Test 2 Corrections

Capstone: Discrete Math

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Absent for Model UN Conference

November 17, 2023

Problem 2. For this problem, prove the Reverse Triangle Inequality, which is as follows: for all real numbers x and y,

$$|x| - |y| \leq |x + y|$$

You may use the Triangle Inequality itself as well as the following lemmas from class freely in your proof, indicating them by name if you apply them:

- **(Lemma 1)** For all real numbers r, $-|r| \leq r \leq |r|$
- **(Lemma 2)** For all real numbers r, $|-r| = |r|$

Hint: $|x| = |x + y + (-y)|$

Proof. We need to prove that for all real numbers x and y, the inequality $|x| - |y| \leq |x + y|$ is true. By the Triangle Inequality, for all real numbers a and b,

$$|a + b| \leq |a| + |b|$$

Applying this to $x + y$ and $-y$, we obtain:

$$\begin{aligned} |(x + y) + (-y)| &\leq |x + y| + |-y| \\ |x + y - y| &\leq |x + y| + |y| \\ |x| &\leq |x + y| + |y| \end{aligned}$$

By Lemma 2, $|-y| = |y|$. Thus,

$$\begin{aligned} |x| &\leq |x + y| + |y| \\ |x| - |y| &\leq |x + y| \end{aligned}$$

Therefore, we have shown that $|x| - |y| \leq |x + y|$ for all real numbers x and y. $\square$

Problem 3. Prove that for all real numbers x, $\lceil x \rceil = -\lfloor -x \rfloor$.

Proof. We need to prove that $\lceil x \rceil = -\lfloor -x \rfloor$ for all real numbers x.
By definition of ceiling, $\lceil x \rceil$ is the smallest integer greater than or equal to x. By definition of floor, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Let n be an integer such that:

$$\begin{aligned} n \leq x &< n + 1 \\ -n - 1 < -x &\leq -n \end{aligned}$$

By definition of ceiling, $\lceil x \rceil = n + 1$. By definition of floor, $\lfloor -x \rfloor = -n - 1$.
Observe,

$$\begin{aligned} \lceil x \rceil &= n + 1 \\ &= -(-n - 1) \\ &= -\lfloor -x \rfloor \end{aligned}$$

Therefore, $\lceil x \rceil = -\lfloor -x \rfloor$ for all real numbers x. $\square$

Problem 4. Let a, b, and c be arbitrary integers and let $r = a \bmod b$. Prove that if $c \mid a$ and $c \mid b$, then $c \mid r$.

Proof. We are given that a, b, and c are arbitrary integers, and that $r = a \bmod b$.
We need to prove that $c \mid r$ given that $c \mid a$ and $c \mid b$.
By definition of mod, $r = a \bmod b$ means that $a = bq + r$ for some integer q.
$c \mid a$ can be written as $a = ck_1$ for some integer k_1.
$c \mid b$ can be written as $b = ck_2$ for some integer k_2.

Substituting these into the equation $a = bq + r$:

$$\begin{aligned} a &= bq + r \\ ck_1 &= (ck_2)q + r \\ r &= ck_1 - (ck_2)q \\ r &= c(k_1 - k_2q) \end{aligned}$$

Since $k_1 - k_2q$ is an integer since it is the sum of products of integers, this means that r is an integer multiple of c. By definition, c divides r.
Therefore, $c \mid r$. $\square$

Problem 5. Prove that for all integers $n > 10$, $n^2 - 100$ is composite.

Proof. We need to prove that $n^2 - 100$ is composite for all integers $n > 10$.
By definition of a composite number, $n^2 - 100$ must be a positive integer that has at least one positive divisor other than 1 and itself.

$$n^2 - 100 = (n - 10)(n + 10)$$

For $n > 10$ and $n \neq 11$:

$$\begin{aligned} (n - 10) &> 1 \\ (n + 10) &> n > 1 \end{aligned}$$

Since $n^2 - 100$ can be written as the product of two integers greater than 1, $(n - 10)$ and $(n + 10)$, it is composite.

For $n = 11$:

$$\begin{aligned} (n - 10) &= 1 \\ (n + 10) &= 21 \\ n^2 - 100 &= 21 \\ &= 1 \cdot 21 \\ &= 3 \cdot 7 \end{aligned}$$

When $n = 11$, since $n^2 - 100$ can be written as the product of two integers other than 1 and itself (21), it is composite.

Thus, for $n > 10$, $n^2 - 100$ is always composite. $\square$

Problem 6. Prove that for all integers n, if $3 \nmid n$, then $3 \mid (n^2 + 2)$. (*Hint:* Quotient-Remainder Theorem will help immensely with this.)

Proof. We need to prove that for all integers n, if $3 \nmid n$, then $3 \mid (n^2 + 2)$.
By Quotient-Remainder Theorem, for any integer n and a positive integer d, there exist unique integers q and r such that:

$$\begin{gathered} n = dq + r \\ 0 \leq r < d \end{gathered}$$

For $d = 3$, and since $3 \nmid n$, the possible remainders for n are $r = 1$ or $r = 2$.

Case 1: If $r = 1$, then $n = 3q + 1$ for some integer q. Squaring both sides:

$$\begin{aligned} n^2 &= (3q + 1)^2 \\ &= 9q^2 + 6q + 1 \\ n^2 + 2 &= 9q^2 + 6q + 3 \\ &= 3(3q^2 + 2q + 1) \end{aligned}$$

Case 2: If $r = 2$, then $n = 3q + 2$. Squaring both sides:

$$\begin{aligned} n^2 &= (3q + 2)^2 \\ &= 9q^2 + 12q + 4 \\ n^2 + 2 &= 9q^2 + 12q + 6 \\ &= 3(3q^2 + 4q + 2) \end{aligned}$$

In both cases, 3 divides $n^2 + 2$, thus proving that for all integers n, if $3 \nmid n$, then $3 \mid (n^2 + 2)$. □