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Problem 1a

Claim. The set of rational numbers ($\mathbb{Q}$) is countable.

Proof. Let $x \in \mathbb{Q}$. Then, by definition, we can write $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$. This fraction can be uniquely mapped to a tuple $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. This means that there exists an injective mapping $\mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ that maps each fraction to a tuple. Thus, we have:

$$|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{N}|$$

By the countability of $\mathbb{N}$, we have shown that $\mathbb{Q}$ is countable. □

Problem 1b

Let E be an event, and let $1_E : \blacksquare \to \{0, 1\}$ be an indicator random variable such that:

$$1_E(\omega) = \begin{cases} 1 & ! \in E \\ 0 & ! \notin E \end{cases}$$

Claim. The expectation of 1_E is equal to the probability that E occurs. In other words:

$$\mathbb{E}[1_E] = \mathbb{P}[E]$$

Proof.

$$\begin{aligned} \mathbb{E}[1_E] &= \mathbb{P}[1_E = 1] \cdot 1 + \mathbb{P}[1_E = 0] \cdot 0 \\ &= \mathbb{P}[1_E = 1] \\ &= \mathbb{P}[\{! \in \blacksquare : 1_E(!) = 1\}] \\ &= \mathbb{P}[E] \end{aligned}$$

□

Problem 2

Claim. $e^{iy} = \cos y + i \sin y$ for $y \in \mathbb{R}$

Proof. We begin by noting that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. Plugging in $x = yi$ for the exponent in e^x gives:

$$\begin{aligned}
e^{yi} &= \sum_{n=0}^{\infty} \frac{(yi)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} + \sum_{n=0}^{\infty} i \cdot \frac{y^{4n+1}}{(4n+1)!} + \sum_{n=0}^{\infty} -1 \cdot \frac{y^{4n+2}}{(4n+2)!} + \sum_{n=0}^{\infty} -i \cdot \frac{y^{4n+3}}{(4n+3)!} \\
&= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} - \frac{y^{4n+2}}{(4n+2)!} + i \left(\sum_{n=0}^{\infty} \frac{y^{4n+1}}{(4n+1)!} - \frac{y^{4n+3}}{(4n+3)!} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \right) \\
&= \cos y + i \sin y
\end{aligned}$$

This equality holds because the series representations of sin and cos are convergent. Thus, $e^{iy} = \cos y + i \sin y$ for $y \in \mathbb{R}$, as required. □

Problem 3

Claim. $\mathsf{P} \neq \mathsf{NP}$

Proof. We have already shown that $\mathsf{P} \subseteq \mathsf{NP}$, so to prove that $\mathsf{NP} \neq \mathsf{P}$ it suffices to show that $\mathsf{NP} \not\subseteq \mathsf{P}$. We begin by fixing a language $L \in \mathsf{NP}$.

$\boxed{\text{TO-DO}}$ $\square$

Problem 4

Claim. Two identical decks of n cards have a k-matching with probability:

$$\pi_k = \frac{1}{k!}\left(1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}\right)$$

Proof. First, we pick and order k cards to be matched. The probability of the selected card orders matching is $\frac{1}{k!}$.

Now, we consider the probability that the remaining $n - k$ cards *do not* match. Similar to the examples from class and the textbook, this is an instance of an indexed union of sets $A_i \in \mathcal{F}$ such that A_i is the set of permutations in which $f(i) = i$. But since we are looking for the probability of this not happening, we consider the probability of the complement, computed as 1 minus the probability of this union of events.

By the Inclusion-Exclusion principle, based on the notes from class, we have that:

$$\mathbb{P}\left[\bigcup_{i=1}^{n-k} A_i\right] = \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}$$

Putting everything together, we get the following, and we're done.

$$\pi_k = \frac{1}{k!}\left(1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}\right)$$

□