# Chapter 5

Sequences, Mathematical Induction, and Recursion

Ojas Chaturvedi

## Contents

# 5.1: Sequences

## Notes

- Sequence: a function whose domain is either all the integers between two given integers, or all the integers greater than or equal to a given integer.
  - Know subscript/index, initial and final term, infinite sequence, general/explicit formula
- Summation Notation:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

  where k is the index, m is the lower limit, and n is the upper limit.
- When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form. Expand the summation notation to first 3 or so, then put ellipsis and then variable form.
- Product Notation:

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$$

- Properties of Summations and Products (aka Theorem 5.1.1)

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k) \tag{1}$$

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k) \tag{2}$$

$$(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k) \tag{3}$$

- When replacing a new variable into a summation or product, make sure to change the index variable to the new variable and the numbers by putting them into the equation of the new variable.
- Factorial: the quantity **n!** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

  and

$$0! = 1$$

  Recursive definition:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n > 0 \end{cases}$$

- $n$ choose $r$: the number of subsets (therefore an integer) of size $r$ that can be chosen from a set of $n$ elements.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

  for all integers $n$ and $r$ with $0 \leq r \leq n$.

## Things to Check

- Recursive definition of summation pg 232

# 5.2: Mathematical Induction I

## Notes

- Principles of Mathematical Induction: Let $P(n)$ be a property that is defined for integers $n$, and let $a$ be a fixed integer. Suppose the following two statements are true:
  1. Basis Step: Show that $P(a)$ is true.
  2. Inductive Step: For all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.
     - To perform this step:
       (a) Suppose that $P(k)$ is true for an arbitrary integer $k \geq a$, which is called the inductive hypothesis.
       (b) Show that $P(k+1)$ is true.
     - Remember that you need to prove each side of the equation separately. Otherwise, the proof is invalid.
  3. Conclusion: Then $P(n)$ is true for all integers $n \geq a$.
- Steps of Proof by Mathematical Induction:
  1. State the theorem to be proved.
     - Let the property $P(n)$ be the equation: problem goes here
  2. Prove the basis step.
     - Show that $P(a)$ is true.
  3. State the inductive hypothesis.
     - Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:
  4. Prove the inductive step.
  5. State the conclusion.
     - Therefore the equation $P(k+1)$ is true [*as was to be shown*]. [*Since we have proved both the basis step and the inductive step, the conclusion follows by the principle of mathematical induction. Therefore the equation $P(n)$ is true for all integers $n \geq 1$.*]
- Sum of the first $n$ integers is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- Geometric sum of the first $n$ integers is

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

## Things to check

- Steps of proof and wording (check examples if need be) pg 247-248
- Check other method for solution to 5.2.4.b pg 283
- Cents problem pg 245

## 5.3: Mathematical Induction II

### Notes

- Enter here.

### Things to check

-

## Template for Mathematical Induction

Let the property $P(n)$ be the equation:

{problem goes here}

First, we must prove that $P(\{$smallest possible number goes here$\})$ is true (basis step).

Show left-hand side = right-hand side of the equation

Thus, $P(\{$smallest possible number goes here$\})$ is true.
Now, suppose that $P(k)$ is true for some integer $k \geq \{$smallest possible number goes here$\}$ (inductive hypothesis). That is,

{problem with k substituted goes here}

We must show that $P(k+1)$ is true (inductive step). That is,

{problem with (k+1) substituted goes here}

The left-hand side of $P(k+1)$ is:

{work with reasoning goes here}

The right-hand side of $P(k+1)$ is:

{work with reasoning goes here}

Thus, the left-hand side of $P(k+1)$ is equal to the right-hand side of $P(k+1)$, and $P(k+1)$ is true.
Since we have proved both the basis step and the inductive step, we conclude that the statement is true using mathematical induction.