Lecture Notes

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1 Speaking Mathematically

1.1 Variables

1.2 The Language of Sets

1.3 The Language of Relations and Functions

2 The Logic of Compound Statements

2.1 Logical Form and Logical Equivalence

2.2 Conditional Statements

2.3 Valid and Invalid Arguments

2.4 Application: Digital Logic Circuits

2.5 Application: Number Systems and Circuits for Addition

3 The Logic of Quantified Statements

3.1 Predicates and Quantified Statements I

3.2 Predicates and Quantified Statements II

3.3 Statements with Multiple Quantifiers

3.4 Arguments with Quantified Statements

4   Elementary Number Theory and Methods of Proof

5 Sequences, Mathematical Induction, and Recursion

5.1 Sequences

5.1.1 Definition

A function whose domain is either all of the integers between two given integers (**finite sequence**)

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

or all of the integers greater than or equal to a given integer (**infinite sequence**)

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

Each individual element a_k (read "a sub k") is called a **term**. The k in a_k is called a **subscript** or **index**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

5.1.2 Alternating Sequence

A sequence whose terms alternate in sign.

$$a_1, -a_2, a_3, -a_4, \ldots$$

Example:

$$c_j = (-1)^j \qquad\qquad \text{for all integers } j \geq 0$$

This sequence will oscillate endlessly between 1 and -1.

5.1.3 Finding an Explicit Formula to fit Given Initial Terms

Note: Any such formula is a guess, but it is very useful to be able to make such guesses.

Problem. Find an explicit formula for a sequence that has the following initial terms:

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots$$

Proof. Denote the general term of the sequence by a_k and suppose the first term of the sequence is a_1. Then observe that the denominator of each term is a perfect square. Thus, the terms can be rewritten as

$$\frac{1}{1^2}, -\frac{1}{2^2}, \frac{1}{3^2}, -\frac{1}{4^2}, \frac{1}{5^2}, -\frac{1}{6^2}, \dots$$
$$a_1, a_2, a_3, a_4, a_5, a_6, \dots$$

Note that the denominator of each term equals the square of the subscript of that term, and that the numerator equals ± 1. Hence

$$a_k = \frac{\pm 1}{k^2}$$

The ± 1 oscillates back and forth between $+1$ and -1, meaning a_k is an alternating sequence. In this specific example, when k is odd, the numerator is $+1$ and when k is even, the numerator is -1. Therefore, the explicit formula of a_k can be written as

$$a_k = \frac{(-1)^{k+1}}{k^2} \qquad\qquad \text{for all integers } k \geq 1$$

If we were to make the first term a_0 instead of a_1, the formula would be

$$a_k = \frac{(-1)^k}{(k+1)^2} \qquad\qquad \text{for all integers } k \geq 0$$

$\square$

6 Set Theory

6.1   Set Theory: Definitions and the Element Method of Proof

6.2   Properties of Sets

6.3   Disproofs, Algebraic Proofs, and Boolean Algebras

6.4   Boolean Algebras, Russell's Paradox, and the Halting Problem