Capstone: Discrete Math

Homework 9

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Homework: 5.3

Problem 12. Prove the statement by mathematical induction:

$$\text{For any integer } n \geq 0, 7^n - 2^n \text{is divisible by 5.}$$

Proof. Let the property $P(n)$ be the statement:

$$7^n - 2^n \text{is divisible by 5}$$

First, we must prove that $P(0)$ is true (basis step).

$$7^0 - 2^0 \text{ is divisible by 5}$$

$$\begin{aligned} 7^0 - 2^0 &= 1 - 1 \\ &= 0 \\ &= 5(0) \end{aligned}$$

Thus, $P(0)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 0$ (inductive hypothesis). That is,

$$7^k - 2^k \text{ is divisible by 5}$$

By the definition of divisibility, for some integer r,

$$7^k - 2^k = 5r$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$7^{k+1} - 2^{k+1} \text{ is divisible by 5}$$

The left-hand side of $P(k+1)$ is:

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7(7^k) - 2(2^k) \\ &= 5 \cdot 7^k + 2 \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k + 2(7^k - 2^k) \\ &= 5 \cdot 7^k + 2(5r) \\ &= 5(7^k + 2r) \end{aligned}$$

$7^k + 2r$ is an integer since it is a sum of the product of integers, so $7^{k+1} - 2^{k+1}$ can be written as $5m$ for some integer $m = (7^k + 2r)$.
By the definition of divisibility, $7^{k+1} - 2^{k+1}$ is divisible by 5, and thus, $P(k+1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. □

Problem 18. Prove the statement by mathematical induction:

$$5^n + 9 < 6^n, \text{ for all integers } n \geq 2.$$

Proof. Let the property $P(n)$ be the inequality:

$$5^n + 9 < 6^n$$

First, we must prove that $P(0)$ is true (basis step).

$$\begin{aligned} 5^2 + 9 &< 6^2 \\ 25 + 9 &< 36 \\ 34 &< 36 \end{aligned}$$

Thus, $P(0)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 2$ (inductive hypothesis). That is,

$$5^k + 9 < 6^k$$

We must show that $P(k+1)$ is true (inductive step). That is,

$$5^{k+1} + 9 < 6^{k+1}$$

The left-hand side of $P(k+1)$ is:

$$\begin{aligned} 5^{k+1} + 9 &= 5(5^k) + 9 \\ &= 5 \cdot (5^k + 9 - 9) + 9 \\ &= 5 \cdot ((5^k + 9) - 9) + 9 \\ &< 5 \cdot (6^k - 9) + 9 \\ &< 5 \cdot 6^k - 45 + 9 \\ &< 5 \cdot 6^k - 36 \\ &< 5 \cdot 6^k \\ &< 6 \cdot 6^k \\ &< 6^{k+1} \end{aligned}$$

Thus, the left-hand side of $P(k+1)$ is less than the right-hand side of $P(k+1)$, and $P(k+1)$ is true. Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. □

Problem 22. Prove the statement by mathematical induction:

$$1 + nx \leq (1+x)^n, \text{ for all real numbers } x > -1 \text{ and integers } n \geq 2.$$

Proof. Let the property $P(n)$ be the inequality:

$$1 + nx \leq (1+x)^n$$

First, we must prove that $P(2)$ is true (basis step).

$$\begin{aligned} 1 + 2x &= 1 + 2x + 0 \\ &\leq 1 + 2x + x^2 \\ &= (1 + x)^2 \\ &= (1 + x)^n \end{aligned}$$

Thus, $P(0)$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq 2$ (inductive hypothesis). That is,

$$1 + kx \leq (1 + x)^k$$

We must show that $P(k + 1)$ is true (inductive step). That is,

$$1 + (k + 1)x \leq (1 + x)^{(k+1)}$$

The right-hand side of $P(k + 1)$ is:

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)^k(1 + x) \\ &\geq (1 + kx)(1 + x) \\ &= (1)(1) + (kx)(1) + (1)(x) + (kx)(x) \\ &= 1 + kx + x + kx^2 \\ &= 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x + 0 \\ &= 1 + (k + 1)x \end{aligned}$$

Thus, the right-hand side of $P(k + 1)$ is greater than or equal to the left-hand side of $P(k + 1)$, and $P(k + 1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction. □

Homework: 5.4

Problem 2. Suppose $b_1, b_2, b_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned} b_1 &= 4 \\ b_2 &= 12 \\ b_k &= b_{k-2} + b_{k-1} && \text{for all integers } k \geq 3. \end{aligned}$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

Proof. We need to prove that b_n is divisible by 4 for all integers $n \geq 1$ in the sequence defined by

$$\begin{aligned} b_1 &= 4 \\ b_2 &= 12 \\ b_k &= b_{k-2} + b_{k-1} && \text{for all integers } k \geq 3. \end{aligned}$$

Basis Step: For $n = 1$, $b_1 = 4$, which is divisible by 4.
For $n = 2$, $b_2 = 12$, which is also divisible by 4.

Inductive Step: Assume the statement is true for some integers k and $k-1$, i.e., both b_k and b_{k-1} are divisible by 4. We must show it is true for $k+1$.
From the recursive definition,

$$b_{k+1} = b_k + b_{k-1}$$

If $b_k = 4m$ and $b_{k-1} = 4n$ for some integers m and n, then

$$b_{k+1} = 4m + 4n = 4(m+n)$$

which is divisible by 4.
Hence, by mathematical induction, b_n is divisible by 4 for all integers $n \geq 1$. □

Problem 3. Suppose that $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 2 \\ c_2 &= 6 \\ c_k &= 3c_{k-3} && \text{for all integers } k \geq 3. \end{aligned}$$

Prove that c_n is even for all integers $n \geq 0$.

Proof. We need to prove that c_n is even for all integers $n \geq 0$ in the sequence defined by

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 2 \\ c_2 &= 6 \\ c_k &= 3c_{k-3} && \text{for all integers } k \geq 3. \end{aligned}$$

Basis Step: For $n = 0$, $c_0 = 2$, which is even.
For $n = 1$, $c_1 = 2$, which is even.
For $n = 2$, $c_2 = 6$, which is even.
Inductive Step: Assume the statement is true for $n = k - 3$, i.e., c_{k-3} is even. We must show it is true for k.
From the recursive definition,

$$c_k = 3c_{k-3}$$

If $c_{k-3} = 2m$ for some integer m, then

$$c_k = 3 \cdot 2m = 2 \cdot (3m)$$

which is divisible by 2, hence even.
Therefore, by mathematical induction, c_n is even for all integers $n \geq 0$. $\square$

Problem 7. Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned} g_1 &= 3 \\ g_2 &= 5 \\ g_k &= 3g_{k-1} - 2g_{k-2} \text{ for all integers } k \geq 3. \end{aligned}$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

Proof. We need to prove that $g_n = 2^n + 1$ for all integers $n \geq 1$ in the sequence defined by

$$\begin{aligned} g_1 &= 3 \\ g_2 &= 5 \\ g_k &= 3g_{k-1} - 2g_{k-2} \text{ for all integers } k \geq 3. \end{aligned}$$

Basis Step: For $n = 1$, $g_1 = 3$ equals $2^1 + 1 = 3$.
For $n = 2$, $g_2 = 5$ equals $2^2 + 1 = 5$.
Inductive Step: Assume the statement is true for $n = k - 1$ and $n = k - 2$, i.e., $g_{k-1} = 2^{k-1} + 1$ and $g_{k-2} = 2^{k-2} + 1$. We must show it is true for k.
From the recursive definition,

$$g_k = 3g_{k-1} - 2g_{k-2}$$

Substituting the inductive hypothesis,

$$g_k = 3(2^{k-1} + 1) - 2(2^{k-2} + 1)$$

Expanding and simplifying,

$$\begin{aligned} g_k &= 3 \cdot 2^{k-1} + 3 - 2^{k-1} - 2 \\ g_k &= 2 \cdot 2^{k-1} + 1 \\ g_k &= 2^k + 1 \end{aligned}$$

Thus, $g_k = 2^k + 1$. Therefore, by mathematical induction, $g_n = 2^n + 1$ for all integers $n \geq 1$. $\square$

Problem 18. Compute $9^0, 9^1, 9^2, 9^3, 9^4$ and 9^5. Make a conjecture about the units digit of 9^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

Proof. Compute the powers of 9:

$$
\begin{aligned}
9^0 &= 1 \\
9^1 &= 9 \\
9^2 &= 81 \\
9^3 &= 729 \\
9^4 &= 6561 \\
9^5 &= 59049
\end{aligned}
$$

Conjecture: The units digit of 9^n alternates between 1 and 9 for positive integers n.

Proof by Strong Mathematical Induction:

Basis Step: The conjecture holds for $n = 0$ and $n = 1$.
Inductive Step: Assume the conjecture holds for all integers less than or equal to k for some $k \geq 1$. We need to show it holds for $k + 1$.

1. If $k + 1$ is even, then k is odd, and by the inductive hypothesis, the units digit of 9^k is 9. Thus, the units digit of $9^{k+1} = 9^k \times 9$ is 1.

2. If $k + 1$ is odd, then k is even, and by the inductive hypothesis, the units digit of 9^k is 1. Thus, the units digit of $9^{k+1} = 9^k \times 9$ is 9.

By strong mathematical induction, the conjecture is true for all integers $n \geq 0$. $\square$

Homework: 5.5

Problem 2. Show that if the predicate is true before entry to the loop, then it is also true after exit from the loop:
Loop:

```
while (m >= 0 and m <= 100)
    m := m + 4
    n := n - 2
end while
```

Predicate: $m + n$ is odd

Proof. We need to show that if the predicate "$m + n$ is odd" is true before entry to the loop, then it is also true after exit from the loop.
The loop performs the following operations while m is between 0 and 100 (inclusive):

```
while (m >= 0 and m <= 100)
    m := m + 4
    n := n - 2
end while
```

The changes to m and n are by even numbers (4 and 2, respectively).
If $m+n$ is odd initially, then adding even numbers to both m and n results in an even change to $m+n$.
Since adding an even number to an odd number results in an odd number, $m + n$ remains odd after each iteration and after the loop exits.
Therefore, if the predicate "$m + n$ is odd" is true before the loop, it remains true after the loop exits.
□

Problem 4. Show that if the predicate is true before entry to the loop, then it is also true after exit from the loop:
Loop:

```
while (n >= 0 and n <= 100)
    n := n + 1
end while
```

Predicate: $2^n < (n + 2)!$

Proof. We need to show that if the predicate $2^n < (n + 2)!$ is true before entry to the loop, then it is also true after exit from the loop.
The loop performs the following operation while n is between 0 and 100 (inclusive):

```
while (n >= 0 and n <= 100)
    n := n + 1
end while
```

Assume the predicate $2^n < (n + 2)!$ is true for some n. We need to show it holds for $n + 1$, i.e., $2^{n+1} < (n + 3)!$
Since $2^n < (n+2)!$, multiplying both sides by 2 gives $2 \cdot 2^n < 2 \cdot (n+2)!$. Also, $2 < (n+3)$ for all $n \geq 0$, hence $2 \cdot (n + 2)! < (n + 3) \cdot (n + 2)! = (n + 3)!$.
Thus, $2^{n+1} < (n + 3)!$, proving the predicate for $n + 1$.

Therefore, if the predicate is true before the loop, it remains true after the loop exits. $\square$

Homework: 5.6

Problem 2. Find the first four terms of the recursively defined sequence:

$$\begin{aligned} b_k &= b_{k-1} + 3k, \text{ for all integers } k \geq 2 \\ b_1 &= 1 \end{aligned}$$

Proof.

$$\begin{aligned} b_1 &= 1 \\ b_2 &= b_1 + 3 \times 2 = 1 + 6 = 7 \\ b_3 &= b_2 + 3 \times 3 = 7 + 9 = 16 \\ b_4 &= b_3 + 3 \times 4 = 16 + 12 = 28 \end{aligned}$$

□

Problem 6. Find the first four terms of the recursively defined sequence:

$$\begin{aligned} t_k &= t_{k-1} + 2t_{k-2}, \text{ for all integers } k \geq 2 \\ t_0 &= -1 \\ t_1 &= 2 \end{aligned}$$

Proof.

$$\begin{aligned} t_0 &= -1 \\ t_1 &= 2 \\ t_2 &= t_1 + 2t_0 = 2 + 2(-1) = 2 - 2 = 0 \\ t_3 &= t_2 + 2t_1 = 0 + 2 \times 2 = 0 + 4 = 4 \end{aligned}$$

□

Problem 8. Find the first four terms of the recursively defined sequence:

$$\begin{aligned} v_k &= v_{k-1} + v_{k-2} + 1, \text{ for all integers } k \geq 3 \\ v_1 &= 1 \\ v_2 &= 3 \end{aligned}$$

Proof.

$$\begin{aligned} v_1 &= 1 \\ v_2 &= 3 \\ v_3 &= v_2 + v_1 + 1 = 3 + 1 + 1 = 5 \\ v_4 &= v_3 + v_2 + 1 = 5 + 3 + 1 = 9 \end{aligned}$$

$\square$

Problem 12. Let $s_0, s_1, s_2, \ldots$ be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$s_k = \frac{-s_{k-1}}{k}$$

Proof. Given that for all integers $n \geq 0$, we need to show that it satisfies $s_k = \frac{-s_{k-1}}{k}$.
Observe,

$$\begin{aligned} s_k &= \frac{(-1)^k}{k!} \\ &= \frac{(-1)^k}{k \cdot (k-1)!} \\ &= \frac{-(-1)^{k-1}}{k \cdot (k-1)!} \\ &= \frac{-1}{k} \cdot \frac{(-1)^{k-1}}{(k-1)!} \\ &= \frac{-s_{k-1}}{k} \end{aligned}$$

Hence, the sequence s_n satisfies the recurrence relation $s_k = \frac{-s_{k-1}}{k}$. $\square$

Problem 14. Let $d_0, d_1, d_2, \ldots$ be defined by the formula $d_n = 3^n - 2^n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation

$$d_k = 5d_{k-1} - 6d_{k-2}$$

Proof. Given that $d_n = 3^n - 2^n$ for all integers $n \geq 0$, we need to show that it satisfies $d_k = 5d_{k-1} - 6d_{k-2}$.
We have:

$$\begin{aligned} d_k &= 3^k - 2^k \\ d_{k-1} &= 3^{k-1} - 2^{k-1} \\ d_{k-2} &= 3^{k-2} - 2^{k-2} \end{aligned}$$

Now, calculate $5d_{k-1} - 6d_{k-2}$:

$$\begin{aligned}
5d_{k-1} - 6d_{k-2} &= 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2}) \\
&= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 6 \cdot 3^{k-2} + 6 \cdot 2^{k-2} \\
&= 3 \cdot 3^{k-2} \cdot 5 - 2 \cdot 2^{k-2} \cdot 5 - 3^{k-2} \cdot 6 + 2^{k-2} \cdot 6 \\
&= 3^{k-2}(3 \cdot 5 - 6) + 2^{k-2}(6 - 2 \cdot 5) \\
&= 3^{k-2} \cdot 9 + 2^{k-2} \cdot (-4) \\
&= 3^k - 2^k
\end{aligned}$$

Thus, $5d_{k-1} - 6d_{k-2} = 3^k - 2^k$, which is the formula for d_k.
Therefore, the sequence d_n satisfies the recurrence relation $d_k = 5d_{k-1} - 6d_{k-2}$. □

Problem 28. Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$, for all integers $k \geq 1$.

Proof. Consider the Fibonacci sequence defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.
We need to prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$ for all integers $k \geq 1$.
The left-hand side of the equation is $F_{k+1}^2 - F_k^2 - F_{k-1}^2$.
Using the Fibonacci recurrence relation, $F_{k+1} = F_k + F_{k-1}$, we have:

$$\begin{aligned}
F_{k+1}^2 &= (F_k + F_{k-1})^2 \\
&= F_k^2 + 2F_kF_{k-1} + F_{k-1}^2
\end{aligned}$$

Substituting this into the left-hand side of the equation gives:

$$\begin{aligned}
F_{k+1}^2 - F_k^2 - F_{k-1}^2 &= F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 - F_k^2 - F_{k-1}^2 \\
&= 2F_kF_{k-1}
\end{aligned}$$

which is equal to the right-hand side of the given equation.
Therefore, $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$ is proven for all integers $k \geq 1$. □

Problem 44. The triangle inequality for absolute value states that for all real numbers a and b, $|a + b| \leq |a| + |b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|$$

Proof. We need to prove by induction that for all positive integers n, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|$$

Basis Step: For $n = 1$, we have $|a_1| \leq |a_1|$, which is trivially true.
Inductive Step: Assume the statement is true for some integer $k \geq 1$, i.e.,

$$\left| \sum_{i=1}^{k} a_i \right| \leq \sum_{i=1}^{k} |a_i|$$

We must show it is true for $k + 1$.
Consider

$$\sum_{i=1}^{k+1} a_i = \sum_{i=1}^{k} a_i + a_{k+1}$$

By the triangle inequality,

$$\left| \sum_{i=1}^{k+1} a_i \right| = \left| \sum_{i=1}^{k} a_i + a_{k+1} \right| \leq \left| \sum_{i=1}^{k} a_i \right| + |a_{k+1}|$$

Using the inductive hypothesis,

$$\left| \sum_{i=1}^{k} a_i \right| \leq \sum_{i=1}^{k} |a_i|$$

Thus,

$$\left| \sum_{i=1}^{k} a_i \right| + |a_{k+1}| \leq \sum_{i=1}^{k} |a_i| + |a_{k+1}| = \sum_{i=1}^{k+1} |a_i|$$

Hence, by mathematical induction, the statement holds for all positive integers n. $\square$