. Let n be any integer greater than or equal to 2.

Let k be any integer greater than or equal to 2 and less than or equal to n .

By definition of factorial, $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and since $2 \leq k \leq n$, k will always be a factor of $n!$.

$$\begin{aligned} n! + k &= n \cdot (n-1) \cdots 2 \cdot 1 + k \\ &= n \cdot (n-1) \cdots (k+1) \cdot k \cdot (k-1) \cdots 2 \cdot 1 + k \\ &= k \cdot (n \cdot (n-1) \cdots (k+1) \cdot (k-1) \cdots 2 \cdot 1 + 1) \end{aligned}$$

Therefore, $n! + k$ can be written as $k \cdot m$ for some integer $m = (n \cdot (n-1) \cdots (k+1) \cdot (k-1) \cdots 2 \cdot 1 + 1)$.

Thus, $n! + k$ is divisible by k by definition of divisibility.

□

Problem 79. Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p .

Proof. Let p be any prime number and r be any integer such that $0 < r < p$. By definition of $\binom{p}{r}$,

$$\binom{p}{r} = \frac{p!}{r! \cdot (p-r)!}$$

By definition of factorial,

$$p! = p \cdot (p-1) \cdots 2 \cdot 1$$

Therefore, $p!$ can be written as $p \cdot n$ for some integer $n = (p-1) \cdots 2 \cdot 1$. Putting $p!$ back into the equation for $\binom{p}{r}$,

$$\binom{p}{r} = \frac{p \cdot n}{r! \cdot (p-r)!}$$

Therefore, $\binom{p}{r}$ can be written as $p \cdot m$ for some integer $m = \frac{n}{r! \cdot (p-r)!}$.

Thus, $\binom{p}{r}$ is divisible by p by definition of divisibility.

□

Homework: 5.2

Problem 7. Prove the statement using mathematical induction.

For all integers $n \geq 1$,

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n - 3)}{2}$$

Proof. Let the property $P(n)$ be the equation:

$$1 + 6 + 11 + 16 + \cdots + (5n - 4) = \frac{n(5n - 3)}{2}$$

First, we must prove that $P(1)$ is true (basis step).

$$1 = \frac{1(5 \cdot 1 - 3)}{2}$$

The left-hand side of the equation is 1, and the right-hand side of the equation is

$$\begin{aligned} \frac{1(5 \cdot 1 - 3)}{2} &= \frac{1(5 - 3)}{2} \\ &= \frac{1(2)}{2} \\ &= 1 \end{aligned}$$

Thus, $P(1)$ is true.

Now, suppose that $P(k)$ is true for some integer $k \geq 1$ (inductive hypothesis). That is,

$$1 + 6 + 11 + 16 + \cdots + (5k - 4) = \frac{k(5k - 3)}{2}$$

We must show that $P(k + 1)$ is true (inductive step). That is,

$$\begin{aligned} 1 + 6 + 11 + 16 + \cdots + (5(k + 1) - 4) &= \frac{(k + 1)(5(k + 1) - 3)}{2} \\ 1 + 6 + 11 + 16 + \cdots + (5k - 1) &= \frac{(k + 1)(5k + 2)}{2} \end{aligned}$$

The left-hand side of $P(k+1)$ is

$$1 + 6 + 11 + 16 + \cdots + (5k-1) = 1 + 6 + 11 + 16 + \cdots + (5k-4) + (5k-1)$$

by making the next-to-last term explicit

$$= \frac{k(5k-3)}{2} + (5k-1)$$

by substitution of the inductive hypothesis

$$= \frac{5k^2 - 3k}{2} + (5k-1)$$

by distribution

$$= \frac{5k^2 - 3k + 10k - 2}{2}$$

by combining like terms

$$= \frac{5k^2 + 7k - 2}{2}$$

by combining like terms

The right-hand side of $P(k+1)$ is

$$\frac{(k+1)(5k+2)}{2} = \frac{5k^2 + 7k + 2}{2}$$

by distribution

$$= \frac{5k^2 + 7k - 2}{2}$$

by combining like terms

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. \square

Problem 9. Prove the statement using mathematical induction.

For all integers $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}$$

Proof. Let $P(n)$ be the equation:

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}$$

First, we must prove that $P(3)$ is true (basis step).

$$4^3 = \frac{4(4^3 - 16)}{3}$$

The left-hand side of the equation is 64, and the right-hand side of the equation is

$$\begin{aligned}\frac{4(4^3 - 16)}{3} &= \frac{4(64 - 16)}{3} \\ &= \frac{4(48)}{3} \\ &= \frac{192}{3} \\ &= 64\end{aligned}$$

Thus, $P(3)$ is true.

Now, suppose that $P(k)$ is true for some integer $k \geq 3$ (inductive hypothesis). That is,

$$4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4(4^k - 16)}{3}$$

We must show that $P(k + 1)$ is true (inductive step). That is,

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}$$

The left-hand side of $P(k + 1)$ is

$$\begin{aligned}4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} &= 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} \\ &\quad \text{by making the next-to-last term explicit} \\ &= \frac{4(4^k - 16)}{3} + 4^{k+1} \\ &\quad \text{by substitution of the inductive hypothesis} \\ &= \frac{4(4^k - 16) + 3(4^{k+1})}{3} \\ &\quad \text{by combining like terms} \\ &= \frac{4 \cdot 4^k - 64 + 3(4 \cdot 4^k)}{3} \\ &\quad \text{by distribution} \\ &= \frac{4 \cdot 4^k - 64 + 12 \cdot 4^k}{3} \\ &\quad \text{by distribution} \\ &= \frac{16 \cdot 4^k - 64}{3} \\ &\quad \text{by combining like terms} \\ &= \frac{4 \cdot (4 \cdot 4^k) - 64}{3} \\ &\quad \text{by factoring} \\ &= \frac{4((4 \cdot 4^k) - 16)}{3} \\ &\quad \text{by factoring} \\ &= \frac{4(4^{k+1} - 16)}{3} \\ &\quad \text{by exponent product rule}\end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. \square

Problem 12. Prove the statement using mathematical induction.

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integers $n \geq 1$.

Proof. Let the property $P(n)$ be the equation:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

First, we must prove that $P(1)$ is true (basis step).

$$\begin{aligned} \frac{1}{1 \cdot 2} &= \frac{1}{1+1} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

Thus, $P(1)$ is true.

Now, suppose that $P(k)$ is true for some integer $k \geq 1$ (inductive hypothesis). That is,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)((k+1)+1)} &= \frac{k+1}{(k+1)+1} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} &= \frac{k+1}{k+2} \\ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k^2+3k+2} &= \frac{k+1}{k+2} \end{aligned}$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
 \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k^2 + 3k + 2} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{k^2 + 3k + 2} \\
 &\text{by making the next-to-last term explicit} \\
 &= \frac{k}{k+1} + \frac{1}{k^2 + 3k + 2} \\
 &\text{by substitution of the inductive hypothesis} \\
 &= \frac{k(k+2) + 1}{k^2 + 3k + 2} \\
 &\text{by combining like terms} \\
 &= \frac{k^2 + 2k + 1}{k^2 + 3k + 2} \\
 &\text{by distribution} \\
 &= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
 &\text{by factoring} \\
 &= \frac{k+1}{k+2} \\
 &\text{by cancelling out like terms}
 \end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. \square

Problem 17. Prove the statement using mathematical induction.

$$\prod_{i=0}^n \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$$

for all integers $n \geq 0$.

Proof. Let the property $P(n)$ be the equation:

$$\prod_{i=0}^n \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$$

First, we must prove that $P(0)$ is true (basis step).

$$\begin{aligned}
\prod_{i=0}^0 \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) &= \frac{1}{(2 \cdot 0 + 2)!} \\
\left(\frac{1}{2 \cdot 0 + 1} \cdot \frac{1}{2 \cdot 0 + 2} \right) &= \frac{1}{2!} \\
\left(\frac{1}{1} \cdot \frac{1}{2} \right) &= \frac{1}{2} \\
\frac{1}{2} &= \frac{1}{2}
\end{aligned}$$

Thus, $P(0)$ is true.

Now, suppose that $P(k)$ is true for some integer $k \geq 0$ (inductive hypothesis). That is,

$$\prod_{i=0}^k \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!}$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$\begin{aligned}
\prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) &= \frac{1}{(2(k+1)+2)!} \\
\prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) &= \frac{1}{(2k+4)!}
\end{aligned}$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
\prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) &= \prod_{i=0}^k \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \cdot \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2} \right) \\
&\quad \text{by writing the } (k+1)\text{st term separately from the first } k \text{ terms} \\
&= \frac{1}{(2k+2)!} \cdot \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2} \right) \\
&\quad \text{by substitution of the inductive hypothesis} \\
&= \frac{1}{(2k+2)!} \cdot \left(\frac{1}{2k+3} \cdot \frac{1}{2k+4} \right) \\
&\quad \text{by distribution} \\
&= \frac{1}{(2k+4) \cdot (2k+3) \cdot (2k+2)!} \\
&= \frac{1}{(2k+4)!} \\
&\quad \text{by definition of factorial}
\end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. \square

Problem 19. Use mathematical induction, the product rule from calculus, and the facts that $\frac{d(x)}{dx} = 1$ and that $x^{k+1} = x \cdot x^k$ to prove that for all integers $n \geq 1$, $\frac{d(x^n)}{dx} = nx^{n-1}$.

Proof. Let the property $P(n)$ be the equation:

$$\frac{d(x^n)}{dx} = nx^{n-1}$$

First, we must prove that $P(1)$ is true (basis step).

$$\begin{aligned}\frac{d(x^1)}{dx} &= 1 \cdot x^{1-1} \\ \frac{d(x)}{dx} &= 1 \cdot x^0 \\ 1 &= 1\end{aligned}$$

Thus, $P(1)$ is true.

Now, suppose that $P(k)$ is true for some integer $k \geq 1$ (inductive hypothesis). That is,

$$\frac{d(x^k)}{dx} = kx^{k-1}$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$\begin{aligned}\frac{d(x^{k+1})}{dx} &= (k+1) \cdot x^{(k+1)-1} \\ \frac{d(x^{k+1})}{dx} &= (k+1) \cdot x^k\end{aligned}$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}\frac{d(x^{k+1})}{dx} &= \frac{d}{dx}(x^k \cdot x) \\ &\quad \text{by power product rule} \\ &= x \cdot \frac{d}{dx}x^k + x^k \cdot \frac{d}{dx}x \\ &\quad \text{by derivative product rule} \\ &= x \cdot k(x^{k-1}) + x^k \cdot 1 \\ &\quad \text{by derivatives} \\ &= k(x^k) + x^k \\ &\quad \text{by combining like terms} \\ &= (k+1) \cdot x^k \\ &\quad \text{by factoring}\end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. \square

Homework: 5.3

Problem 9. Prove the statement using mathematical induction: $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

Proof. Enter proof here.

□

Problem 10. Prove the statement using mathematical induction: $n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

Proof. Enter proof here.

□

Problem 17. Prove the statement using mathematical induction: $1 + 3n \leq 4^n$, for every integer $n \geq 0$.

Proof. Enter proof here.

□

Problem 20. Prove the statement using mathematical induction: $2^n < (n + 2)!$, for every integer $n \geq 0$.

Proof. Enter proof here.

□

Problem 36. In a round-robin tournament each team plays every other team exactly once. If the teams are labeled T_1, T_2, \dots, T_n , then the outcome of such a tournament can be represented by a drawing, called a directed graph, in which the teams are represented as dots and an arrow is drawn from one dot to another if, and only if, the team represented by the first dot beats the team represented by the second dot. For example, the directed graph below shows one outcome of a round-robin tournament involving five teams, A, B, C, D, and E.

Use mathematical induction to show that in any round-robin tournament involving n teams, where $n \geq 2$, it is possible to label the teams T_1, T_2, \dots, T_n so that T_i beats T_{i+1} for all $i = 1, 2, \dots, n - 1$. For instance, one such labeling in the example above is $T_1 = A, T_2 = B, T_3 = C, T_4 = E, T_5 = D$. (Hint: Given $k + 1$ teams, pick one – say T' beats T_1 , T' loses to the first m teams (where $1 \leq m \leq k - 1$) and beats the $(m + 1)$ st team, and T' loses to all the other teams.)

Proof. Enter proof here.

