
5.8.6: Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } k \geq 2$$

where C and D are real numbers. Show that for any choice of C and D,

$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2$$

Proof. $\boxed{\text{TO-DO}}$ □

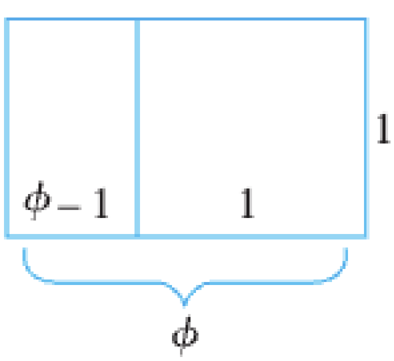
5.8.10: Suppose a sequence of the form $1.t.t^2.t^3 \ldots t^n \ldots$ where $t \neq 0$, satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of t. Suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

$$c_k = c_{k-1} + 6c_{k-2} \qquad \text{for all integers } k \geq 2$$
$$c_0 = 0 \quad c_1 = 3$$

5.8.15: Suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for the sequence.

$$t_k = 6t_{k-1} - 9t_{k-2} \qquad \text{for all integers } k \geq 2$$
$$t_0 = 1 \quad t_1 = 3$$

5.8.24: The numbers $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ that appear in the explicit formula for the Fibonacci sequence are related to a quantity called the *golden ration* in Greek mathematics. Consider a rectangle of length ϕ units and height 1, where $\phi > 1$.
Divide the rectangle into a rectangle and a square as shown in the preceding diagram. The square



is 1 unit on each side, and the rectangle has sides of length 1 and $\phi - 1$.

5.9.4b: The set of arithmetic expressions over the real numbers can be defined recursively as follows:

I. BASE: Each real number r is an arithmetic expression.

II. RECURSION: If u and v are arithmetic expressions, then the following are also arithmetic expressions:

   a. $(+u)$
   b. $(-u)$
   c. $(u + v)$
   d. $(u - v)$
   e. $(u \cdot v)$
   f. $\left(\frac{u}{v}\right)$

III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

(Note that the *expression* $\left(\frac{u}{v}\right)$ is legal even though the value of v may be 0.) Give derivations showing that each of the following is an arithmetic expression.

$$\left(\frac{(9 \cdot (6.1 + 2))}{((4 - 7) \cdot 6)}\right)$$

5.9.6: Define a set S recursively as follows:

I. BASE: $a \in S$

II. RECURSION: If $s \in S$, then

   a. $sa \in S$
   b. $sb \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S begins with an a.

5.9.11: Define a set S recursively as follows:

I. BASE: $0 \in S$

II. RECURSION: If $s \in S$, then

   a. $s + 3 \in S$
   b. $s - 3 \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every integer in S is divisible by 3.

5.9.16: Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

5.9.18: Give a recursive definition for the set of all strings of a's and b's that contain exactly one a.

6.1.6: Let $A = \{x \in \mathbf{Z} \mid x = 5a + 2 \text{ for some integer } a\}$, $B = \{y \in \mathbf{Z} \mid y = 10b - 3 \text{ for some integer } b\}$, and $C = \{z \in \mathbf{Z} \mid z = 10c + 7 \text{ for some integer } c\}$. Prove or disprove each of the following statements.

a. $A \subseteq B$

b. $B \subseteq A$

c. $B = C$

6.1.20: Let $B_i = \{x \in \mathbf{R} \mid 0 \leq x \leq i\}$ for all integers $i = 1, 2, 3, 4$.

a. $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$

b. $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$

c. Are B_1, B_2, B_3, and B_4 mutually disjoint? Explain.

6.1.23: Let $V_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} \leq x \leq \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

a. $\bigcup_{i=1}^{4} V_i = ?$

b. $\bigcap_{i=1}^{4} V_i = ?$

c. Are $V_1, V_2, V_3, \ldots$ mutually disjoint? Explain.

d. $\bigcup_{i=1}^{n} V_i = ?$

e. $\bigcap_{i=1}^{n} V_i = ?$

f. $\bigcup_{i=1}^{\infty} V_i = ?$

g. $\bigcap_{i=1}^{\infty} V_i = ?$

6.1.33:

a. Find $\mathscr{P}(\emptyset)$.

b. Find $\mathscr{P}(\mathscr{P}(\emptyset))$.

c. Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$.

6.2.10: Use an element argument to prove the statement. Assume that all sets are subsets of a universal set U. For all sets A, B, and C,

$$(A - B) \cap (C - B) = (A \cap C) - B.$$

6.2.14: Use an element argument to prove the statement. Assume that all sets are subsets of a universal set U. For all sets A, B, and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

6.2.32: Use the element method for proving a set equals the empty set to prove the statement. Assume that all sets are subsets of a universal set U. For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

6.2.39: Prove the statement. For all integers $n \geq 1$, if $A_1, A_2, A_3, \ldots$ and B are any sets, then

$$\bigcap_{i=1}^{n}(A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B.$$
